NCERT-8.1.ex.5

EE24BTECH11065 - Spoorthi yellamanchali

Question:
Find the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ and the ordinates $x = 0$ and $x = ae$, where, $b^2 = a^2\left(1 - e^2\right)$ and $e < 1$.
Theoretical Solution: On substituting $x = ae$ and $b^2 = a^2\left(1 - e^2\right)$,we get,

$$\frac{(ae)^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \tag{0.1}$$

$$e^2 + \frac{y^2}{a^2(1-e^2)} = 1 \tag{0.2}$$

$$\frac{y^2}{a^2(1-e^2)} = 1 - e^2 \tag{0.3}$$

$$y^2 = a^2\left(1-e^2\right)^2 \tag{0.4}$$

$$y = \left|a\left(1-e^2\right)\right| \tag{0.5}$$

$$y = \pm a\left(1-e^2\right) \tag{0.6}$$

From the equation of ellipse , we can write the expression for y as,

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2} \tag{0.7}$$

Area A enclosed is equal to

$$A = 2\int_0^{ae} \frac{b}{a}\sqrt{a^2 - x^2}\,dx \tag{0.8}$$

$$= \frac{2b}{a}\left[ae\sqrt{a^2 - (ae)^2} + a^2\sin^{-1} e\right] \tag{0.9}$$

$$= ab\left[e\sqrt{1 - e^2} + \sin^{-1} e\right] \tag{0.10}$$

Solution by using the trapezoidal rule:
In the trapezoidal rule,we calculate the area of the curve by breaking the whole area into trapeziums of small areas and adding them up.

Under trapezoidal rule,Area A is approximated by,

$$J = \int_a^b f(x)\,dx \approx h\left(\frac{1}{2}\left(f(a) + f(x_1)\right) + \frac{1}{2}\left(f(x_1) + f(x_2)\right) + ... + \frac{1}{2}\left(f(x_{n-1}) + f(b)\right)\right) \tag{0.11}$$

$$J = \int_a^b f(x)\,dx \approx h\left(\frac{1}{2}f(a) + f(x_1) + f(x_2) + ... + f(x_{n-1} + \frac{1}{2}f(b)\right) \tag{0.12}$$

Where h is assumed to be the distance between parallel sides of trapezium and its value is close to zero.

$$h = \frac{b-a}{2} \tag{0.13}$$

and, On observing equation (0.12), we can see that,
If $A(x_n)$ is area enclosed by the curve $f(x)$, from $x = x_0$ to $x = x_n$,then,

$$A(x_n) = A(x_{n-1}) + \frac{1}{2}h\left(f(x_{n-1}) + f(n)\right) \tag{0.14}$$

From the method of finite differences , we know that,

$$x_n = x_{n-1} + h \tag{0.15}$$

$$y_n = y_{n-1} + h\left(\frac{dy}{dx}|_{x=x_{n-1}}\right) \tag{0.16}$$

On differentiating equation (0.7), and substituting the expression for $\frac{dy}{dx}$ in equation (0.16), we get the difference equation for the curve which is,

$$y_n = y_{n-1} + h\left(\frac{-bx}{a\sqrt{a^2 - x^2}}\right) \tag{0.17}$$

$$A_{n+1} = A_n + \frac{1}{2}h\left((y_n + hy'_n) + y_n\right) \tag{0.18}$$

$$A_{n+1} = A_n + \frac{1}{2}h\left(2y_n + hy'_n\right) \tag{0.19}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{0.20}$$

$$\tag{0.21}$$

In the given question, $y_n = \frac{b}{a}\sqrt{a^2 - x_n^2}$ and $y'_n = \frac{-bx_n}{\left(a\sqrt{a^2 - x_n^2}\right)}$

The general difference equation will be given by

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{0.22}$$

$$A_{n+1} = A_n + h\left(\frac{b}{a}\sqrt{a^2 - x_n^2}\right) + \frac{1}{2}h^2\left(\frac{-bx_n}{\left(\sqrt{a^2 - x_n^2}\right)}\right) \tag{0.23}$$

$$\tag{0.24}$$

We know that,

$$x_0 = a \tag{0.25}$$

$$x_n = ae \tag{0.26}$$

On assuming a value for h,close to zero and substituting in equation and on iterating till we reach $x_n = ae$ will return an area.
$\therefore$ our required area is twice the returned area.
For our plot, let us take $a = 4$,$e = \frac{\sqrt{7}}{4}$ and $h = 0.01$.
In this case we get,
Theoretical area = 14.625751423656318
Area calculated by using the trapezoidal rule = 14.644422362727173