NCERt - 11.16.1.2

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Problem Statement:
A fair six-sided die is rolled twice. Find the Probability Mass Function (PMF) and the Moment-Generating Function (MGF) using the Z-Transform. Verify the PMF through simulation.

Solution:
Let X be a discrete random variable representing the sum of two independent die rolls:

$$X = X_1 + X_2 \tag{1}$$

where X_1, X_2 are independent uniform discrete random variables:

$$X_1, X_2 \sim \text{Uniform}(\{1, 2, 3, 4, 5, 6\}) \tag{2}$$

Probability Mass Function (PMF):
The probability mass function (PMF) is given by:

$$p_X(n) = P(X = n) = \frac{\text{number of ways to obtain } n}{36} \tag{3}$$

where $36 = 6 \times 6$ is the total number of possible outcomes.
Explicit PMF Calculation:

$$p_X(n) = \begin{cases} \frac{1}{36}, & n = 2 \\ \frac{2}{36}, & n = 3 \\ \frac{3}{36}, & n = 4 \\ \frac{4}{36}, & n = 5 \\ \frac{5}{36}, & n = 6 \\ \frac{6}{36}, & n = 7 \\ \frac{5}{36}, & n = 8 \\ \frac{4}{36}, & n = 9 \\ \frac{3}{36}, & n = 10 \\ \frac{2}{36}, & n = 11 \\ \frac{1}{36}, & n = 12 \end{cases} \tag{4}$$

Moment-Generating Function (MGF) Using the Z-Transform: The Z-transform of the PMF is given by:

$$M_{X_i}(z) = \sum_{n=-\infty}^{\infty} p_{X_i}(n) z^{-n} \tag{5}$$

Since X_i takes values from 1 to 6 with equal probability:

$$M_{X_i}(z) = \sum_{k=1}^{6} \frac{1}{6} z^{-k} \tag{6}$$

which simplifies to:

$$M_{X_i}(z) = \frac{1}{6}\left(z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6}\right) \tag{7}$$

Since X_1 and X_2 are independent, their total MGF is:

$$M_X(z) = M_{X_1}(z) M_{X_2}(z) \tag{8}$$

$$= \left(\frac{1}{6}\sum_{k=1}^{6} z^{-k}\right) \times \left(\frac{1}{6}\sum_{k=1}^{6} z^{-k}\right) \tag{9}$$

Expanding this multiplication gives:

$$M_X(z) = \frac{1}{36}\sum_{m=2}^{12} c_m z^{-m} \tag{10}$$

where c_m represents the number of ways to obtain sum m.

Simulation:

We simulate this process by generating uniform random numbers representing die rolls. The algorithm follows these steps:

1) Generate a uniform random integer between 1 and 6 for two dice.
2) Compute their sum.
3) Repeat this process for 10^5 simulations.
4) Count occurrences of each possible value (2 through 12).
5) Divide by the total number of trials to get the probability estimate.

The graph below shows the comparison between the theoretically calculated and simulated PMF of the given random variable.

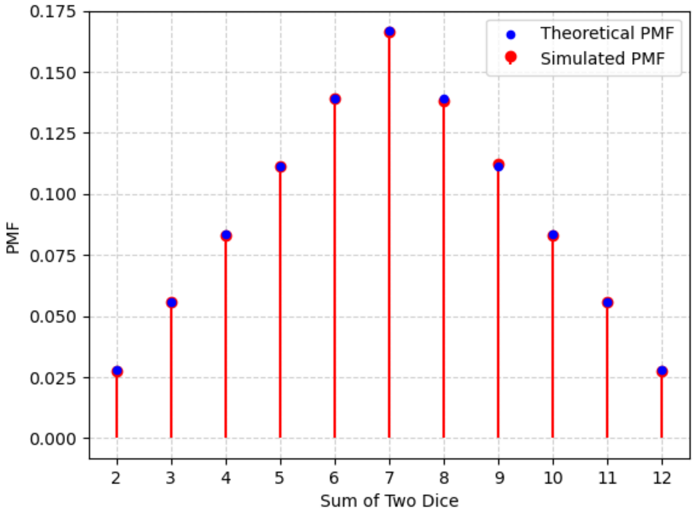

Fig. 1: Theoretical vs Simulated PMF of Sum of Two Dice Rolls