① Let the no., $a_i = k_i \bmod n$
where k_i is an arbitrary remainder.
$\Rightarrow k_i \in [0, n-1)$ $\therefore$ n distinct k_i

~~Case~~ $\overset{n}{}$, If $k_i = k_j \Rightarrow k_i - k_j = 0$
$\Rightarrow a_i = k_i \bmod n$
& $a_j = k_j \bmod n$
$\cancel{\Rightarrow (k_i - k_j}$
$\Rightarrow a_i - a_j = (k_i - k_j) \bmod n$
$= 0 \bmod n$
$\therefore$ $\cancel{a_i - a_j}$ $n \mid a_i - a_j$

~~Case (b)~~,
& There are $(n+1)$ distinct no. & only
n distinct k_i
$\therefore$ Acc. to Pigeon hole principle, there
exist a k_i pair.

② Let all points at distance $d \geq 1/7$.

$\therefore$ max n(points) in Row $= 1/1/7 = 7$

Illy, max n (points) in Column $= 1/1/7 = 7$

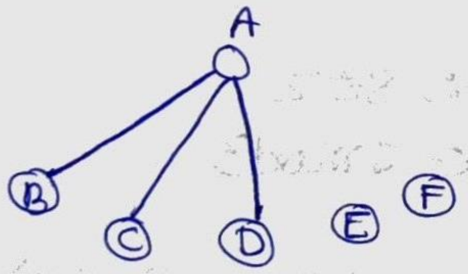
$\Rightarrow$ max n(points) $= 7 \times 7 = 49$.

$\Rightarrow$ There can only be 49 points or less greater than 1/7 distance.

$\because$ Circle can consist of ~~two~~ 2/7 distace $\Rightarrow$ 2 dots & Due to Pigeon hole principle, 2 dots will be at distance closer than 1/7.

$\Rightarrow$ There exist three pair.

③ Let Assume A having different possibility



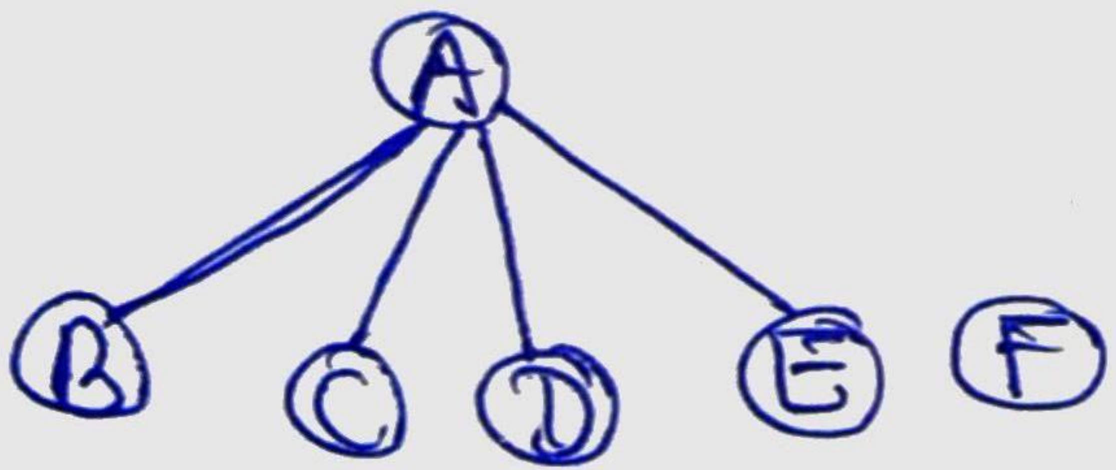
Either He can be friend with 3+ People or He is not friend with 3+ People.

~~Acc. to~~

~~For minimal Case~~,

If A is ~~friend with~~ Knows BCD / stranger with ~~AEF~~ BCD then AEF will prove that $\exists$ 3 who are ~~Friend~~ Known/Stranger respectively.

If A ~~find~~ knows with 4+,



Now for B, C, D, E, if any of them knows eachother,
We will get Δ, A XY, [X, Y $\in$ B,C,D,E]

& If B, C, D, E doesn't know each other, then we will get 3 ~~pairs of~~ strangers who doesn't know each other.

4. Soln - If we connect all 5 comp. with all 5 routers directly, $n(\text{Connection}) = 5 \times 5 = 25$

$n(\text{Other Connection}) = 5$ {from com. to network}

$\Rightarrow n(\text{Total Connection}) = 25 + 5 = 30$

5. Soln - ~~Let~~ Consider ~~n+1~~ no. of form 1, 11, 111, ...

$1 . \overset{(n-1)}{1} 1$.

If we divide it by n, Acc. to Pigeon hole priciple, there must be atleast two same reminder. & would be n digits apart.

~~as 1 = K mod n~~ $K_1 = p \bmod n = tn + p$

~~⇒ 11 = 11 (K mod n)~~ $K_2 = p \bmod n = (t+s)n + p$

⇒ Their difference = Multiple of n = sn

Hence proved.

6 Soln - In 2n nos there are n even and n odd numbers

For ∀ even no. ∃ a pair ∴ there are n/2 pairs.

(∵ for $k < n$, ∃ $k \leq 2n$)

~~Even~~ No. $= 2^p \times K$, where K is odd no.

∵ There are only n odd K ∴ There can only be n distinct no. with different factors but there are n+1 no.

∴ ∃ a no. dividing another no.

{According to Pigeon hole Principle}.

7. Sol^n - (a) A is Countable

$\Rightarrow \exists$ bijective $f : N \longrightarrow A$

$\Rightarrow \exists$ Surjective $f : N \rightarrow A$ (b) Proved

$\because f : N \longrightarrow A$ is Surjective & Injective

$\Rightarrow \exists f^{-1} : A \rightarrow N$ is bijective

$\therefore f^{-1} \equiv g : A \rightarrow N$ is injective (c) Proved

Hence, $a \equiv b \equiv c$

8. Soln :-

Q \ P	1	2	3	4 ......
1	1	2	3	4
2	1/2	1	3/2	2 .....
3	1/3	2/3	1	4/3 .....
4	1/4	1/2	3/2	1 ...
5	1/5	⋮	⋮	⋮

∵ All Rational⊕ no. can be represented as P/Q grid of Natural no.

∴ We can count using particular pattern

⇒ Countable

9 Sol^n: $\because$ A is Countable, B is Countable

$\Rightarrow \exists f : N \to A \mid f$ is Bijective

Ily, $\exists g : N \to B \mid g$ is Bijective

$\therefore \exists K : (N \times N) \to (A \times B) \mid K$ is bijective

$\therefore$ $A \times B$ is Countable Sn.

Or,

	a_1	a_2	a_3	...
b_1	(a_1, b_1)	(a_2, b_1)	(a_3, b_1)	...
b_2	$(a_1 b_2)$	$(a_2 b_2)$	$(a_3 b_2)$	...
b_3	$(a_1 b_3)$	$(a_2 b_3)$	$(a_3 b_3$	...
b_4	(a_1, b_4)			

10 Sol^n - (a) Assume $x \in (A \cup B) \Rightarrow x \in A$ or $x \in B$

$f(x) \in f(A)$ or $f(B)$

$f(x) \in f(A) \cup f(B)$ — ①

Case ① If $x \in A$ & $x \notin B \Rightarrow x \in A \cup B \therefore \cancel{f(A \cup B)}$
$f(x) \subset f(A \cup B)$

Case ② Ily, If $x \in B$ & $x \notin B \Rightarrow x \in A \cup B \therefore f(x) \subset f(A \cup B)$ — ②

$\Rightarrow f(x) = f(A) \cup f(B)$

{from ① and ②}

ⓑ $f(A \cap B) \subseteq f(A) \cap f(B)$

Let Assume $x \in A \cap B \Rightarrow x \in A$ & $x \in B$

$\Rightarrow f(x) \in f(A)$ and $f(x) \in f(B)$

$\therefore f(x) \in f(A) \cap f(B)$

$\Rightarrow f(A \cap B) \subseteq f(A) \cap f(B)$

ⓒ $f(A-B) \subseteq f(A) - f(B)$, If f is injective

Let $x \in A-B \Rightarrow x \in A$ & $\notin B$

$\therefore f(x) \in f(A) \notin f(B)$

If f is not injective,

$f(x) \in f(A)$, but $f(y) \in f(B)$, for another value than
$\underline{\text{Can be}}$ B. | $f(x) = f(y)$

$\therefore \forall f$ injective, $f(x) \in f(A) - f(B)$

11. ⓐ $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

$\Rightarrow f(f^{-1}(A \cup B)) = f(f^{-1}(A) \cup f^{-1}(B))$

$\Rightarrow A \cup B = f(f^{-1}(A)) \cup f(f^{-1}(B))$ {ⓐ Property}

$\therefore A \cup B = A \cup B$ {LHS = RHS} Hence, Proved

ⓑ $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

$\Rightarrow f(f^{-1}(A \cap B)) = f(f^{-1}(A) \cap f^{-1}(B))$.

$\Rightarrow A \cap B \subseteq f(f^{-1}(A)) \cap f(f^{-1}(B))$ {ⓑ property}

$\Rightarrow A \cap B \subseteq A \cap B$ ($\because f$ is bijective)

ⓒ $f^{-1}(A-B) = f^{-1}(A) - f^{-1}(B)$

$\Rightarrow f(f^{-1}(A-B)) \subseteq f(f^{-1}(A)) - f(f^{-1}(B))$ ($\because f$ is injective as Inverse exists)

$\Rightarrow (A-B) \subseteq (A-B)$

(13) (a) $f(x) = \frac{x}{1+x^2}$

Domain: $1+x^2 \neq 0$ (Not Possible)

$\therefore$ ~~Domain $\in$ R~~ Domain $= R$

Range: $f(x) = \frac{1}{1/x + x}$

$\because \left(\frac{1}{x} + x\right) \in [-\infty, -2] \cup [2, \infty)$

$\therefore f(x) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

(b) $2 - |x-5|$

Domain: R

Range: $\because |x-5| \geq 0$

$\therefore$ ~~Range $\in$~~ $f(x) \in (-\infty, 2]$

(c) $f(x) = \frac{x^2}{1+x^2}$

Domain: R

Range: $f(x) = \frac{1}{\frac{1}{x^2} + 1}$

~~$x^2 \in [0, \infty)$~~

$x \in (-\infty, \infty) \Rightarrow x^2 \in [0, \infty)$

$\therefore 1/x^2 \in (0, \infty)$

$\Rightarrow \frac{1}{x^2} + 1 \in (1, \infty)$

$\therefore f(x) \in (0, 1)$

(d) $f(x) = \frac{(x-2)(x-1)}{(x-2)}$

Domain: $x \neq 2$ $\because f(x) = \frac{K}{0}$

$\Rightarrow$ Undefined

Range:- $f(x) = (x-1)$ $\left(\because \frac{\cancel{(x-2)}(x-1)}{\cancel{(x-2)}}\right)$

$\because x \in (-\infty, \infty)$

$\therefore x - 1 \in (-\infty, \infty)$

$\Rightarrow f(x) \in (-\infty, \infty)$

(e) $f(x) = \log_2 (x - x^2)$

(1) $x - x^2 \neq 0 \Rightarrow x(1-x) \neq 0 \therefore x \neq 0, x \neq 1$

Domain

(2) $\cancel{x - x^2 \in (0, \infty)}$

$x - x^2 > 0$

$\therefore x(1-x) > 0$

$\Rightarrow x(x-1) < 0$

$\Rightarrow$ + (0) − (1) +

$\therefore x(x-1) \in [0, 1]$

~~But $x(x-1) \in [1, \infty)$~~

② Range

$\because x(1-x) \in \cancel{(\tfrac{1}{4},} (-\infty, \tfrac{1}{4})$

$\therefore \log_2(x(1-x)) \in (\log_2 0^+, \log_2 \tfrac{1}{4})$

$\therefore f(x) \in (-\infty, -2)$

$\because$ Domain $\in (0, 1)$

12. Soln - Case (a) $Ka \geq Kb$

$f(Ka) > f(Kb)$ {for Increasing fn}

$f(Ka) < f(Kb)$ {for decreasing fn}

$\Rightarrow \forall\, Ka > Kb,\ f(Ka) \neq f(Kb)$ & Vice-versa
Contraindic

Case (b) $Ka < Kb$

$f(Ka) < f(Kb)$ {for Increasing fn}

$f(Ka) > f(Kb)$ {for decreasing fn}

$\Rightarrow \forall\, Ka < Kb,\ f(Ka) \neq f(Kb)$ & Vice versa
Contraindica

$\therefore \exists\, f(Ka) = f(Kb) \Leftrightarrow Ka = Kb$

$\therefore$ f is injective (Hence proved)

1~~4~~. Soln - (a) $f(2/3) = |2/3 - 1| = |-1/3| = 1/3$
(a)

(b) $goh(1/2)$:

$h(1/2) = 2 \times \frac{1}{2} + 1 = 2$

$g(h(1/2)) = g(2) = \frac{4}{4-3} = 4$

(c) $fof(-2)$:

$f(-2) = |-2-1| = |-3| = 3$

$fof(-2) = f(3) = |3-1| = |2| = 2$

(d) $foh(x)$:

$h(x) = 2x+1$, ~~$g(h(x)) = g(2x+1)$~~

$f(h(x)) = f(2x+1)$

$= |(2x+1) - 1|$

$= |2x|$

~~$= \frac{2 \times (2x+1)}{(2x+1)^2 - 3}$~~

~~$= \frac{2(2x+1)}{4x^2+4x-2}$~~

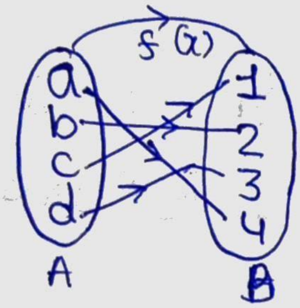
~~$= \frac{2x+1}{2x^2+2x+1}$~~

(e) $fog(x)$ & $g(x) = \frac{2x}{x^2-3}$

$fog(x) = f\left(\frac{2x}{x^2-3}\right)$

$= \left|\frac{2x}{x^2-3} - 1\right| = \left|\frac{2x - x^2 + 3}{x^2-3}\right| = \left|\frac{(x+1)}{(x+3)}\right|$

15. Sol^n - (a)



$\because \forall\, a_i \in A,\ f(a_i) \neq f(a_j)$

$\Rightarrow$ Injective

$\forall\, b \in B,\ \nexists\, b \mid f(a) \neq b\ \forall\, a \in A.$

$\therefore$ Surjective

(b) Let $f(x_1) = f(x_2)$.

$\Rightarrow \frac{(x_1)^2 + 1}{(x_1)^2 + 2} = \frac{(x_2)^2 + 1}{(x_2)^2 + 2}$

$\Rightarrow \cancel{(x_1)^2(x_2)^2} + 2(x_1)^2 + (x_2)^2 + \cancel{2} = \cancel{(x_2)^2(x_1)^2} + (x_1)^2 + 2(x_2)^2 + \cancel{2}$

$\Rightarrow 2\{(x_1)^2 - (x_2)^2\} = ((x_1)^2 - (x_2)^2)$

$\Rightarrow (x_1)^2 - (x_2)^2 = 0 \Rightarrow (x_1)^2 = (x_2)^2 \Rightarrow (x_1 - x_2)(x_1 + x_2) = 0$

$\Rightarrow x_1 = -x_2$ or $x_1 = x_2$

$\therefore$ Not Injective

$f(x) = 1 - \frac{1}{(x^2+2)}$ $\quad \because x^2 + 2 \in [2, \infty)$

$\Rightarrow \frac{1}{x^2+2} \in \left(0, \frac{1}{2}\right]$

$\Rightarrow f(x) \in \left(1, \frac{1}{2}\right] \neq R$

$\therefore$ Not Surjective

(c) For, $n = 2, 3$, $f(n) = \left[\frac{2}{2}\right] = 1$, $\left[\frac{3}{2}\right] = 1$

$\Rightarrow f(2) = f(3)$, but $2 \neq 3$ $\therefore$ Not Injective

$f(n) = [n/2]$ $\because n \in Z$ $\therefore n/2 \in I$ & $[n/2] \in Z$

$\Rightarrow f(n) \in Z$

$\therefore$ Surjective

(d) $f: R \to R$, $f(x) = e^x$

$\because$ e^x is Increasing fn $\therefore$ e^x is Injective

Range of $e^x \in (0, \infty) \neq R$ $\therefore$ e^x is not Surjective

(e) $f: R \to R^+$, $f(x)\ e^x$

$\because$ e^x is always Increasing fn $\therefore$ e^x is Injective

Range of $e^x \in (0, \infty) = R^+$ $\therefore$ e^x is Surjective

16. (a) let $f(x_1) = f(x_2)$

$\Rightarrow (x_1)^2 = (x_2)^2 \Rightarrow (x_1)^2 - (x_2)^2 = 0 \Rightarrow (x_1 - x_2)(x_1 + x_2) = 0$

$\therefore x_1 = x_2$ or $x_1 = -x_2$

$\Rightarrow$ Non-Injective.

$\forall\ x \in N$, $f(x) = x^2 \in N \equiv$ Codomain

$\therefore$ Surjective $\Rightarrow$ Right Invertible

~~(b) $f: Z \to Z$ | $f(x) = \begin{cases} n+4, & n = 3K, K \in Z \\ -n-3, & n = 3K+1, K \in Z \\ n+1, & n = 3K+2, K \in Z \end{cases}$~~

~~$\therefore f(K) = \begin{cases} 3K+4, & K \in Z \\ -(3K+4); & K \in Z \\ 3K+3, & K \in Z \end{cases}$~~ $\because$ they are linear $\Rightarrow$ ~~strictly $\uparrow$ or $\downarrow$ fn~~

inc. or dec.-fn

~~$\therefore$ Individually, Injective~~

~~For pairs, If $n+4 = -n-3 \Rightarrow n = -7/2 \notin Z$~~

~~$-n-3 = n+1$~~ $\Rightarrow n =$

(c) $\because$ $\ln(x)$ is increasing fn $\therefore$ It is injective

~~& $\ln(x) \neq 0$~~

but for $x = 1$, $\ln(x) = 0$; for $x = 0$, $f(x) = 0$

$\Rightarrow f(x) = 0$

$\therefore f(x) = 0 \Rightarrow x = 0, 1$ $\therefore$ Not Injective

Range:- $0 \cup \{\text{Range of } \ln(x)\}$

$\Rightarrow 0 \cup (-\infty, \infty) \Rightarrow (-\infty, \infty) \equiv$ Codomain $\therefore$ Surjective

Hence, Right Invertible

(17). (a) Range = $\{2,3\}$, Domain = $\{1,2,3,4,5\}$

$\Rightarrow$ Combination = $2^5 = 32$ ($\because$ Each $a \in$ Domain have 2 choices $\{2,3\}$)

(b) $\left({}^5C_2 \times {}^3C_2 = {}^5C_3 \times {}^3C_2\right)$

$2^3 \times {}^5C_2$
$= 80$

1 have two PI. for 2 & C ~~2 have 2 PI~~

(c) $3^4 = 81$ ($\because \{2,3,4,5\}$ have 3 options)

(d) 0, $\forall$ One-One, n (Elements in Domain) < n (Elements in Codomain)

(e) $\because$ All fn going to be manyone,
Total no. of fns = $3^5 = 243$

(16) (b) $f(x) = \begin{cases} n+4, & n \equiv 0 \bmod 3 \\ -n-3, & n \equiv 1 \bmod 3 \\ n+1, & n \equiv 2 \bmod 3 \end{cases}$

$\because$ A no. Z can be only K mod 3, where K is unique
$\therefore$ We can check Bijectivity of fn Exclusively.

$\because$ $n+4, -n-3, n+1$, are linear fn $\Rightarrow$ Strictly increasing / decreasing fn $\therefore$ Injective

Range: $n+4$, $n = 3K$, $K \in Z$
$\Rightarrow 3K+4 \equiv 3S+1$, $(S = K+1)$ $\therefore S \in Z$
~~$\therefore$ 1 mod 3~~ $\therefore y \in 1 \bmod 3$
~~$\Rightarrow$ Range $\in$ {i = All~~

|||y, $-n-3 = -(3K+1)-3 = -3K-4 = -3(K+2)+2$
$= -3S+2 \equiv 3t+2$, $(K,S,t \in Z)$
$\therefore y \in 2 \bmod 3$

|||y, $n+1 = 3K+2+1$
$= 3K+3 = 3S$ $(K,S \in Z)$ $\therefore y \in 0 \bmod 3$

$\Rightarrow$ Range contains $(0 \bmod 3 \cup 1 \bmod 3 \cup 2 \bmod 3) \Rightarrow Z$
$\therefore$ Surjective

$$\Rightarrow f(x) = \begin{cases} n+4 & , n = 0 \bmod 3 \\ -n-3 & , n = 1 \bmod 3 \\ n+1 & , n = 2 \bmod 3 \end{cases}$$

3 $\Rightarrow f^{-1}($[illegible]$) =$

$$\begin{cases} y-4 & , y = \overset{0 \bmod 3 + 4 \,=}{\cancel{0 \bmod 3 + 4}} \; 1 \bmod 3 \\ -(y+3) & , y = -1 \bmod 3 - 3 = 2 \bmod 3 \\ (y-1 & , y = 2 \bmod 3 + 1 = 0 \bmod 3 \end{cases}$$