BLM1612 Circuit Theory

Capacitors and Inductors

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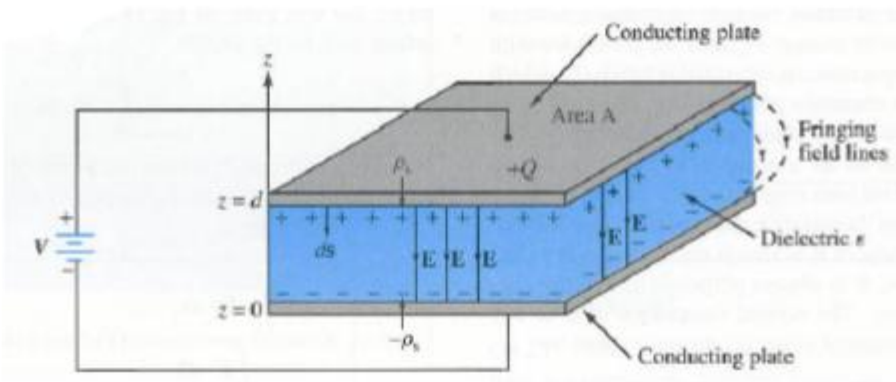
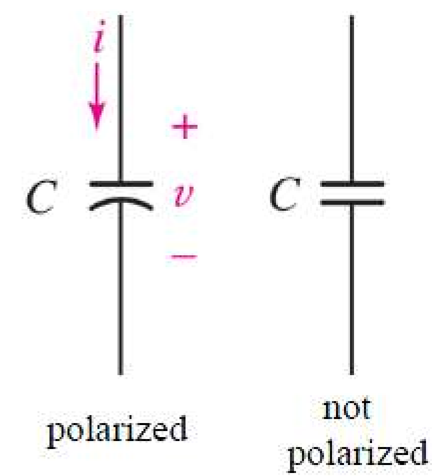
Capacitance

A **capacitor** is a linear circuit element which stores energy in the **electric field** in the space between two conducting bodies occupied by a material with permittivity ε.





- *charged* by applying current (for a finite amount of time, from another source) to its terminals

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(t')\cdot dt' + v(t_0)$$

- *discharged* when it provides current (for a finite amount of time, to a circuit) from its terminals

Capacitor Current & Voltage

$$v(t)=\frac{1}{C}\int_{t_0}^{t} i(\tau)\cdot d\tau+v(t_0)$$

same equation

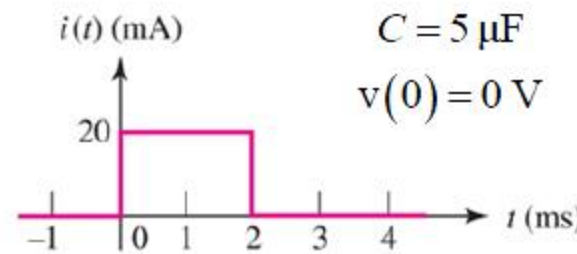
$$i(t)=C\frac{dv(t)}{dt} \qquad q(t)=C\cdot v(t)$$

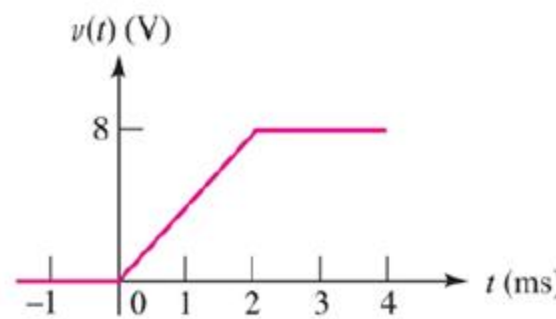
$$C=Q/V$$

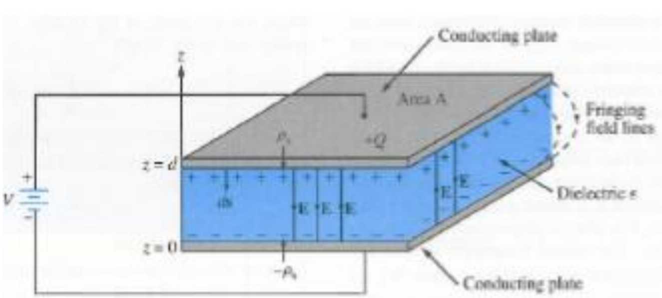
unit of capacitance = farad
1 F = 1 C / V

$C = 5\ \mu F$

$v(0)=0\ V$



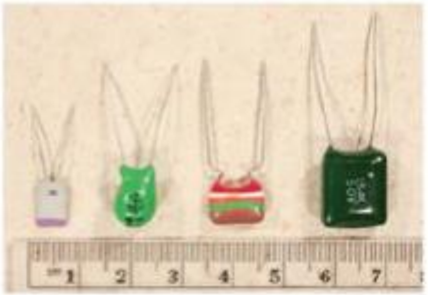




$$C=\varepsilon\frac{A}{d}$$

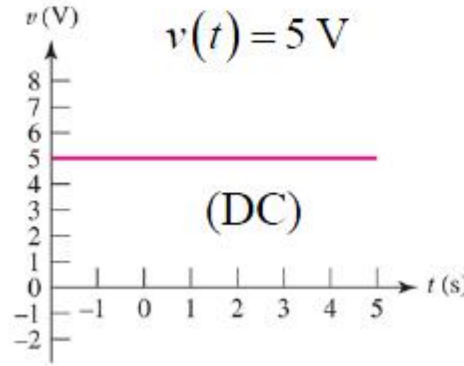
ε = permittivity

Capacitor Characteristics

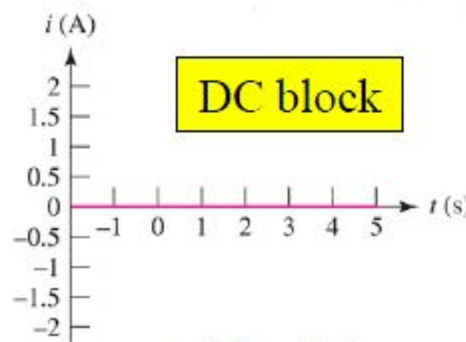


$$i = C\frac{dv}{dt}$$

$$v(t) = 5\ \text{V}$$

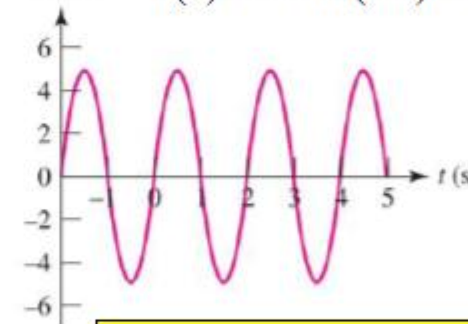


$C = 2\ \mu\text{F}$

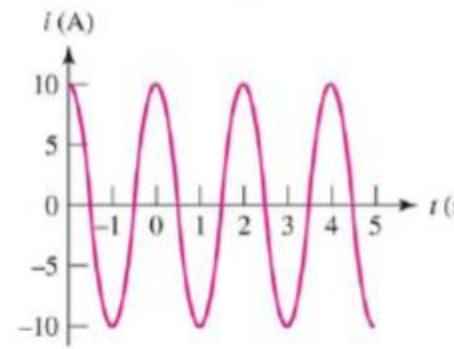


$$i(t) = 0\ \text{A}$$

$$v(t) = 5\sin(\omega t)\ \text{V}$$



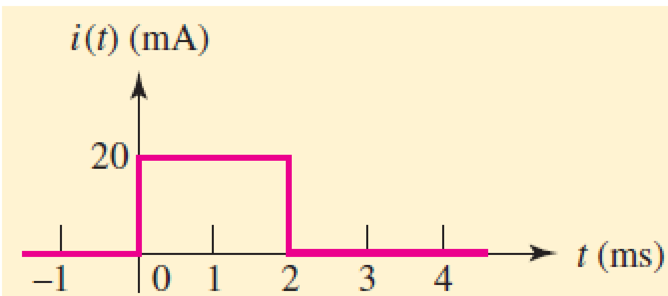
current "leads" voltage,
voltage "lags" current

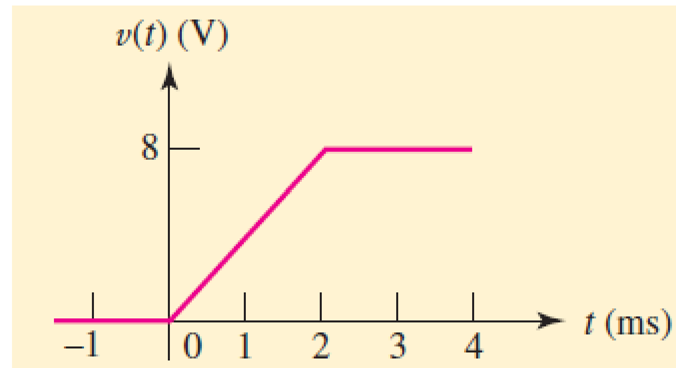


$$i(t) = 10\cos(\omega t)\ \mu\text{A}$$

Example 7.2

- Find the capacitor voltage that is associated with the current shown graphically in below Figure. The value of the capacitance is 5 μF.





$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(t')\,dt' + v(t_0)$$

$$v(t) = \frac{1}{5 \times 10^{-6}}\int_{0}^{t} 20 \times 10^{-3}\,dt' + v(0)$$

Since $v(0) = 0$,

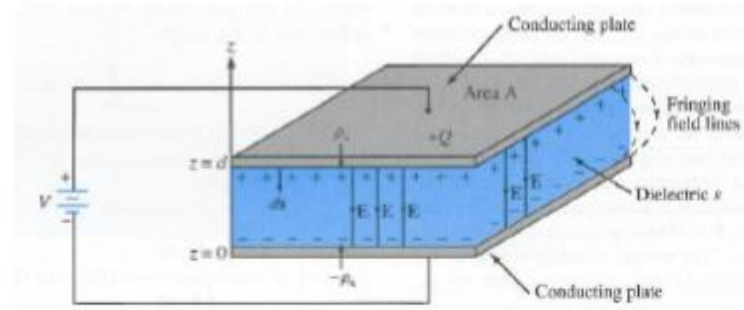
$$v(t) = 4000t \qquad 0 \le t \le 2 \text{ ms}$$

Capacitor Energy Storage

$$p = i \cdot v$$

$$i = C\frac{dv}{dt}$$

$$\begin{aligned}
w &= \int_0^t p(\tau) \cdot d\tau \\
&= \int_{v(0)}^{v(t)} C\frac{dv}{d\tau} \cdot v \cdot d\tau \\
&= C \cdot \int_0^t v \cdot dv \\
&= C \cdot \frac{1}{2} v^2 \Big|_0^t \\
&= C \cdot \frac{1}{2}\left[v(t)^2 - v(0)^2\right] \longrightarrow
\end{aligned}$$

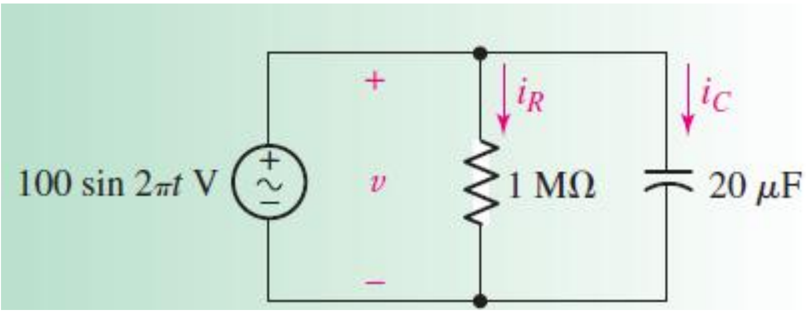


electrical energy stored in a capacitor with voltage v across its plates:

$$\boxed{w = \frac{1}{2} C \cdot v^2}$$

Example 7.3

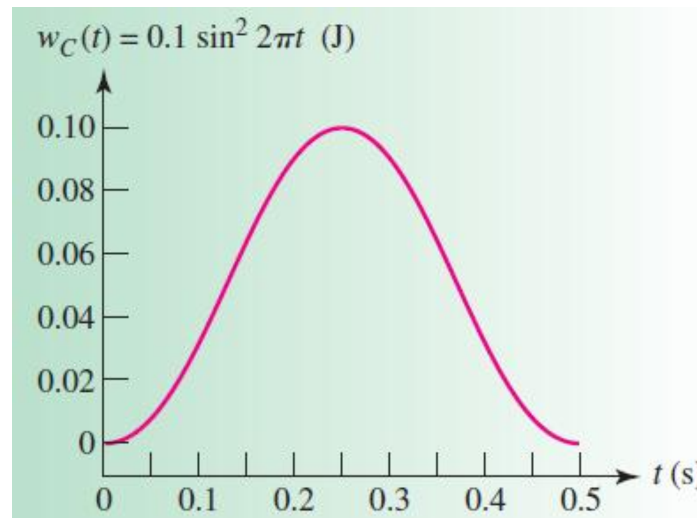
Find the maximum energy stored in the capacitor and the energy dissipated in the resistor over the interval 0 < t < 0.5 s.



$$w_C = \frac{1}{2} C \cdot v^2$$

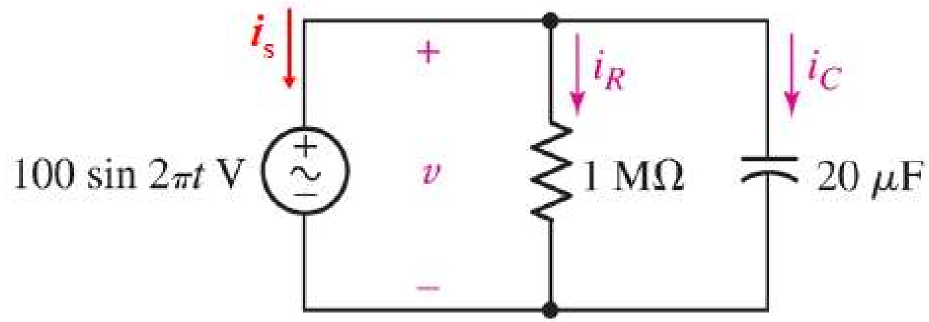
$$w_C = \frac{1}{2}(20\ \mu\text{F}) \cdot \{100 \sin(2\pi t)\ \text{V}\}^2$$

$$= 100 \sin^2(2\pi t)\ \text{mJ}$$



Example 7.3 (Cont.)

$$i_C = C \cdot \frac{dv}{dt} = (20\,\mu) \cdot \frac{d}{dt} 100 \sin(2\pi t)$$
$$= (2\text{ m}) \cdot 2\pi \cdot \cos(2\pi t) = 4\pi \cdot \cos(2\pi t)\text{ mA}$$



$$i_R = \frac{v}{R} = \frac{100 \sin(2\pi t)\text{ V}}{10^6\ \Omega} = 0.1 \sin(2\pi t)\text{ mA}$$

$$i_s + i_R + i_C = 0$$

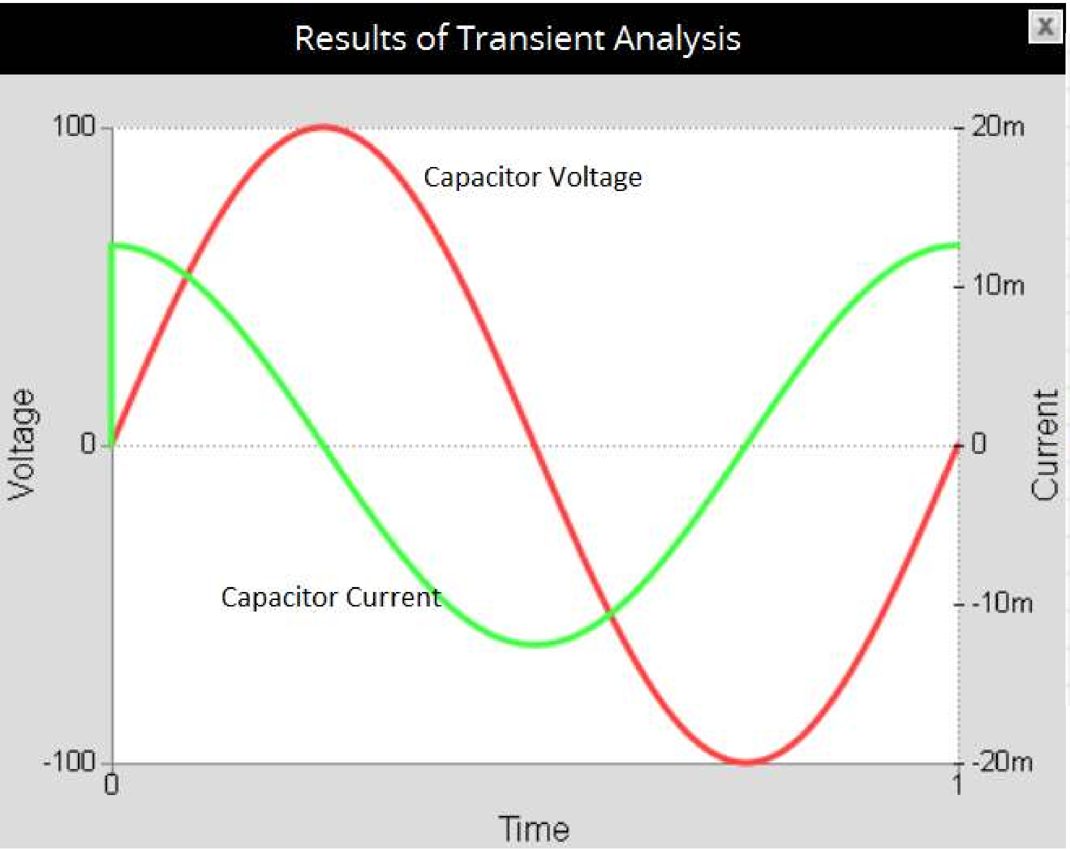
$$i_s = -0.1 \sin(2\pi t) - 4\pi \cdot \cos(2\pi t)\text{ mA}$$

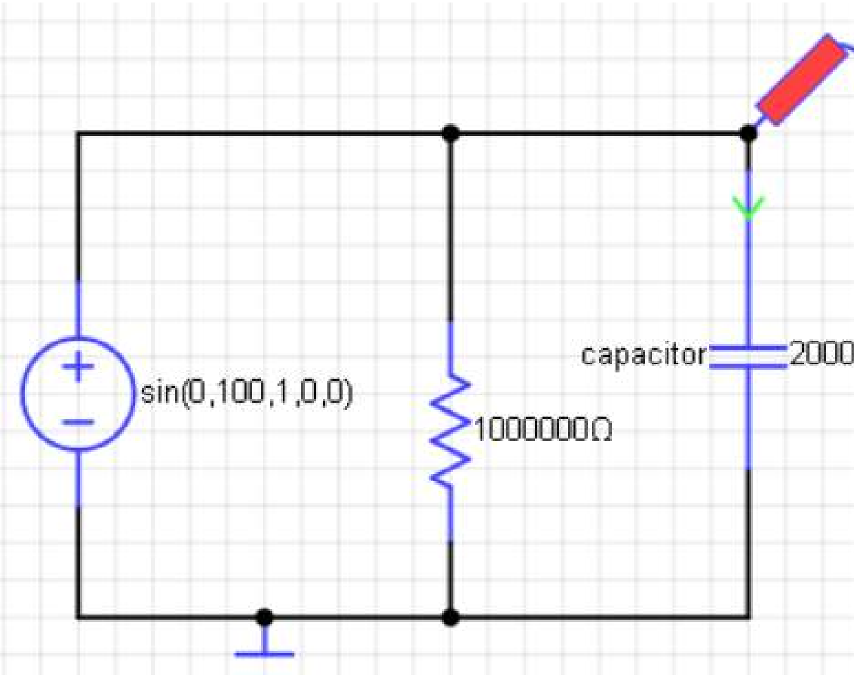
$$p_R = i_R^2 R = (10^{-8})(10^6) \sin^2 2\pi t$$

so that the energy dissipated in the resistor between 0 and 0.5 s is

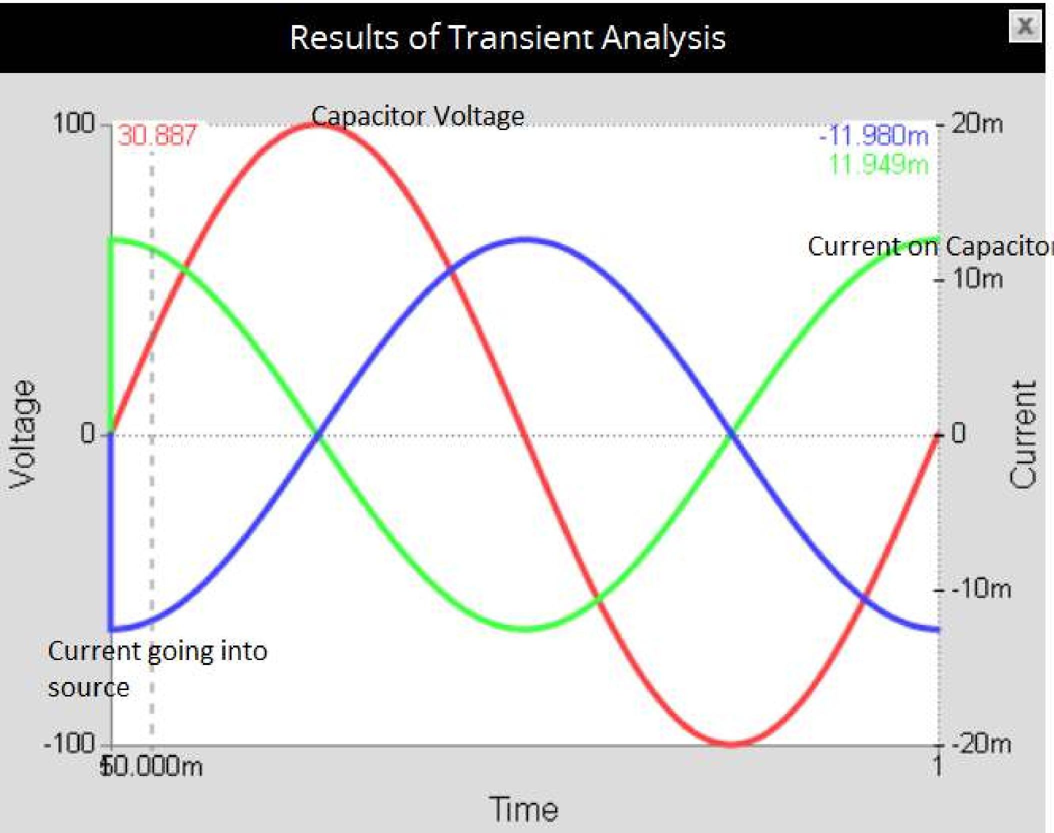
$$w_R = \int_0^{0.5} p_R\, dt = \int_0^{0.5} 10^{-2} \sin^2 2\pi t\, dt \qquad \text{J}$$

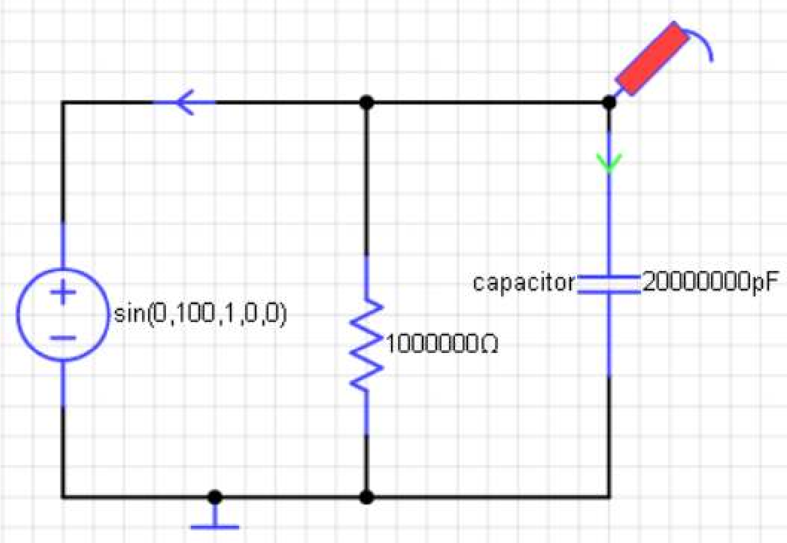
Example 7.3 (Cont.)

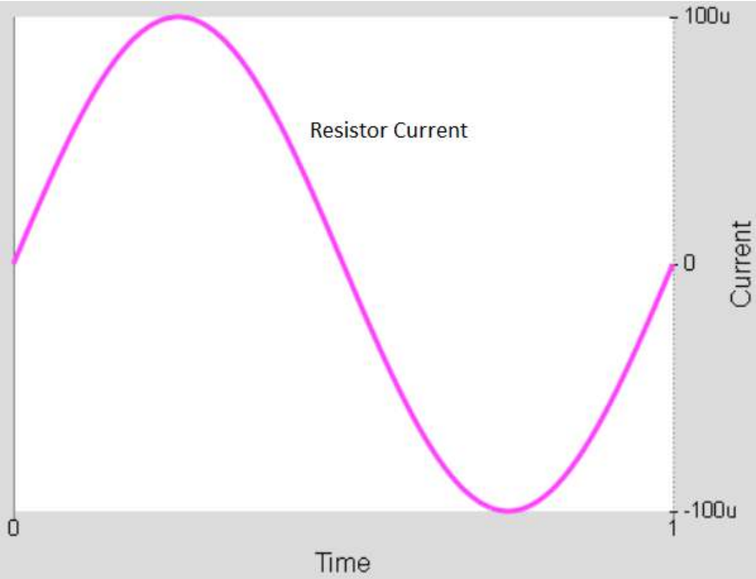




Example 7.3 (Cont.)





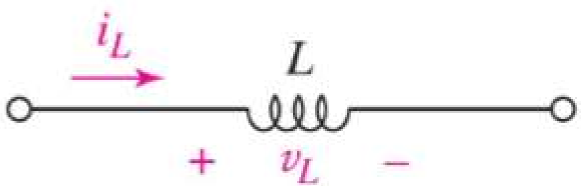


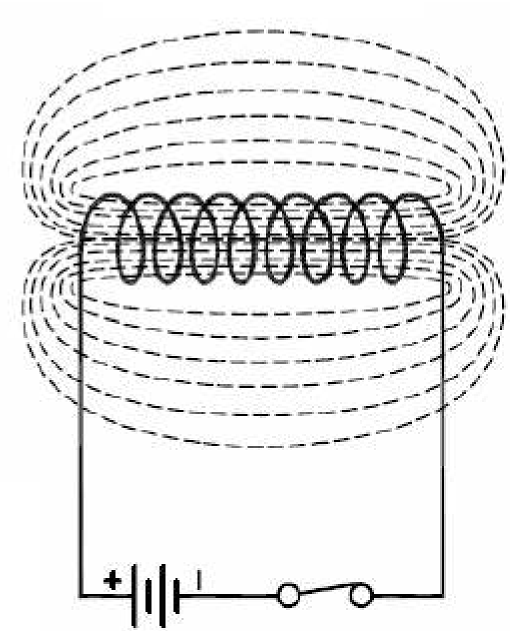
Important Characteristics of an Ideal Capacitor

1. There is no current through a capacitor if the voltage across it is not changing with time. A capacitor is therefore an ***open circuit*** *to* ***DC***.
2. A finite amount of energy can be stored in a capacitor even if the current through the capacitor is zero, such as when the voltage across it is constant.
3. It is impossible to change the voltage across a capacitor by a finite amount in zero time, as this requires an infinite current through the capacitor.
4. A capacitor never dissipates energy, but only stores it. Although this is true for the *mathematical model, it is not true for a physical* capacitor due to finite resistances associated with the dielectric as well as the packaging.

Inductance

An **inductor** is a linear circuit element which stores energy in the **magnetic field** in the space between current-carrying wires occupied by a material with permeability μ .





- *charged* by applying current (for a finite amount of time, from another source) to its terminals

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(\tau)\cdot d\tau + i(t_0)$$

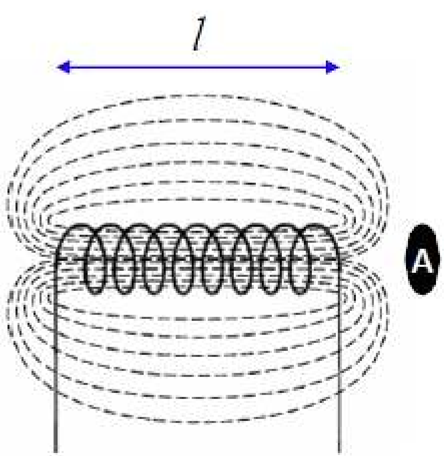
- *discharged* when it provides current (for a finite amount of time, to a circuit) from its terminals

Inductor Current & Voltage

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(\tau)\cdot d\tau + i(t_0)$$

same equation

$$v(t) = L\frac{di(t)}{dt}$$



$$L = \mu\frac{N^2 A}{l}$$

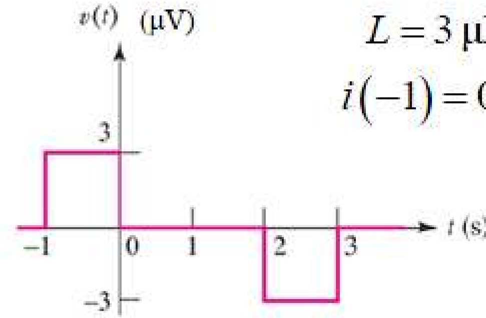
N = # of turns

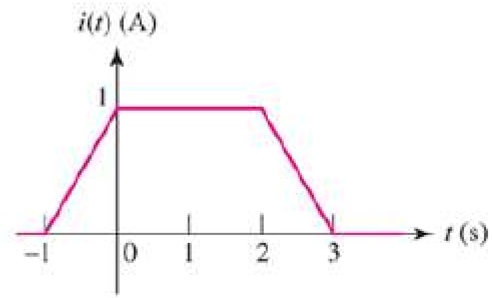
μ = permeability

unit of inductance = henry
1 H = 1 V-s / A

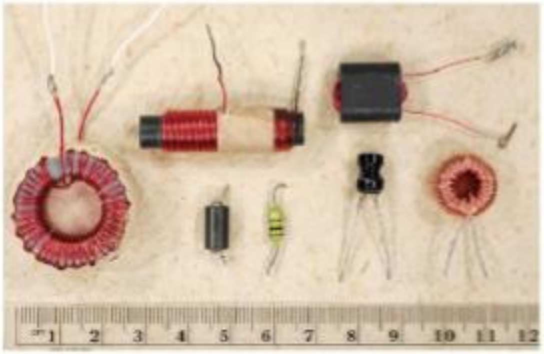
$L = 3\ \mu\text{H}$

$i(-1) = 0\ \text{A}$



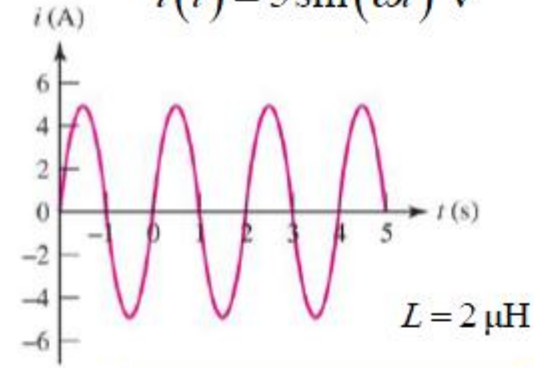


Inductor Characteristics

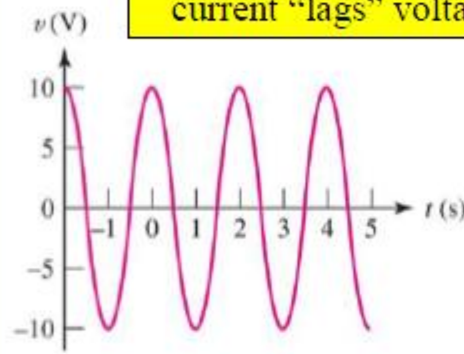


$$v = L\frac{di}{dt}$$

$$i(t) = 5\sin(\omega t)\ \mathrm{V}$$



voltage "leads" current,
current "lags" voltage



$$v(t) = 10\cos(\omega t)\ \mu\mathrm{A}$$

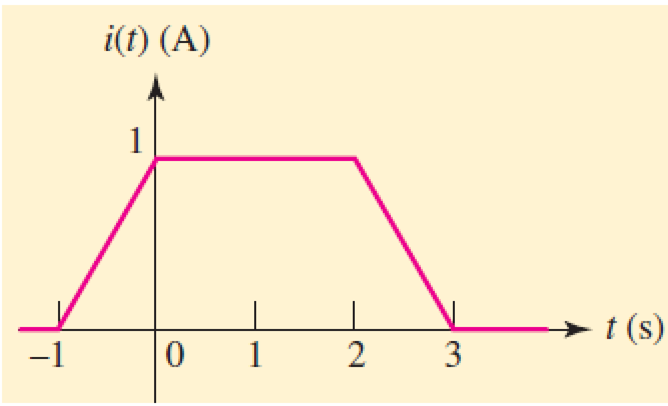
similar to the capacitor,
but the relationships
between voltage &
current are reversed

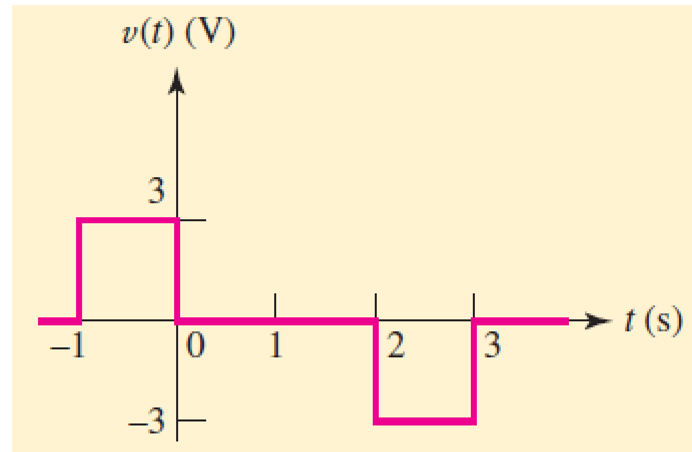
$$v_L = L\frac{di_L}{dt}$$

$$i_C = C\frac{dv_C}{dt}$$

Example 7.4

- Given the waveform of the current in a 3 H inductor as shown in below Figure. Determine the inductor voltage and sketch it.





$$v = 3\frac{di}{dt}$$

t < -1 sec   i = 0 ampere

0 sec < t < 2 sec   i = 1 ampere

t > 3 sec   i = 0 ampere

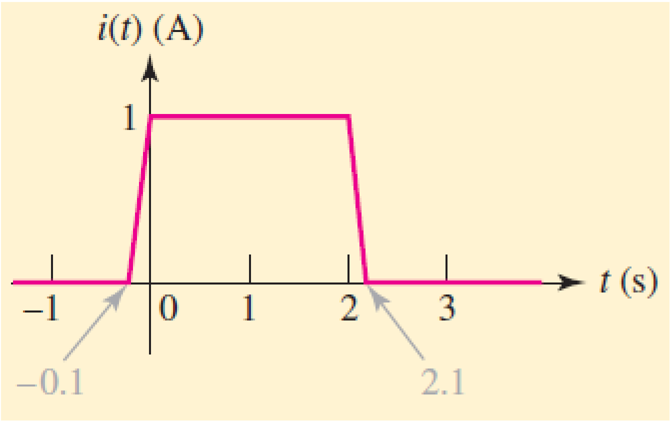
the current is constant, *v* = 0

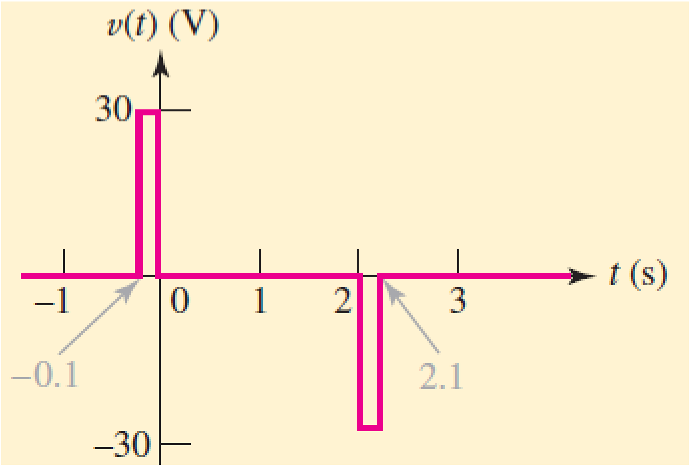
-1 sec <= t <= 0 sec di/dt = 1 A/s
v = 3 volt

2 sec <= t <= 3 sec di/dt = -1 A/s
v = -3 volt

Example 7.5

• Given the waveform of the current in a 3 H inductor as shown in below Figure. Determine the inductor voltage and sketch it.



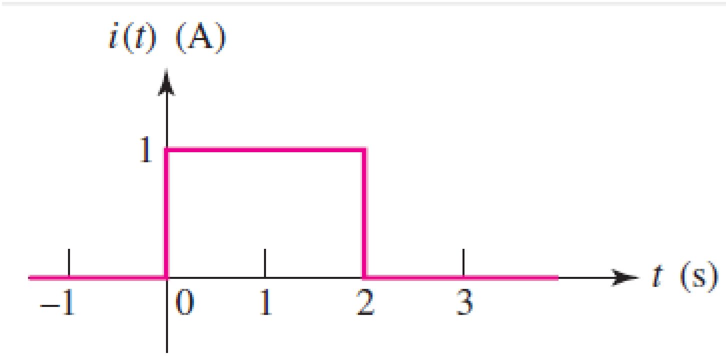


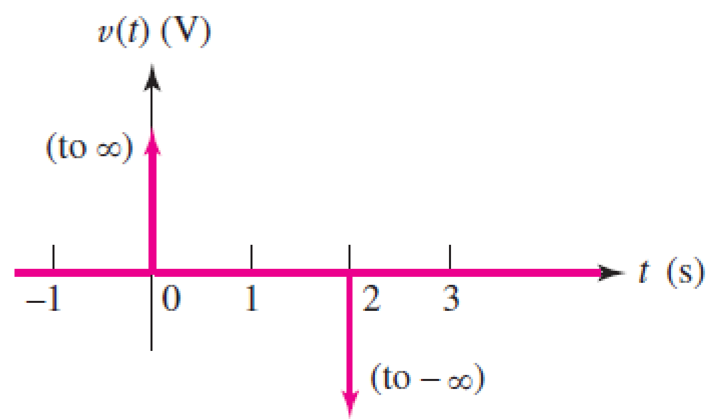
• The intervals for the rise and fall have decreased to 0.1 s. Thus, the magnitude of each derivative will be 10 times larger.

• It is interesting to note that the area under each voltage pulse is 3 V× s.

Example 7.5 (Cont.)

- A further decrease in the rise and fall times of the current waveform will produce a proportionally larger voltage magnitude, but only within the interval in which the current is increasing or decreasing.
- An abrupt change in the current will cause the infinite voltage “spikes” (each having an area of 3 V×s)





- This is useful in the **ignition system of some automobiles**, where the current through the spark coil is interrupted by the distributor and the arc appears across the spark plug.

Example 7.6

- The voltage across a 2 H inductor is known to be 6 *cos 5t* V. Determine the resulting inductor current if $i(t = -\pi/2) = 1$A.

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(\tau)\cdot d\tau + i(t_0)$$

$$i(t) = \frac{1}{2}\int_{t_0}^{t} 6\cos 5t'\, dt' + i(t_0)$$

$$\begin{aligned} i(t) &= \frac{1}{2}\left(\frac{6}{5}\right)\sin 5t - \frac{1}{2}\left(\frac{6}{5}\right)\sin 5t_0 + i(t_0) \\ &= 0.6\sin 5t - 0.6\sin 5t_0 + i(t_0) \end{aligned}$$

If we accept $t_0 = -\pi/2$

$$i(t) = 0.6\sin 5t - 0.6\sin(-2.5\pi) + 1$$

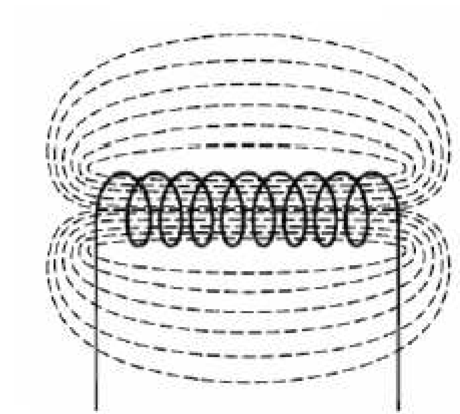
$$i(t) = 0.6\sin 5t + 1.6$$

Inductor Energy Storage

$$p = i \cdot v$$

$$v = L\frac{di}{dt}$$

$$
\begin{aligned}
w &= \int_0^t p(\tau)\cdot d\tau \\
&= \int_{v(0)}^{v(t)} i \cdot L\frac{di}{d\tau}\cdot d\tau \\
&= L\cdot\int_0^t i \cdot di \\
&= L\cdot\frac{1}{2}i^2\Big|_0^t \\
&= L\cdot\frac{1}{2}\left[i(t)^2 - i(0)^2\right] \longrightarrow \boxed{w = \frac{1}{2}L\cdot i^2}
\end{aligned}
$$

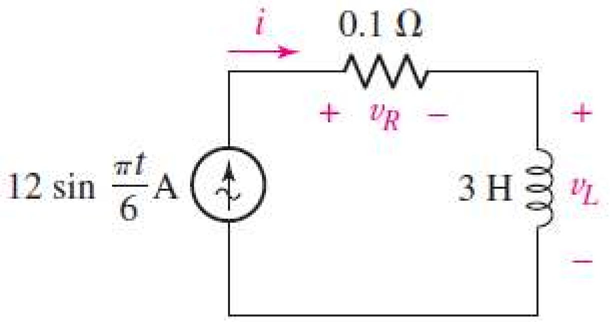


magnetic energy stored in an inductor with current i through its coils:

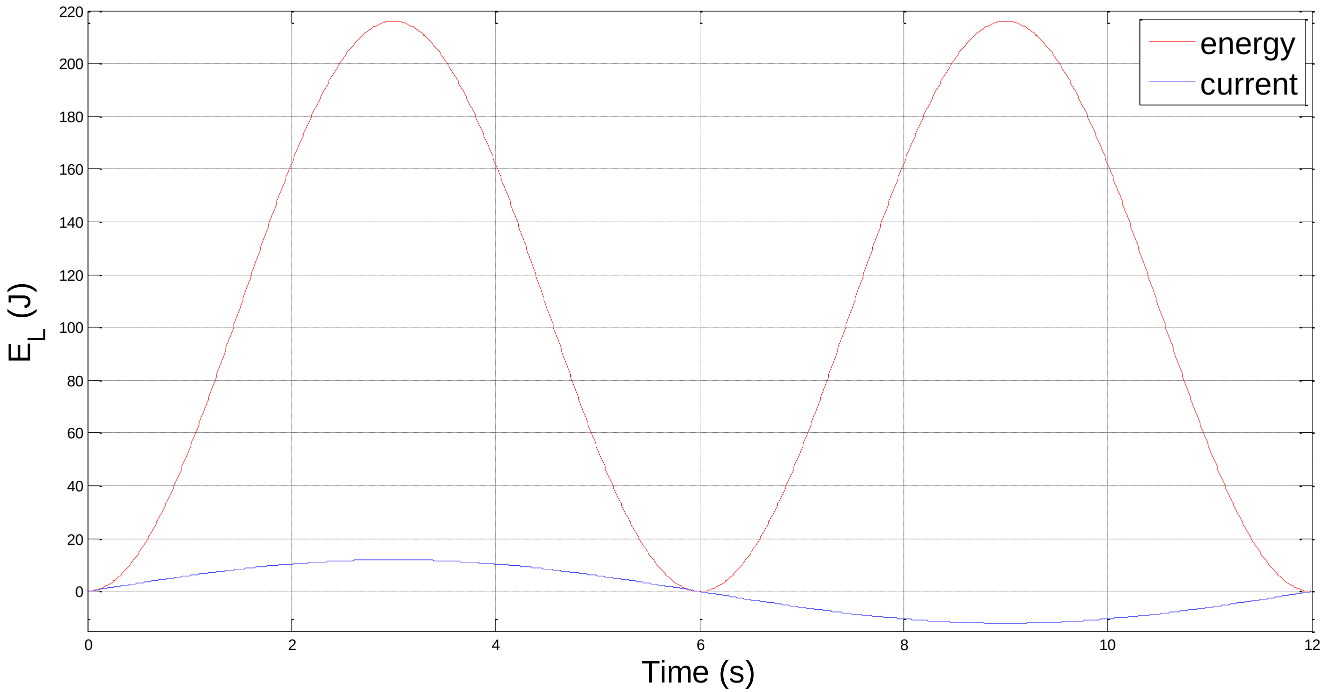
$$\boxed{w = \frac{1}{2}L\cdot i^2}$$

Example 7.7

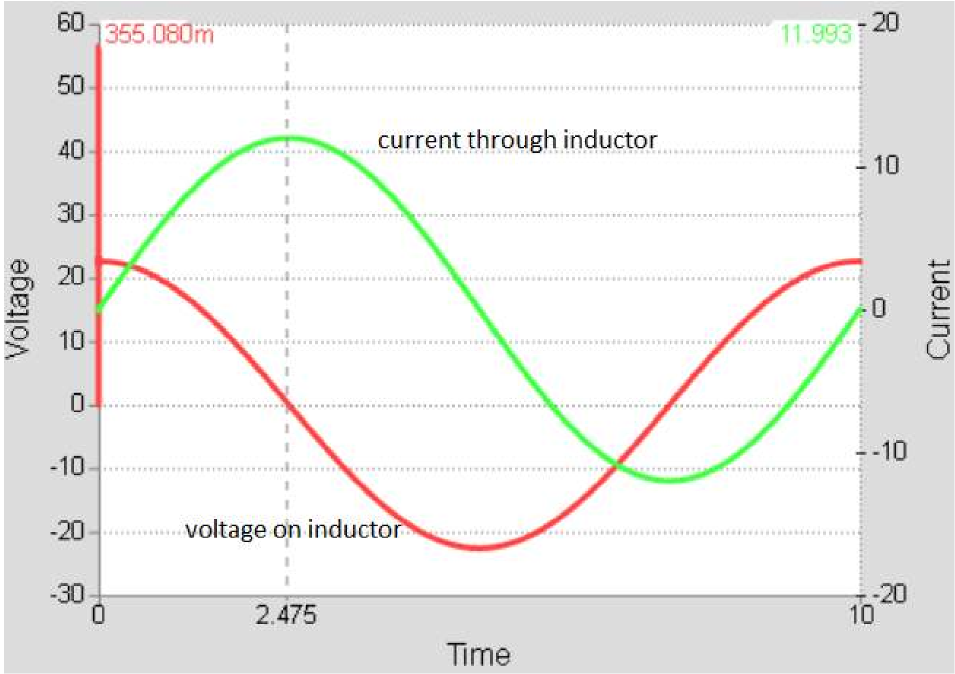
- Find the maximum energy stored in the and calculate how much energy is dissipated in the resistor in the time during which the energy is being stored in, and then recovered from, the inductor.

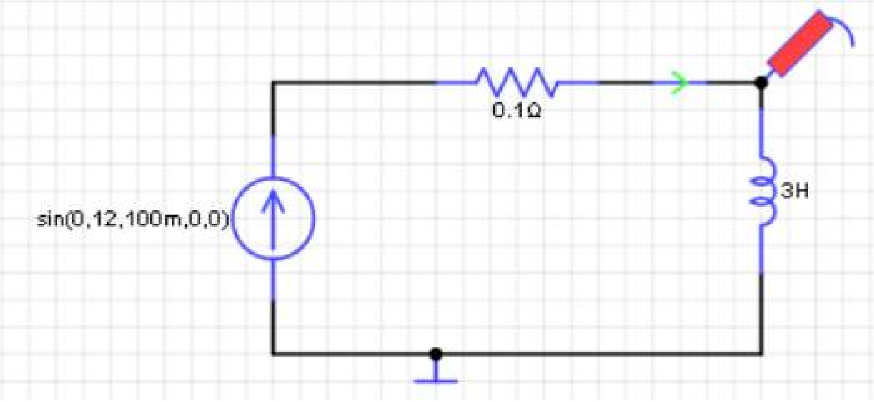


$$w_L = \frac{1}{2}Li^2 = 216 \sin^2 \frac{\pi t}{6} \qquad \text{J}$$



Example 7.7 (Cont.)





Current and voltage has 90 degrees phase shift. Voltage **is leading** the current.

Example 7.7 (Cont.)

- The power dissipated in the resistor is easily found as

$$p_R = i^2 R = 14.4 \sin^2 \frac{\pi t}{6} \qquad \text{W}$$

- the energy converted into heat in the resistor within this 6 s interval is

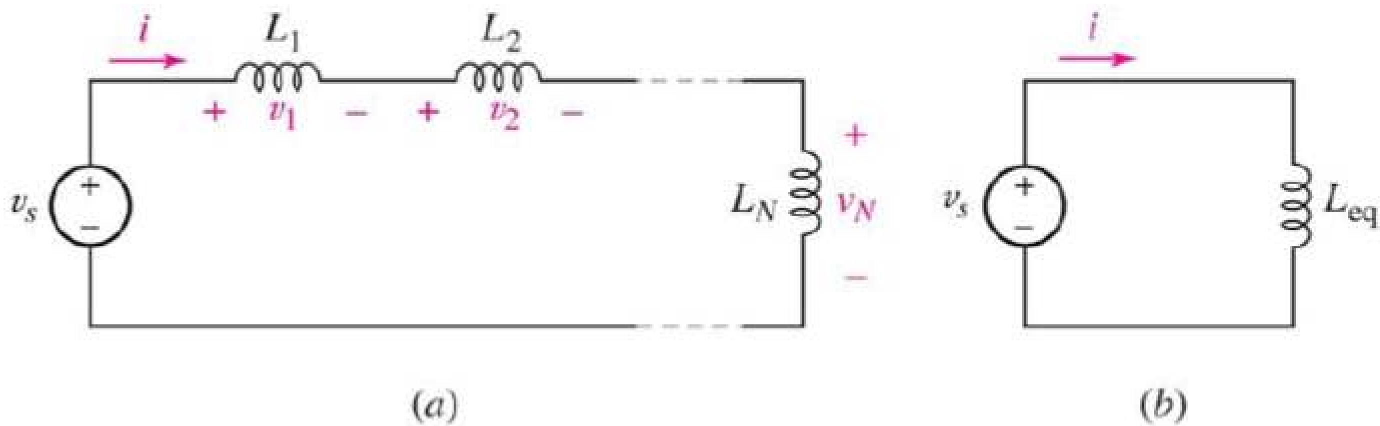
$$w_R = \int_0^6 p_R \, dt = \int_0^6 14.4 \sin^2 \frac{\pi}{6} t \, dt$$

$$w_R = \int_0^6 14.4 \left(\frac{1}{2}\right) \left(1 - \cos \frac{\pi}{3} t\right) dt = 43.2 \text{ J}$$

Important Characteristics of an Ideal Inductor

1. There is no voltage across an inductor if the current through it is not changing with time. An inductor is therefore a ***short circuit*** to dc.
2. A finite amount of energy can be stored in an inductor even if the voltage across the inductor is zero, such as when the current through it is constant.
3. It is impossible to change the current through an inductor by a finite amount in zero time, for this requires an infinite voltage across the inductor.
4. The inductor never dissipates energy, but only stores it. Although this is true for the *mathematical model, it is not true for a physical* inductor due to series resistances.

Equivalent Series Inductance



$$-v_s + \sum_{n=1}^{N} v_n = 0 \qquad v_n = L_n \frac{di}{dt}$$

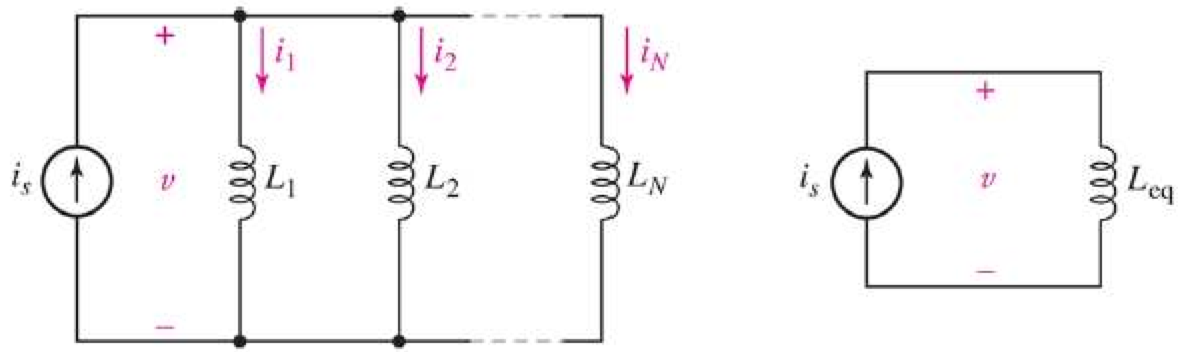
$$v_s = \sum_{n=1}^{N} L_n \cdot \frac{di}{dt} = \frac{di}{dt} \sum_{n=1}^{N} L_n$$

$$v_s = L_{eq} \frac{di}{dt}$$

$$\frac{di}{dt} \sum_{n=1}^{N} L_n = L_{eq} \frac{di}{dt}$$

$$\boxed{\sum_{n=1}^{N} L_n = L_{eq}}$$

Equivalent Parallel Inductance



(a) (b)

$$-i_s + \sum_{n=1}^{N} \left\{ \frac{1}{L_n} \int_{t_0}^{t} v \cdot dt' + i_n(t_0) \right\} = 0 \qquad v = L_n \frac{di_n}{dt}$$

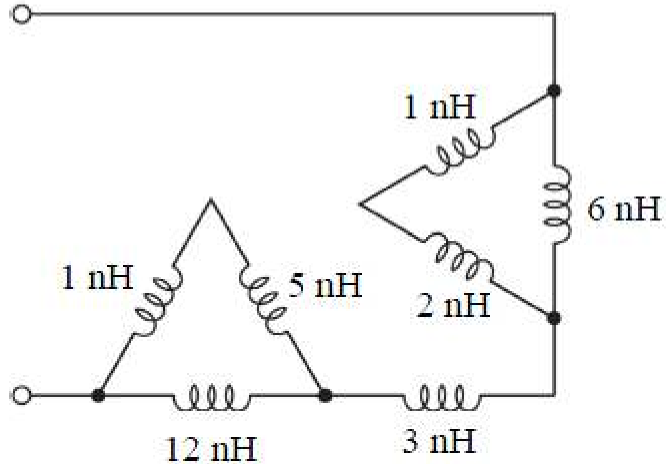
$$i_s = \sum_{n=1}^{N} \frac{1}{L_n} \left\{ \int_{t_0}^{t} v \cdot dt' \right\} + \sum_{n=1}^{N} i_n(t_0)$$

$$i_s = \frac{1}{L_{eq}} \int_{t_0}^{t} v \cdot dt' + i_s(t_0)$$

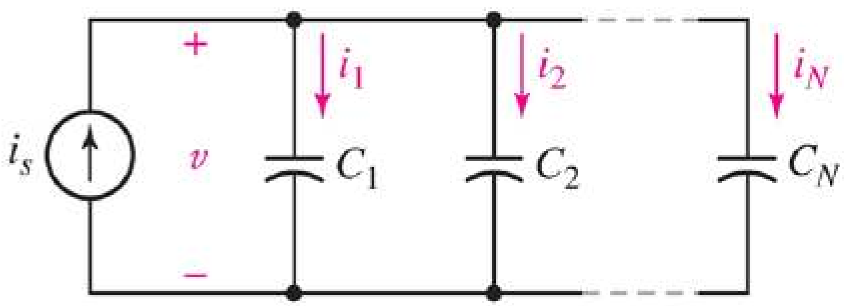
$$\sum_{n=1}^{N} \frac{1}{L_n} \left\{ \int_{t_0}^{t} v \cdot dt' \right\} + \sum_{n=1}^{N} i_n(t_0) = \frac{1}{L_{eq}} \int_{t_0}^{t} v \cdot dt' + i_s(t_0) \Rightarrow \boxed{\sum_{n=1}^{N} \frac{1}{L_n} = \frac{1}{L_{eq}}}$$

Example

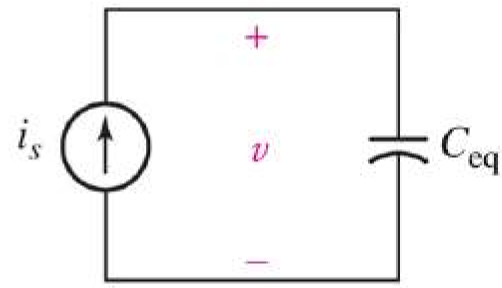
Determine the equivalent inductance (L_{eq}) *at the open-circuit terminals.*



Equivalent Parallel Capacitance



(a)



(b)

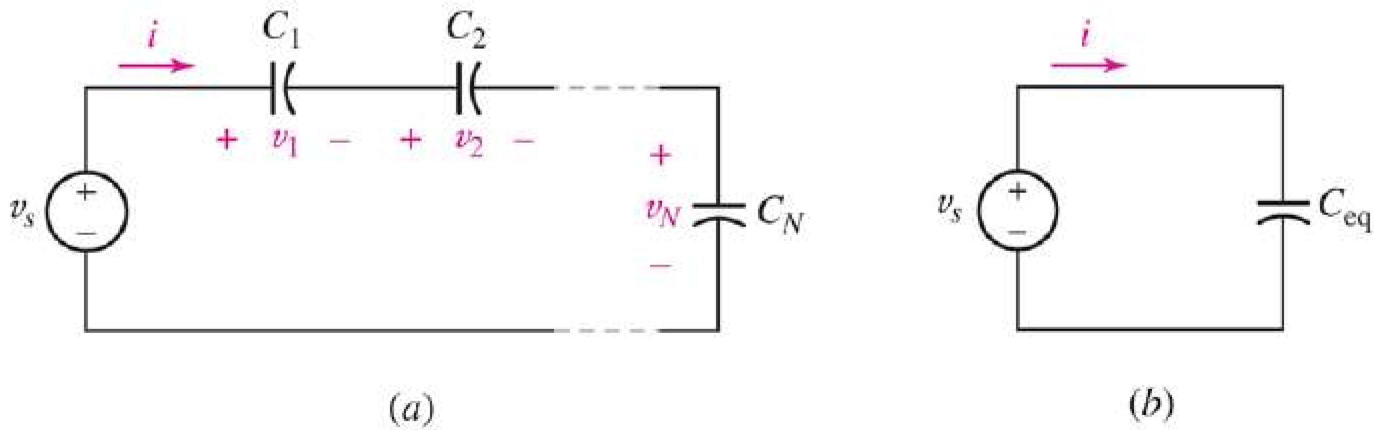
$$-i_s + \sum_{n=1}^{N} i_n = 0 \qquad i_n = C_n \frac{dv}{dt}$$

$$i_s = \sum_{n=1}^{N} C_n \cdot \frac{dv}{dt} = \frac{dv}{dt} \sum_{n=1}^{N} C_n$$

$$i_s = C_{eq} \frac{dv}{dt}$$

$$\frac{dv}{dt} \sum_{n=1}^{N} C_n = C_{eq} \frac{dv}{dt} \quad \Rightarrow \quad \boxed{\sum_{n=1}^{N} C_n = C_{eq}}$$

Equivalent Series Capacitance



$$-v_s + \sum_{n=1}^{N} \left\{ \frac{1}{C_n} \int_{t_0}^{t} i \cdot dt' + v_n(t_0) \right\} = 0 \qquad i = C_n \frac{dv_n}{dt}$$

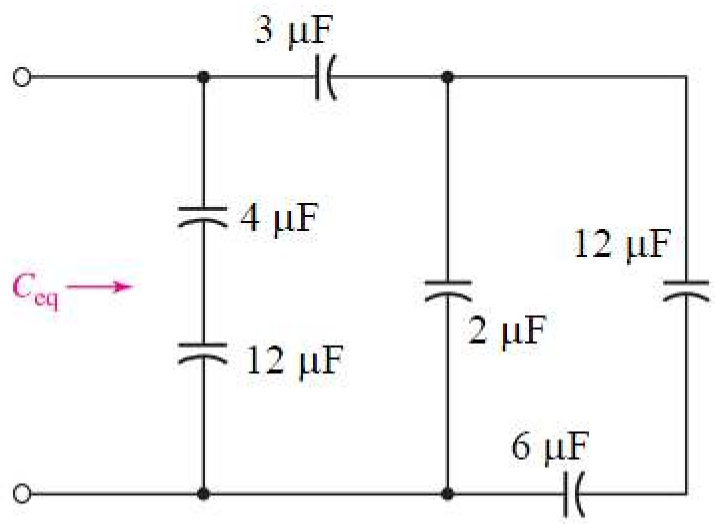
$$v_s = \sum_{n=1}^{N} \frac{1}{C_n} \left\{ \int_{t_0}^{t} i \cdot dt' \right\} + \sum_{n=1}^{N} v_n(t_0)$$

$$v_s = \frac{1}{C_{eq}} \int_{t_0}^{t} i \cdot dt' + v_s(t_0)$$

$$\sum_{n=1}^{N} \frac{1}{C_n} \left\{ \int_{t_0}^{t} i \cdot dt' \right\} + \sum_{n=1}^{N} v_n(t_0) = \frac{1}{C_{eq}} \int_{t_0}^{t} i \cdot dt' + v_s(t_0) \Rightarrow \boxed{\sum_{n=1}^{N} \frac{1}{C_n} = \frac{1}{C_{eq}}}$$

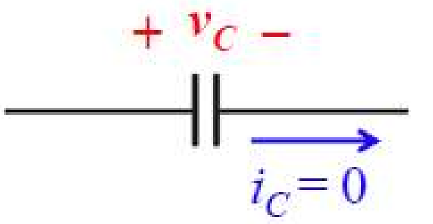
Example

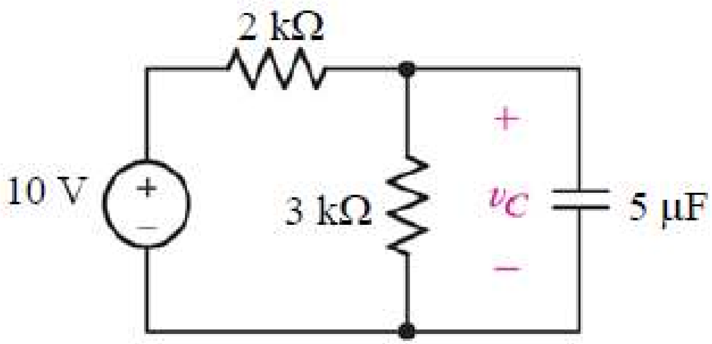
Determine the equivalent capacitance C_{eq}.

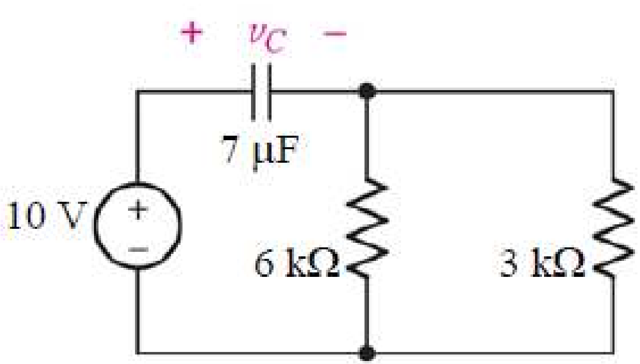


DC Capacitor Circuits (Cont.)

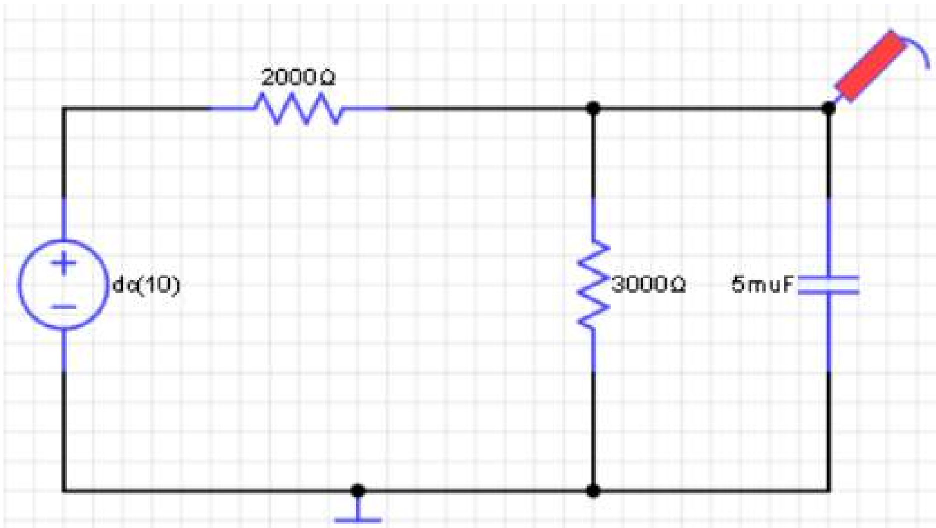
A **capacitor** that has been sitting in a DC circuit for a long time acts as an **open circuit**.

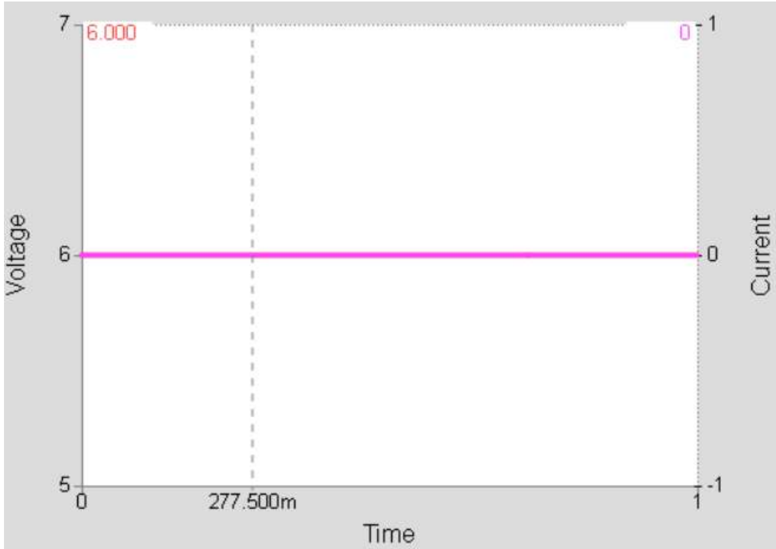


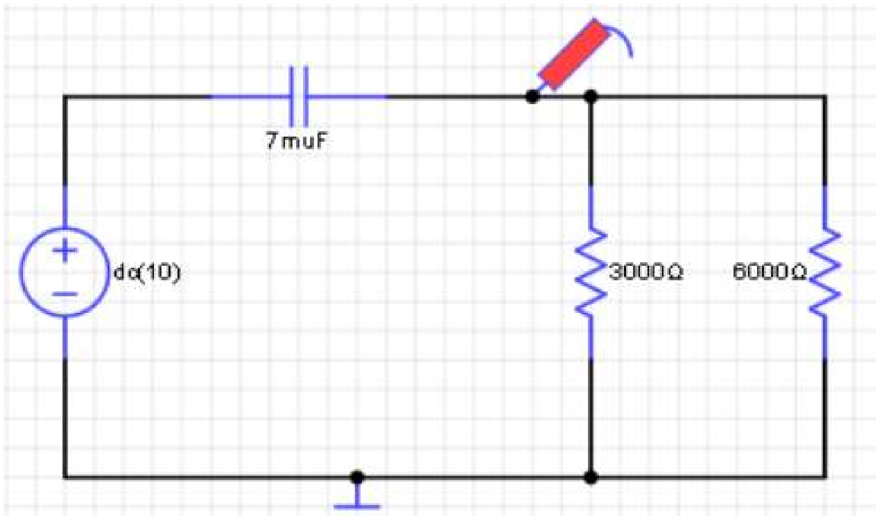


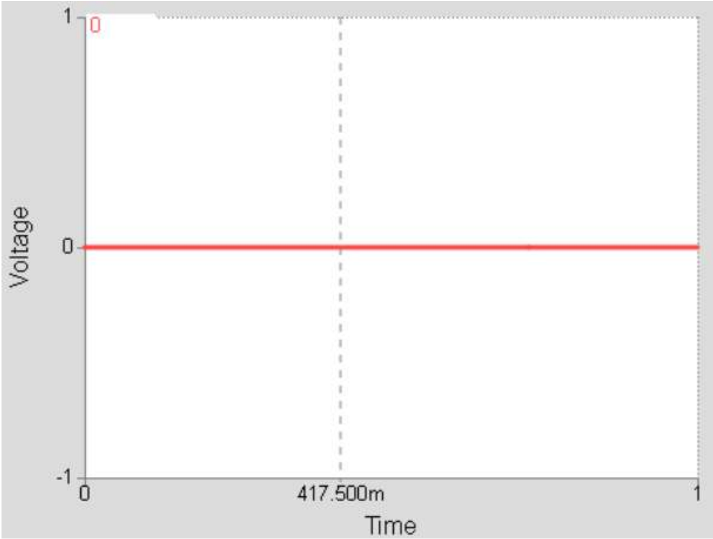


DC Capacitor Circuits



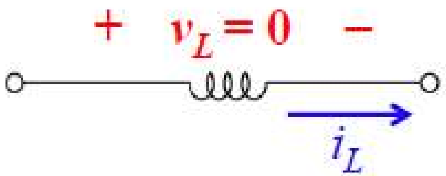


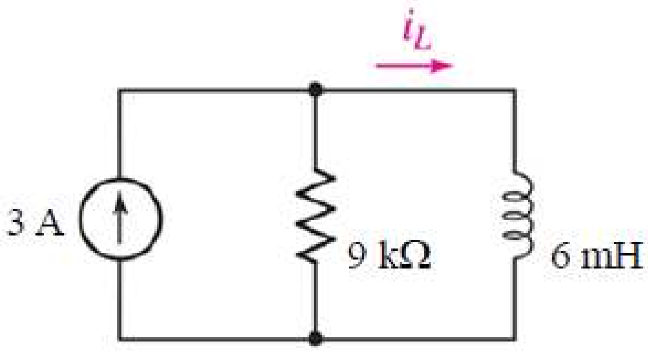


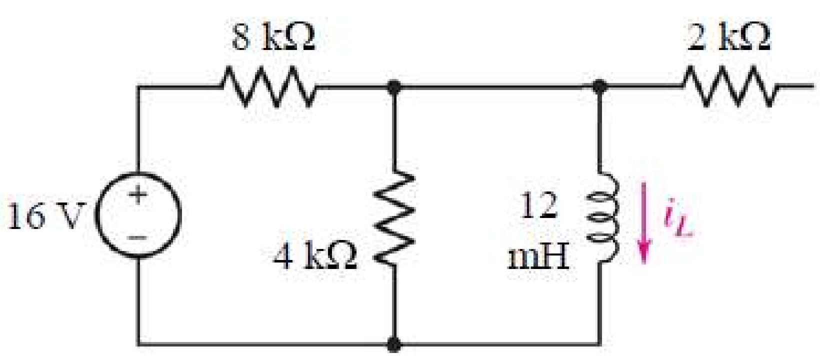


DC Inductor Circuits

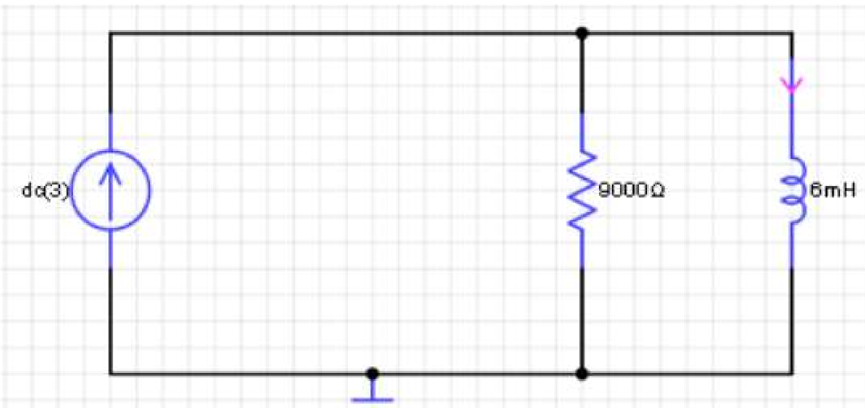
An **inductor** that has been sitting in a DC circuit for a long time acts as a **short circuit**.

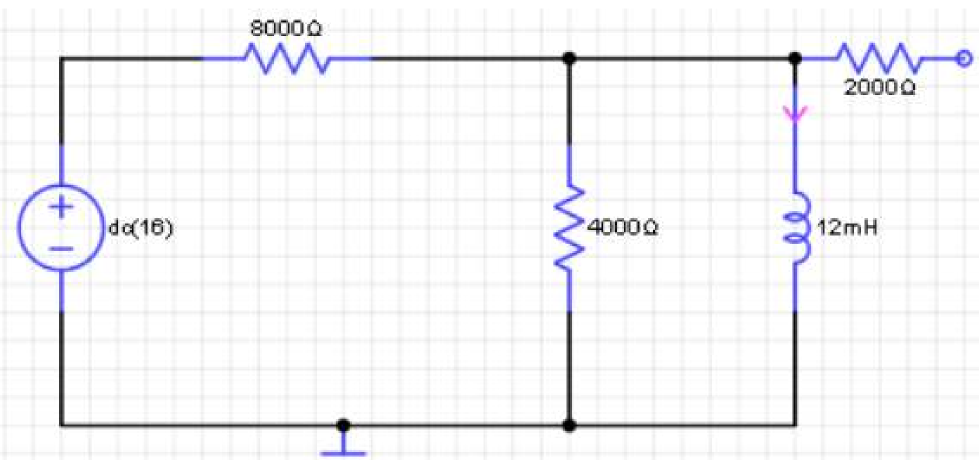


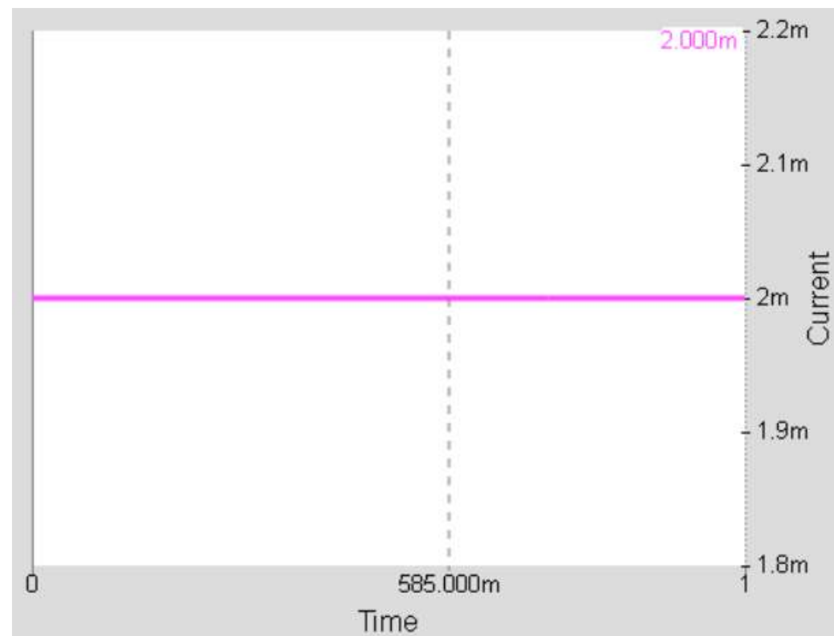




DC Inductor Circuits (Cont.)







CONSEQUENCES OF LINEARITY

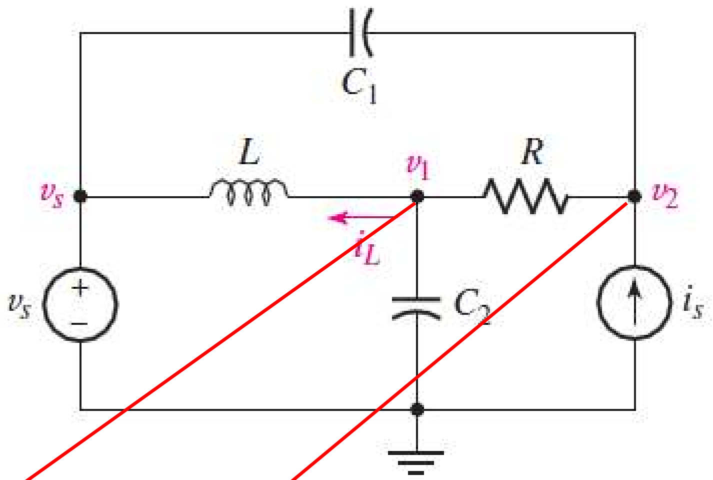
• In RLC circuits, we can safely apply **Kirchhoff's laws**, we can apply them in writing a set of equations that are both sufficient and independent. They will be constant coefficient **linear integro-differential** equations.

•The principle of **superposition** can be applied to RLC circuits, but it should be emphasized that initial inductor currents and capacitor voltages must be treated as independent sources in applying the superposition principle; each initial value must take its turn in being rendered inactive.

• All linear circuits that contain any combinations of independent voltage and current sources, linear dependent voltage and current sources, and linear resistors, inductors, and capacitors can be analyzed with the use of **Thévenin's and Norton's theorems**.

EXAMPLE 7.9

- Write appropriate nodal equations for the circuit.



KCL at v_1 node

$$\frac{1}{L}\int_{t_0}^{t}(v_1 - v_s)\,dt' + i_L(t_0) + \frac{v_1 - v_2}{R} + C_2\frac{dv_1}{dt} = 0$$

KCL at v_2 node

$$C_1\frac{d(v_2 - v_s)}{dt} + \frac{v_2 - v_1}{R} - i_s = 0$$

When we rewrite these equations

$$\frac{v_1}{R} + C_2\frac{dv_1}{dt} + \frac{1}{L}\int_{t_0}^{t} v_1\,dt' - \frac{v_2}{R} = \frac{1}{L}\int_{t_0}^{t} v_s\,dt' - i_L(t_0)$$

$$-\frac{v_1}{R} + \frac{v_2}{R} + C_1\frac{dv_2}{dt} = C_1\frac{dv_s}{dt} + i_s$$

linear integro-differential equations

EXAMPLE 7.9 (Cont.)

- We will not attempt the solution of integro-differential equations here. It is worthwhile pointing out, however, that when the voltage forcing functions are **sinusoidal functions** of time, it will be possible to define a voltage-current ratio (called ***impedance***) or a current-voltage ratio **(**called ***admittance*****)** for each of the three passive elements.

- The factors operating on the two node voltages in the preceding equations will then become simple multiplying factors, and the equations will be **linear algebraic equations** once again.

- These we may solve by **determinants or a simple elimination of variables** as before.

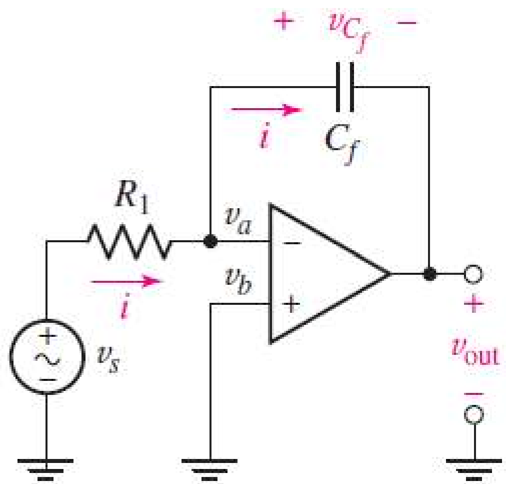
SIMPLE OP AMP CIRCUITS WITH CAPACITORS AND INDUCTORS

- In amplifier circuits based on the ideal Op-Amp. In almost every case, we found that the output was related to the input voltage by some combination of resistance ratios.

- If we replace one or more of these resistors with a capacitor, it is possible to obtain some interesting circuits in which the output is proportional to either the **derivative or integral** of the input voltage.

- Such circuits find widespread use in practice. For example, a velocity sensor can be connected to an Op-Amp circuit that provides a signal proportional to the acceleration, or an output signal can be obtained that represents the total charge incident on a metal electrode during a specific period of time by simply integrating the measured current.

AN INTEGRATOR CIRCUIT



Performing nodal analysis at the inverting input,

$$0 = \frac{v_a - v_s}{R_1} + i$$

We can relate the current i to the voltage across the capacitor,

$$i = C_f \frac{dv_{C_f}}{dt}$$

resulting in

$$0 = \frac{v_a - v_s}{R_1} + C_f \frac{dv_{C_f}}{dt}$$

Invoking ideal op amp rule 2, we know that $v_a = v_b = 0$, so

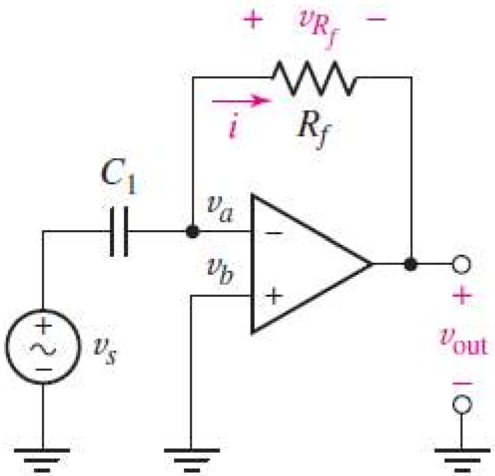
$$0 = \frac{-v_s}{R_1} + C_f \frac{dv_{C_f}}{dt}$$

Integrating and solving for v_{out}, we obtain

$$v_{C_f} = v_a - v_{out} = 0 - v_{out} = \frac{1}{R_1 C_f} \int_0^t v_s \, dt' + v_{C_f}(0)$$

$$v_{out} = -\frac{1}{R_1 C_f} \int_0^t v_s \, dt' - v_{C_f}(0)$$

EXAMPLE 7.10



We begin by writing a nodal equation at the inverting input pin, with $v_{C_1} \triangleq v_a - v_s$:

$$0 = C_1 \frac{dv_{C_1}}{dt} + \frac{v_a - v_{\text{out}}}{R_f}$$

Invoking ideal op amp rule 2, $v_a = v_b = 0$. Thus,

$$C_1 \frac{dv_{C_1}}{dt} = \frac{v_{\text{out}}}{R_f}$$

Solving for v_{out},

$$v_{\text{out}} = R_f C_1 \frac{dv_{C_1}}{dt}$$

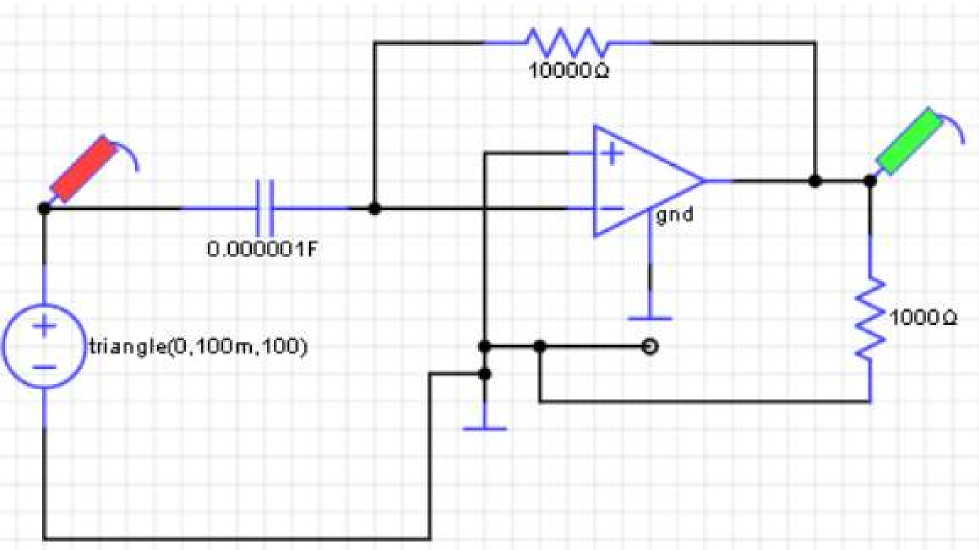
Since $v_{C_1} = v_a - v_s = -v_s$,

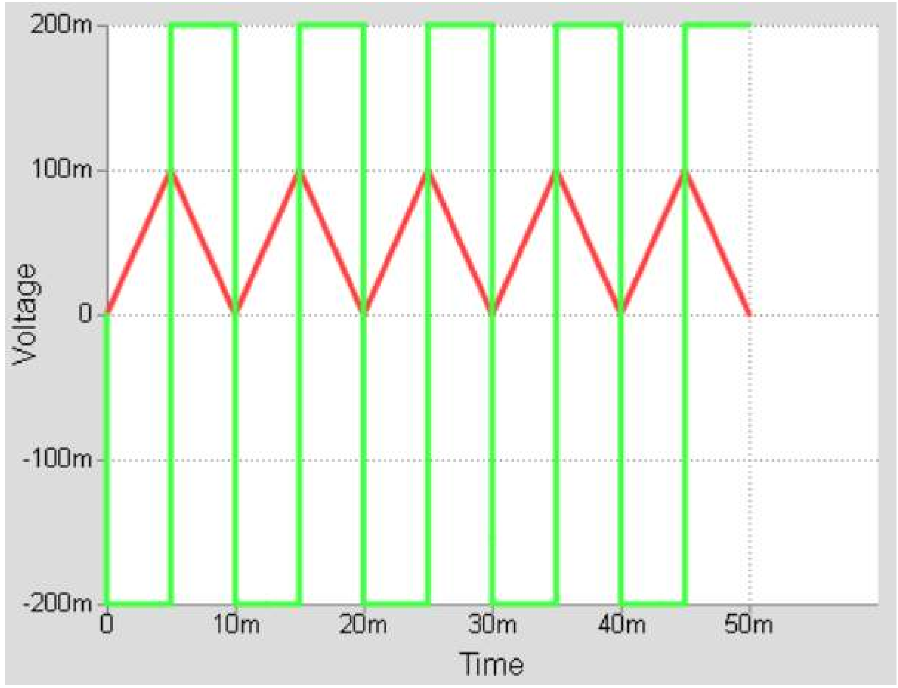
$$v_{\text{out}} = -R_f C_1 \frac{dv_s}{dt}$$

EXAMPLE 7.10 (A real Implementation)

C = 1 µF
R = 10 KΩ
RC = 0.01 (sec)

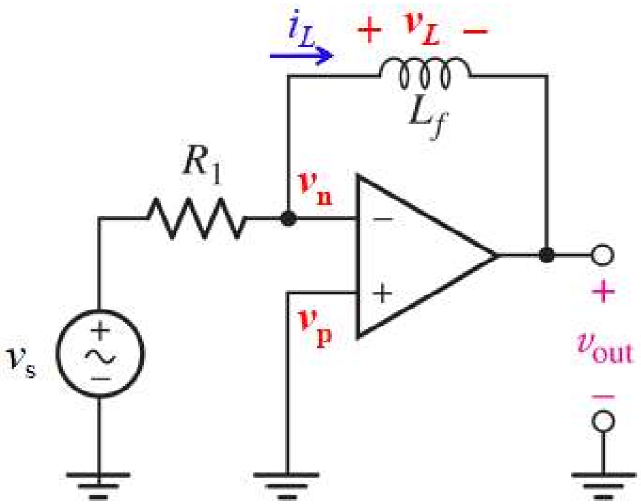
dv/dt = 100 mV/5 msec = 20 V/sec





PRACTICE

Derive an expression for v_{out} *in terms of* v_s *for the circuit.*



$$v_n = v_p \quad , \quad i_n = i_p = 0$$

$$\frac{v_s - v_n}{R_1} - i_L = 0$$

$$\frac{v_s - v_n}{R_1} - \frac{1}{L_f}\int_{t_0}^{t} v_L \cdot dt' + i_n(t_0) = 0$$

$$\frac{v_s - v_n}{R_1} - \frac{1}{L_f}\int_{t_0}^{t} (v_n - v_{out}) \cdot dt' + i_n(t_0) = 0$$

$$\int_{t_0}^{t} v_{out} \cdot dt' + L_f \cdot i_n(t_0) = -\frac{L_f}{R_1} v_s$$

$$v_{out} = \frac{d}{dt}\left(-\frac{L_f}{R_1} v_s\right) =$$

Summary & Review

- **Capacitors**
  - store energy in *electric* fields: the energy is charged up by applying transient current

$$v_C(t) = \frac{1}{C}\int_{t_0}^{t} i_C(t')\cdot dt' + v_C(t_0) \quad \Rightarrow \quad i_C = C\frac{dv_C}{dt} \qquad w_C = \frac{1}{2}C\cdot v_C^{\,2}$$

  - *C* sitting for a long time in a DC circuit → **open circuit**
  - capacitors in *parallel* are added like resistors in *series* (and vice versa)

- **Inductors**
  - store energy in *magnetic* fields: the energy is charged up by applying transient voltage

$$i_L(t) = \frac{1}{L}\int_{t_0}^{t} v_L(t')\cdot dt' + i_L(t_0) \quad \Rightarrow \quad v_L = L\frac{di_L}{dt} \qquad w_L = \frac{1}{2}L\cdot i_L^{\,2}$$

  - *L* sitting for a long time in a DC circuit → **short circuit**
  - inductors in *series* are added like resistors in *series* (and vice versa)

- **Linear Circuit Elements**
  - KVL, KCL, nodal/mesh analysis, superposition, Thevenin/Norton **all apply**