9/10/23

Non-homogeneous Recurrences
(Linear, First / Second Order)

$$\boxed{C_1 a_n + C_2 a_{n-1} + C_3 a_{n-2} = f(n)}$$

(Today)
1. Method of undetermined coefficients
(Later)
2. Method of generating functions

A particular solution $\lambda a_n^{(P)}$
Solve the underlying homogeneous recurrence $a_n^{(P)}$
General solution
$\lambda_1 a_n^{(H)} + \lambda a_n^{(P)}$

Example $a_n = 3a_{n-1} + n \qquad a_n = 1$

$a_n^{(1)} = An + B$

$An + B = 3(A(n-1) + B) + \cancel{B}\, n$

$\Rightarrow n(A - 3A - 1] + B[1-3] + 3A = 0$

$A = -1/2 \qquad -2B - 3/2 = 0 \qquad B = -3/4$

$$\boxed{a_n^{(P)} = -\frac{1}{2}n - 3/4}$$

$a_n^{(H)} = AB^n$

$a_0 = A - 3/4 = 1, \; A = 7/4$

$$\boxed{a_n = \frac{7}{4}3^n - \frac{n}{2} - \frac{3}{4}}$$

Ex $a_n = 5a_{n-1} + 2\cdot 7^n$

$f(n) = Kr^n$

Case 1 r^n is not a solution in $a_n^{(h)}$

$a_n(P) = Ar^n$

Case 2 If r^n is a soln of $a_n^{(h)}$

$a_n(P) = Bnr^n$

Ex $a_n = 5a_{n-1} + 2\cdot 7^n$ $a_0 = 1$

$a_n^{(P)} = K\cdot 7^n$

$K\cdot 7^n = 5K7^{n-1} + 2\cdot 7^n$

$K\cdot 7 = 5K + 14 \Rightarrow K = 7$

$$\boxed{a_n^{(P)} = 7^{n+1}}$$

$a_n^{(h)} = A\cdot 5^n$

$a_n = A5^n + 7^{n+1}$

$a_0 = 1 = A + 7$; $A = -6$

$$\boxed{a_n = -6\cdot 5^n + 7^{n+1}}$$

Ex $a_0 = 1$ $a_n = 3a_{n-1} + 7\cdot 3^n$

$a_n^{(H)} = A\cdot 3^n$

$a_n^{(P)} = K\cdot n\cdot 3^n$

$K\cdot n\cdot 3^n = 3K(n-1)3^{n-1} + 7\cdot 3^n$

$$3nK = 3Kn - 3K + 21$$
$$K = 7$$
$$a_n = A \cdot 3^n + 7n\, 3^n$$
$$a_0 = 1 = A$$
$$\boxed{a_1 = 3^n + 7n\,3^n = (7n+1)\,3^n}$$

Ex: $a_n = 2a_{n-1} + 3a_{n-2} + 4 \cdot 5^n$

$$r^n = 2r^{n-1} + 3r^{n-2}$$
$$r^2 - 2r - 3 = 0$$
$$r = \frac{2 \pm \sqrt{4+12}}{2} = 3 \text{ or } -1$$
$$a_n^{(H)} = A \cdot 3^n + B(-1)^n$$
$$a_n^{(P)} = K \cdot 5^n = (5^{n+2})/3$$
$$K \cdot 5^n = 2 \cdot K \cdot 5^{n-1} + 3K5^{n-2} + 4 \cdot 5^n$$

$$a_n = A \cdot 3^n + B(-1)^n + \frac{1}{3} 5^{n+2}$$
$$a_0 = 1 = A + B + \frac{25}{3} \Rightarrow A + B = -22/3 \quad (1)$$

$$a_1 = 2 = 3A - B + \frac{125}{3} \Rightarrow 3A - B = -\frac{119}{3} \quad (2)$$

$$4A = -\frac{141}{3} = -47$$

$$B = -\frac{22}{3} + \frac{47}{4} = \frac{53}{12}$$

$$\boxed{a_n = \left(\frac{-47}{4}\right) 3^n + \left(\frac{53}{12}\right)(-1)^n + \frac{1}{3} \cdot 5^{n+2}}$$

Ex: $a_0 = 0,\ a_1 = 1$

$$a_n = a_{n-1} + a_{n-2} + 4\cdot\left(\frac{1-\sqrt5}{2}\right)^n$$

$$a_n^{(h)} = A\left(\frac{1+\sqrt5}{2}\right)^n + B\left(\frac{1+\sqrt5}{2}\right)^n$$

$$a_n^{(p)} = Kn\left(\frac{1-\sqrt5}{2}\right)^n$$

$$Kn\left(\frac{1-\sqrt5}{2}\right)^n = K(n-1)(\quad)^{n-1} + K(n-2)$$

Solve for K

$$Kn\left(\frac{1\pm\sqrt5}{2}\right)^n = K(n-1)\left(\frac{1-\sqrt5}{2}\right) + K(n-2) + 4\left(\frac{1-\sqrt5}{2}\right)^2$$

$a_0 = 1,\ a_1 = 2$

$$a_n = 2a_{n-1} - a_{n-2} + 3$$

$$a_n^{(h)} = A(1^n) + Bn(1^n)$$

$$A_n^{(p)} = C\cdot n^2(1^n) = \frac{3}{2}n^2$$

$$Cn^2 = 2C(n-1)^2 - C(n-2)^2 + 3$$

$$Cn^2 = 2Cn^2 - 4Cn + 2C - Cn^2 + 4Cn - 4C + 3$$

$$\Rightarrow \boxed{C = 3/2}$$

$$a_n = A + Bn + \frac{3}{2}n^2$$

put $a_0 = 1,\ a_1 = 2$

$$\boxed{a_n = 1 - \frac{n}{2} + \frac{3}{2}n^2}$$

$C_1 a_n + C_2 a_{n-1} + C_3 a_{n-2} = f(x)$

(same)

Given $a_n^{(P)}$ = form of $f(n)$

Substitute and solve for coefficients in $a_n^{(P)}$

$$\boxed{a_n = A\, a_n^{(H)} + a_n^{(P)}}$$