MAT1320-Linear Algebra
Lecture Notes

Eigenvalues and Eigenvectors, Characteristic Polynomial, Diagonalization, Cayley-Hamilton Theorem

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Eigenvalues and Eigenvectors

Definition
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Note: Note that an eigenvector cannot be $\overrightarrow{\mathbf{0}}$, but an eigenvalue can be $0 \in \mathbb{R}$. If 0 is an eigenvalue of A, then there must be some nontrivial vector $\overrightarrow{\mathbf{x}}$ for which $A\overrightarrow{\mathbf{x}} = 0\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{0}}$ which implies that A is not invertible.

$A.\vec{x} = \vec{0} \rightarrow A^{-1}A.\vec{x} = A^{-1}\vec{0}$

$\boxed{\vec{x} = \vec{0}}$

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$V_\lambda = \{ \vec{x} ; \ A \cdot \vec{x} = \lambda \cdot \vec{x} \}$

Note: The eigenspace of the $n \times n$ matrix A corresponding to the eigenvalue λ of A is the set of all eigenvectors of A corresponding to λ.

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Let A be an $n \times n$ square matrix. Then we have

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Let A be an $n \times n$ square matrix. Then the equation $\det(A - \lambda I_n) = 0$ is called the characteristic equation of the matrix A and the result of the determinant $\det(A - \lambda I_n)$ is polynomial of the form $P_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$ is called characteristic polynomial of A.

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Note: $P_A(\lambda) = (-1)^n\lambda^n + (-1)^{n-1}Tr(A)\lambda^{n-1} + \ldots + \underbrace{\det(A)}_{=0.}$.
For example, for a 2×2 square matrix A

$$P_A(\lambda) = \lambda^2 - Tr(A)\lambda + \det(A).$$

$\rightarrow \lambda^2 - \lambda + 2$ $\quad Tr(A) = -1$ $\quad \rightarrow |A| = -2$

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Note 1: The eigenvalues of a triangular matrix are the entries on its main diagonal.

$$\begin{pmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix} \longrightarrow \underline{\text{Eigenvalues}: 1, 2, 3}.$$

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Note 1: The eigenvalues of a triangular matrix are the entries on its main diagonal.

Note 2: If $P_A(\lambda)$ has multiple roots, then there exists multiple eigenvalues.

$P_A(\lambda) = (x-1)^2(x+1)$ eigenvalues : $1, -1,$

Example

Let $A = \begin{pmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{pmatrix}$. Find

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$$\underline{A-\lambda . I_3} = \begin{pmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{pmatrix} - \lambda . \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3-\lambda & 6 & -8 \\ 0 & -\lambda & 6 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

$$\Rightarrow P_A(\lambda) = \begin{vmatrix} 3-\lambda & 6 & -8 \\ 0 & -\lambda & 6 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (3-\lambda).(-1)^{1+1} . \begin{vmatrix} -\lambda & 6 \\ 0 & 2-\lambda \end{vmatrix}$$

$$= \boxed{(3-\lambda) . -\lambda . (2-\lambda) = P_A(\lambda)}$$

characteristic polynomial.

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$(A-\lambda I_n).\vec{x} = \vec{0}$

1. characteristic polynomial,
2. eigenvalues,
3. eigenvectors.

$$\Rightarrow P_A(\lambda) = (3-\lambda)(2-\lambda).-\lambda = 0$$

$$\Rightarrow \boxed{\lambda_1 = 0, \ \lambda_2 = 2, \ \lambda_3 = 3}$$

eigenvalues.

$$\underline{\lambda_1 = 0}: \quad (A - 0 . I_3).\vec{x} = \vec{0} \Rightarrow A . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1+6x_2-8x_3 \\ 6x_3 \\ 2x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_3 = 0 \Rightarrow 3x_1 + 6x_2 = 0 \Rightarrow x_1 = -2x_2$$

$\{(-2x_2, x_2, 0) : x_2 \in \mathbb{R}\}$ eigenvectors corresponding $\lambda_1 = 0$.

Solution (1)
$P_A(\lambda) = \det(A - \lambda I_n)$

$$A - \lambda I_3 = \begin{pmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\underline{\lambda_2 = 2}$: $(A - 2.I_3)\vec{X} = \vec{0} \Rightarrow \begin{pmatrix} 1 & 6 & -8 \\ 0 & -2 & 6 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 + 6x_2 - 8x_3 \\ -2x_2 + 6x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$-2x_2 + 6x_3 = 0 \Rightarrow x_2 = 3x_3 \Rightarrow x_1 + 6.3.x_3 - 8.x_3 = x_1 + 10x_3 = 0$
$\Rightarrow x_1 = -10x_3$

$(-10x_3, 3x_3, x_3) = x_3.(-10, 3, 1)$

$\{ t.(-10, 3, 1) : t \in \mathbb{R} \}$ eigenvectors corresponding to $\lambda = 2$.

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$\lambda_3 = 3$:

$$A - \lambda I_3 = \begin{pmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3-\lambda & 6 & -8 \\ 0 & -\lambda & 6 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

$(A - 3.I_3).\vec{X} = \vec{0}$

$\begin{pmatrix} 0 & 6 & -8 \\ 0 & -3 & 6 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow -x_3 = 0$
$\to -3x_2 + 6x_3 = -3x_2 = 0 \to x_2 = 0$

$(x_1, 0, 0) = x_1.(1, 0, 0)$

$\{ t.(1,0,0) : t \in \mathbb{R} \}$ eigenvectors corresponding $\lambda_3 = 3$.

Solution (1)
$P_A(\lambda) = \det(A - \lambda I_n)$

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&= \begin{pmatrix} 3-\lambda & 6 & -8 \\ 0 & -\lambda & 6 \\ 0 & 0 & 2-\lambda \end{pmatrix} \\
&\Rightarrow \quad P_A(\lambda) = -\lambda(3-\lambda)(2-\lambda).
\end{aligned}$$

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Since, eigenvalues are the roots of $P_A(\lambda)$, *we have*

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$$\begin{aligned} P_A(\lambda) &= -\lambda(3-\lambda)(2-\lambda) = 0 \\ &\Rightarrow \lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 3. \end{aligned}$$

Solution (3)
For $\lambda_1 = 0$,

$$(A - 0I_3) \quad = \quad A = \begin{pmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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&\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, t \in \mathbb{R} \Rightarrow \overrightarrow{\mathbf{v}}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.
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For $\lambda_2 = 2$,

$$(A - 2I_3) = \begin{pmatrix} 1 & 6 & -8 \\ 0 & -2 & 6 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

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For $\lambda_3 = 3$,

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$$\overrightarrow{\mathbf{x}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow (A - 3I_n)\,\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{0}}$$

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Cayley-Hamilton Theorem

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Theorem (Cayley-Hamilton Theorem) $P_A(\lambda) = |A - \lambda I_n|$.
Every matrix A is a root of its characteristic polynomial, i.e.,
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$a_0 = |A|$

$$\begin{aligned} P_A(A) &= A^n + a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 I_n = 0 \\ &\Rightarrow A^n = -\left(a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 I_n\right) \end{aligned}$$

$A^n + a_{n-1} \cdot A^{n-1} + \dots + a_1 A + a_0 \cdot I_n = 0$, $a_0 \neq 0 \left(|A| \neq 0\right)$

$$\frac{a_0 \cdot I_n}{a_0} = \frac{-\left(A^n + a_{n-1} \cdot A^{n-1} + \dots + a_1 A\right)}{a_0} = A \cdot \frac{-\left(A^{n-1} + a_{n-1} A^{n-2} + \dots + a_1 I_n\right)}{a_0}$$

$$I_n = A \cdot \boxed{\frac{-1}{a_0} \cdot \left(A^{n-1} + a_{n-1} A^{n-2} + \dots + a_1 I_n\right)} \to \underline{A^{-1}}$$

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If $a_0 \neq 0$, then

$$I_n = A \underbrace{\frac{-1}{a_0}\left(A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_1 I_n\right)}_{A^{-1}}$$

$A-\lambda.I_2=\begin{pmatrix}1&4\\2&3\end{pmatrix}-\lambda\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1-\lambda&4\\2&3-\lambda\end{pmatrix}\Rightarrow P_A(\lambda)=\begin{vmatrix}1-\lambda&4\\2&3-\lambda\end{vmatrix}$

$=(1-\lambda)(3-\lambda)-8$

$=3+\lambda^2-4\lambda-8=\lambda^2-4\lambda-5$

$\Rightarrow \boxed{P_A(\lambda)=\lambda^2-4\lambda-5}$ characteristic polynomial.

Example

Let $A=\begin{pmatrix}1&4\\2&3\end{pmatrix}_{2\times 2}$. Find

$(A-\lambda I_n).\vec{x}=\vec{0}$

1. characteristic polynomial,

$\Rightarrow P_A(\lambda)=\lambda^2-4\lambda-5=(\lambda-5)(\lambda+1)=0$

$\Rightarrow \lambda_1=5,\ \lambda_2=-1$ (eigenvalues)

$\underline{\lambda_1=5}$: $(A-5.I_2).\vec{x}=\vec{0}$

$\begin{pmatrix}-4&4\\2&-2\end{pmatrix}.\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

$\Rightarrow -4x_1+4x_2=0 \rightarrow x_1=x_2$

$(x_1,x_1)=x_1.(1,1)$

$\{t.(1,1):t\in\mathbb{R}\}$ eigenvectors for $\lambda=5$

$\underline{\lambda_2=-1}$: $(A+1.I_2).\vec{x}=\vec{0}$

$\begin{pmatrix}2&4\\2&4\end{pmatrix}.\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

$\Rightarrow 2x_1+4x_2=0 \rightarrow x_1=-2x_2$

$(-2x_2,x_2)=x_2.(-2,1)$

$\{t.(-2,1):t\in\mathbb{R}\}$ eigenvectors for $\lambda=-1$.

Example

Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Find

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1. characteristic polynomial,
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4. matrices P and D such that $A = PDP^{-1}$, if any,

5) $P_A(\lambda) = \lambda^2 - 4\lambda - 5 \Rightarrow P_A(A) = A^2 - 4A - 5I_2 = O_{2\times 2}$

$A^2 - 4A = 5.I_2 \Rightarrow A.(A - 4.I_2) = 5.I_2$

$\Rightarrow A.\underbrace{\frac{(A-4.I_2)}{5}}_{A^{-1}} = I_2 \Rightarrow A^{-1} = \frac{1}{5}.(A - 4I_2)$

$\boxed{A^{-1} = \frac{1}{5}.\begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}}$.

Example

Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ $-4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Find

1. characteristic polynomial,
2. eigenvalues,
3. eigenvectors,
4. ~~matrices P and D such that $A = PDP^{-1}$, if any,~~
5. $\cancel{A^{-1}}$ and A^5 (by using Cayley-Hamilton Theorem).

$A^2 - 4A - 5I_2 = 0 \Rightarrow A^2 = 4A + 5.I_2 \Rightarrow A^4 = A^2A^2 = (4A + 5I_2)(4A + 5I_2)$

$= (16A^2 + 40A + 25I_2) = 16(4A + 5I_2) + 40A + 25I_2 = 104A + 105I_2$.

$\Rightarrow A^5 = AA^4 = A(104A + 105I_2) = 104A^2 + 105A = 104(4A + 5I_2) + 105A$

$A^5 = \underline{521A + 520I_2}$

Solution (1)
$P_A(\lambda) = \det(A - \lambda I_n)$

$$A - \lambda I_2 = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Solution (1)
$P_A(\lambda) = \det(A - \lambda I_n)$

$$\begin{aligned} A - \lambda I_2 &= \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix} \end{aligned}$$

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Solution (2)
The eigenvalues are the roots of $P_A(\lambda)$ such that

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Solution (2)
The eigenvalues are the roots of $P_A(\lambda)$ *such that*

$$\begin{aligned} P_A(\lambda) &= \lambda^2 - 4\lambda - 5 = 0 \\ &\Rightarrow \lambda_1 = -1, \lambda_2 = 5. \end{aligned}$$

Solution (3)
For $\lambda_1 = -1$,

$$(A + I_2) \quad = \quad \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

Finding Eigenvalues and Eigenvectors

Solution (3)
For $\lambda_1 = -1$,

$$
\begin{aligned}
(A + I_2) &= \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \\
\overrightarrow{\mathbf{x}} &= \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow (A + I_2)\,\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{0}}
\end{aligned}
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&\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \end{pmatrix}, t \in \mathbb{R} \Rightarrow \overrightarrow{\mathbf{v}}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.
\end{aligned}$$

Solution (3)
For $\lambda_2 = 5$,

$$(A - 5I_2) = \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

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&\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, t \in \mathbb{R} \Rightarrow \overrightarrow{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\end{aligned}$$

Solution (4)

$P = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ *and* $D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \Rightarrow$

$A = PDP^{-1}$.

Solution (5)

$$P_A(A) = A^2 - 4A - 5I_2 = 0$$

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$$\begin{aligned} P_A(A) &= A^2 - 4A - 5I_2 = 0 \\ &\Rightarrow \underbrace{A\frac{1}{5}(A - 4I_2)}_{A^{-1}} = I_2 \\ A^{-1} &= \frac{1}{5}\left(\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\right) = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix} \end{aligned}$$

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$$A^2 = 4A + 5I_2$$

Solution (5)

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\end{aligned}
$$

$$
\begin{aligned}
A^2 &= 4A + 5I_2 \\
A^4 &= (4A + 5I_2)(4A + 5I_2)
\end{aligned}
$$

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$$\begin{aligned} A^2 &= 4A + 5I_2 \\ A^4 &= (4A + 5I_2)(4A + 5I_2) \\ A^5 &= A(4A + 5I_2)(4A + 5I_2) \end{aligned}$$

?