Question-1-1.11-11

EE24BTECH11035 - KOTHAPALLI AKHIL

Question:
The scalar product of the vector $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of the vector $\mathbf{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\mathbf{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to 1. Find the value of λ and hence find the unit vector along $\mathbf{b} + \mathbf{c}$.
Solution:

Point	Coordinates
A	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
B	$\begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}$

TABLE 0: given vectors

Given vector $\mathbf{c}$ is:

$$\mathbf{c} = \begin{pmatrix} \lambda \\ 2 \\ 3 \end{pmatrix} \tag{1}$$

The sum of vectors $\mathbf{b}$ and $\mathbf{c}$ is:

$$\mathbf{b} + \mathbf{c} = \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix} + \begin{pmatrix} \lambda \\ 2 \\ 3 \end{pmatrix} \tag{2}$$

$$= \begin{pmatrix} 2 + \lambda \\ 4 + 2 \\ -5 + 3 \end{pmatrix} \tag{3}$$

$$= \begin{pmatrix} 2 + \lambda \\ 6 \\ -2 \end{pmatrix} \tag{4}$$

Now, we find the norm of $\mathbf{b} + \mathbf{c}$. The norm is given by:

$$\begin{aligned}
\|\mathbf{b}+\mathbf{c}\| &= \sqrt{(\mathbf{b}+\mathbf{c})^\top(\mathbf{b}+\mathbf{c})} && (5)\\
&= \sqrt{(2+\lambda)^2+6^2+(-2)^2} && (6)\\
&= \sqrt{(2+\lambda)^2+36+4} && (7)\\
&= \sqrt{(2+\lambda)^2+40} && (8)
\end{aligned}$$

The unit vector along $\mathbf{b}+\mathbf{c}$ is:

$$\hat{\mathbf{u}} = \frac{\mathbf{b}+\mathbf{c}}{\|\mathbf{b}+\mathbf{c}\|} \tag{9}$$

Given,The scalar product of $\mathbf{a}$ with this unit vector as 1:

$$\mathbf{a}^\top\hat{\mathbf{u}} = 1 \tag{10}$$

Substituting the expressions for $\mathbf{a}$ and $\hat{\mathbf{u}}$, we get:

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix} \cdot \frac{1}{\sqrt{(2+\lambda)^2+40}}\begin{pmatrix}2+\lambda\\ 6\\ -2\end{pmatrix} = 1 \tag{11}$$

$$\begin{aligned}
\frac{1}{\sqrt{(2+\lambda)^2+40}}\left((2+\lambda)+6-2\right) &= 1 && (12)\\
\frac{1}{\sqrt{(2+\lambda)^2+40}}(\lambda+6) &= 1 && (13)
\end{aligned}$$

Squaring both sides:

$$\frac{(\lambda+6)^2}{(2+\lambda)^2+40} = 1 \tag{14}$$

$$(\lambda+6)^2 = (2+\lambda)^2+40 \tag{15}$$

$$\lambda^2+12\lambda+36 = \lambda^2+4\lambda+4+40 \tag{16}$$

Simplifying:

$$\begin{aligned}
12\lambda+36 &= 4\lambda+44 && (17)\\
8\lambda &= 8 && (18)\\
\lambda &= 1 && (19)
\end{aligned}$$

Thus, $\lambda = 1$. Substituting $\lambda = 1$ into $\mathbf{b} + \mathbf{c}$:

$$\mathbf{b} + \mathbf{c} = \begin{pmatrix} 2+1 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix} \tag{20}$$

The norm of $\mathbf{b} + \mathbf{c}$ is:

$$\|\mathbf{b} + \mathbf{c}\| = \sqrt{3^2 + 6^2 + (-2)^2} = \sqrt{9 + 36 + 4} = \sqrt{49} = 7 \tag{21}$$

Thus, the unit vector along $\mathbf{b} + \mathbf{c}$ is:

$$\hat{\mathbf{u}} = \frac{1}{7} \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \\ \frac{-2}{7} \end{pmatrix} \tag{22}$$

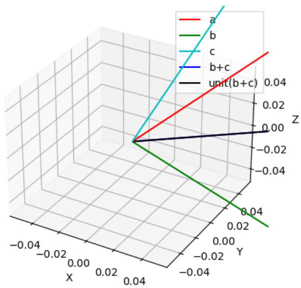

Fig. 0.1: $\mathbf{a}, \mathbf{b}, \mathbf{c}$,unit vector in the direction of $\mathbf{b} + \mathbf{c}$