BLM3620 Digital Signal Processing

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Lecture #10 – Discrete Fourier Transform and Properties

- Discrete Fourier Transform
- Examples
- Solution using Properties
- MATLAB Applications

Course Materials

Important Materials:

- James H. McClellan, R. W. Schafer, M. A. Yoder, *DSP First Second Edition*, Pearson, 2015.

- Lizhe Tan, Jean Jiang, *Digital Signal Processing: Fundamentals and Applications*, Third Edition, Academic Press, 2019.

Auxilary Materials:

- Prof. Sarp Ertürk, *Sayısal İşaret İşleme*, Birsen Yayınevi.

- Prof. Nizamettin Aydin, DSP Lecture Notes.

- J. G. Proakis, D. K. Manolakis, *Digital Signal Processing Fourth Edition*, Peason, 2014.

- J. K. Perin, *Digital Signal Processing*, *Lecture Notes,* Standford University, 2018.

Syllabus

Week	Lectures
1	Introduction to DSP and MATLAB
2	Sinuzoids and Complex Exponentials
3	Spectrum Representation
4	Sampling and Aliasing
5	Discrete Time Signal Properties and Convolution
6	Convolution and FIR Filters
7	Frequency Response of FIR Filters
8	Midterm Exam
9	Discrete Time Fourier Transform and Properties
10	Discrete Fourier Transform and Properties
11	Fast Fourier Transform and Windowing
12	z- Transforms
13	FIR Filter Design and Applications
14	IIR Filter Design and Applications
15	Final Exam

For more details -> Bologna page: http://www.bologna.yildiz.edu.tr/index.php?r=course/view&id=5730&aid=3

Recap: Discrete Time Fourier Transform

Definition of the **DTFT**: $$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

- Always periodic with a period of 2π

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn}$$ ⇨ DFT

Why do we need Discrete Fourier Transform?

Answer: To compute Fourier Transform of a Discrete-Time signal on computer systems. The number of points in DTFT is infinite.

How can we do it?

Answer: (1) Finite signal length (N), (2) Finite numnber of frequencies.

Discrete Fourier Transform

DFT can be obtained by sampling of DTFT.

DTFT

$$X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\hat{\omega}n}$$

$$\hat{\omega} = \frac{2\pi}{N}k$$

DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

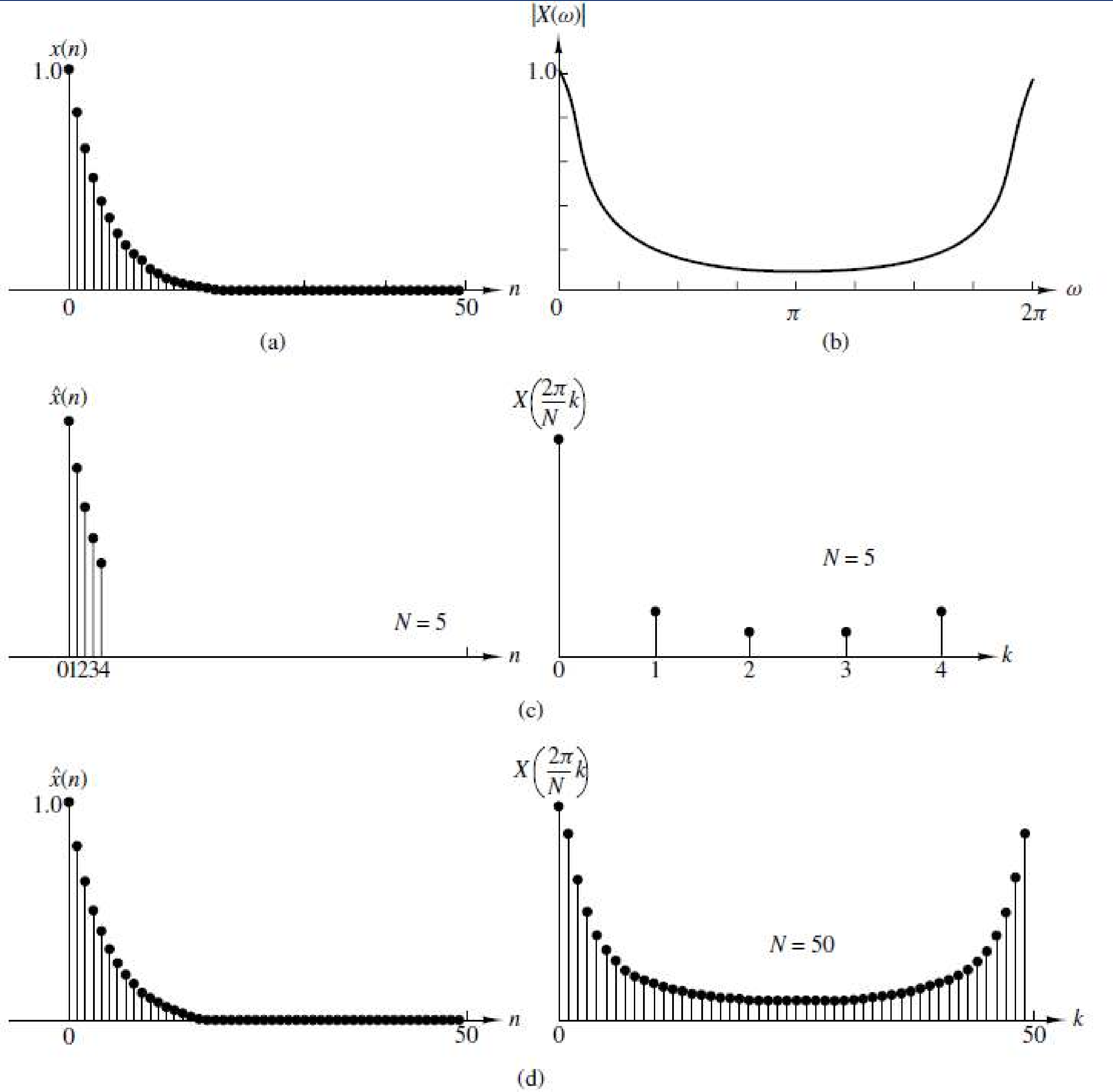
In DTFT, X is a continuous function of w whereas in DFT X is discrete. k -> freq. index

Inverse DFT Transform:

$$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}$$

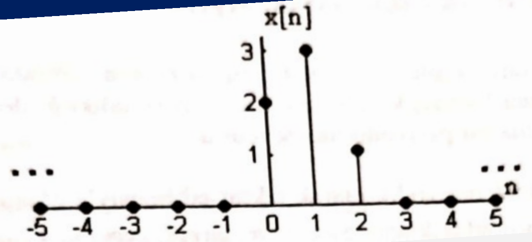
$$\text{Periodic}: X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}}) \quad \Rightarrow \quad X[k+N] = X[k]$$

Effects of *N* Value on Result



Should be greater or equal than the number of samples.

Example from Sarp Erturk's book:



$x[n] = 2\delta[n] + 3\delta[n-1] + \delta[n-2]$ find DFT of x[n].

İşaretin sadece üç değeri sıfırdan farklı olduğu için $N = 3$ olarak alınabilmektedir. İşaretin ayrık Fourier dönüşümü

$k = 0$ için

$$X[0] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)0n} = \sum_{n=0}^{2} x[n] = x[0] + x[1] + x[2] = 6$$

$k = 1$ için

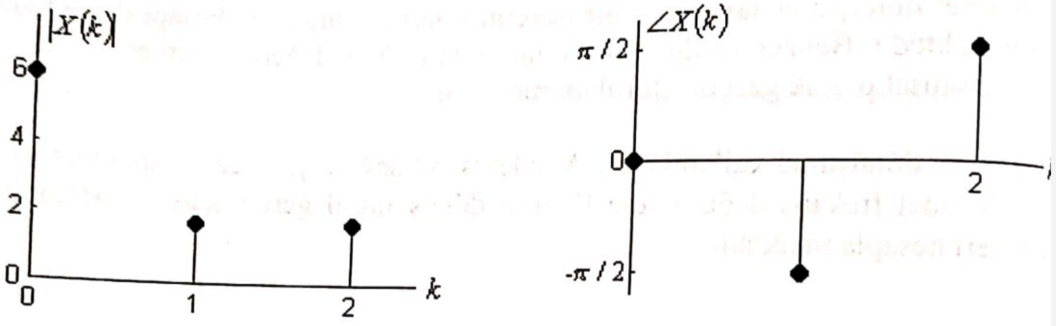
$$X[1] = \sum_{n=0}^{2} x[n]e^{-j(2\pi/3)n} = x[0] + x[1]e^{-j(2\pi/3)} + x[2]e^{-j(4\pi/3)}$$

$$X[1] = 2 + 3e^{-j(2\pi/3)} + e^{-j(4\pi/3)} = -j1.7321 = 1.7321e^{-j\pi/2}$$

$k = 2$ için

$$X[2] = \sum_{n=0}^{2} x[n]e^{-j(2\pi/3)2n} = x[0] + x[1]e^{-j(4\pi/3)} + x[2]e^{-j(8\pi/3)}$$

$$X[2] = 2 + 3e^{-j(4\pi/3)} + e^{-j(8\pi/3)} = j1.7321 = 1.7321e^{j\pi/2}$$



Şekil 4. 9. Örnek 4.12 için ayrık Fourier dönüşümünün genliği ve fazı.

4-pt DFT: Numerical Example

- Take the 4-pt DFT of the following signal

$$x[n] = \delta[n] + \delta[n-1] \qquad \{x[n]\} = [1, 1, 0, 0]$$

$$X[0] = x[0]e^{-j0} + x[1]e^{-j0} + x[2]e^{-j0} + x[3]e^{-j0} = 1 + 1 + 0 + 0 = 2$$

$$X[1] = x[0]e^{-j0} + x[1]e^{-j\pi/2} + x[2]e^{-j2\pi/2} + x[3]e^{-j3\pi/2}$$
$$= 1 - j = \sqrt{2}e^{-j\pi/4}$$

$$X[2] = x[0]e^{-j0} + x[1]e^{-j\pi} + x[2]e^{-j2\pi} + x[3]e^{-j3\pi} = 1 - 1 + 0 + 0 = 0$$

$$X[3] = x[0]e^{-j0} + x[1]e^{-j3\pi/2} + x[2]e^{-j3\pi} + x[3]e^{-j9\pi/2}$$
$$= 1 + j = \sqrt{2}e^{j\pi/4}$$

4-pt iDFT: Numerical Example

Example 66-8: Short-Length IDFT

The 4-point DFT in Example 66-7 is $X[k] = \{2, \sqrt{2}e^{-j\pi/4}, 0, \sqrt{2}e^{j\pi/4}\}$. If we compute the 4-point IDFT of the sequence $X[k]$, we should recover $x[n]$ when we apply the IDFT summation (66.52) for each value of $n = 0, 1, 2, 3$. As before, the exponents in (66.52) will all be integer multiples of $\pi/2$ when $N = 4$.

$$
\begin{aligned}
x[0] &= \tfrac{1}{4}\left(X[0]e^{j0} + X[1]e^{j0} + X[2]e^{j0} + X[3]e^{j0}\right) \\
&= \tfrac{1}{4}\left(2 + \sqrt{2}e^{-j\pi/4} + 0 + \sqrt{2}e^{j\pi/4}\right) = 1 \\
x[1] &= \tfrac{1}{4}\left(X[0]e^{j0} + X[1]e^{j\pi/2} + X[2]e^{j\pi} + X[3]e^{j3\pi/2}\right) \\
&= \tfrac{1}{4}\left(2 + \sqrt{2}e^{j(-\pi/4+\pi/2)} + 0 + \sqrt{2}e^{j(\pi/4+3\pi/2)}\right) = \tfrac{1}{4}(2 + (1+j) + (1-j)) = 1 \\
x[2] &= \tfrac{1}{4}\left(X[0]e^{j0} + X[1]e^{j\pi} + X[2]e^{j2\pi} + X[3]e^{j3\pi}\right) \\
&= \tfrac{1}{4}\left(2 + \sqrt{2}e^{j(-\pi/4+\pi)} + 0 + \sqrt{2}e^{j(\pi/4+3\pi)}\right) = \tfrac{1}{4}(2 + (-1+j) + (-1-j)) = 0 \\
x[3] &= \tfrac{1}{4}\left(X[0]e^{j0} + X[1]e^{j3\pi/2} + X[2]e^{j3\pi} + X[3]e^{j9\pi/2}\right) \\
&= \tfrac{1}{4}\left(2 + \sqrt{2}e^{j(-\pi/4+3\pi/2)} + 0 + \sqrt{2}e^{j(\pi/4+9\pi/2)}\right) = \tfrac{1}{4}(2 + (-1-j) + (-1+j)) = 0
\end{aligned}
$$

Thus we recover the signal $x[n] = \{1, 1, 0, 0\}$ from its DFT coefficients, $X[k] = \{2, \sqrt{2}e^{-j\pi/4}, 0, \sqrt{2}e^{j\pi/4}\}$.

DFT Properties

Table 8-2 Basic discrete Fourier transform properties.

Table of DFT Properties		
Property Name	***Time-Domain: $x[n]$***	***Frequency-Domain: $X[k]$***
Periodic	$x[n] = x[n+N]$	$X[k] = X[k+N]$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
Conjugate Symmetry	$x[n]$ is real	$X[N-k] = X^*[k]$
Conjugation	$x^*[n]$	$X^*[N-k]$
Time-Reversal	$x[((N-n))_N]$	$X[N-k]$
Delay	$x[((n-n_d))_N]$	$e^{-j(2\pi k/N)n_d} X[k]$
Frequency Shift	$x[n]e^{j(2\pi k_0/N)n}$	$X[k-k_0]$
Modulation	$x[n]\cos((2\pi k_0/N)n)$	$\frac{1}{2}X[k-k_0] + \frac{1}{2}X[k+k_0]$
Convolution	$\sum_{m=0}^{N-1} h[m]x[((n-m))_N]$	$H[k]X[k]$
Parseval's Theorem	$\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \frac{1}{N}\sum_{k=0}^{N-1} \lvert X[k]\rvert^2$	

DFT Properties

Özellik Adı	İşaret $x_1[n], x_2[n]$	N noktalı Ayrık Fourier Dönüşümü $X_1[k], X_2[k]$
Periyodiklik	$x_1[n] = x_1[n+N]$	$X_1[k] = X_1[k+N]$
Zamanda Tersleme	$x_1[-n] = x_1[N-n]$	$X_1[-k] = X_1[N-k]$
Lineerlik	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
Konjuge simetriği	x[n], reel ise	$X[N-k] = X^*[k]$
Çifteşlik	$X[n]$	$Nx[-k]_{mod\ N}$
Dairesel öteleme	$x_1[n-n_0]_{mod\ N}, n_0$ tamsayı	$e^{-j(2\pi k/N)n_0}X[k]$
Frekansta dairesel öteleme	$e^{j\left(\frac{2\pi k}{N}\right)l}x_1[n]$, l tamsayı	$X_1[k-l]_{mod\ N}$
Dairesel Konvolüsyon	$\sum_{m=0}^{N-1} x_1[m]x_2[n-m]_{mod\ N}$	$X_1[k]X_2[k]$
Dairesel Modülasyon	$x_1[n]x_2[n]$	$\frac{1}{N}\sum_{l=0}^{N-1} X_1[l]X_2[k-l]_{mod\ N}$

DFT periodic in k (frequency domain)

- Since DTFT is periodic in frequency, the DFT must also be periodic in k

$$X[k] = X(e^{j(2\pi/N)k})$$

$$X[k+N] = X(e^{j(2\pi/N)(k+N)}) = X(e^{j(2\pi/N)(k)+\cancel{(2\pi/N)N}}) = X(e^{j(2\pi/N)k})$$

- What about Negative indices and Conjugate Symmetry?

$$X(e^{-j(2\pi/N)k}) = X^*(e^{j(2\pi/N)k})$$

$$\Rightarrow X[-k] = X^*[k]$$

$$X[N-k] = X^*[k]$$

$$N = 32 \quad \Rightarrow$$

$$X[31] = X^*[1]$$

$$X[30] = X^*[2]$$

$$X[29] = X^*[3]$$

Circular Convolution for Periodic DT Signals

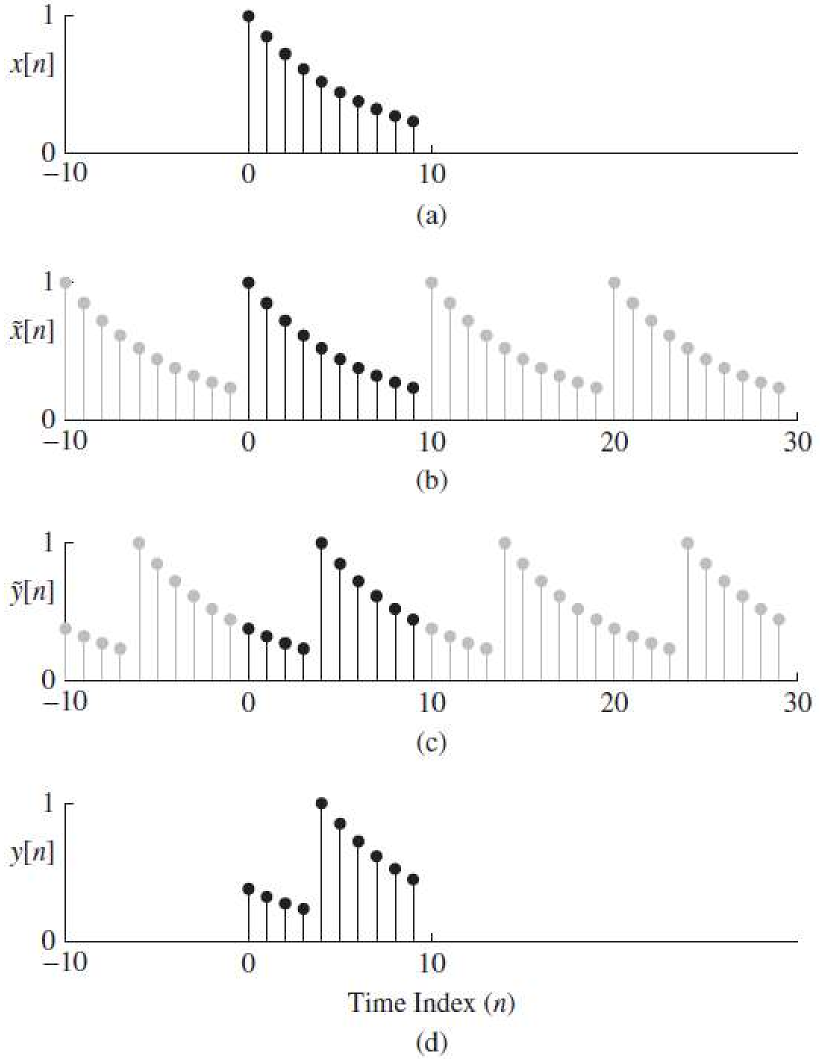


Figure 8-8 Illustration of the time-shift property of the DFT. (a) A finite-length sequence $x[n]$ of length 10. (b) The inherent periodic sequence $\tilde{x}[n]$ for a 10-point DFT representation. (c) Time-shifted periodic sequence $\tilde{y}[n] = \tilde{x}[n-4]$ which is also equal to the IDFT of $Y[k] = e^{-j(2\pi k/10)(4)}X[k]$. (d) The sequence $y[n]$ obtained by evaluating the 10-point IDFT of $Y[k]$ only in the interval $0 \le n \le 9$.

Circular Convolution for Periodic DT Signals

$x_1[n] = [1 \;\; 2 \;\; 3 \;\; 4]$ ve $x_2[n] = [1 \;\; 1\,]$ ise bu iki işaretin dairesel konvolüsyonunu bulunuz.

$$x_3[n] = \sum_{m=0}^{3} x_1[m]x_2[n-m]_{mod\ 4}$$

Öncelikle sona sıfır eklenerek işaret uzunlukları eşitlenmelidir:

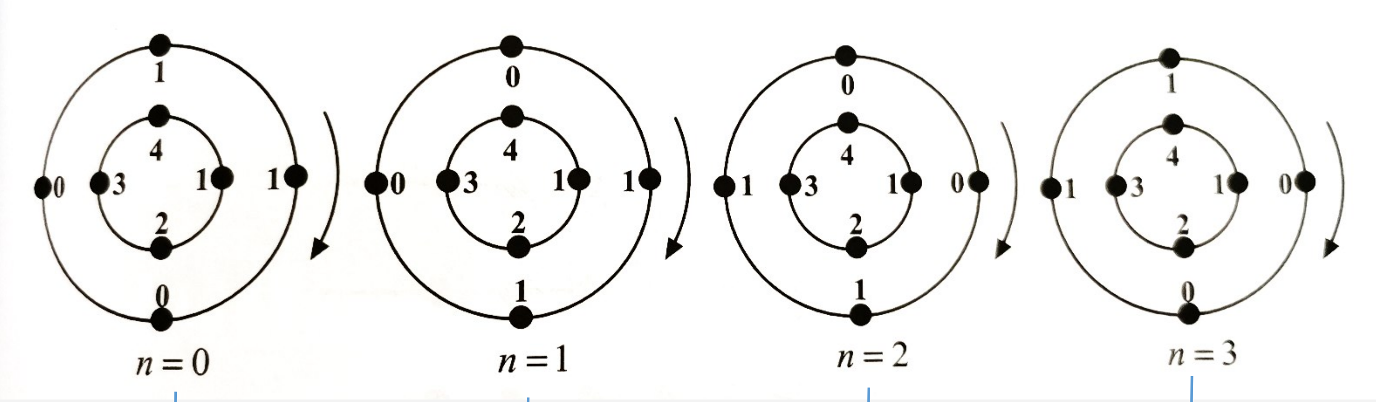
$$x_3[0] = \sum_{m=0}^{3} x_1[m]x_2[0-m]_{mod\ 4} = x_1[0]x_2[0] + x_1[1]x_2[3] + x_1[2]x_2[2] + x_1[3]x_2[1] = 5$$

$$x_3[1] = \sum_{m=0}^{3} x_1[m]x_2[1-m]_{mod\ 4} = x_1[0]x_2[1] + x_1[1]x_2[0] + x_1[2]x_2[3] + x_1[3]x_2[2] = 3$$

Tamamı hesaplanırsa : $x_3[n] = [5 \;\;\; 3 \;\; 5 \;\;\; 7]$

Circular Convolution

Or…



$x_3[0] = 1 \times 1 + 4 \times 1 = 5$ $x_3[1] = 3$ $x_3[2] = 5$ $x_3[3] = 5$

Duality (Çifteşlik)

$$x[n] \leftrightarrow X[k] \qquad \text{ise} \qquad X[n] \leftrightarrow Nx[-k]_{mod\ N}$$

$x[n] = 2\delta[n] + 3\delta[n-1] + \delta[n-2]$ işaretinin AFDsi X[k] = [6 -1.7321j +1.732j] olduğu biliniyor.

Buna göre, x[n] = [6 -1.7321j +1.732j] işaretinin Ayrık Fourier Dönüşümü X[k] nedir?

$$x[-k]_{mod\ 3} = x[3-n]_{mod\ 3} \Rightarrow \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}$$

Buradan:

$$X[n] \leftrightarrow Nx[-k]_{mod\ N} = 3 \times [2 \quad 1 \quad 3] = [6 \quad 3 \quad 9]$$

olacaktır.

Örnek – Konjuge Simetri:

Konjuge simetriği	x[n], reel ise	$X[N-k] = X^*[k]$

$x[n] = [2 \quad 1 \quad 2 \quad 0 \quad 1 \quad 1\,]$ işaretinin Ayrık Fourier Dönüşümünü simetri özelliğinden yararlanarak hesaplayalım.

N = 6 olduğundan $X[6-k] = X^*[k]$ olacaktır. Buna göre:

k=1 için -> $X[5] = X^*[1]$

k=2 için -> $X[4] = X^*[2]$

k=3 için -> $X[3] = X^*[3]$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

olmaktadır. Eğer X[1], X[2] bilinirse X[4] ve X[5] i hesaplamadan bulabiliriz.

$x[n] = [2 \quad 1 \quad 2 \quad 0 \quad 1 \quad 1\,]$ işaretinin Ayrık Fourier Dönüşümünü simetri özelliğinden yararlanarak hesaplayalım.

$$X[0] = \sum_{n=0}^{5} x[n]\, e^{-j\frac{2\pi}{5}.0.n} = 7 \qquad X[1] = \sum_{n=0}^{5} x[n]\, e^{-j\frac{2\pi}{5}.1.n} = 1.5 - j0.86 \qquad X[2] = \sum_{n=0}^{5} x[n]\, e^{-j\frac{2\pi}{5}.2.n} = 0.5 - j0.86$$

$$X[3] = \sum_{n=0}^{5} x[n]\, e^{-j\frac{2\pi}{5}.2.n} = 3 \qquad X[4] = X^*[2] = 0.5 + j0.86 \qquad X[5] = X^*[1] = 1.5 + j0.86$$

k=1 için -> $X[5] = X^*[1]$

k=2 için -> $X[4] = X^*[2]$

$$X[k] = [7 \quad 1.5 - j0.86 \quad 0.5 - j0.86 \quad 3 \quad 0.5 + j0.86 \quad 1.5 + j0.86]$$

MATLAB Code for DFT

```
clc; clear all;
%%
x = [1 2 2 1 1 2 3 4];
N = length(x);
X = zeros(1,N);

for k = 0:N-1
    for n = 0:N-1
        X(k+1) = X(k+1) + x(n+1)*exp(-j*(2*pi/N)*k*n);
    end
end

X
fft(x)
```

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

Matrix Form for N-pt DFT

- In MATLAB, NxN DFT matrix is `dftmtx(N)`
  - Obtain DFT by `X = dftmtx(N)*x`
  - Or, **<u>more efficiently by</u>** `X = fft(x,N)`

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j2\pi/N} & e^{-j4\pi/N} & \cdots & e^{-j2(N-1)\pi/N} \\ 1 & e^{-j4\pi/N} & e^{-j8\pi/N} & \cdots & e^{-j4(N-1)\pi/N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j2(N-1)\pi/N} & e^{-j4(N-1)\pi/N} & \cdots & e^{-j2(N-1)(N-1)\pi/N} \end{bmatrix}}_{\text{DFT matrix}} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\text{Signal vector}}$$

Understanding DFT Matrix

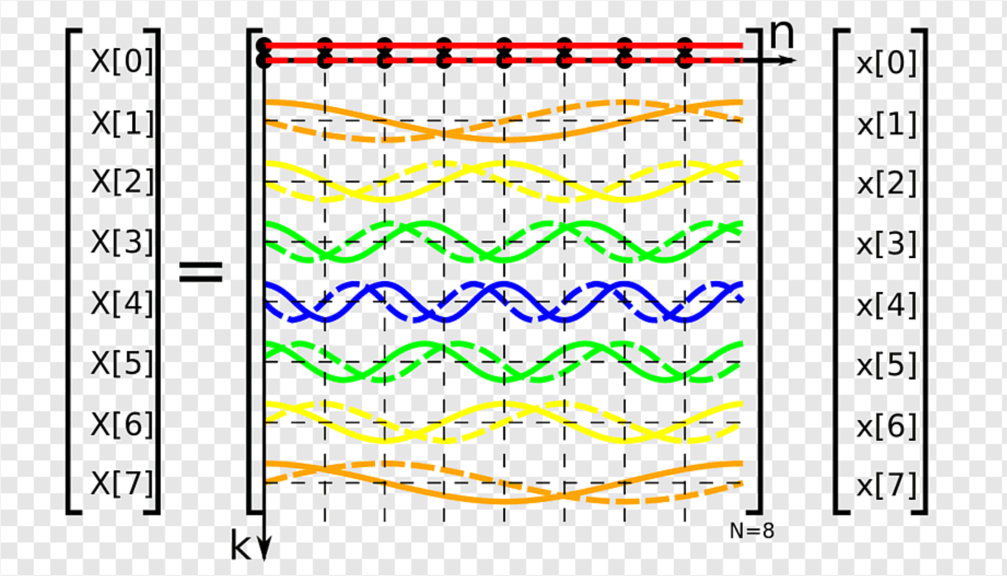
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-2\pi i/4} & e^{-4\pi i/4} & e^{-6\pi i/4} \\ 1 & e^{-4\pi i/4} & e^{-8\pi i/4} & e^{-12\pi i/4} \\ 1 & e^{-6\pi i/4} & e^{-12\pi i/4} & e^{-18\pi i/4} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

```
n = 0:0.01:5; x = cos(1.8*pi*n);
N = length(x); X = zeros(1,N);

DFTM = dftmtx(N);
figure(1);
for i =1:8
    plot(real(DFTM(i,:)));      hold on;
    plot(x,'r');      hold off;      pause;
end

figure(2); plot(abs(fft(x,N)));
```



Example about Conv. Property

Given x = [1 1 0 0] and h = [0 0 1 1], compute the output by using convolution property.

Convolution	$\sum_{m=0}^{N-1} h[m]x[((n-m))_N]$	$H[k]X[k]$

1- Compute H[k] using 4-pt DFT,

2- Compute X[k] using 4-pt DFT,

3- Product them in freq. Domain,

4- Compute y[n] by using 4-pt IDFT

```
x = [1 2 3 4];
h = [1 1 0 0];

X = myDFT(x);
H = myDFT(h);

Y = X.*H;
y = myIDFT(Y)
```

Circular convolution!

Linear vs. Circular Convolution

- Note that the results of linear and circular convolution are different. This is a problem! Why?
- All LTI systems are based on the principle of linear convolution, as the output of an LTI system is the linear convolution of the system impulse response and the input to the system, which is equivalent to the product of the respective DTFTs in the frequency domain.
  - However, if we use DFT instead of DTFT (so that we can compute it using a computer), then the result appear to be invalid:
    - DTFT is based on linear convolution, and DFT is based on circular convolution, and they are not the same!!!
    - For starters, they are not even of equal length: For two sequences of length N and M, the linear convolution is of length N+M-1, whereas circular convolution of the same two sequences is of length max(N,M), where the shorter sequence is zero padded to make it the same length as the longer one.
    - Is there any relationship between the linear and circular convolutions? Can one be obtained from the other? OR can they be made equivalent?

Linear vs. Circular Convolution

- YES!, rather easily, as a matter of fact!
  - **FACT:** If we *zero pad* both sequences x[n] and h[n], so that they are both of length N+M-1, then linear convolution and circular convolution result in identical sequences
  - **Furthermore:** If the respective DFTs of the zero padded sequences are X[k] and H[k], then the inverse DFT of X[k]·H[k] is equal to the linear convolution of x[n] and h[n]
  - Note that, normally, the inverse DFT of X[k].H[k] is the circular convolution of x[n] and h[n]. If they are zero padded, then the inverse DFT is the linear convolution of the two.

With Zero Padding

Conv. Length = N + M – 1 ->   CL = 2*N-1,   Zero-Pad signals with N+1

If N = 4, then

```
x = [1 2 3 4 0 0 0 0 0];
h = [1 1 0 0 0 0 0 0 0];

X = myDFT(x);
H = myDFT(h);

Y = X.*H;
y = real(myIDFT(Y))

conv(x,h)
```

DFT of an Image

The 2D DFT is

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi(ux/M+vy/N)}$$

Fourier basis element

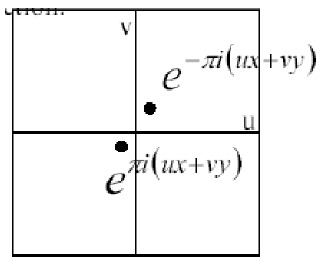
$$e^{-i2\pi(ux+vy)}$$

example, real part

$F^{u,v}(x,y)$

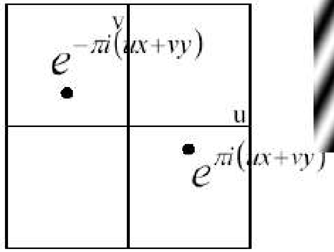
$F^{u,v}(x,y)$=const. for $(ux+vy)$=const.

Vector (u,v)
- Magnitude gives frequency
- Direction gives orientation.



Slide credit: S. Thrun

Here u and v are larger than in the previous slide.



Slide credit: S. Thrun

The 2D DFT is

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi(ux/M+vy/N)}$$

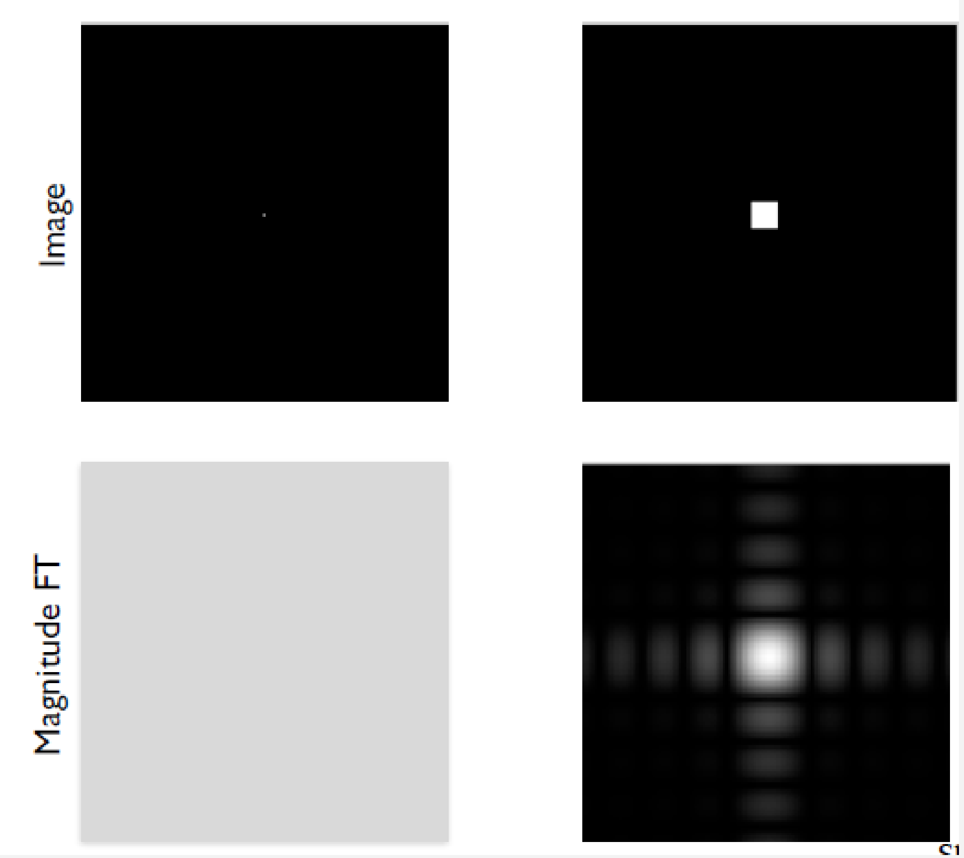
Sinusoid with frequency = 1 and its FFT

Sinusoid with frequency = 10 and its FFT

DFT of an Image

The 2D DFT is

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\,e^{-j2\pi(ux/M+vy/N)}$$

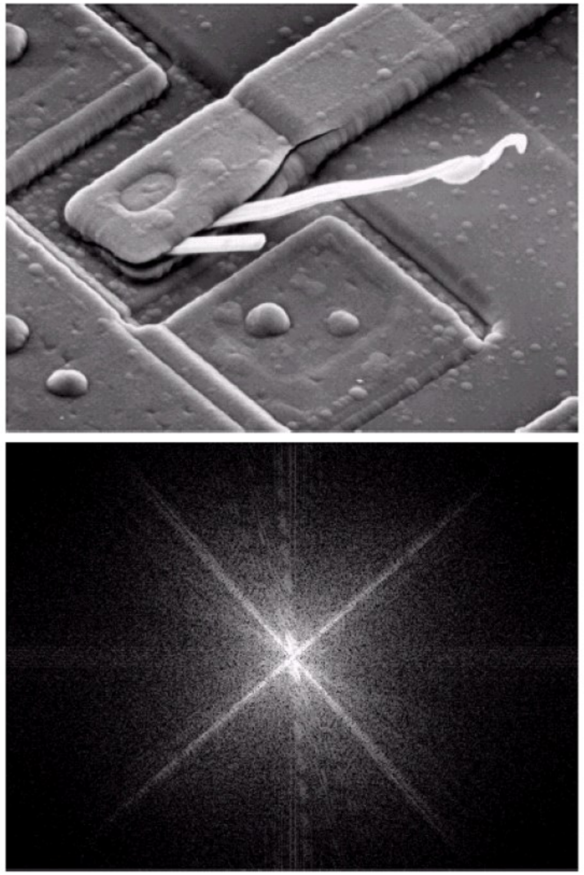


DFT of an Image

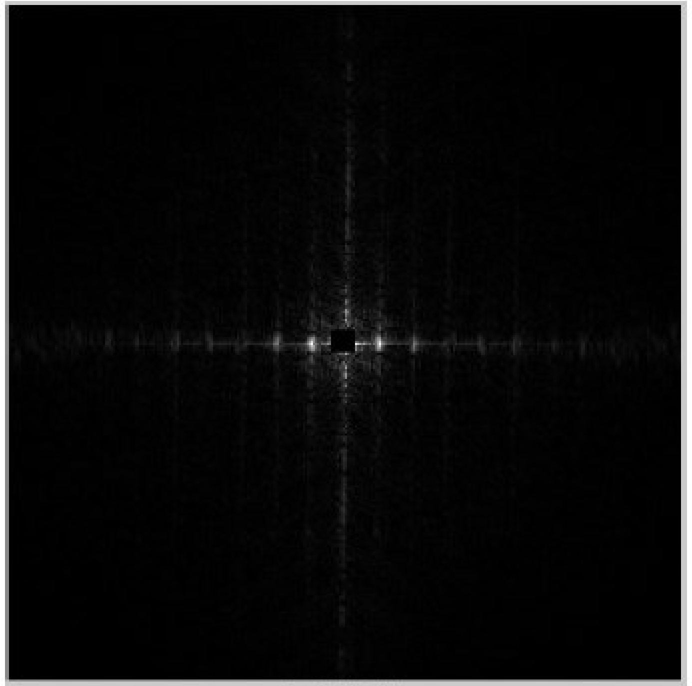
The 2D DFT is

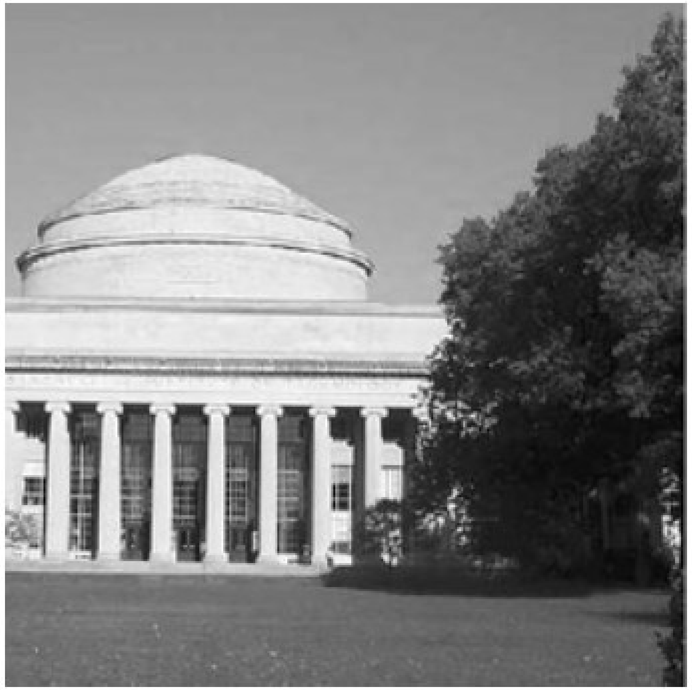
$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi(ux/M + vy/N)}$$

```
I = double(imread('moon.tif'));
imshow(I, []);
F = fft2(I);
figure, imshow(log(abs(fftshift(F))),[]);
```



Pop-up Quiz



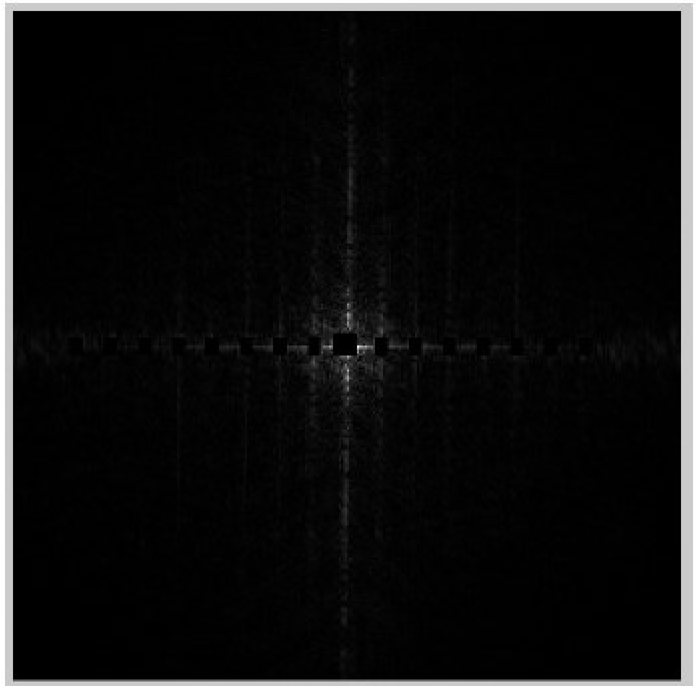


What in the image causes the dots?

Slide credit: B. Freeman and A. Torralba ba

Pop-up Quiz

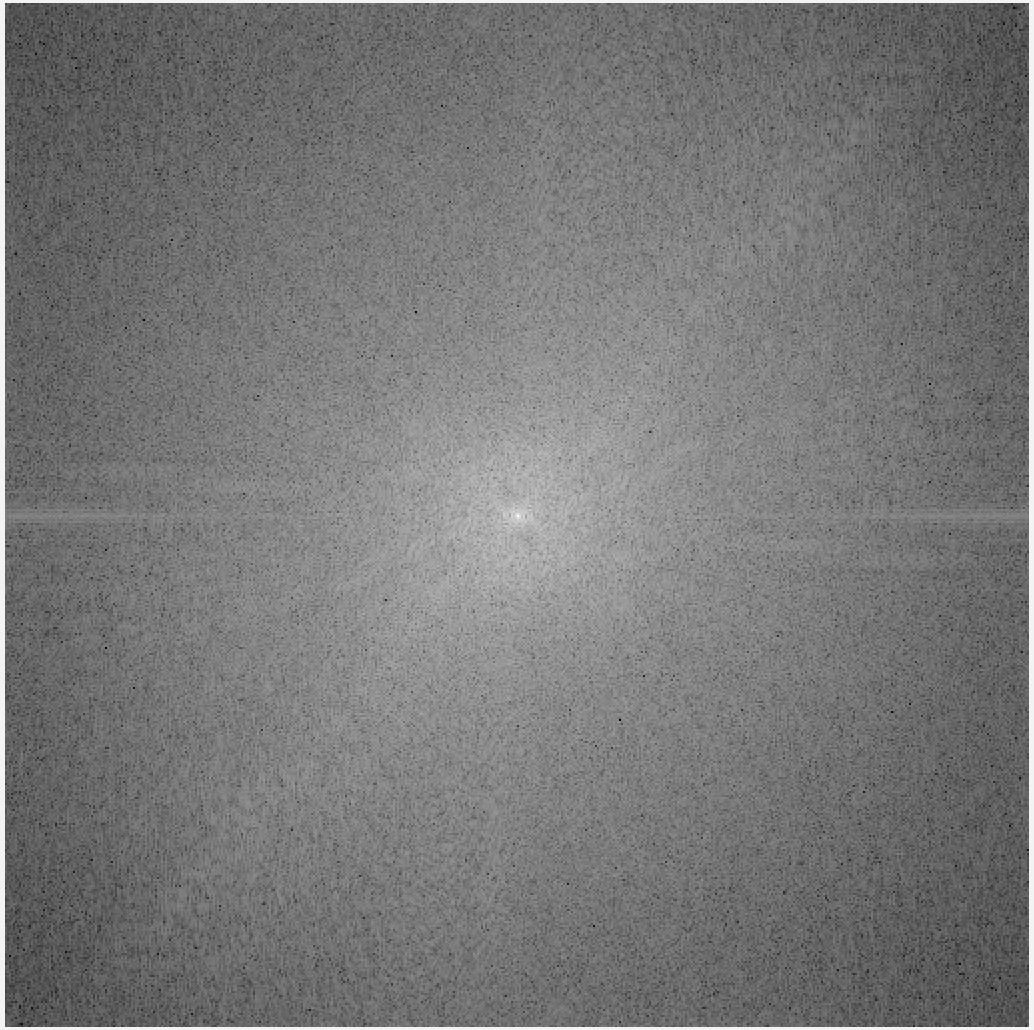
Masking out the fundamental and harmonics from periodic pillars



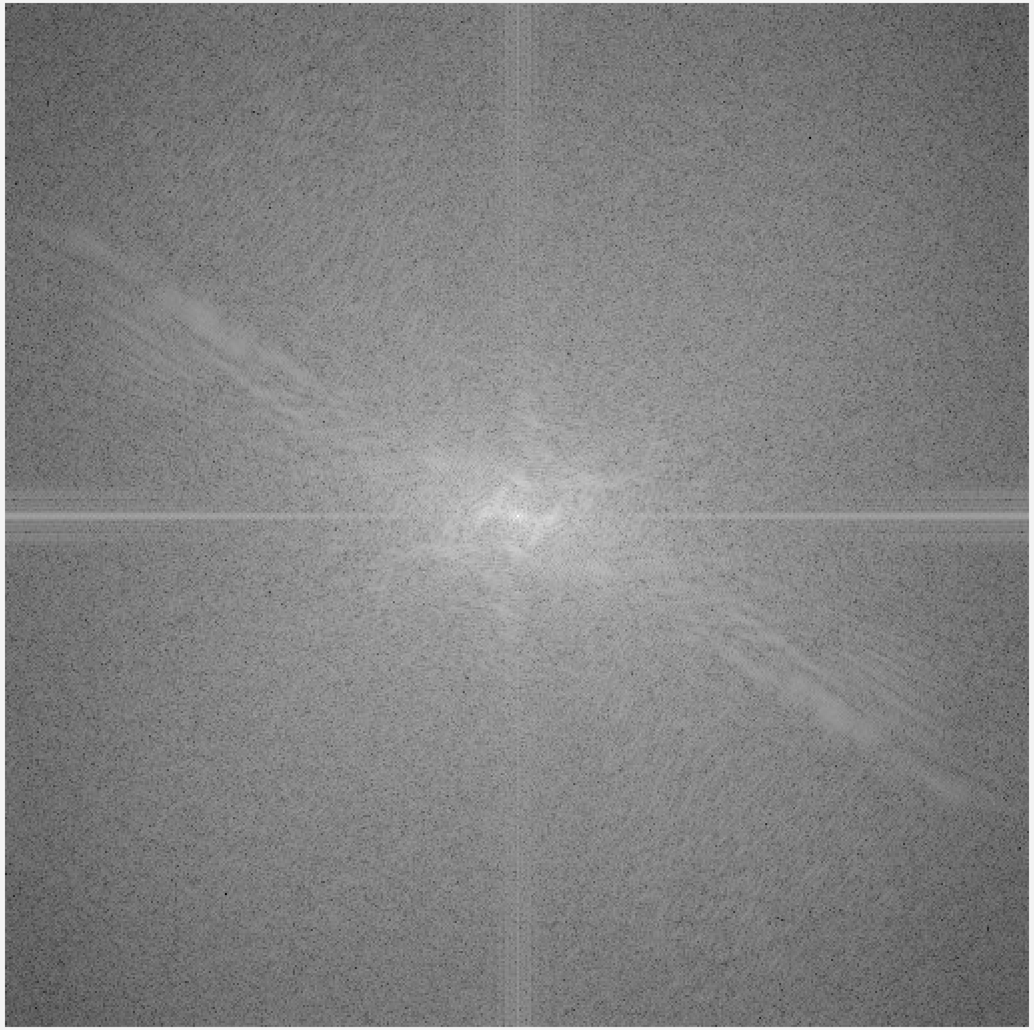
Range [0, 3.29e+005]
Dims [256, 256]

Range [0.000551, 297]
Dims [256, 256]

An Interesting Experiment: Cheetah vs Zebra



An Interesting Experiment: Cheetah vs Zebra



Reconstruction with zebra phase, cheetah magnitude