Graph Theory

Monsoon 2023

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0 Outline

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Definition (Complete Graphs)

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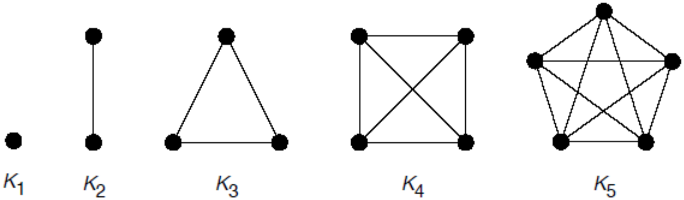

Definition (Bipartite Graphs)

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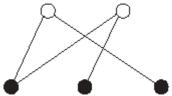

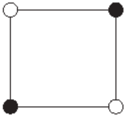

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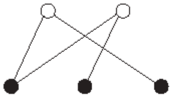

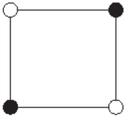

Example

What is a smallest possible simple graph that is not bipartite?

Theorem

A bipartite graph cannot have any self-loops.

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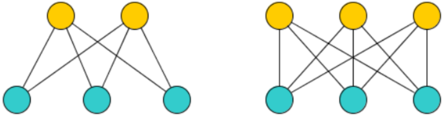

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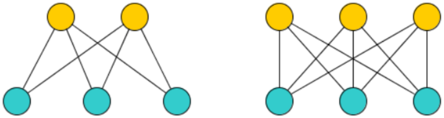

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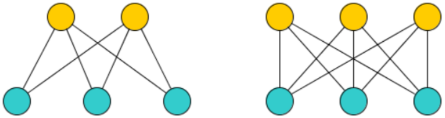

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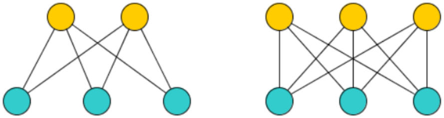

Answer the following:

1. What are m, n in above graphs?
2. What are the number of edges? Is there a formula to count number of edges?
3. What could be the applications of bipartite graphs?

Definition (Regular Graph)

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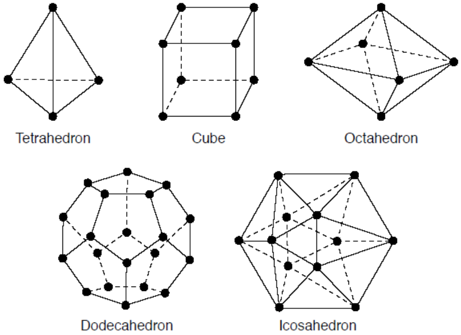

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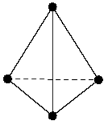
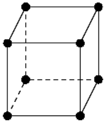
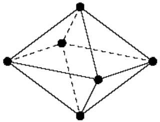


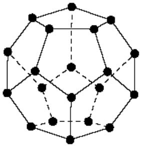
Tetrahedron



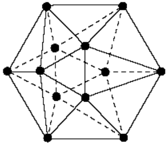
Cube



Octahedron



Dodecahedron



Icosahedron

Example

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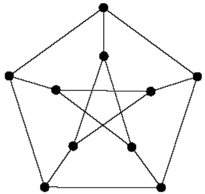

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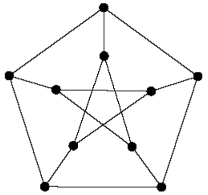

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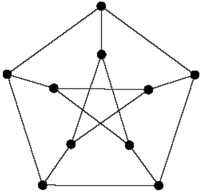

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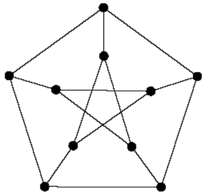

Answer the following:

1. Is peterson graph a complete graph? Is it bipartite?
2. Is the peterson graph k regular? What is k?
3. Is peterson graph bipartite? If yes, then show the partition of vertices.

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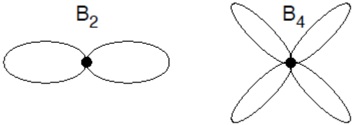

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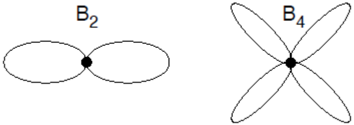

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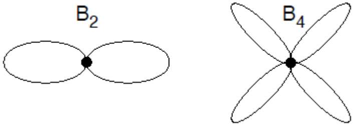

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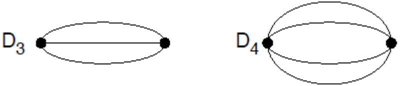

Definition (Path Graph)

A path graph P is a simple graph with $|V_P| = |E_P| + 1$ that can be drawn so that all of its vertices and edges lie on a single straight line. A path graph with n vertices and $n - 1$ edges is denoted P_n.

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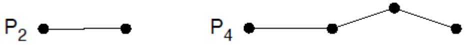

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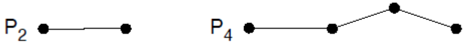

Definition (Cycle Graph)

A cycle graph is a single vertex with a self-loop or a simple graph C with $V_C = E_C$ that can be drawn so that all of its vertices and edges lie on a single circle. An n-vertex cycle graph is denoted C_n.

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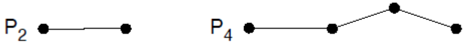

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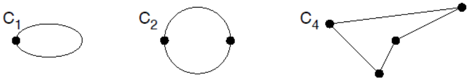

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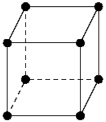

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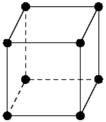

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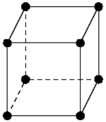

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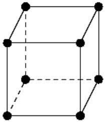

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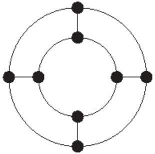

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To the group of integers $Z_n = \{0, 1, \ldots, n-1\}$ under addition modulo n and a set $S \subset \{1, \ldots, n-1\}$, we associate the circulant graph $circ(n : S)$ whose vertex set is Z_n, such that two vertices i and j are adjacent if and only if there is a number $s \in S$ such that $i + s = j \bmod n$ or $j + s = i \bmod n$. In this regard, the elements of the set S are called connections.

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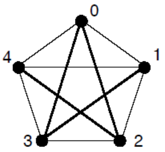

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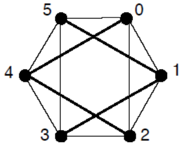

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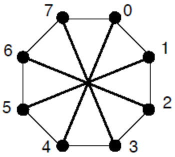

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Remark

Notice that circulant graphs are simple graphs.

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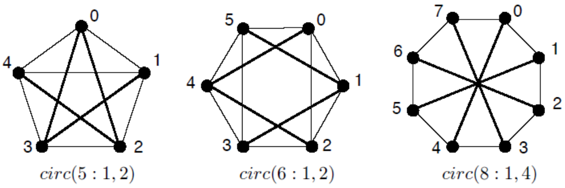

Remark

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A simple graph G with vertex-set $V_G = \{v_1, v_2, \ldots, v_n\}$ is an intersection graph if there exists a family of sets $F = \{S_1, S_2, \ldots, S_n\}$ such that vertex v_i is adjacent to v_j if and only if $i \neq j$ and $S_i \cap S_j \neq \phi$.

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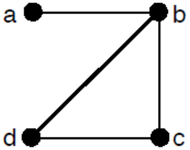

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(Archeology) Suppose that a collection of artifacts was found at the site of a town known to have existed from 1000 BC to AD 1000. In the graph shown, the vertices correspond to artifact types, and two vertices are adjacent if some grave contains artifacts of both types. It is reasonable to assume that artifacts found in the same grave have overlapping time intervals during which they were in use. If the graph is an interval graph, then there is an assignment of subintervals of the interval (-1000; 1000) (by suitable scaling, if necessary) that is consistent with the archeological find.

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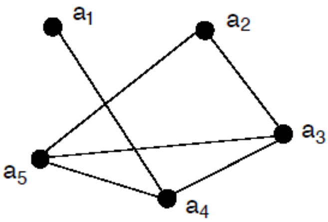

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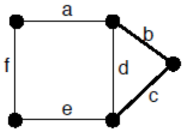
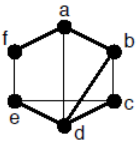


G



L(G)

Answer the following:

1. Verify that $L(G)$ is the intersection graph.

Example (Personnel-Assignment Problem)

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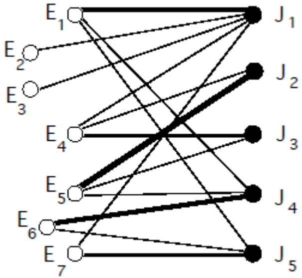

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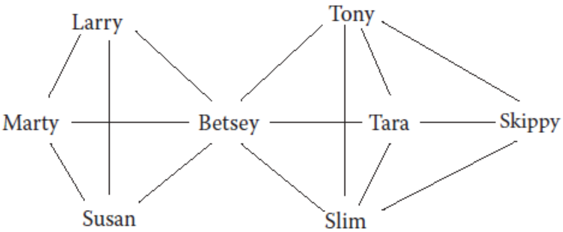

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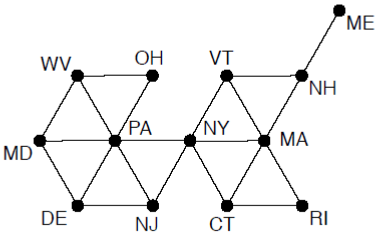

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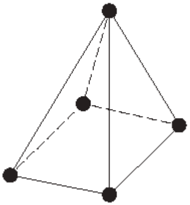

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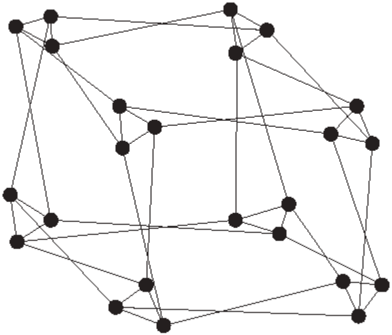

Example (Roadways between States)

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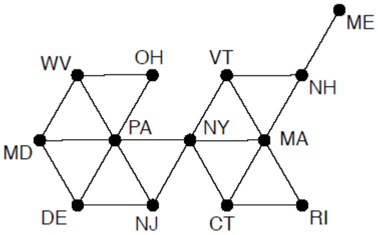

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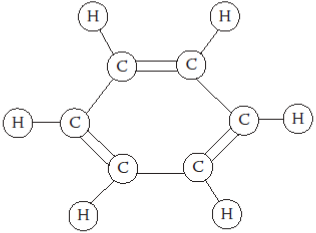

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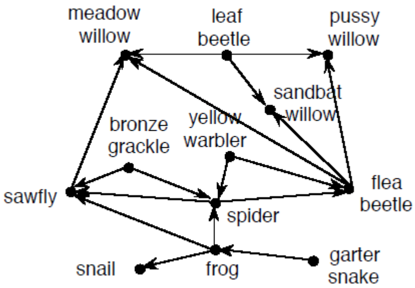

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In large projects, often there are some tasks that cannot start until certain others are completed. Figure below shows a digraph model of the precedence relationships among some tasks for building a house. Vertices correspond to tasks. An arc from vertex u to vertex v means that task v cannot start until task u is completed. To simplify the drawing, arcs that are implied by transitivity are not drawn. This digraph is the cover diagram of a partial ordering of the tasks.

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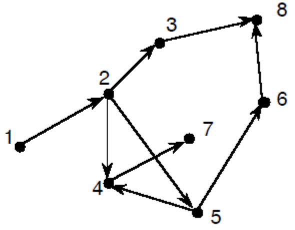
	Activities
1	Foundation
2	Walls and Ceilings
3	Roof
4	Electrical wiring
5	Windows
6	Siding
7	Paint interior
8	Paint exterior



Example (Markov Diagrams)

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Suppose that the inhabitants of some remote area purchase only two brands of breakfast cereal, O's and W's. The consumption patterns of the two brands are encapsulated by the transition matrix shown in Figure below. For instance, if someone just bought O's, there is a 0.4 chance that the person's next purchase will be W's and a 0.6 chance it will be O's. In a Markov process, the transition probability of going from one state to another depends only on the current state. Here, states "O" and "W" correspond to whether the most recent purchase was O's or W's, respectively. The digraph model for this Markov process, called a Markov diagram, is shown in Figure below. Each arc is labeled with the transition probability of moving from the state at the tail vertex to the state at the head. Thus, the probabilities on the outgoing edges from each vertex must sum to 1. This Markov diagram is an example of a weighted graph.

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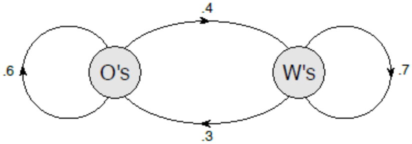

	O's	W's
O's	.6	.4
W's	.3	.7