BLM2041 Signals and Systems

Syllabus

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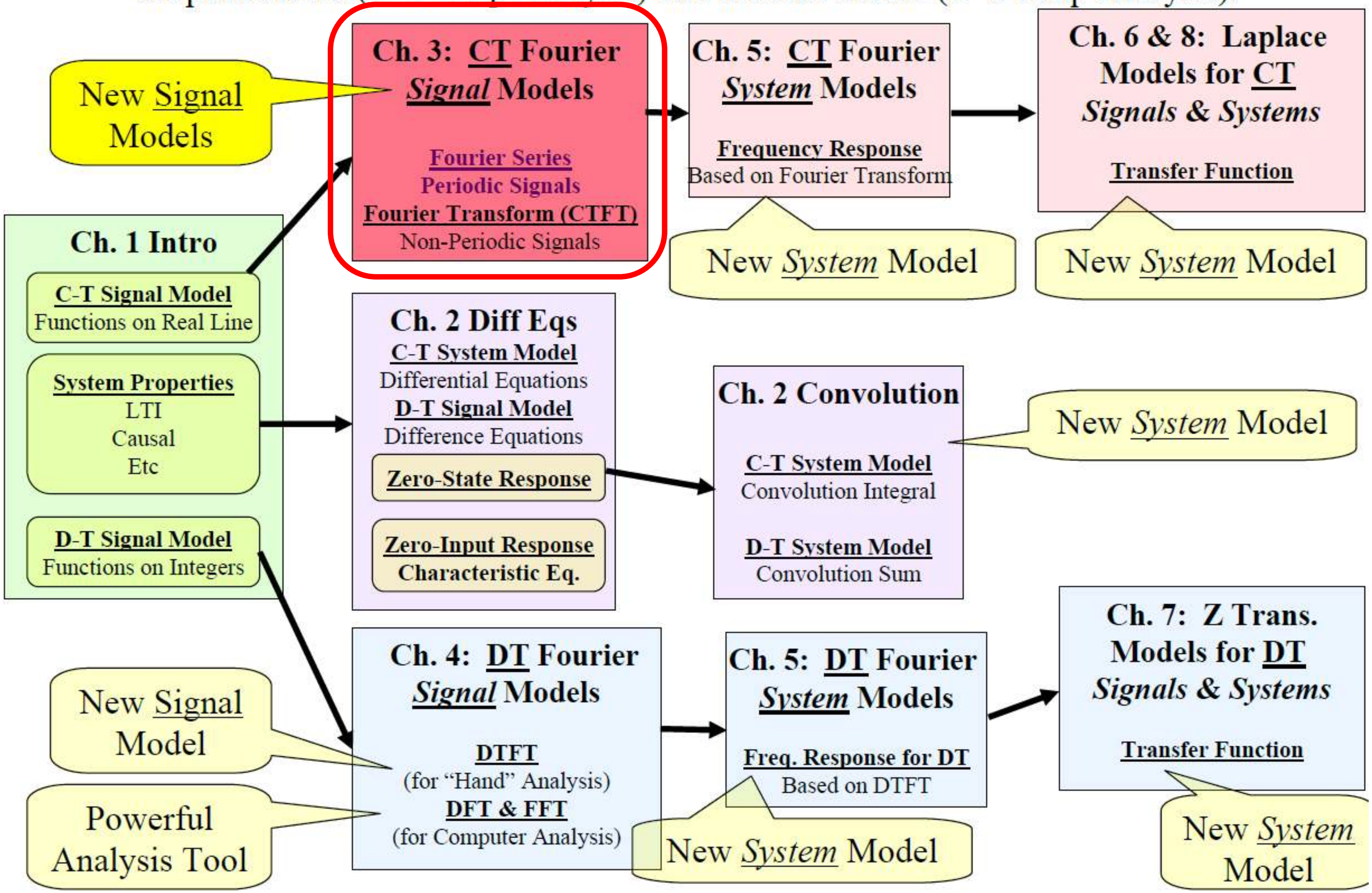
Where are we now?

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



Fourier Transform

Recall: Fourier Series represents a periodic signal as a sum of sinusoids

or complex sinusoids $e^{jk\omega_0 t}$

Note: Because the FS uses "harmonically related" frequencies $k\omega_0$, it can only create periodic signals

Q: Can we modify the FS idea to handle non-periodic signals?

A: Yes!!

What about $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_k t}$?

With *arbitrary* discrete frequencies… NOT harmonically related

This will give some non-periodic signals but not all signals of interest!!

The problem with this is that it cannot include all possible frequencies!

No matter how close we try to choose the discrete frequencies ω_k there are always some left out of the sum!!!

We need some way to include ALL frequencies!!

Fourier Transform

How about:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

Yes… this will work for any *practical* non-periodic signal!!

Called the "Fourier Integral" also, more commonly, called the "**Inverse Fourier Transform**"

Plays the role of c_k

Plays the role of $e^{jk\omega_0 t}$

Integral replaces sum because it can "add up over the ***continuum*** of frequencies"!

Okay… given $x(t)$ how do we get $X(\omega)$?

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Called the "**Fourier Transform**" of $x(t)$

Note: $X(\omega)$ is complex-valued function of $\omega \in (-\infty, \infty)$

$|X(\omega)|$ $\quad\quad$ $\angle X(\omega)$

Need to use two plots to show it

Fourier Transform

Comparison of FT and FS

Fourier Series: Used for periodic signals

Fourier Transform: Used for non-periodic signals (although we will see later that it can also be used for periodic signals)

	Synthesis	**Analysis**
Fourier Series	$x(t)=\sum_{n=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ **Fourier Series**	$c_k=\frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}dt$ **Fourier Coefficients**
Fourier Transform	$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$ **Inverse Fourier Transform**	$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ **Fourier Transform**

FS coefficients c_k are a complex-valued function of integer k

FT $X(\omega)$ is a complex-valued function of the variable $\omega \in (-\infty, \infty)$

Fourier Transform Defined

- For non-periodic signals

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis
(Inverse Transform)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis
(Forward Transform)

$$\text{Time - Domain} \Leftrightarrow \text{Frequency - Domain}$$

$$x(t) \Leftrightarrow X(j\omega)$$

Fourier Transform

Synthesis Viewpoints:

FS: $x(t) = \sum_{n=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

$|c_k|$ shows how much there is of the signal at frequency $k\omega_0$

$\angle c_k$ shows how much phase shift is needed at frequency $k\omega_0$

We need two plots to show these

FT: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

$|X(\omega)|$ shows how much there is in the signal at frequency ω

$\angle X(\omega)$ shows how much phase shift is needed at frequency ω

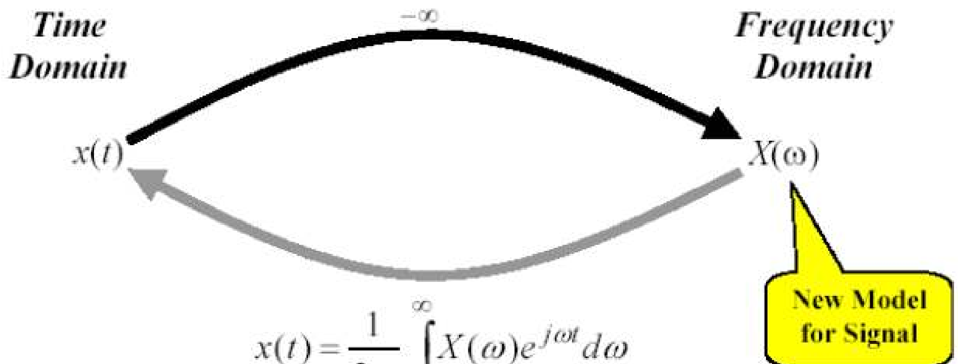
We need two plots to show these

Fourier Transform

Fourier Transform Viewpoint

View FT as a transformation into a new "domain"

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$



$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$x(t)$ is the "time domain" description of the signal
$X(\omega)$ is the "frequency domain" description of the signal

Alternate Notations

1. $x(t) \leftrightarrow X(\omega)$

2. $X(\omega) = \mathfrak{F}\{x(t)\}$

   $\Rightarrow$ $\mathfrak{F}\{\ \}$ is an "operator on" $x(t)$ to give $X(\omega)$

3. $x(t) = \mathfrak{F}^{-1}\{X(\omega)\}$

   $\Rightarrow$ $\mathfrak{F}^{-1}\{\}$ is an "operator on" $X(\omega)$ to give $x(t)$

Analogy: Looking at $X(\omega)$ is "like" looking at an x-ray of the signal- in the sense that an x-ray lets you see what is inside the object… shows what stuff it is made from.

In this sense: $X(\omega)$ shows what is "inside" the signal – it shows how much of each complex sinusoid is "inside" the signal

Note: $x(t)$ completely determines $X(\omega)$

$X(\omega)$ completely determines $x(t)$

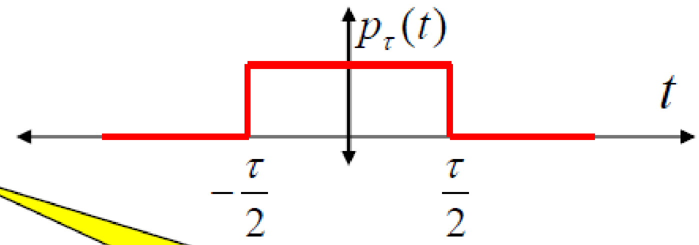
There are some advanced mathematical issues that can be hurled at these comments… we'll not worry about them

Fourier Transform

Example: FT of a Rectangular pulse

τ = pulse width

Given: a rectangular pulse signal $p_\tau(t)$



Find: $P_\tau(\omega)$… the FT of $p_\tau(t)$

Note the Notational Convention: lower-case for time signal and corresponding upper-case for its FT

Recall: we use this symbol to indicate a rectangular pulse with width τ

Solution: (Here we'll directly do the integral… but later we'll use the "FT Table")

Note that

$$p_\tau(t) = \begin{cases} 1, & -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0, & otherwise \end{cases}$$

Fourier Transform

Now apply the definition of the FT:

$$P_\tau(\omega) = \int_{-\infty}^{\infty} p_\tau(t) e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt$$

Limit integral to where $p_\tau(t)$ is non-zero… and use the fact that it is 1 over that region

$$= \frac{-1}{j\omega}\left[e^{-j\omega t}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \frac{2}{\omega}\underbrace{\left[\frac{e^{j\frac{\omega\tau}{2}} - e^{-j\frac{\omega\tau}{2}}}{j2}\right]}_{= \sin\left(\frac{\omega\tau}{2}\right)}$$

Artificially inserted 2 in numerator and denominator

Use Euler's Formula

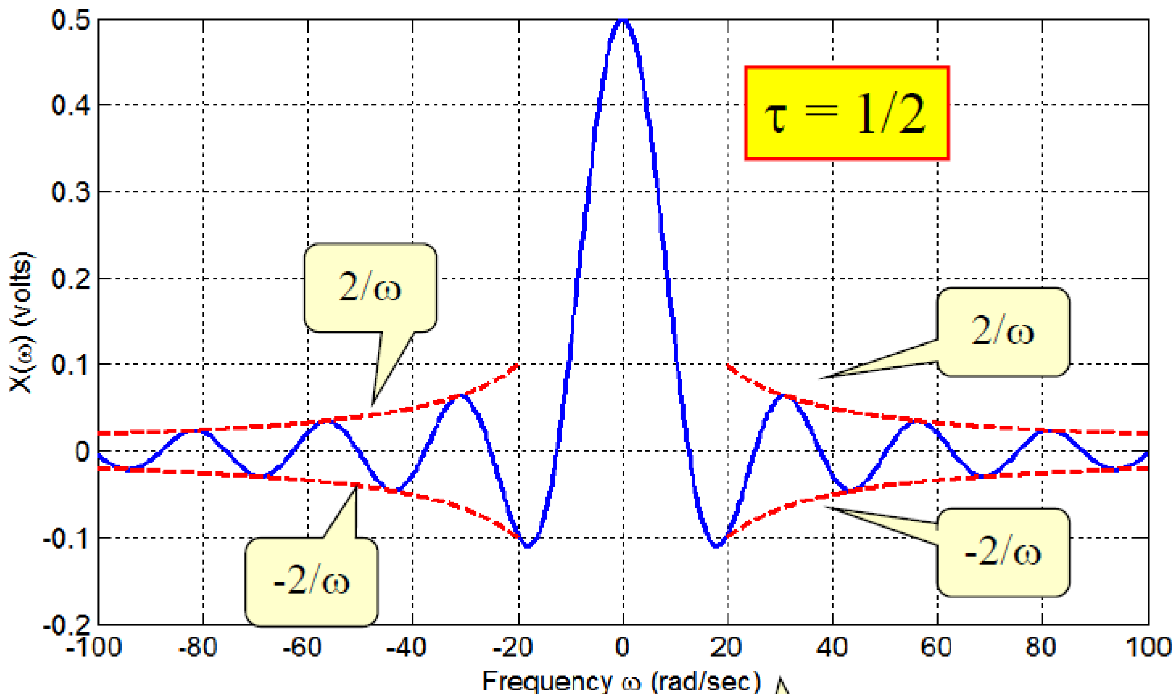
$$\boxed{P_\tau(\omega) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega}}$$

sin goes up and down between -1 and 1

1/ω decays down as |ω| gets big… this causes the overall function to decay down

Fourier Transform

For ***this*** case the FT is real valued so we can plot it using a single plot (shown in solid blue here):



$$P_\tau(\omega) = \frac{2\sin\left(\dfrac{\omega\tau}{2}\right)}{\omega}$$

The sine wiggles up & down "between ±2/ω"

Fourier Transform

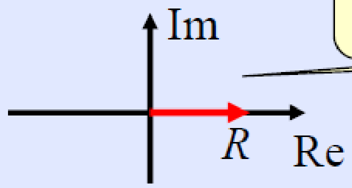
Now… let's think about how to make a magnitude/phase plot…

Even though this FT is real-valued we can still plot it using magnitude and phase plots:

We can view any real number as a complex number that has zero as its imaginary part

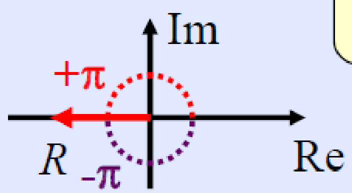
A *positive* real number R will have:

$|R| = R \qquad \angle R = 0$



A *negative* real number R will have:

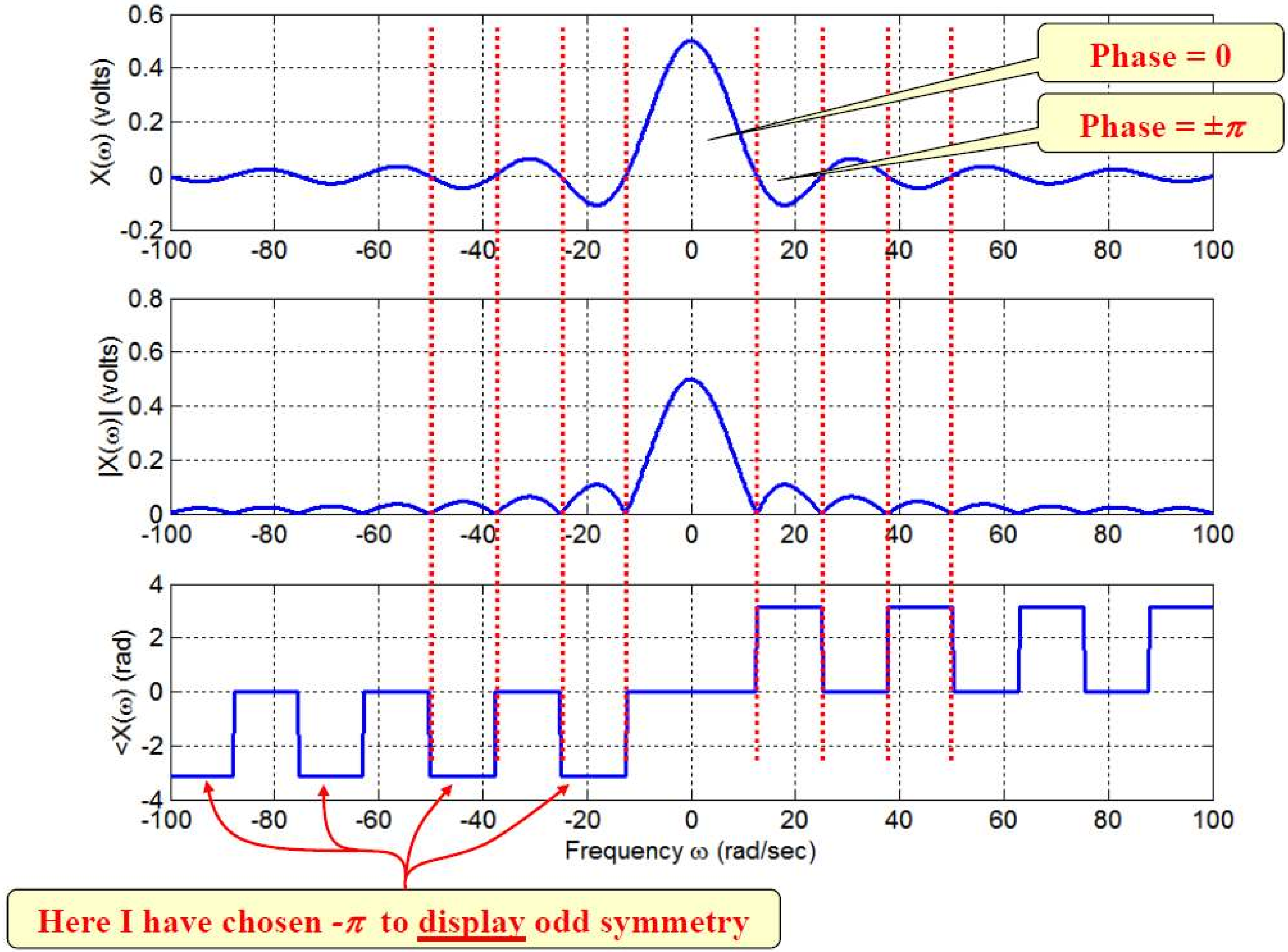
$|R| = -R \qquad \angle R = \pm\pi$



Can use either one!!

Fourier Transform

Applying these Ideas to the Real-valued FT $P_\tau(\omega)$



Fourier Transform

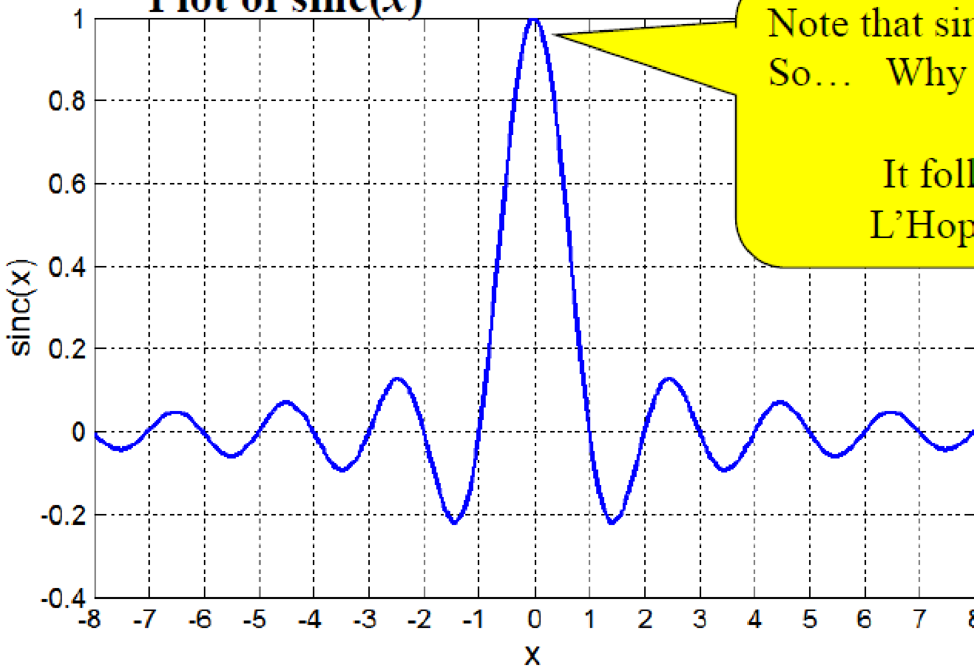
Definition of "Sinc" Function

The result we just found had this mathematical form: $P_\tau(\omega) = \dfrac{2\sin\left(\dfrac{\omega\tau}{2}\right)}{\omega}$

This structure shows up enough that we define a special function to capture it:

Define: $\boxed{\operatorname{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}}$

Plot of sinc(x)



Note that sinc(0) = 0/0.
So… Why is sinc(0) = 1?

It follows from
L'Hopital's Rule

Fourier Transform

With a little manipulation we can re-write the FT result for a pulse in terms of the sinc function:

Recall:

$$\boxed{\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}}$$

Need π times something…

$$P_\tau(\omega) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega} = \frac{2\sin\left(\boxed{\frac{\pi}{\pi}}\frac{\omega\tau}{2}\right)}{\omega} = \frac{2\sin\left(\pi\boxed{\frac{\omega\tau}{2\pi}}\right)}{\omega}$$

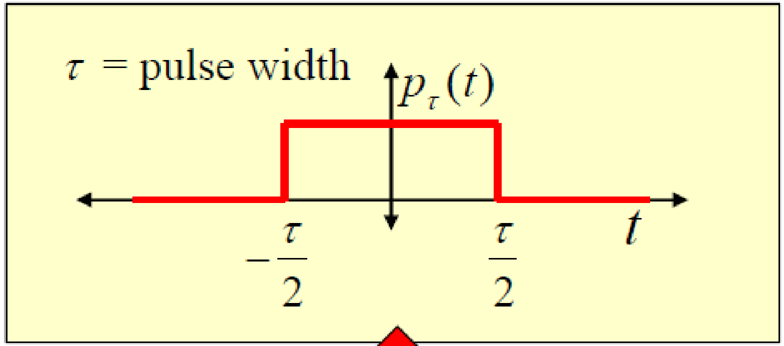
Now we need the same thing down here as inside the sine…

$$= \boxed{\cancel{\pi\frac{\tau}{2\pi}}}\,\frac{\cancel{2}\sin\left(\pi\frac{\omega\tau}{2\pi}\right)}{\boxed{\pi\frac{\tau}{2\pi}}\omega} = \tau\,\frac{\sin\left(\pi\frac{\omega\tau}{2\pi}\right)}{\pi\frac{\omega\tau}{2\pi}} = \tau\operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

$$\Rightarrow \boxed{P_\tau(\omega) = \tau\operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)}$$

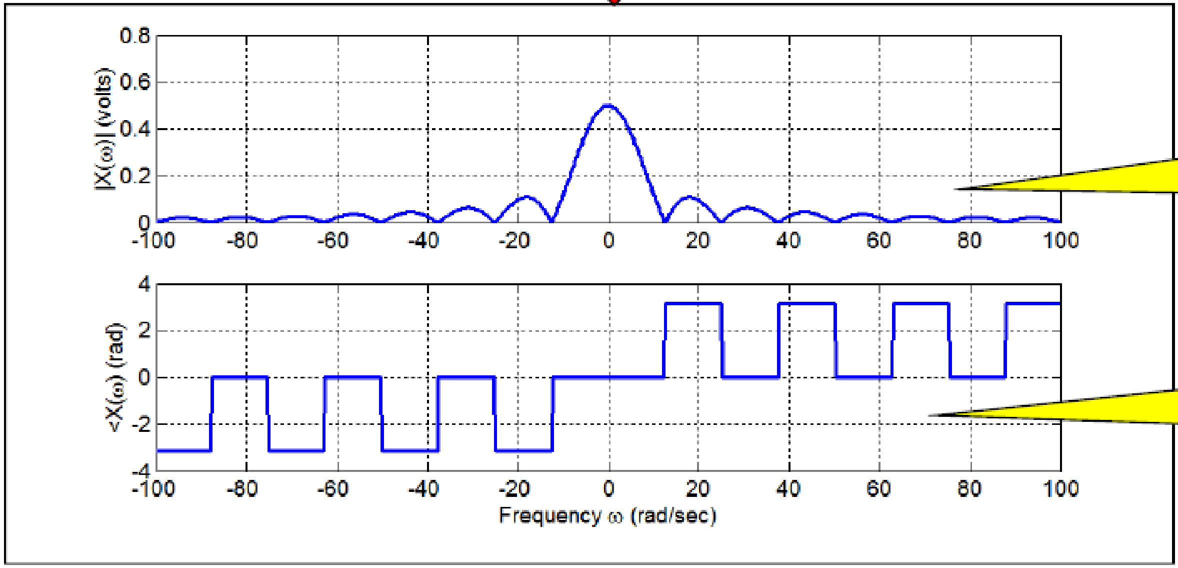
Fourier Transform

FT of Rect. Pulse = Sinc Function



Time-Domain View

Frequency-Domain View



Tells what **amplitude** is needed at each frequency

Tells what **phase** is needed at each frequency

Example 1

$$\boxed{x(t) = e^{-at}u(t)}$$

$$X(j\omega) = \int_0^\infty e^{-at}e^{-j\omega t}dt = \int_0^\infty e^{-(a+j\omega)t}dt$$

$$\boxed{a > 0}$$

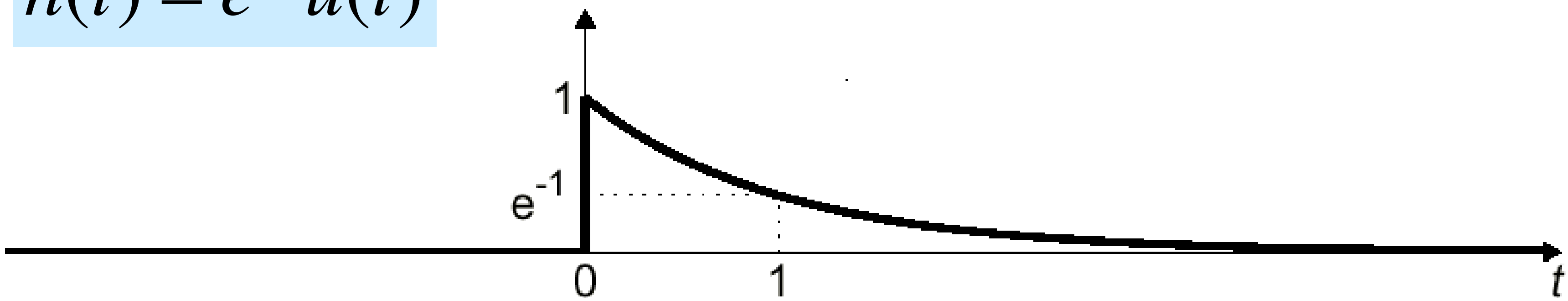
$$X(j\omega) = -\left.\frac{e^{-at}e^{-j\omega t}}{a+j\omega}\right|_0^\infty = \frac{1}{a+j\omega}$$

$$\boxed{X(j\omega) = \frac{1}{a+j\omega}}$$

Example 1 - Frequency Response

- Fourier Transform of $h(t)$ is
  - the Frequency Response

$$h(t) = e^{-t}u(t)$$

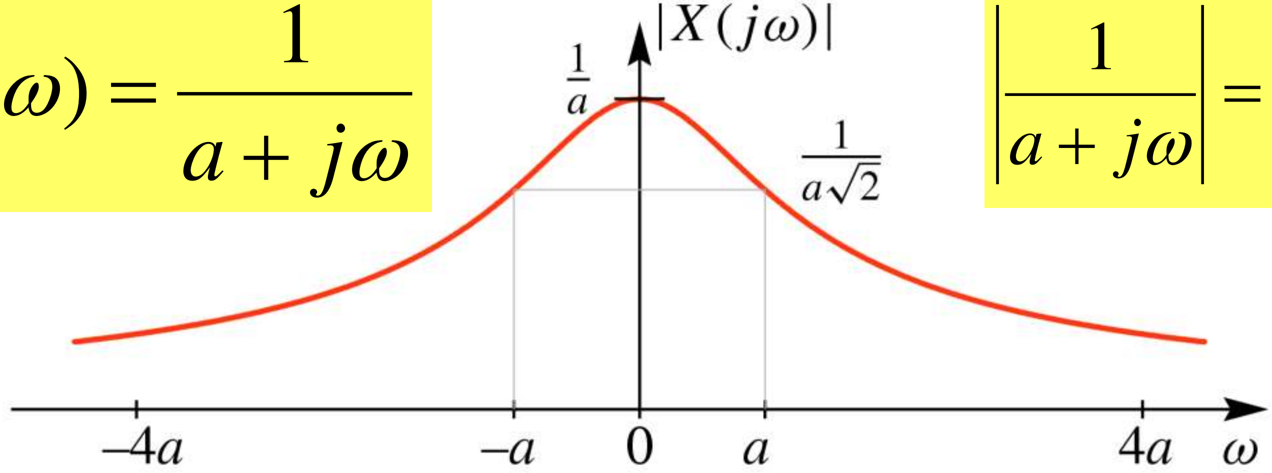


$$h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1 + j\omega}$$

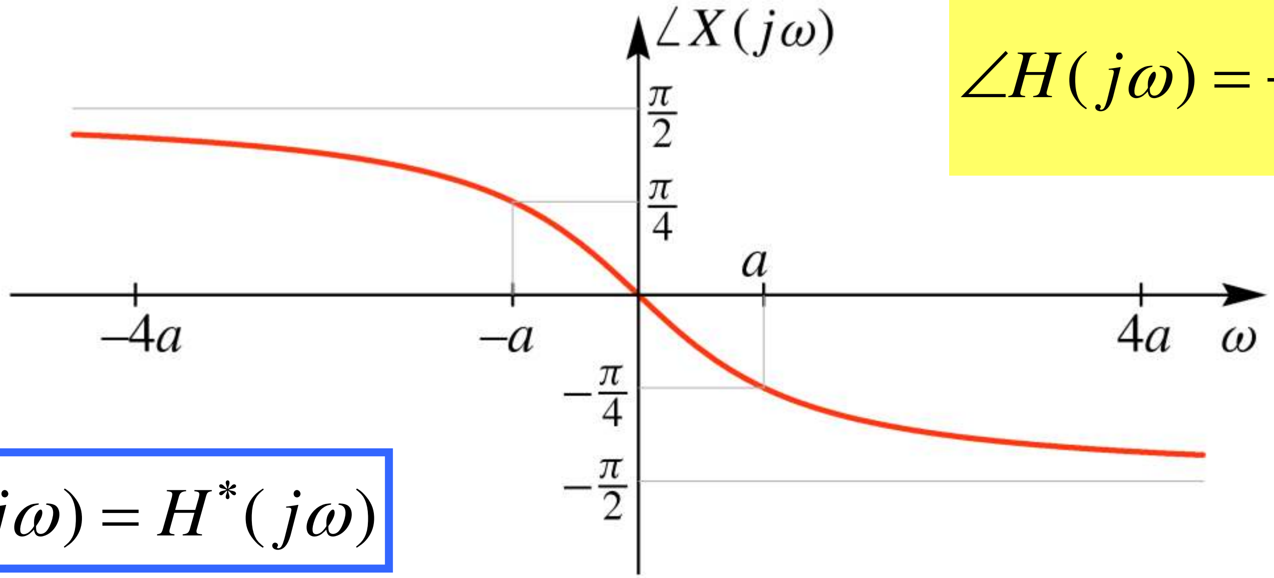
Example 1 - Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{a + j\omega}$$

$$\left| \frac{1}{a + j\omega} \right| = \left| \frac{1}{\sqrt{a^2 + \omega^2}} \right|$$



$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$



$$\boxed{H(-j\omega) = H^*(j\omega)}$$

Example 2

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}$$

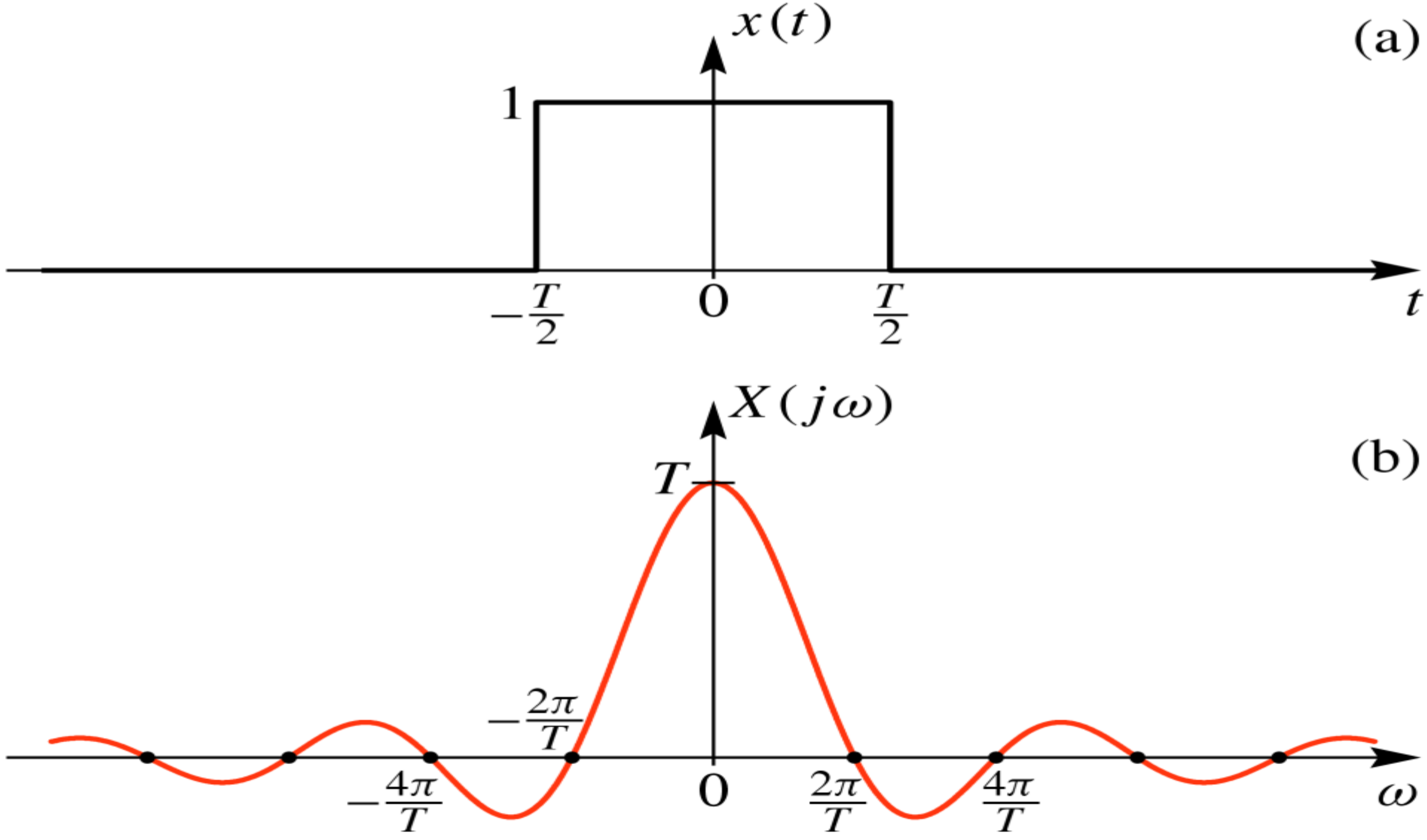
$$X(j\omega) = \int_{-T/2}^{T/2} (1) e^{-j\omega t} dt = \int_{-T/2}^{T/2} e^{-j\omega t} dt$$

$$X(j\omega) = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-T/2}^{T/2} = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

Example 2

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \quad \Leftrightarrow \quad X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$



Example 3

$$\boxed{X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}}$$

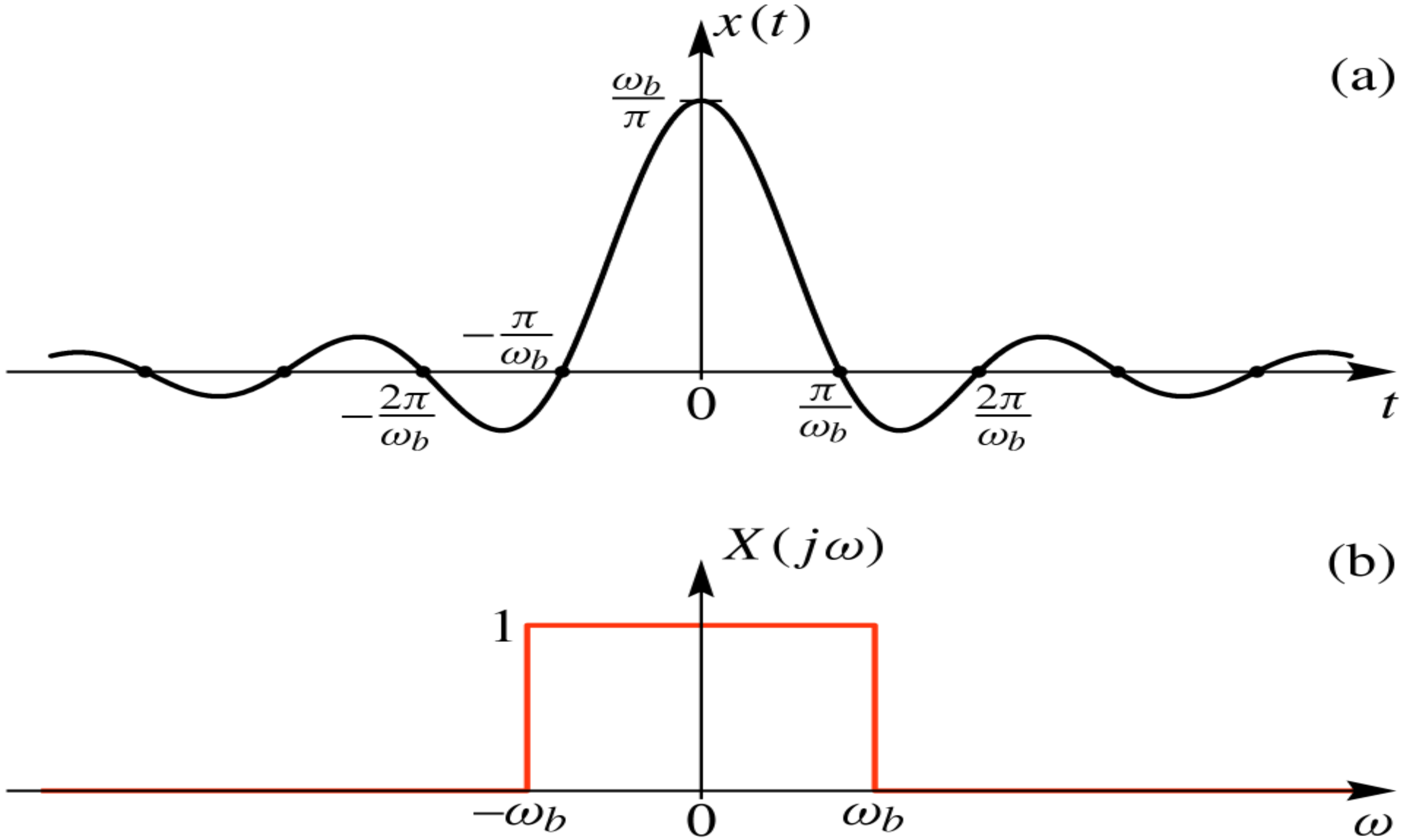
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_b}^{\omega_b} 1e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\frac{e^{j\omega t}}{jt}\Bigg|_{-\omega_b}^{\omega_b} = \frac{1}{2\pi}\frac{e^{j\omega_b t} - e^{-j\omega_b t}}{jt}$$

$$\boxed{x(t) = \frac{\sin(\omega_b t)}{\pi t}}$$

Example 3

$$x(t) = \frac{\sin(\omega_b t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$



Example 4

$$\boxed{x(t) = \delta(t - t_0)}$$

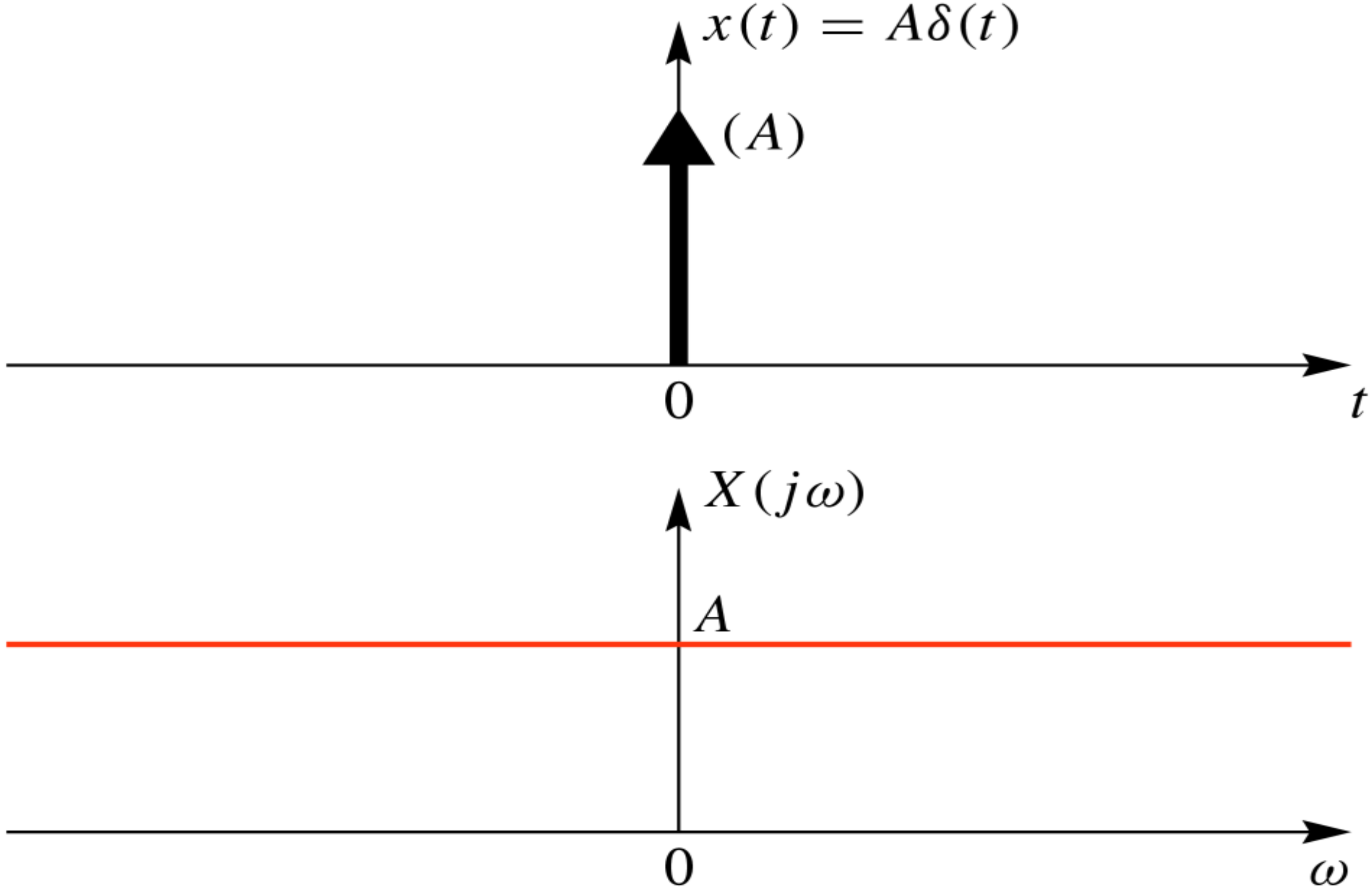
$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0}}$$

Shifting Property of the Impulse

Impulse function – Time and Frequency domains

$$\boxed{x(t) = \delta(t) \Leftrightarrow X(j\omega) = 1}$$



Example 5

$$\boxed{X(j\omega) = 2\pi\delta(\omega - \omega_0)}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\,d\omega = e^{j\omega_0 t}$$

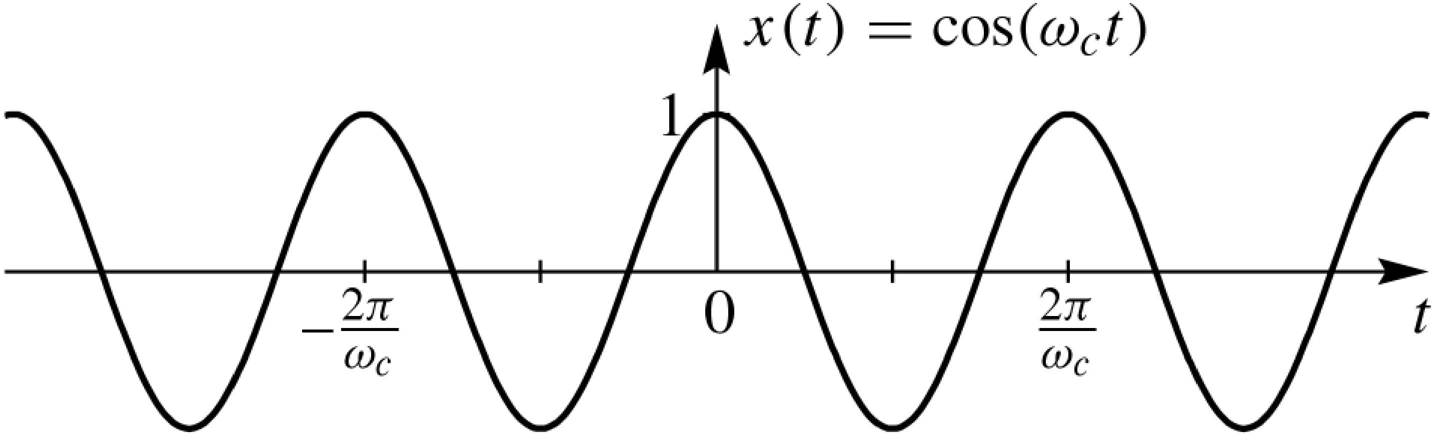
$$\boxed{x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)}$$

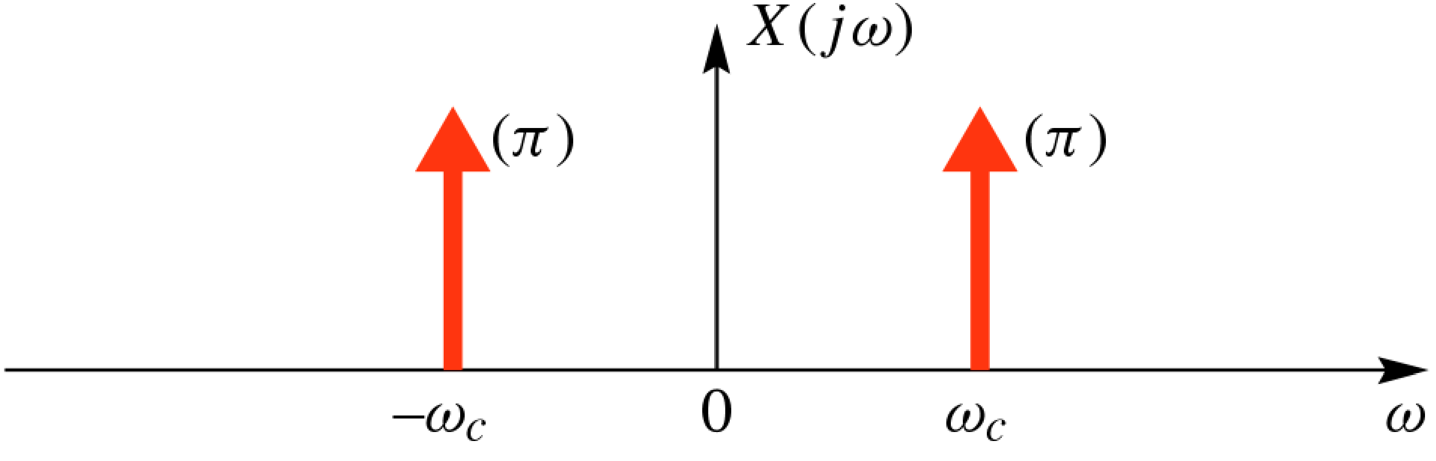
$$\boxed{x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)}$$

$$\boxed{\begin{aligned} &x(t) = \cos(\omega_0 t) \Leftrightarrow \\ &\qquad X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) \end{aligned}}$$

Example 5

$$x(t)=\cos(\omega_c t) \Leftrightarrow X(j\omega)=\pi\delta(\omega-\omega_c)+\pi\delta(\omega+\omega_c)$$





Fourier Transform Table

We have just found the FT for a common signal…

$$p_\tau(t) = \begin{cases} 1, & -\dfrac{\tau}{2} \le t \le \dfrac{\tau}{2} \\ 0, & \textit{otherwise} \end{cases} \longleftrightarrow P_\tau(\omega) = \tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

We derived that result by directly applying the integral form of the FT to the given signal equation.

For the common "textbook" signals this has already been done… and the results are available in tables published in books and on-line

You should study the table provided…

- If you encounter a time signal or FT that is on this table you should recognize that it is on the table without being told that it is there.
- You should be able to recognize entries in graphical form as well as in equation form.
- Later we'll learn about some "FT properties" that will expand your ability to apply these entries on the FT Table

In the real-world, engineers use these table results to understand basic ideas and concepts and to think through how things work in principle!

So… next we'll look at some of the more important entries in the table provided…

Fourier Transform Table

Decaying Exponential

As we'll see later… this signal naturally occurs in lots of real-world places!

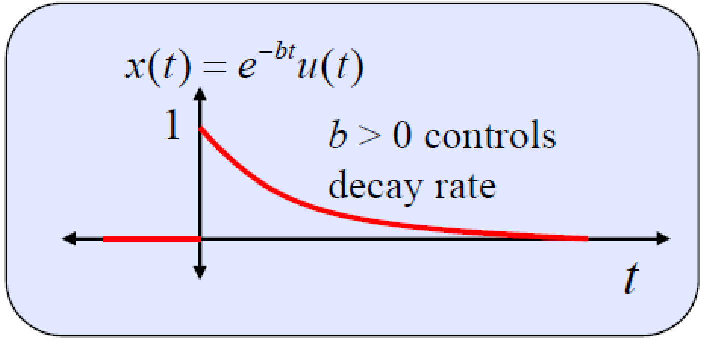
$$\boxed{x(t) = e^{-bt}u(t)} \longleftrightarrow \boxed{X(\omega) = \frac{1}{b + j\omega}}$$

For $b > 0$

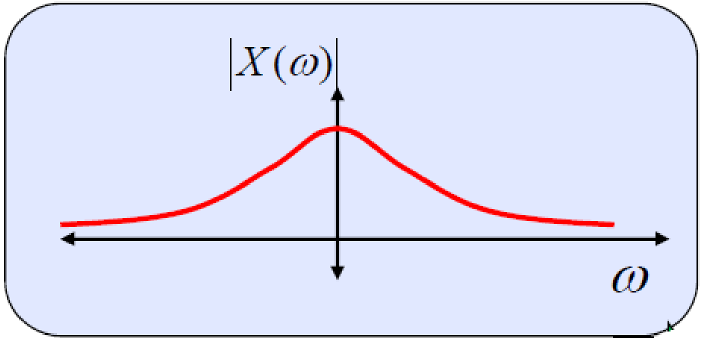
(Complex Valued)

$$|X(\omega)| = \frac{1}{\sqrt{b^2 + \omega^2}}$$ Magnitude

$$\angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{b}\right)$$ Phase

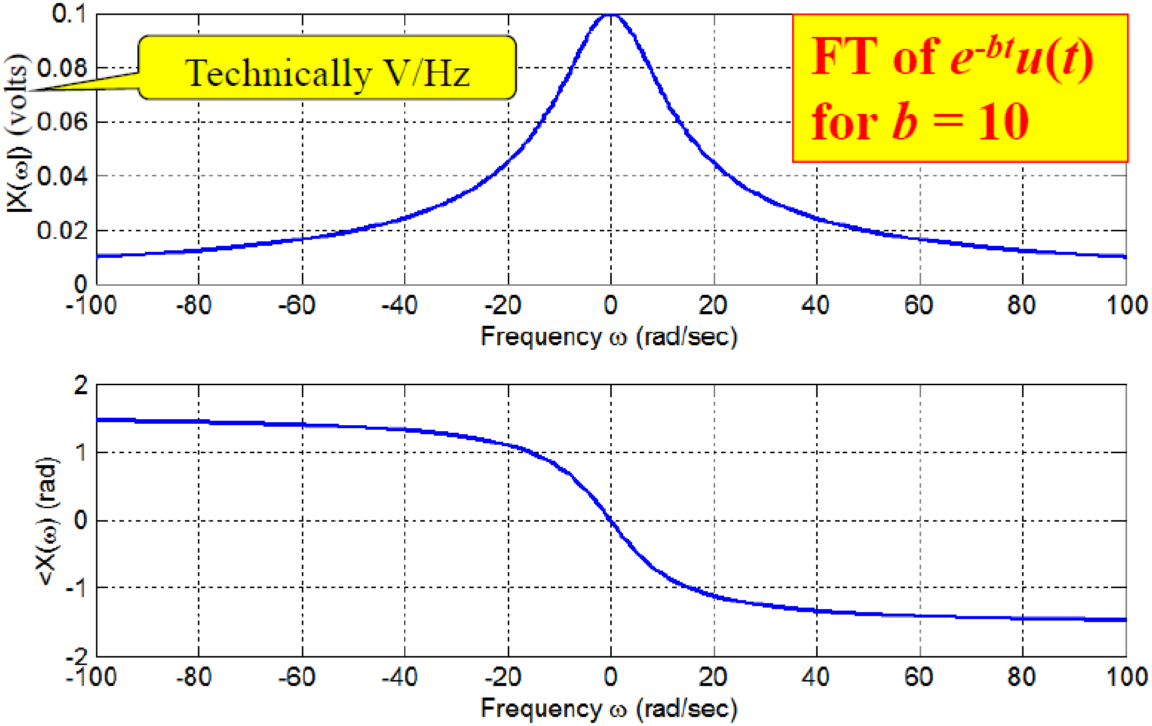


$\longleftrightarrow$



Fourier Transform Table

Can Use Matlab to Make Plots of FT Results



Note that magnitude plot has even symmetry

Note that phase plot has odd symmetry

True for every real-valued signal

MATLAB Commands to Compute FT

```
w=-100:0.2:100;
b=10;
X=1./(b+j*w);
```

Plotting Commands

```
subplot(2,1,1); plot(w,abs(X))
xlabel('Frequency \omega (rad/sec)')
ylabel('|X(\omega|) (volts)'); grid
subplot(2,1,2); plot(w,angle(X))
xlabel('Frequency \omega (rad/sec)')
ylabel('<X(\omega) (rad)'); grid
```

Fourier Transform Table

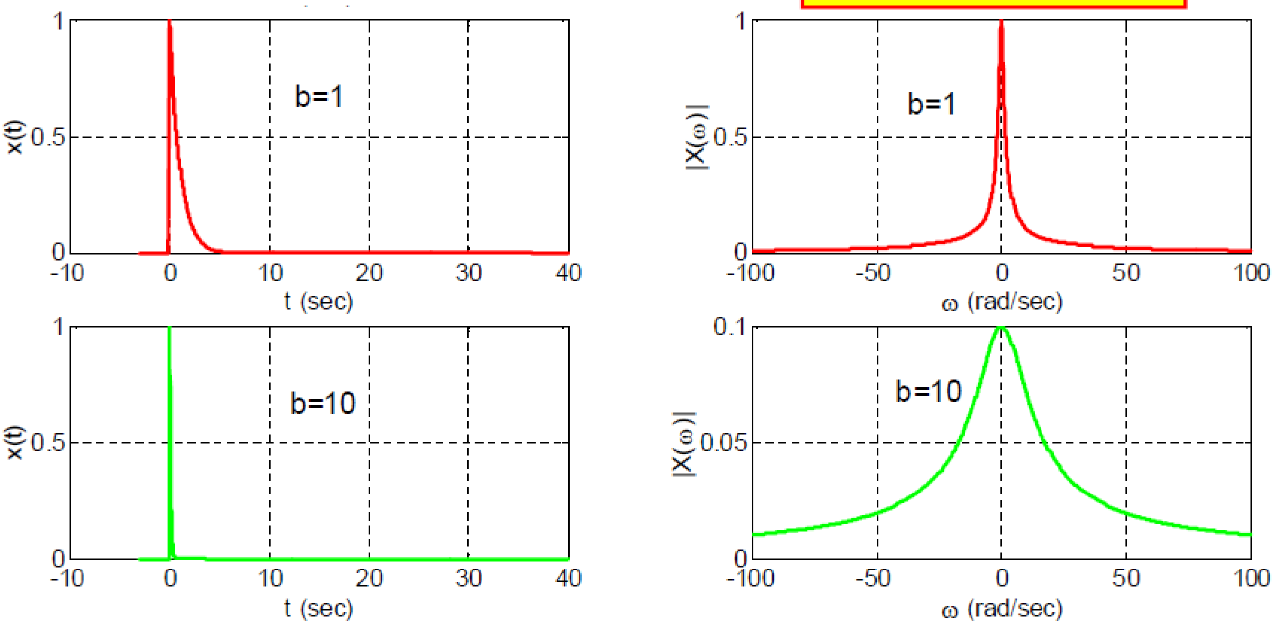
Effect of Exp. Decay Rate b on FT Magnitude

Time Signal

$$x(t) = e^{-bt}u(t)$$

FT Magnitude

$$|X(\omega)| = \frac{1}{\sqrt{b^2 + \omega^2}}$$



Short Signals have FTs that spread more into High Frequencies!!!

Note: As *b* increases…

1. **Decay rate in time signal increases**
2. **High frequencies in Fourier transform are more prominent.**

Fourier Transform Table

Some Important Signals & Their FTs (see Table for More!)

$$1, \quad -\infty < t < \infty \Leftrightarrow 2\pi\delta(\omega)$$

$$u(t) \Leftrightarrow \pi\delta(\omega) + 1/j\omega$$

$$-0.5 + u(t) \Leftrightarrow 1/j\omega$$

$$\delta(t) \Leftrightarrow 1, \quad -\infty < \omega < \infty$$

$$\cos(\omega_o t) \Leftrightarrow \pi[\delta(\omega+\omega_o) + \delta(\omega-\omega_o)]$$

$$\sin(\omega_o t) \Leftrightarrow j\pi[\delta(\omega+\omega_o) - \delta(\omega-\omega_o)]$$

$$e^{j\omega_o t} \Leftrightarrow 2\pi\delta(\omega-\omega_o), \quad \omega_o \text{ real}$$

Fourier Transform Table

Time Signal	Fourier Transform
$1, \quad -\infty < t < \infty$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$1/j\omega$
$u(t)$	$\pi\delta(\omega) + 1/j\omega$
$\delta(t)$	$1, \quad -\infty < \omega < \infty$
$\delta(t-c), \quad c$ real	$e^{-j\omega c}, \quad c$ real
$e^{-bt}u(t), \quad b > 0$	$\frac{1}{j\omega + b}, \quad b > 0$
$e^{j\omega_o t}, \quad \omega_o$ real	$2\pi\delta(\omega - \omega_o), \quad \omega_o$ real
$p_\tau(t)$	$\tau \operatorname{sinc}[\tau\omega/2\pi]$
$\tau \operatorname{sinc}[\tau t/2\pi]$	$2\pi p_\tau(\omega)$
$\left[1 - \frac{2\lvert t\rvert}{\tau}\right] p_\tau(t)$	$\frac{\tau}{2} \operatorname{sinc}^2[\tau\omega/4\pi]$
$\frac{\tau}{2} \operatorname{sinc}^2[\tau t/4\pi]$	$2\pi\left[1 - \frac{2\lvert\omega\rvert}{\tau}\right] p_\tau(\omega)$
$\cos(\omega_o t)$	$\pi[\delta(\omega + \omega_o) + \delta(\omega - \omega_o)]$
$\cos(\omega_o t + \theta)$	$\pi[e^{-j\theta}\delta(\omega + \omega_o) + e^{j\theta}\delta(\omega - \omega_o)]$
$\sin(\omega_o t)$	$j\pi[\delta(\omega + \omega_o) - \delta(\omega - \omega_o)]$
$\sin(\omega_o t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega + \omega_o) - e^{j\theta}\delta(\omega - \omega_o)]$

Property Name	**Property**	
Linearity	$ax(t)+bv(t)$	$aX(\omega)+bV(\omega)$
Time Shift	$x(t-c)$	$e^{-j\omega c}X(\omega)$
Time Scaling	$x(at), \quad a\neq 0$	$\frac{1}{a}X(\omega/a), \quad a\neq 0$
Time Reversal	$x(-t)$	$X(-\omega)$ $\overline{X(\omega)}$ if $x(t)$ is real
Multiply by t^n	$t^n x(t), \quad n=1,2,3,\dots$	$j^n \dfrac{d^n}{d\omega^n}X(\omega), \quad n=1,2,3,\dots$
Multiply by Complex Exponential	$e^{j\omega_o t}x(t), \quad \omega_o$ real	$X(\omega-\omega_o), \quad \omega_o$ real
Multiply by Sine	$\sin(\omega_o t)x(t)$	$\dfrac{j}{2}[X(\omega+\omega_o)-X(\omega-\omega_o)]$
Multiply by Cosine	$\cos(\omega_o t)x(t)$	$\dfrac{1}{2}[X(\omega+\omega_o)+X(\omega-\omega_o)]$
Time Differentiation	$\dfrac{d^n}{dt^n}x(t), \quad n=1,2,3,\dots$	$(j\omega)^n X(\omega), \quad n=1,2,3,\dots$
Time Integration	$\displaystyle\int_{-\infty}^{t} x(\lambda)d\lambda$	$\dfrac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$
Convolution in Time	$x(t)*h(t)$	$X(\omega)H(\omega)$
Multiplication in Time	$x(t)w(t)$	$\dfrac{1}{2\pi}X(\omega)*W(\omega)$
Parseval's Theorem (General)	$\displaystyle\int_{-\infty}^{\infty} x(t)\overline{v(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\overline{V(\omega)}d\omega$	
Parseval's Theorem (Energy)	$\displaystyle\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$ if $x(t)$ is real $\displaystyle\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$	
Duality: If $x(t) \leftrightarrow X(\omega)$	$X(t)$	$2\pi x(-\omega)$

Fourier Transform Table

FT of Periodic Signal

Note that we have now used the FT to analyze cosine and sine… which are *PERIODIC* signals!!! Before we used the Fourier *Series* to analyze *periodic* signals… Now we see that we can also use the Fourier *Transform*!

If $x(t)$ is periodic then we can write the FS of it as: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

Now we can take the FT of both sides of this: $\mathfrak{F}\{x(t)\} = \mathfrak{F}\left\{ \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right\}$

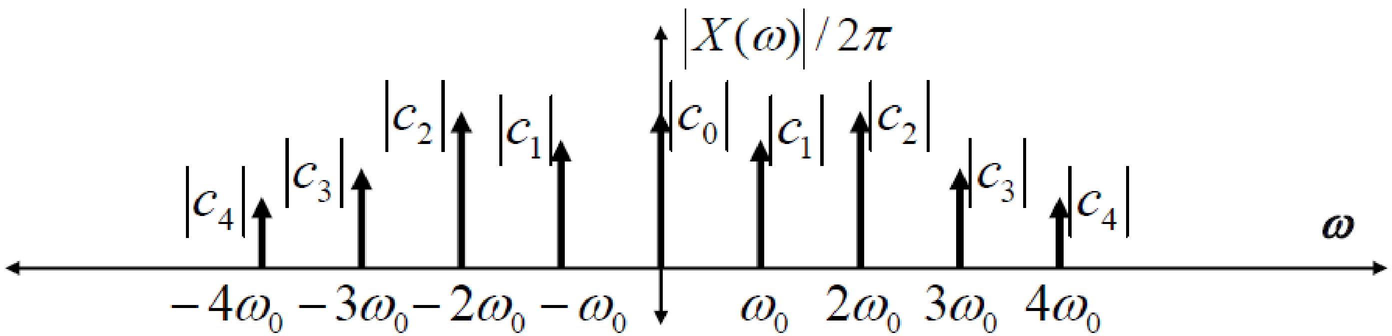
$$= \sum_{k=-\infty}^{\infty} c_k \underbrace{\mathfrak{F}\left\{ e^{jk\omega_0 t} \right\}}_{2\pi\delta(\omega - k\omega_0)}$$

FT of a Periodic Signal

$$\boxed{X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega - k\omega_0)}$$

Note: the FT is a bunch of delta functions with “weights” given by the FS coefficients!

Fourier Transform Table



So the FT of a periodic signal is zero except at multiples of the fundamental frequency ω_0, where you get impulses.

We call these spikes "Spectral Lines"

Note that if we start with the Amplitude-Phase Trig form we end up with the same result for the FT

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(k\omega_0 t + \theta_k\right)$$

For each cosine term we get two deltas (a positive frequency & negative frequency):

$$\cos(\omega_o t + \theta) \Leftrightarrow \pi\left[e^{-j\theta}\delta(\omega + \omega_o) + e^{j\theta}\delta(\omega - \omega_o)\right]$$

Fourier Transform Properties

These properties are useful for two main things:

1. They help you apply the table to a wider class of signals
2. They are often the key to understanding how the FT can be used in a given application.

So… even though these results may at first seem like "just boring math" they are important tools that let signal processing engineers understand how to build things like cell phones, radars, mp3 processing, etc.

Here… we will only cover the most important properties.

See the available table for the complete list of properties!

In this note set we simply learn these most-important properties… in the next note set we'll see how to use them.

Fourier Transform Properties

1. **Linearity** **(Supremely Important)**

Gets used virtually all the time!!

If $x(t) \leftrightarrow X(\omega)$ & $y(t) \leftrightarrow Y(\omega)$

then $[ax(t) + by(t)] \leftrightarrow [aX(\omega) + bY(\omega)]$

Another way to write this property:

$$\boxed{\mathfrak{F}\{ax(t) + by(t)\} = a\mathfrak{F}\{x(t)\} + b\mathfrak{F}\{y(t)\}}$$

To see why: $\mathfrak{F}\{ax(t) + by(t)\} = \int_{-\infty}^{\infty} [ax(t) + by(t)] e^{-j\omega t} dt$

Use Defn of FT

$$= a\underbrace{\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt}_{= X(\omega)} + b\underbrace{\int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt}_{= Y(\omega)}$$

By standard Property of Integral of sum of functions

By Defn of FT

Fourier Transform Properties

<u>2. Time Shift</u> (Really Important!)

Used often to understand <u>practical</u> issues that arise in <u>audio</u>, <u>communications</u>, <u>radar</u>, etc.

$$\text{If} \quad x(t) \leftrightarrow X(\omega) \quad \text{then} \quad x(t-c) \leftrightarrow X(\omega)e^{-jc\omega}$$

<u>Note</u>: If $c > 0$ then $x(t-c)$ is a **<u>delay</u>** of $x(t)$

So… what does this *<u>mean</u>*??

<u>First</u>… it does nothing to the magnitude of the FT: $\left|X(\omega)e^{-j\omega c}\right| = \left|X(\omega)\right|$

That means that a shift doesn't change "how much" we need of each of the sinusoids we build with

<u>Second</u>… it does change the <u>phase</u> of the FT: $\angle\{X(\omega)e^{-jc\omega}\} = \angle X(\omega) + \angle e^{-jc\omega}$

$$= \angle X(\omega) + \underbrace{c\omega}$$

Line of slope $-c$

Phase shift increases linearly as the frequency increases

This gets added to original phase

<u>Shift of Time Signal</u> $\Leftrightarrow$ "<u>Linear" Phase Shift of Frequency Components</u>

Fourier Transform Properties

3. Time Scaling (Important)

Q: If $x(t) \leftrightarrow X(\omega)$, then $x(at) \leftrightarrow ???$ for $a \neq 0$

A: $\boxed{x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)}$

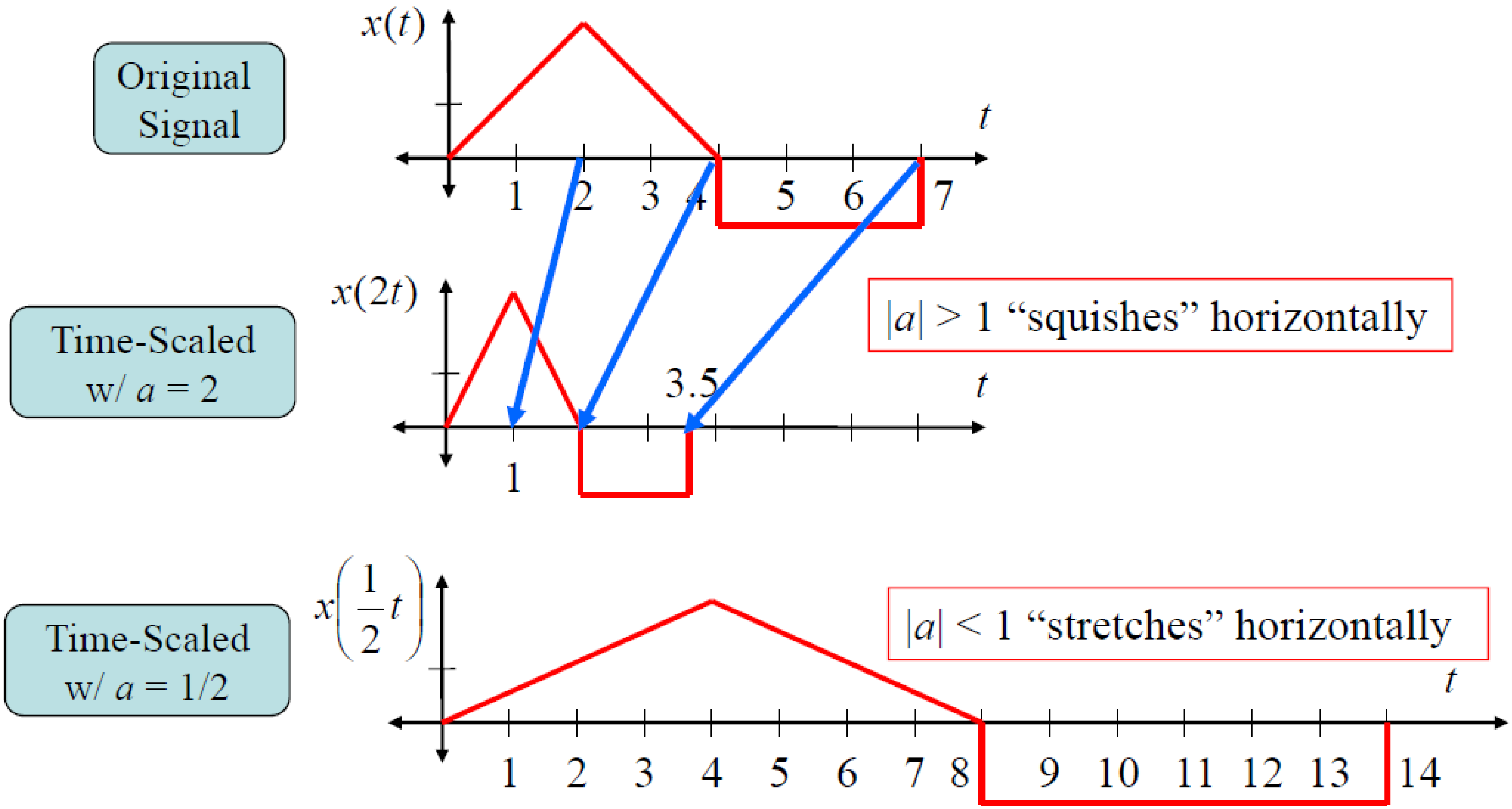
If the time signal is
Time Scaled by *a*

Then… The FT is
Freq. Scaled by 1/a

An interesting "duality"!!!

Fourier Transform Properties

To explore this FT property…first, what does $x(at)$ look like?



$|a| > 1$ makes it "wiggle" faster $\Rightarrow$ need <u>more</u> high frequencies

$|a| < 1$ makes it "wiggle" slower $\Rightarrow$ need <u>less</u> high frequencies

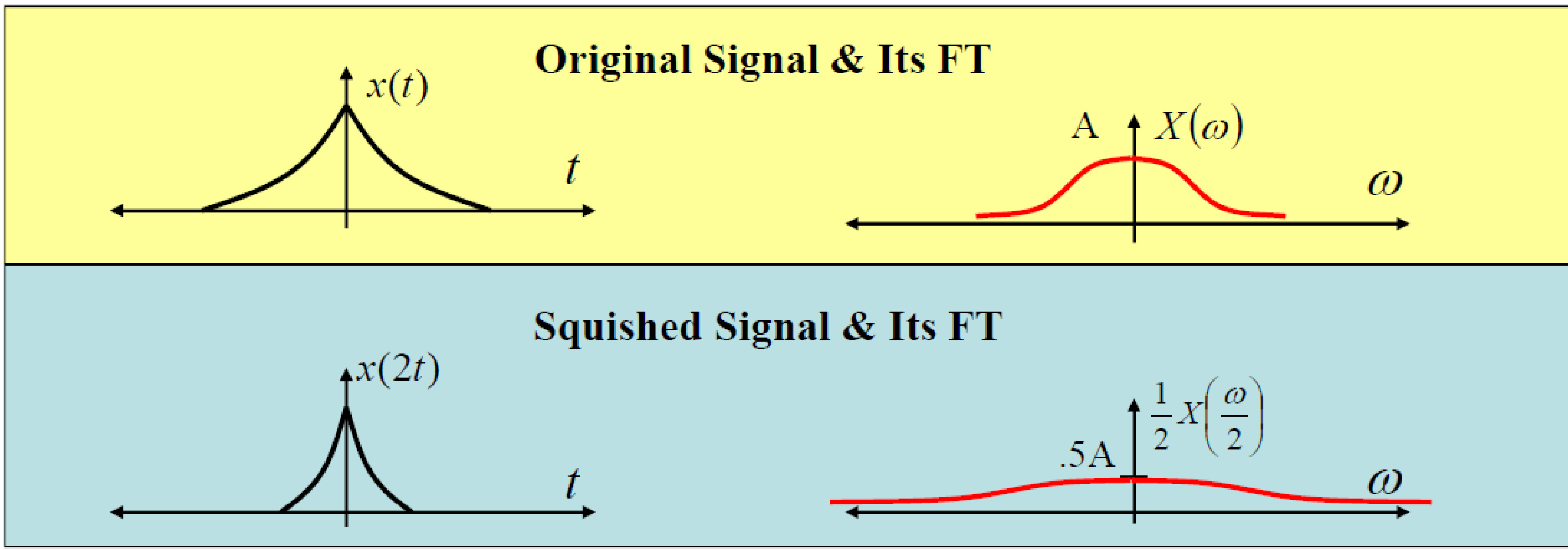
Fourier Transform Properties

When $|a| > 1 \;\Rightarrow\; |1/a| < 1$

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Time Signal is Squished

FT is Stretched Horizontally and Reduced Vertically



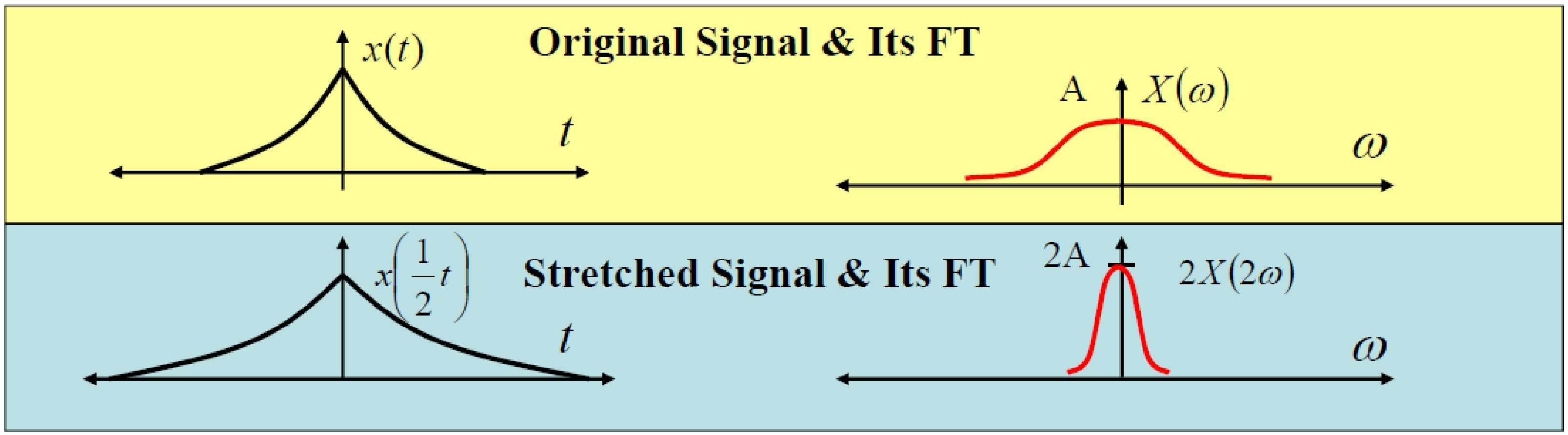
Fourier Transform Properties

When $|a| < 1 \;\Rightarrow\; |1/a| > 1$

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Time Signal is Stretched

FT is Squished Horizontally
and Increased Vertically



Rough Rule of Thumb we can extract from this property:

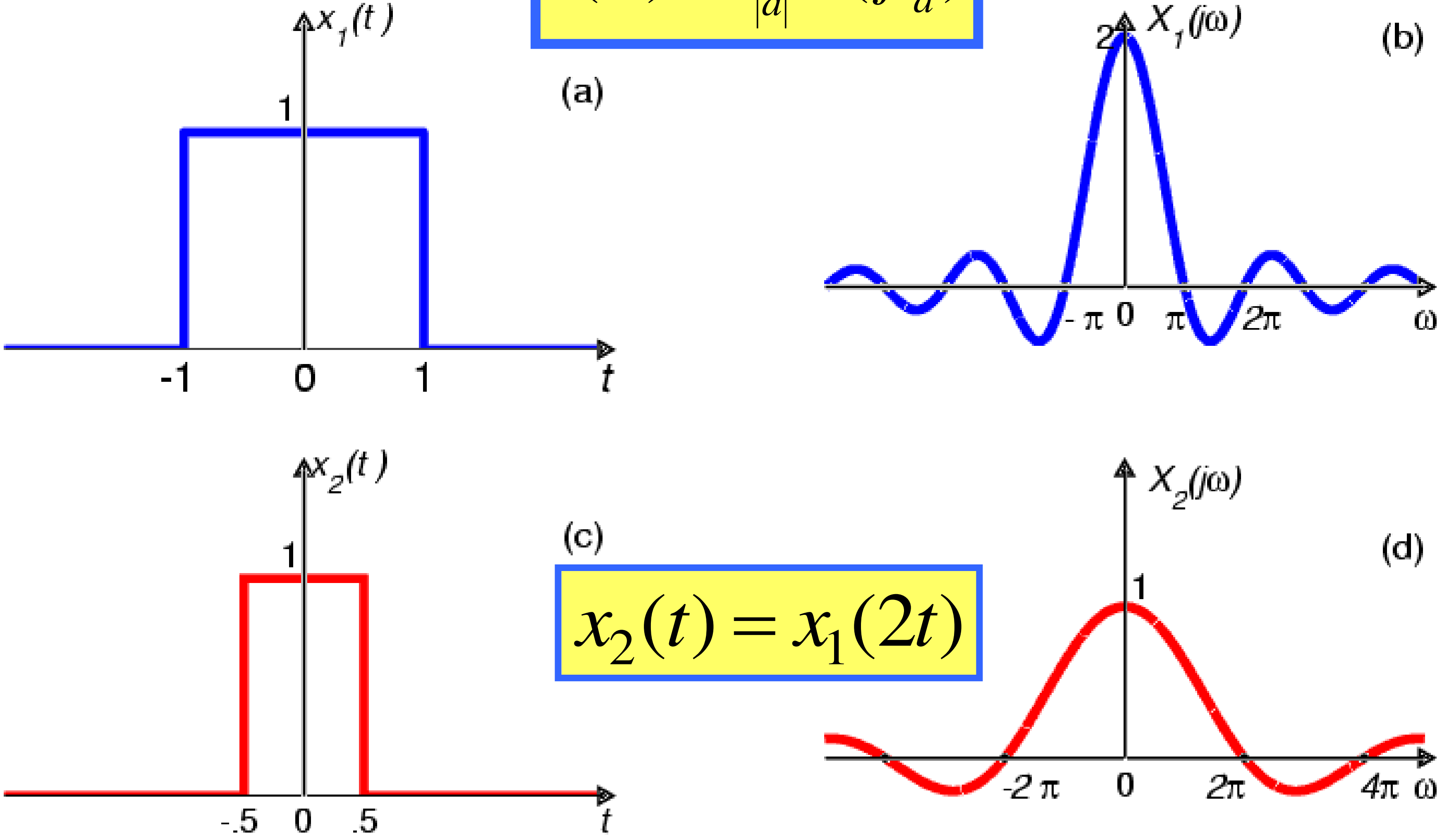
$\uparrow$ Duration $\Rightarrow$ $\downarrow$ Bandwidth

$\downarrow$ Duration $\Rightarrow$ $\uparrow$ Bandwidth

Very Short Signals *tend* to take up Wide Bandwidth

Scaling Property

$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$



$$x_2(t) = x_1(2t)$$

Fourier Transform Properties

<u>4. Time Reversal</u> (Special case of time scaling: $a = -1$)

$$\boxed{x(-t) \leftrightarrow X(-\omega)}$$

Note: $X(-\omega) = \int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}dt = \overline{\overline{\int_{-\infty}^{\infty} x(t)e^{+j\omega t}dt}}$ ← double conjugate = "No Change"

$$= \overline{\int_{-\infty}^{\infty} \overline{x(t)}\,\overline{e^{+j\omega t}}dt}$$

Conjugate changes to $-j$

$\overline{x(t)}$ = $x(t)$ if $x(t)$ is real

$$= \overline{\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt} = \overline{X(\omega)}$$

Recall: conjugation doesn't change abs. value but negates the angle

So if $x(t)$ is <u>real</u>, then we get the <u>special case</u>:

$$\boxed{x(-t) \leftrightarrow \overline{X(\omega)}}$$

$$\boxed{\begin{gathered} \left|\overline{X(\omega)}\right| = \left|X(\omega)\right| \\ \angle\overline{X(\omega)} = -\angle X(\omega) \end{gathered}}$$

Fourier Transform Properties

5. Modulation Property Super important!!!

Essential for understanding practical issues that arise in communications, radar, etc.

There are two forms of the modulation property…

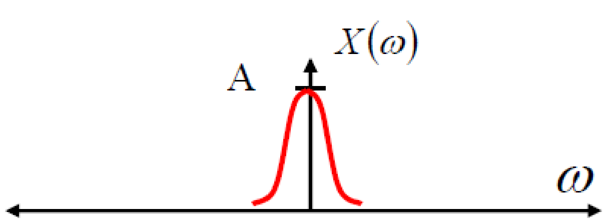
1. **Complex Exponential Modulation** … simpler mathematics, doesn't *directly* describe real-world cases
2. **Real Sinusoid Modulation**… mathematics a bit more complicated, directly describes real-world cases

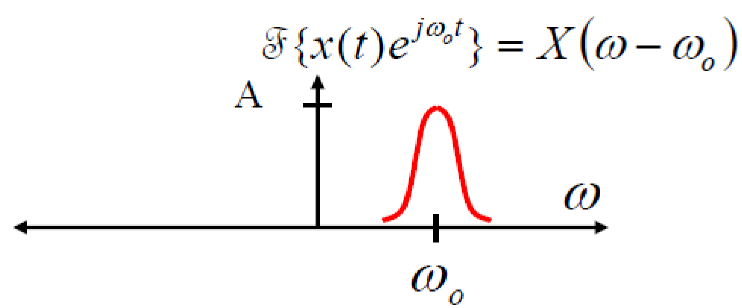
Euler's formula connects the two… so you often can use the Complex Exponential form to analyze real-world cases

Complex Exponential Modulation Property: $\boxed{x(t)e^{j\omega_0 t} \leftrightarrow X(\omega-\omega_0)}$

Multiply signal by a complex sinusoid

Shift the FT in frequency





Fourier Transform Properties

Real Sinusoid Modulation

Based on Euler, Linearity property, & the Complex Exp. Modulation Property

$$\mathfrak{F}\{x(t)\cos(\omega_0 t)\} = \mathfrak{F}\left\{\frac{1}{2}\left[x(t)e^{j\omega_0 t} + x(t)e^{-j\omega_0 t}\right]\right\}$$

Euler's Formula

$$= \frac{1}{2}\left[\mathfrak{F}\left\{\left[x(t)e^{j\omega_0 t}\right]\right\} + \mathfrak{F}\left\{\left[x(t)e^{-j\omega_0 t}\right]\right\}\right]$$

Linearity of FT

$$= \frac{1}{2}\left[X(\omega - \omega_o) + X(\omega + \omega_o)\right]$$

Comp. Exp. Mod.

The Result:

$$\boxed{x(t)\cos(\omega_0 t) \quad \leftrightarrow \quad \frac{1}{2}\left[X(\omega + \omega_0) + X(\omega - \omega_0)\right]}$$

Shift Down Shift Up

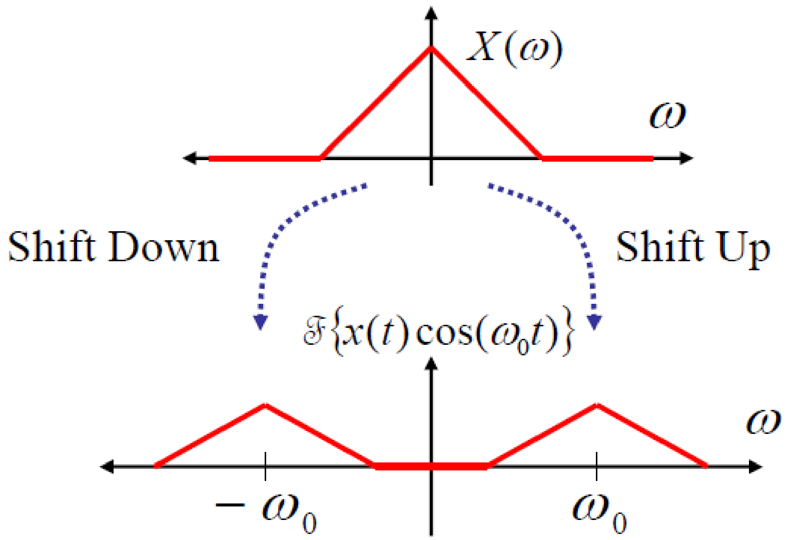
Related Result:

$$\boxed{x(t)\sin(\omega_0 t) \quad \leftrightarrow \quad \frac{j}{2}\left[X(\omega + \omega_0) - X(\omega - \omega_0)\right]}$$

Fourier Transform Properties

Visualizing the Result

$$\boxed{x(t)\cos(\omega_0 t) \quad \leftrightarrow \quad \frac{1}{2}\left[\underbrace{X(\omega-\omega_0)}_{\text{Shift up}} + \underbrace{X(\omega+\omega_0)}_{\text{Shift down}}\right]}$$



Interesting… This tells us how to move a signal's spectrum up to higher frequencies without changing the shape of the spectrum!!!

What is that good for??? Well… only <u>high</u> frequencies will radiate from an antenna and propagate as electromagnetic waves and then induce a signal in a receiving antenna….

Fourier Transform Properties

6. Convolution Property **(The Most Important FT Property!!!)**

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad \leftrightarrow \quad Y(\omega) = X(\omega)H(\omega)}$$

In the next Note Set we will explore the real-world use of the right side of this result!

7. Parseval's Theorem (Recall Parseval's Theorem for FS!)

$$\underbrace{\int_{-\infty}^{\infty} |x(t)|^2 dt}_{\text{Energy computed in time domain}} = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega}_{\text{Energy computed in frequency domain}}$$

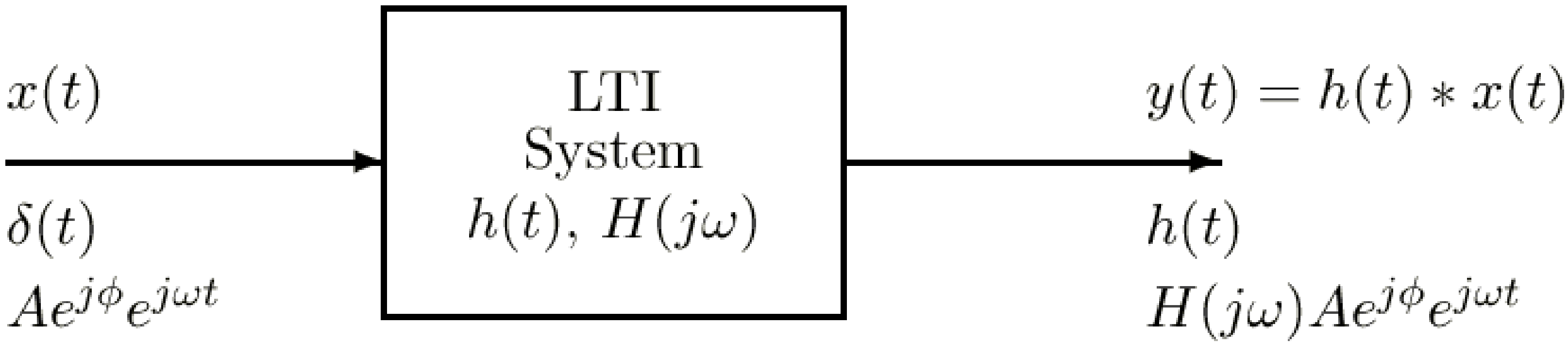
$|x(t)|^2 dt$

= energy at time t

$|X(\omega)|^2 \dfrac{d\omega}{2\pi}$

= energy at freq. ω

Convolution Property



- Convolution in the time-domain

$$\boxed{y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau}$$

corresponds to **MULTIPLICATION** in the frequency-domain

$$\boxed{Y(j\omega) = H(j\omega)X(j\omega)}$$

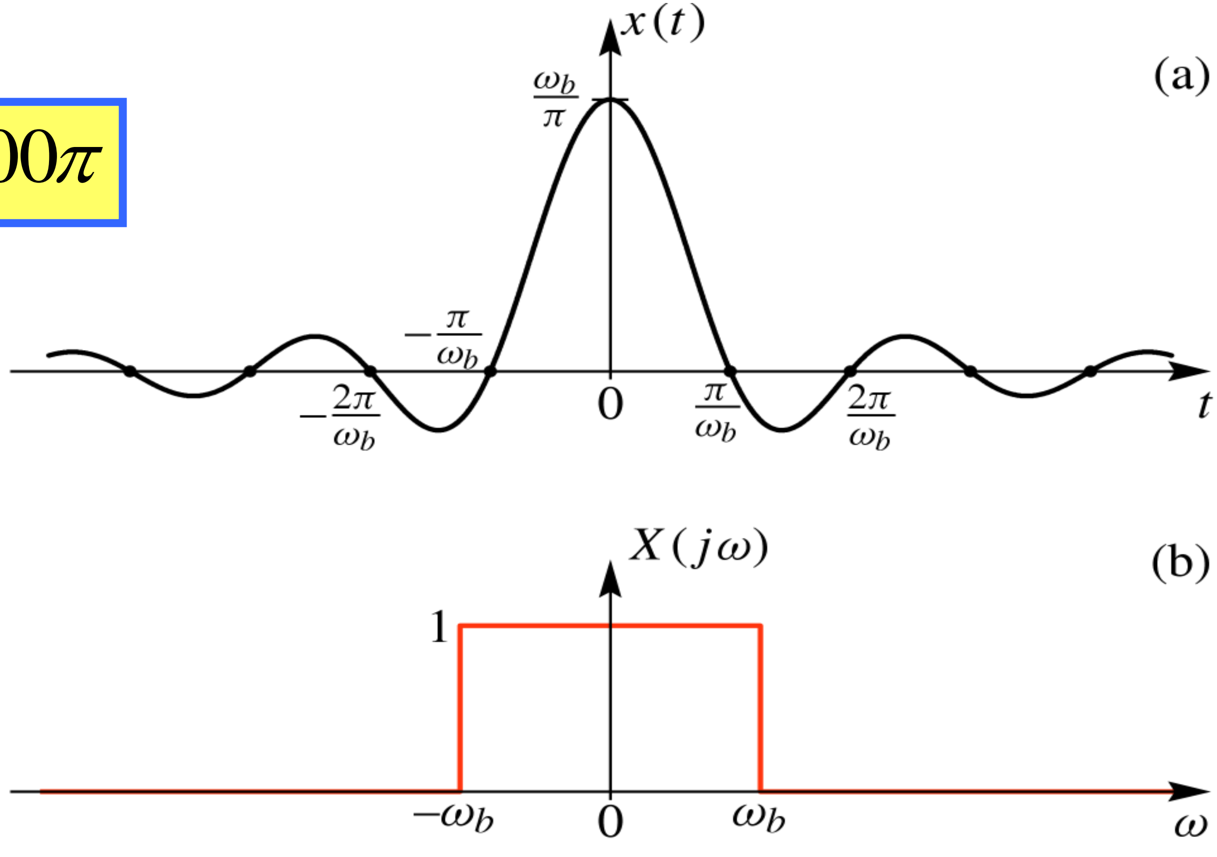
Convolution Example

- Bandlimited **Input** Signal
  - "sinc" function
- Ideal LPF (Lowpass Filter)
  - $\boldsymbol{h(t)}$ is a "sinc"
- **Output** is Bandlimited
  - Convolve "sincs"

Ideally Bandlimited Signal

$$x(t) = \frac{\sin(100\pi t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < 100\pi \\ 0 & |\omega| > 100\pi \end{cases}$$

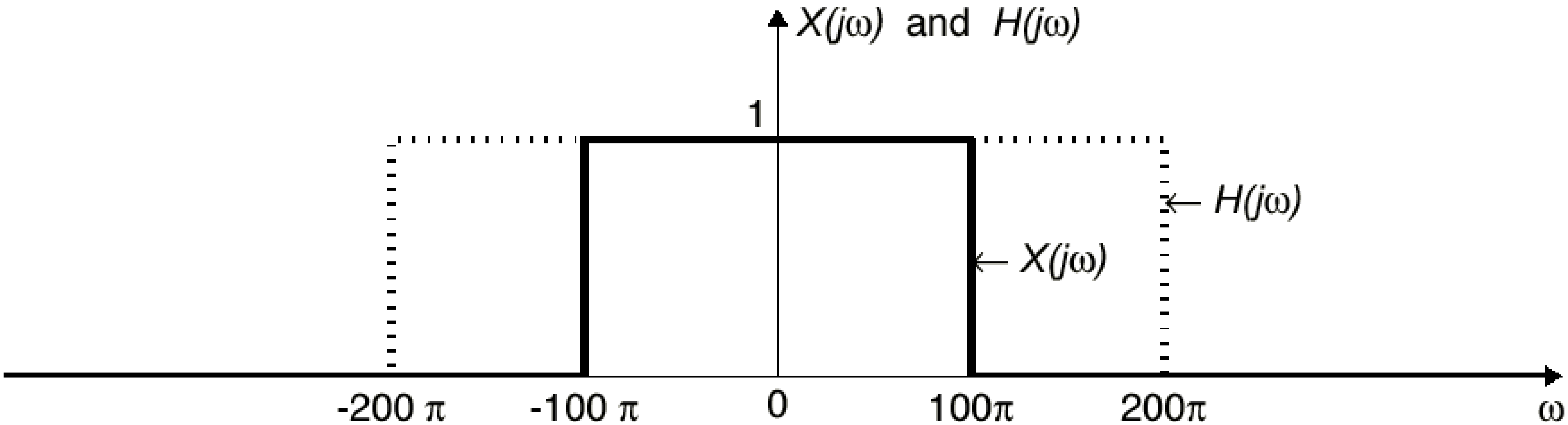
$$\omega_b = 100\pi$$



Convolution Example 1

$$\boxed{x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)}$$

$$\frac{\sin(100\pi t)}{\pi t} * \frac{\sin(200\pi t)}{\pi t} = \frac{\sin(100\pi t)}{\pi t}$$



Cosine Input to LTI System

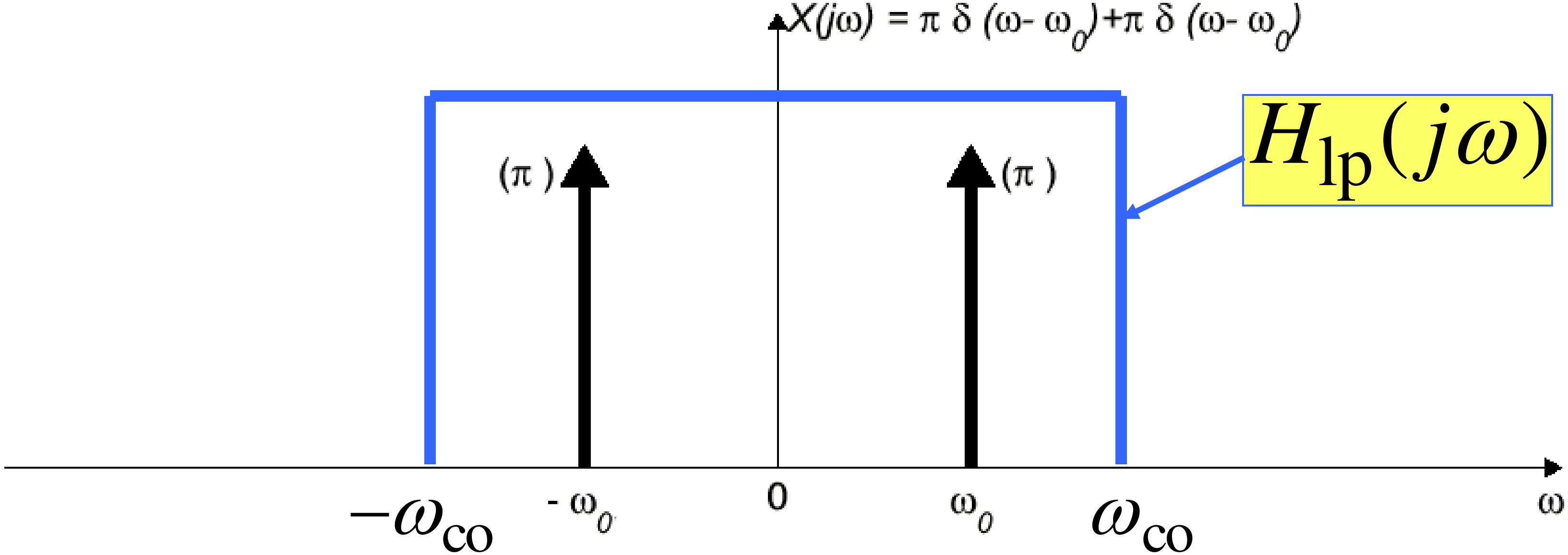
$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$= H(j\omega)[\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$

$$= H(j\omega_0)\pi\delta(\omega - \omega_0) + H(-j\omega_0)\pi\delta(\omega + \omega_0)$$

$$\begin{aligned} y(t) &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H(-j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H^*(j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= \left|H(j\omega_0)\right|\cos(\omega_0 t + \angle H(j\omega_0)) \end{aligned}$$

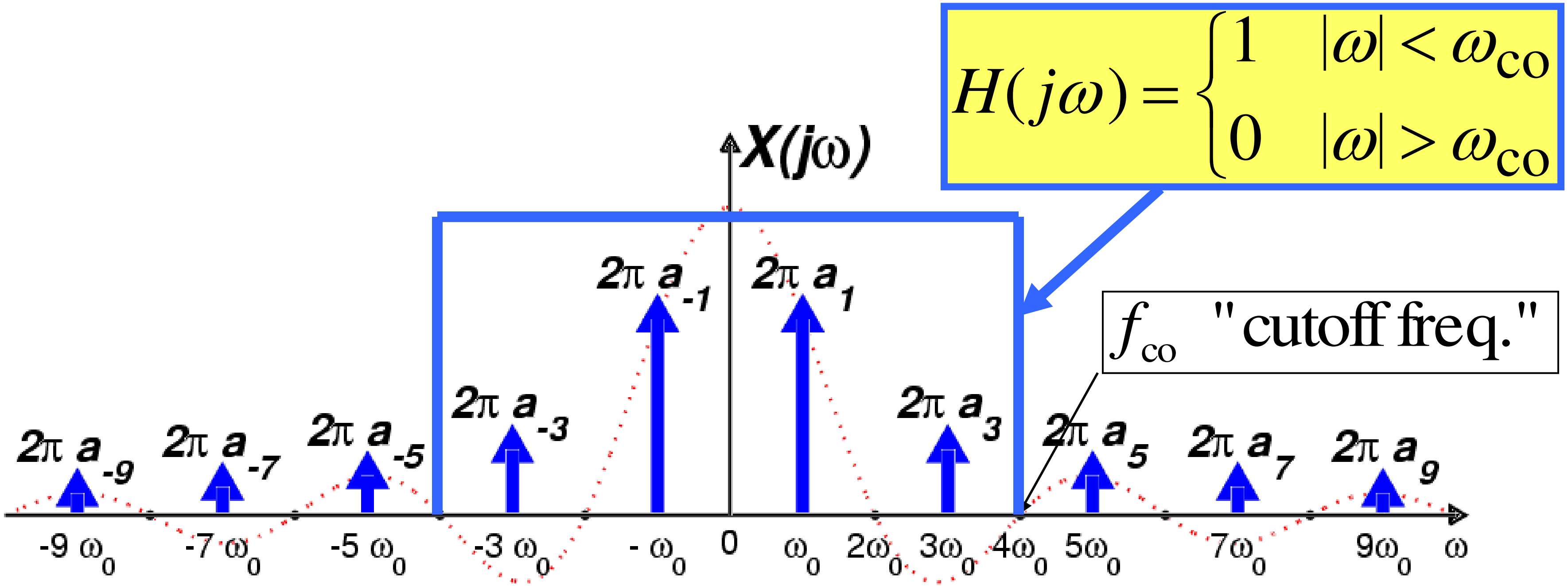
Ideal Lowpass Filter

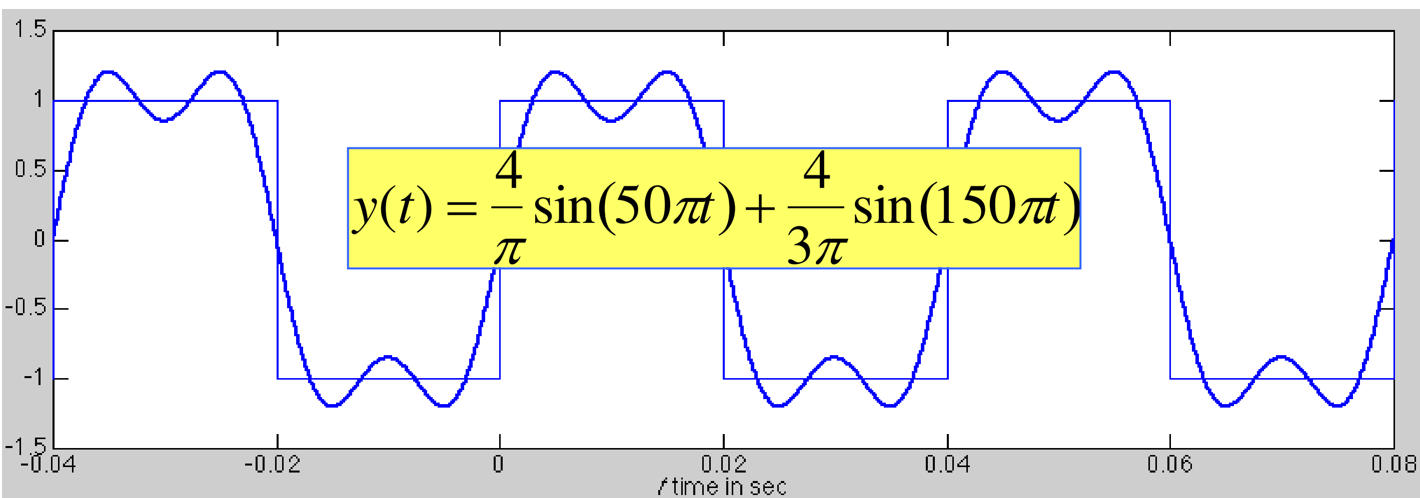


$$y(t) = x(t) \qquad \text{if } \omega_0 < \omega_{\mathrm{co}}$$

$$y(t) = 0 \qquad \text{if } \omega_0 > \omega_{\mathrm{co}}$$

Ideal Lowpass Filter





Fourier Transform Properties

8. Duality:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$x(t) \rightleftarrows X(\omega)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

Both FT & IFT are pretty much the "same machine": $c\int_{-\infty}^{\infty} f(\lambda) e^{\pm j\lambda\xi}\, d\lambda$

So if there is a "time-to-frequency" property we would expect a virtually similar "frequency-to-time" property

Illustration: Delay Property: $x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$

Modulation Property: $x(t)e^{j\omega_0 t} \leftrightarrow X(\omega-\omega_0)$

Other Dual Properties: (Multiply by t^n) vs. (Diff. in time domain)

(Convolution) vs. (Mult. of signals)

Fourier Transform Properties

Also, this duality structure gives FT pairs that show duality.

Suppose we have a FT table that a FT Pair A… we can get the dual Pair B using the <u>general</u> Duality Property:

1. Take the FT side of (known) Pair A and replace ω by t and move it to the time-domain side of the table of the (unknown) Pair B.
2. Take the time-domain side of the (known) Pair A and replace t by –ω, multiply by 2π, and then move it to the FT side of the table of the (unknown) Pair B.

Here is an example… We found the FT pair for the pulse signal:

Pair A $\boxed{p_\tau(t)} \longleftrightarrow \boxed{\tau \,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)}$

Step 1 **Step 2**

Pair B $\boxed{\tau \,\mathrm{sinc}\left(\frac{\tau t}{2\pi}\right)} \longleftrightarrow \boxed{2\pi p_\tau(\omega)}$

Here we have used the fact that $p_\tau(-\omega) = p_\tau(\omega)$

Fourier Transform Properties

Differentiation in the Time Domain:

$$\frac{dx(t)}{dt} \leftrightarrow j\omega X(\omega) \tag{5.55}$$

Equation (5.55) shows that the effect of differentiation in the time domain is the multiplication of $X(\omega)$ by $j\omega$ in the frequency domain (Prob. 5.28).

Differentiation in the Frequency Domain:

$$(-jt)x(t) \leftrightarrow \frac{dX(\omega)}{d\omega} \tag{5.56}$$

Equation (5.56) is the dual property of Eq. (5.55).

Alternatively, we can apply the Fourier transform directly to the differential equation [use the differentiation property: $\frac{d^n y}{dt^n} \leftrightarrow (j\omega)^n Y(\omega)$]

Fourier Transform Properties

Using the Fourier transform, redo Prob. 2.25.

The system is described by

$$y'(t) + 2y(t) = x(t) + x'(t)$$

Taking the Fourier transforms of the above equation, we get

$$j\omega Y(\omega) + 2Y(\omega) = X(\omega) + j\omega X(\omega)$$

$$(j\omega + 2)Y(\omega) = (1 + j\omega)X(\omega)$$

Hence, by Eq. (5.67) the frequency response $H(\omega)$ is

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1 + j\omega}{2 + j\omega} = \frac{2 + j\omega - 1}{2 + j\omega} = 1 - \frac{1}{2 + j\omega}$$

Taking the inverse Fourier transform of $H(\omega)$, the impulse response $h(t)$ is

$$\boxed{h(t) = \delta(t) - e^{-2t}u(t)}$$

Fourier Transform Properties

Example. Determine the *frequency response function*, $H(\omega)$, for the following stable linear system,

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = f(t)$$

Alternatively, we can apply the Fourier transform directly to the differential equation [use the differentiation property: $\frac{d^n y}{dt^n} \leftrightarrow (j\omega)^n Y(\omega)$] and obtain,

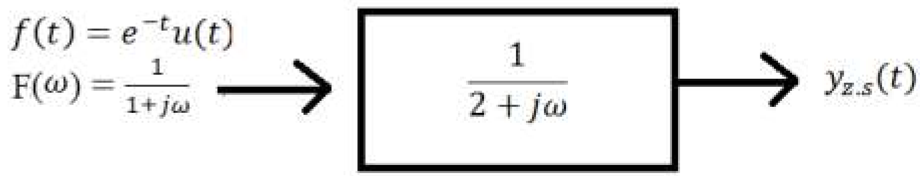
$$[(j\omega)^2 + 3(j\omega) + 2]Y(\omega) = F(\omega)$$

or,

$$H(\omega) = \frac{Y(\omega)}{F(\omega)} = \frac{1}{-\omega^2 + 3j\omega + 2}$$

Fourier Transform Properties

Example. Determine the zero-state response for the following stable system.



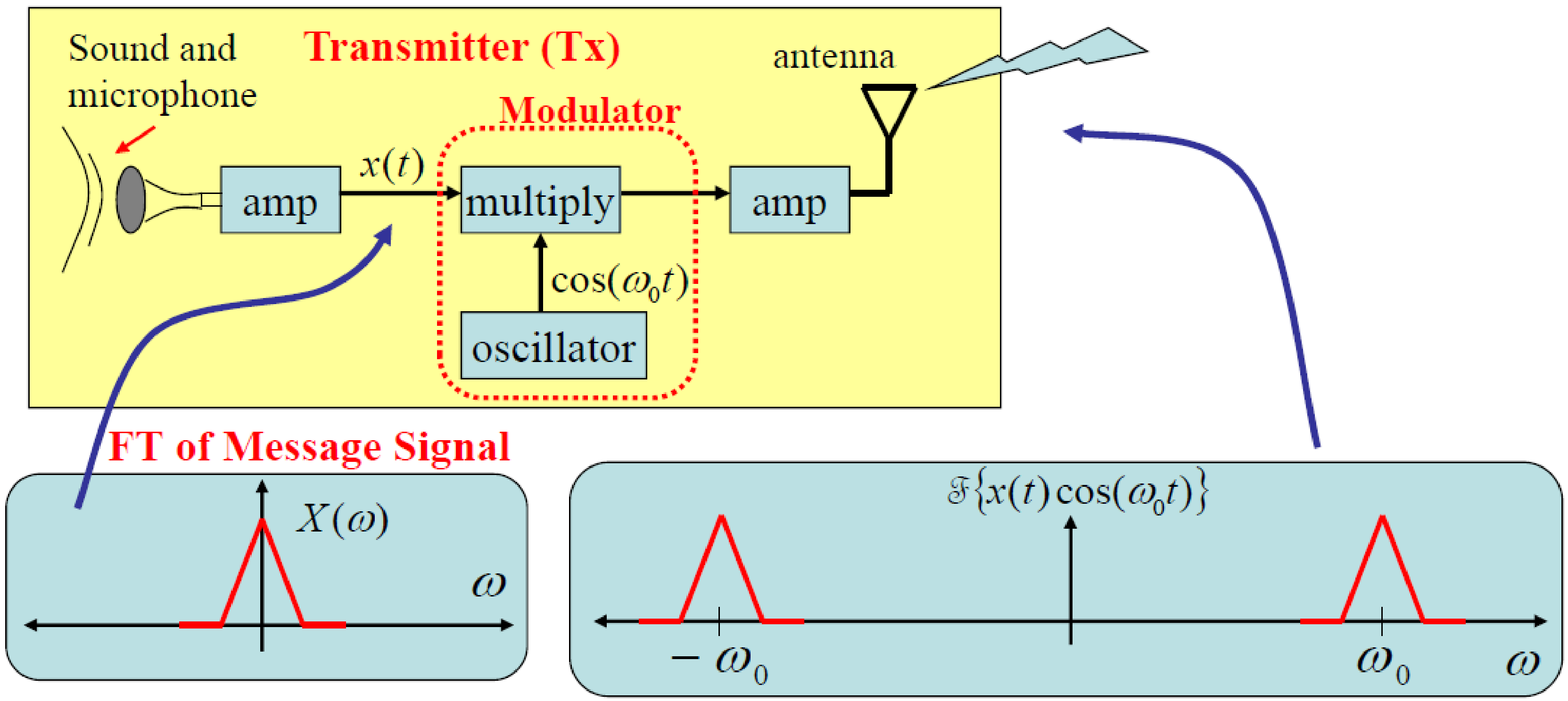
$$y_{zs}(t) = F^{-1}\left\{\frac{1}{2+j\omega}\cdot\frac{1}{1+j\omega}\right\} = F^{-1}\left\{\frac{1}{1+j\omega} - \frac{1}{2+j\omega}\right\}$$

$$y_{zs}(t) = e^{-t}u(t) - e^{-2t}u(t)$$

Application Example

Application of Modulation Property to Radio Communication

FT theory tells us what we need to do to make a **simple** radio system… *then* electronics can be built to perform the operations that the FT theory calls for:



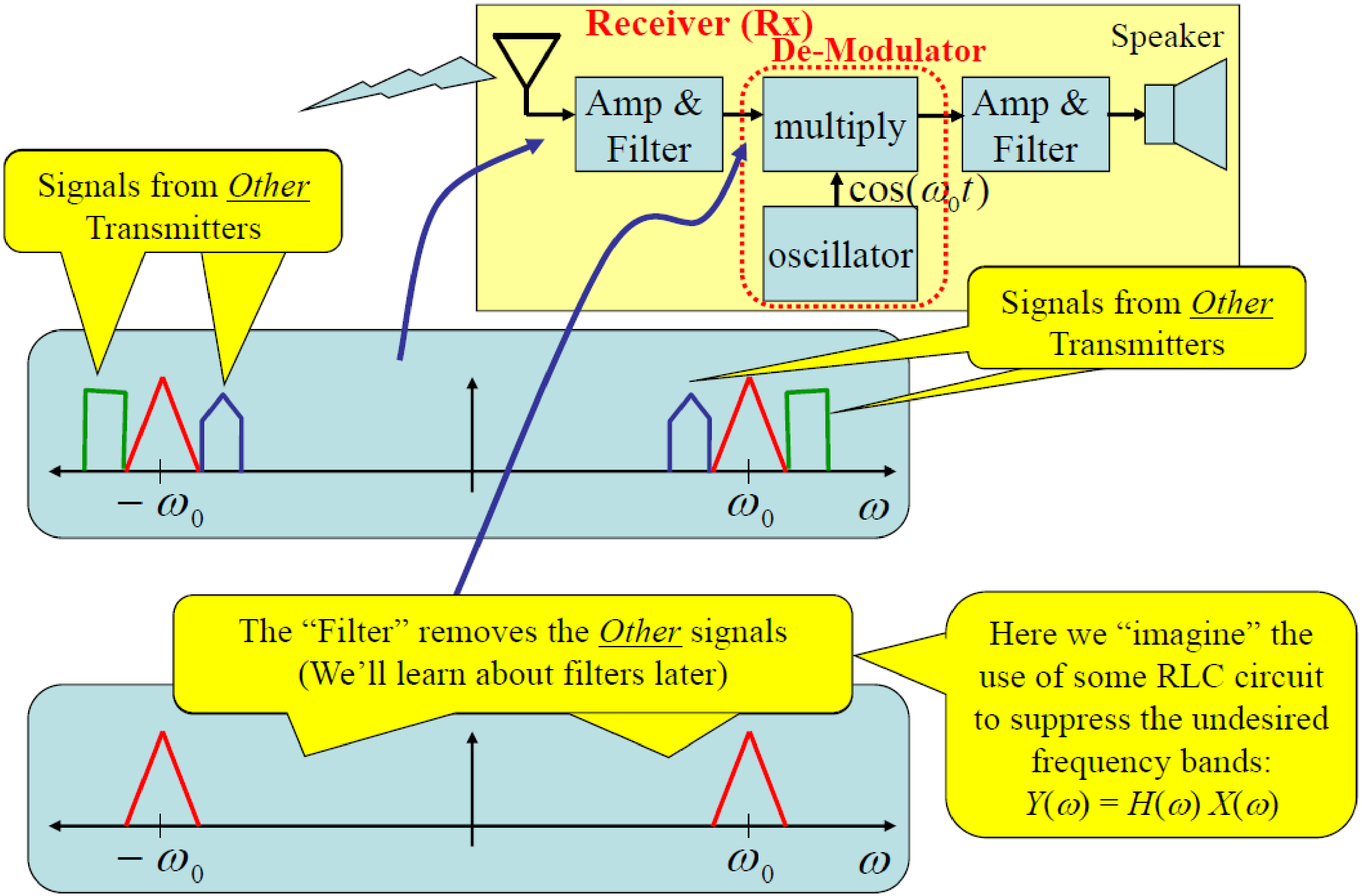
Choose $f_0 > 10$ kHz to enable efficient radiation (with $\omega_0 = 2\pi f_0$)

AM Radio: around 1 MHz FM Radio: around 100 MHz

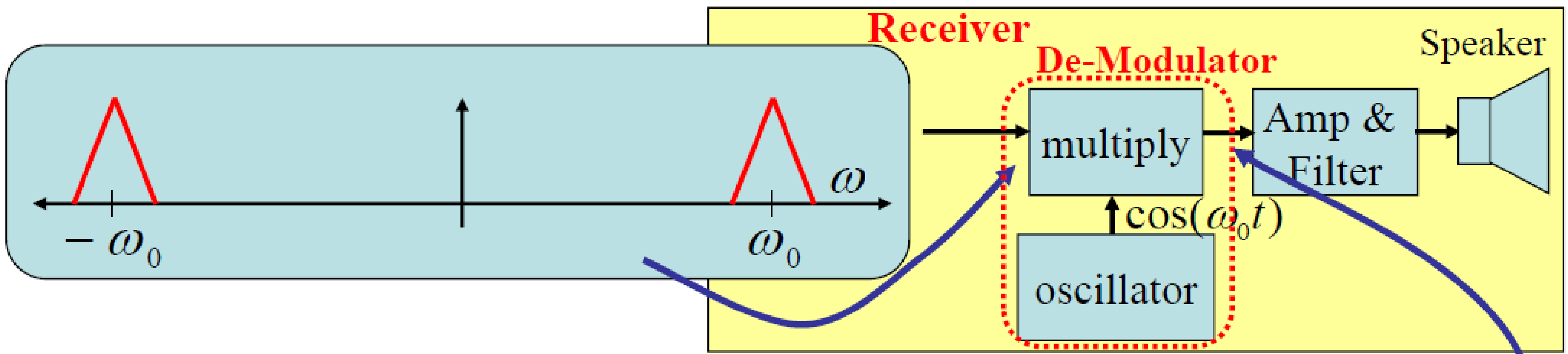
Cell Phones: around 900 MHz, around 1.8 GHz, around 1.9 GHz etc.

Application Example

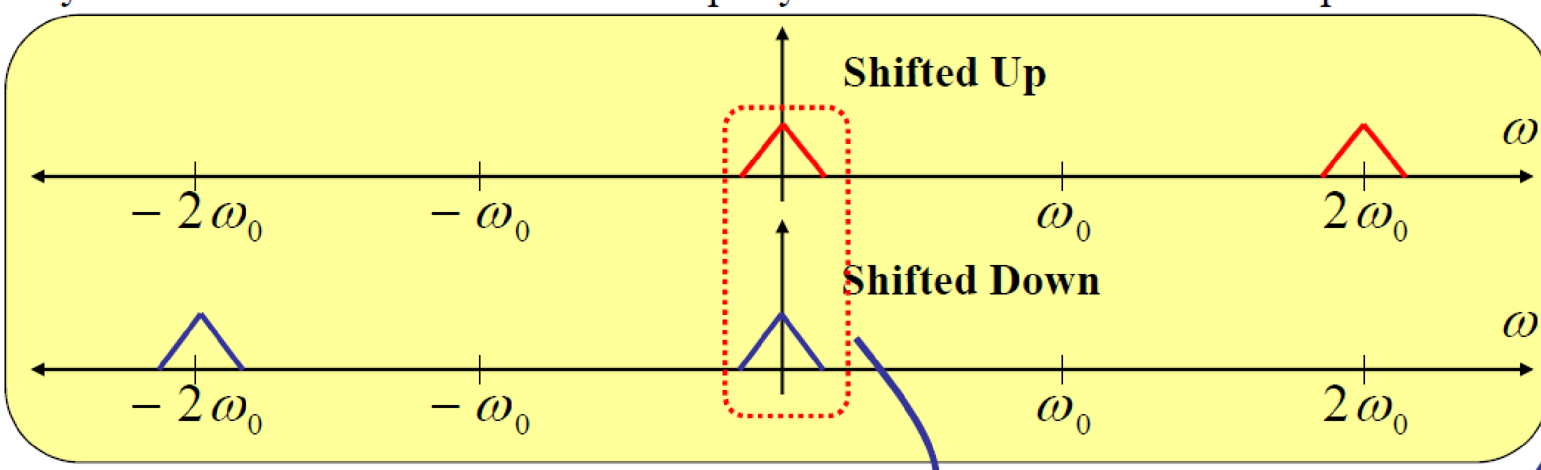
The next several slides show how these ideas are used to make a receiver:

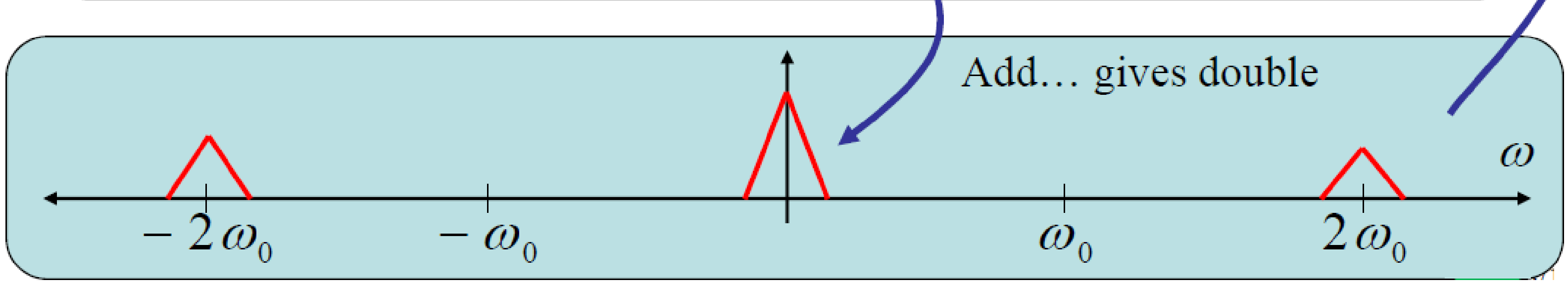


Application Example

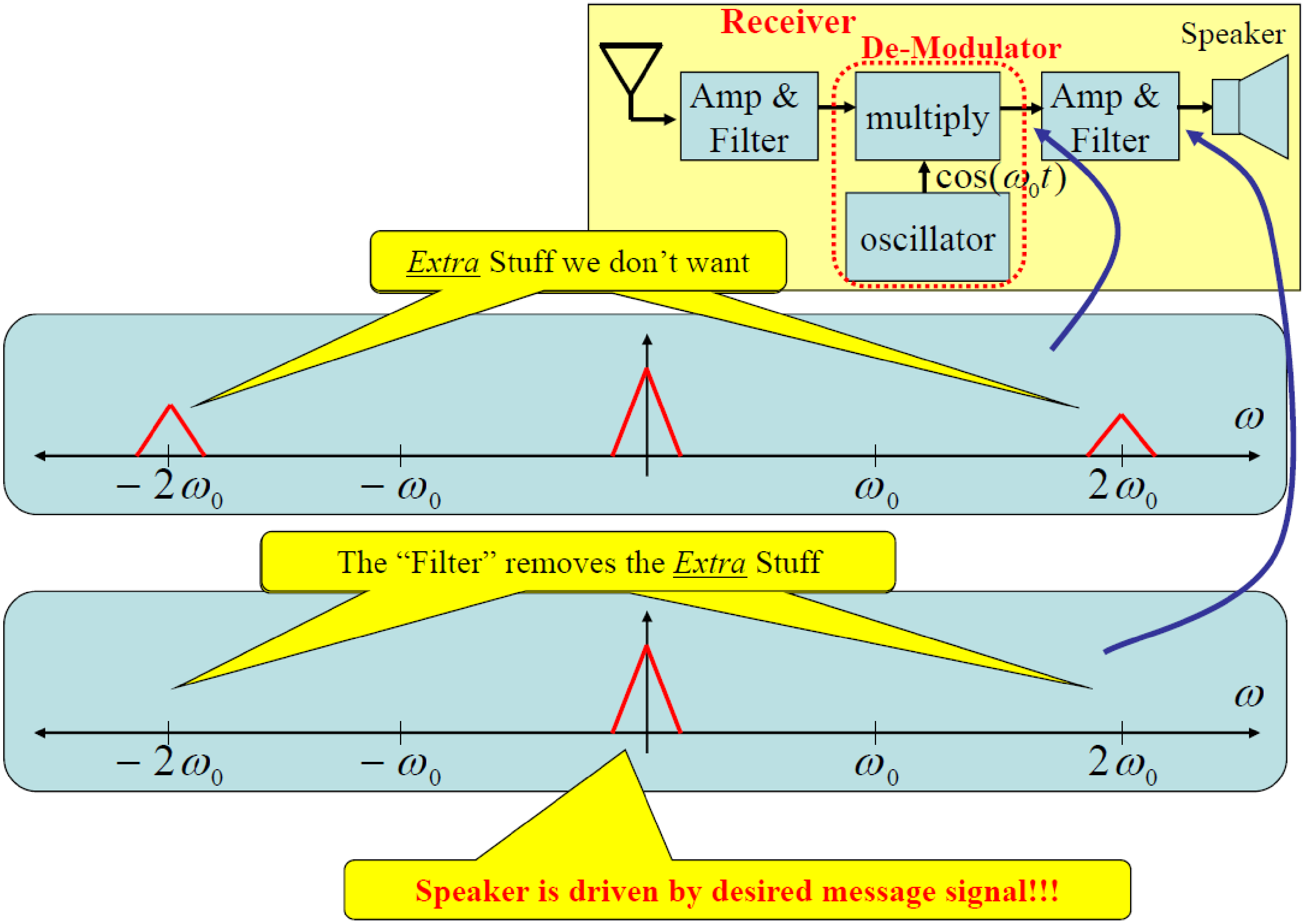


By the Real-Sinusoid Modulation Property… the De-Modulator shifts up & down:





Application Example



Application Example

So… what have we seen in this example:

Using the Modulation property of the FT we saw…

1. Key Operation at Transmitter is up-shifting the message spectrum:
    a) FT Modulation Property tells the theory then we can build…
    b) "modulator" = oscillator and a multiplier circuit
2. Key Operation at Recevier is down-shifting the received spectrum
    a) FT Modulation Property tells the theory then we can build…
    b) "de-modulator" = oscillator and a multiplier circuit
    c) But… the FT modulation property theory also shows that we need filters to get rid of "extra spectrum" stuff
        i. So… one thing we still need to figure out is how to deal with these filters…
        ii. Filters are a specific "system" and we still have a lot to learn about Systems…
        iii. That is the subject of much of the rest of this course!!!