



BLM3620 Digital Signal Processing*

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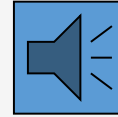
*Based on lecture notes from Ali Can Karaca & Ahmet Elbir

Lecture #2 – Sinusoids and Complex Exponentials

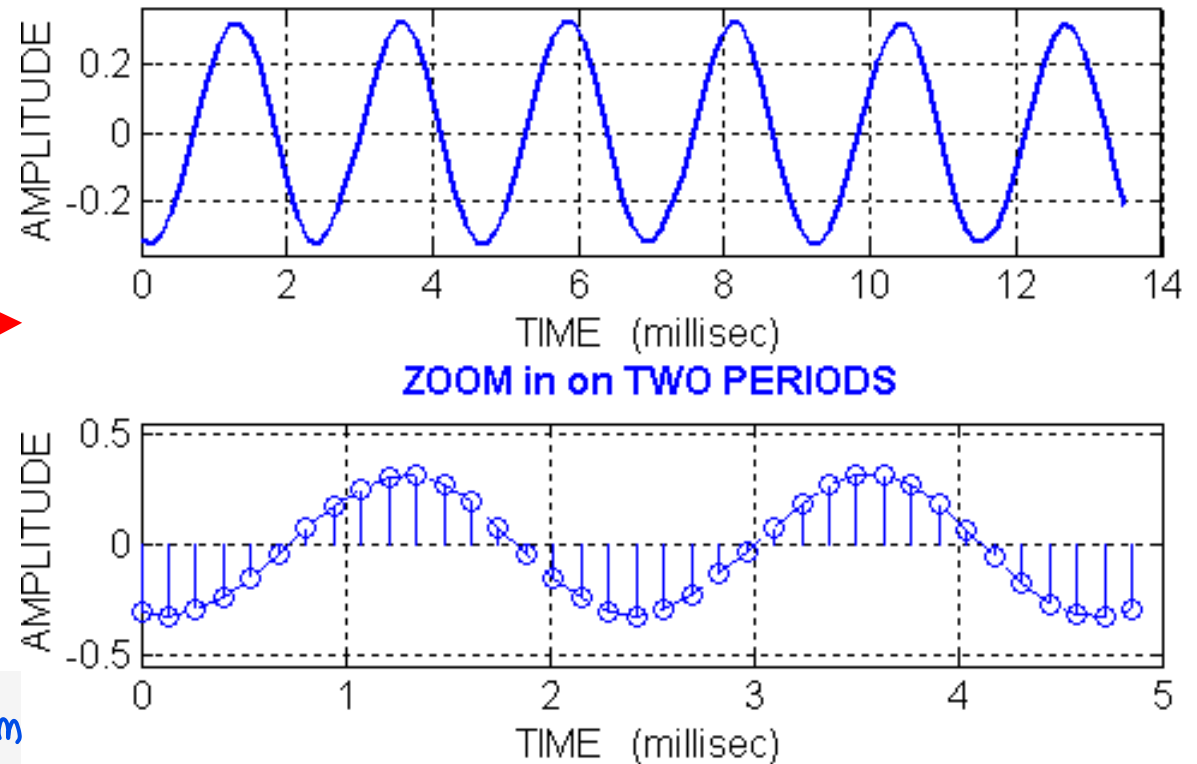
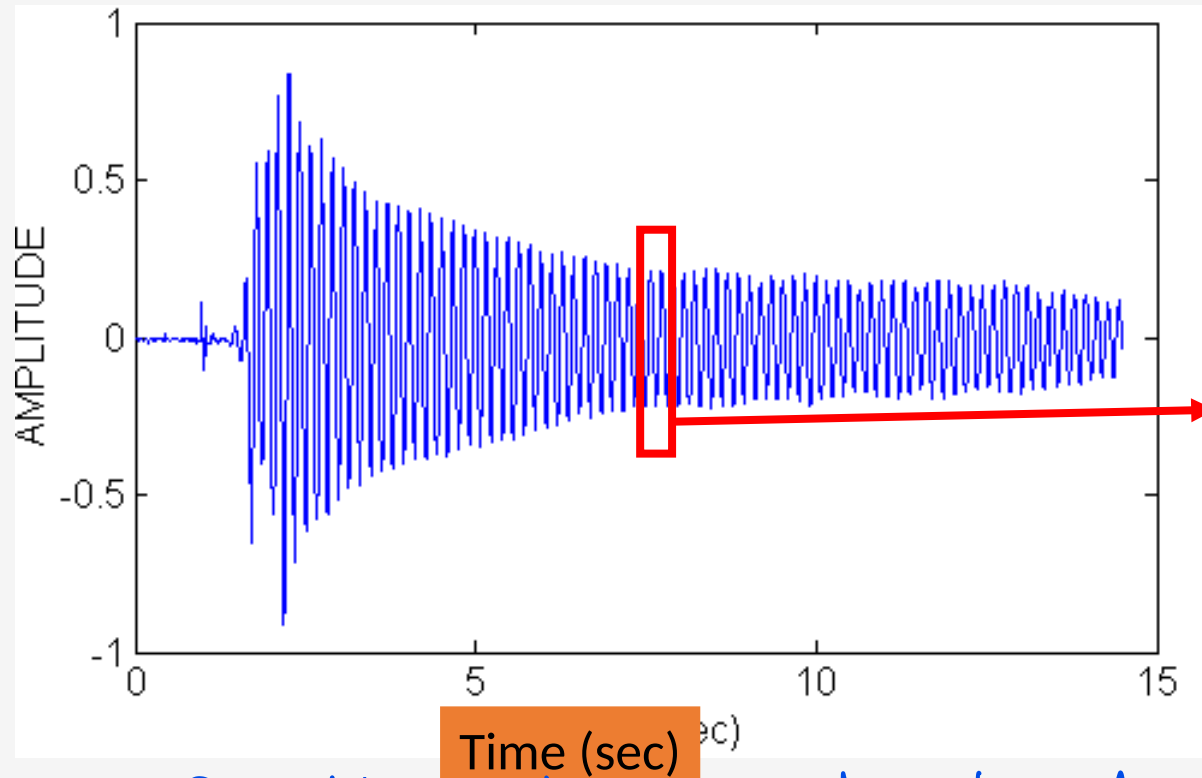
- Sinusoidal Signals Farklı sinüsoidal sinyaller birbirine ortogondur.
- Frequency, Period, Phase and Amplitude
- Complex Exponential Signals Bu problemleri çözeblmek için frekans dizlemine geçmemiz gerekir. Bunun için ise bu problemlerdeki sinyali frekansını bildiğimiz, sonsuz defa dönebilen temel fonk. cinsinden yazabilmemiz gerekir. (kadin - erkek sesi ayırt etme)
- Phasor Addition
- MATLAB Applications

Recall: Tuning Fork

Sinusoids are important part of our world.



TUNING FORK A-440



Doğadaki sesler sönümlenerek alınım
yapar. Genel e^x yapısındadırlar.

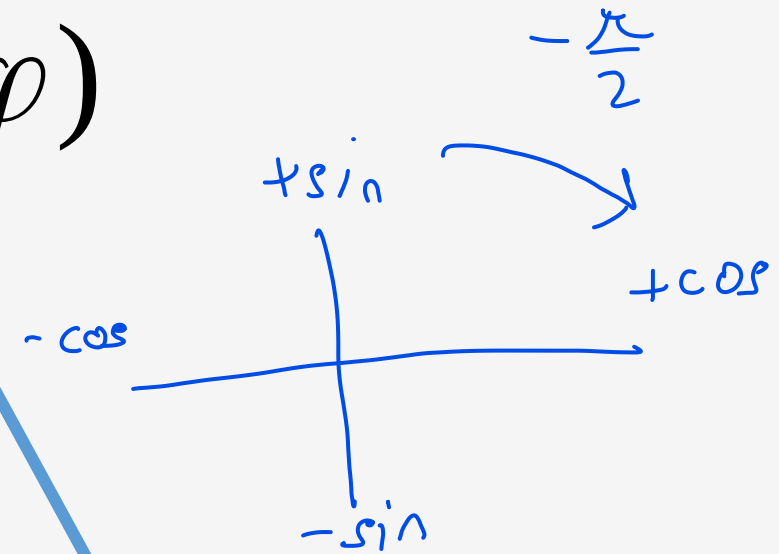
SINES and COSINES



- Always use the COSINE FORM

$$A \cos(2\pi(440)t + \varphi)$$

karmaşık sayı düzlemine geçerken yardımcı oluyor.



- Sine is a special case:

$$\sin(\omega t) = \cos(\omega t - \frac{\pi}{2})$$

Sinusoid Signal



$$A \cos(\omega t + \varphi)$$

B DC bileşen
 ω FREQUENCY
 • Radians/sec
 • Hertz (cycles/sec)
 $\omega = (2\pi) f$
 • AMPLITUDE
 • Magnitude
 $\omega \cdot t = \text{açı}$

φ PHASE
 $T = \frac{1}{f} = \frac{2\pi}{\omega}$
 • PERIOD (in sec)
 bir tam dönüşte(t) 2 π alca
 tıyor. (sinusoidal)
 Böylece $\omega \cdot T = 2\pi \rightarrow \omega = \frac{2\pi}{T}$
 rad/s ω 'nin birimi olur.

Some Trigonometric Identities



Fourier Serisi sadece periyodikler için geçerlidir.

Number	Equation
1	$\sin^2 \theta + \cos^2 \theta = 1$
2	$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
3	$\sin 2\theta = 2 \sin \theta \cos \theta$
4	$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$
5	$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$

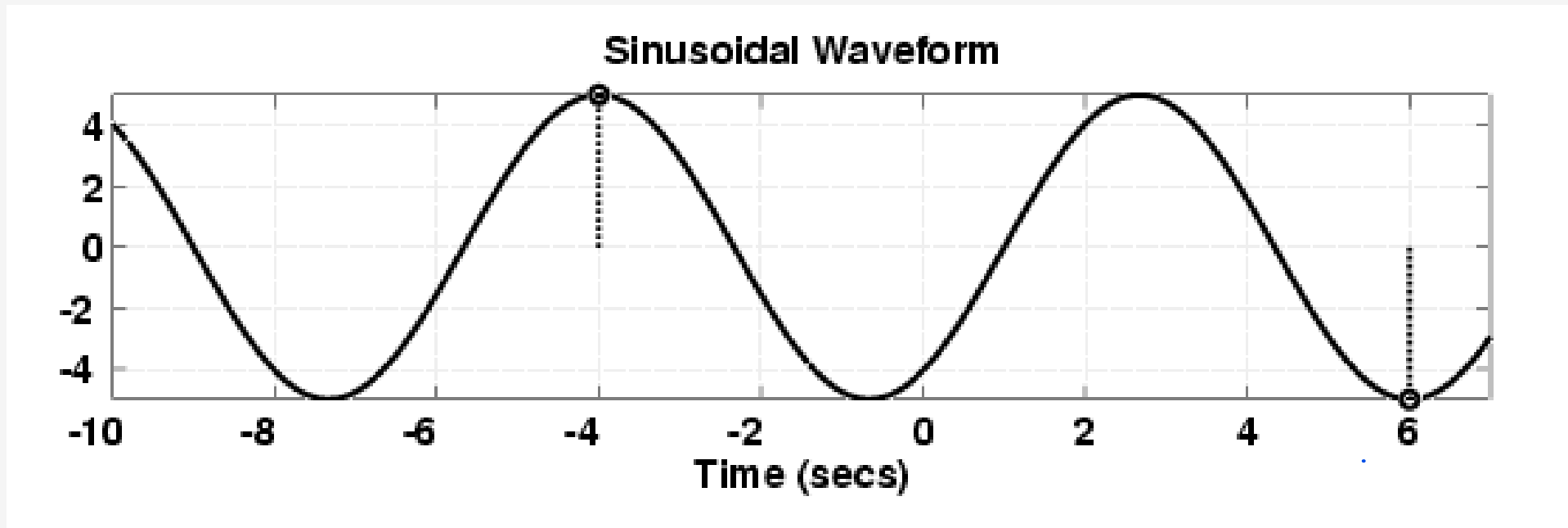
EXAMPLE of SINUSOID



- Given the Formula

$$5\cos(0.3\pi t + 1.2\pi)$$

- Make a plot



PLOT COSINE SIGNAL



$$5\cos(0.3\pi t + 1.2\pi)$$

- Formula defines A , ω , and ϕ

$$A = 5$$

$$\omega = 0.3\pi$$

$$\phi = 1.2\pi$$

PLOTTING COSINE SIGNAL from the FORMULA



$$5\cos(0.3\pi t + 1.2\pi)$$

- Determine period:

$$T = 2\pi / \omega = 2\pi / 0.3\pi = 20 / 3$$

- Determine a peak location by solving

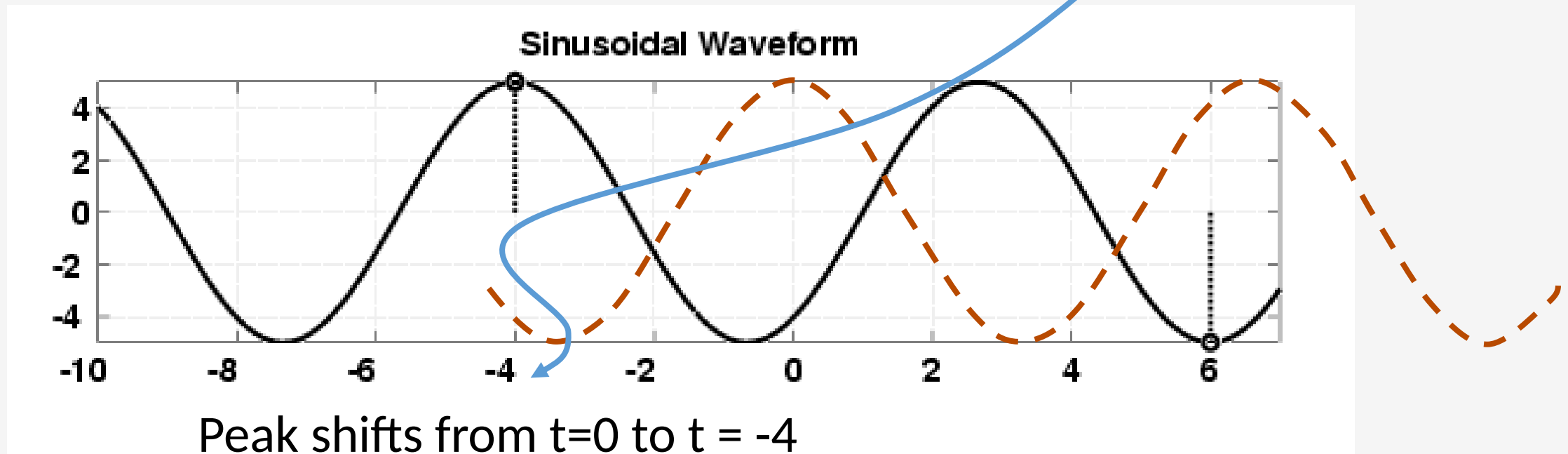
$$(\omega t + \varphi) = 0 \Rightarrow (0.3\pi t + 1.2\pi) = 0$$

- Zero crossing is $T/4$ before or after
- Positive & Negative peaks spaced by $T/2$

Time-shifted Sinusoid

$$x(t) = 5\cos(0.3\pi t) \quad \text{One peak at } t = 0$$

$$x(t + 4) = 5\cos(0.3\pi(t + 4)) = 5\cos(0.3\pi(t - (-4)))$$



How to determine Amplitude, Phase and Period from a plot



- Measure the period, T
 - Between peaks or zero crossings
- Compute frequency: $\omega = 2\pi/T$
- Measure time of a peak: t_m
 - Compute phase: $\phi = -\omega t_m$
- Measure height of positive peak: A

3 steps

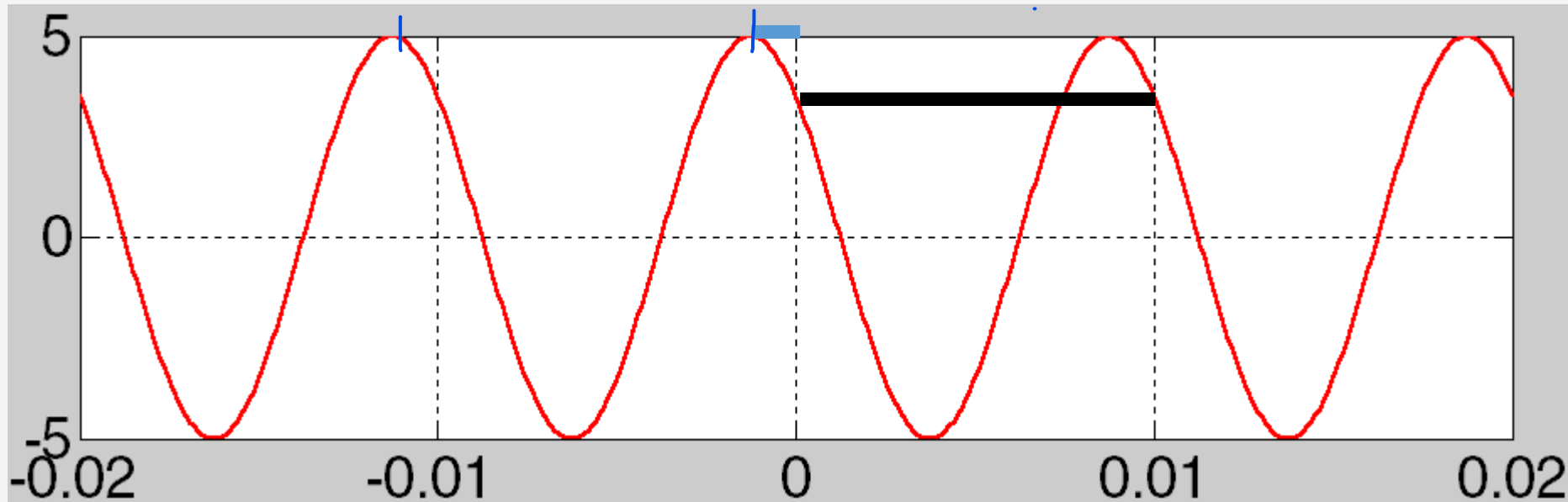
A blue box containing the text '3 steps' is positioned to the right of the list. Three blue curved lines originate from the box and point to the three main steps of the process: 'Measure the period, T', 'Compute frequency: ω = 2π/T', and 'Measure time of a peak: t_m'.

(A, ω , ϕ) from a PLOT

$$\tau = 10^{-2} \quad f = 100$$



$$+ 5 \cos(100 \cdot 2\pi \cdot t + \phi)$$



$$T = \frac{0.01 \text{ sec}}{1 \text{ period}} = \frac{1}{100} \longrightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{0.01} = 200\pi$$

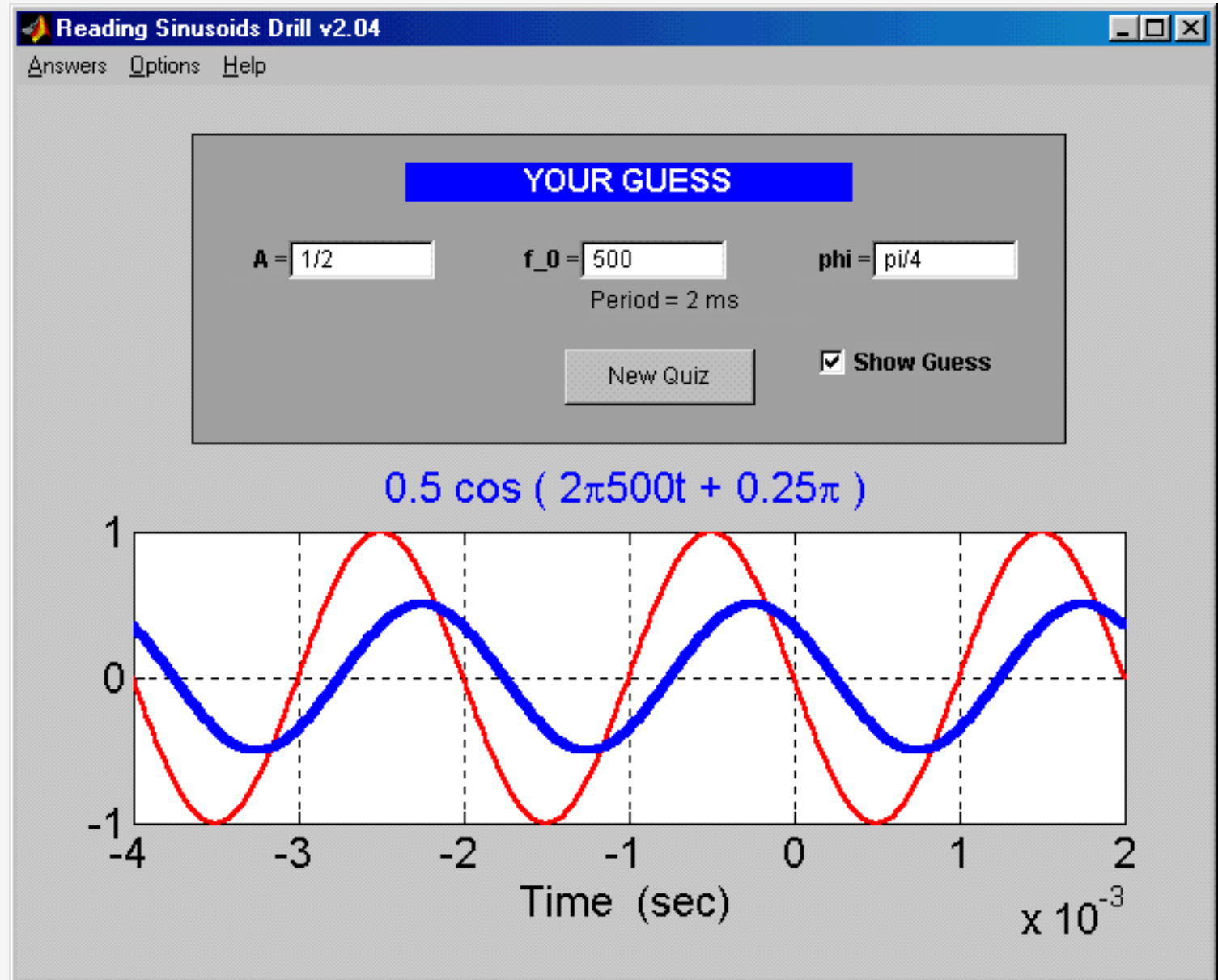
$$t_m = -0.00125 \text{ sec} \longrightarrow \phi = -\omega t_m = -(200\pi)(t_m) = 0.25\pi$$

SINE DRILL (MATLAB GUI)

<https://dspfirst.gatech.edu/matlab/#sindrill>

SinDrill is a program that tests the users ability to determine basic parameters of a sinusoid.

After a plot of a sinusoid is displayed, the user must correctly guess its amplitude, frequency, and phase.



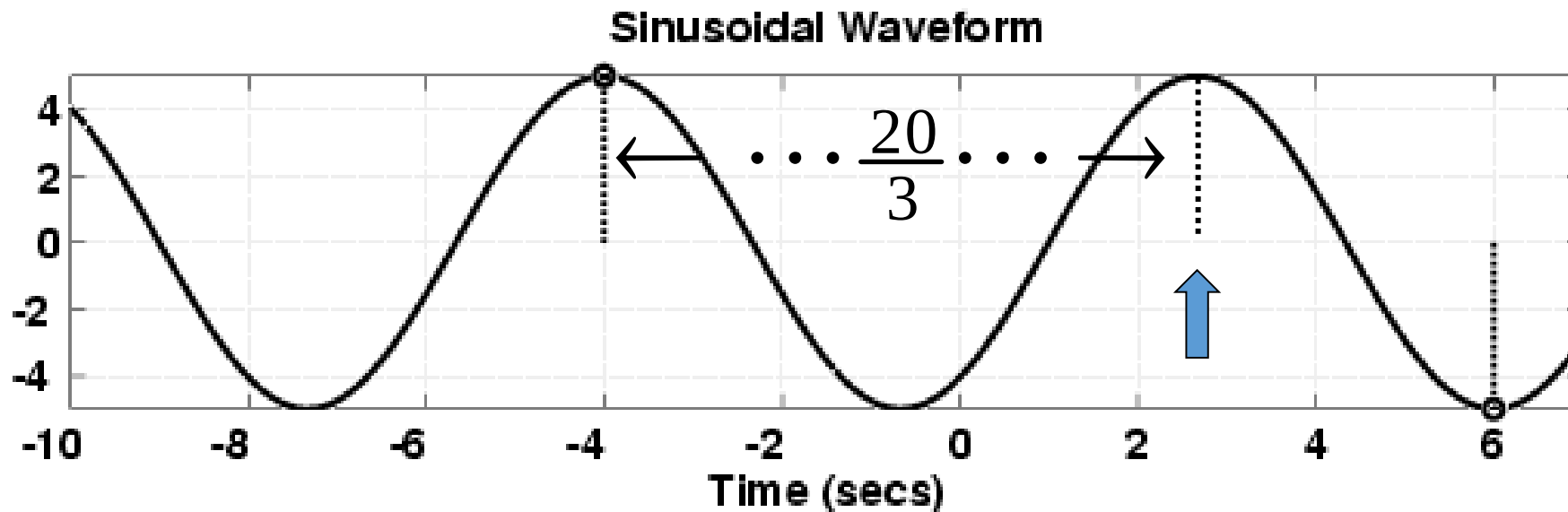
Phase is Ambiguous

The cosine signal is periodic

– Period is 2π

$$A \cos(\omega t + \varphi + 2\pi) = A \cos(\omega t + \varphi)$$

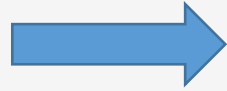
$$5 \cos(0.3\pi t + 1.2\pi) = 5 \cos(0.3\pi t - 0.8\pi)$$



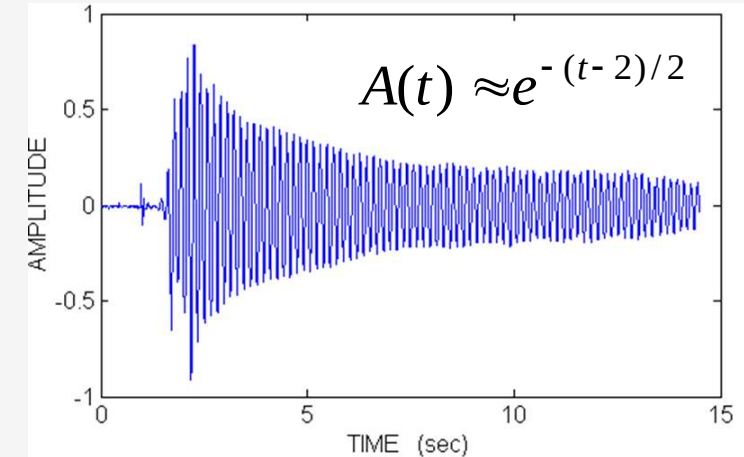
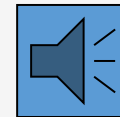
Attenuation: Amplitude Varies with Time (Fade Out?)



$$x(t) = A \cos(\omega t + \varphi)$$



$$A(t) = A e^{-t/\alpha}$$



```
fs = 8000;  
% define array tt for time  
% time runs from -1s to +3.2s  
% sampled at an interval of 1/fs  
tt = 0: 1/fs : 3.2;  
xx = 2.1 * cos(2*pi*440*tt + 0.4*pi);
```

```
soundsc (xx, fs)
```

$$x(t) = 2.1 \cos(880\pi t + 0.4\pi)$$

```
fs = 8000;  
tt = 0: 1/fs : 3.2;  
yy = exp(-tt*1.2); % exponential decay  
yy = xx.*yy;
```

```
soundsc(yy, fs)
```

$$y(t) = 2.1 e^{-1.2t} \cos(880\pi t + 0.4\pi)$$

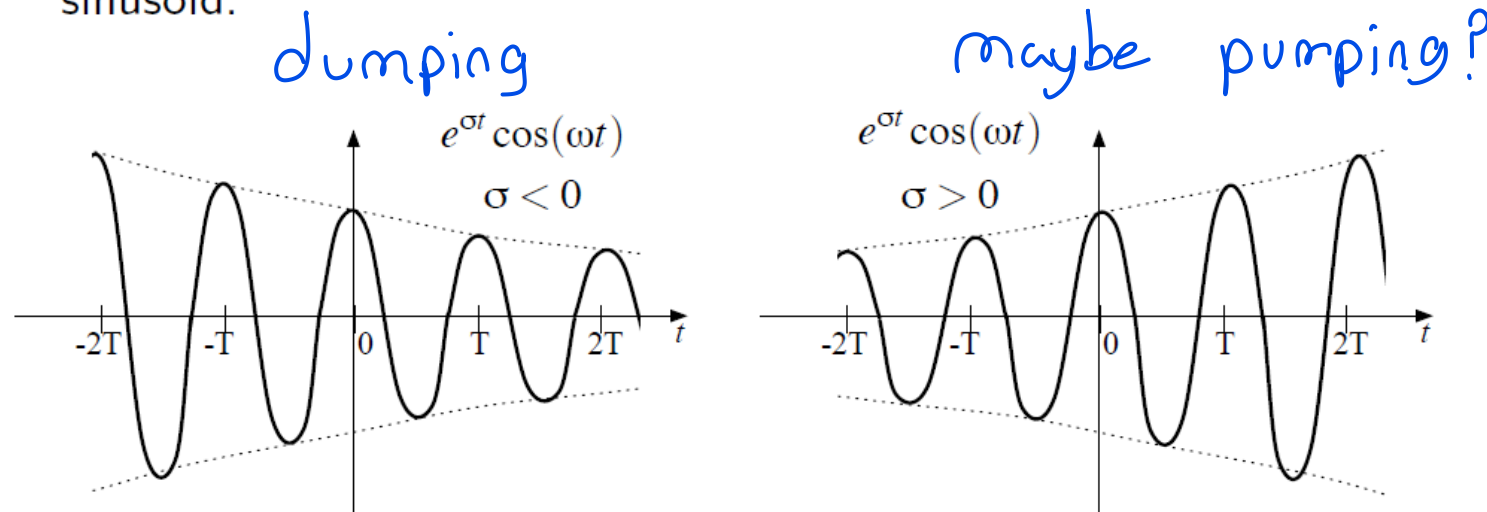
Growing Sinusoid? (Exponential Sinusoid)

Damped or Growing Sinusoids

- A damped or growing sinusoid is given by

$$x(t) = e^{\sigma t} \cos(\omega t + \theta)$$

- Exponential growth ($\sigma > 0$) or decay ($\sigma < 0$), modulated by a sinusoid.



Remember: Complex Numbers

if $x=0 \rightarrow z = \text{pure imaginary}$

Cartesian Coordinate System

- To solve: $z^2 = -1$

- $z = j$

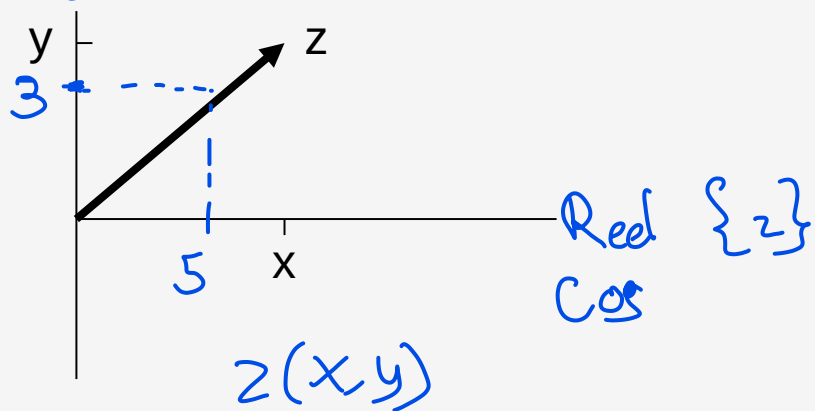
- Math and Physics use $z = i$

Engineering use $z = j$, cuz $\text{amp} = i$

- Complex number: $z = x + jy$

$\text{Im}\{z\}$

$z = 5 + 3j$



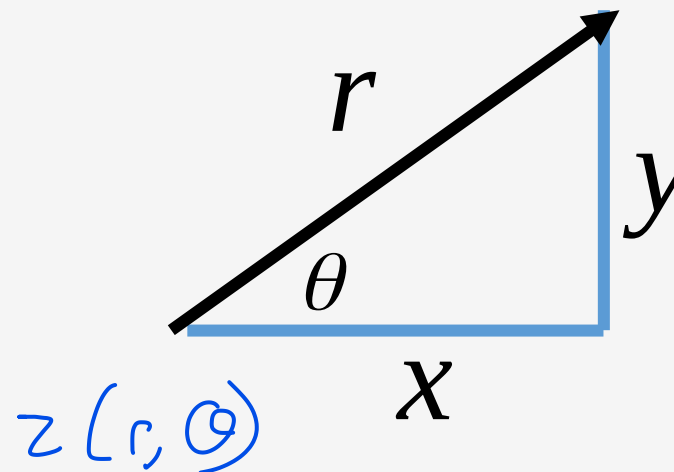
Polar Coordinate System

$$r^2 = x^2 + y^2$$

$$x = r \cos \theta$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

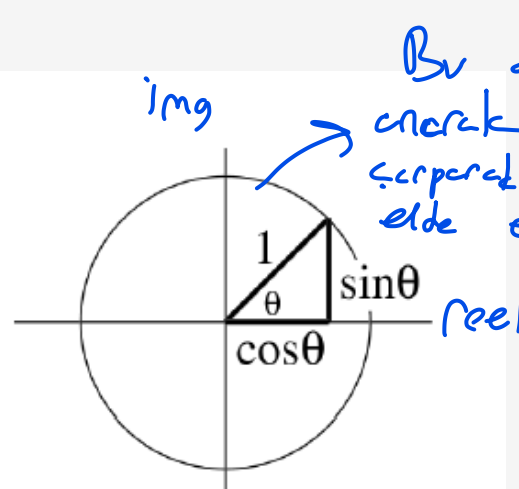
$$y = r \sin \theta$$



Euler's Formula (Important!!)

• Complex Exponential

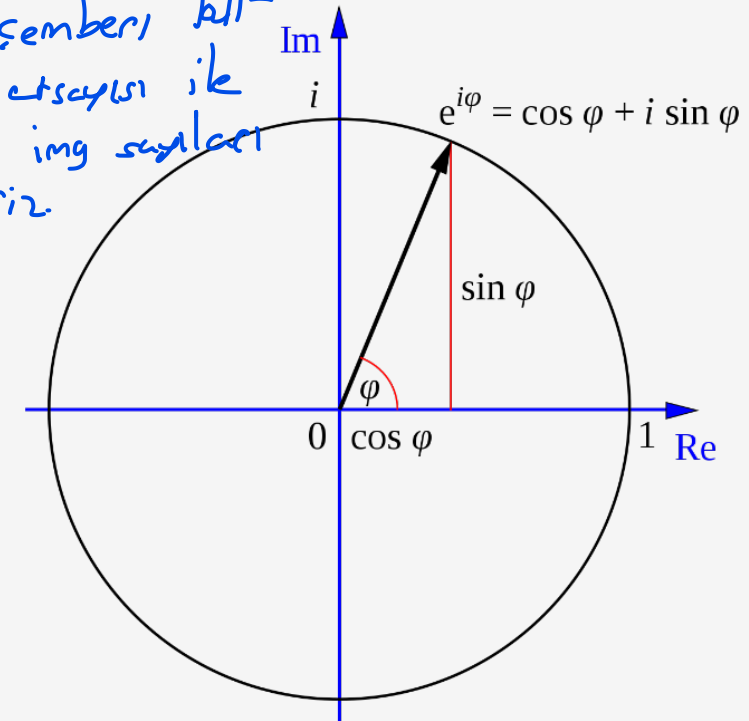
- Real part is cosine
- Imaginary part is sine
- Magnitude is one



$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

$$r e^{j\theta} = r \cos(\theta) + j r \sin(\theta)$$

$$r \cdot e^{-j\theta} = r \cdot \cos \theta - j \cdot r \cdot \sin \theta$$



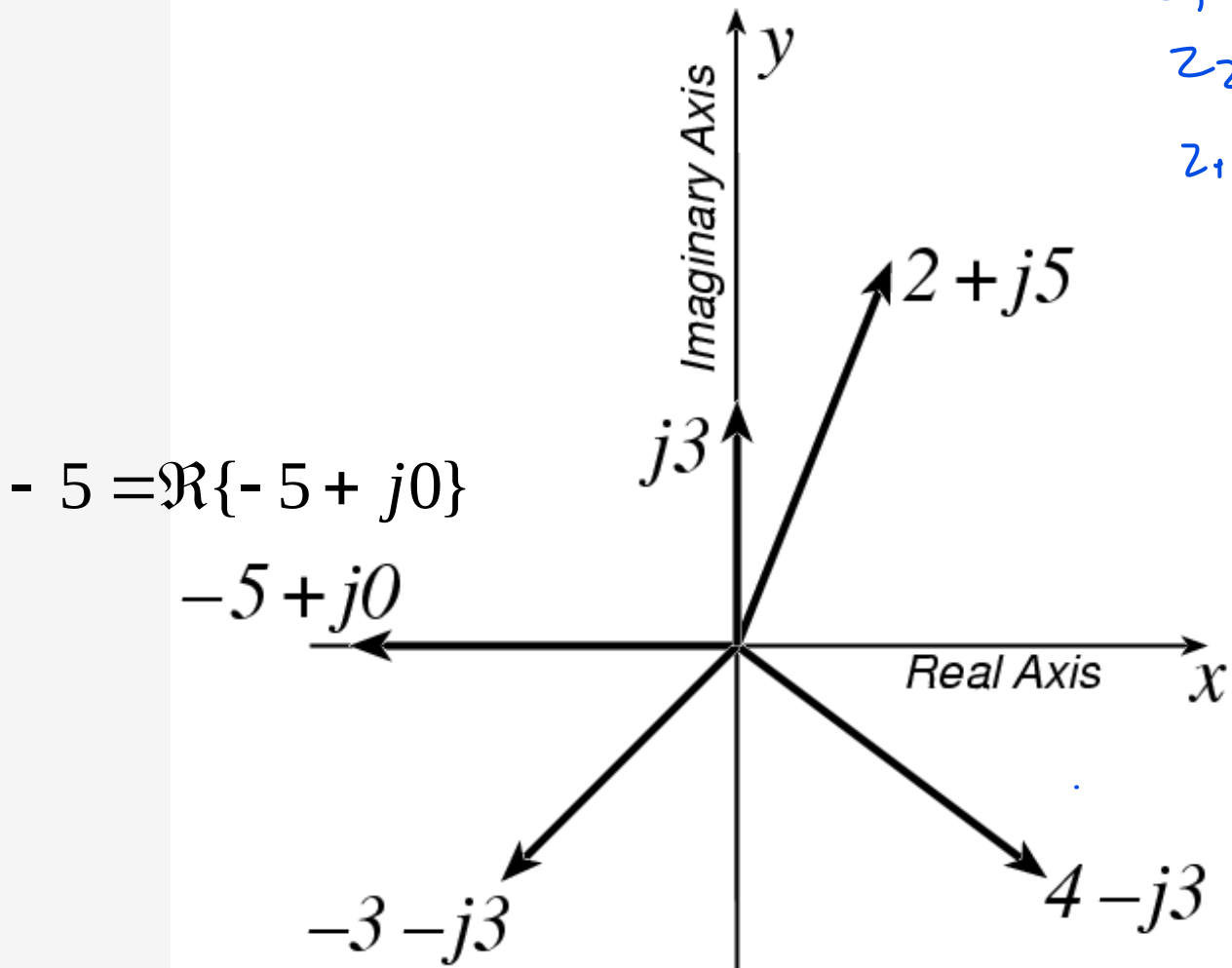
$$e^{j\theta} = z$$

$$z = \cos \theta + j \cdot \sin \theta$$

$$e^{j\theta} = \cos \theta + j \cdot \sin \theta$$

$$r \cdot e^{j\theta} = r \cdot \cos \theta + j \cdot r \cdot \sin \theta$$

Remember: Complex Numbers



$$z_1 = 3 \cdot e^{j2}$$

$$z_2 = 2 \cdot e^{j3}$$

$$z_1 \cdot z_2 = 6 \cdot e^{j5}$$

Handwritten notes:
 z_1 and z_2 are circled in blue.
 6 is circled in red.
 e^{j5} is circled in red.
 Below the equations, there are handwritten notes in Turkish: "r'in karşılığı" (correspondence of r) and "açıların toplamı" (sum of angles).

Real part:

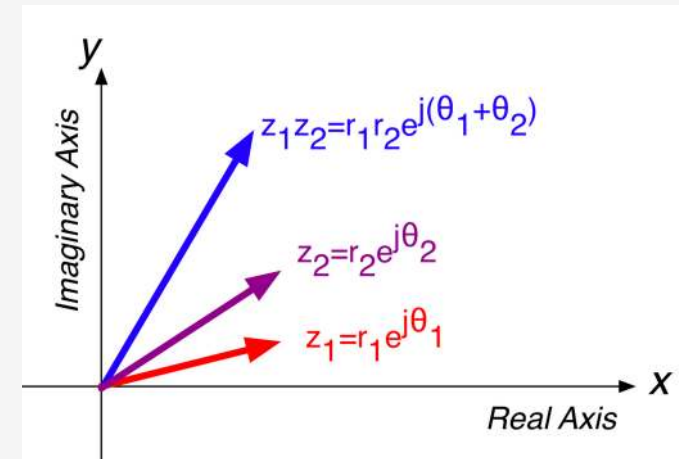
$$x = \Re\{z\}$$

Imaginary part:

$$y = \Im\{z\}$$

Complex addition?

Complex multiplication?



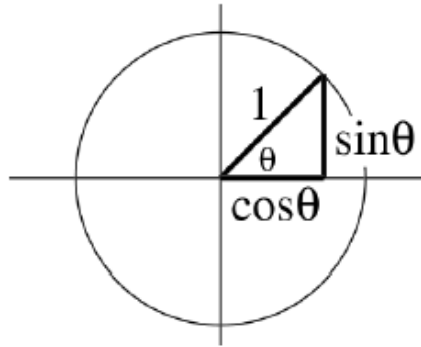
Zdrill tool

<https://dspfirst.gatech.edu/matlab/#zdrill>

Euler's Formula (Important!!)

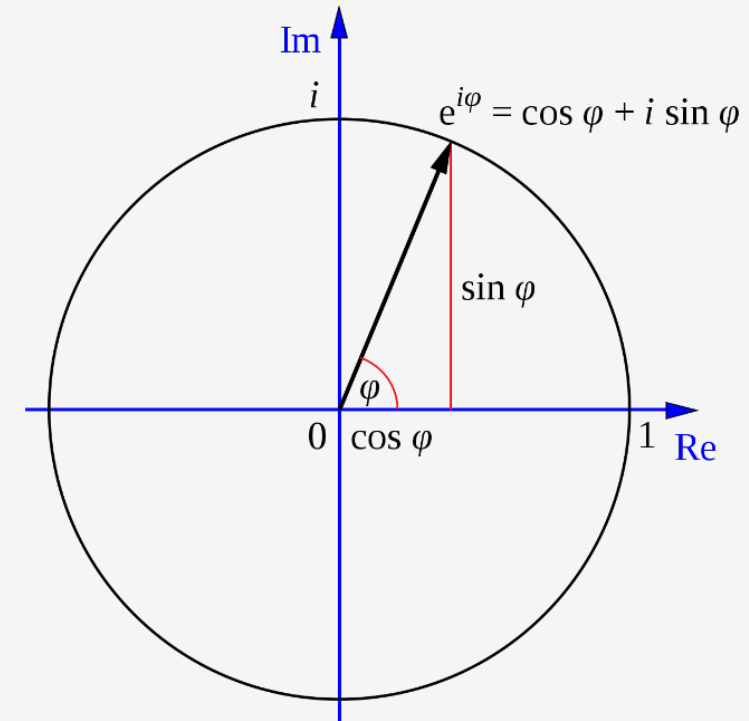
- **Complex Exponential**

- Real part is cosine
- Imaginary part is sine
- Magnitude is one



$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

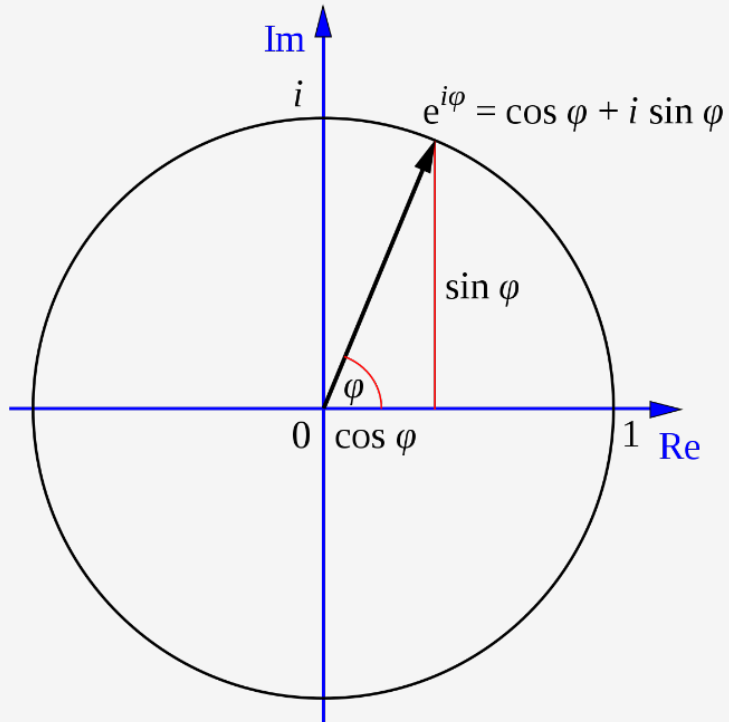
$$re^{j\theta} = r \cos(\theta) + jr \sin(\theta)$$



What happens if we write variable instead of Theta?

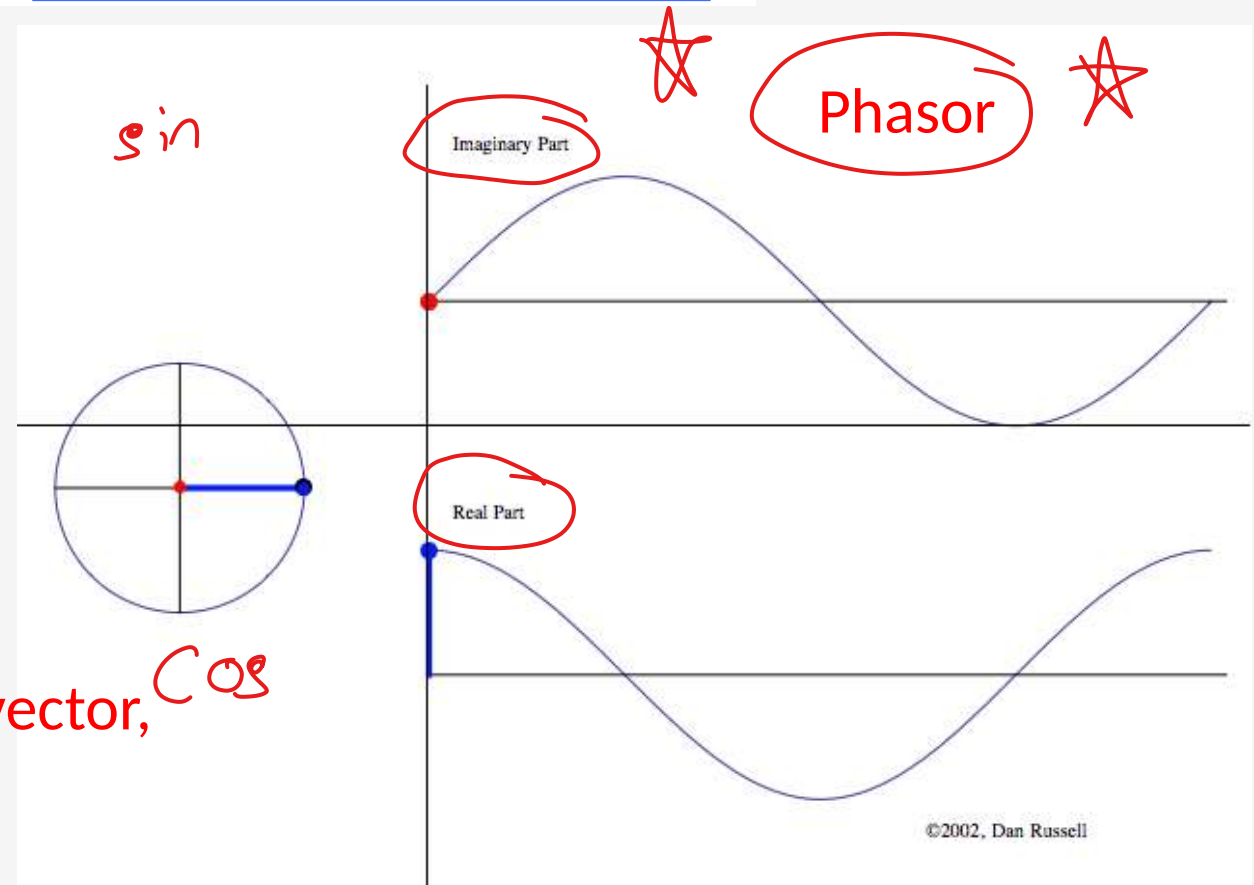
$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

Euler's Formula (Important!!)



What happens if we write variable instead of Theta?

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$



Complex Exponential includes a rotating vector,
= complex summation of sinuzoids

Euler's Formula Reversed

- Solve for cosine (or sine)

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j \sin(-\omega t)$$

$$e^{-j\omega t} = \overset{\text{çift}}{\cos(\omega t)} - \overset{\text{tek}}{j \sin(\omega t)}$$

$$e^{j\omega t} + e^{-j\omega t} = 2 \cos(\omega t)$$

$$\cos(\omega t) = \frac{1}{2} (e^{j\omega t} + e^{-j\omega t})$$

INVERSE Euler's Formula



- Solve Euler's formula for **cosine** (or sine)

$$\cos(\omega t) = \frac{1}{2} (e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t})$$

$$z = A \cdot \cos(\omega t + \phi) + 0 \cdot A \cdot \sin(\omega t + \phi)$$

$$\begin{aligned} &= A \cdot e^{j(\omega t + \phi)} \rightarrow A \cdot e^{j\omega t} \cdot e^{j\phi} \rightarrow \text{ancak img taraf katsayısı } 0 \text{ o halde} \\ &= \mathcal{R}\{ \underbrace{A \cdot e^{j\omega t}}_{\text{sabit}} \cdot \underbrace{e^{j\phi}}_{\text{sabit}} \} \rightarrow \text{zamanla değişken} \end{aligned}$$

Phasor Form of A Cosine



$$A \cos(\omega t + \varphi) = \Re \{ (Ae^{j\varphi}) e^{j\omega t} \}$$

Handwritten annotations:

- $Ae^{j\varphi}$ is circled in pink and labeled "genlik" (amplitude).
- $e^{j\omega t}$ is circled in blue and labeled "phase angle".
- $Ae^{j\varphi}$ is circled in blue and labeled "Complex Amplitude: Constant".
- $e^{j\omega t}$ is circled in green and labeled "Varies with time".
- A pink arrow points from the text "phasor ve genlik çarpımından oluşur." (formed from phasor and amplitude multiplication) to the $Ae^{j\varphi}$ term.

- Find the COMPLEX AMPLITUDE for:

$$x(t) = \sqrt{3} \cos(77\pi t + 0.5\pi)$$

$$x(t) = \Re \{ \sqrt{3} e^{j(77\pi t + 0.5\pi)} \}$$

- Use EULER's FORMULA:

$$= \Re \{ \underbrace{\sqrt{3} e^{j0.5\pi}}_{\text{genlik bilgisi}} \underbrace{e^{j77\pi t}}_{\text{frekans bilgisi}} \}$$

$$X = \sqrt{3} e^{j0.5\pi}$$

POP QUIZ



- Determine the 60-Hz sinusoid whose COMPLEX AMPLITUDE is:

$$X = \sqrt{3} + j3$$

- Convert X to **POLAR**:

$$\begin{aligned} x(t) &= \Re\{(\sqrt{3} + j3)e^{j(120\pi t)}\} \\ &= \Re\{\sqrt{12}e^{j\pi/3}e^{j120\pi t}\} \end{aligned}$$

$$\Rightarrow x(t) = \sqrt{12} \cos(120\pi t + \pi/3)$$

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Want to Add Sinusoids with same frequency



Adding sinusoids of common frequency results in sinusoid with SAME frequency

$$x(t) = \sum_{k=1}^N A_k \cos(\omega_0 t + \varphi_k)$$

Handwritten annotations: "değişken" (variable) with arrows pointing to A_k and φ_k ; "sabit" (constant) with an arrow pointing to ω_0 .

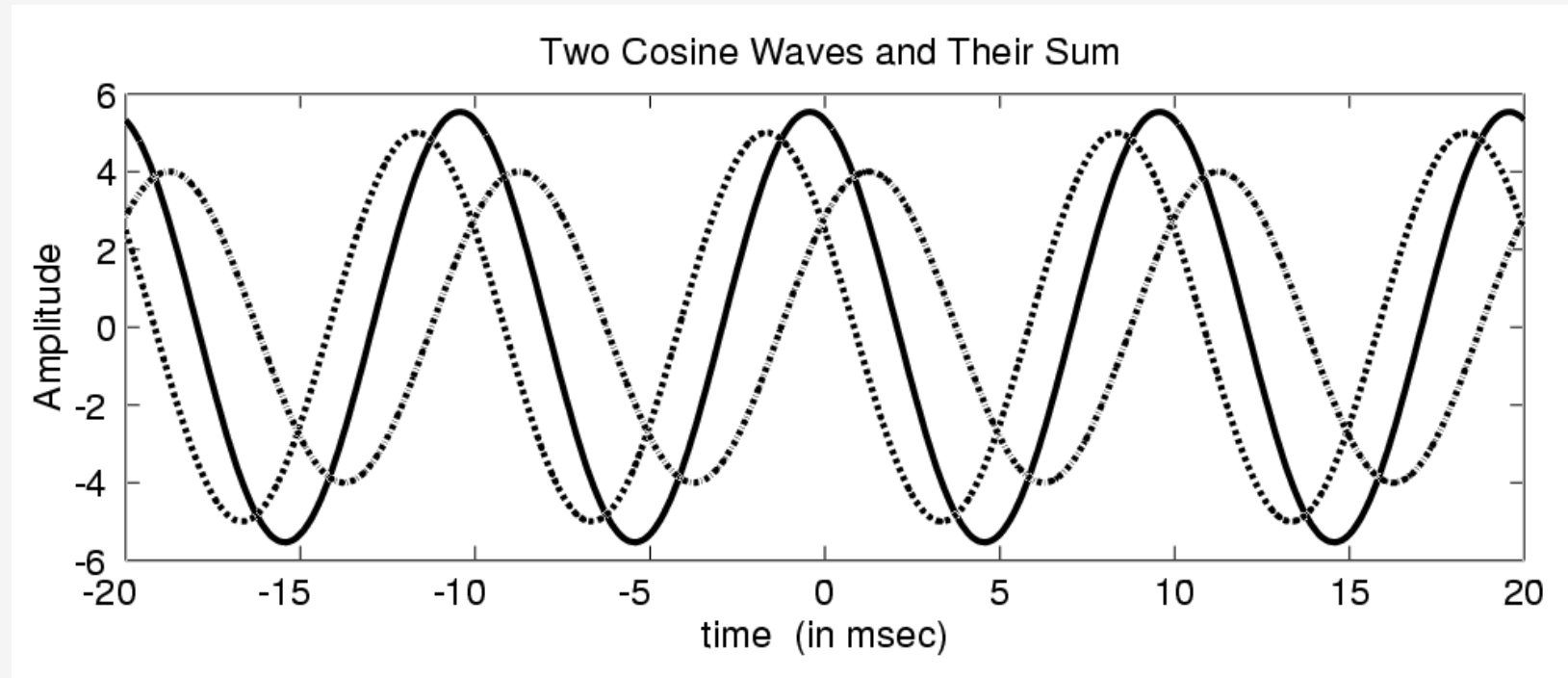
$$= A \cos(\omega_0 t + \varphi)$$

Get the new complex amplitude by complex addition

$$\sum_{k=1}^N A_k e^{j\varphi_k} = A e^{j\varphi}$$

Want to Add Sinusoids with same frequency

Adding sinusoids of common frequency results in sinusoid with SAME frequency



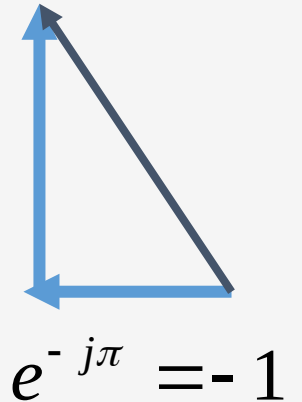
Want to Add Sinusoids with same frequency

- ADD THESE 2 SINUSOIDS:

$$x_1(t) = \cos(77\pi t - \pi) = 1 \cdot e^{-j\pi} \cdot e^{j77\pi t}$$

$$x_2(t) = \sqrt{3} \cos(77\pi t + 0.5\pi) = \sqrt{3} \cdot e^{j0.5\pi} \cdot e^{j77\pi t}$$

$$\sqrt{3}e^{j\pi/2} = j\sqrt{3}$$



- COMPLEX (PHASOR) ADDITION:

$$-1 + j\sqrt{3} = 2e^{j2\pi/3}$$

$$\begin{aligned} & (1e^{-j\pi} + \sqrt{3}e^{j0.5\pi}) \cdot e^{j77\pi t} \\ & (-1 + j\sqrt{3}) \cdot e^{j77\pi t} \rightarrow 2 \cdot e^{j\frac{2\pi}{3}} \cdot e^{j77\pi t} \end{aligned}$$

$$x_3(t) = 2 \cos(77\pi t + \frac{2\pi}{3})$$

Phasor Addition

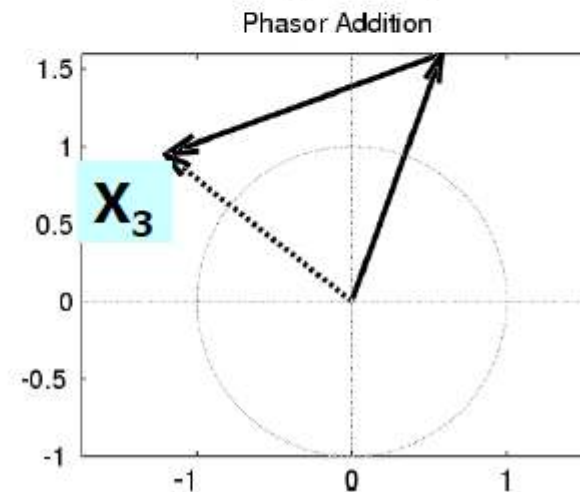
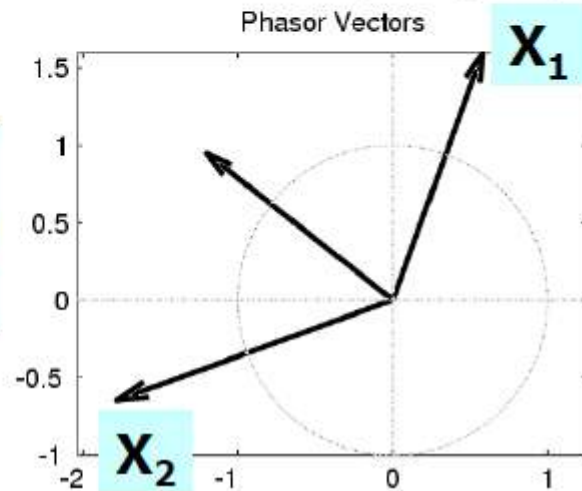
$$x_1(t) = 1.7 \cos(2\pi(10)t + 70\pi/180)$$

$$x_2(t) = 1.9 \cos(2\pi(10)t + 200\pi/180)$$

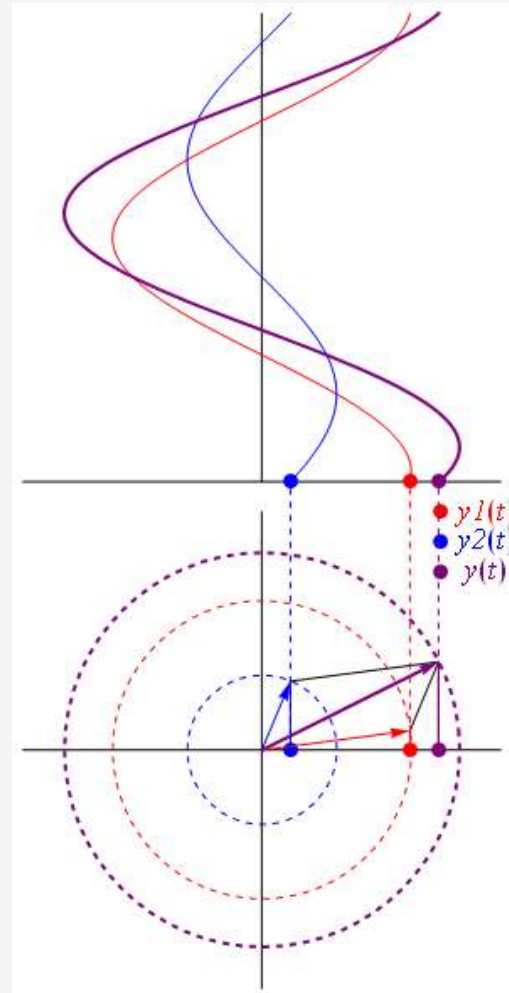
$$x_3(t) = x_1(t) + x_2(t)$$

$$= 1.532 \cos(2\pi(10)t + 141.79\pi/180)$$

VECTOR
(PHASOR)
ADD



Sum of Phasors and Fourier Series



Plotting A Complex Exponential in MATLAB



```
% Plot signal
tt = 0: 1/10000 : 3.2;
xx = 2.1*exp(2*pi*10*tt*1j);
xx2 = 0.5*exp(2*pi*10*tt*1j);

figure(1); plot (tt,real(xx)); xlim([0 0.01]);
figure(2); plot (tt,imag(xx)); xlim([0 0.01]);

% Simulate Phasor
close all;
figure(1);

for i = 1:length(tt)

    x = real(xx(i));    y = imag(xx(i));

    plot([0 x],[0 y]);
    xlim([-4 4]);      ylim([-4 4]);    drawnow;

end
```

```
% Simulate sum of Phasor-2
close all;
figure(1);

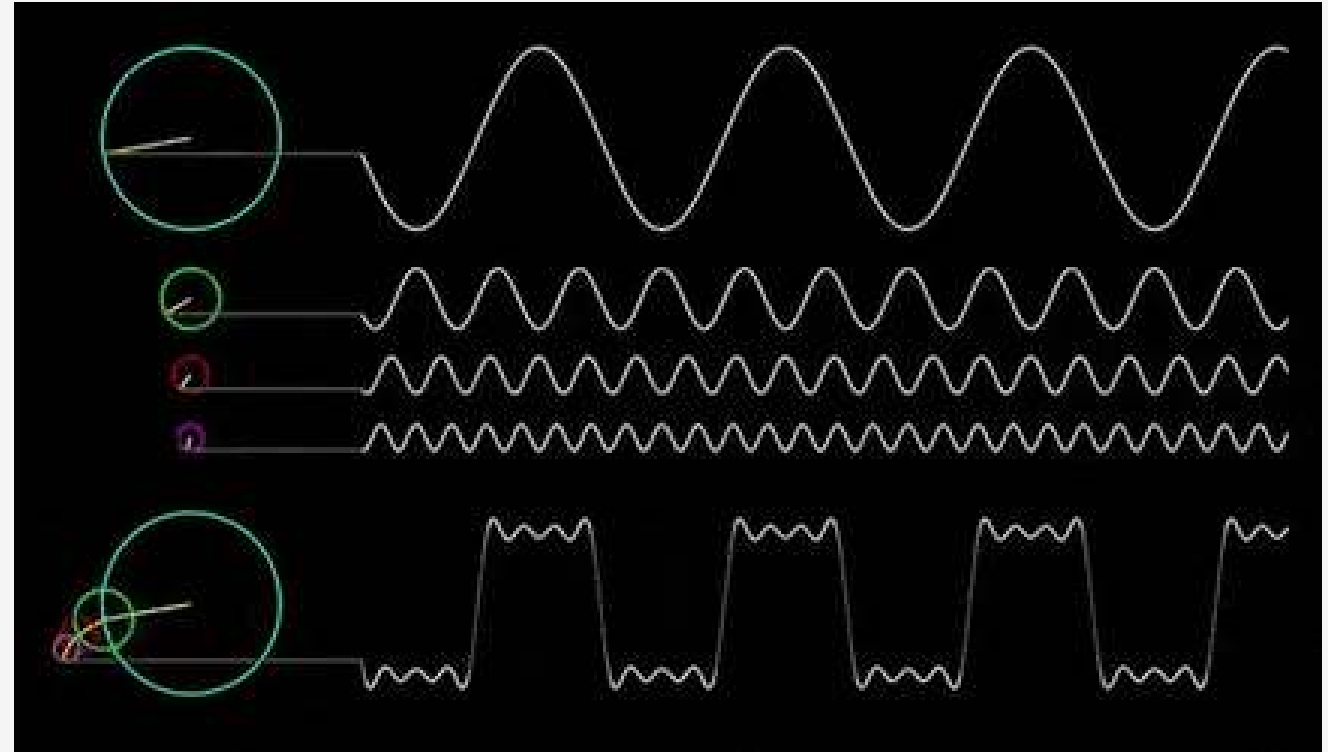
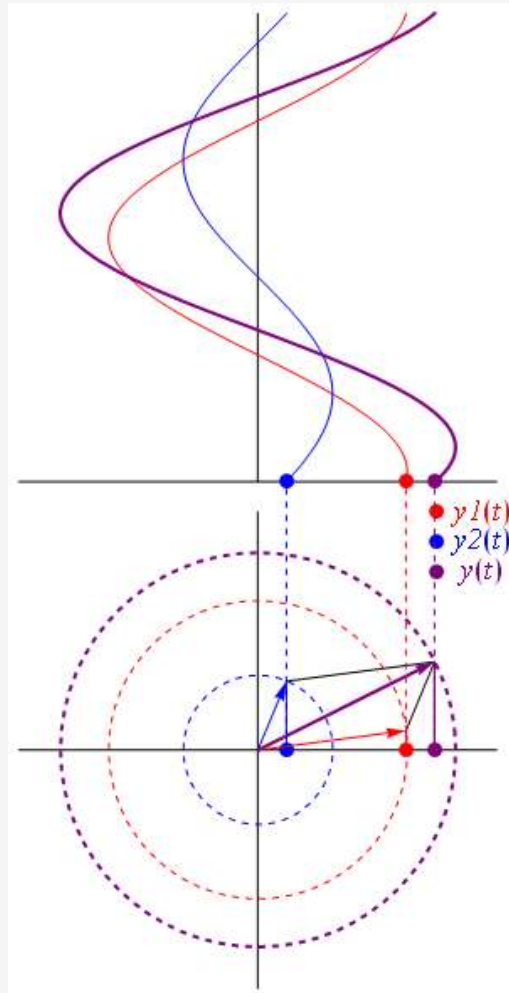
for i = 1:length(tt)

    x = real(xx(i));
    y = imag(xx(i));
    x2 = real(xx2(i));
    y2 = imag(xx2(i));

    plot([0 x],[0 y], 'r'); hold on;
    plot([x x+x2],[y y+y2], 'b');
    plot([0 x+x2],[0 y+y2], 'k');
    xlim([-4 4]);      ylim([-4 4]);
    drawnow; hold off;

end
```

Sum of Phasors and Fourier Series

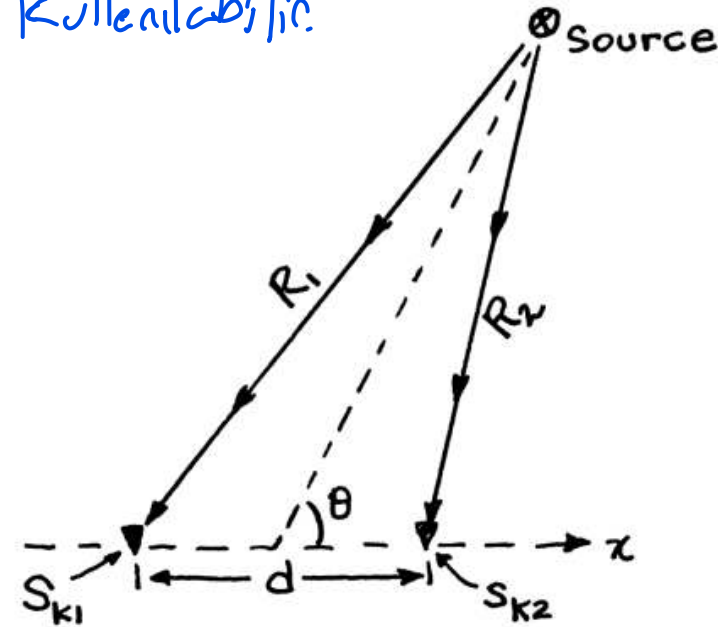
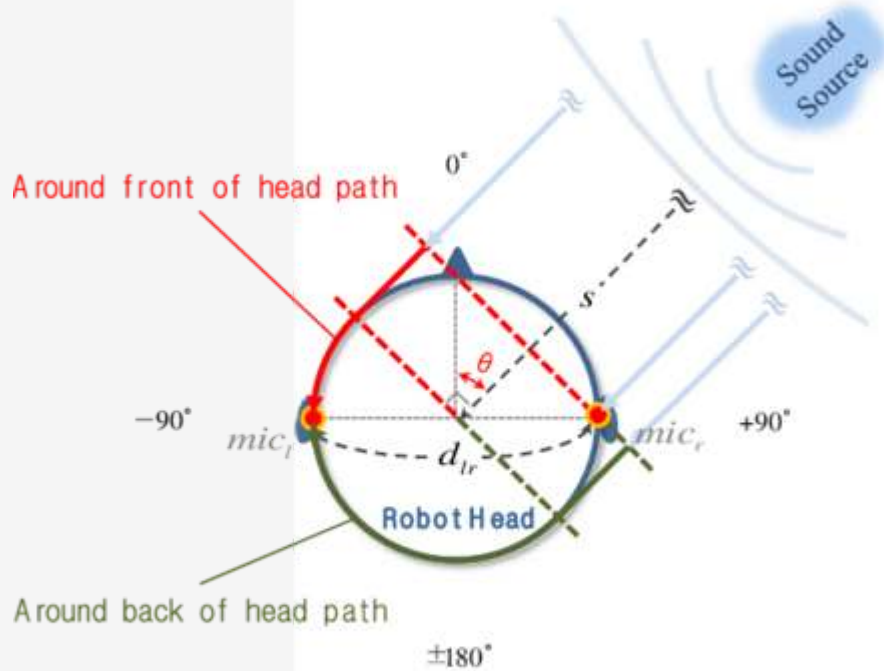


$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Demo Link: <https://dspfirst.gatech.edu/chapters/02sines/demos/phasors/index.html>

Where Can We Use Phase Info: Binaural Sound Localization

Gelen ses dalgasının nereden ve ne kadar uzaktan olduğunu belirlemek için faz farkı kullanılabilir.



$$\Delta \tau = \frac{d}{c} \cos \theta$$

$$\Delta \tau = \tau_{k_1} - \tau_{k_2}$$

$$\text{Sensor } S_{k_1}: r_{k_1}(t) = s(t - \tau_{k_1})$$

$$\text{Sensor } S_{k_2}: r_{k_2}(t) = s(t - \tau_{k_2})$$

Exercise - 1

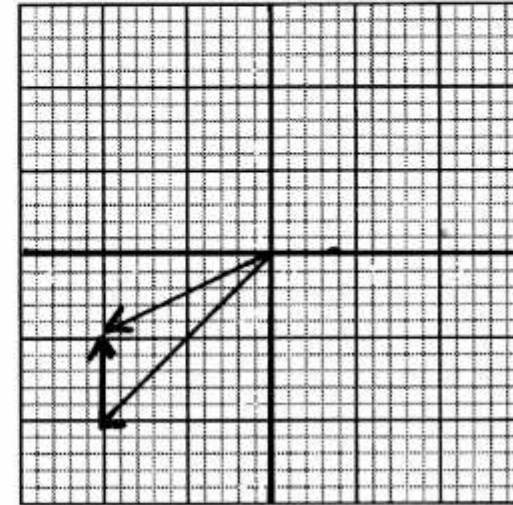
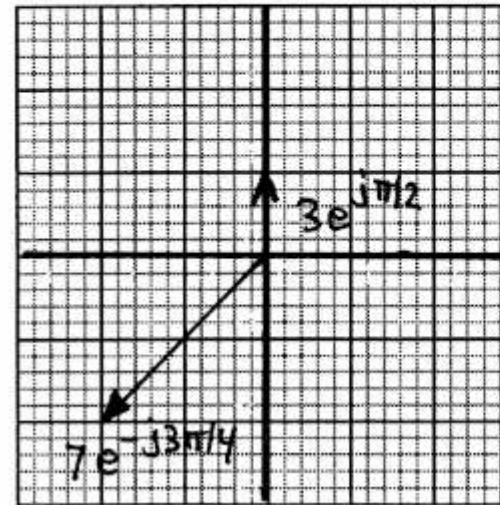
Define $x(t)$ as

$$x(t) = 7 \cos(100\pi t - 3\pi/4) + 3 \cos(100\pi(t + 0.005))$$

- (a) Use phasor addition to express $x(t)$ in the form $x(t) = A \cos(\omega_0 t + \phi)$ by finding the numerical values of A and ϕ , as well as ω_0 .

$$\begin{aligned} x(t) &= 7 \cos(100\pi t - 3\pi/4) + 3 \cos(100\pi t + \pi/2) \\ &= \operatorname{Re} \left\{ 7e^{-j3\pi/4} e^{j100\pi t} + 3e^{j\pi/2} e^{j100\pi t} \right\} \\ &= \operatorname{Re} \left\{ \underbrace{\left(7e^{-j3\pi/4} + 3e^{j\pi/2} \right)}_{5.3199 e^{-j0.8806\pi}} e^{j100\pi t} \right\} \\ &= \operatorname{Re} \left\{ 5.3199 e^{-j0.8806\pi} \cdot e^{j100\pi t} \right\} \\ &= 5.3199 \cos(100\pi t - 0.8806\pi) \end{aligned}$$

- (b) Make two complex plane plots to illustrate how complex amplitudes (phasors) were used to solve part (a). On the first plot, show the two complex amplitudes being added; on the second plot, show your solution as a vector and the addition of the two complex amplitudes as vectors (head-to-tail).



Exercise - 2

Bir karmaşık sayıyı genliği 1 olan başka bir karmaşık sayı ile çarparsak o vektörü sadece döndürürüz

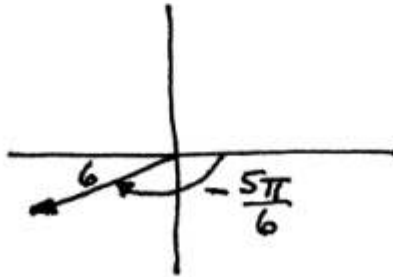


Simplify the following complex-valued expressions. In each case reduce the answers to a **simple** numerical form. Let

$$V = -3 + j3\sqrt{3}.$$

(a) Express jV in polar form. In addition plot jV as a vector.

$$\begin{aligned} jV &= -3j - 3\sqrt{3} \\ &= 6e^{-j\frac{5\pi}{6}} \end{aligned}$$



(d) Express $\Re\{j^3 V e^{j15t}\}$ in the standard "cosine" form.

$$\begin{aligned} \Re\{j^3 V e^{j15t}\} &= \Re\{e^{-j\frac{\pi}{2}} \cdot 6e^{j\frac{2\pi}{3}} e^{j15t}\} = \Re\{6e^{j\frac{\pi}{6}} e^{j15t}\} \\ &= \boxed{6 \cos(15t + \frac{\pi}{6})} \end{aligned}$$

Exercise - 3



The phase of a sinusoid can be related to time shift: $x(t) = A \cos(2\pi f_0 t + \phi) = A \cos(2\pi f_0 (t - t_1))$
In the following parts, assume that the period of the sinusoidal wave is $T = 20$ sec.

- (a) "When $t_1 = 5$ sec, the value of the phase is $\phi = 3\pi/2$."

Explain whether this is TRUE or FALSE.

$$\phi = -2\pi(t_1/T)$$

$$t_1 = 5 \Rightarrow \phi = -2\pi(5/20) = -\pi/2$$

BUT YOU CAN ADD 2π , SO $\phi = -\pi/2 + 2\pi = 3\pi/2$

TRUE

- (b) "When $t_1 = -5$ sec, the value of the phase is $\phi = \pi/4$."

Explain whether this is TRUE or FALSE.

$$\phi = -2\pi(-5/20) = +\pi/2$$

FALSE

$\pi/2 - \pi/4 = \pi/4$ IS NOT MULTIPLE of 2π

Sample Q



P-2.10 Define $x(t)$ as

$$x(t) = 2 \sin(\omega_0 t + \pi/4) + \cos(\omega_0 t)$$

- (a) Express $x(t)$ in the form $x(t) = A \cos(\omega_0 t + \phi)$.
- (b) Find a complex-valued signal $z(t)$ such that $x(t) = \Re\{z(t)\}$.

P-2.7 Simplify the following expressions:

- (a) $3e^{j\pi/3} + 4e^{-j\pi/6}$
- (b) $(\sqrt{3} - j3)^{10}$
- (c) $(\sqrt{3} - j3)^{-1}$
- (d) $(\sqrt{3} - j3)^{1/3}$
- (e) $\Re\{je^{-j\pi/3}\}$

Give the answers in *both* Cartesian form ($x + jy$) and polar form ($re^{j\theta}$).

P-2.11 Define $x(t)$ as

$$x(t) = 5 \cos(\omega t) + 5 \cos(\omega t + 120^\circ) + 5 \cos(\omega t - 120^\circ)$$

Simplify $x(t)$ into the standard sinusoidal form: $x(t) = A \cos(\omega t + \phi)$. Use phasors to do the algebra, but also provide a plot of the vectors representing each of the three phasors.