Question-9-9.2-33

EE24BTECH11035 - KOTHAPALLI AKHIL

Question:
Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$.
Solution:
We will express both the parabola and the line equations in matrix form and use the given method to calculate the area between them.

The general conic form for a parabola $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$ can be represented by matrices:

$$V = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad u = \begin{pmatrix} d \\ e \end{pmatrix}, \quad f \tag{1}$$

For the parabola $x^2 = y$, the matrix representation becomes:

$$V_{\text{parabola}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad u_{\text{parabola}} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, \quad f_{\text{parabola}} = 0 \tag{2}$$

The line equation $y = x + 2$ can also be expressed in matrix form as:

$$h^T x + m = 0 \tag{3}$$

Where h is the vector of coefficients and m is the constant.
For $y = x + 2$, we have:

$$h_{\text{line}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad m_{\text{line}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4}$$

To find the intersection points of the parabola $x^2 = y$ and the line $y = x + 2$, we substitute $y = x + 2$ into the parabola equation:

$$x^2 = x + 2 \tag{5}$$

$$x^2 - x - 2 = 0 \tag{6}$$

$$(x - 2)(x + 1) = 0 \tag{7}$$

Thus, $x = 2$ or $x = -1$.
Substitute these x-values back into the line equation to get the corresponding y-values:

$$y(2) = 2 + 2 = 4, \tag{8}$$

$$y(-1) = -1 + 2 = 1 \tag{9}$$

So, the points of intersection are $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$.
The area between the parabola and the line is given by the integral of the difference between the two curves.
The general form for calculating the area between two curves y_1 and y_2 from x_1 to x_2 is:

$$Area = \int_{x_1}^{x_2} (y_1 - y_2)\, dx \tag{10}$$

Here, $y_1 = x + 2$ and $y_2 = x^2$.

Thus, the area is:

$$\text{Area} = \int_{-1}^{2} \left((x + 2) - x^2 \right) dx \tag{11}$$

$$\int_{-1}^{2} \left(x + 2 - x^2 \right) dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} \tag{12}$$

Substitute the limits:

$$= \left[\frac{2^2}{2} + 2(2) - \frac{2^3}{3} \right] - \left[\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3} \right] \tag{13}$$

$$= \left[\frac{4}{2} + 4 - \frac{8}{3} \right] - \left[\frac{1}{2} - 2 + \frac{1}{3} \right] \tag{14}$$

$$= \left[2 + 4 - \frac{8}{3} \right] - \left[\frac{1}{2} - 2 + \frac{1}{3} \right] \tag{15}$$

$$= \left(6 - \frac{8}{3} \right) - \left(-\frac{3}{2} + \frac{1}{3} \right) \tag{16}$$

$$= \frac{10}{3} + \frac{11}{6} = \frac{60 + 22}{18} = \frac{82}{18} = \frac{41}{9} \tag{17}$$

Thus, the area of the region enclosed by the parabola and the line is $\frac{41}{9}$ square units.

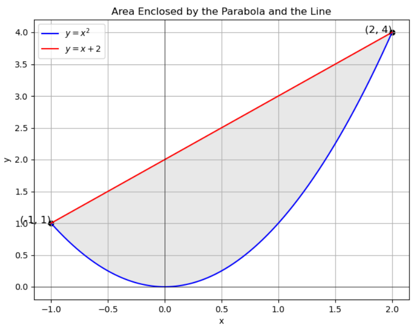

Fig. 0: Area enclosed between parabola and Line