BLM2041 Signals and Systems

Week 8

The Instructors:

Prof. Dr. Nizamettin Aydın
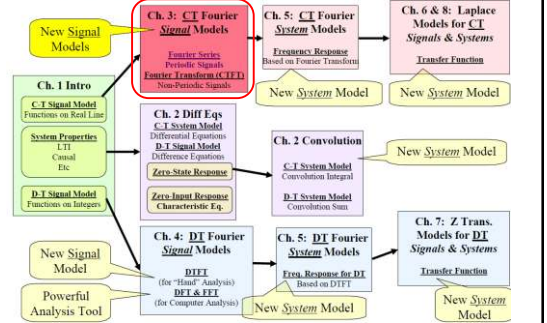
naydin@yildiz.edu.tr

Asist. Prof. Dr. Ferkan Yilmaz

ferkan@yildiz.edu.tr

Where are we now?

Fourier Transform

Recall: Fourier Series represents a periodic signal as a sum of sinusoids

or complex sinusoids $\quad e^{jk\omega_0 t}$

Note: Because the FS uses "harmonically related" frequencies $k\omega_0$, it can only create periodic signals

Q: Can we modify the FS idea to handle non-periodic signals?

A: Yes!!

What about $\quad x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_k t}$?

With *arbitrary* discrete frequencies… NOT harmonically related

This will give some non-periodic signals but not all signals of interest!!

The problem with this is that it cannot include all possible frequencies!

No matter how close we try to choose the discrete frequencies ω_k there are always some left out of the sum!!!

We need some way to include ALL frequencies!!

Fourier Transform

How about:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$$

Yes… this will work for any *practical* non-periodic signal!!

Called the "Fourier Integral" also, more commonly, called the "Inverse Fourier Transform"

Plays the role of c_k

Plays the role of $e^{jk\omega_0 t}$

Integral replaces sum because it can "add up over the *continuum* of frequencies"!

Okay… given $x(t)$ how do we get $X(\omega)$?

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

Called the "Fourier Transform" of $x(t)$

Note: $X(\omega)$ is complex-valued function of $\omega \in (-\infty, \infty)$

$|X(\omega)| \qquad \angle X(\omega)$

Need to use two plots to show it

Fourier Transform

Comparison of FT and FS

Fourier Series: Used for periodic signals

Fourier Transform: Used for non-periodic signals (although we will see later that it can also be used for periodic signals)

	Synthesis	Analysis
Fourier Series	$x(t) = \sum_{n=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ Fourier Series	$c_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}\, dt$ Fourier Coefficients
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$ Inverse Fourier Transform	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$ Fourier Transform

FS coefficients c_k are a complex-valued function of integer k

FT $X(\omega)$ is a complex-valued function of the variable $\omega \in (-\infty, \infty)$

Fourier Transform

Synthesis Viewpoints:

FS: $\quad x(t) = \sum_{n=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

$|c_k|$ shows how much there is of the signal at frequency $k\omega_0$

$\angle c_k$ shows how much phase shift is needed at frequency $k\omega_0$

We need two plots to show these

FT: $\quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$

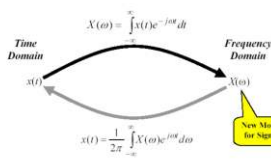$|X(\omega)|$ shows how much there is in the signal at frequency ω

$\angle X(\omega)$ shows how much phase shift is needed at frequency ω

We need two plots to show these

Fourier Transform

Slide 7

Fourier Transform Viewpoint
View FT as a transformation into a new "domain"

Time Domain $x(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Frequency Domain $X(\omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

New Model for Signal

$x(t)$ is the "time domain" description of the signal
$X(\omega)$ is the "frequency domain" description of the signal

Analogy: Looking at $X(\omega)$ is "like" looking at an x-ray of the signal- in the sense that an x-ray lets you see what is inside the object... shows what stuff it is made from.

In this sense: $X(\omega)$ shows what is "inside" the signal – it shows how much of each complex sinusoid is "inside" the signal

Note: $x(t)$ completely determines $X(\omega)$

$X(\omega)$ completely determines $x(t)$

There are some advanced mathematical issues that can be hurled at these comments... we'll not worry about them

Alternate Notations

1. $x(t) \leftrightarrow X(\omega)$
2. $X(\omega) = \mathfrak{F}\{x(t)\}$
$\Rightarrow \mathfrak{F}\{\ \}$ is an "operator on" $x(t)$ to give $X(\omega)$
3. $x(t) = \mathfrak{F}^{-1}\{X(\omega)\}$
$\Rightarrow \mathfrak{F}^{-1}\{\}$ is an "operator on" $X(\omega)$ to give $x(t)$

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Example: FT of a Rectangular pulse

τ = pulse width

Given: a rectangular pulse signal $p_\tau(t)$

Find: $P_\tau(\omega)$... the FT of $p_\tau(t)$

Note the Notational Convention: lower-case for time signal and corresponding upper-case for its FT

Recall: we use this symbol to indicate a rectangular pulse with width τ

Solution: (Here we'll directly do the integral... but later we'll use the "FT Table")

Note that

$$p_\tau(t) = \begin{cases} 1, & -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0, & otherwise \end{cases}$$

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Now apply the definition of the FT:

$$P_\tau(\omega) = \int_{-\infty}^{\infty} p_\tau(t)e^{-j\omega t}dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t}dt$$

Limit integral to where $p_\tau(t)$ is non-zero... and use the fact that it is 1 over that region

$$= \frac{-1}{j\omega}\left[e^{-j\omega t}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \frac{2}{\omega}\left[\frac{e^{j\frac{\omega\tau}{2}} - e^{-j\frac{\omega\tau}{2}}}{j2}\right]$$

Artificially inserted 2 in numerator and denominator

$$= \sin\left(\frac{\omega\tau}{2}\right)$$
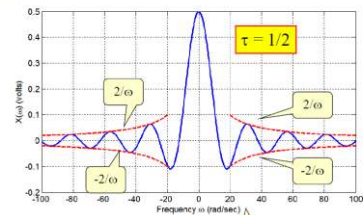
Use Euler's Formula

$$P_\tau(\omega) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega}$$

sin goes up and down between -1 and 1

$1/\omega$ decays down as $|\omega|$ gets big... this causes the overall function to decay down

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For **this** case the FT is real valued so we can plot it using a single plot (shown in solid blue here):

$\tau = 1/2$

$2/\omega$ $2/\omega$
$-2/\omega$ $-2/\omega$

$$P_\tau(\omega) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega}$$
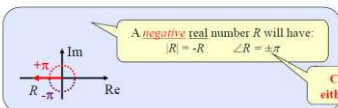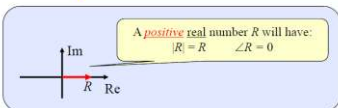
The sine wiggles up & down "between $\pm 2/\omega$"

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Now... let's think about how to make a magnitude/phase plot...

Even though this FT is real-valued we can still plot it using magnitude and phase plots:

We can view any real number as a complex number that has zero as its imaginary part

A *positive* real number R will have:
$|R| = R \qquad \angle R = 0$

A *negative* real number R will have:
$|R| = -R \qquad \angle R = \pm\pi$

$+\pi$
$-\pi$

Can use either one!!

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Applying these Ideas to the Real-valued FT $P_\tau(\omega)$

Phase = 0
Phase = $\pm\pi$
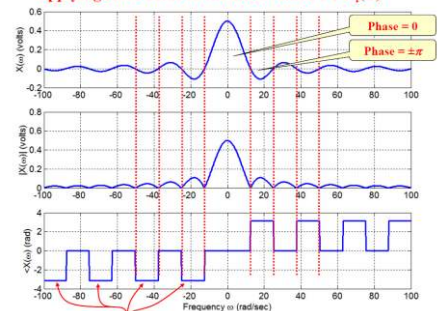
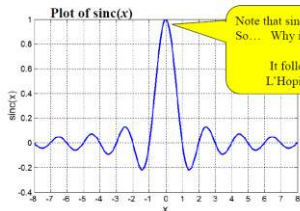Here I have chosen $-\pi$ to display odd symmetry

Fourier Transform

Definition of "Sinc" Function

The result we just found had this mathematical form: $P_\tau(\omega) = \dfrac{2\sin\left(\dfrac{\omega\tau}{2}\right)}{\omega}$

This structure shows up enough that we define a special function to capture it:

Define: $\text{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$

Plot of sinc(x)

Note that sinc(0) = 0/0.
So... Why is sinc(0) = 1?

It follows from L'Hopital's Rule

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Fourier Transform

With a little manipulation we can re-write the FT result for a pulse in terms of the sinc function:

Recall: $\text{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$

Need π times something...

Now we need the same thing down here as inside the sine...

$P_\tau(\omega) = \dfrac{2\sin\left(\dfrac{\omega\tau}{2}\right)}{\omega} = \dfrac{2\sin\left(\dfrac{\pi}{\pi}\dfrac{\omega\tau}{2}\right)}{\omega} = \dfrac{2\sin\left(\pi\dfrac{\omega\tau}{2\pi}\right)}{\omega}$
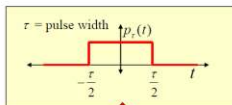
$= \pi\dfrac{\tau}{2\pi}\dfrac{2\sin\left(\pi\dfrac{\omega\tau}{2\pi}\right)}{\pi\dfrac{\tau}{2\pi}\omega} = \tau\dfrac{\sin\left(\pi\dfrac{\omega\tau}{2\pi}\right)}{\pi\dfrac{\omega\tau}{2\pi}} = \tau\,\text{sinc}\left(\dfrac{\omega\tau}{2\pi}\right)$

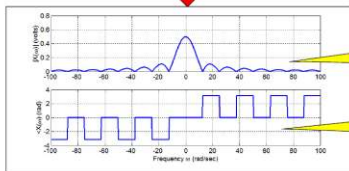$P_\tau(\omega) = \tau\,\text{sinc}\left(\dfrac{\omega\tau}{2\pi}\right)$

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Fourier Transform

FT of Rect. Pulse = Sinc Function

τ = pulse width $\quad p_\tau(t)$

Time-Domain View

$-\dfrac{\tau}{2} \qquad \dfrac{\tau}{2}$

Frequency-Domain View

Tells what **amplitude** is needed at each frequency

Tells what **phase** is needed at each frequency

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Fourier Transform Table

We have just found the FT for a common signal...

$$p_\tau(t) = \begin{cases} 1, & -\dfrac{\tau}{2} \le t \le \dfrac{\tau}{2} \\ 0, & otherwise \end{cases} \quad\longleftrightarrow\quad P_\tau(\omega) = \tau\,\text{sinc}\left(\dfrac{\omega\tau}{2\pi}\right)$$

We derived that result by directly applying the integral form of the FT to the given signal equation.

For the common "textbook" signals this has already been done... and the results are available in tables published in books and on-line

You should study the table provided...
*   If you encounter a time signal or FT that is on this table you should recognize that it is on the table without being told that it is there.
*   You should be able to recognize entries in graphical form as well as in equation form.
*   Later we'll learn about some "FT properties" that will expand your ability to apply these entries on the FT Table

In the real-world, engineers use these table results to understand basic ideas and concepts and to think through how things work in principle!

So... next we'll look at some of the more important entries in the table provided...

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Fourier Transform Table

Decaying Exponential
As we'll see later... this signal naturally occurs in lots of real-world places!

$$x(t) = e^{-bt}u(t) \quad\longleftrightarrow\quad X(\omega) = \dfrac{1}{b + j\omega}$$

For $b > 0$

(Complex Valued)

$|X(\omega)| = \dfrac{1}{\sqrt{b^2 + \omega^2}}$ Magnitude

$\angle X(\omega) = -\tan^{-1}\left(\dfrac{\omega}{b}\right)$ Phase

$x(t) = e^{-bt}u(t)$

$b > 0$ controls decay rate

$|X(\omega)|$
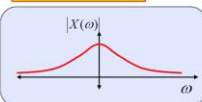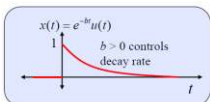
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Fourier Transform Table

Can Use Matlab to Make Plots of FT Results

Technically V/Hz

FT of $e^{-bt}u(t)$ for $b = 10$
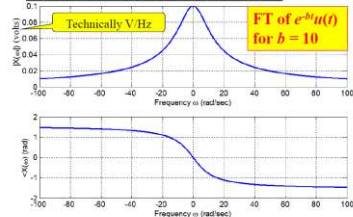
Note that magnitude plot has **even** symmetry

Note that phase plot has **odd** symmetry

True for every real-valued signal

MATLAB Commands to Compute FT
```
w=-100:0.2:100;
b=10;
X=1./(b+j*w);
```

Plotting Commands
```
subplot(2,1,1); plot(w,abs(X))
xlabel('Frequency \omega (rad/sec)')
ylabel('X(\omega) (volts)'); grid
subplot(2,1,2); plot(w,angle(X))
xlabel('Frequency \omega (rad/sec)')
ylabel('<X(\omega) (rad)'); grid
```

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Fourier Transform Table

Effect of Exp. Decay Rate b on FT Magnitude

Time Signal	FT Magnitude
$x(t) = e^{-bt}u(t)$	$|X(\omega)| = \dfrac{1}{\sqrt{b^2 + \omega^2}}$

Note: As b increases…
1. Decay rate in time signal increases
2. High frequencies in Fourier transform are more prominent.

Short Signals have FTs that spread more into High Frequencies!!!

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Fourier Transform Table

Some Important Signals & Their FTs (see Table for More!)

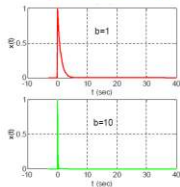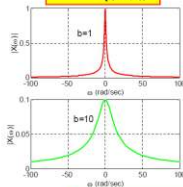$$1, \quad -\infty < t < \infty \iff 2\pi\delta(\omega)$$

$$u(t) \iff \pi\delta(\omega) + 1/j\omega$$

$$-0.5 + u(t) \iff 1/j\omega$$

$$\delta(t) \iff 1, \quad -\infty < \omega < \infty$$

$$\cos(\omega_o t) \iff \pi[\delta(\omega + \omega_o) + \delta(\omega - \omega_o)]$$

$$\sin(\omega_o t) \iff j\pi[\delta(\omega + \omega_o) - \delta(\omega - \omega_o)]$$

$$e^{j\omega_o t} \iff 2\pi\delta(\omega - \omega_o), \quad \omega_o \text{ real}$$

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Fourier Transform Table

Time Signal	Fourier Transform
$1, \quad -\infty < t < \infty$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$1/j\omega$
$u(t)$	$\pi\delta(\omega) + 1/j\omega$
$\delta(t)$	$1, \quad -\infty < \omega < \infty$
$\delta(t-c), \quad c \text{ real}$	$e^{-j\omega c}, \quad c \text{ real}$
$e^{-bt}u(t), \quad b > 0$	$\dfrac{1}{j\omega + b}, \quad b > 0$
$e^{j\omega_o t}, \quad \omega_o \text{ real}$	$2\pi\delta(\omega - \omega_o), \quad \omega_o \text{ real}$
$p_\tau(t)$	$\tau \,\mathrm{sinc}[\tau\omega/2\pi]$
$\tau \,\mathrm{sinc}[\tau t/2\pi]$	$2\pi p_\tau(\omega)$
$\left[1 - \frac{2|t|}{\tau}\right]p_\tau(t)$	$\frac{\tau}{2}\mathrm{sinc}^2[\tau\omega/4\pi]$
$\frac{\tau}{2}\mathrm{sinc}^2[\tau t/4\pi]$	$2\pi\left[1 - \frac{2|t|}{\tau}\right]p_\tau(\omega)$
$\cos(\omega_o t)$	$\pi[\delta(\omega + \omega_o) + \delta(\omega - \omega_o)]$
$\cos(\omega_o t + \theta)$	$\pi\left[e^{-j\theta}\delta(\omega + \omega_o) + e^{j\theta}\delta(\omega - \omega_o)\right]$
$\sin(\omega_o t)$	$j\pi[\delta(\omega + \omega_o) - \delta(\omega - \omega_o)]$
$\sin(\omega_o t + \theta)$	$j\pi\left[e^{-j\theta}\delta(\omega + \omega_o) - e^{j\theta}\delta(\omega - \omega_o)\right]$

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Fourier Transform Table

Property Name	Property	
Linearity	$ax(t) + bv(t)$	$aX(\omega) + bV(\omega)$
Time Shift	$x(t-c)$	$e^{-j\omega c}X(\omega)$
Time Scaling	$x(at), \quad a \neq 0$	$\frac{1}{|a|}X(\omega/a), \quad a \neq 0$
Time Reversal	$x(-t)$	$X(-\omega)$
		$\overline{X(\omega)}$ if $x(t)$ is real
Multiply by t^n	$t^n x(t), \quad n = 1, 2, 3, \dots$	$j^n \frac{d^n}{d\omega^n}X(\omega), \quad n = 1, 2, 3, \dots$
Multiply by Complex Exponential	$e^{j\omega_o t}x(t), \quad \omega_o \text{ real}$	$X(\omega - \omega_o), \quad \omega_o \text{ real}$
Multiply by Sine	$\sin(\omega_o t)x(t)$	$\frac{j}{2}[X(\omega + \omega_o) - X(\omega - \omega_o)]$
Multiply by Cosine	$\cos(\omega_o t)x(t)$	$\frac{1}{2}[X(\omega + \omega_o) + X(\omega - \omega_o)]$
Time Differentiation	$\frac{d^n}{dt^n}x(t), \quad n = 1, 2, 3, \dots$	$(j\omega)^n X(\omega), \quad n = 1, 2, 3, \dots$
Time Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda$	$\frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in Time	$x(t) * h(t)$	$X(\omega)H(\omega)$
Multiplication in Time	$x(t)w(t)$	$\frac{1}{2\pi}X(\omega)*W(\omega)$
Parseval's Theorem (General)	$\int_{-\infty}^{\infty} x(t)\overline{v(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\overline{V(\omega)}d\omega$	
Parseval's Theorem (Energy)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$ if $x(t)$ is real	
	$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$	
Duality: If $x(t) \leftrightarrow X(\omega)$	$X(t)$	$2\pi x(-\omega)$

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Fourier Transform Table

FT of Periodic Signal

Note that we have now used the FT to analyze cosine and sine… which are **PERIODIC** signals!!! Before we used the Fourier *Series* to analyze *periodic* signals… Now we see that we can also use the Fourier *Transform*!

If $x(t)$ is periodic then we can write the FS of it as: $\quad x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

Now we can take the FT of both sides of this: $\quad \mathfrak{F}\{x(t)\} = \mathfrak{F}\left\{\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right\}$

$$= \sum_{k=-\infty}^{\infty} c_k \mathfrak{F}\left\{e^{jk\omega_0 t}\right\}$$

$$2\pi\delta(\omega - k\omega_0)$$

FT of a Periodic Signal

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega - k\omega_0)$$

Note: the FT is a bunch of delta functions with "weights" given by the FS coefficients!

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Fourier Transform Table
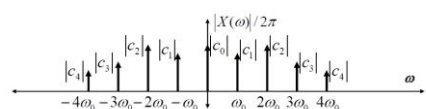
So the FT of a periodic signal is zero except at multiples of the fundamental frequency ω_0, where you get impulses.

We call these spikes "Spectral Lines"

Note that if we start with the Amplitude-Phase Trig form we end up with the same result for the FT

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

For each cosine term we get two deltas (a positive frequency & negative frequency):

$$\cos(\omega_o t + \theta) \iff \pi\left[e^{-j\theta}\delta(\omega + \omega_o) + e^{j\theta}\delta(\omega - \omega_o)\right]$$

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Fourier Transform Properties

These properties are useful for two main things:

1. They help you apply the table to a wider class of signals
2. They are often the key to understanding how the FT can be used in a given application.

So… even though these results may at first seem like "just boring math" they are important tools that let signal processing engineers understand how to build things like cell phones, radars, mp3 processing, etc.

Here… we will only cover the most important properties.

See the available table for the complete list of properties!

In this note set we simply learn these most-important properties… in the next note set we'll see how to use them.

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Fourier Transform Properties

1. **Linearity** (Supremely Important) — Gets used virtually all the time!!

If $x(t) \leftrightarrow X(\omega)$ & $y(t) \leftrightarrow Y(\omega)$

then $[ax(t) + by(t)] \leftrightarrow [aX(\omega) + bY(\omega)]$

Another way to write this property:

$\mathcal{F}\{ax(t) + by(t)\} = a\mathcal{F}\{x(t)\} + b\mathcal{F}\{y(t)\}$

To see why: $\mathcal{F}\{ax(t) + by(t)\} = \int_{-\infty}^{\infty} [ax(t) + by(t)] e^{-j\omega t} dt$ — Use Defn of FT

$= a\underbrace{\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt}_{= X(\omega)} + b\underbrace{\int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt}_{= Y(\omega)}$

By standard Property of Integral of sum of functions

By Defn of FT

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Fourier Transform Properties

2. **Time Shift** (Really Important!) — Used often to understand _practical_ issues that arise in _audio_, _communications_, _radar_, etc.

If $x(t) \leftrightarrow X(\omega)$ then $x(t-c) \leftrightarrow X(\omega)e^{-j c\omega}$

Note: If $c > 0$ then $x(t-c)$ is a **delay** of $x(t)$

So… what does this _mean_??

First… it does nothing to the magnitude of the FT: $|X(\omega)e^{-j\omega c}| = |X(\omega)|$

That means that a shift doesn't change "how much" we need of each of the sinusoids we build with

Second… it does change the phase of the FT: $\angle\{X(\omega)e^{-jc\omega}\} = \angle X(\omega) + \angle e^{-jc\omega}$

$= \angle X(\omega) + c\omega$

Line of slope $-c$

Phase shift increases linearly as the frequency increases

This gets added to original phase

Shift of Time Signal $\Leftrightarrow$ **"Linear" Phase Shift of Frequency Components**

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Fourier Transform Properties

3. **Time Scaling** (Important)

Q: If $x(t) \leftrightarrow X(\omega)$, then $x(at) \leftrightarrow$??? for $a \neq 0$

A: $x(at) \leftrightarrow \dfrac{1}{|a|}X\left(\dfrac{\omega}{a}\right)$

If the time signal is Time Scaled by a

Then… The FT is Freq. Scaled by $1/a$

An interesting "duality"!!!

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Fourier Transform Properties

To explore this FT property…first, what does $x(at)$ look like?

Original Signal

$x(t)$

Time-Scaled w/ $a = 2$

$x(2t)$
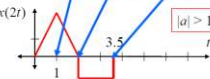
$|a| > 1$ "squishes" horizontally

Time-Scaled w/ $a = 1/2$

$x\left(\dfrac{1}{2}t\right)$
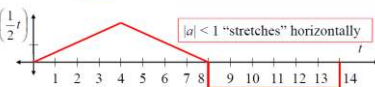
$|a| < 1$ "stretches" horizontally

$|a| > 1$ makes it "wiggle" faster $\Rightarrow$ need _more_ high frequencies

$|a| < 1$ makes it "wiggle" slower $\Rightarrow$ need _less_ high frequencies

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Fourier Transform Properties

When $|a| > 1 \Rightarrow |1/a| < 1$

$x(at) \leftrightarrow \dfrac{1}{|a|}X\left(\dfrac{\omega}{a}\right)$

Time Signal is Squished

FT is Stretched Horizontally and Reduced Vertically

Original Signal & Its FT

$x(t)$ A $\uparrow X(\omega)$

Squished Signal & Its FT

$x(2t)$ .5A $\dfrac{1}{2}X\left(\dfrac{\omega}{2}\right)$

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Fourier Transform Properties

When $|a| < 1 \Rightarrow |1/a| > 1$

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

Time Signal is Stretched

FT is Squished Horizontally and Increased Vertically

Original Signal & Its FT

$x(t)$ ... t ... $A \uparrow X(\omega)$ ... ω

Stretched Signal & Its FT

$x\left(\frac{1}{2}t\right)$ ... t ... $2A \downarrow 2X(2\omega)$ ... ω
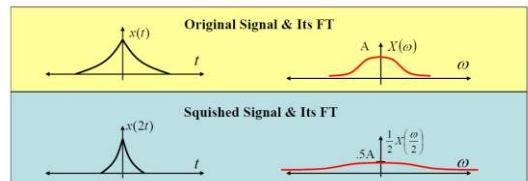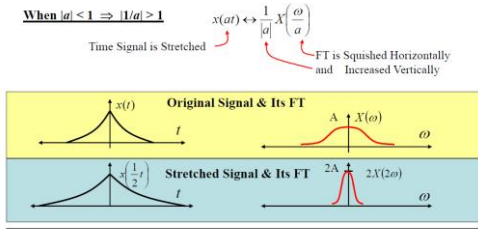
Rough Rule of Thumb we can extract from this property:

$\uparrow$ Duration $\Rightarrow$ $\downarrow$ Bandwidth

$\downarrow$ Duration $\Rightarrow$ $\uparrow$ Bandwidth

Very Short Signals *tend* to take up Wide Bandwidth

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Fourier Transform Properties

4. Time Reversal (Special case of time scaling: $a = -1$)

$$x(-t) \leftrightarrow X(-\omega)$$

double conjugate

Note: $X(-\omega) = \int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}dt = \overline{\int_{-\infty}^{\infty} \overline{x(t)}e^{+j\omega t}dt}$ = "No Change"

$= \overline{\int_{-\infty}^{\infty} \overline{x(t)}e^{j\omega t}dt}$ Conjugate changes to $-j$

$= x(t)$ if $x(t)$ is real

$= \overline{\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt} = \overline{X(\omega)}$

So if $x(t)$ is <u>real</u>, then we get the **special case**:

$$x(-t) \leftrightarrow \overline{X(\omega)}$$

Recall: conjugation doesn't change abs. value but negates the angle

$$\left|\overline{X(\omega)}\right| = |X(\omega)|$$
$$\angle\overline{X(\omega)} = -\angle X(\omega)$$

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Fourier Transform Properties

5. Modulation Property | Super important!!!

Essential for understanding practical issues that arise in communications, radar, etc.

There are two forms of the modulation property…

1. **Complex Exponential Modulation** … simpler mathematics, doesn't *directly* describe real-world cases
2. **Real Sinusoid Modulation**… mathematics a bit more complicated, directly describes real-world cases

Euler's formula connects the two… so you often can use the Complex Exponential form to analyze real-world cases

Complex Exponential Modulation Property: $x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$

Multiply signal by a complex sinusoid

Shift the FT in frequency

$A \uparrow X(\omega)$ ... ω

$\mathcal{F}\{x(t)e^{j\omega_o t}\} = X(\omega - \omega_o)$ ... A ... ω ... ω_o
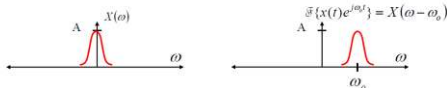
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Fourier Transform Properties

Real Sinusoid Modulation

Based on Euler, Linearity property, & the Complex Exp. Modulation Property

$$\mathcal{F}\{x(t)\cos(\omega_0 t)\} = \mathcal{F}\left\{\frac{1}{2}\left[x(t)e^{j\omega_0 t} + x(t)e^{-j\omega_0 t}\right]\right\}$$ Euler's Formula

$$= \frac{1}{2}\left[\mathcal{F}\{x(t)e^{j\omega_0 t}\} + \mathcal{F}\{x(t)e^{-j\omega_0 t}\}\right]$$ Linearity of FT

$$= \frac{1}{2}\left[X(\omega - \omega_o) + X(\omega + \omega_o)\right]$$ Comp. Exp. Mod.

The Result: $x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}\left[X(\omega + \omega_0) + X(\omega - \omega_0)\right]$

Shift Down   Shift Up

Related Result: $x(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}\left[X(\omega + \omega_0) - X(\omega - \omega_0)\right]$
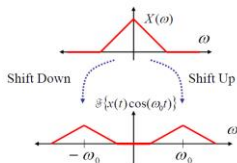
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Fourier Transform Properties

Visualizing the Result

$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}\left[X(\omega - \omega_0) + X(\omega + \omega_0)\right]$

Shift up   Shift down

$X(\omega)$ ... ω

Shift Down   Shift Up

$\mathcal{F}\{x(t)\cos(\omega_0 t)\}$

$-\omega_0$ ... ω_0 ... ω

Interesting… This tells us how to move a signal's spectrum up to higher frequencies without changing the shape of the spectrum!!!

What is that good for??? Well… only <u>high</u> frequencies will radiate from an antenna and propagate as electromagnetic waves and then induce a signal in a receiving antenna….

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Fourier Transform Properties

6. Convolution Property (The Most Important FT Property!!!)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \leftrightarrow Y(\omega) = X(\omega)H(\omega)$$

In the next Note Set we will explore the real-world use of the right side of this result!

7. Parseval's Theorem (Recall Parseval's Theorem for FS!)

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Energy computed in time domain   Energy computed in frequency domain

$|x(t)|^2 dt$ = energy at time t

$|X(\omega)|^2 \frac{d\omega}{2\pi}$ = energy at freq. ω

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Fourier Transform Properties

8. Duality:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$x(t) \longleftrightarrow X(\omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

Both FT & IFT are pretty much the "same machine": $c\int_{-\infty}^{\infty} f(\lambda)e^{\pm j\lambda\xi}\,d\lambda$

So if there is a "time-to-frequency" property we would expect a virtually similar "frequency-to-time" property

Illustration: Delay Property: $\quad x(t-c) \longleftrightarrow X(\omega)e^{-j\omega c}$

Modulation Property: $\quad x(t)e^{j\omega_0 t} \longleftrightarrow X(\omega - \omega_0)$

Other Dual Properties: (Multiply by t^n) vs. (Diff. in time domain)
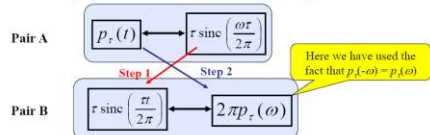
(Convolution) vs. (Mult. of signals)

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Fourier Transform Properties

Also, this duality structure gives FT pairs that show duality.

Suppose we have a FT table that a FT Pair A… we can get the dual Pair B using the general Duality Property:

1. Take the FT side of (known) Pair A and replace ω by t and move it to the time-domain side of the table of the (unknown) Pair B.

2. Take the time-domain side of the (known) Pair A and replace t by $-\omega$, multiply by 2π, and then move it to the FT side of the table of the (unknown) Pair B.

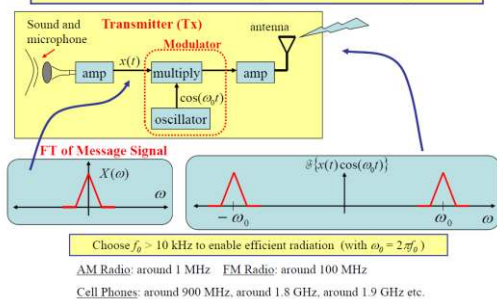Here is an example… We found the FT pair for the pulse signal:

Pair A $\quad p_\tau(t) \longleftrightarrow \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$

Step 1 — Step 2

Here we have used the fact that $p_\tau(-\omega) = p_\tau(\omega)$

Pair B $\quad \tau\,\mathrm{sinc}\left(\frac{t\tau}{2\pi}\right) \longleftrightarrow 2\pi p_\tau(\omega)$

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Application Example

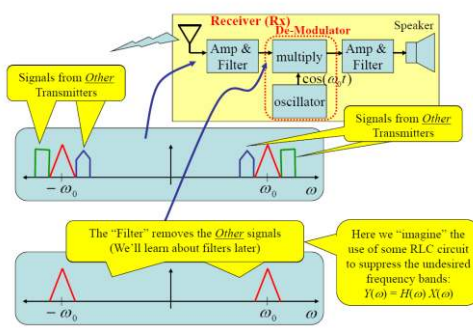**Application of Modulation Property to Radio Communication**

FT theory tells us what we need to do to make a **simple** radio system… *then* electronics can be built to perform the operations that the FT theory calls for:
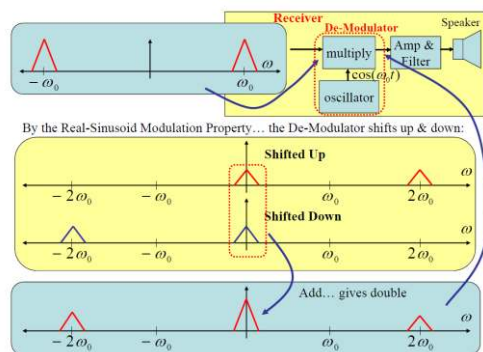
Sound and microphone — **Transmitter (Tx)** — Modulator — antenna

amp — $x(t)$ — multiply — amp

$\cos(\omega_0 t)$ — oscillator

FT of Message Signal

$X(\omega)$

$\mathscr{F}\{x(t)\cos(\omega_0 t)\}$

Choose $f_0 > 10$ kHz to enable efficient radiation (with $\omega_0 = 2\pi f_0$)

AM Radio: around 1 MHz   FM Radio: around 100 MHz

Cell Phones: around 900 MHz, around 1.8 GHz, around 1.9 GHz etc.

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Application Example

The next several slides show how these ideas are used to make a receiver:

Receiver (Rx) — De-Modulator — Speaker

Amp & Filter — multiply — Amp & Filter

$\cos(\omega_0 t)$ — oscillator

Signals from *Other* Transmitters

Signals from *Other* Transmitters

The "Filter" removes the *Other* signals (We'll learn about filters later)

Here we "imagine" the use of some RLC circuit to suppress the undesired frequency bands:
$Y(\omega) = H(\omega)\,X(\omega)$

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Application Example

Receiver — De-Modulator — Speaker

Amp & Filter — multiply — Amp & Filter

$\cos(\omega_0 t)$ — oscillator

By the Real-Sinusoid Modulation Property… the De-Modulator shifts up & down:

Shifted Up

Shifted Down

Add… gives double

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Application Example

Receiver — De-Modulator — Speaker

Amp & Filter — multiply — Amp & Filter

$\cos(\omega_0 t)$ — oscillator

Extra Stuff we don't want

The "Filter" removes the *Extra* Stuff

Speaker is driven by desired message signal!!!

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Application Example

So… what have we seen in this example:

Using the Modulation property of the FT we saw…

1. Key Operation at Transmitter is up-shifting the message spectrum:
    a) FT Modulation Property tells the theory then we can build…
    b) "modulator" = oscillator and a multiplier circuit
2. Key Operation at Recevier   is down-shifting the received spectrum
    a) FT Modulation Property tells the theory then we can build…
    b) "de-modulator" = oscillator and a multiplier circuit
    c) But… the FT modulation property theory also shows that we need filters to get rid of "extra spectrum" stuff
        i. So… one thing we still need to figure out is how to deal with these filters…
        ii. Filters are a specific "system" and we still have a lot to learn about Systems…
        iii. That is the subject of much of the rest of this course!!!