Graph Theory

Monsoon 2023

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September 2023

0 Outline

1. Isomorphism
2. Graph Terminologies
3. Graph Operations
4. Planarity
5. Coloring

1 Outline

1. Isomorphism
2. Graph Terminologies
3. Graph Operations
4. Planarity
5. Coloring

Definition (Structural Equivalence for Simple Graphs)

Let G and H be two simple graphs. A vertex function $f : V_G \to V_H$ preserves adjacency if for every pair of adjacent vertices u and v in graph G, the vertices $f(u)$ and $f(v)$ are adjacent in graph H. Similarly, f preserves non-adjacency if $f(u)$ and $f(v)$ are non-adj whenever u and v are non-adj.

Definition

A vertex bijection $f : V_G \to v_H$ between two two simple graphs G and H is structure-preserving if it preserves adjacency and non-adjacency. That is, for every pair of vertices in G, u and v are adj in G iff $f(u)$ and $f(v)$ are adj in H. This leads us to a formal mathematical definition of what we mean by the "same" graph.

Definition (Isomorphic)

Two simple graphs G and H are isomorphic, denoted $G \cong H$, if $\exists$ a **structure-preserving** bijection $f : V_G \to V_H$. Such a function f is called an isomorphism from G to H.

- When we regard a vertex function $f : V_G \to V_H$ as a mapping from one graph to another, we may write $f : G \to H$.
- Two graphs related by isomorphism differs only by the names of the vertices and edges. There is a complete structural equivalence between two such graphs.

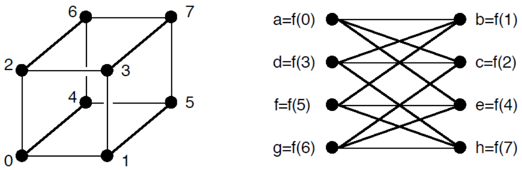
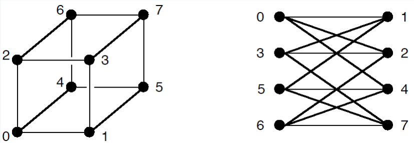


Figure: Two graphs are isomorphic because there is a bijection f from the vertex set of first graph to the second, and adjacency and non-adjacency is preserved.

Example (Isomorphism)

Alternatively, one may relabel the vertices of the codomain graph with names of vertices in the domain, as in



Example (Isomorphism)

The vertex function $j \to j + 4$ depicted in Fig below is bijective and adjacency-preserving, but it is not an isomorphism, since it does not preserve non-adjacency. In particular, the non-adjacent pair $\{0; 2\}$ maps to the adjacent pair $\{4; 6\}$

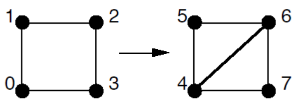


Figure: Bijective and adj-preserving, but not an isomorphism

Example (Isomorphism)

The vertex function $j \to j \bmod 2$ depicted in Figure below is structure-preserving, since it preserves adjacency and non-adjacency, but it is not an isomorphism since it is not bijective.

Figure: Preserves adj and non-adj, but not bijective

Example (Isomorphism)

The mapping $f : V_G \to V_H$ between the vertex-sets of the two graphs shown in Figure below be given by

$$f(i) = i, \quad i = 1, 2, 3$$

preserves adjacency and non-adjacency, but the two graphs are clearly not structurally equivalent. How should we extend our definition of isomorphism to general graphs?

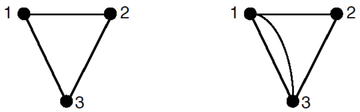


Figure: These graphs are not structurally equivalent. Why?

Definition

A vertex bijection $f : V_G \to V_H$ between two graphs G and H, simple or general, is structure-preserving if

1. the $\#$ of edges (even if 0) between every pair of distinct vertices u and v in graph G equals the $\#$ of edges between their images $f(u)$ and $f(v)$ in graph H, and
2. the $\#$ of self-loops at each vertex x in G equals the $\#$ of self-loops at the vertex $f(x)$ in H.

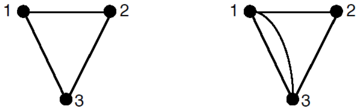


Figure: These graphs are not structurally equivalent

Definition

Two graphs G and H (simple or general) are isomorphic graphs if $\exists$ structure-preserving vertex bijection $f : V_G \to V_H$. This relationship is denoted by $G \cong H$.

Definition

For isomorphic graphs G and H, a pair of bijections

$$f_V : V_G \to V_H \quad \text{and} \quad f_E : E_G \to E_H$$

is consistent if for every edge $e \in E_G$, the function f_V maps the endpoints of e to the endpoints of the edge $f_E(e)$.

Theorem

$G \cong H$ iff there is a consistent pair of bijections

$$f_V : V_G \to V_H, \quad \textit{and} \quad f_E : E_G \to E_H$$

Remark

If G and H are isom simple graphs, then every structure preserving vertex bijection $f : V_G \to V_H$ induces a unique consistent edge bijection, by the rule: $uv \to f(u)f(v)$.

Definition (Isomorphism for General Graphs)

If G and H are graphs with multi-edges, then an isomorphism from G to H is specified by giving a consistent pair of bijections $f_V : V_G \to V_H$ and $f_E : E_G \to E_H$.

Example

Both of the structure-preserving vertex bijections $G \to H$ in Fig below have six consistent edge-bijections.

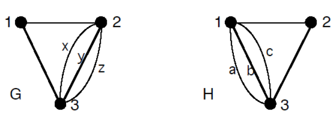


Figure: There are 12 distinct isoms from G to H.

Theorem

Let G and H be isomorphic graphs. Then they have the same number of vertices and edges.

Theorem

Let $f : G \to H$ be a graph isomorphism and let $v \in V_G$. Then $deg(f(v)) = deg(v)$.

Corollary

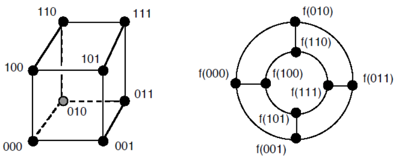
Let G and H be isomorphic graphs. Then they have the same degree sequence.

Corollary

Let $f : G \to H$ be a graph isom and $e \in E_G$. Then the endpoints of edge $f(e)$ have the same degrees as the endpoints of e.

Example (Are the graphs isomorphic?)

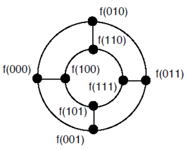
Are the following graphs isomorphic?

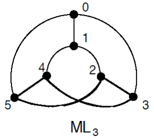


1. We observe that Q_3 and CL_4 both have 8 vertices and 12 edges and are 3-regular.
2. The vertex labelings specify a vertex bijection. A careful examination reveals that this vertex bijection is structure-preserving.
3. It follows that Q_3 and CL_4 are isomorphic graphs.

Definition (Mobius Ladder)

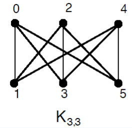
The Mobius ladder ML_n is a graph obtained from the circular ladder CL_n by deleting from the circular ladder two of its parallel curved edges and replacing them with two edges that cross-match their endpoints.

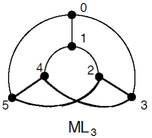




Example (Isomorphic Graphs)

Are the following two graphs isomorphic?



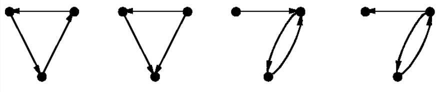


Definition (Isomorphism of Digraphs)

Two digraphs G and H are isomorphic if there is an isomorphism f between their underlying graphs that preserves the direction of each edge.

Example

Which of the graphs below are isomorphic? Are the underlying graphs isomorphic?



Definition (Graph Isomorphism Problem)

The graph-isomorphism problem is to devise a practical general algorithm to decide graph isomorphism, or, alternatively, to prove that no such algorithm exists.

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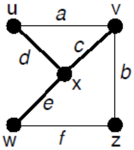
Definition (Subgraphs)

A subgraph of a graph G is a graph H whose vertices and edges are all in G. If H is subgraph of G, we may also say that G is a supergraph of H.

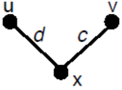
Definition (Proper Subgraph)

A proper subgraph H of G is a subgraph such that V_H is a proper subset of V_G or E_H is a proper subset of E_G.

G

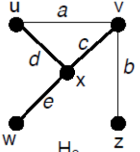


edge	a	b	c	d	e	f
endpts	u	v	v	u	x	w
	v	z	x	x	w	z



H_1

edge	c	d
endpts	v	u
	x	x



H_2

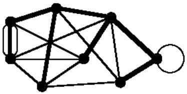
edge	a	b	c	d	e
endpts	u	v	v	u	x
	v	z	x	x	w

Definition (Spanning Subgraphs)

A subgraph H is said to span a graph G if $V_H = V_G$.

Definition (Spanning Tree)

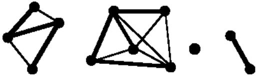
A spanning tree is a spanning subgraph that is a tree.



Definition (Forest)

An acyclic graph is called a forest.

G



Definition (Clique)

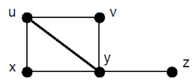
A subset S of V_G is called a clique if every pair of vertices in S is joined by at least one edge, and no proper superset of S has this property.

Definition (Clique Number)

The clique number of a graph G is the number $\omega(G)$ of vertices in a largest clique in G.

Example (Clique)

In Fig below, the vertex subsets, $\{u, v, y\}$, $\{u, x, y\}$ and $\{y, z\}$ induce complete subgraphs, and $\omega(G) = 3$.



Definition (Independent Set)

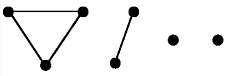
A subset S of V_G is said to be an independent set if no pair of vertices in S is joined by an edge.

Definition (Component)

A component of G is a maximal connected subgraph of G.

Example

The 7-vertex graph in Figure below has four components.



Definition (Reachable Vertex)

Recall: A vertex v is said to be reachable from vertex u if there is a walk from u to v.

Definition (component of a vertex)

In a graph G, the component of a vertex v, denoted $C(v)$, is the subgraph induced by the subset of all vertices reachable from v.

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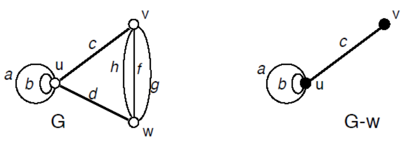
Definition (vertex-deletion subgraph)

The vertex-deletion subgraph $G - v$ is the subgraph induced by the vertex-set $V_G - \{v\}$. That is,

$$V_{G-v} = V_G - \{v\} \quad \text{and}$$
$$E_{G-v} = \{e \in E_G : v \notin endpts(e)\}$$

Example

Result of deleting vertex w from graph G:



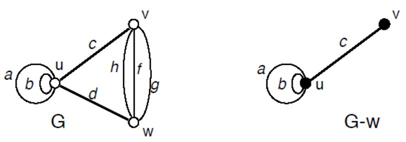
Definition (edge-deletion subgraph)

The edge-deletion subgraph $G - e$ is the subgraph induced by the vertex-set $E_G - \{e\}$. That is,

$$V_{G-e} = V_G \quad \text{and}$$
$$E_{G-e} = E_G - \{e\}$$

Example

Result of deleting edge f from graph G :

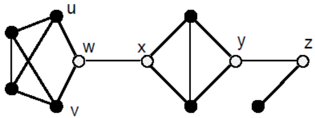


Definition (Vertex Cut)

A vertex-cut in a graph G is a vertex-set U such that $G - U$ has more components than G.

Definition (Cut-Vertex)

A cut-vertex (or cutpoint) is a vertex-cut consisting of a single vertex.



Definition (Edge Cut)

An edge-cut in a graph G is a set of edges D such that $G - D$ has more components than G.

Definition (Cut-Edge)

A cut-edge (or bridge) is an edge-cut consisting of a single edge.

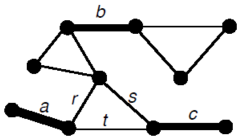


Figure: A graph with three cut edges

Definition (Adding Edges or Vertices)

Adding an edge between two vertices u and w of a graph G means creating a supergraph, denoted $G \cup \{e\}$, with vertex-set V_G and edgeset $E_G \cup \{e\}$, where e is a new edge with endpoints u and w.

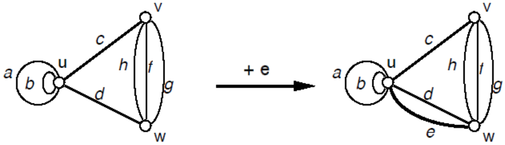


Figure: Adding an edge e with endpoints u and w.

Definition (Adding a Vertex)

Adding a vertex v to a graph G, where v is a new vertex not already in V_G, means creating a supergraph, denoted $G \cup \{v\}$, with vertex-set $V_G \cup \{v\}$ and edge-set E_G.

Definition (Edge Complement)

Let G be a simple graph. Its edge-complement (or com- plement) G has the same vertex-set, but two vertices are adjacent in $\bar{G}$ iff they are not adjacent in G.

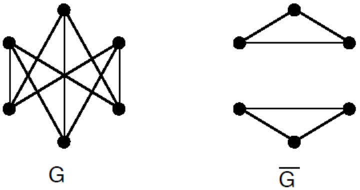


Figure: A graph and its complement.

Remark

The edge-complement of the edge-complement is the original graph, i.e., $\bar{\bar{G}} = G$.

Definition

A graph invariant (or digraph invariant) is a property of graphs (digraphs) that is preserved by isomorphisms.

Remark

We established in before that the number of vertices, the number of edges, and the degree sequence are the same for any two isomorphism graphs, so they are graph invariants.

Theorem

Let $f : G \to H$ be a graph isomorphism, and let $v \in V_G$. Then the multiset of degrees of the neighbors of v equals the multiset of degrees of the neighbors of $f(v)$.

Figure: Non-isom graphs with the same degree seq.

Definition (Distance Invariance)

Let

$$W = \langle v_0, e_1, v_1, \ldots, e_n, v_n \rangle$$

be a walk in the domain G of a graph isom $f : G \to H$. Then the image of walk W is the walk

$$f(W) = \langle f(v_0), f(e_1), f(v_1), \ldots, f(e_n), f(v_n) \rangle$$

is the codomain graph H.

Theorem

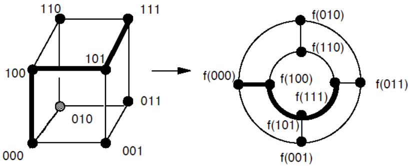
The isomorphic image of a graph walk W is a walk of the same length.

Corollary

The isomorphic image of a trail, path, or cycle is a trail, path, or cycle, respectively, of the same length.

Corollary

For each integer ℓ, two isomorphic graphs must have the same $\#$ of trails (paths) (cycles) of length ℓ.



Corollary

The diameter, the radius, and the girth are graph invariants.

Theorem

Let G and H both be simple graphs. They are isomorphic iff their edge-complements are isomorphic.

Example

The edge-complement of the left graph consists of two disjoint 4-cycles, and the edge-complement of the right graph is an 8-cycle.

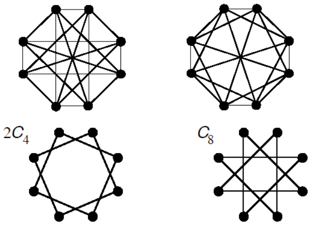


Figure: Two relatively dense, non-isomorphic 5-regular graphs and their edge- complements.

3 Characterizations of Trees

Definition

In a undirected tree, a leaf is a vertex of degree 1.

Theorem

Every tree with at least one edge has at least two leaves.

3 Properties of Trees Continued...

Theorem

If the minimum degree of a graph is at least 2, then that graph must contain a cycle.

Theorem

Every tree on n vertices has exactly $n - 1$ edges.

Definition

An acyclic graph is called a forest. A tree is also a forest!

Notation for number of components

The number of components of a graph G is denoted $c(G)$.

Corollary

A forest G on n vertices has $n - c(G)$ edges.

Corollary

Any graph G on n vertices has at least $n - c(G)$ edges.

3 Six Characterizations of a Tree

Theorem

Let T be a graph with n vertices. Then the following statements are equivalent.

1. *T is a tree.*
2. *T contains no cycles and has $n - 1$ edges.*
3. *T is connected and has $n - 1$ edges.*
4. *T is connected, and every edge is a cut-edge.*
5. *Any two vertices of T are connected by exactly one path.*
6. *T contains no cycles, and for any new edge e, the graph $T + e$ has exactly one cycle.*

3 Rooted, Ordered, and Binary Trees

Definition

A directed tree is a directed graph whose underlying graph is a tree.

Definition

A rooted tree is a tree with a designated vertex called the root. Each edge is implicitly directed away from the root.

3 Rooted Tree Terminology

Definition

In a rooted tree, the depth or level of a vertex v is its distance from the root, i.e., the length of the unique path from the root to v. Thus, the root has depth 0.

Definition

The height of a rooted tree is the length of a longest path from the root (or the greatest depth in the tree).

Definition

If vertex v immediately precedes vertex w on the path from the root to w, then v is parent of w and w is child of v.

Definition

Vertices having the same parent are called siblings.

Definition

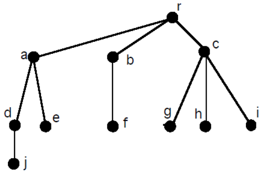
A vertex w is called a descendant of a vertex v (and v is called an ancestor of w), if v is on the unique path from the root to w. If, in addition, $w \neq v$, then w is a proper descendant of v (and v is a proper ancestor of w).

Definition

A leaf in a rooted tree is any vertex having no children.

Definition

An internal vertex in a rooted tree is any vertex that has at least one child. The root is internal, unless the tree is trivial (i.e., a single vertex).



The height of this tree is 3. Also,

- r, a, b, c, and d are the internal vertices
- vertices e, f, g, h, i, and j are the leaves
- vertices g, h, and i are siblings
- vertex a is an ancestor of j and
- j is a descendant of a

Definition

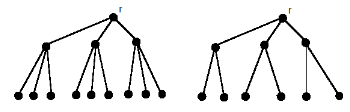
An *m*-ary tree ($m \geq 2$) is a rooted tree in which every vertex has m or fewer children.

Definition

A complete m-ary tree is an m-ary tree in which every internal vertex has exactly m children and all leaves have the same depth.

Example

Fig below shows two ternary (3-ary) trees; the one on the left is complete; the other one is not.



Definition

Two rooted trees are said to be isomorphic as rooted trees if there is a graph isomorphism between them that maps root to root.

3 Ordered Trees

Definition

An ordered tree is a rooted tree in which the children of each vertex are assigned a fixed ordering.

Definition

In a standard plane drawing of an ordered tree,

1. the root is at the top,
2. the vertices at each level are horizontally aligned, and
3. the left-to-right order of the vertices agrees with their prescribed order.

3 Tree Traversals

Remark

In an ordered tree, the prescribed local ordering of the children of each vertex extends to several possible global orderings of the vertices of the tree. One of them, the level order, is equivalent to reading the vertex names top-to-bottom, left-to-right in a standard plane drawing.

Definition

A binary tree is an ordered 2-ary tree in which each child is designated either a left-child or a right-child.

Definition

The left (right) subtree of a vertex v in a binary tree is the binary subtree spanning the left (right)-child of v and all of its descendants.

Theorem

The complete binary tree of height h has $2^{h+1} - 1$ vertices.

Corollary

Every binary tree of height h has at most $2^{h+1} - 1$ vertices.

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Definition (Planar Drawings)

A planar drawing of a graph is a drawing of the graph in the plane without edge-crossings.

Definition (Planar Graph)

A graph is said to be planar if there exists a planar drawing of it.

Figure 7.1.1 A nonplanar drawing and a planar drawing of K_4.

Example (A puzzle)

An instance of the problem of determining whether a given graph is planar occurs in the form of a well-known puzzle, called the utilities problem, in which three houses are on one side of a street and three utilities (electricity, gas, and water) are on the other. The objective of the puzzle is to join each of the three houses to each of the three utilities without having any crossings of the utility lines.
Is this possible? See next slide then justify.

Theorem

Every drawing of the complete graph K_5 in the plane contains at least one edge-crossing.

Theorem

Every drawing of the complete bipartite graph $K_{3,3}$ in the plane contains at least one edge-crossing.

Theorem

If either K_5 or $K_{3,3}$ is a subgraph of a graph G, then every drawing of G in the plane contains at least one edge-crossing.

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Definition (Chromatic Number)

The chromatic number of a graph is the minimal number of colours needed to colour the vertices in such a way that no two adjacent vertices have the same colour.