Question-1-1.11-11

EE24BTECH11035 - KOTHAPALLI AKHIL

Question:
The scalar product of the vector $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of the vector $\mathbf{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\mathbf{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to 1. Find the value of λ and hence find the unit vector along $\mathbf{b} + \mathbf{c}$.
Solution:
Given vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{0.1}$$

$$\mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}, \tag{0.2}$$

$$\mathbf{c} = \begin{pmatrix} \lambda \\ 2 \\ 3 \end{pmatrix} \tag{0.3}$$

The sum of vectors $\mathbf{b}$ and $\mathbf{c}$ is:

$$\mathbf{b} + \mathbf{c} = \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix} + \begin{pmatrix} \lambda \\ 2 \\ 3 \end{pmatrix} \tag{0.4}$$

$$= \begin{pmatrix} 2 + \lambda \\ 4 + 2 \\ -5 + 3 \end{pmatrix} \tag{0.5}$$

$$= \begin{pmatrix} 2 + \lambda \\ 6 \\ -2 \end{pmatrix} \tag{0.6}$$

Now, we find the norm of $\mathbf{b} + \mathbf{c}$. The norm is given by:

$$\|\mathbf{b} + \mathbf{c}\| = \sqrt{(\mathbf{b} + \mathbf{c})^\top (\mathbf{b} + \mathbf{c})} \tag{0.7}$$

$$= \sqrt{(2 + \lambda)^2 + 6^2 + (-2)^2} \tag{0.8}$$

$$= \sqrt{(2 + \lambda)^2 + 36 + 4} \tag{0.9}$$

$$= \sqrt{(2 + \lambda)^2 + 40} \tag{0.10}$$

The unit vector along $\mathbf{b} + \mathbf{c}$ is:

$$\hat{\mathbf{u}} = \frac{\mathbf{b} + \mathbf{c}}{||\mathbf{b} + \mathbf{c}||} \tag{0.11}$$

Given,The scalar product of **a** with this unit vector as 1:

$$\mathbf{a}^\top \hat{\mathbf{u}} = 1 \tag{0.12}$$

Substituting the expressions for **a** and $\hat{\mathbf{u}}$, we get:

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{(2+\lambda)^2 + 40}} \begin{pmatrix} 2+\lambda \\ 6 \\ -2 \end{pmatrix} = 1 \tag{0.13}$$

$$\frac{1}{\sqrt{(2+\lambda)^2 + 40}} \left((2+\lambda) + 6 - 2\right) = 1 \tag{0.14}$$

$$\frac{1}{\sqrt{(2+\lambda)^2 + 40}} (\lambda + 6) = 1 \tag{0.15}$$

Squaring both sides:

$$\frac{(\lambda + 6)^2}{(2+\lambda)^2 + 40} = 1 \tag{0.16}$$

$$(\lambda + 6)^2 = (2+\lambda)^2 + 40 \tag{0.17}$$

$$\lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 4 + 40 \tag{0.18}$$

Simplifying:

$$12\lambda + 36 = 4\lambda + 44 \tag{0.19}$$

$$8\lambda = 8 \tag{0.20}$$

$$\lambda = 1 \tag{0.21}$$

Thus, $\lambda = 1$. Substituting $\lambda = 1$ into $\mathbf{b} + \mathbf{c}$:

$$\mathbf{b} + \mathbf{c} = \begin{pmatrix} 2+1 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix} \tag{0.22}$$

The norm of $\mathbf{b} + \mathbf{c}$ is:

$$\|\mathbf{b} + \mathbf{c}\| = \sqrt{3^2 + 6^2 + (-2)^2} = \sqrt{9 + 36 + 4} = \sqrt{49} = 7 \tag{0.23}$$

Thus, the unit vector along $\mathbf{b} + \mathbf{c}$ is:

$$\hat{\mathbf{u}} = \frac{1}{7}\begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \\ \frac{-2}{7} \end{pmatrix} \tag{0.24}$$

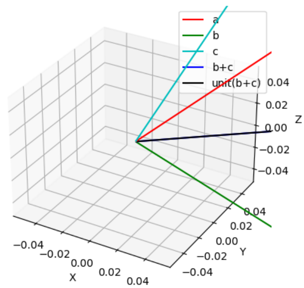

Fig. 0.1: $\mathbf{a}, \mathbf{b}, \mathbf{c}$,unit vector in the direction of $\mathbf{b} + \mathbf{c}$