CENG 222
Statistical Methods for Computer Engineering

Week 6

Chapter 5

Computer Simulations and Monte Carlo Methods

Outline

- Generation of random numbers from specific distributions
  - Discrete distributions
  - Continuous distributions
- Chebyshev's inequality (3.3.7)
- Solving problems by Monte Carlo methods
  - Estimating probabilities
  - Estimating means and standard deviations

Uniform Random Numbers

- Tables of random numbers
- Pseudo-random number generators
  - Long sequences of random-looking numbers
  - Seed: starting location in the sequence
    - May use system time as seed
- Many systems provide standard uniform random number generators
  - Uniform(0,1)
- Question: Can we generate random numbers from any distribution using Uniform(0,1) rvs?

Bernoulli

- Let U be Uniform(0,1)
- $X = \begin{cases} 1, & \text{if } U < p \\ 0, & \text{if } U \geq p \end{cases}$
- $P\,(\text{ success }) = P\,(\,U < p\,) = p$

Binomial

- Sum of n independent Bernoulli variables.
- Example

```
n = 20; p = 0.68;
U = rand(n,1);
        % generates an nx1 vector
        % of uniform random numbers
X = sum(U < p);
```

Geometric

- Iterate and count the number of generated rvs until first success
- Example:

```
p = 0.16; X = 1;
while rand > p;
    X = X+1;
end;
X
```

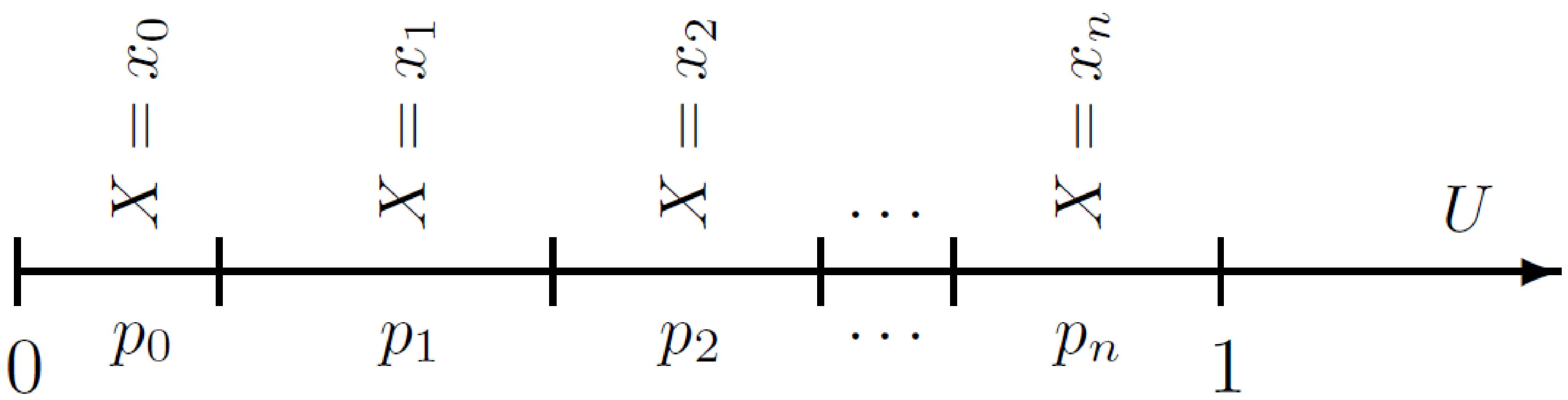
Negative Binomial

- Generate k independent Geometric(p) random numbers and sum them to get a NegativeBinomial(k,p) number.
- Example:

```
p = 0.16; X = 0; i = 0;
while i < k;
    G = 1;
    while rand > p;
          G = G+1;
    end;
    X = X+G;
end;
```

- How efficient is generating a Binomial, a Geometric, or a Negative Binomial random number?

Arbitrary discrete distributions



Algorithm 5.1

1. Divide the interval [0,1] into subintervals A_i as follows:
   – $A_i = [p_0+p_1+..+p_{i-1}, p_0+p_1+..+p_i)$
2. Generate U, a standard uniform number
3. If U belongs to A_i then $X = x_i$

- How efficient is this method?
  – If you want to generate many Xs, efficiency is important.
    - O(n), O(log n), O(1)?
    - Check out the *Alias Method,* if you need an O(1) method.

Poisson

- Using Algorithm 5.1 to generate Poisson numbers.
- Example:

```
lambda 5;
U = rand; i = 0;
F = exp(-lambda);    % F(0)
while (U >= F);
        i = i + 1;
        F = F+exp(-lambda)*lamda^i/gamma(i+1);
end;
X = i;
```

Inverse transform method

- Theorem: $U = F_X(X)$ is Uniform(0,1)
- Proof:
  - Note that the standard uniform cdf is $F_U(u) = u$ (i.e., $F'_U(u) = f_U(u) = 1$). We will try to show this fact using the given definition of $U = F_X(X)$
  - $F_U(u) = P(U \leq u)$

$$
\begin{aligned}
&= P(\, F_X(X) \leq u \,) \\
&= P(\, X \leq F_X^{-1}(u) \,) \\
&= F_X(\, F_X^{-1}(u) \,) \\
&= u
\end{aligned}
$$

Inverse transform method

- If $U = F_X(X)$ then $X = F_X^{-1}(U)$
- The method:
  - Generate a uniform random number
  - Plug it in F_X^{-1} to generate X (i.e. solve for X).
- Example 5.10 (Exponential):
  - $F_X(X) = 1 - e^{-\lambda X} = U$
  - $\rightarrow X = -\frac{1}{\lambda}\ln(1-U)$
  - Can also use $X = -\frac{1}{\lambda}\ln(U)$ since 1-U is also Uniform(0,1).

Inverse transform method

- Difficult to use if the inverse of the cdf is not easy to compute
- For example, for discrete distributions, $F_X^{-1}(U)$ does not exist. $U = F_X(X)$ has no roots, because X (hence $F_X(X)$) is finite and countable; whereas U is continuous.
- Therefore, for discrete rvs, we use the inverse method with a slight modification:
  - $X = \min\{x \in S$ such that $F(x) > U\}$ where S is the set of possible values of X.

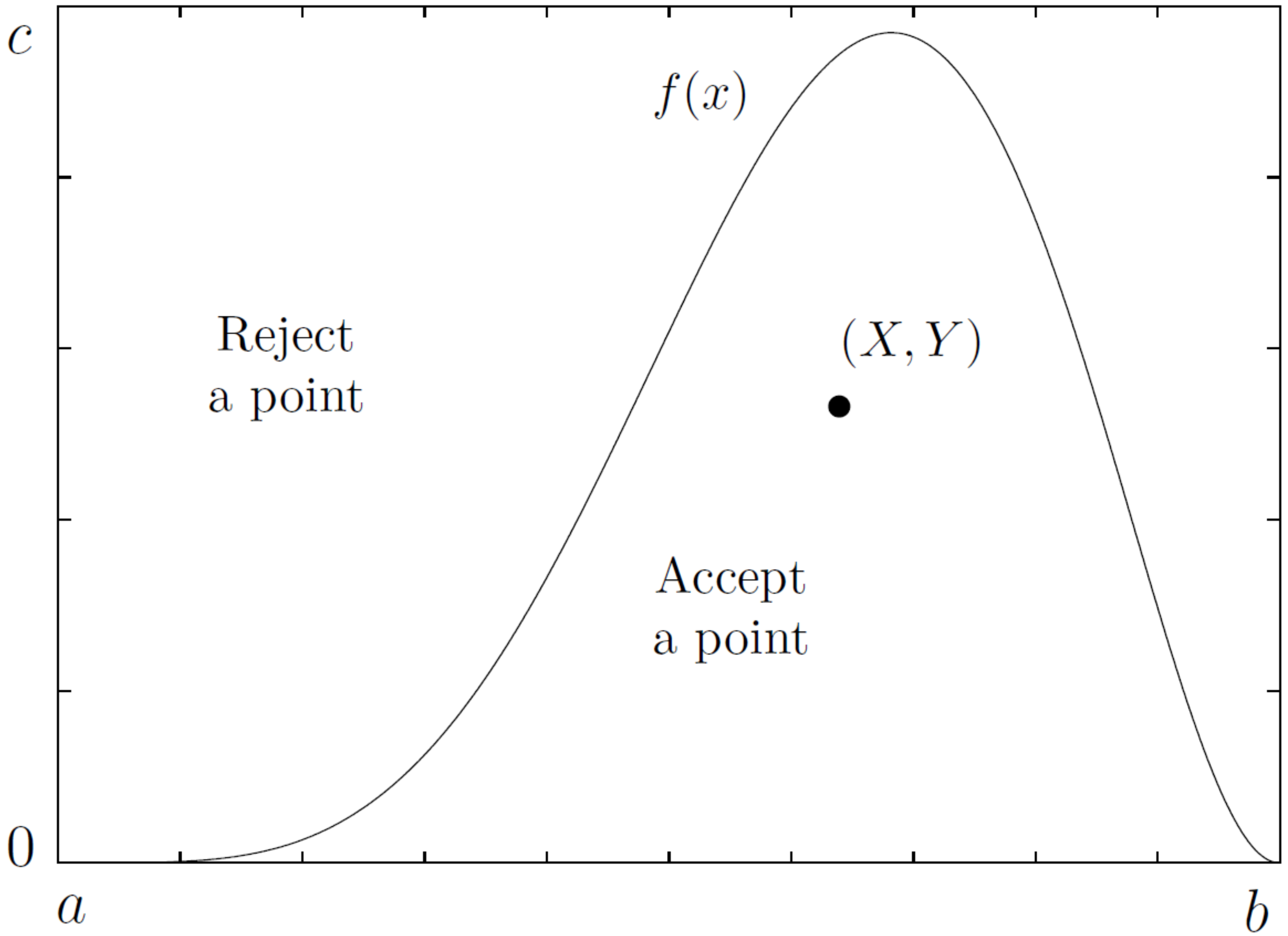
Example 5.12

- Using the inverse transform method for generating Geometric variables
- $X = \left\lceil \frac{\ln(1-U)}{\ln(1-p)} \right\rceil$
- The geometric variable is the ceiling of the exponential variable with $\lambda = -\ln(1-p)$
  - Exponential is the continuous analogue of geometric
  - Both have the memoryless property.

Rejection method

- When the cdf is difficult to solve for X and the pdf f_X is available, the rejection method can be used to generate random numbers from f_X.
- Idea:
  – Generate 2D uniform coordinates (X,Y) in the bounding box of f_X and if $Y \leq f_X(X)$ output X.

Rejection method

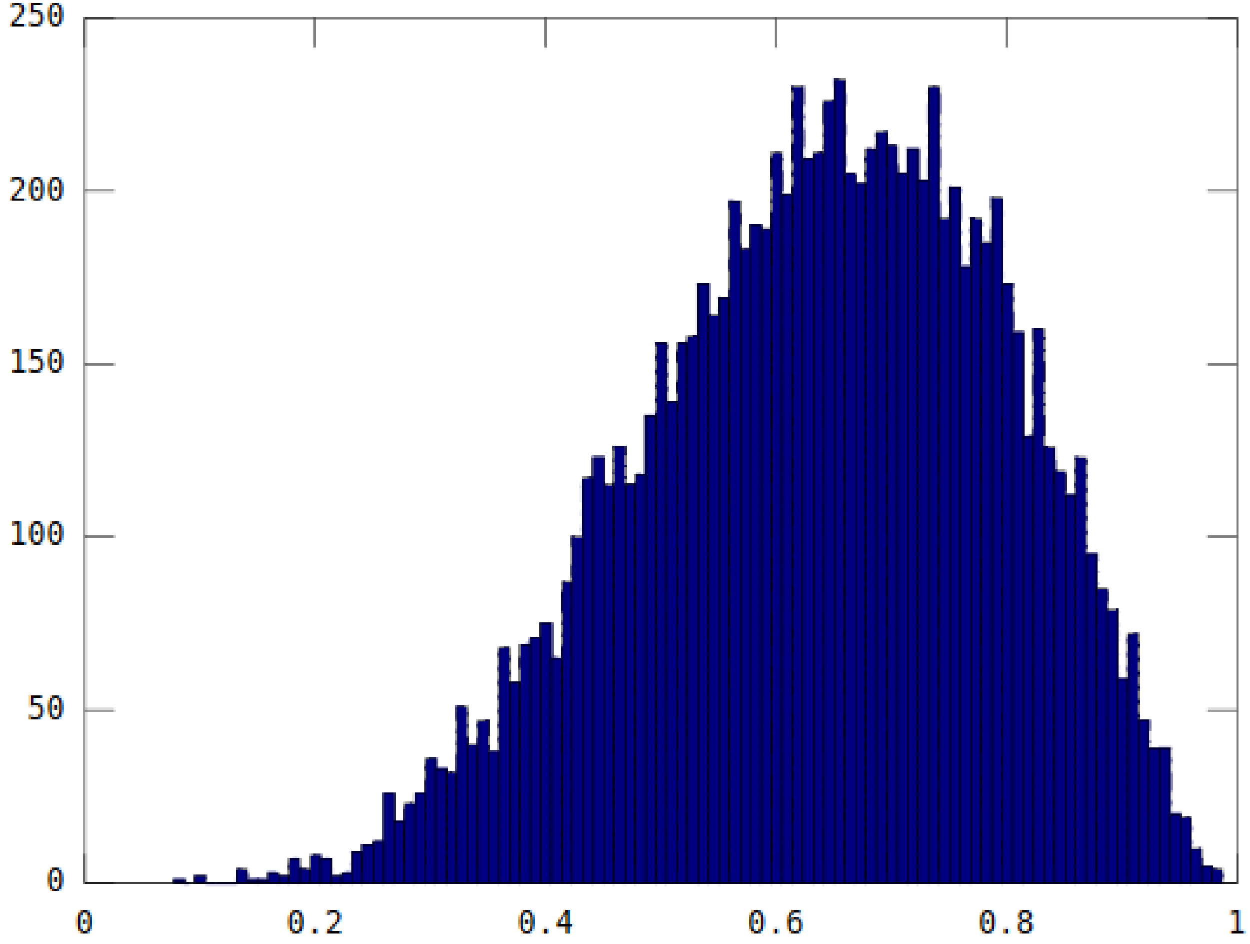


Example

- The figure in the previous slide is the pdf of Beta(α=5.5,β=3.1)
  - $f_X = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ for $0 \le x \le 1$
- Bounding box: $m = 2.5$, $s = 0$, $t = 1$.

```
a=5.5; b=3.1; s=0; t=1; m=2.5;
X = 0; Y = m;
F = gamma(a+b)/gamma(a)/gamma(b)*X^(a-1)*(1-X)^(b-1);
while (Y > F);
        U = rand; V = rand;
        X = s+(t-s)*U; Y = m*V;
        F = … % same as above;
end; X
```

Example



Monte Carlo methods

- Generate many random variables from a distribution and estimate probabilities, means, standard deviations, etc. by simulating what happens in the long run.
- Question: How many numbers needed for acceptable results?
  - i.e., What will be the "size" of the Monte Carlo experiment?
  - Revisit Chebyshev's Inequality

Chebyshev's Inequality (3.3.7)

- For any distribution with expectation μ and variance σ^2 and for any positive ε
  - $P(|X - \mu| > \varepsilon) \leq \left(\frac{\sigma}{\varepsilon}\right)^2$
  - In other words: any random variable *X* from the distribution is within ε distance of the μ with probability of at least $1 - (\sigma/\varepsilon)^2$

Estimating probabilities

- The probability $p=P(X \in A)$ can be estimated as $\hat{p}$ by generating N random numbers and computing the proportion of random numbers that are in A.
- How accurate is the estimator?
  - What is $\mathbf{E}(\hat{p})$ and $\mathrm{Std}(\hat{p})$?
  - The number of X_i that are in A among the N generated random numbers is Binomial(N,p) with expectation Np and variance $Np(1-p)$

$\rightarrow$ $\mathbf{E}(\hat{p}) = p$ (unbiased estimator)

$\mathrm{Std}(\hat{p}) = \sqrt{\frac{p(1-p)}{N}}$ the error in $\hat{p}$ decreases with $1/\sqrt{N}$

How large should *N* be?

- Given the error ε and the probability, α, to exceed this error limit
- If an intelligent guess p^* on the value of p is available:
  - $N \geq p^*(1-p^*)\left(\frac{z_{\alpha/2}}{\varepsilon}\right)^2$
- If not:
  - $N \geq 0.25\left(\frac{z_{\alpha/2}}{\varepsilon}\right)^2$

Example 5.14

How large should *N* be?

- If the *N* returned by these equations are not large enough for Binomial approximation, we may use Chebyshev's inequality:
  - If an intelligent guess p^* on the value of p is available:
    - $N \geq \frac{p^*(1-p^*)}{\alpha\varepsilon^2}$
  - If not:
    - $N \geq \frac{1}{4\alpha\varepsilon^2}$

Estimating means and standard deviations

- $\bar{X} = \frac{1}{N}(X_1 + \cdots + X_N)$
  - also unbiased and its error decreases with $1/\sqrt{N}$
- $s^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})^2$
  - $1/N$-1 needed so that $\mathbf{E}(s^2) = \sigma^2$