

# BLM2041 Signals and Systems

## Syllabus

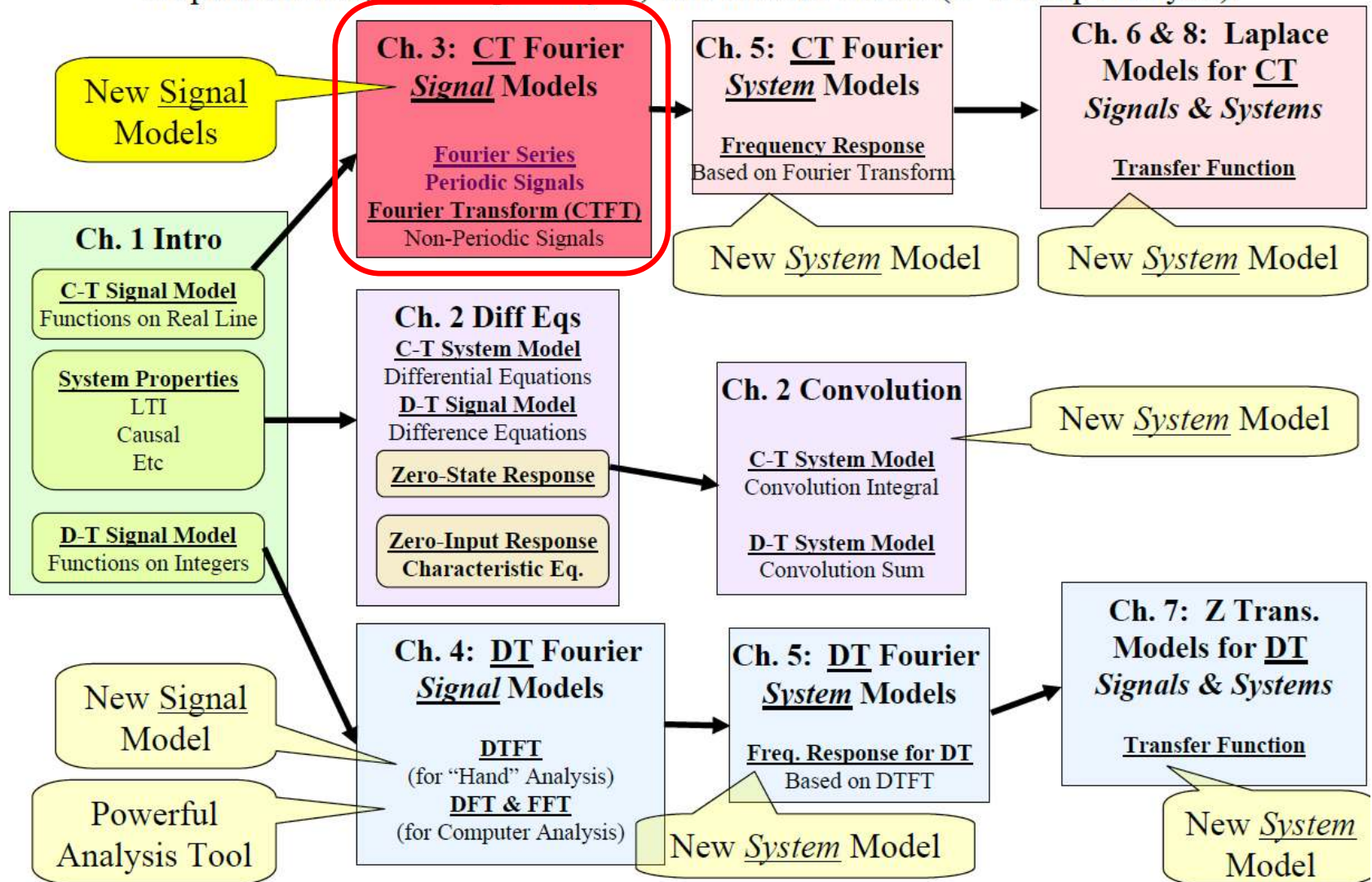
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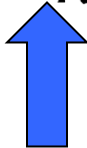
# Where are we now?

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



# LECTURE OBJECTIVES

- Sinusoids with **DIFFERENT** Frequencies
  - **SYNTHESIZE** by Adding Sinusoids

$$x(t) = \sum_{k=1}^N A_k \cos(2\pi f_k t + \varphi_k)$$


- **SPECTRUM** Representation
  - Graphical **Form** shows **DIFFERENT** Freqs

# LECTURE OBJECTIVES

- Signals with HARMONIC Frequencies
  - Add Sinusoids with  $f_k = kf_0$

$$x(t) = A_0 + \sum_{k=1}^N A_k \cos(2\pi k f_0 t + \varphi_k)$$

FREQUENCY can change **vs. TIME**

Chirps:

$$x(t) = \cos(\alpha t^2)$$

Introduce Spectrogram Visualization (`specgram.m`)  
(`plotspec.m`)

# LECTURE OBJECTIVES

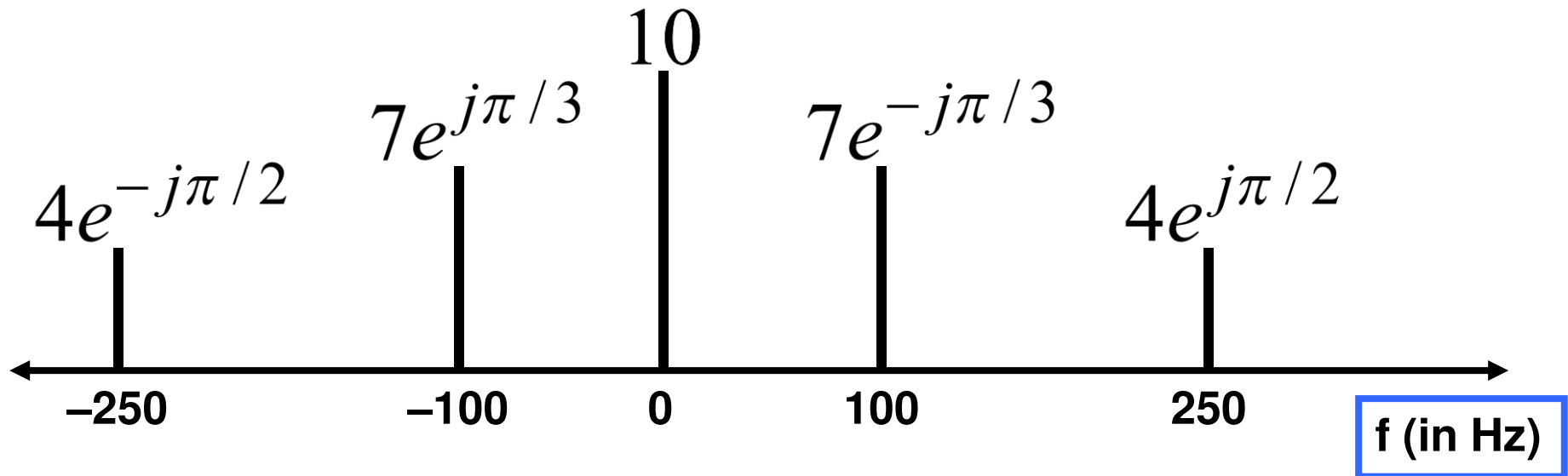
- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi k / T_0)t} dt$$

- ANALYSIS via Fourier Series
  - For PERIODIC signals:  $x(t+T_0) = x(t)$
- SPECTRUM from Fourier Series
  - $a_k$  is Complex Amplitude for  $k$ -th Harmonic

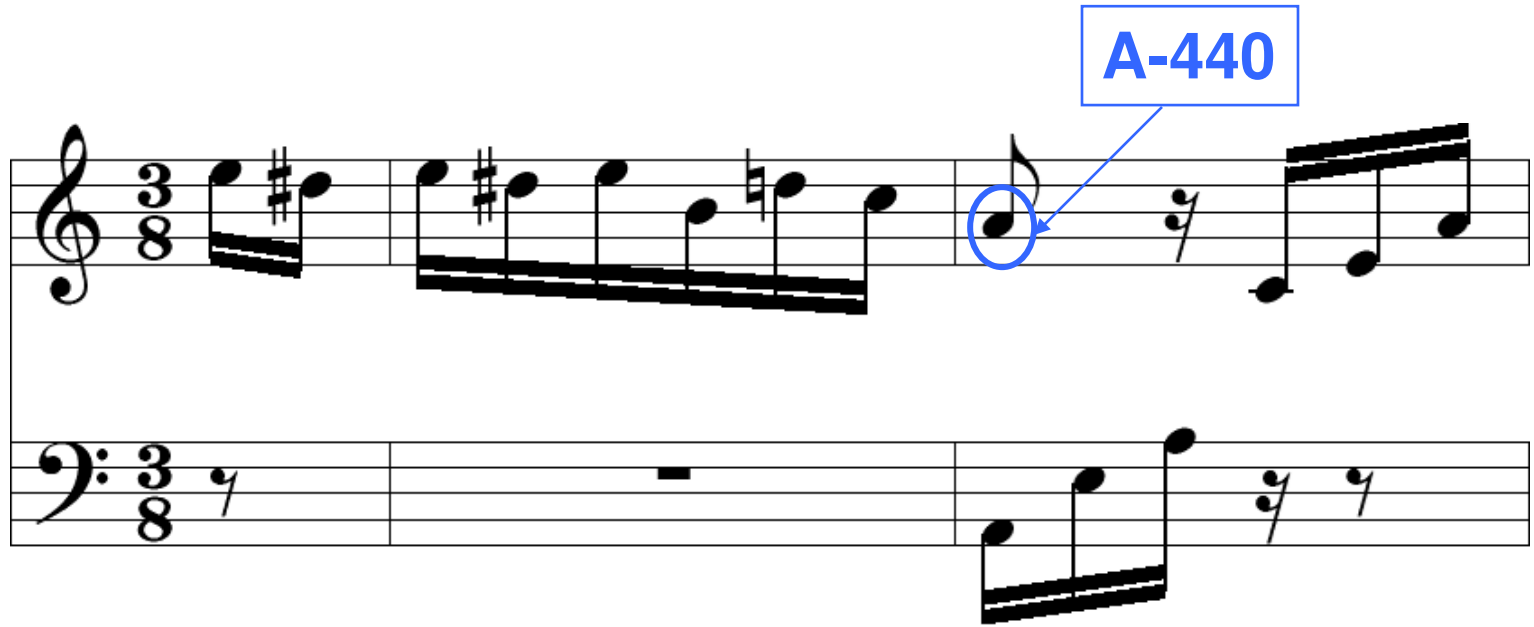
# FREQUENCY DIAGRAM

- Plot Complex Amplitude vs. Freq



# Another FREQ. Diagram

*Frequency is the vertical axis*



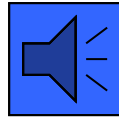
**Figure 3.18** Sheet-music notation is a time–frequency diagram.

**Time is the horizontal axis**

# MOTIVATION

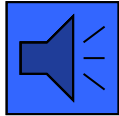
- Synthesize **Complicated** Signals

- Musical Notes



- Piano uses 3 strings for many notes
    - Chords: play several notes simultaneously

- Human Speech



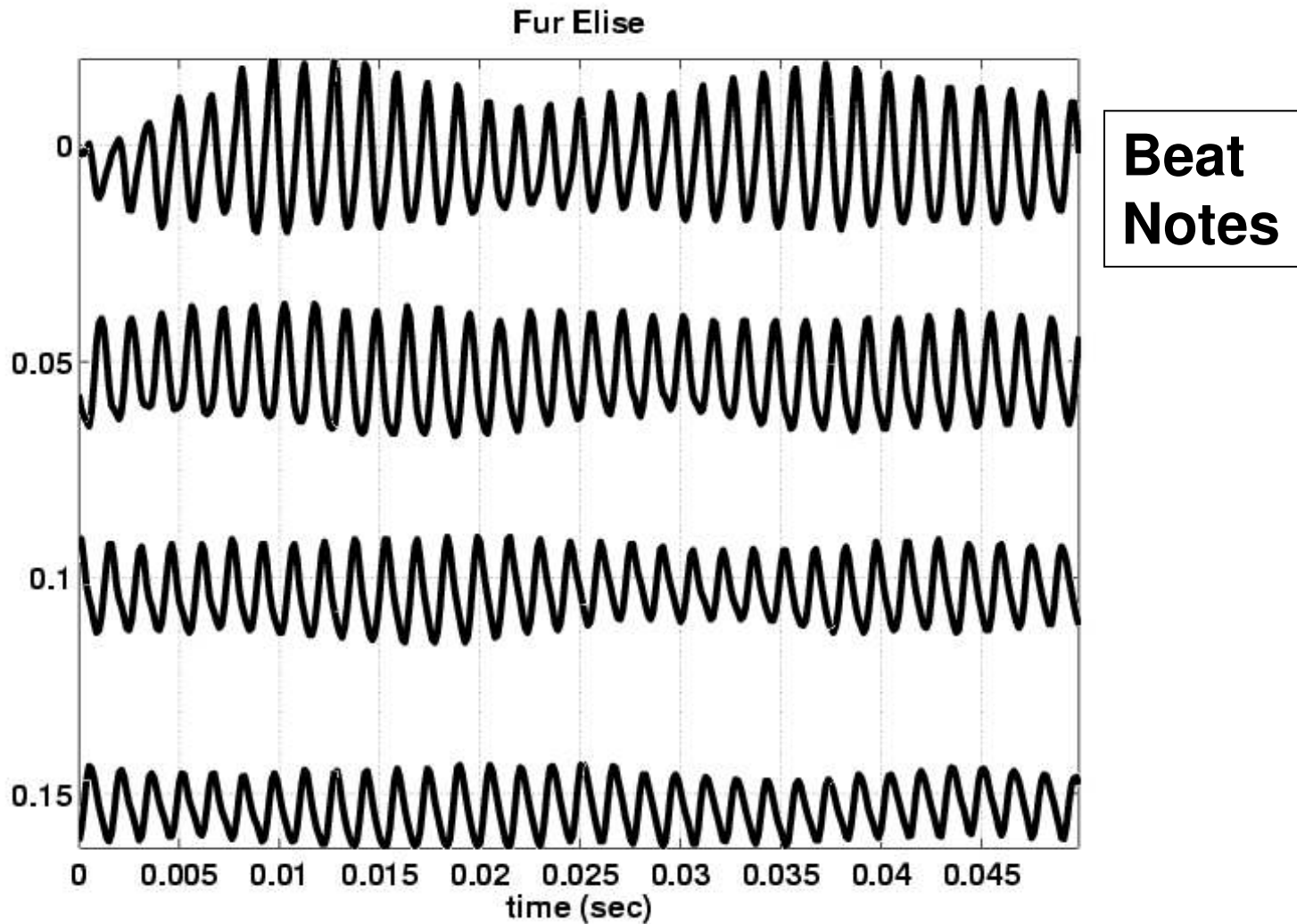
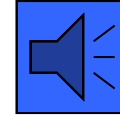
- Vowels have dominant frequencies
    - Application: computer generated speech

- Can **all** signals be generated this way?

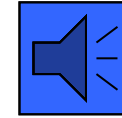
- Sum of sinusoids?



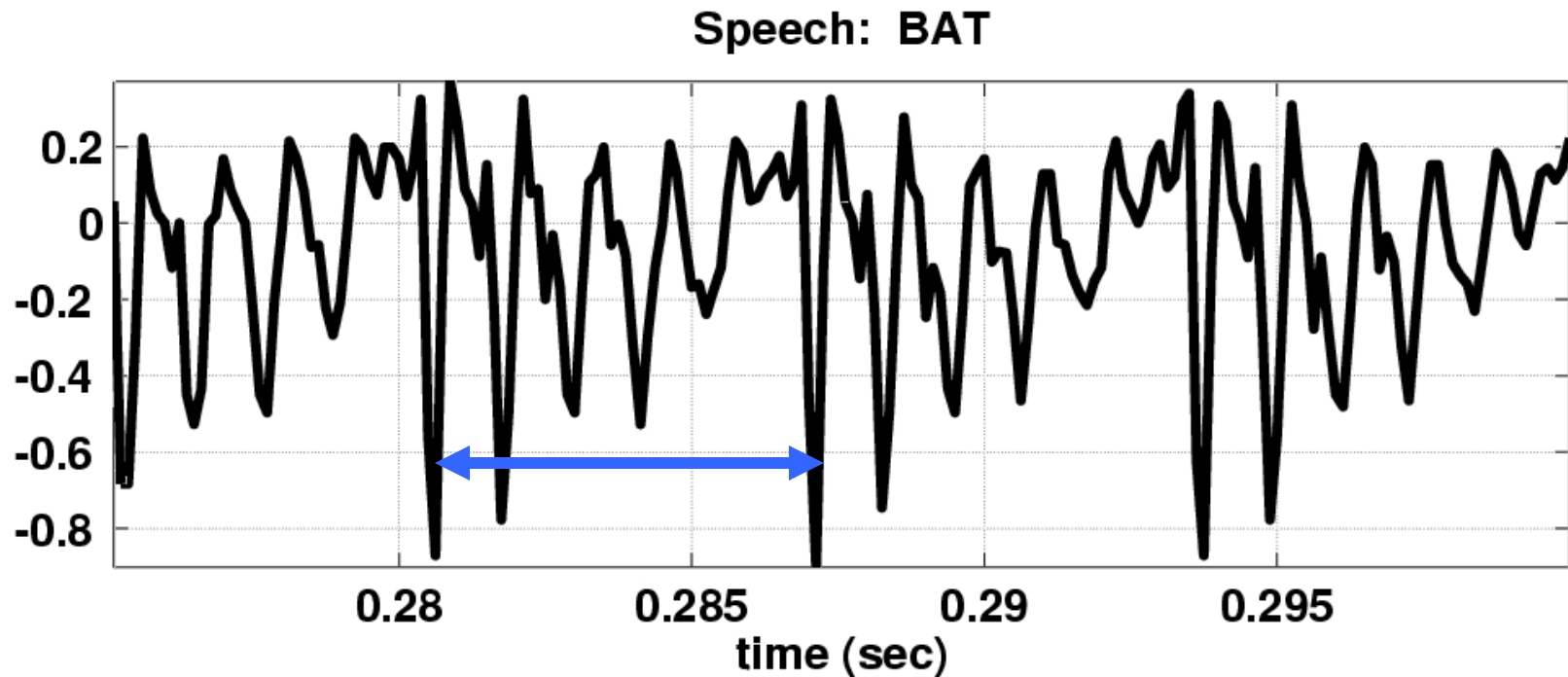
# Fur Elise WAVEFORM



# Speech Signal: BAT



- Nearly **Periodic** in Vowel Region
  - Period is (Approximately)  $T = 0.0065$  sec



# Euler's Formula Reversed

- Solve for **cosine** (or sine)

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j \sin(-\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j \sin(\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2 \cos(\omega t)$$

$$\cos(\omega t) = \frac{1}{2} (e^{j\omega t} + e^{-j\omega t})$$

# INVERSE Euler's Formula

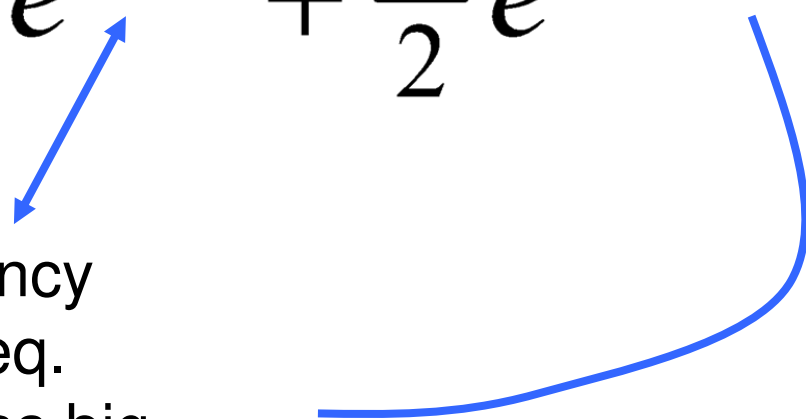
- Solve for **cosine** (or sine)

$$\cos(\omega t) = \frac{1}{2} (e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t})$$

# SPECTRUM Interpretation

- Cosine = sum of 2 complex exponentials:

$$A \cos(7t) = \frac{A}{2} e^{j7t} + \frac{A}{2} e^{-j7t}$$


One has a positive frequency  
The other has **negative** freq.  
Amplitude of each is half as big

# NEGATIVE FREQUENCY

- Is negative frequency real?
- Doppler Radar provides an example
  - Police radar measures speed by using the Doppler shift principle
  - Let's assume 400Hz  $\square$   $\square$  60 mph
  - +400Hz means towards the radar
  - -400Hz means away (opposite direction)
  - Think of a train whistle

# SPECTRUM of SINE

- Sine = sum of 2 complex exponentials:

$$A \sin(7t) = \frac{A}{2j} e^{j7t} - \frac{A}{2j} e^{-j7t}$$
$$= \frac{1}{2} A e^{-j0.5\pi} e^{j7t} + \frac{1}{2} A e^{j0.5\pi} e^{-j7t}$$

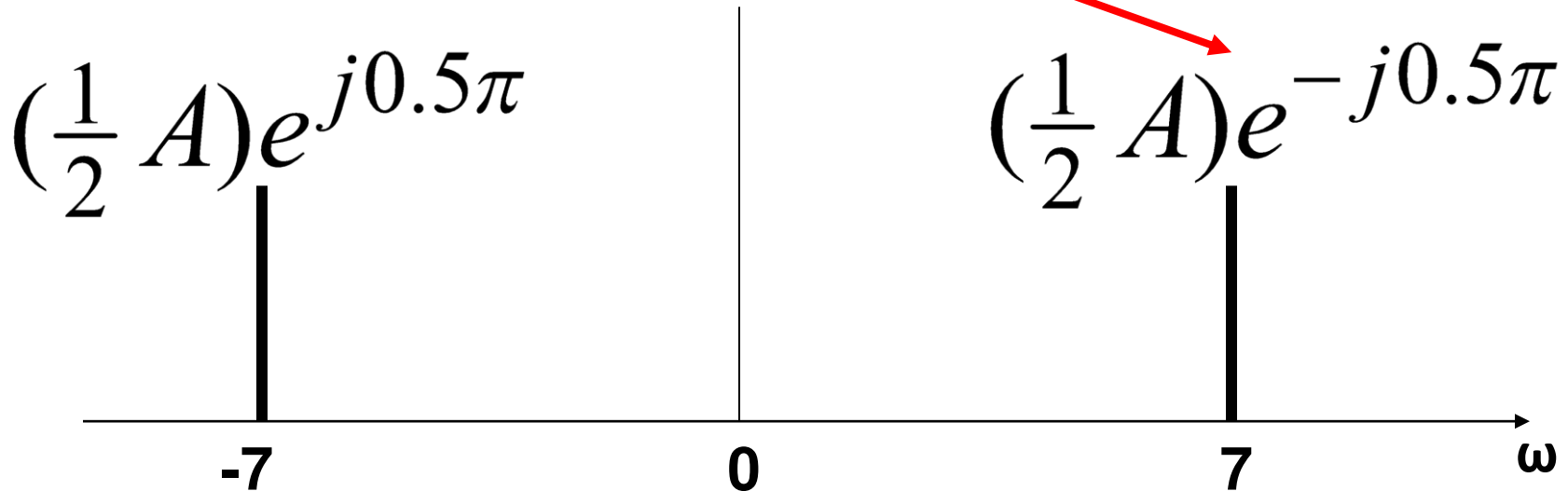
$$\frac{-1}{j} = j = e^{j0.5\pi}$$

- Positive freq. has phase =  $-0.5\pi$
- Negative freq. has phase =  $+0.5\pi$

# GRAPHICAL SPECTRUM

EXAMPLE of SINE

$$A \sin(7t) = \frac{1}{2} A e^{-j0.5\pi} e^{j7t} + \frac{1}{2} A e^{j0.5\pi} e^{-j7t}$$

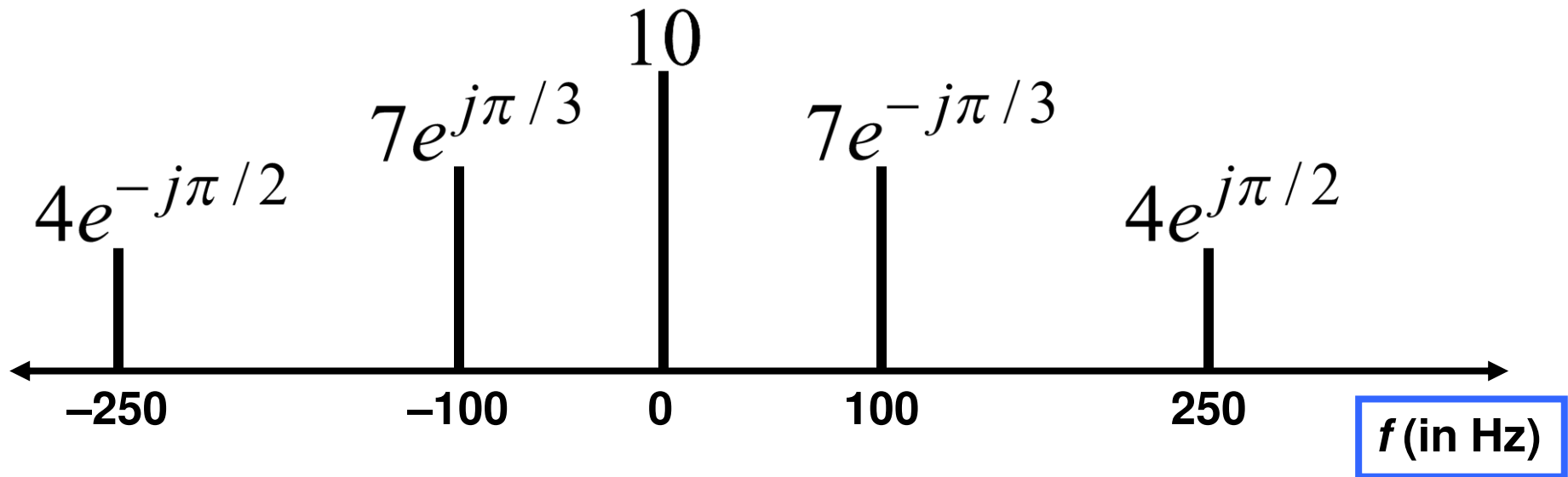


AMPLITUDE, PHASE & FREQUENCY are shown




# SPECTRUM ---> SINUSOID

- Add the spectrum components:



**What is the formula for the signal  $x(t)$ ?**

# Gather ( $A, \omega, \phi$ ) information

- | • Frequencies: | • Amplitude & Phase |
|----------------|---------------------|
| – -250 Hz      | – 4 $-\pi/2$        |
| – -100 Hz      | – 7 $+\pi/3$        |
| – 0 Hz         | – 10      0         |
| – 100 Hz       | – 7 $-\pi/3$        |
| – 250 Hz       | – 4 $+\pi/2$        |
- 

Note the **conjugate phase**

**DC** is another name for zero-freq component

**DC** component always has  $\phi=0$  or  $\pi$  (for real  $\mathbf{x}(t)$ )  
)

# Add Spectrum Components-1

- Frequencies:**

- -250 Hz
- -100 Hz
- 0 Hz
- 100 Hz
- 250 Hz

- Amplitude & Phase**

- 4       $-\pi/2$
- 7       $+\pi/3$
- 10     0
- 7       $-\pi/3$
- 4       $+\pi/2$

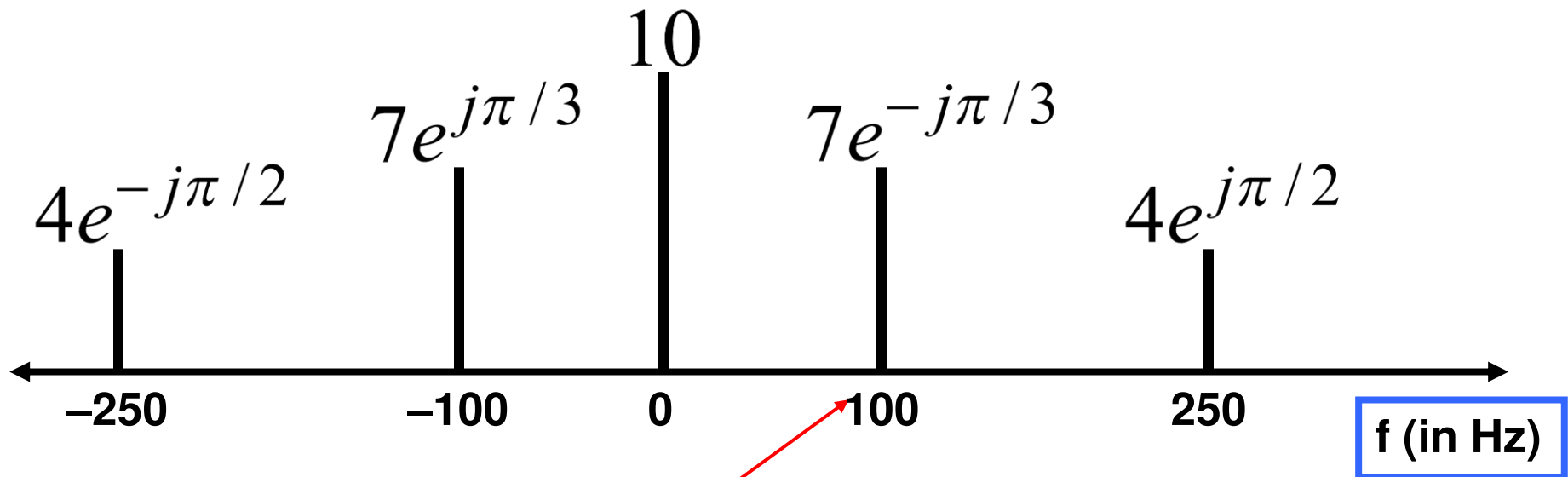


$$x(t) = 10 +$$

$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$

$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

# Add Spectrum Components-2



$$x(t) = 10 +$$

$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} \\ 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

# Simplify Components

$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} \\ 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$


Use Euler's Formula to get **REAL** sinusoids:

$$A \cos(\omega t + \varphi) = \frac{1}{2} A e^{-j\varphi} e^{j\omega t} + \frac{1}{2} A e^{-j\varphi} e^{-j\omega t}$$

# FINAL ANSWER

$$x(t) = 10 + 14 \cos(2\pi(100)t - \pi / 3) \\ + 8 \cos(2\pi(250)t + \pi / 2)$$

So, we get the general form:

$$x(t) = A_0 + \sum_{k=1}^N A_k \cos(2\pi f_k t + \varphi_k)$$


# Summary: GENERAL FORM

$$x(t) = A_0 + \sum_{k=1}^N A_k \cos(2\pi f_k t + \varphi_k)$$

$$x(t) = X_0 + \sum_{k=1}^N \Re\{X_k e^{j2\pi f_k t}\}$$

$$X_k = A_k e^{j\varphi_k}$$

Frequency =  $f_k$

$$\Re\{z\} = \frac{1}{2} z + \frac{1}{2} z^*$$

$$x(t) = X_0 + \sum_{k=1}^N \left\{ \frac{1}{2} X_k e^{j2\pi f_k t} + \frac{1}{2} X_k^* e^{-j2\pi f_k t} \right\}$$

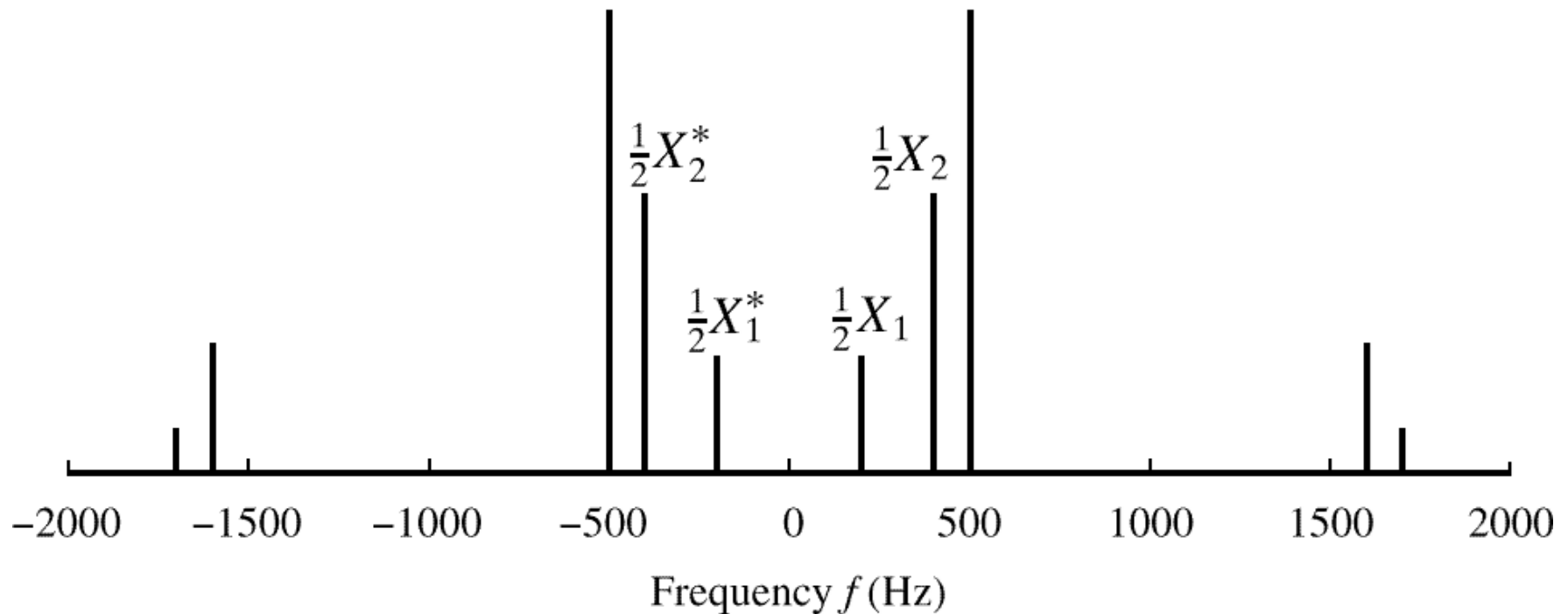
# Example: Synthetic Vowel

- Sum of 5 Frequency Components
  - Complex amplitudes for harmonic signal that approximates the vowel sound «ah»

$f_k$ (Hz)	$X_k$	Mag	Phase (rad)
200	$(771 + j12202)$	12,226	1.508
400	$(-8865 + j28048)$	29,416	1.876
500	$(48001 - j8995)$	48,836	-0.185
1600	$(1657 - j13520)$	13,621	-1.449
1700	$4723 + j0$	4723	0

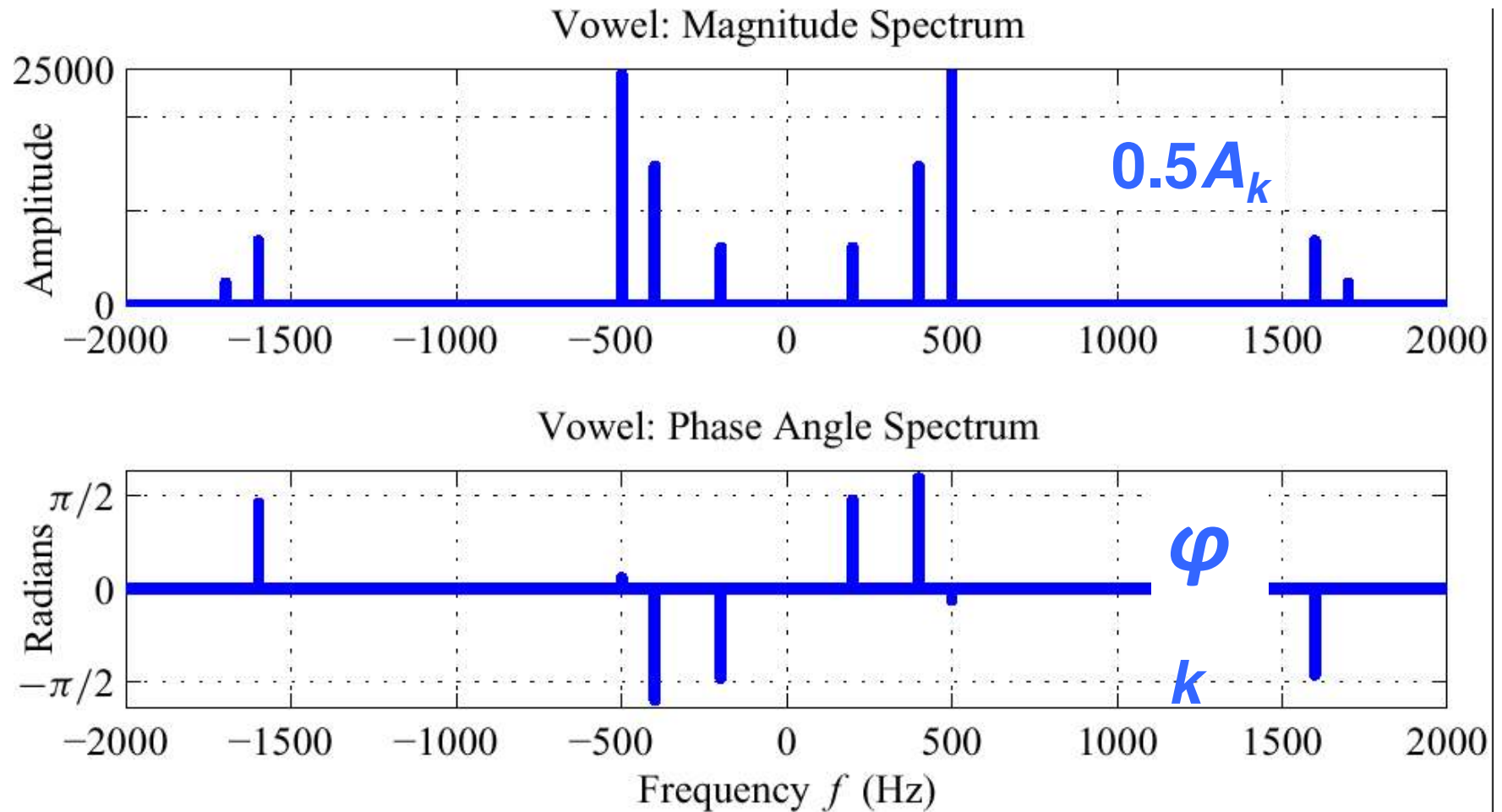


# SPECTRUM of VOWEL



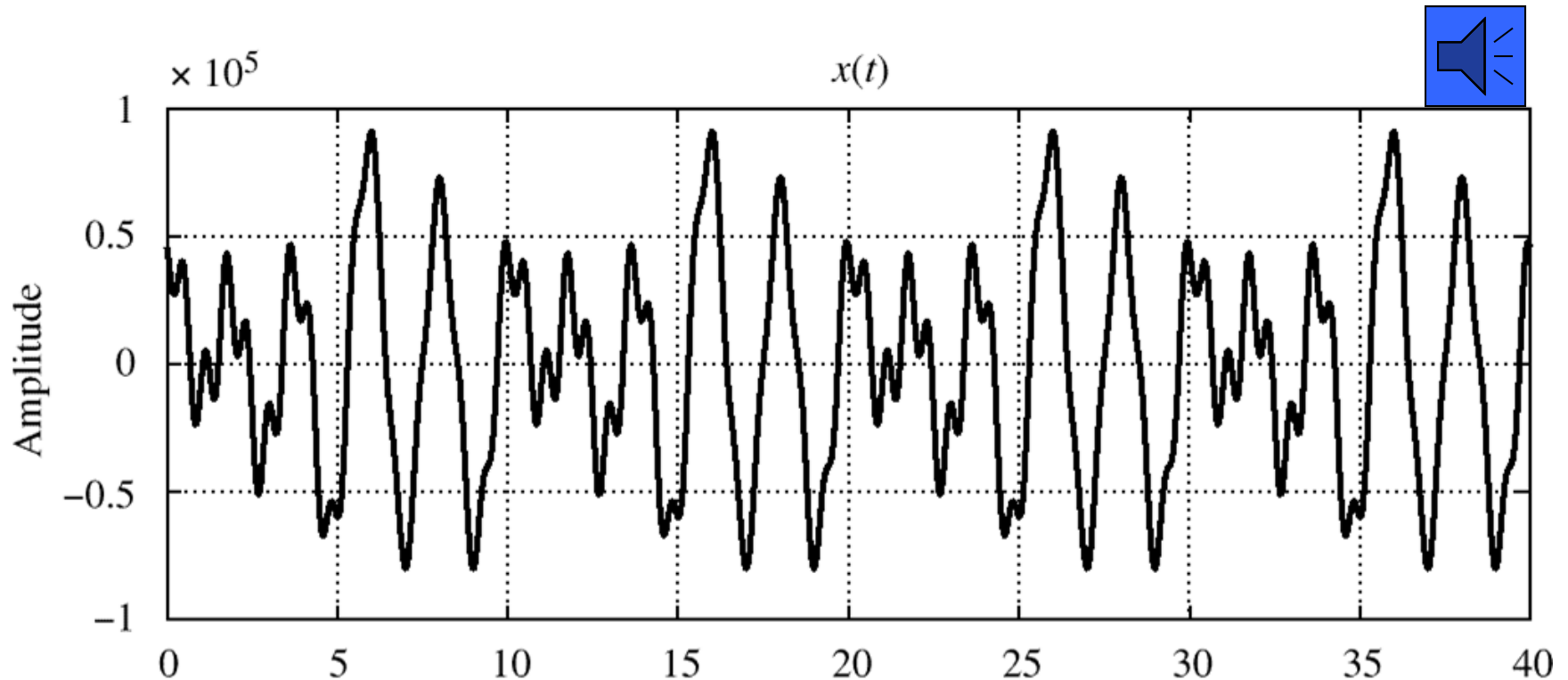
- Note: Spectrum has  $0.5X_k$  (except  $X_{DC}$ )
- Conjugates in negative frequency

# SPECTRUM of VOWEL (Polar Format)



# Vowel Waveform (sum of all 5 components)

- Sum of all of the signals in the previous slides



– Note that the period is 10 ms, which equals  $1/f_0$

# Fourier Series Motivation

“Fourier Series” allows us to write “virtually any” real-world PERIODIC signal as a sum of sinusoids with appropriate amplitudes and phases.

So... we can think of “building a periodic signal from sinusoidal building blocks”.

Later we will extend that idea to also build many non-periodic signals from sinusoidal building blocks!

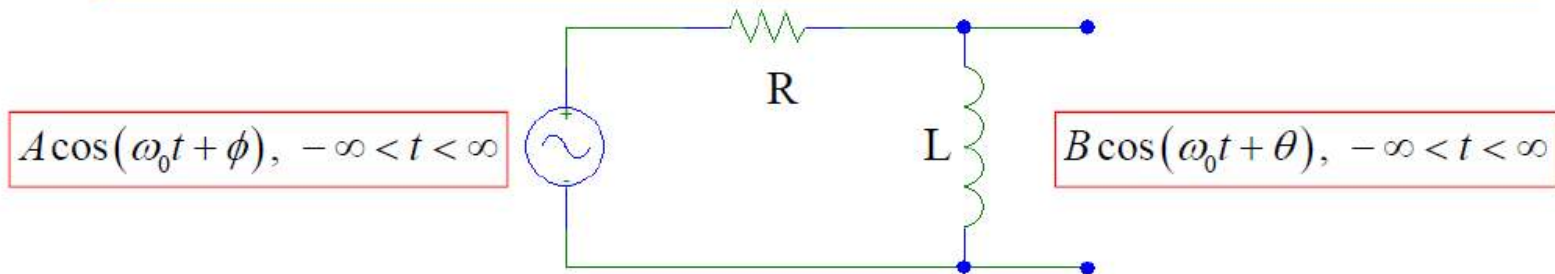
Thus, it is very common for engineers to think about “virtually any” signal as being made up of “sinusoidal components”.

Q: Why all this attention to sinusoids?

A: Recall from Circuits... “sinusoidal analysis” of RLC circuits:

Fundamental Result: Sinusoid In  $\Rightarrow$  Sinusoid Out

(Same Frequency, Different Amplitude & Phase)



# Fourier Series Motivation

This “sinusoid in, sinusoid out” result holds for Constant-Coefficient, Linear Differential Equations as well as any LTI system. We’ll only motivate this result for this Diff. Eq.:

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = x(t)$$

If the input  $x(t)$  is a sinusoid  $A \cos(\omega_0 t + \phi)$ ,  $-\infty < t < \infty$

... then the solution  $y(t)$  must be such that it and its derivatives can be combined to give the input sinusoid.

So... suppose the solution is  $y(t) = B \cos(\omega_0 t + \theta)$ ,  $-\infty < t < \infty$

$$\omega_0^2 B \cos(\omega_0 t + \theta) + a_1 \omega_0 B \sin(\omega_0 t + \theta) + a_0 B \cos(\omega_0 t + \theta) = A \cos(\omega_0 t + \phi)$$

By slogging through lots of algebra and trig identities we can show this can be met with a proper choice of  $B$  and  $\theta$ .

But it makes sense that to add up to a sinusoid we’d need all the terms on the left to be sinusoids of some sort!!!

So... we have reason to believe this:

Fundamental Result: Sinusoid In  $\Rightarrow$  Sinusoid Out

(Same Frequency, Different Amplitude & Phase)



# Fourier Series Motivation

Now... if our input is the linear combination of sinusoids:

$$x(t) = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) + A_3 \cos(\omega_3 t + \phi_3) + \dots, \quad -\infty < t < \infty$$

By linearity (i.e., superposition) we know that we can simply handle each term separately... and we know that each input sinusoid term gives an output sinusoid term:

$$y(t) = B_1 \cos(\omega_1 t + \theta_1) + B_2 \cos(\omega_2 t + \theta_2) + B_3 \cos(\omega_3 t + \theta_3) + \dots, \quad -\infty < t < \infty$$

**So... breaking a signal into sinusoidal parts makes the job of solving a Diff. Eq. EASIER!! (This was Fourier's big idea!!)**

**But.... What kind of signals can we use this trick on?**

**Or in other words...**

**What kinds of signals can we build by adding together sinusoids??!!!**

# What Can We Build with Sinusoids?

Let  $\omega_0$  be some given “fundamental” frequency

Q: What can I build from building blocks that looks like:

$$A_k \cos(\underbrace{k\omega_0}_{\text{integer multiples of } \omega_0} + \theta_k) \quad ?$$

Only frequencies that are integer multiples of  $\omega_0$

Ex.:  $\omega_0 = 30$  rad/sec then consider 0, 30 60, 90, ...

We can explore this by choosing a few different cases of values for the  $A_k$  and  $\theta_k$

On the next slide we limit ourselves to looking at three cases where we limit ourselves to having only three terms...

For this example let  $\omega_0 = 2\pi$  rad/sec and look at a sum for  $k = 1, 2, 3$ :

$$x(t) = A_1 \cos(2\pi t + \phi_1) + A_2 \cos(2 \times 2\pi t + \phi_2) + A_3 \cos(3 \times 2\pi t + \phi_3)$$

# What Can We Build with Sinusoids?

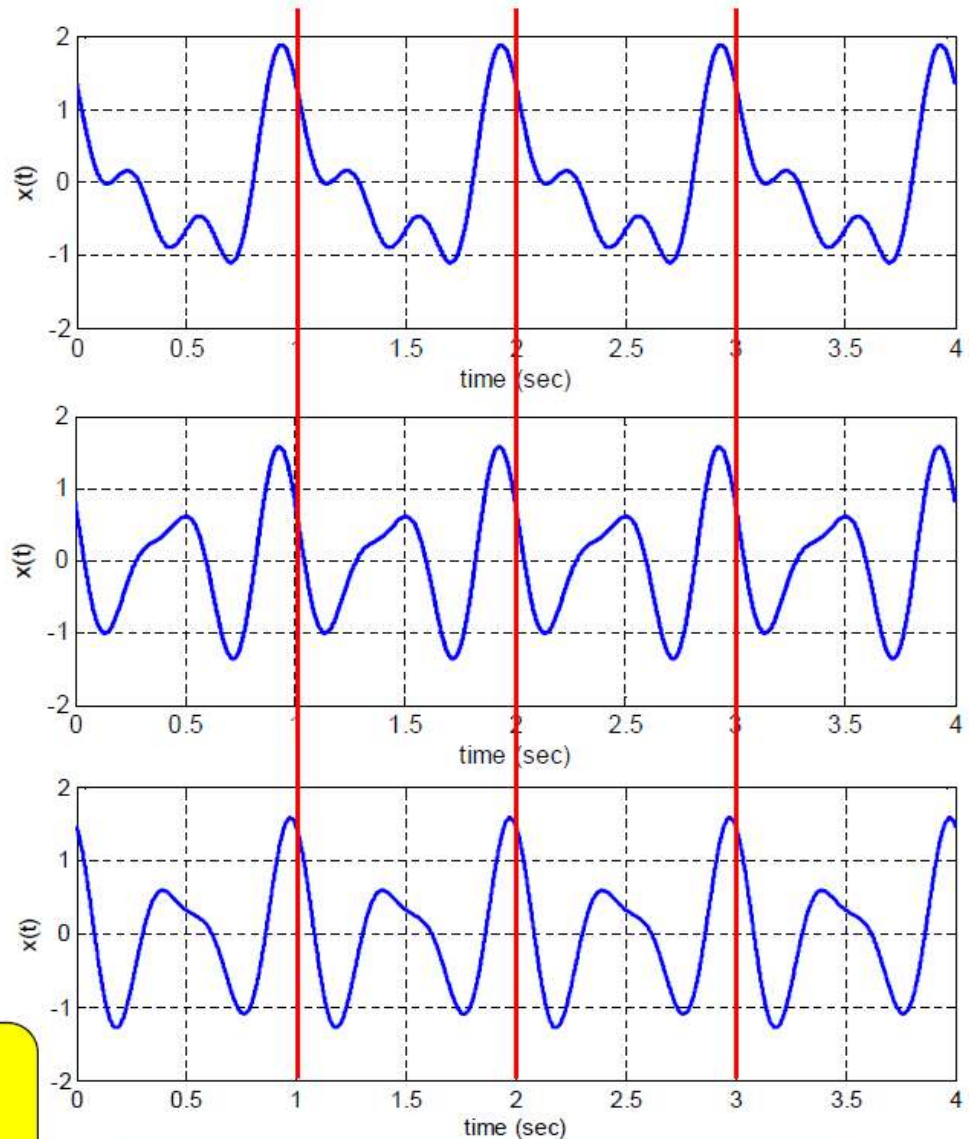
$$\begin{array}{ll} A_1 = 1.0 & \theta_1 = 0 \\ A_2 = 0.5 & \theta_2 = \pi/4 \\ A_3 = 0.5 & \theta_3 = \pi/2 \end{array}$$



$$\begin{array}{ll} A_1 = 0.1 & \theta_1 = 0 \\ A_2 = 1.0 & \theta_2 = \pi/4 \\ A_3 = 0.5 & \theta_3 = \pi/2 \end{array}$$



$$\begin{array}{ll} A_1 = 0.1 & \theta_1 = 0 \\ A_2 = 1.0 & \theta_2 = \pi/7 \\ A_3 = 0.5 & \theta_3 = \pi/14 \end{array}$$



## Note:

1. All are periodic with period of 1s
2. All are “centered” vertically @ 0

In one period: Area Above = Area Below



# What Can We Build with Sinusoids?

Why do these all have period of 1 s???

$$x(t) = A_1 \cos(2\pi t + \phi_1) + A_2 \cos(2 \times 2\pi t + \phi_2) + A_3 \cos(3 \times 2\pi t + \phi_3)$$

Repeats every 1 s

Repeats every 1/2 s  
... so it also repeats  
every 1 s

Repeats every 1/3 s  
... so it also repeats  
every 1 s

This motivates the following general statement:

A sum of sinusoids with frequencies that are integer multiples of some lowest “fundamental” frequency  $\omega_o$  will give a periodic signal with period  $T = 2\pi/\omega_o$  seconds.

So... we can now think about adding together any number of harmonically-related sinusoids... even infinitely many!

$$x(t) = \sum_{k=1}^{\infty} A_k \cos(k\omega_o t + \phi_k), \quad -\infty < t < \infty$$

i.e., all frequencies are an integer multiple of fund. freq.  $\omega_o$

# What Can We Build with Sinusoids?

Why are these all centered vertically @ 0???

$$x(t) = \underbrace{A_1 \cos(2\pi t + \phi_1)}_{\text{Centered @ 0}} + \underbrace{A_2 \cos(2 \times 2\pi t + \phi_2)}_{\text{Centered @ 0}} + \underbrace{A_3 \cos(3 \times 2\pi t + \phi_3)}_{\text{Centered @ 0}}$$

---

This motivates the following general statement:

Unless we have a constant term added, a sum of sinusoids (with frequencies at  $\omega_o$ ,  $2\omega_o$ ,  $3\omega_o$ , ...) will be centered vertically at 0

So... we can now add a constant term

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_o t + \phi_k), \quad -\infty < t < \infty$$

**Note:** for  $k = 0$  we have  $A_0 \cos(0 \times \omega_o t) = A_0$  so we can think of the constant term as a cosine with frequency = 0 and phase = 0

# Fourier Series... A Way to Build a Periodic Signal


$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_o t + \phi_k), \quad -\infty < t < \infty$$

This signal has Period  $T = 2\pi/\omega_o$

**Big Idea:** We can think of (virtually) any real-world **periodic** signal as being made up of (possibly infinitely) many sinusoids whose frequencies are all an integer multiple of a fundamental frequency  $\omega_o$ .

Once we set  $\omega_o$  all we have to do is specify all the amplitudes ( $A_k$ ) and phases ( $\theta_k$ ) and we get some periodic signal with period  $T = 2\pi/\omega_o$ .

But... if we are **GIVEN a periodic signal** how do we determine the correct:

- Fundamental Frequency  $\omega_o$  (rad/sec)
  - Amplitudes ( $A_k$ )
  - Phases ( $\theta_k$ )
- 

**Easy:**  $\omega_o = 2\pi/T$

**Need to Learn How!!**

# Three Forms of Fourier Series

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_o t + \phi_k)$$

“Amplitude & Phase”  
Form

The equation above is just one of three (totally equivalent!) different forms of the Fourier Series.

Each one contains the same information but presents it differently.

Which form you use in a particular setting depends....

- Partly on your preference
- Partly on what you are trying to do

Both of these come  
with experience...

We can easily find the other two by applying trig identities to the terms in the above form.

# Convert to Complex Exponential Form

$$x(t) = A_0 + \boxed{A_1 \cos(1\omega_0 t + \phi_1)} + \boxed{A_2 \cos(2\omega_0 t + \phi_2)} + \dots$$

“Amplitude  
&  
Phase”  
Form

**Euler's Formula**  
 $\cos(\theta) = \frac{1}{2} [e^{j\theta} + e^{-j\theta}]$

$$x(t) = \underbrace{A_0}_{\triangleq c_0} + \left[ \underbrace{\frac{A_1}{2} e^{j\phi_1}}_{\triangleq c_1} e^{j1\omega_0 t} + \underbrace{\frac{A_1}{2} e^{-j\phi_1}}_{\triangleq c_{-1}} e^{j(-1)\omega_0 t} \right] + \left[ \underbrace{\frac{A_2}{2} e^{j\phi_2}}_{\triangleq c_2} e^{j2\omega_0 t} + \underbrace{\frac{A_2}{2} e^{-j\phi_2}}_{\triangleq c_{-2}} e^{j(-2)\omega_0 t} \right] + \dots$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

“Complex Exponential”  
Form



# Convert to Sine-Cosine Form

$$x(t) = A_0 + \boxed{A_1 \cos(1\omega_0 t + \phi_1)} + \boxed{A_2 \cos(2\omega_0 t + \phi_2)} + \dots$$

“Amplitude  
&  
Phase”  
Form

**Trig Identity**

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$x(t) = \underbrace{A_0}_{\triangleq a_0} + \left[ \underbrace{A_1 \cos(\phi_1)}_{\triangleq a_1} \cos(1\omega_0 t) - \underbrace{A_1 \sin(\phi_1)}_{\triangleq b_1} \sin(1\omega_0 t) \right] + \left[ \underbrace{A_2 \cos(\phi_2)}_{\triangleq a_2} \cos(2\omega_0 t) - \underbrace{A_2 \sin(\phi_2)}_{\triangleq b_2} \sin(2\omega_0 t) \right] + \dots$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)]$$

“Sine-Cosine”  
Form

# Three (Equivalent) Forms of FS and Their Relationships

Best for “thinking about real-world ideas”

Trig Form: Amplitude & Phase

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

$$A_0 = c_0$$

$$\left. \begin{aligned} A_k &= 2|c_k| \\ \theta_k &= \angle c_k \end{aligned} \right\} k = 1, 2, 3, \dots$$

Best for “doing math”  
( $c_k$  are like phasors!!)

Exponential Form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$c_0 = A_0$$

$$\left. \begin{aligned} c_k &= \frac{1}{2} A_k e^{j\theta_k} \\ c_{-k} &= \frac{1}{2} A_k e^{-j\theta_k} \end{aligned} \right\} k = 1, 2, 3, \dots$$

$$c_0 = a_0$$

$$\left. \begin{aligned} c_k &= \frac{1}{2} (a_k - jb_k) \\ c_{-k} &= \frac{1}{2} (a_k + jb_k) \end{aligned} \right\} k = 1, 2, 3, \dots$$

Best for some  
“special scenarios”

Trig Form: Sine-Cosine

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)]$$

$$a_0 = c_0$$

$$a_k = 2 \operatorname{Re}\{c_k\}, \quad k = 1, 2, 3, \dots$$

$$b_k = -2 \operatorname{Im}\{c_k\}, \quad k = 1, 2, 3, \dots$$

$$\left. \begin{aligned} A_0 &= a_0 \\ A_k &= \sqrt{a_k^2 + b_k^2} \\ \theta_k &= \tan^{-1}\left(\frac{-b_k}{a_k}\right) \end{aligned} \right\}$$

$$\left. \begin{aligned} a_0 &= c_0 \\ a_k &= A_k \cos(\theta_k) \\ b_k &= -A_k \sin(\theta_k) \end{aligned} \right\}$$

**Example:** Consider  $x(t) = \cos(t) + 0.5 \cos(4t + \pi / 3) + 0.25 \cos(8t + \pi / 2)$

which is already in **Amp-Phase Form** of the Fourier Series with  $\omega_0 = 1$  :

$$A_1 = 1 \qquad A_4 = 0.5 \qquad A_8 = 0.25 \qquad (\text{all other } A_k \text{ are } 0)$$

$$\theta_1 = 0 \qquad \theta_4 = \pi/3 \qquad \theta_8 = \pi/2$$

Using the conversion results on the previous slide we can re-write this in **Complex Exponential Form** of the FS as:

$$c_1 = 0.5 \qquad c_4 = 0.25e^{j\pi/3} \qquad c_8 = 0.125e^{j\pi/2} \qquad (\text{all other } c_k \text{ are } 0)$$

$$c_{-1} = 0.5 \qquad c_{-4} = 0.25e^{-j\pi/3} \qquad c_{-8} = 0.125e^{-j\pi/2}$$

$$c_0 = A_0$$

$$c_k = \frac{1}{2} A_k e^{j\theta_k}$$

$$c_{-k} = \frac{1}{2} A_k e^{-j\theta_k}$$

$$x(t) = \left[ 0.5e^{jt} + 0.5e^{-jt} \right] + \left[ 0.25e^{j\pi/3}e^{j4t} + 0.25e^{-j\pi/3}e^{-j4t} \right] + \left[ 0.125e^{j\pi/2}e^{j8t} + 0.125e^{-j\pi/2}e^{-j8t} \right]$$

Using the conversion results on the previous slide we can re-write this in **Sine-Cosine Form** of the FS as:

$$a_1 = 1 \qquad a_4 = 0.25 \qquad a_8 = 0 \qquad (\text{all other } a_k, b_k \text{ are } 0)$$

$$b_1 = 0 \qquad b_4 = 0.43 \qquad b_8 = 0.25$$

$$a_0 = c_0$$

$$a_k = A_k \cos(\theta_k)$$

$$b_k = -A_k \sin(\theta_k)$$

$$x(t) = \left[ \cos(t) \right] + \left[ 0.25 \cos(4t) - 0.43 \sin(4t) \right] + \left[ 0.25 \sin(8t) \right]$$



# Analytically Finding FS Coefficients

**Q: How do we find the Exponential Form FS Coefficients?**

**A: Use this: (it can be proved but we won't do that here!)**

$$c_k = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt$$

Integrate over any complete period

Some books use only  $t_0 = 0$ .

where:  $T$  = fundamental period of  $x(t)$  (in seconds)

$\omega_0$  = fundamental frequency of  $x(t)$  (in rad/second)  
 $= 2\pi/T$

$t_0$  = any time point (you pick  $t_0$  to ease calculations)

$k \in$  all integers (... -3, -2, -1, 0, 1, 2, 3, ...)

Looks like we have to do this integral infinitely many times!!!  
But... Usually you can do the integral in terms of arbitrary  $k$ !

Comment: Note that for  $k = 0$  this gives

$$c_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt$$

$c_0$  is the “DC offset”, which is the time-average over one period

# Analytically Finding FS Coefficients

**Q: How do we find the Sine-Cosine Form FS Coefficients?**

**A: Use these: (can be proved but we won't do that here!)**

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt$$

$a_0$  is the “DC offset”, which is the time-average over one period

$$a_k = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos(k\omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin(k\omega_0 t) dt$$

**Integrate over  
any complete  
period**

where:  $T$  = fundamental period of  $x(t)$  (in seconds)

$\omega_0$  = fundamental frequency of  $x(t)$  (in rad/second)

$$= 2\pi/T$$

$t_0$  = any time point (you pick  $t_0$  to ease calculations)

$k \in$  all integers

# Analytically Finding FS Coefficients

**Q: How do we find the Amplitude-Phase Form FS Coefficients?**

**A: No easy direct way! So convert from one of the other forms!**

$$A_0 = a_0$$

$$A_k = \sqrt{a_k^2 + b_k^2}$$

$$\theta_k = \tan^{-1}\left(\frac{-b_k}{a_k}\right)$$

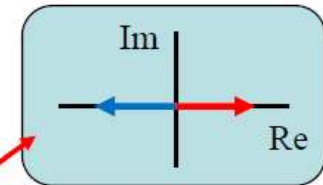
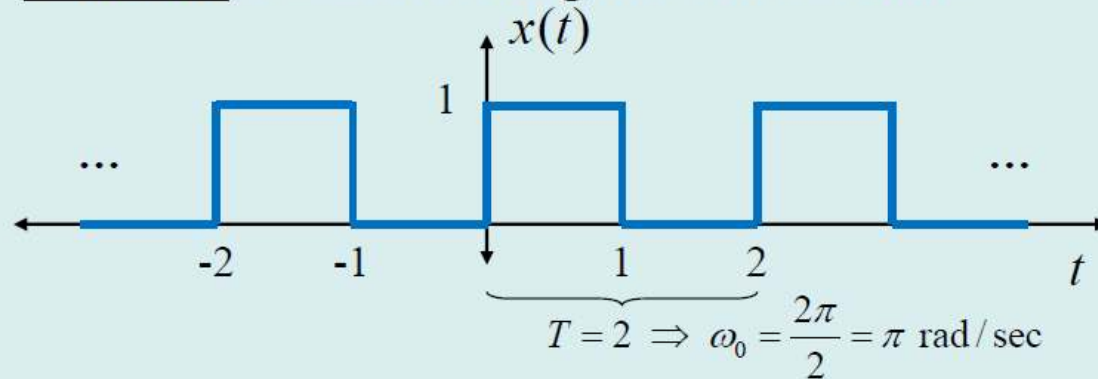
$$A_0 = c_0$$

$$\left. \begin{array}{l} A_k = 2|c_k| \\ \theta_k = \angle c_k \end{array} \right\} k = 1, 2, 3, \dots$$

- Recall... you can convert from any form into any other form using some simple equations!
- Thus... I tend to always find the  $c_k$  and then convert to other forms if needed.
- Why do I prefer to find the  $c_k$ ?
  - Only one integral to actually do (although it is complex valued!)
  - Integrals involving exponential are usually easier than for sinusoids!

# Analytically Finding FS Coefficients

## Example: FS of Rectangular Pulse Train



$$c_k = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt$$

choose  $t_0 = 0$

$$= \frac{1}{2} \int_0^2 x(t) e^{-jk\pi t} dt$$

$$= \frac{1}{2} \left[ \int_0^1 1 e^{-jk\pi t} dt + \int_1^2 0 \times e^{-jk\pi t} dt \right]$$

$$= \frac{1}{2} \int_0^1 e^{-jk\pi t} dt$$

$$= \frac{1}{2} \left[ \frac{1}{-jk\pi} e^{-jk\pi t} \right]_0^1$$

Not valid for  $k=0$ ... so have to do that case separately!

$$= \frac{j}{2(k\pi)} [e^{-jk\pi} - 1]$$

$$c_k = \begin{cases} 0, & k \text{ even, } \neq 0 \\ \frac{-j}{k\pi}, & k \text{ odd} \end{cases}$$

$$= \begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd} \end{cases}$$

$$c_0 = \frac{1}{2} \int_0^1 1 e^{-j0\pi t} dt = \frac{1}{2} \int_0^1 1 dt$$

$$c_0 = \frac{1}{2}$$

DC Level (also called DC Offset)

# Analytically Finding FS Coefficients

So... we've found the exponential FS to be:

$$x(t) = \cdots + \frac{-j}{-3\pi} e^{-j3\omega_o t} + \frac{-j}{-1\pi} e^{-j1\omega_o t} + \frac{1}{2} + \frac{-j}{1\pi} e^{j1\omega_o t} + \frac{-j}{3\pi} e^{j3\omega_o t} + \cdots$$

$$c_k = \begin{cases} \frac{1}{2}, & k = 0 \\ 0, & k \text{ even}, \neq 0 \\ \frac{-j}{k\pi}, & k \text{ odd} \end{cases}$$

$$a_0 = c_0$$

$$a_k = 2 \operatorname{Re}\{c_k\}, \quad k = 1, 2, 3, \dots$$

$$b_k = -2 \operatorname{Im}\{c_k\}, \quad k = 1, 2, 3, \dots$$

$$a_k = \begin{cases} \frac{1}{2}, & k = 0 \\ 0, & k \neq 0 \end{cases}$$
$$b_k = \begin{cases} 0, & k \text{ even} \\ \frac{2}{k\pi}, & k \text{ odd} \end{cases}$$

$$x(t) = \frac{1}{2} + \frac{2}{1\pi} \sin(1\omega_o t) + \frac{2}{3\pi} \sin(3\omega_o t) + \frac{2}{5\pi} \sin(5\omega_o t) + \cdots$$

# Analytically Finding FS Coefficients

So... we've found the exponential FS to be:

$$x(t) = \cdots + \frac{-j}{-3\pi} e^{-j3\omega_o t} + \frac{-j}{-1\pi} e^{-j1\omega_o t} + \frac{1}{2} + \frac{-j}{1\pi} e^{j1\omega_o t} + \frac{-j}{3\pi} e^{j3\omega_o t} + \cdots$$

$$c_k = \begin{cases} \frac{1}{2}, & k = 0 \\ 0, & k \text{ even}, \neq 0 \\ \frac{-j}{k\pi}, & k \text{ odd} \end{cases}$$

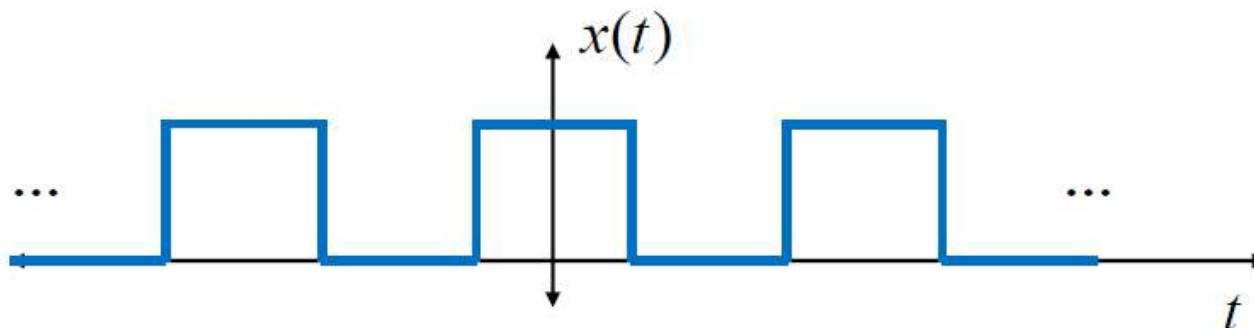
$$\left. \begin{aligned} A_0 &= c_0 \\ A_k &= 2|c_k| \\ \theta_k &= \angle c_k \end{aligned} \right\} k = 1, 2, 3, \dots$$

$$A_k = \begin{cases} \frac{1}{2}, & k = 0 \\ 0, & k \text{ even} \\ \frac{2}{k\pi}, & k \text{ odd} \end{cases} \quad \theta_k = \begin{cases} \text{N/A}, & k = 0 \\ \text{N/A}, & k \text{ even} \\ -\frac{\pi}{2}, & k \text{ odd} \end{cases}$$

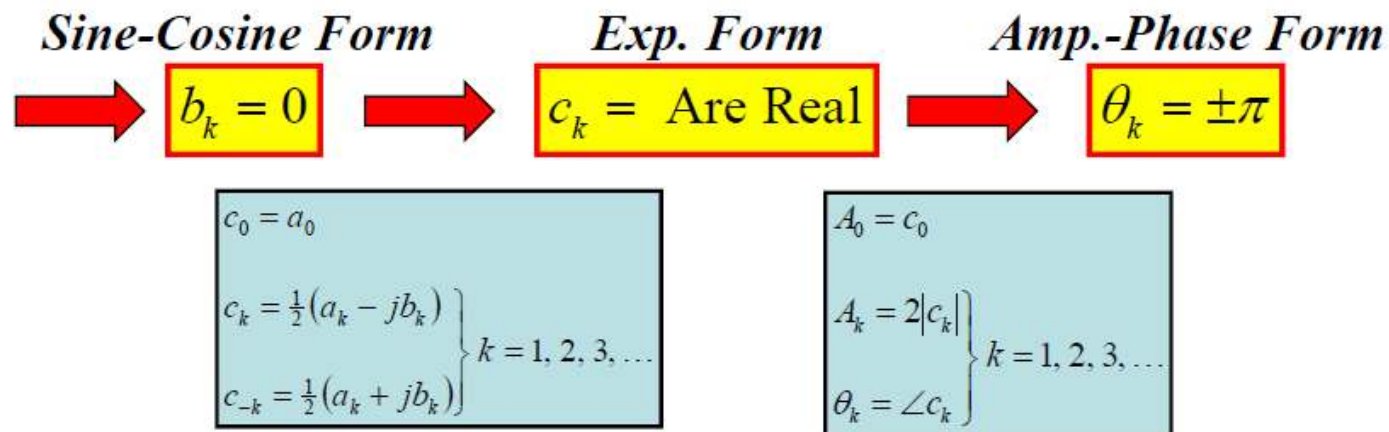
$$x(t) = \frac{1}{2} + \frac{2}{1\pi} \cos(1\omega_o t - \pi/2) + \frac{2}{3\pi} \cos(3\omega_o t - \pi/2) + \frac{2}{5\pi} \cos(5\omega_o t - \pi/2) + \cdots$$

# Symmetry “Tricks” for Finding FS Coefficients

**Even Symmetry**:  $x(-t) = x(t)$  (“flipping” around  $t = 0$  does nothing)

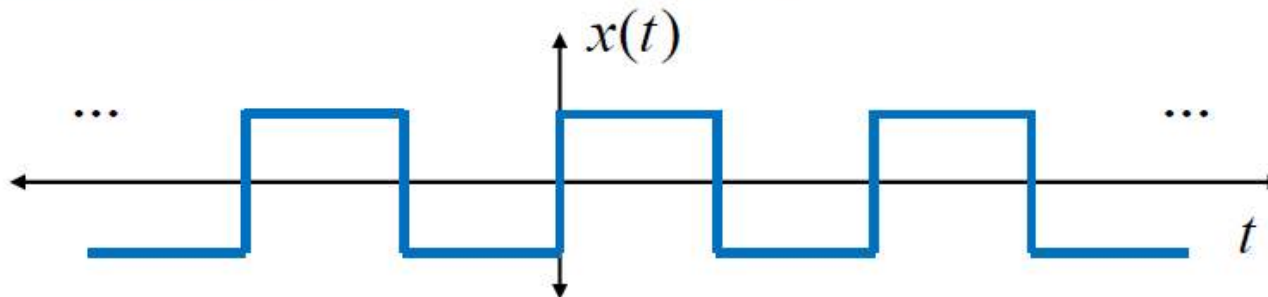


Noting that cosines have even symmetry and sines have odd symmetry it is not surprising that an even  $x(t)$  needs only cosine components in the Sine-Cosine Form:

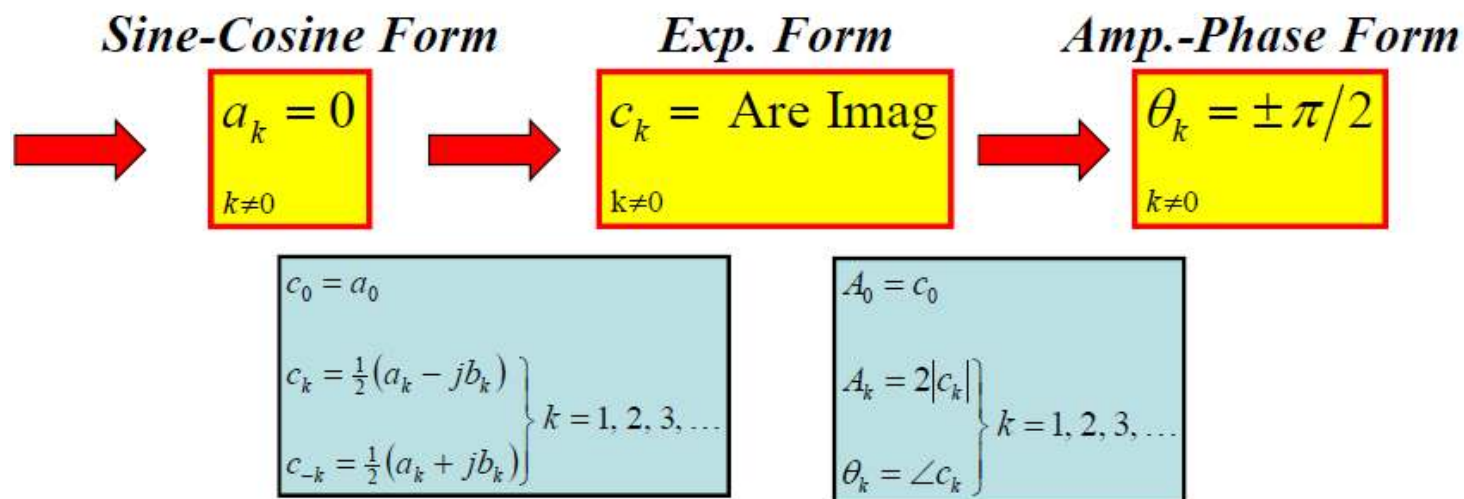




**Odd Symmetry:**  $x(-t) = -x(t)$  (“flipping” around  $t = 0$  negates  $x(t)$ )

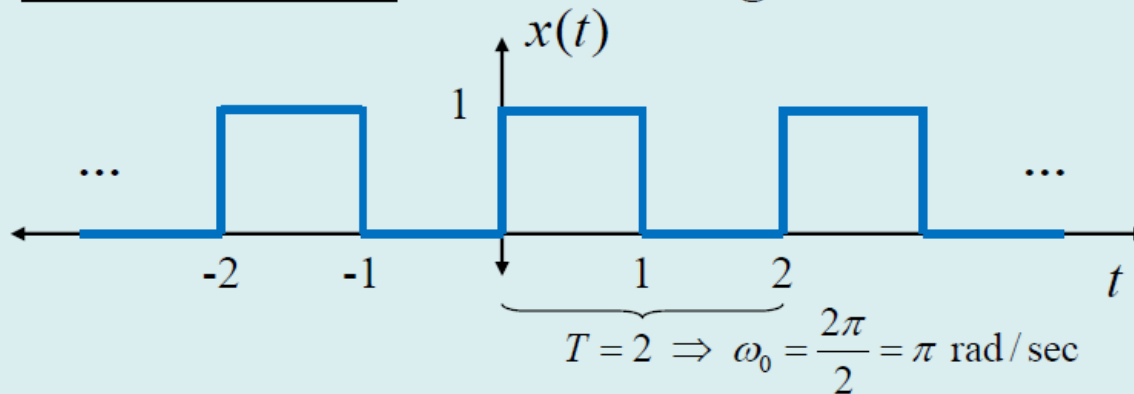


Noting that cosines have even symmetry and sines have odd symmetry it is not surprising that an ODD  $x(t)$  needs only sine components in the Sine-Cosine Form:





## Recall Example: FS of Rectangular Pulse Train



*Sine-Cosine Form*

$$a_k = 0$$

$k \neq 0$

*Exp. Form*

$$c_k = \text{Are Imag}$$

$k \neq 0$

*Amp.-Phase Form*

$$\theta_k = \pm \pi/2$$

$k \neq 0$

$$a_k = \begin{cases} \frac{1}{2}, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

$$b_k = \begin{cases} 0, & k \text{ even} \\ \frac{2}{k\pi}, & k \text{ odd} \end{cases}$$

$$c_k = \begin{cases} 0, & k \text{ even, } \neq 0 \\ -j, & k \text{ odd} \end{cases}$$

$$\theta_k = \begin{cases} \text{N/A}, & k = 0 \\ \text{N/A}, & k \text{ even} \\ -\frac{\pi}{2}, & k \text{ odd} \end{cases}$$

# Fourier Series Spectrum

## Trig Form “Spectrum”... Is “Single Sided”

Best for “thinking about real-world ideas”

Trig Form: Amplitude & Phase

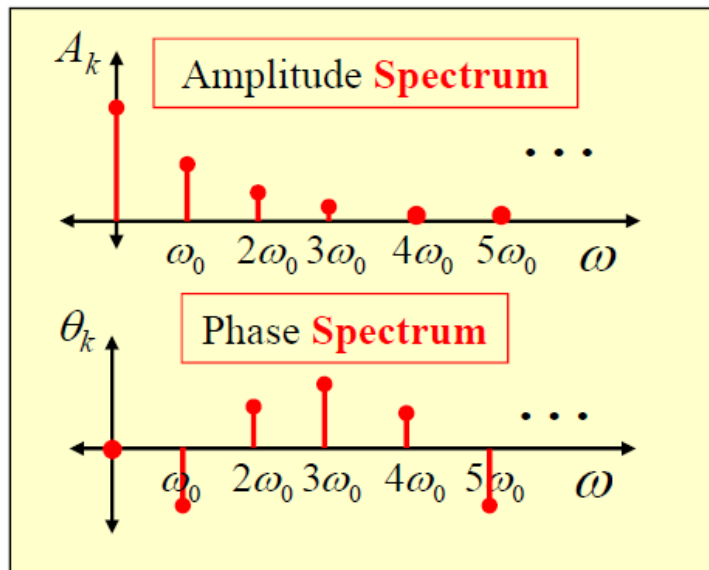
$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

Need  $A_k$  and  $\theta_k$   
for  $k = 0, 1, 2, \dots$

$A_k$  = Amplitude  
 $\theta_k$  = Phase

So... to describe a signal via FS we specify:  
“Amplitude & Phase @ Each Frequency”

A good way to “see” the FS coefficients is by plotting them vs. frequency:



For this form of FS:

- Do not need negative freqs  
→ “Single Sided” Spectrum

# Fourier Series Spectrum

## Exp Form “Spectrum”... Is “Double Sided”

Best for “doing math” ( $c_k$  are like phasors!!)

### Exponential Form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

So... to describe a signal via FS we specify:  
“Magnitude & Phase @ Each Frequency”

Need  $c_k$  (complex!)  
for  $k = \dots -2, -1, 0, 1, 2 \dots$

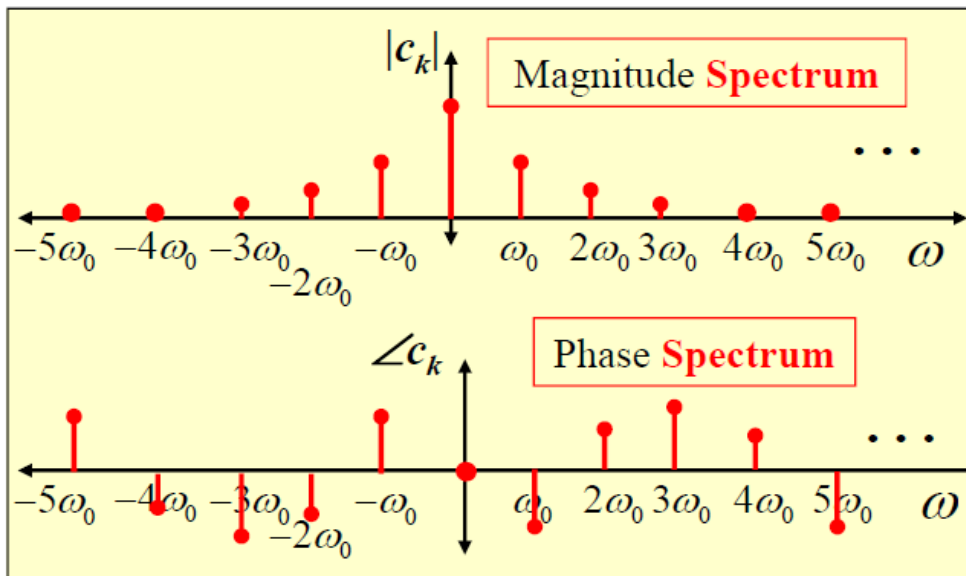
$|c_k|$  = Magnitude  
 $\angle c_k$  = Phase

$$c_k e^{jk\omega_0 t} = [|c_k| e^{j\angle c_k}] e^{jk\omega_0 t}$$

$$= |c_k| e^{j(k\omega_0 t + \angle c_k)}$$

For this form of FS:

- Do need negative freqs  
→ “Double Sided” Spectrum



# Fourier Series Spectrum

## Spectrum Characteristics

### Trig Form: Amplitude & Phase

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

For Trig Form of FS Spectrum:

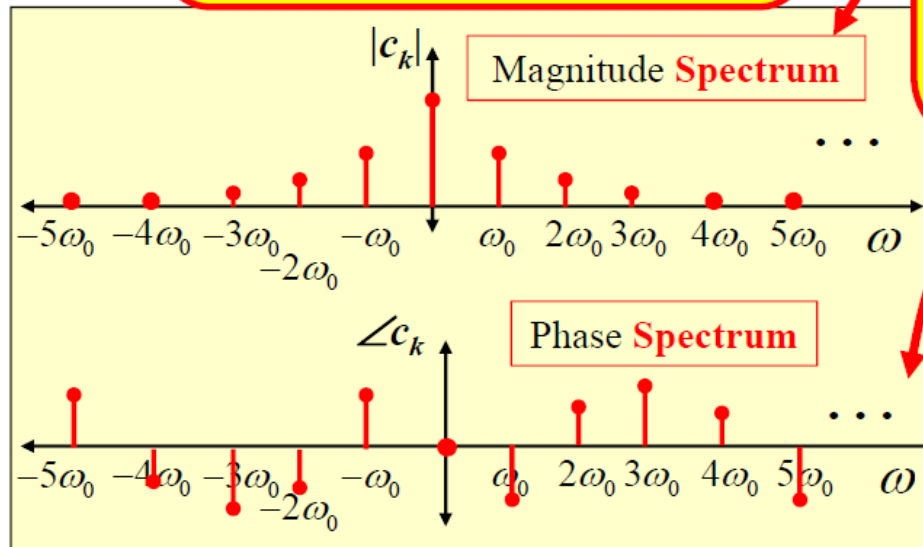
- “Single Sided” Spectrum
- $A_k \geq 0$  for  $k > 0$ 
  - $A_0$ : positive or negative
- $\theta_k$  is in **radians**  $\theta_0 = 0$

### Exponential Form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

For Exp Form of FS Spectrum:

- “Double Sided” Spectrum
- $|c_k| \geq 0$  for all  $k$ 
  - *Even Symmetry for Magn.*
- $\angle c_k$  is in **radians**
- $\angle c_0 = 0$  or  $\pm\pi$
- $\angle c_k = -\angle c_{-k}$ 
  - *Odd Symmetry for Phase*



$$\left. \begin{aligned} c_k &= \frac{1}{2} A_k e^{j\theta_k} \\ c_{-k} &= \frac{1}{2} A_k e^{-j\theta_k} \end{aligned} \right\} k = 1, 2, 3, \dots$$

# Fourier Series Spectrum

## Parseval's Theorem

We saw earlier how to compute the average power of a periodic signal if we are given its time-domain model:

$$P = \frac{1}{T} \int_{t_0}^{T+t_0} x^2(t) dt$$

**Q: Can we compute the average power from the frequency domain model**

**A: Parseval's Theorem says... Yes!**

$$\{c_k\}, \quad k = 0, \pm 1, \pm 2, \dots$$

Parseval's theorem says that the avg. power can be computed this way:

$$P = \sum_{k=-\infty}^{\infty} |c_k|^2$$



$$\frac{1}{T} \int_{t_0}^{t_0+T} x^2(t) dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

$c_k$  are the Exp. Form FS coefficients

Left side is clearly finite for real-world signals...

**Thus, the  $|c_k|$  must decay fast enough as  $k \rightarrow \pm\infty$**

# Fourier Series Spectrum

## Interpreting Parseval's Theorem

$$\underbrace{\frac{1}{T} \int_{t_0}^{t_0+T} x^2(t) dt}_{\text{"sum" of squares in time-domain model}} = \underbrace{\sum_{k=-\infty}^{\infty} |c_k|^2}_{\text{"sum" of squares in freq.-domain model}}$$

"sum" of squares in time-domain model

"sum" of squares in freq.-domain model

$x^2(t)$  = power at time  $t$  (includes effects of all frequencies)

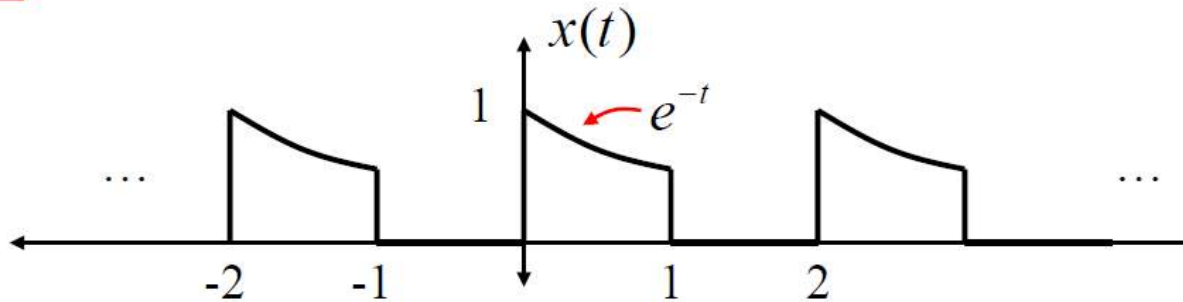
We can find the power in the time domain by "adding up" all the "powers at each time"

$|c_k|^2$  = power at frequency  $k\omega_0$  (includes effects of all times)

We can find the power in the frequency domain by adding up all the "powers at each frequency"

# Fourier Series Example

## Example #1



choose

$$T = 2 \Rightarrow \omega_0 = \frac{2\pi}{2} = \pi \text{ rad/sec}$$

$$c_k = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{2} \int_0^2 x(t) e^{-jk\pi t} dt$$

$$= \frac{1}{2} \left[ \int_0^1 e^{-t} e^{-jk\pi t} dt + \int_1^2 0 \times e^{-jk\pi t} dt \right]$$

$$= \frac{1}{2} \int_0^1 e^{-(1+jk\pi)t} dt$$

$$= \frac{1}{2} \left[ \frac{-1}{1+jk\pi} e^{-(1+jk\pi)t} \right]_0^1$$

$$= \frac{-1}{2(1+jk\pi)} [e^{-(1+jk\pi)} - 1]$$

$$= \frac{1 - e^{-1} e^{jk\pi}}{2(1+jk\pi)}$$

Note:  $e^{-jk\pi} = \begin{cases} 1, & \text{even } k \\ -1, & \text{odd } k \end{cases}$

or equivalently  $e^{-jk\pi} = (e^{-j\pi})^k = (-1)^k$

So...

$$c_k = \frac{1 - e^{-1} (-1)^k}{2(1+jk\pi)}$$



Now we can use Matlab to plot  $|c_k|$  &  $\angle c_k$

Spectrum

