CENG 222
Statistical Methods for Computer Engineering

Week 10

Chapter 9

9.4 Hypothesis Testing

Testing Hypotheses

- Hypothesis H_0 and the alternative H_A are two mutually exclusive statements about some unknown parameter θ.
- Testing steps:
  - Collect data
  - Compute a test statistic
  - State if there is sufficient evidence to reject H_0 in favor of H_A
- Examples: 9.22, 9.23, and 9.24

Type I and Type II errors and level of significance

	Result of the test	
	Reject H_0	Accept H_0
H_0 is true	*Type I error*	correct
H_0 is false	correct	*Type II error*

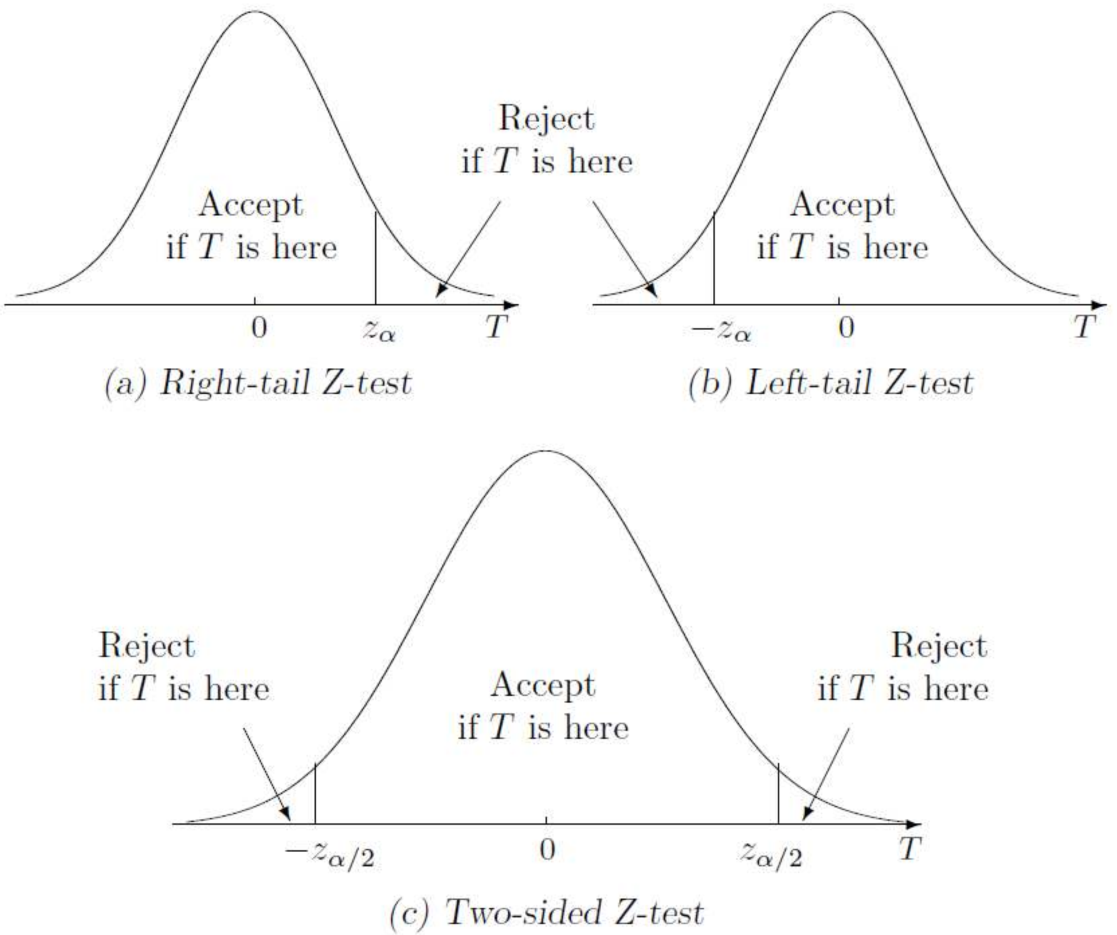
- We generally want to limit Type I errors (also called false positives) while minimizing Type II errors (also called false negatives)
- Significance level is the probability to observe Type I errors.

Type I and Type II errors and level of significance

- We generally want to limit Type I errors (also called false positives) while minimizing Type II errors (also called false negatives)
- $\alpha = \boldsymbol{P}(\text{reject } H_0 | H_0 \text{ is true})$
- Power of the test is the probability to avoid Type II error
  - Power = sensitivity = recall = True Positive Rate
  - $p(\theta) = \boldsymbol{P}(\text{reject } H_0 | \theta; H_A \text{ is true})$
- See:
  - https://en.wikipedia.org/wiki/Sensitivity_and_specificity

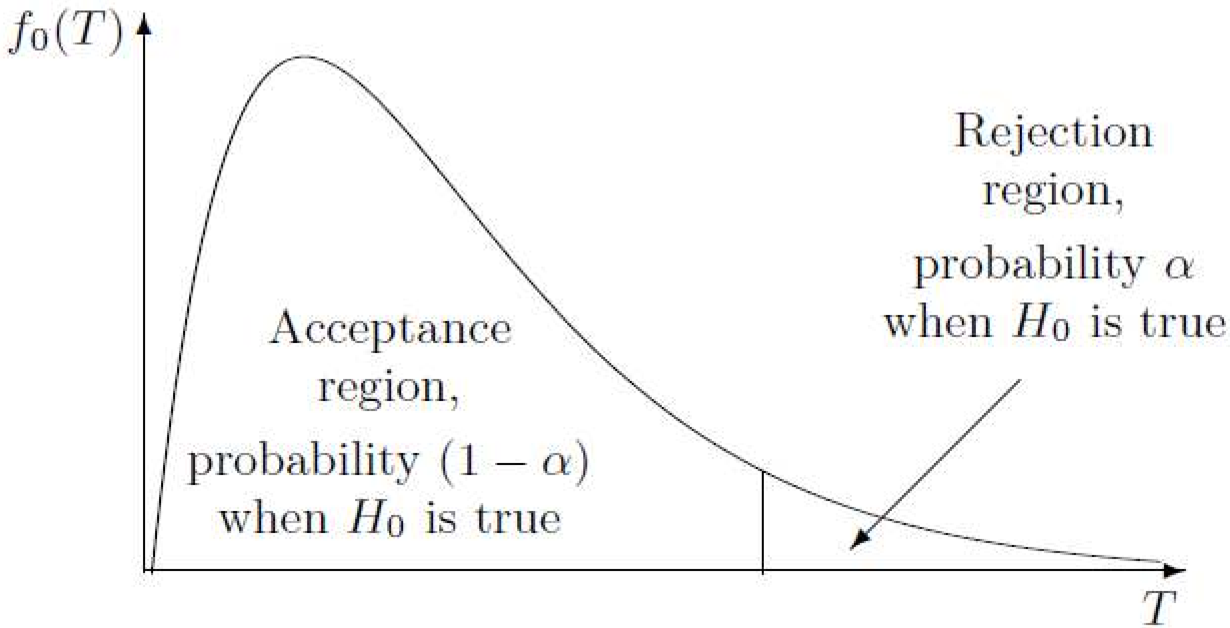
Types of alternatives

- Two sided, one-sided left tail, and one-sided right tail alternatives



Null distribution and acceptance/rejection regions

- The distribution of the test statistic T is called the null distribution.



Z-test

- If the null distribution of the test statistic is Standard Normal, the tests are called Z-tests.
- Z-tests are used when we know population variance.
- T-tests are used for unknown population variance
- Tests can be performed for one sample, two samples (e.g., when comparing two populations)
- Common hypotheses are about population means, proportions, and differences.

Two-tail Z-test

- Data: $X_1, \dots, X_n$ from Normal(μ, σ) with unknown μ and known σ
- Test $H_0 : \mu = \mu_0$ versus $H_A : \mu \neq \mu_0$
  1. Find $\pm z_{\alpha/2}$. Acceptance region is $[-z_{\alpha/2}, z_{\alpha/2}]$
  2. Compute the test statistic
  $$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$
  3. If Z belongs to the acceptance region do not reject H_0
  Otherwise, reject H_0

One-sided right tail Z-test

- Data: $X_1, \ldots, X_n$ from Normal(μ, σ) with unknown μ and known σ
- Test $H_0 : \mu = \mu_0$ versus $H_A : \mu > \mu_0$
  1. Find z_α. Acceptance region is $(-\infty, z_\alpha]$
  2. Compute the test statistic

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

  3. If Z belongs to the acceptance region do not reject H_0
     Otherwise, reject H_0

One-sided left tail Z-test

- Data: $X_1, \dots, X_n$ from Normal(μ, σ) with unknown μ and known σ
- Test $H_0 : \mu = \mu_0$ versus $H_A : \mu < \mu_0$
  1. Find z_α. Acceptance region is $[-z_\alpha, +\infty)$
  2. Compute the test statistic
  $$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$
  3. If Z belongs to the acceptance region do not reject H_0
  Otherwise, reject H_0

Unknown variance: T-test

- We use the estimator for variance

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

- and use the t-distribution with $n - 1$ degrees of freedom

Z-test summary

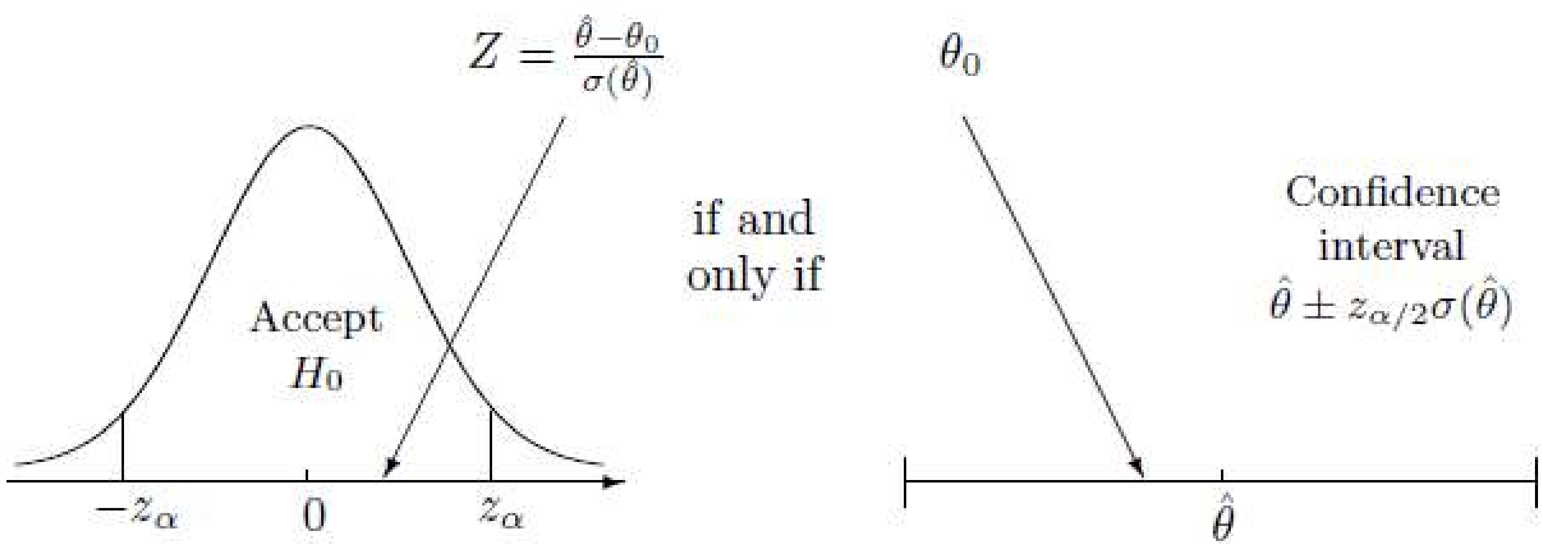
Null hypothesis H_0	Parameter, estimator $\theta, \hat{\theta}$	If H_0 is true: $\mathbf{E}(\hat{\theta})$	If H_0 is true: $\mathrm{Var}(\hat{\theta})$	Test statistic $Z = \dfrac{\hat{\theta} - \theta_0}{\sqrt{\mathrm{Var}(\hat{\theta})}}$
One-sample Z-tests for means and proportions, based on a sample of size n				
$\mu = \mu_0$	$\mu, \bar{X}$	μ_0	$\dfrac{\sigma^2}{n}$	$\dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$
$p = p_0$	$p, \hat{p}$	p_0	$\dfrac{p_0(1-p_0)}{n}$	$\dfrac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$
Two-sample Z-tests comparing means and proportions of two populations, based on independent samples of size n and m				
$\mu_X - \mu_Y = D$	$\mu_X - \mu_Y$, $\bar{X} - \bar{Y}$	D	$\dfrac{\sigma_X^2}{n} + \dfrac{\sigma_Y^2}{m}$	$\dfrac{\bar{X} - \bar{Y} - D}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}$
$p_1 - p_2 = D$	$p_1 - p_2$, $\hat{p}_1 - \hat{p}_2$	D	$\dfrac{p_1(1-p_1)}{n} + \dfrac{p_2(1-p_2)}{m}$	$\dfrac{\hat{p}_1 - \hat{p}_2 - D}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n} + \frac{\hat{p}_2(1-\hat{p}_2)}{m}}}$
$p_1 = p_2$	$p_1 - p_2$, $\hat{p}_1 - \hat{p}_2$	0	$p(1-p)\left(\dfrac{1}{n} + \dfrac{1}{m}\right)$, where $p = p_1 = p_2$	$\dfrac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n} + \frac{1}{m}\right)}}$ where $\hat{p} = \dfrac{n\hat{p}_1 + m\hat{p}_2}{n+m}$

T-test summary

Hypothesis H_0	Conditions	Test statistic t	Degrees of freedom
$\mu = \mu_0$	Sample size n; unknown σ	$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$	$n - 1$
$\mu_X - \mu_Y = D$	Sample sizes n, m; unknown but equal standard deviations, $\sigma_X = \sigma_Y$	$t = \frac{\bar{X} - \bar{Y} - D}{s_p\sqrt{\frac{1}{n} + \frac{1}{m}}}$	$n + m - 2$
$\mu_X - \mu_Y = D$	Sample sizes n, m; unknown, unequal standard deviations, $\sigma_X \neq \sigma_Y$	$t = \frac{\bar{X} - \bar{Y} - D}{\sqrt{\frac{s_X^2}{n} + \frac{s_Y^2}{m}}}$	Satterthwaite approximation, formula (9.12)

Confidence intervals versus two-sided level-α tests

- A level-α two sided test is the same as testing whether a given test statistic is in the $(1-\alpha)100\%$ confidence interval $[a,b]$



- Examples: 9.31, 9.35 (one-sided test)

P-value

- Instead of a fixed significance level α, we can compute the boundary level for acceptance/rejection for the computed test statistic
  - This computed value is called the p-value
  - It is the probability that another sample results in a more extreme test static.