11.16.3.8.2

EE24BTECH11035 -K.Akhil

Question:
If three coins are tossed once, what is the probability of getting exactly 2 heads?
Solution:
Define a discrete random variable X = number of heads
We will take our random variable as a sum of outcomes of three Bernoulli random variables

$$X = X_1 + X_2 + X_3 \tag{0.1}$$

Where

$$X_i = \begin{cases} 1, & \text{Outcome in Heads} \\ 0, & \text{Outcome in Tails} \end{cases} \tag{0.2}$$

$$p_{X_i}(n) = \begin{cases} 1 - p, & n = 0 \\ p, & n = 1 \end{cases} \tag{0.3}$$

Where $p = \frac{1}{2}$
Using properties of Z-Transform of PMF

$$M_X(z) = M_{X_1}(z) M_{X_2}(z) M_{X_3}(z) \tag{0.4}$$

$$M_{X_1}(z) = \sum_{n=-\infty}^{\infty} p_{X_1}(n) z^{-n} = (1 - p) + pz^{-1} \tag{0.5}$$

$$M_{X_2}(z) = \sum_{n=-\infty}^{\infty} p_{X_2}(n) z^{-n} = (1 - p) + pz^{-1} \tag{0.6}$$

$$M_{X_3}(z) = \sum_{n=-\infty}^{\infty} p_{X_3}(n) z^{-n} = (1 - p) + pz^{-1} \tag{0.7}$$

$$M_X(z) = ((1 - p) + pz^{-1})^3 \tag{0.8}$$

$$= \sum_{n=-\infty}^{\infty} {}^3C_n (1 - p)^{3-n} p^n z^{-n} \tag{0.9}$$

$$p_X(n) = {}^3C_n p^n (1 - p)^{3-n} \tag{0.10}$$

$$p_X(n) = \frac{{}^3C_n}{8} \tag{0.11}$$

The Probability Mass Function (PMF) for the given random variable is

$$p_X(n) = \begin{cases} \frac{1}{8}, & n = 0 \\ \frac{3}{8}, & n = 1 \\ \frac{3}{8}, & n = 2 \\ \frac{1}{8}, & n = 3 \end{cases} \tag{0.12}$$

The probability of getting exactly 2 heads is

$$p_X(2) = \frac{3}{8} \tag{0.13}$$

The Cumulative Distribution Function (CDF) for the given random variable is

$$F_X(n) = \sum_{k=-\infty}^{n} p_X(k) = \begin{cases} 0, & n < 0 \\ \frac{1}{8}, & 0 \le n < 1 \\ \frac{4}{8}, & 1 \le n < 2 \\ \frac{7}{8}, & 2 \le n < 3 \\ 1, & n \ge 3 \end{cases} \tag{0.14}$$

To find the probability of getting exactly 2 heads using the CDF:

$$P(X = 2) = F_X(2) - F_X(1) \tag{0.15}$$

$$= \frac{7}{8} - \frac{4}{8} \tag{0.16}$$

$$= \frac{3}{8} \tag{0.17}$$

The probability of getting exactly 2 heads is $\frac{3}{8}$.

Simulation Process:

To run a simulation we need to generate random numbers with uniform probability, which is done as shown below:

1) Generate a random number by calling `rand()`. It generates a random number between 0 and `RAND_MAX`.
2) Divide the generated number by `RAND_MAX` so that it becomes a real number in the range [0, 1).
3) If the number is less than p, take it as the event happened (heads), otherwise, the event did not happen (tails).

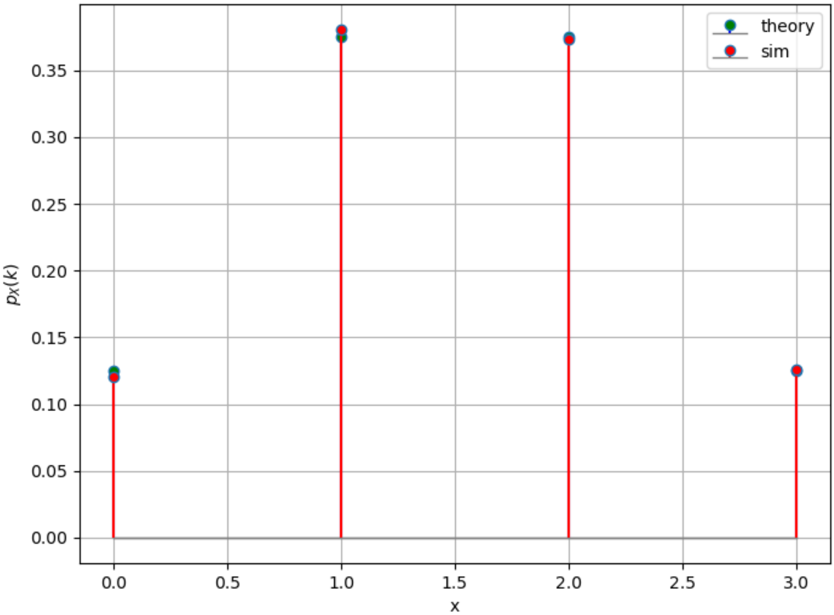

Fig. 3.1: Probability Mass Function of given Random variable

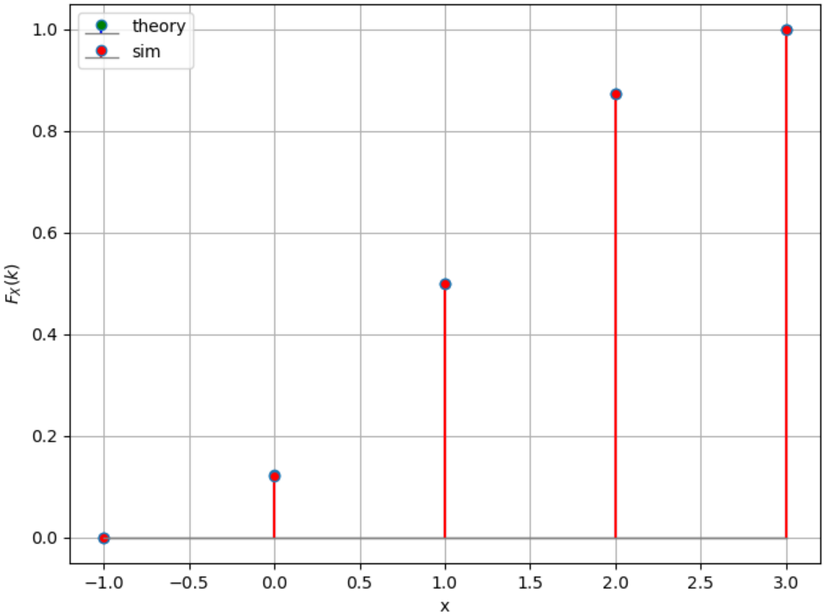

Fig. 3.2: Cumulative Distribution Function of given Random variable