NCERT-12.8.ex.14

EE24BTECH11043 - Murra Rajesh Kumar Reddy

Question:
Find the area of the given region :
$\left\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\right\}$
Solution:
Theoretical Solution:
Given regions are,

$$A_1 = \left\{(x, y) : 0 \le y \le x^2 + 1\right\} \tag{0.1}$$

$$A_2 = \{(x, y) : 0 \le y \le x + 1\} \tag{0.2}$$

$$A_3 = \{(x, y) : 0 \le x \le 2\} \tag{0.3}$$

Let's say the area we need to find is of function $f(x)$ within the limits of a and b along x-axis then the required area is given by

$$Area = \int_a^b f(x)\, dx \tag{0.4}$$

From the equation (0.3) we can say

$$a = 0 \tag{0.5}$$

$$b = 2 \tag{0.6}$$

From equations (0.1) and (0.2) we can say point of intersection of A_1 and A_2 is **Point of intersection**
Expressing the equation of parabola in matrix form $g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$,

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 1 = 0 \tag{0.7}$$

The general form of a line equation can be expressed as

$$\mathbf{m}^\top \mathbf{x} = c \tag{0.8}$$

For $y = x + 1$

$$\mathbf{m} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad c = 1 \tag{0.9}$$

Intersection of a line and a conic is given by,

$$\kappa_i = \frac{-\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{\left[\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})\right]^2 - g(h)\left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right)}}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \tag{0.10}$$

On substituting and solving
The intersection point is $(1, 2)$ and $f(x)$ will be

$$f(x) = x^2 + 1 \qquad (0 \le x \le 1) \tag{0.11}$$

$$f(x) = x + 1 \qquad (1 \le x \le 2) \tag{0.12}$$

$$Area = \int_0^1 \left(x^2 + 1\right) dx + \int_1^2 (x + 1)\, dx \tag{0.13}$$

By computing each integral we get

$$\int_0^1 \left(x^2 + 1\right) dx = \left[\frac{x^3}{3} + x\right]_0^1 = \left[\frac{4}{3} - 0\right] = \frac{4}{3} \tag{0.14}$$

$$\int_1^2 (x + 1)\, dx = \left[\frac{x^2}{2} + x\right]_1^2 = \left[4 - \frac{3}{2}\right] = \frac{5}{2} \tag{0.15}$$

From the equations (0.14) and (0.15)

$$Area = \frac{23}{6} \tag{0.16}$$

Total area of given regions is $\frac{23}{6}$
Computational Solution:
Logic:
According trapezoidal rule the given integral will be:

$$\int_a^b f(x)\, dx \approx \sigma_{k=1}^{N} \frac{f(x_{k+1} + f(x_k))}{2} h \qquad h = \frac{b - a}{N} \tag{0.17}$$

$\therefore$ The equation obtained is

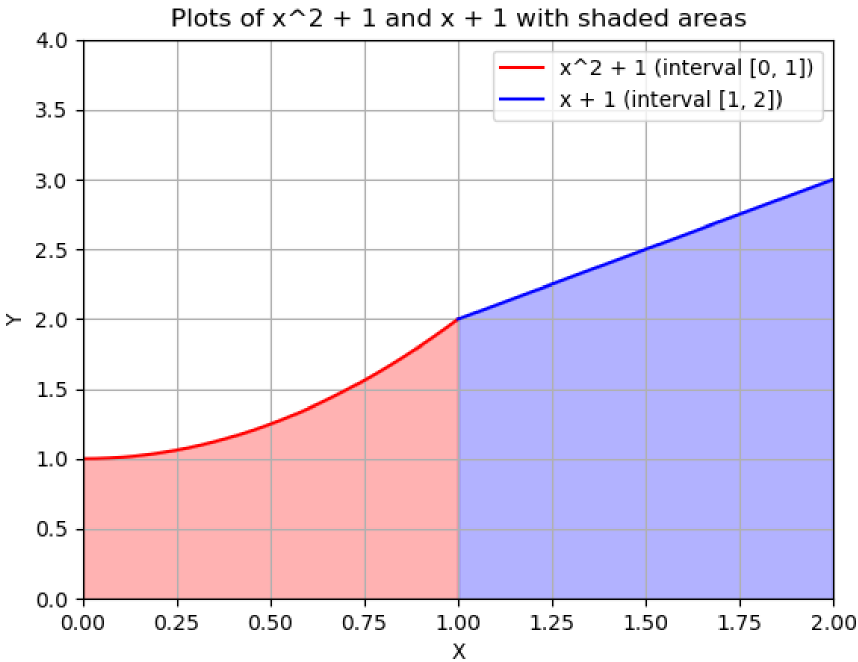
$$Area = \int_a^b f(x)\, dx \approx h\left(\frac{1}{2} f(a) + f(x_1) + f(x_2) + \cdots + f(x_n - 1) + \frac{1}{2} f(b)\right) \tag{0.18}$$

$$Area = j_n \quad \left(j_{i+1} = j_i + h\frac{f(x_{i+1} + f(x_i))}{2}\right) \tag{0.19}$$

$$x_{i+1} = x_i + h \tag{0.20}$$

$$N = 100000 \tag{0.21}$$

Using the code the answer we get is


Plots of x^2 + 1 and x + 1 with shaded areas
x^2 + 1 (interval [0, 1])
x + 1 (interval [1, 2])
Y
X
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00