

Question-1-1.11-11

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Question:

The scalar product of the vector $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of the vector $\mathbf{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\mathbf{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to 1. Find the value of λ and hence find the unit vector along $\mathbf{b} + \mathbf{c}$.

Solution:

Given vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} are:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad (0.1)$$

$$\mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}, \quad (0.2)$$

$$\mathbf{c} = \begin{pmatrix} \lambda \\ 2 \\ 3 \end{pmatrix} \quad (0.3)$$

The sum of vectors \mathbf{b} and \mathbf{c} is:

$$\mathbf{b} + \mathbf{c} = \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix} + \begin{pmatrix} \lambda \\ 2 \\ 3 \end{pmatrix} \quad (0.4)$$

$$= \begin{pmatrix} 2 + \lambda \\ 4 + 2 \\ -5 + 3 \end{pmatrix} \quad (0.5)$$

$$= \begin{pmatrix} 2 + \lambda \\ 6 \\ -2 \end{pmatrix} \quad (0.6)$$

Now, we find the norm of $\mathbf{b} + \mathbf{c}$. The norm is given by:

$$\|\mathbf{b} + \mathbf{c}\| = \sqrt{(\mathbf{b} + \mathbf{c})^\top (\mathbf{b} + \mathbf{c})} \quad (0.7)$$

$$= \sqrt{(2 + \lambda)^2 + 6^2 + (-2)^2} \quad (0.8)$$

$$= \sqrt{(2 + \lambda)^2 + 36 + 4} \quad (0.9)$$

$$= \sqrt{(2 + \lambda)^2 + 40} \quad (0.10)$$

The unit vector along $\mathbf{b} + \mathbf{c}$ is:

$$\hat{\mathbf{u}} = \frac{\mathbf{b} + \mathbf{c}}{\|\mathbf{b} + \mathbf{c}\|} \quad (0.11)$$

Given, The scalar product of \mathbf{a} with this unit vector as 1:

$$\mathbf{a}^\top \hat{\mathbf{u}} = 1 \quad (0.12)$$

Substituting the expressions for \mathbf{a} and $\hat{\mathbf{u}}$, we get:

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{(2+\lambda)^2 + 40}} \begin{pmatrix} 2+\lambda \\ 6 \\ -2 \end{pmatrix} = 1 \quad (0.13)$$

$$\frac{1}{\sqrt{(2+\lambda)^2 + 40}} ((2+\lambda) + 6 - 2) = 1 \quad (0.14)$$

$$\frac{1}{\sqrt{(2+\lambda)^2 + 40}} (\lambda + 6) = 1 \quad (0.15)$$

Squaring both sides:

$$\frac{(\lambda + 6)^2}{(2 + \lambda)^2 + 40} = 1 \quad (0.16)$$

$$(\lambda + 6)^2 = (2 + \lambda)^2 + 40 \quad (0.17)$$

$$\lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 4 + 40 \quad (0.18)$$

Simplifying:

$$12\lambda + 36 = 4\lambda + 44 \quad (0.19)$$

$$8\lambda = 8 \quad (0.20)$$

$$\lambda = 1 \quad (0.21)$$

Thus, $\lambda = 1$. Substituting $\lambda = 1$ into $\mathbf{b} + \mathbf{c}$:

$$\mathbf{b} + \mathbf{c} = \begin{pmatrix} 2+1 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix} \quad (0.22)$$

The norm of $\mathbf{b} + \mathbf{c}$ is:

$$\|\mathbf{b} + \mathbf{c}\| = \sqrt{3^2 + 6^2 + (-2)^2} = \sqrt{9 + 36 + 4} = \sqrt{49} = 7 \quad (0.23)$$

Thus, the unit vector along $\mathbf{b} + \mathbf{c}$ is:

$$\hat{\mathbf{u}} = \frac{1}{7} \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \\ \frac{-2}{7} \end{pmatrix} \quad (0.24)$$