4.2.1.1

EE24BTECH11004 - Ankit Jainar

Question Find the roots of quadratic equation:

$$x^2 - 3x - 10 = 0 \tag{0.1}$$

Solution

The given equation can be solved using analytical and numerical methods. Let us explore both approaches.

Quadratic Formula

The standard quadratic equation is:

$$ax^2 + bx + c = 0 \tag{0.2}$$

Here, $a = 1, b = -3, c = -10$. The roots are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{0.3}$$

Substitute the values of a, b, and c:

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-10)}}{2(1)} \tag{0.4}$$

$$x = \frac{3 \pm \sqrt{9 + 40}}{2} \tag{0.5}$$

$$x = \frac{3 \pm \sqrt{49}}{2} \tag{0.6}$$

Simplify further:

$$x_1 = \frac{3+7}{2} = 5, \quad x_2 = \frac{3-7}{2} = -2 \tag{0.7}$$

Thus, the roots of the equation are:

$$x_1 = 5, \quad x_2 = -2 \tag{0.8}$$

Solution using Matrix Approach by finding eigen values

Matrix-Based Method

For a polynomial equation of the form:

$$x^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0 = 0 \tag{0.9}$$

we construct a matrix called the *companion matrix*, which is defined as:

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-1} \end{bmatrix} \tag{0.10}$$

The eigenvalues of this matrix are the roots of the given polynomial equation.

For the quadratic equation $x^2 - 3x - 10 = 0$, we can write it as:

$$x^2 + (-3)x + (-10) = 0 \tag{0.11}$$

The coefficients are:

$$b_1 = -3, \quad b_0 = -10$$

The companion matrix for this equation is:

$$\Lambda = \begin{bmatrix} 0 & 1 \\ -(-10) & -(-3) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 10 & 3 \end{bmatrix} \tag{0.12}$$

Eigenvalue Computation

The eigenvalues of Λ are obtained by solving:

$$\det(\Lambda - \lambda I) = 0 \tag{0.13}$$

Substitute Λ:

$$\begin{vmatrix} 0-\lambda & 1 \\ 10 & 3-\lambda \end{vmatrix} = 0 \tag{0.14}$$

Simplify the determinant:

$$(-\lambda)(3-\lambda) - (10)(1) = 0 \tag{0.15}$$

$$\lambda^2 - 3\lambda - 10 = 0 \tag{0.16}$$

This is the original quadratic equation, so the eigenvalues are:

$$\lambda_1 = 5, \quad \lambda_2 = -2 \tag{0.17}$$

Solution using Fixed Point Iteration

We rewrite the equation as:

$$x = g(x) \tag{0.18}$$

A possible choice for $g(x)$ is:

$$g(x) = \sqrt{3x + 10} \tag{0.19}$$

The iterative update becomes:

$$x_{n+1} = \sqrt{3x_n + 10} \tag{0.20}$$

Starting with an initial guess $x_0 = 2$, the iterations are as follows:

$$x_1 = \sqrt{3(2) + 10} = \sqrt{16} = 4 \tag{0.21}$$

$$x_2 = \sqrt{3(4) + 10} = \sqrt{22} \approx 4.69 \tag{0.22}$$

$$x_3 = \sqrt{3(4.69) + 10} \approx 5.14 \tag{0.23}$$

$$\vdots \tag{0.24}$$

Observation: The iterations converge to $x = 5$, one of the roots of the quation. For $x_2 = -2$, a similar setup with $g(x) = -\sqrt{3x + 10}$ would be used.

Newton-Raphson Method

The Newton-Raphson method is defined as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{0.25}$$

Here:

$$f(x) = x^2 - 3x - 10, \quad f'(x) = 2x - 3 \tag{0.26}$$

Substitute into the formula:

$$x_{n+1} = x_n - \frac{x_n^2 - 3x_n - 10}{2x_n - 3} \tag{0.27}$$

Example: Starting with an initial guess $x_0 = 3$:

$$x_1 = 3 - \frac{3^2 - 3(3) - 10}{2(3) - 3} = 3 - \frac{9 - 9 - 10}{6 - 3} = 3 + \frac{10}{3} \approx 6.33 \tag{0.28}$$

$$x_2 = 6.33 - \frac{6.33^2 - 3(6.33) - 10}{2(6.33) - 3} \approx 5.02 \tag{0.29}$$

Observation: The iterations quickly converge to $x = 5$. Similarly, starting with $x_0 = -1$ converges to $x = -2$.

Computational Approach

The following results were obtained using a computational method:

$$\text{Running Fixed Point Iterations Method:Root 1: } 5$$
$$\text{Root 2: } -2$$
$$\text{Running Newton-Raphson Method:}$$
$$\text{Root 1: } 5$$
$$\text{Root 2: } -2$$

Conclusion

The roots of the quadratic equation $x^2 - 3x - 10 = 0$ are:

$$x_1 = 5, \quad x_2 = -2 \tag{0.30}$$

Both numerical and analytical methods confirm these results. Here are the roots of the Quadratic Equation

$$x^2 - 3x - 10 = 0 \tag{0.31}$$

in plotted below.