

Lecture 7 – Binary Logic

Dr. Aftab M. Hussain,
Associate Professor, PATRIOT Lab, CVEST

Chapter 2

Postulates of Boolean algebra – Duality principle

- The *duality principle* states that every algebraic expression deducible from the postulates of Boolean algebra remains valid if the operators and identity elements are interchanged, i.e., AND is changed to OR and vice versa
- In a two-valued Boolean algebra, the identity elements and the elements of the set S are the same: 1 and 0
- Thus, if the *dual* of an algebraic expression is desired, we simply interchange OR and AND operators and replace 1's by 0's and 0's by 1's



$$(a) \quad x + 0 = x$$

$$(a) \quad x + x' = 1$$

$$(a) \quad x + x = x$$

$$(a) \quad x + 1 = 1$$

$$(b) \quad x \cdot 1 = x$$

$$(b) \quad x \cdot x' = 0$$

$$(b) \quad x \cdot x = x$$

$$(b) \quad x \cdot 0 = 0$$

Theorems of Boolean algebra

- Theorem: $x + x = x$

$$\begin{aligned}x + x &= (x + x) \cdot 1 \\&= (x + x) \cdot (x + x') \\&= x + x \cdot x' \\&= x + 0 \\&= x\end{aligned}$$

- Its dual: $x \cdot x = x$

Theorems of Boolean algebra

- Theorem: $x + 1 = 1$

$$\begin{aligned}x + 1 &= 1 \cdot (x + 1) \\&= (x + x') \cdot (x + 1) \\&= x + x' \cdot 1 \\&= x + x' \\&= 1\end{aligned}$$

- Its dual: $x \cdot 0 = 0$

Theorems of Boolean algebra

- Theorem: $x + xy = x$ (absorption theorem)

$$\begin{aligned}x + xy &= x \cdot 1 + xy \\&= x \cdot (1 + y) \\&= x \cdot 1 \\&= x\end{aligned}$$

- Its dual: $x \cdot (x + y) = x$

Theorems of Boolean algebra

- Theorem: $(x + y)' = x' \cdot y'$ (first DeMorgan's theorem)
- A simple way to prove the theorem is to prove that $(x+y)$ and $x'y'$ are complements of each other so that $(x+y)'$ is the same as $x'y'$
- So, we need to prove that $(x + y) + x'y' = 1$ and $(x + y) \cdot x'y' = 0$. If they are both true, then $(x + y)' = x'y'$

$$\begin{aligned}(x + y) + x'y' &= x + y + x'y' \\ &= x + (y + x')(y + y') \\ &= x + (y + x') \cdot 1 \\ &= x + y + x' \\ &= y + (x + x') \\ &= y + 1 \\ &= 1\end{aligned}$$

Theorems of Boolean algebra

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$$\begin{aligned}(x + y) \cdot x'y' &= x \cdot x' \cdot y' + y \cdot x' \cdot y' \\&= (x \cdot x') \cdot y' + x' \cdot (y \cdot y') \\&= 0 \cdot y' + x' \cdot 0 \\&= 0 + 0 \\&= 0\end{aligned}$$

- Because both are true together, the theorem is proved

Operator precedence

- The operator precedence for evaluating Boolean expressions is (1) parentheses, (2) NOT, (3) AND, and (4) OR. This is similar to BODMAS
- In other words, expressions inside parentheses must be evaluated before all other operations
- The next operation that holds precedence is the complement, and then follows the AND and, finally, the OR
- As an example, consider one of DeMorgan's theorems
- The left side of the expression is $(x + y)'$. Therefore, the expression inside the parentheses is evaluated first and the result then complemented
- The right side of the expression is $x'y'$, so the complement of x and the complement of y are both evaluated first and the result is then ANDed

Boolean functions

- In normal algebra, we define functions on real number variables using addition, subtraction, exponents, etc.
- A Boolean function described by a Boolean expression consists of binary variables, the constants 0 and 1, and the logic operation symbols
- For a given value of the binary variables, the function can be equal to either 1 or 0 (closure on the Boolean set)
- As an example, consider the Boolean function: $F = x + y \cdot z$
- A Boolean function expresses the logical relationship between binary variables and is evaluated by determining the binary value of the expression for all possible values of the variable (truth table of the function)

Boolean functions

- A Boolean function can be represented in a truth table
- The number of rows in the truth table is 2^n , where n is the number of variables in the function
- An easy way for obtaining binary combinations for the truth table are by counting the binary numbers from 0 through $(2^n - 1)$
- For example, there are eight possible binary combinations for assigning bits to the three variables x , y , and z

Boolean functions

- There is only one way that a Boolean function can be represented in a truth table
- However, when the function is in algebraic form, it can be expressed in a variety of ways, all of which have equivalent logic
- Here is a key fact that motivates our use of Boolean algebra: By manipulating a Boolean expression according to the rules of Boolean algebra, it is sometimes possible to obtain a simpler expression for the same logic function, and thus reduce the complexity of the circuit representing that logic
- Consider, for example, the following Boolean functions:

$$F_1 = x'yz + xy'z + xyz$$

$$F_2 = x'yz + yz$$

- Can we simplify these?

Boolean functions

- A simplified function also has a simplified circuit implementation
- Therefore, the two circuits have the same outputs for all possible binary combinations of inputs (truth table)
- Thus, each circuit implements the same identical function, but the one with fewer components is preferable
- In general, there are many equivalent circuit representations of a logic function
- Finding the most economic representation of the logic is an important design task

Boolean functions

- The manipulation of Boolean algebra consists mostly of reducing an expression for the purpose of obtaining a simpler circuit
- Functions of up to five variables can be simplified by the map method which will be discussed later
- For complex Boolean functions and many different outputs, designers of digital circuits use computer minimization programs that are capable of producing optimal circuits
- The manual method available is a procedure employing the basic relations and other manipulation techniques that become familiar with use, but remain relatively less in use
- Example: Simplify $F = xy + x'z + yz$ (consensus theorem)