Graph Theory

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0 Outline

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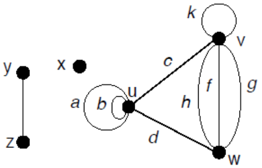
Definition (Degree of a Vertex (Textbook, Page 7,8))

1. **Adjacent vertices** are two vertices that are joined by an edge
2. **Adjacent edges** are two distinct edges that have an endpoint in common
3. If vertex v is an endpoint of edge e, then v is said to be **incident** on e, and e is **incident** on v
4. The **degree (or valence)** of a vertex v in a graph G, denoted $deg(v)$, is the number of proper edges incident on v plus twice the number of self-loops
5. A vertex of degree d is also called a **d-valent** vertex
6. The **degree sequence** of a graph is the sequence formed by arranging the vertex degrees in non-increasing order



- Are x and y adjacent vertices?
- Which edges are incident on vertex y?
- What is the degree sequence of the graph on left?
- Is there any 3-valent vertex?
- Can two different graphs have same degree sequence?

Theorem (Theorem-1)

A non-trivial simple graph G must have at least one pair of vertices whose degrees are equal.

Proof.

Textbook, Page 8, Prop. 1.1.1. $\square$

Theorem (Theorem-2)

[Euler's Degree-Sum Theorem] The sum of the degrees of the vertices of a graph is twice the number of edges.

Proof.

Textbook, Page 8, Th. 1.1.2 □

Theorem (Corollary)

In a graph, there is an even number of vertices having odd degree

Proof.

Textbook, Page 8, Cor. 1.1.3 $\square$

Theorem (Theorem-3)

Suppose that $\langle d_1, d_2, \ldots, d_n \rangle$ *is a sequence of nonnegative integers whose sum is even. Then there exists a graph with vertices* $v_1, v_2, \ldots, v_n$ *such that* $deg(v_i) = d_i$*, for* $i = 1, \ldots, n$.

Proof.

Textbook, Page 9, Th. 1.1.4. $\square$

Example (Graph corresponding to a degree sequence)

Construct a graph whose degree sequence is $\langle 5, 4, 3, 3, 2, 1, 0 \rangle$.

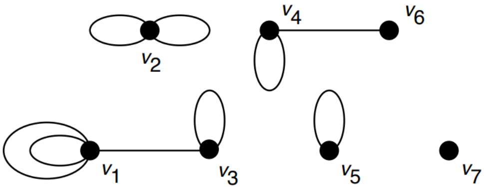


Figure: Graph corresponding to given degree sequence

Theorem (Graphic Sequence)

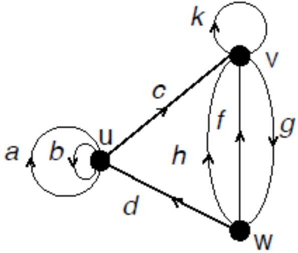
A non-increasing sequence $\langle d_1, d_2, \ldots, d_n \rangle$ *is said to be graphic if it is the degree sequence of some simple graph. That simple graph is said to realize the sequence. If* $\langle d_1, d_2, \ldots, d_n \rangle$ *is the degree sequence of a simple graph, then, clearly,* $d_1 \leq n - 1$.

Proof.

Textbook, Page 9, Remark. □

Definition (Indegree and Outdegree in a Digraph)

The indegree of a vertex v in a digraph is the number of arcs directed to v; the outdegree of vertex v is the number of arcs directed from v : Each self-loop at v counts one toward the indegree of v and one toward the outdegree.



vertex	u	v	w
indegree	3	4	1
outdegree	3	2	3

Theorem

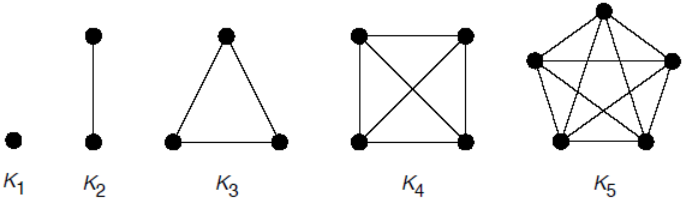
[Eulers Degree Sum] In a digraph, the sum of the indegrees and the sum the outdegrees both equal the number of edges.

Proof.

Textbook, Page 11, Th. 1.1.7. □

Definition (Complete Graphs)

A complete graph is a simple graph such that every pair of vertices is joined by an edge. Any complete graph on n vertices is denoted K_n



Definition (Bipartite Graphs)

A bipartite graph G is a graph whose vertex-set V can be partitioned into two subsets U and W, such that each edge of G has one endpoint in U and one endpoint in W. The pair U, W is called a (vertex) bipartition of G, and U and W are called the bipartition subsets.

Example

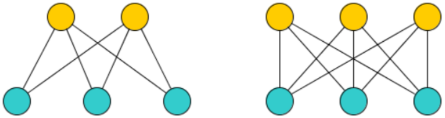
What is a smallest possible simple graph that is not bipartite?

Theorem

A bipartite graph cannot have any self-loops.

Definition (Complete Bipartite Graph)

A complete bipartite graph is a simple bipartite graph such that every vertex in one of the bipartition subsets is joined to every vertex in the other bipartition subset. Any complete bipartite graph that has m vertices in one of its bipartition subsets and n vertices in the other is denoted $K_{m,n}$.



Answer the following:

1. What are m, n in above graphs?
2. What are the number of edges? Is there a formula to count number of edges?
3. What could be the applications of bipartite graphs?