# Graph Theory

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Pawan Kumar

International Institute of Information Technology, Hyderabad

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# 0 Outline

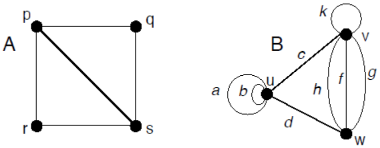
# 1 Outline

# 1 Terminology for graphical objects



### Definition (Graph)

A graph $G = (V; E)$ is a mathematical structure consisting of two finite sets $V$ and $E$. The elements of $V$ are called vertices (or nodes), and the elements of $E$ are called edges. Each edge has a set of one or two vertices associated to it, which are called its endpoints.




1 Answer the following:

- $V_A =$ $\qquad$ $E_A =$
- $V_B =$ $\qquad$ $E_B =$

# 1 Terminology for graphical objects

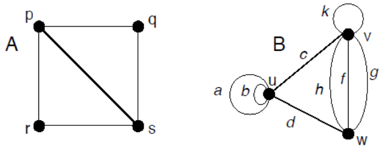


### Definition (Neighbor of Graph)

An edge is said to join its endpoints. A vertex joined by an edge to a vertex $v$ is said to be a neighbor of $v$.

### Definition (Neighborhood)

The (open) neighborhood of a vertex $v$ in a graph $G$, denoted $N(v)$, is the set of all the neighbors of $v$ : The closed neighborhood of $v$ is given by $N[v] = N(v) \cup \{v\}$.




1. Answer the following:
   - $N(p) =$ $\qquad$ $N(w) =$
   - $N[p] =$ $\qquad$ $N[w] =$

### Definition (Simple Graph and Multigraph)

1. A proper edge is an edge that joins two distinct vertices.
2. A self-loop is an edge that joins a single endpoint to itself
3. A multi-edge is a collection of two or more edges having identical endpoints. The edge multiplicity is the number of edges within the multi-edge
4. A simple graph has neither self-loops nor multi-edges
5. A loopless graph (or multi-graph) may have multi-edges but no selfloops
6. A (general) graph may have self-loops and/or multi-edges

# 1 Terminology for graphical objects



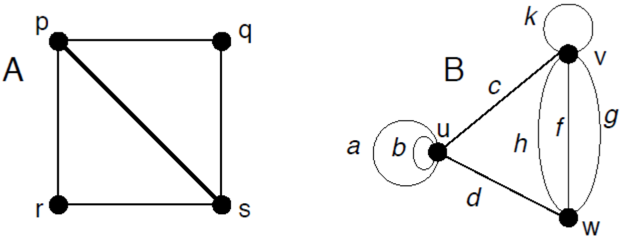



Figure 1.1: ***Simple graph*** $A$; ***graph*** $B$.

Answer the following:

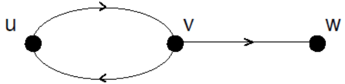
1. How many self loops are there in graph $B$?
2. How many multiedges are there in graph $A$?
3. Make graph $B$ simple by removing edges

**Definition (Digraph: Directed Graphs)**

1. A directed edge (or arc) is an edge, one of whose endpoints is designated as the tail, and whose other endpoint is designated as the head
2. An arc is said to be directed from its tail to its head
3. Two arcs between a pair of vertices are said to be oppositely directed if they do not have the same head and tail
4. A multi-arc is a set of two or more arcs having the same tail and same head. The arc multiplicity is the number of arcs within the multi-arc
5. A directed graph (or digraph) is a graph each of whose edges is directed
6. A digraph is simple if it has neither self-loops nor multi-arcs

# 1 Terminology for graphical objects






1 Identify directed edges: identify head and tail
2 Identify arcs that are opposite
3 Identify multi-arcs in the graph
4 Is the above graph directed graph?
5 Is the above graph a digraph?

# 1 Terminology for graphical objects



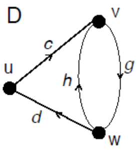
## Definition (Null Graphs and Trivial Graphs)

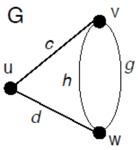
1. A null graph is a graph whose vertex- and edge-sets are empty
2. A trivial graph is a graph consisting of one vertex and no edges

# 1 Digraph, Mixed Graphs, Underlying Graphs...

## Definition (Digraph, Mixed Graphs, etc)

1. A **mixed graph** (or partially directed graph) is a graph that has both undirected and directed edges
2. The underlying graph of a directed or mixed graph G is the graph that results from removing all the designations of head and tail from the directed edges of G (i.e., deleting all the edge-directions).



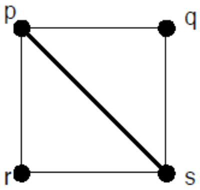




Answer the following:

1. Which of the graphs on left is a mixed graph?

## Definition (Formal Specification of Graphs and Digraphs)

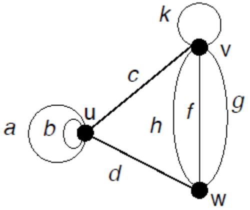
1. A formal specification of a simple graph is given by an adjacency table with a row for each vertex, containing the list of neighbors of that vertex




| | | | |
|---|---|---|---|
| $p:$ | $q$ | $r$ | $s$ |
| $q:$ | $p$ | $s$ | |
| $r:$ | $p$ | $s$ | |
| $s:$ | $p$ | $q$ | $r$ |

## Definition (Formal Specification of Graphs and Digraphs)

2 A formal specification of a general graph $G = (V; E; \text{endpts})$ consists of a list of its vertices, a list of its edges, and a two-row incidence table (specifying the endpts function) whose columns are indexed by the edges. The entries in the column corresponding to edge $e$ are the endpoints of $e$. The same endpoint appears twice if $e$ is a self-loop. (An isolated vertex will appear only in the vertex list.)

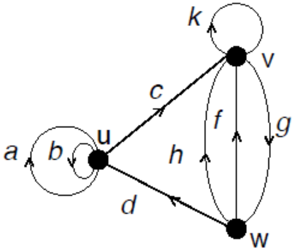



$V = \{u, v, w\}$ and $E = \{a, b, c, d, f, g, h, k\}$

| edge | $a$ | $b$ | $c$ | $d$ | $f$ | $g$ | $h$ | $k$ |
|---|---|---|---|---|---|---|---|---|
| endpts | $u$ | $u$ | $u$ | $w$ | $v$ | $v$ | $w$ | $v$ |
| | $u$ | $u$ | $v$ | $u$ | $w$ | $w$ | $v$ | $v$ |

## Definition (Specification of a general digraph)

A formal specification of a general digraph or a mixed graph $D = (V; E; endpts; head; tail)$ is obtained from the formal specification of the underlying graph by adding the functions $head : E_G \to V_G$ and $tail : E_G \to V_G$, which designate the head vertex and tail vertex of each arc.

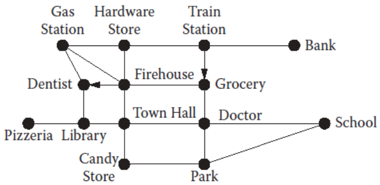



| edge | $a$ | $b$ | $c$ | $d$ | $f$ | $g$ | $h$ | $k$ |
|---|---|---|---|---|---|---|---|---|
| endpts | $u$ | $u^h$ | $u$ | $w$ | $v^h$ | $v$ | $w$ | $v$ |
| | $u^h$ | $u$ | $v^h$ | $u^h$ | $w$ | $w^h$ | $v^h$ | $v^h$ |

$$head(a) = tail(a) = head(b) = tail(b) = head(d) = tail(c) = u;$$
$$head(c) = head(h) = head(f) = tail(g) = head(k) = tail(k) = v;$$
$$head(g) = tail(d) = tail(h) = tail(f) = w.$$

## Example (Example-1)

The mixed graph in Figure on the right is a model for a roadmap. The vertices represent landmarks, and the directed and undirected edges represent the one-way and two-way streets, respectively.




## Example (Example-2)

The digraph in Figure on the right represents the hierarchy within a company. This illustrates how, beyond physical networks, graphs and digraphs are used to model social relationships.

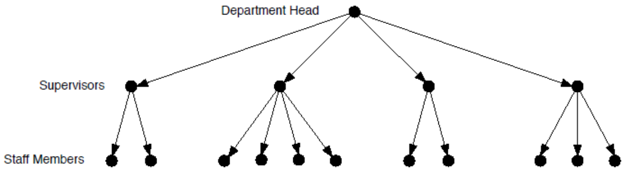



**Figure 1.1.8** A corporate hierarchy.