Graph Theory

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0 Outline

Definition (Walks)

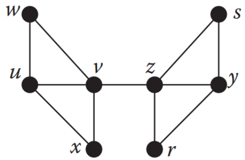
In a graph G, a walk from vertex v_0 to vertex v_n is an alternating sequence $W = \langle v_0, e_1, v_1, e_2, \ldots, v_n, e_n, v_n \rangle$ of vertices and edges, such that $endpts(e_i) = \{v_{i-1}, v_i\}$ for $i = 1, \ldots, n$. If G is a digraph (or mixed graph), then W is a directed walk if each edge e_i is directed from vertex v_{i-1} to vertex v_i, i.e., $tail(e_i) = v_{i-1}$ and $head(e_i) = v_i$.

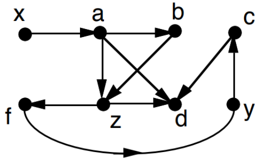
In first graph on right:

- How many walks of length 2 are there?
- Are there any walks of length 10?
- Can there be walks of arbitrary large lengths?

In second graph on right:

- How many walks of length 2 are there?
- Are there any walks of length 10?





- In a simple graph, there is only one edge between two consecutive vertices of a walk, so one could abbreviate the representation as a vertex sequence

$$W = \langle v_0, v_1, \dots, v_n \rangle$$

- In a general graph, one might abbreviate the representation as an edge sequence from the starting vertex to the destination vertex

$$W = \langle v_0, e_1, e_2, \dots, e_n, v_n \rangle$$

> Is $\langle y, z, y, z \rangle$ a valid walk?
> Is $\langle u, a, b, b, c, f, g, v \rangle$ a valid walk?
> Can there exist a walk of arbitrary length?

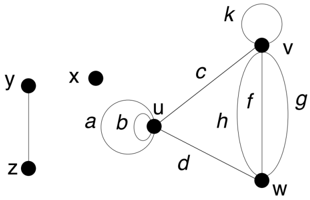


Figure: Walk on General Graphs

- The length of a walk or directed walk is the number of edge-steps in the walk sequence.
- A walk of length zero, i.e., with one vertex and no edges, is called a trivial walk.
- A closed walk (or closed directed walk) is a nontrivial walk (or directed walk) that begins and ends at the same vertex. An open walk (or open directed walk) begins and ends at different vertices.
- The distance $d(s,t)$ from a vertex s to a vertex t in a graph G is the length of a shortest $s-t$ walk if one exists; otherwise, $d(s,t) = \infty$.

- What is the length of the walk $\langle y, z, y, z \rangle$?
- What is the length $\langle u, a, b, b, c, f, g, v \rangle$?
- Is $\langle u, c, f, v \rangle$ a closed walk?
- What is a trivial walk?
- What is the value $d(x, y)$?
- What is $d(u, w)$?

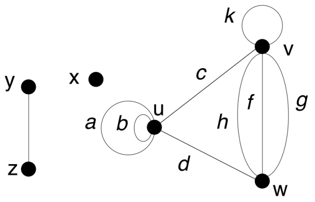


Figure: Walk on General Graphs

Definition (eccentricity)

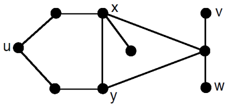
The eccentricity of a vertex v, denoted $ecc(v)$, is the distance from v to a vertex farthest from v. That is,

$$ecc(v) = \max_{x \in V_G}\{d(v,x)\}$$

Definition (diameter)

The diameter of a graph is the max of its eccentricities, or, equivalently, the max distance between two vertices, i.e.,

$$diam(G) = \max_{x \in V_G}\{ecc(x)\} = \max_{x,y \in V_G}\{d(x,y)\}$$



Answer the following:

1. What is the diameter?
2. What are the eccentries of vertices x and y?

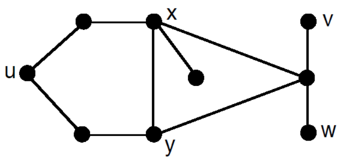
Definition (radius)

The radius of a graph G, denoted rad(G), is the min of the vertex eccentricities. That is,

$$rad(G) = \min_{x \in V_G} \{ecc(x)\}$$

Definition (central vertex)

A central vertex v of a graph G is a vertex with min eccentricity. Thus, $ecc(v) = rad(G)$.



Answer the following:

1. What is the radius of the graph?
2. What are central vertices?

Definition

Vertex v is reachable from vertex u if there is a walk from u to v.

Definition

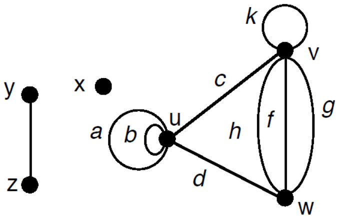
A graph is connected if for every pair of vertices u and v, there is a walk from u to v.

Definition

A digraph is connected if its underlying graph is connected.

Definition

The non-connected graph in Figure below is made up of connected pieces called components.



Definition (Trail)

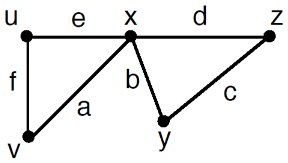
A trail is a walk with no repeated edges.

Definition (Path)

A path is a trail with no repeated vertices (except possibly the initial and final vertices).

Definition (Trivial: walk, path, trail)

A walk, trail, or path is trivial if it has only one vertex and no edges.



$W = \langle v, a, e, f, a, d, z \rangle$
$T = \langle v, a, b, c, d, e, u \rangle$

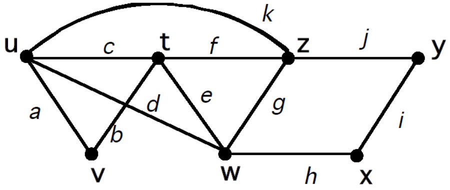
Is the walk W trail? Is trail T, a path?

Definition (cycle)

A nontrivial closed path is called a cycle. It is called an odd cycle or an even cycle, depending on the parity of its length.

Definition (acyclic)

An acyclic graph is a graph that has no cycles.



Answer the following:

1. Identify the cycles. How many cycles are there?
2. How many odd and even cycles?

Definition (Eulerian trail)

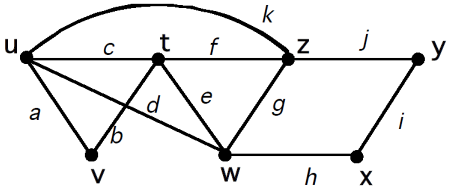
An Eulerian trail in a graph is a trail that contains every edge of that graph.

Definition (Eulerian tour)

An Eulerian tour is a closed Eulerian trail.

Definition (Eulerian graph)

An Eulerian graph is a graph that has an Eulerian tour.



Answer the following:

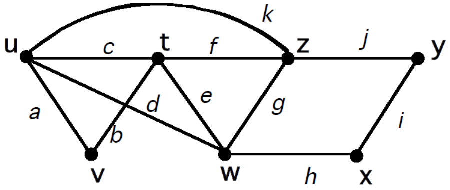
1. Is this a Eulerian graph?

Definition (Hamiltonian cycle)

A cycle that includes every vertex of a graph is call a hamiltonian cycle.

Definition (Hamiltonian graph)

A hamiltonian graph is a graph that has a hamiltonian cycle.

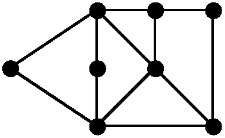


Answer the following:

1. Is this a Hamiltonian graph?

Definition (Girth)

The girth of a graph with at least one cycle is the length of a shortest cycle. The girth of an acyclic graph is undefined.

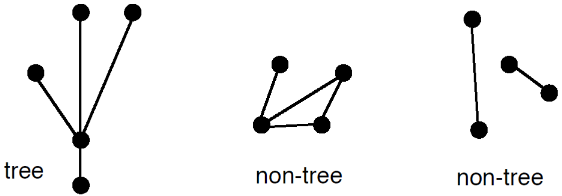


Answer the following:

1. What is the girth of the graph?

Definition (Tree)

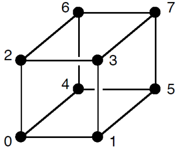
A tree is a connected graph that has no cycles.

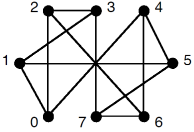


Answer the following:

1. Which of these are trees?

0 How to identify same graphs?





Are these same graphs?
They are (clearly) the "same" because each vertex v has the exact same set of neighbors in both graphs.

0.	1	2	4
1.	0	3	5
2.	0	4	6
3.	1	2	7
4.	0	5	6
5.	1	4	6
6.	2	4	7
7.	3	5	6

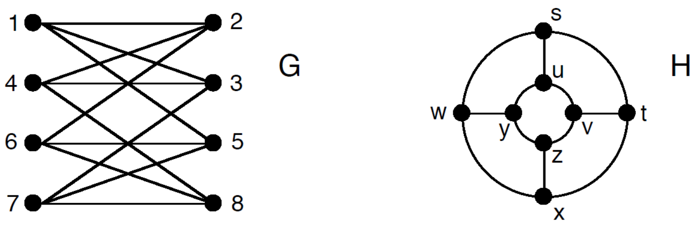


Figure: Are these same graphs?

These two graphs are the "same" because instead of having the same set of vertices, this time we have a bijection $V_G \to V_H$

$$1 \to s \quad 2 \to t \quad 3 \to u \quad 4 \to v$$
$$5 \to w \quad 6 \to x \quad 7 \to y \quad 8 \to z$$

between the two vertex sets, such that neighborhoods map bijectively to neighborhoods.