MATHEMATICS

February 4, 2025

1. For a Comple number z, let Re(z) denote the real part of z. Let S be the set of all complwx numbers z satisfying $z^4 - |z|^4 = 4iz^2$, where $i = \sqrt{-1}$.Then the minimum possible value of $|z_1 - z_2|^2$, where $z_1, z_2 \in S$ with $Re(z_1) > 0$ and $Re(z_2) < 0$, is ___________.

2. The probability that a missile hits a target successfully is 0.75. In order to destroy the target completely, at least three successful hits are required. Then the minimum number of missiles that have to be fired so that the probability of completely destroying the target is **NOT** less than 0.95, is ___________.

3. Let O be the centre of the circle $x^2 + y^2 = r^2$,where $r > \frac{\sqrt{5}}{2}$. Suppose PQ is a chord of this circle and the equation of the line passing through P and Q is $2x + 4y = 5$. If the centre of the circumcircle of the triangle OPQ lies on the lies on the line $x + 2y = 4$, then the value of r is ___________.

4. The trace of a square matrix is defined to be the sum of its diagonal entries. If A is a 2 $\times$ 2 matrix, such that the trace of A is 3 and the trace of $A^3 is - -18$, then the value of the determinant of A is ___________.

5. Let the functions $f : (-1, 1) \rightarrow Randg : (-1, 1) \rightarrow ((-)\, 1, 1)$ be defined by $f(x) =\mid 2x - 1 \mid + \mid 2x + 1 \mid$ and $g(x) = x - [x]$,where $[x]$ denotes the greatest integer lessthan or equal to x. Let $f^\circ g : (-1, 1) \rightarrow R$ bethe composite function defined by $(f^\circ g)(x) = f(g(x))$.Suppose c is the number of points in the interval (-1,1) at which $f^\circ g$ is **NOT** continuous, and suppose d is the number of points in the interval (-1,1) at which $f^\circ g$ is **NOT** differentiable. Then the value of $c + d$ is ___________.

6. The value of the limit
   $lim_{x \rightarrow \frac{\pi}{2}} \frac{4\sqrt{2}(sin3x+sinx)}{(2sin2xsin\frac{3x}{2}+cos\frac{5x}{2})-(\sqrt{2}+\sqrt{2}cos2x+cos\frac{3x}{2}}$ is ______.

7. Let b be a nonzero real number. Suppose f: R $\rightarrow$ R is a differentiable function such that f(0) = 1. If the derivative f' of f satisfies the equation
   $f'(x) = \frac{f(x)}{b^2+x^2}$
   for all x $\in$ R, then which of the following statements is/are TRUE?

(A) If $b > 0$, then f is an increasing function
(B) If $b < 0$, then f is a decreasing function
(C) $f(x)f(-x) = 1$ for all x $\in$ R
(D) $f(x) - f(-x) = 0$ for all x $\in$ R

8. Let a and b be positive real numbers such that $a > 1$ and $b < a$. Let P be a point in the first quadrant that lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2}$. Suppose the tangent to the hyperbola at P passes through the point (1, 0), and suppose the normal to the hyperbola at P cuts off equal intercepts on the coordinate axes. Let $\triangle$ denote the area of the triangle formed by the tangent at P, the normal at P and the x-axis. If e denotes the eccentricity of the hyperbola, then which of the following statements is/are TRUE?

   (A)$1 < e < \sqrt{2}$
   (B)$\sqrt{2} < e < 2$
   (C)$\triangle = a^4$
   (D)$\triangle = b^4$

9. Let f:R $\to$ R and g: R $\to$ R be functions satisfying

   $f(x + y) = f(x) + f(y) + f(x)f(y)$ and $f(x) = xg(x)$ for all x, y $\in$ R. If $\lim_{x\to 0} g(x) = 1$,then which of the following statements is/are TRUE?

   (A) f is differentiable at every x $\in$ R (B) If g(0) = 1, then g is differentiable at every x$\in$ R (C) The derivative f'(1) is equal to 1 (D) The derivative f'(0) is equal to 1

10. Let $\alpha, \beta, \gamma, \delta$ be real numbers such that $\alpha^2 + \beta^2 + \gamma^2 \neq 0$ and $\alpha + \gamma = 1$. Suppose the point (3,2,-1) is themirror image of the point (1,0,-1) with respect to theplane $\alpha x + \beta y + \gamma z = \delta$.Then which of the following statements is/are TRUE?

    (A)$\alpha + \beta = 2$

    (B)$\delta - \gamma = 3$

    (C)$\delta + \beta = 4$

    (D)$\alpha + \beta + \gamma = \delta$

11. Let a and b be positive real numbers. Suppose $\overrightarrow{PQ} = a\hat{i} + b\hat{j}$ and $\overrightarrow{PQ} = a\hat{i} - b\hat{j}$ are adjacent sides of a parallelogram PQRS. Let $\overrightarrow{u}$ and $\overrightarrow{v}$ be the projection vectors of $\overrightarrow{w} = \hat{i} + \hat{j}$ along $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ respectively. If $|\overrightarrow{u}| + |\overrightarrow{v}| = |\overrightarrow{w}|$ and if the area ofthe parallelogram PQRS is 8, then which of the following statements is/are TRUE?

    (A) $a + b = 4$ (B) $a - b = 2$ (C) The length of the diagonal PR of the parallelogram PQRS is 4 (D) $\overrightarrow{w}$ is an angle bisector of the vectors $\overrightarrow{PS}$ and $\overrightarrow{PS}$

12. For nonnegative integers s and r, let

$$\binom{s}{r} = \begin{cases} \frac{s!}{r!(s-r)!}, & \text{if} r \leq s, \\ 0, & \text{if} r > s. \end{cases}$$

For positive integers m and n, let g

$g(m, n) = \sum_{p=0}^{m+n} f(m, n, p) \binom{n+p}{p}$

where for any nonnegative integer p,

$f(m, n, p) = \sum_{i=0}^{p} \binom{m}{i} \binom{n+i}{p} \binom{p+n}{p-i}$

Then which of the following statements is/are TRUE?

(A) $(g(m, n) = g(n, m))$ for all positive integers (m, n)

(B) $(g(m, n+1) = g(m+1, n))$ for all positiveintegers (m, n)

(C) $(g(2m, 2n) = 2g(m, n))$ for all positive integers (m, n)

(D) $(g(2m, 2n) = (g(m, n))^2)$ for all positive integers (m, n)

13. An engineer is required to visit a factory for exactly four days during the first 15 days of every month and it is mandatory that **no** two visits take place on consecutive days. Then the number of all possibleways in which such visits to the factory can be made by the engineer during 1-15 June 2021 is ___________.

14. In a hotel, four rooms are available. Six persons are to be accommodated in these four rooms in such a way that each of these rooms contains at least one person and at most two persons. Then the number of all possible ways in which this can be done is ___________.

15. Two fair dice, each with faces numbered 1, 2, 3, 4, 5 and 6, are rolled together and the sum of the numbers on the faces is observed.This process is repeated till the sum is either a prime number or a perfect square. Suppose the sum turns out to be a perfect square before it turns out to be a prime number. If is the probability that this perfect square is an odd number, then the value of 14p is ___________.

16. Let the function $f : left[0, right1] \rightarrow R$ be defined by

$f(x) = \frac{4^x}{4^x+2}$

Then the value of

$f\left(\frac{1}{40}\right) + f\left(\frac{2}{40}\right) + f\left(\frac{3}{40}\right) + \cdots + f\left(\frac{39}{40}\right) - f\left(\frac{1}{2}\right)$ is ___________.

17. Let f: R $\rightarrow$ R be a differentiable function such that its derivative f' is continuous and $f(\pi) = --6. If F : [0, \pi] \rightarrow R$ is defined by $F(x) = \int_0^x f(t)\,dt$

$\int (f'(x) + F(x)) \cos x \, dx = 2$

then the value of f(0) is ___________.

18. Let the function $f : (0, \pi) \to R$ be defined by $f(\theta) = (sin\theta + cos\theta)^2 + (sin\theta - cos\theta)^4$. Suppose the function f has a local minimum at θ precisely when $\theta \in$

$\lambda_1 \pi, \ldots, \lambda_r \pi$, where $0 < \lambda_1 < \cdots < \lambda_r < 1$. Then the value of $\lambda_1 + \cdots + \lambda_r$ is ______.