6/10/23

Second order linear homogeneous Constant coefficients Recurrences

$$\boxed{C_1 a_n + C_2 a_{n-1} + C_3 a_{n-2} = 0}$$

~~Ar^n~~ Guess $A_n = r^n$

$$C_1 r^n + C_2 r^{n-1} + C_3 r^{n-2} = 0$$

$$C_1 r^2 + C_2 r + C_3 = 0$$

$$r = \frac{-C_2 \pm \sqrt{C_2^2 - 4C_1C_3}}{2C_1}$$

$$r_1 = \frac{-C_2 + \sqrt{C_2^2 - 4C_1C_3}}{2C_1}$$

$$r_2 = \frac{-C_2 - \sqrt{C_2^2 - 4C_1C_3}}{\cancel{2C_2}\ 2C_1}$$

r_1, r_2 Distinct real roots

$$\boxed{A_n = A r_1^n + B r_2^n}$$

If r_1 and r_2 are complex

$$a_n = A r_1^n + B r_2^n$$

$\gamma_1 = \gamma e^{i\theta}$

$\gamma_2 = \gamma e^{-i\theta}$

$$a_n = A\gamma^n e^{in\theta} + B\gamma^n e^{-in\theta}$$
$$= \gamma^n \left[(A+B)\cos n\theta + i(A-B)\sin n\theta \right]$$
$$= \boxed{\gamma^n (K_1 \cos n\theta + K_2 \sin n\theta)}$$

If Repeated roots

$$\boxed{a_n = A\gamma^n + Bn\gamma^n}$$

$\cancel{g(n) \cdot \gamma^n}$

$C_1 g(n) \cancel{\gamma^2 + C_2 g(n-1)\gamma + C_2 g(n-2)} = 0$

Example #1

Solve: $a_n = a_{n-1} + a_{n-2}$; $a_0 = 0$ $a_1 = 1$

Assume : $a_n = \gamma^n$

$$\gamma^n = \gamma^{n-1} + \gamma^{n-2}$$

$$\gamma^2 - \gamma - 1 = 0 \qquad \gamma = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\gamma_1 = \frac{1+\sqrt{5}}{2} \qquad \gamma_2 = \frac{1-\sqrt{5}}{2}$$

$$a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

① $a_0 = 0 = A + B$

② $a_1 = 1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$

$$A = 1/\sqrt{5}, \; B = -1/\sqrt{5}$$

$$\boxed{a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

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$$\boxed{a_n = \frac{r_1^n - r_2^n}{r_1 - r_2}}$$

Example #2

$a_n = 2a_{n-1} - a_{n-2}$, $\quad a_0 = 0$, $a_1 = 1$

$r^n = 2r^{n-1} - r^{n-2}$

$r^2 - 2r + 1 = 0$

$$r = \frac{2 \pm \sqrt{4-4}}{2} = 1$$

$a_n = A r^n + B n r^n$

$= A + Bn$

$a_0 = 0 \quad A = 0$

$a_1 = 1 \quad B = 1$

$\overline{a_n = n}$

0, 1, 2, 3, 4, 5, 6

If $a_0 = 1$ $a_1 = 5$

$a_n = A r^n + B n r^n$

$= A + Bn$

$a_0 = 1 = A \quad a_1 = 5 = A + B \Rightarrow B = 4$

$a_n = 4n$

Example

$a_n = 2a_{n-1} + 2a_{n-2}$ $\quad a_0 = 0$ $a_1 = 1$

$a_2 = 2a_1 - a_0$

$a_3 = 2a_2 - a_1$

$r^n = 2r^{n-1} + 2r^{n-2}$

$r^2 - 2r - 2 = 0 \quad r = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$

$r_1 = 1 + \sqrt{3}$

$r_2 = 1 - \sqrt{3}$

$a_0 = 0 = A + B$

$a_1 = 1 = A(1+\sqrt{3}) - A(1-\sqrt{3})$

$A = 1/2\sqrt{3}$; $B = -1/2\sqrt{3}$

Example

$a_n = 2a_{n-1} - 2a_{n-2}$

$r^n - 2r^{n-1} + 2r^{n-2}$

$r^2 - 2r + 2 = 0 \quad r = 1 \pm i$

$r = \sqrt{2} e^{\pm i\pi/4}$

$a_n = (\sqrt{2})^n \left[A e^{2n\pi/4} + B e^{-in\pi/4} \right]$

$(\sqrt{2})^n \left[A\cos\left(\frac{n\pi}{2}\right) + Ai\sin\frac{n\pi}{4} + B\cos\left(\frac{n\pi}{4}\right) - Bi\sin(n\pi/4) \right]$

$$\boxed{(\sqrt{2})^n \left[K_1 \cos(n\pi/4) + K_2 \sin(n\pi/2) \right]}$$

$K_1 = 2, \; K_2 = 3$

$a_0 = 2 = K_1$

$a_1 = 5 = \sqrt{2}\left[2\cos\left(\frac{\pi}{4}\right) + K_2 \sin\left(\frac{\pi}{4}\right) \right] = 2 + K_2 = 5$