Question-9-9.2-23

EE24BTECH11035 - KOTHAPALLI AKHIL

Question:
Find the area of the region enclosed by the parabola $y^2 = 4x$ and the line $x = 3$.
Solution:

The general conic form for a parabola $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$ can be represented by matrices:

$$V = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad u = \begin{pmatrix} d \\ e \end{pmatrix}, \quad f \tag{1}$$

For the parabola $y^2 = 4x$,

$$V_{\text{parabola}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad u_{\text{parabola}} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \quad f_{\text{parabola}} = 0 \tag{2}$$

The line equation can be expressed in matrix form as:

$$h^T x + m = 0 \tag{3}$$

Where h is the vector of coefficients and m is the constant.
For $x = 3$,

$$h_{\text{line}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad m_{\text{line}} = -3 \tag{4}$$

Substituting $x = 3$ into the parabola equation $y^2 = 4x$:

$$y^2 = 4(3) = 12 \tag{5}$$

$$y = \pm 2\sqrt{3} \tag{6}$$

The points of intersection are $\begin{pmatrix} 3 \\ 2\sqrt{3} \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -2\sqrt{3} \end{pmatrix}$.
Calculating area between parabola and line from $y = -2\sqrt{3}$ to $y = 2\sqrt{3}$ is:

$$\text{Area} = 2\int_0^{2\sqrt{3}} \left(3 - \frac{y^2}{4}\right) dy \tag{7}$$

$$\int \left(3 - \frac{y^2}{4}\right) dy = 3y - \frac{y^3}{12} \tag{8}$$

Substituting the limits,

$$\left[3y - \frac{y^3}{12}\right]_0^{2\sqrt{3}} = 3(2\sqrt{3}) - \frac{(2\sqrt{3})^3}{12} \tag{9}$$

$$= 6\sqrt{3} - \frac{8(3\sqrt{3})}{12} = 6\sqrt{3} - 2\sqrt{3} = 4\sqrt{3} \tag{10}$$

Thus, the area of the region enclosed by the parabola and the line is:

$$\text{Area} = 2 \times 4\sqrt{3} = 8\sqrt{3} \text{ square units.} \tag{11}$$

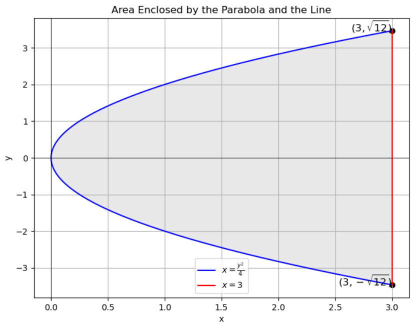

Fig. 0: Area enclosed between parabola and Line