PDAs Accept
Context-Free Languages

Theorem:

$$\left\{ \begin{array}{c} \text{Context-Free} \\ \text{Languages} \\ \text{(Grammars)} \end{array} \right\} = \left\{ \begin{array}{c} \text{Languages} \\ \text{Accepted by} \\ \text{PDAs} \end{array} \right\}$$

Proof - Step 1:

$$\left\{ \begin{array}{c} \text{Context-Free} \\ \text{Languages} \\ \text{(Grammars)} \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{Languages} \\ \text{Accepted by} \\ \text{PDAs} \end{array} \right\}$$

Convert any context-free grammar G to a PDA M with: $L(G) = L(M)$

Proof - Step 2:

$$\left\{ \begin{array}{c} \text{Context-Free} \\ \text{Languages} \\ \text{(Grammars)} \end{array} \right\} \supseteq \left\{ \begin{array}{c} \text{Languages} \\ \text{Accepted by} \\ \text{PDAs} \end{array} \right\}$$

Convert any PDA M to a context-free grammar G with: $L(G) = L(M)$

Proof - step 1

Convert

Context-Free Grammars
to
PDAs
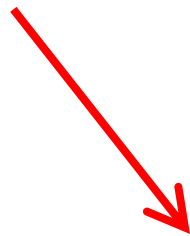
Take an arbitrary context-free grammar G

We will convert G to a PDA M such that:

$$L(G) = L(M)$$
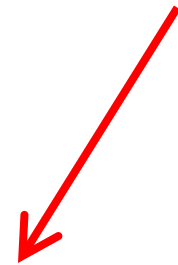
Conversion Procedure:

For each production in G

For each terminal in G

$$A \rightarrow w$$

$$a$$

Add transitions

$$\varepsilon, A \rightarrow w$$

$$a, a \rightarrow \varepsilon$$

$$\varepsilon, \varepsilon \rightarrow S \qquad \varepsilon, \$ \rightarrow \$$$

$q_0 \qquad q_1 \qquad q_2$
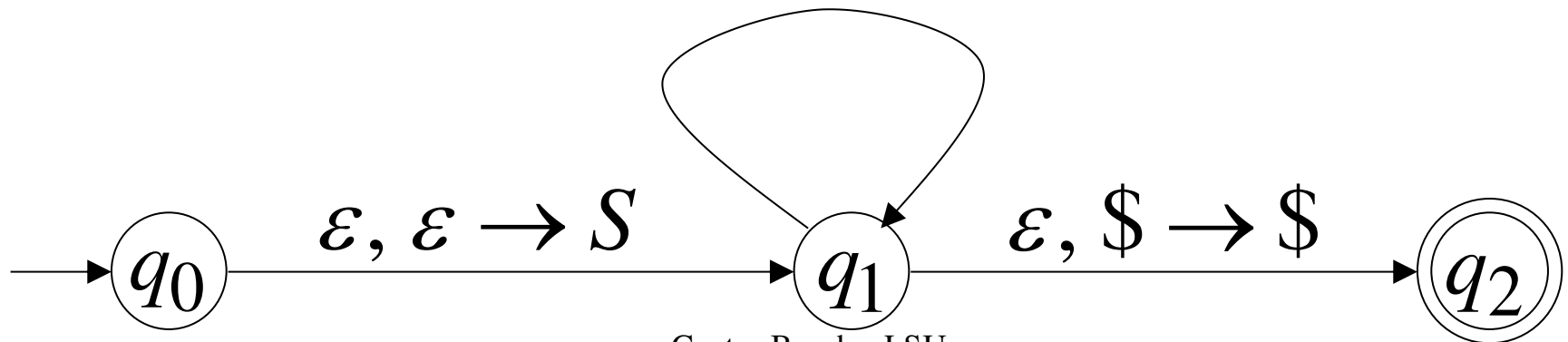
Example

Grammar

$$S \rightarrow aSTb$$

$$S \rightarrow b$$

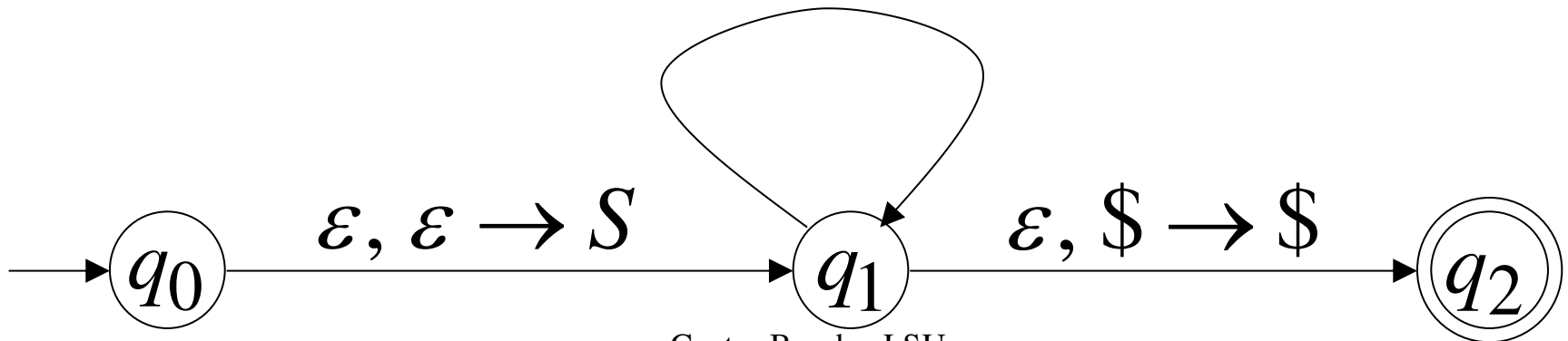$$T \rightarrow Ta$$

$$T \rightarrow \varepsilon$$

PDA

$$\varepsilon, S \rightarrow aSTb$$

$$\varepsilon, S \rightarrow b$$

$$\varepsilon, T \rightarrow Ta \qquad a, a \rightarrow \varepsilon$$

$$\varepsilon, T \rightarrow \varepsilon \qquad b, b \rightarrow \varepsilon$$

$$\xrightarrow{\quad} \ q_0 \ \xrightarrow{\ \varepsilon, \varepsilon \rightarrow S\ } \ q_1 \ \xrightarrow{\ \varepsilon, \$ \rightarrow \$\ } \ q_2$$

PDA simulates leftmost derivations

Grammar
Leftmost Derivation

S

$\Rightarrow \cdots$

$\Rightarrow \boxed{\sigma_1 \cdots \sigma_k} \boxed{X_1 \cdots X_m}$

$\Rightarrow \cdots$

$\Rightarrow \sigma_1 \cdots \sigma_k \sigma_{k+1} \cdots \sigma_n$
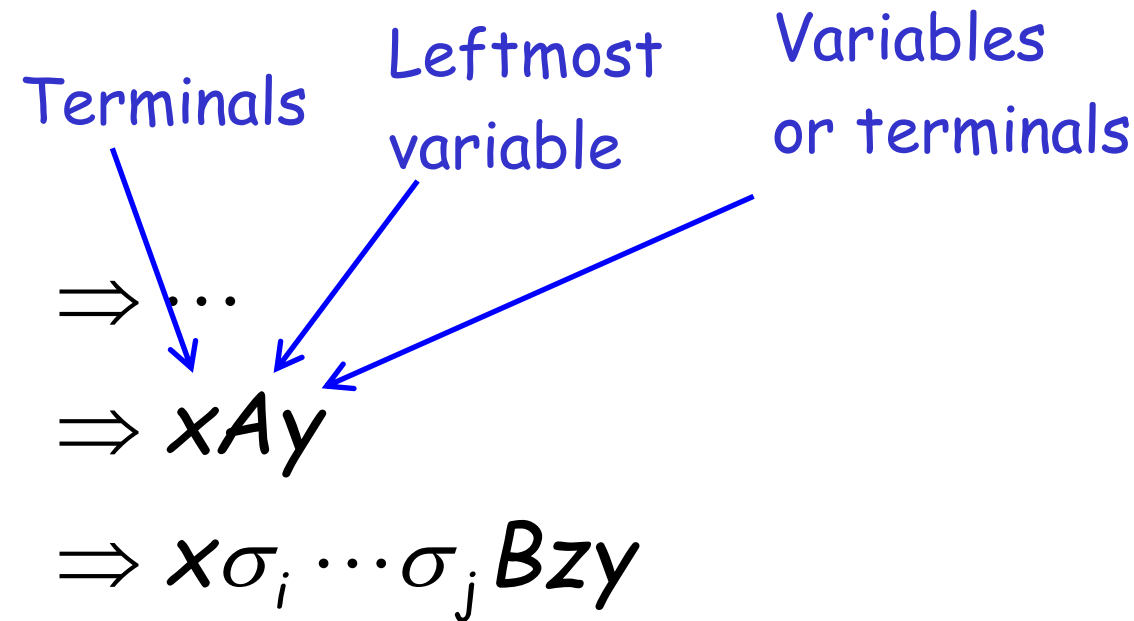
PDA Computation

$(q_0, \sigma_1 \cdots \sigma_k \sigma_{k+1} \cdots \sigma_n, \$)$

$\succ (q_1, \boxed{\sigma_1 \cdots \sigma_k} \sigma_{k+1} \cdots \sigma_n, S\$)$

$\succ \cdots$

$\succ (q_1, \sigma_{k+1} \cdots \sigma_n, \boxed{X_1 \cdots X_m} \$)$

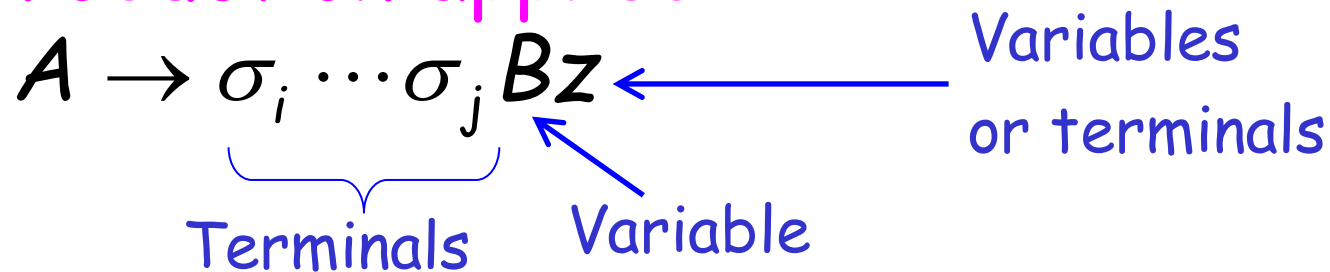$\succ \cdots$

$\succ (q_2, \varepsilon, \$)$

Scanned
symbols

Stack
contents

Grammar
Leftmost Derivation

Terminals

Leftmost variable

Variables or terminals

$$\Rightarrow \cdots$$

$$\Rightarrow xAy$$

$$\Rightarrow x\sigma_i \cdots \sigma_j Bzy$$

Production applied

$$A \rightarrow \sigma_i \cdots \sigma_j Bz$$

Variables or terminals

Terminals

Variable

Grammar
Leftmost Derivation

$\Rightarrow \cdots$

$\Rightarrow xAy$

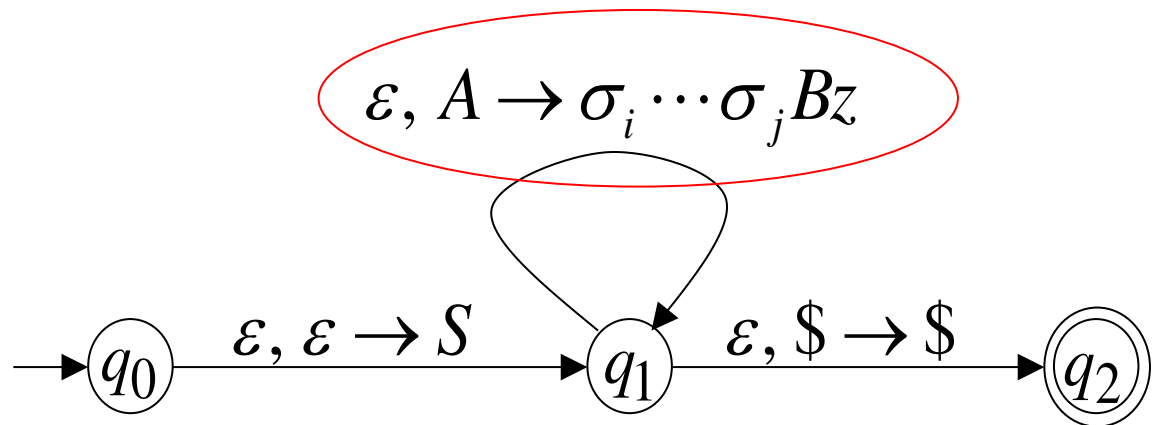$\Rightarrow x\sigma_i \cdots \sigma_j Bzy$

PDA Computation

$\succ \cdots$

$\succ (q_1, \sigma_i \cdots \sigma_n, Ay\$)$

$\succ (q_1, \sigma_i \cdots \sigma_n, \sigma_i \cdots \sigma_j Bzy\$)$

Production applied

$A \rightarrow \sigma_i \cdots \sigma_j Bz$

Transition applied

$\varepsilon, A \rightarrow \sigma_i \cdots \sigma_j Bz$

$\rightarrow (q_0) \xrightarrow{\varepsilon, \varepsilon \rightarrow S} (q_1) \xrightarrow{\varepsilon, \$ \rightarrow \$} ((q_2))$

Grammar Leftmost Derivation

$\Rightarrow \cdots$

$\Rightarrow xAy$

$\Rightarrow x\sigma_i \cdots \sigma_j Bzy$

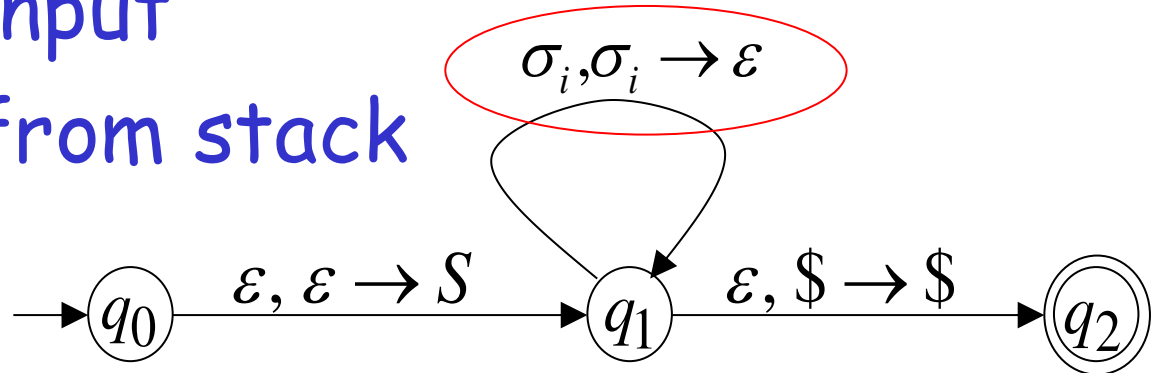# PDA Computation

$\succ \cdots$

$\succ (q_1, \sigma_i \cdots \sigma_n, Ay\$)$

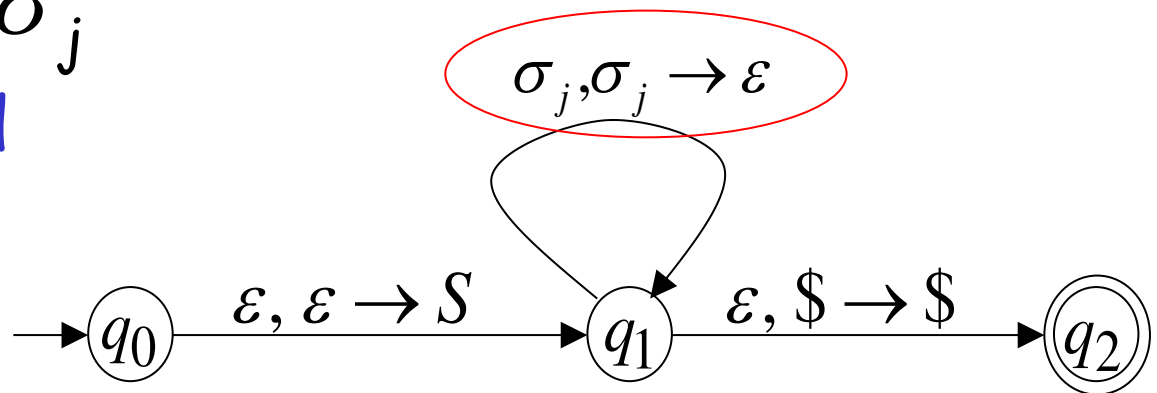$\succ (q_1, \sigma_i \cdots \sigma_n, \sigma_i \cdots \sigma_j Bzy\$)$

$\succ (q_1, \sigma_{i+1} \cdots \sigma_n, \sigma_{i+1} \cdots \sigma_j Bzy\$)$

Read σ_i from input and remove it from stack

Transition applied

$\sigma_i, \sigma_i \rightarrow \varepsilon$

$\rightarrow (q_0) \xrightarrow{\varepsilon, \varepsilon \rightarrow S} (q_1) \xrightarrow{\varepsilon, \$ \rightarrow \$} (q_2)$

Grammar
Leftmost Derivation

$\Rightarrow \cdots$

$\Rightarrow xAy$

$\Rightarrow x\sigma_i \cdots \sigma_j Bzy$

PDA Computation

$\succ \cdots$

$\succ (q_1, \sigma_i \cdots \sigma_n, Ay\$)$

$\succ (q_1, \sigma_i \cdots \sigma_n, \sigma_i \cdots \sigma_j Bzy\$)$

$\succ (q_1, \sigma_{i+1} \cdots \sigma_n, \sigma_{i+1} \cdots \sigma_j Bzy\$)$

$\succ \cdots$

$\succ (q_1, \sigma_{j+1} \cdots \sigma_n, Bzy\$)$

Last Transition applied

All symbols $\sigma_i \cdots \sigma_j$

have been removed
from top of stack

$$\sigma_j, \sigma_j \to \varepsilon$$

$\rightarrow (q_0) \xrightarrow{\varepsilon, \varepsilon \to S} (q_1) \xrightarrow{\varepsilon, \$ \to \$} (q_2)$

The process repeats with the next leftmost variable

$\Rightarrow \cdots$

$\Rightarrow xAy$

$\Rightarrow x\sigma_i \cdots \sigma_j Bzy$

$\Rightarrow x\sigma_i \cdots \sigma_j \sigma_{j+1} \cdots \sigma_k Cpzy$

$\succ \cdots$

$\succ (q_1, \sigma_{j+1} \cdots \sigma_n, Bzy\$)$

$\succ (q_1, \sigma_{j+1} \cdots \sigma_n, \sigma_{j+1} \cdots \sigma_k Cpzy\$)$

$\succ \cdots$

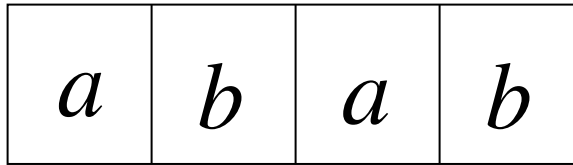$\succ (q_1, \sigma_{k+1} \cdots \sigma_n, Cpzy\$)$

Production applied

$$B \rightarrow \sigma_{j+1} \cdots \sigma_k Cp$$

And so on……

Example:

Input

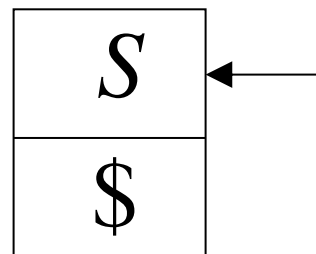| a	b	a	b

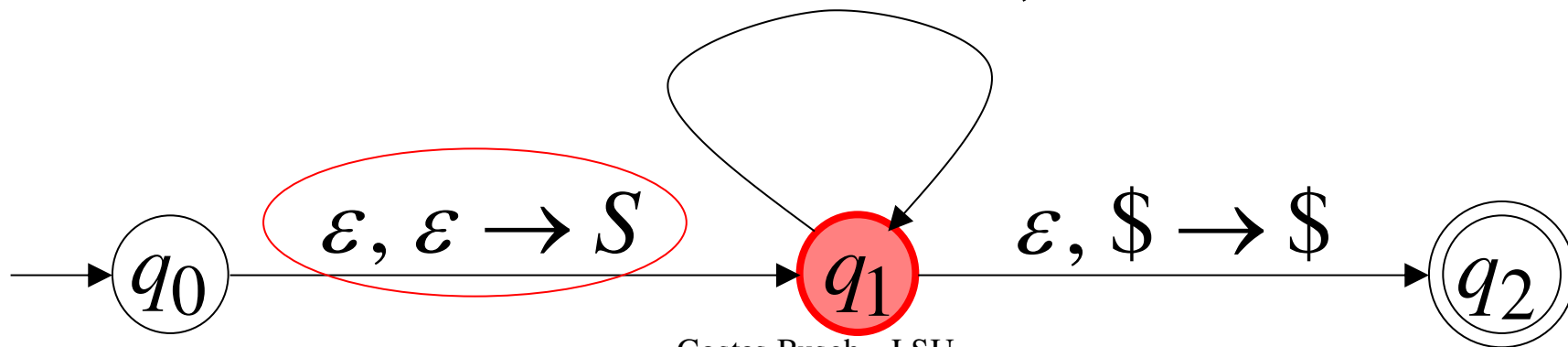Time 0

$$\varepsilon, S \rightarrow aSTb$$

$$\varepsilon, S \rightarrow b$$

$$\varepsilon, T \rightarrow Ta \qquad a, a \rightarrow \varepsilon$$

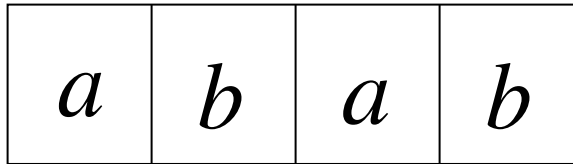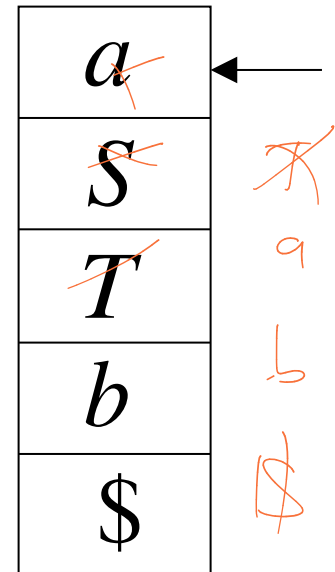$$\varepsilon, T \rightarrow \varepsilon \qquad b, b \rightarrow \varepsilon$$
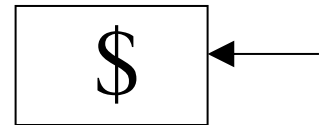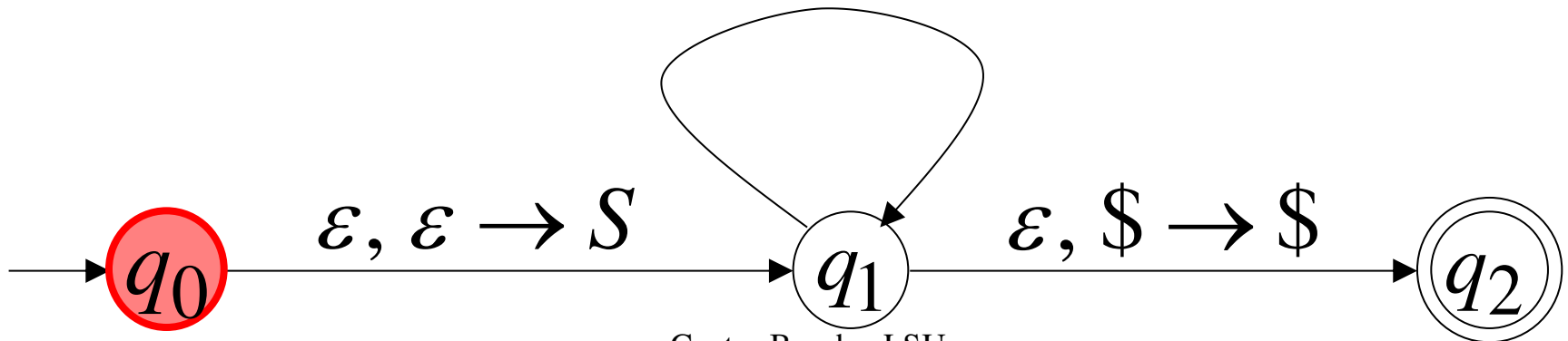
$\$$

Stack

$$\varepsilon, \varepsilon \rightarrow S \qquad\qquad \varepsilon, \$ \rightarrow \$$$

q_0 $\qquad$ q_1 $\qquad$ q_2

Derivation: S

Input

a	b	a	b

Time 1

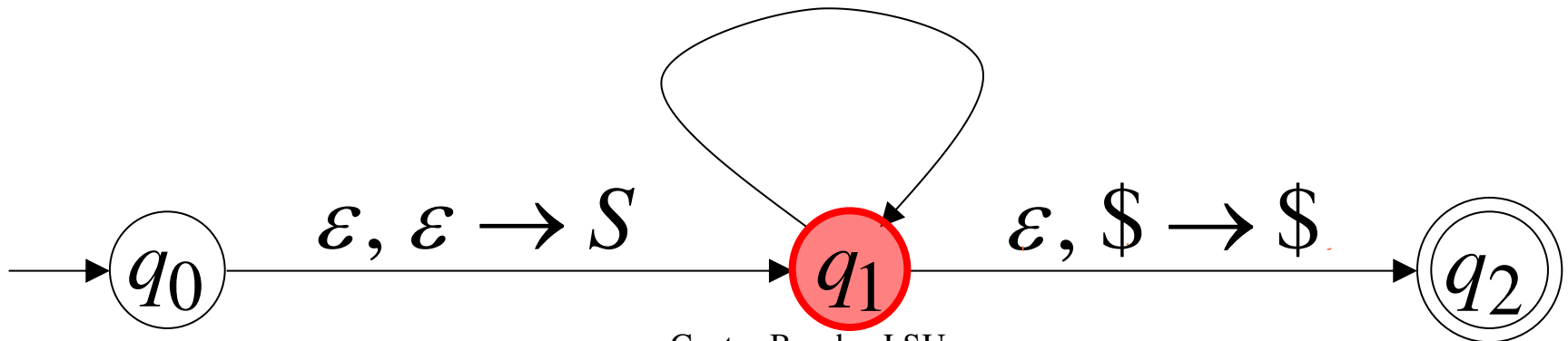$$\varepsilon, S \rightarrow aSTb$$

$$\varepsilon, S \rightarrow b$$

$$\varepsilon, T \rightarrow Ta \qquad a, a \rightarrow \varepsilon$$

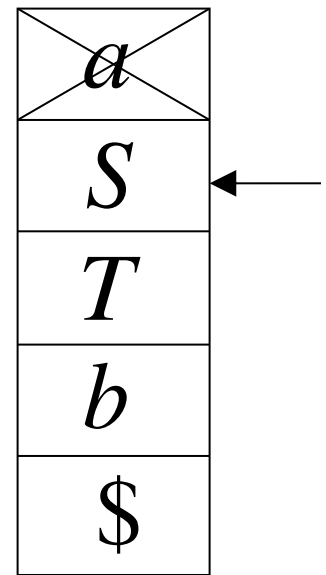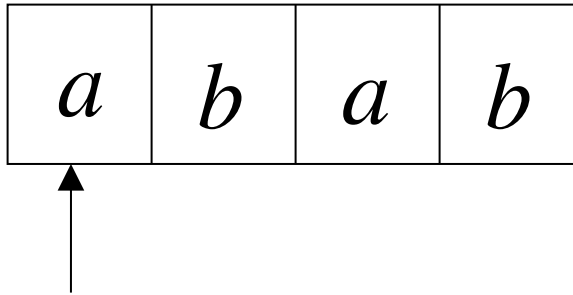$$\varepsilon, T \rightarrow \varepsilon \qquad b, b \rightarrow \varepsilon$$

S
$\$$
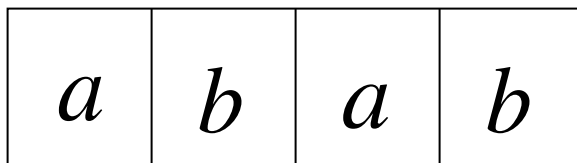
Stack

$$q_0 \quad \varepsilon, \varepsilon \rightarrow S \quad q_1 \quad \varepsilon, \$ \rightarrow \$ \quad q_2$$

Derivation: $S \Rightarrow aSTb$

Input

a	b	a	b

Stack

a
S
T
b
$\$$

Time 2

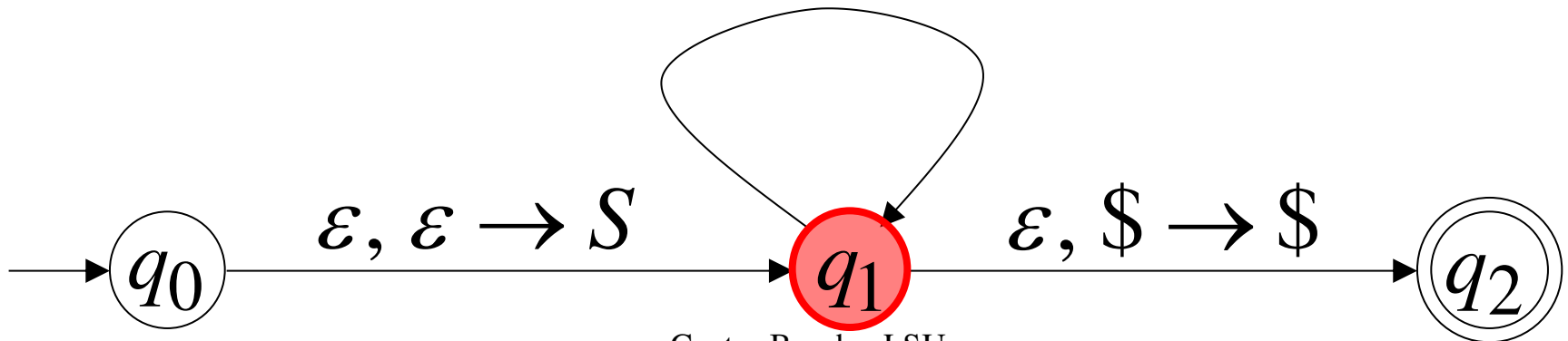$\varepsilon, S \rightarrow aSTb$

$\varepsilon, S \rightarrow b$

$\varepsilon, T \rightarrow Ta \qquad a, a \rightarrow \varepsilon$
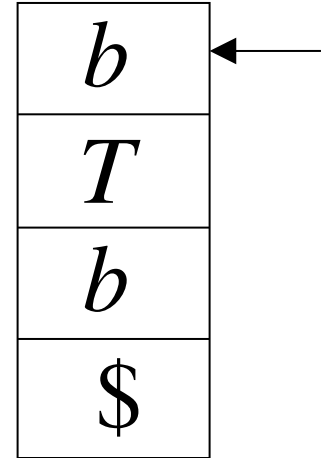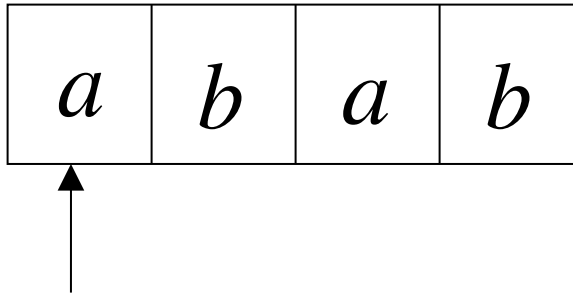
$\varepsilon, T \rightarrow \varepsilon \qquad b, b \rightarrow \varepsilon$

$q_0 \xrightarrow{\ \varepsilon, \varepsilon \rightarrow S\ } q_1 \xrightarrow{\ \varepsilon, \$ \rightarrow \$\ } q_2$

Derivation: $S \Rightarrow aSTb$

Input

a	b	a	b

Stack

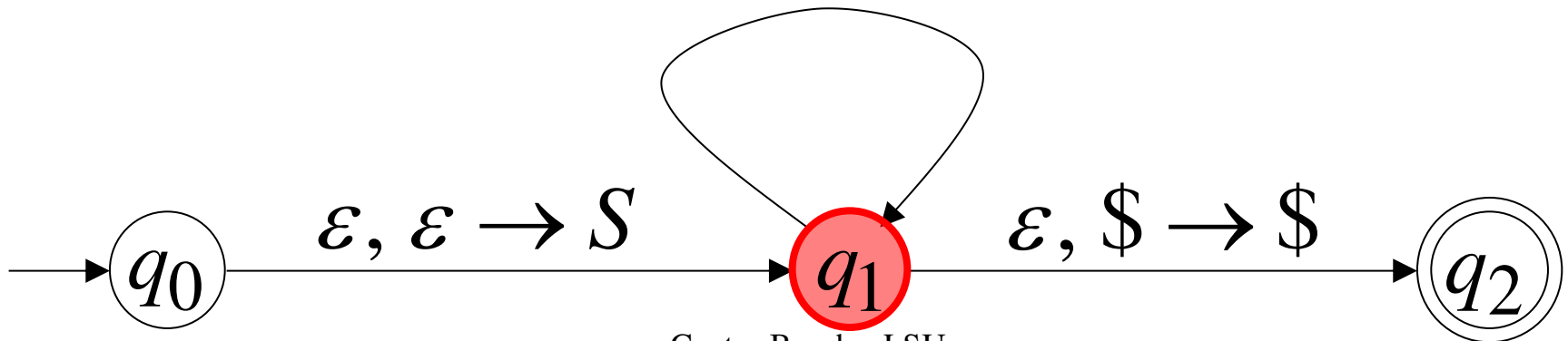| a |
|-----|
| S |
| T |
| b |
| $\$$ |

Time 3

$$\varepsilon, S \rightarrow aSTb$$

$$\varepsilon, S \rightarrow b$$

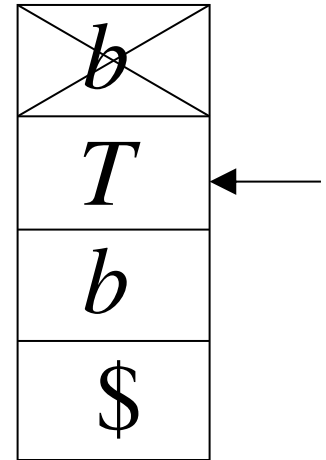$$\varepsilon, T \rightarrow Ta \qquad a, a \rightarrow \varepsilon$$

$$\varepsilon, T \rightarrow \varepsilon \qquad b, b \rightarrow \varepsilon$$

$$\varepsilon, \varepsilon \rightarrow S \qquad q_1 \qquad \varepsilon, \$ \rightarrow \$$$

$q_0 \qquad q_1 \qquad q_2$

Derivation: $S \Rightarrow aSTb \Rightarrow abTb$

Input

a	b	a	b

Stack

b
T
b
$\$$

Time 4

$\varepsilon, S \rightarrow aSTb$

$\varepsilon, S \rightarrow b$

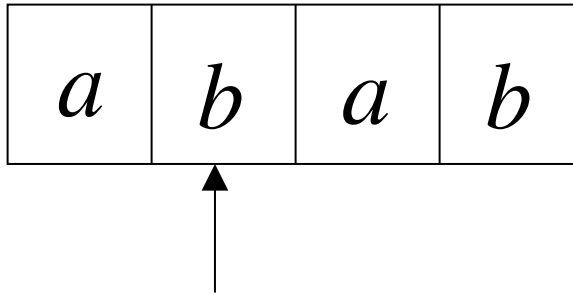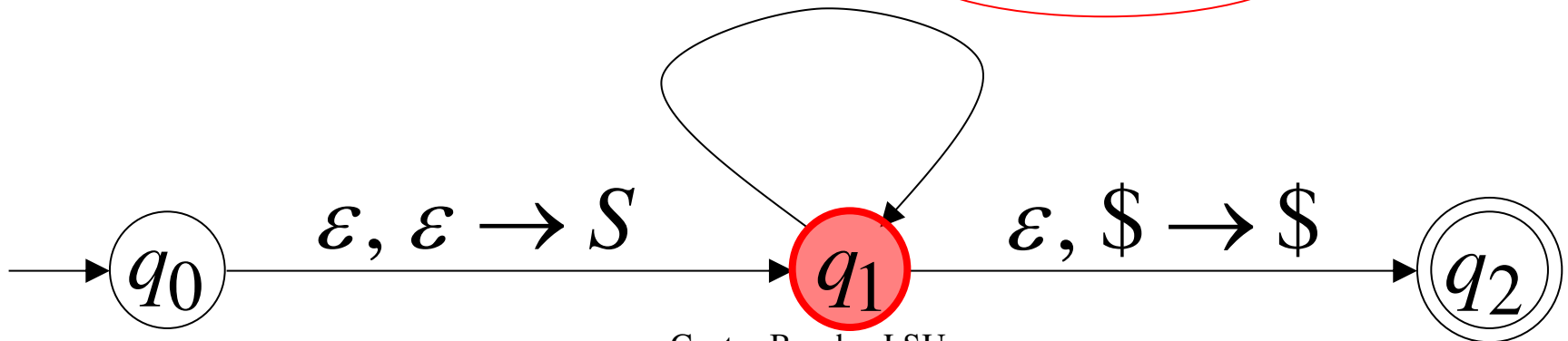$\varepsilon, T \rightarrow Ta \qquad a, a \rightarrow \varepsilon$

$\varepsilon, T \rightarrow \varepsilon \qquad b, b \rightarrow \varepsilon$

$q_0 \xrightarrow{\varepsilon, \varepsilon \rightarrow S} q_1 \xrightarrow{\varepsilon, \$ \rightarrow \$} q_2$

Derivation: $S \Rightarrow aSTb \Rightarrow abTb$

Input

a	b	a	b

Time 5

$\varepsilon, S \rightarrow aSTb$

$\varepsilon, S \rightarrow b$

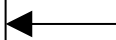$\varepsilon, T \rightarrow Ta \qquad a, a \rightarrow \varepsilon$

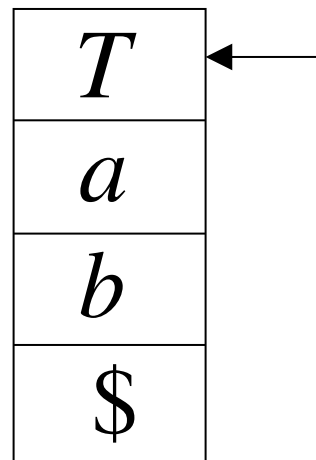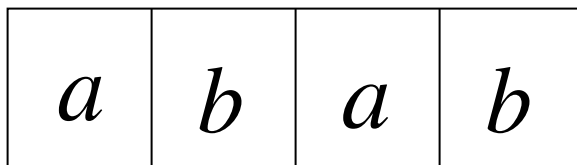$\varepsilon, T \rightarrow \varepsilon \qquad b, b \rightarrow \varepsilon$

Stack

b
T
b
$\$$

$\varepsilon, \varepsilon \rightarrow S$ q_0 → q_1 $\varepsilon, \$ \rightarrow \$$ → q_2

Derivation: $S \Rightarrow aSTb \Rightarrow abTb \Rightarrow abTab$

Input

a	b	a	b

Stack

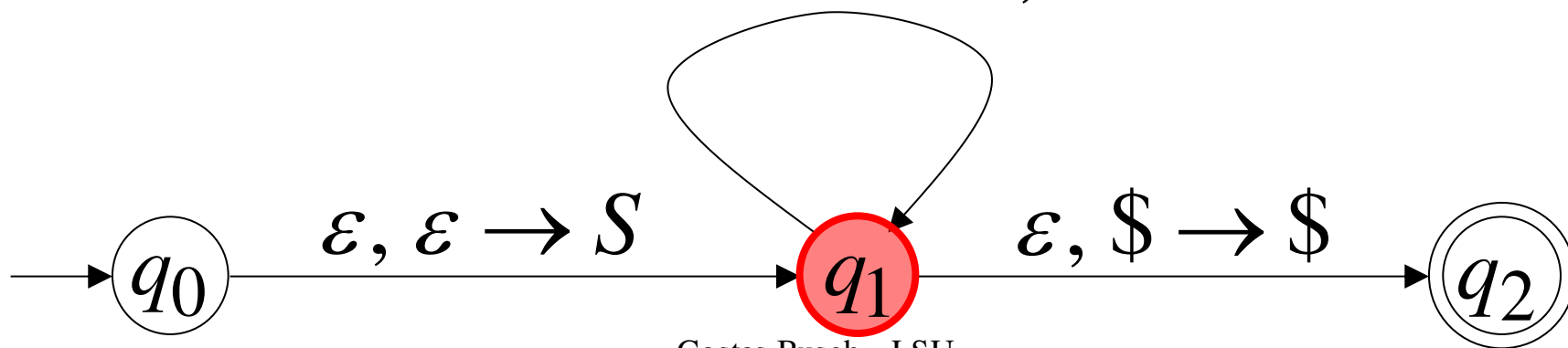| T |
|---|
| a |
| b |
| $\$ |

Time 6

$\varepsilon, S \rightarrow aSTb$

$\varepsilon, S \rightarrow b$

$\varepsilon, T \rightarrow Ta$ $a, a \rightarrow \varepsilon$

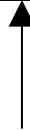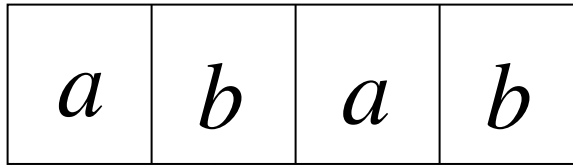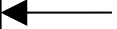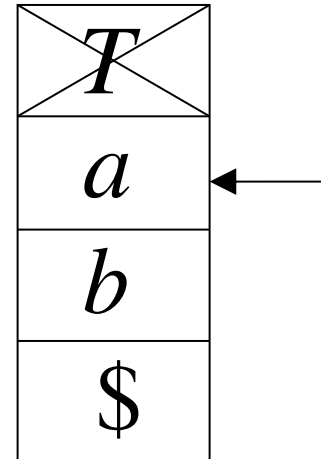$\varepsilon, T \rightarrow \varepsilon$ $b, b \rightarrow \varepsilon$

q_0 —— $\varepsilon, \varepsilon \rightarrow S$ —— q_1 —— $\varepsilon, \$ \rightarrow \$$ —— q_2

Derivation: $S \Rightarrow aSTb \Rightarrow abTb \Rightarrow abTab \Rightarrow abab$

Input

a	b	a	b

Stack
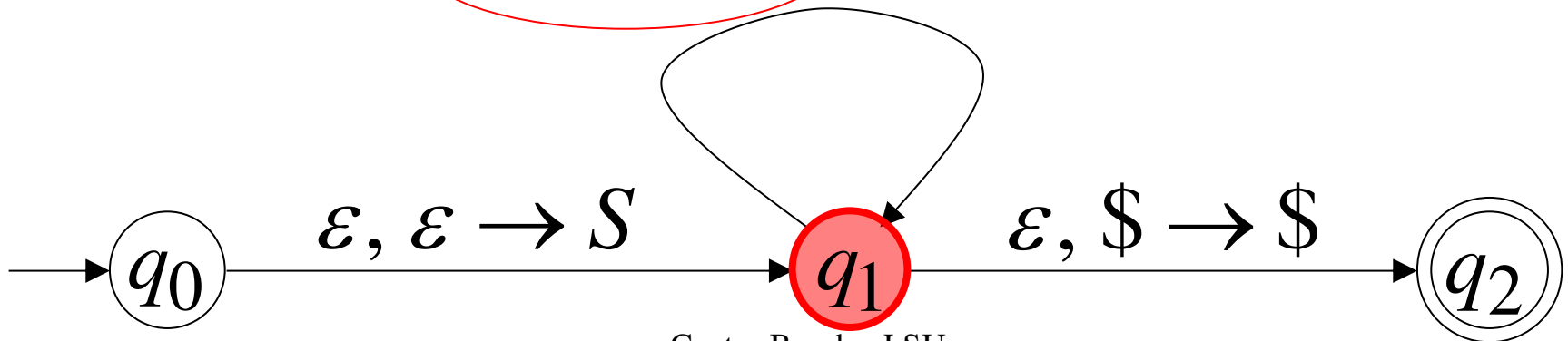
T
a
b
$\$$

Time 7

$\varepsilon, S \rightarrow aSTb$

$\varepsilon, S \rightarrow b$

$\varepsilon, T \rightarrow Ta \qquad a, a \rightarrow \varepsilon$

$\varepsilon, T \rightarrow \varepsilon \qquad b, b \rightarrow \varepsilon$

$q_0 \qquad \varepsilon, \varepsilon \rightarrow S \qquad q_1 \qquad \varepsilon, \$ \rightarrow \$ \qquad q_2$

Derivation: $S \Rightarrow aSTb \Rightarrow abTb \Rightarrow abTab \Rightarrow abab$

Input

a	b	a	b

Time 8

$\varepsilon, S \rightarrow aSTb$

$\varepsilon, S \rightarrow b$

$\varepsilon, T \rightarrow Ta \qquad a, a \rightarrow \varepsilon$

$\varepsilon, T \rightarrow \varepsilon \qquad b, b \rightarrow \varepsilon$

Stack

a
b
$\$$

$\varepsilon, \varepsilon \rightarrow S \qquad\qquad q_1 \qquad \varepsilon, \$ \rightarrow \$$

$q_0 \qquad\qquad\qquad\qquad q_1 \qquad\qquad\qquad\qquad q_2$

Derivation: $S \Rightarrow aSTb \Rightarrow abTb \Rightarrow abTab \Rightarrow abab$

Input

a	b	a	b

Time 9

$\varepsilon, S \rightarrow aSTb$

$\varepsilon, S \rightarrow b$

$\varepsilon, T \rightarrow Ta$ $a, a \rightarrow \varepsilon$

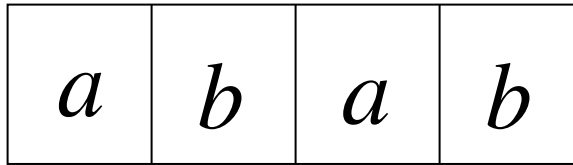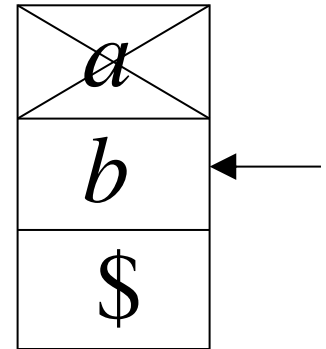$\varepsilon, T \rightarrow \varepsilon$ $b, b \rightarrow \varepsilon$
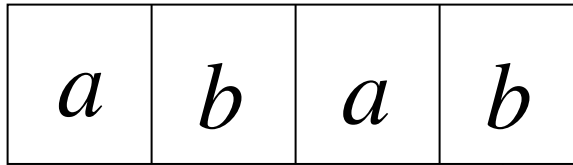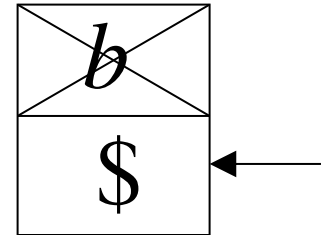
Stack

$\cancel{b}$
$\$$

$\varepsilon, \varepsilon \rightarrow S$ $\varepsilon, \$ \rightarrow \$$

q_0 q_1 q_2

Derivation: $S \Rightarrow aSTb \Rightarrow abTb \Rightarrow abTab \Rightarrow abab$

Input

a	b	a	b

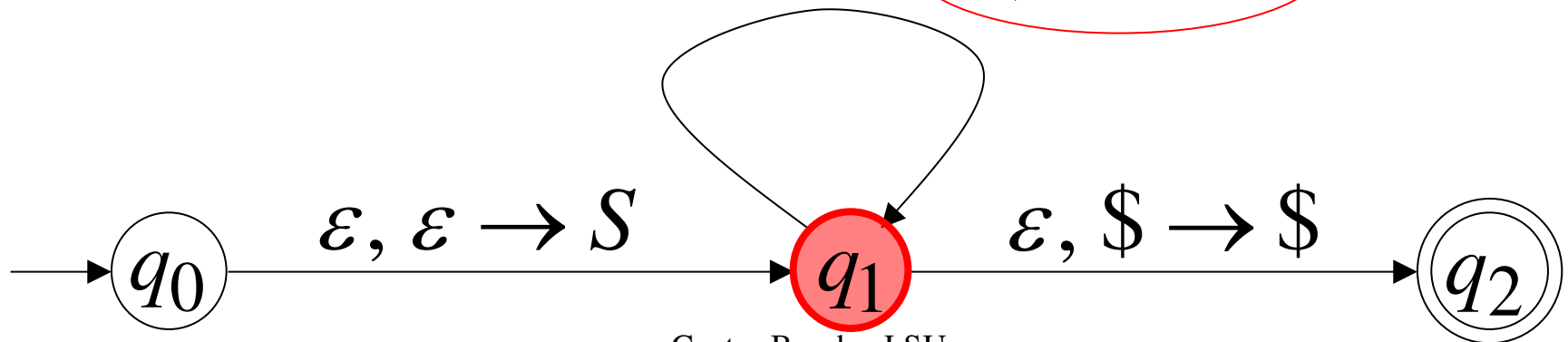**Time 10**

$\varepsilon, S \rightarrow aSTb$

$\varepsilon, S \rightarrow b$

$\varepsilon, T \rightarrow Ta \qquad a, a \rightarrow \varepsilon$

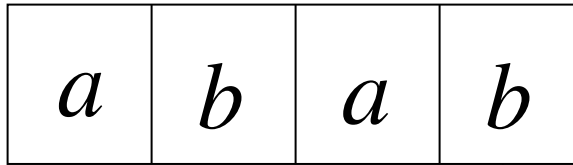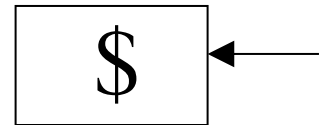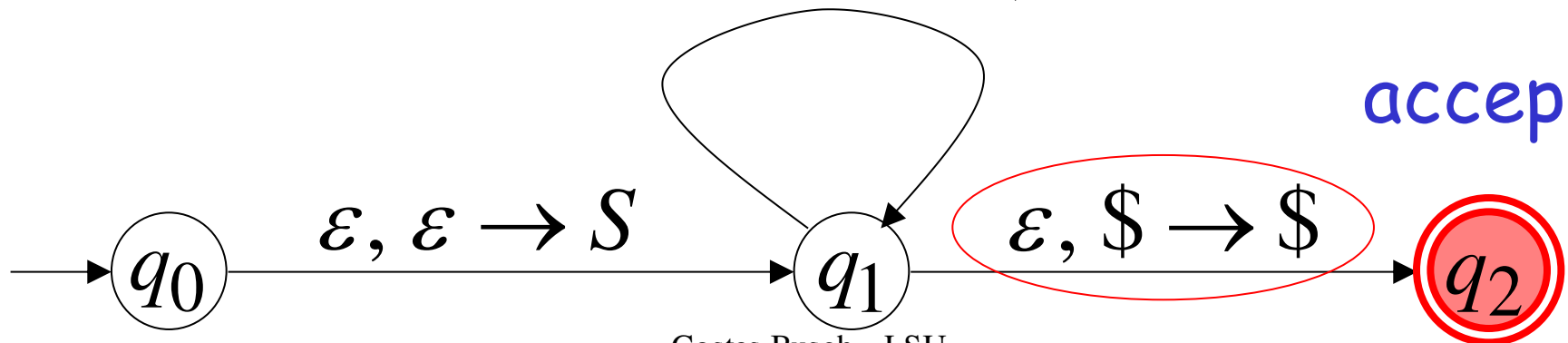$\varepsilon, T \rightarrow \varepsilon \qquad b, b \rightarrow \varepsilon$

$$\boxed{\$}$$

Stack

accept

$q_0 \quad \varepsilon, \varepsilon \rightarrow S \quad q_1 \quad \varepsilon, \$ \rightarrow \$ \quad q_2$

Grammar
Leftmost Derivation

S

$\Rightarrow aSTb$

$\Rightarrow abTb$

$\Rightarrow abTab$

$\Rightarrow abab$

PDA Computation

$(q_0, abab, \$)$

$\succ (q_1, abab, S\$)$

$\succ (q_1, bab, STb\$)$

$\succ (q_1, bab, bTb\$)$

$\succ (q_1, ab, Tb\$)$

$\succ (q_1, ab, Tab\$)$

$\succ (q_1, ab, ab\$)$
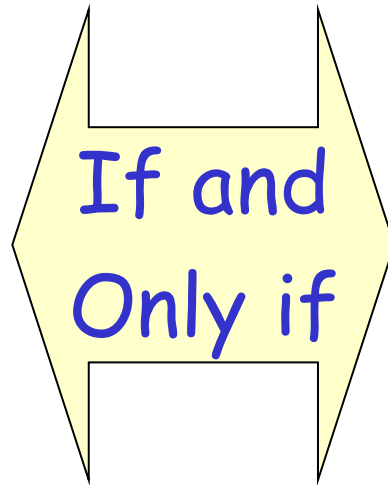
$\succ (q_1, b, b\$)$

$\succ (q_1, \varepsilon, \$)$

$\succ (q_2, \varepsilon, \$)$

In general, it can be shown that:

Grammar G
generates
string w

$$S \overset{*}{\Rightarrow} w$$

If and
Only if

PDA M
accepts w

$$(q_0, w, \$) \overset{*}{\succ} (q_2, \varepsilon, \$)$$

Therefore $L(G) = L(M)$

Proof - step 2

Convert

PDAs
to
Context-Free Grammars

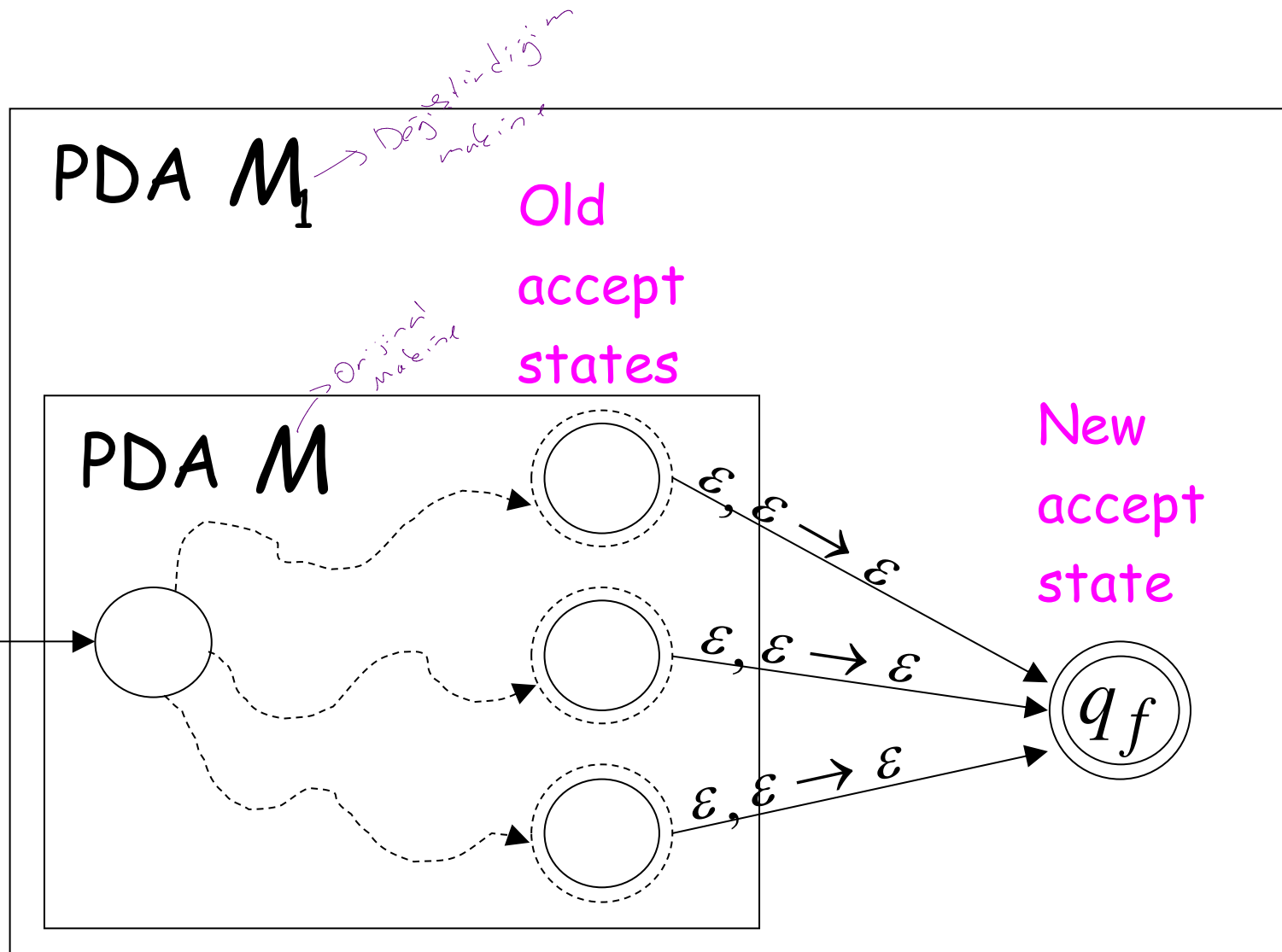Take an arbitrary PDA M

We will convert M
to a context-free grammar G such that:
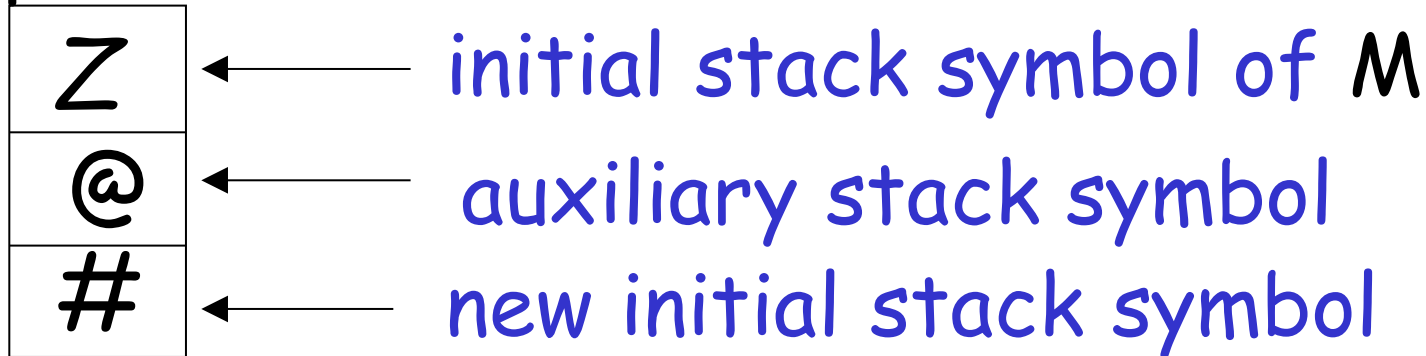
$$L(M) = L(G)$$

First modify PDA M so that:

1. The PDA has a single accept state

2. Use new initial stack symbol #

3. On acceptance the stack contains only stack symbol # (this symbol is not used in any transition)

4. Each transition either pushes a symbol or pops a symbol but not both together

1. The PDA has a single accept state
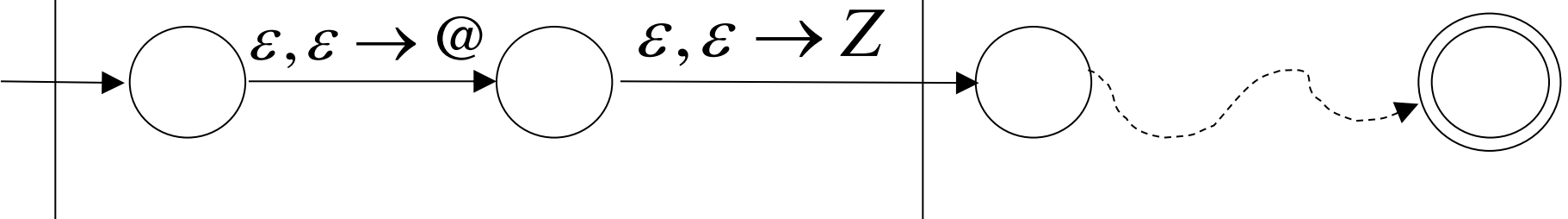

PDA M_1 → Değiştirdiğim machine

Old accept states

→ Original machine

PDA M

New accept state

$\varepsilon, \varepsilon \to \varepsilon$

$\varepsilon, \varepsilon \to \varepsilon$

$\varepsilon, \varepsilon \to \varepsilon$

q_f

2. Use new initial stack symbol

Top of stack

Z	← initial stack symbol of M
@	← auxiliary stack symbol
#	← new initial stack symbol

PDA M_2

PDA M_1

$\varepsilon, \varepsilon \to @$ $\varepsilon, \varepsilon \to Z$

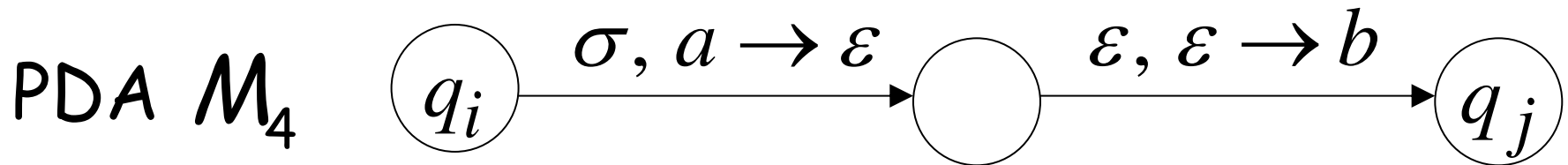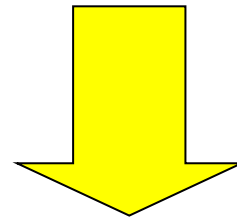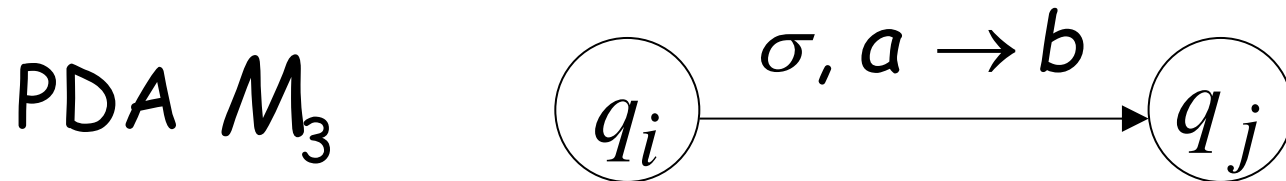M_1 still thinks that Z is the initial stack

3. On acceptance the stack contains only stack symbol
(this symbol is not used in any transition)

PDA M_3

Empty stack

$\forall x \in \Gamma - \{@, \#\}$

PDA M_2

Old accept state

$\varepsilon, x \to \varepsilon$

New accept state

$\varepsilon, \varepsilon \to \varepsilon$

$\varepsilon, @ \to \varepsilon$

q_f

4. Each transition either pushes a symbol or pops a symbol but not both together

PDA M_3

$q_i \xrightarrow{\;\sigma,\, a \to b\;} q_j$

PDA M_4

$q_i \xrightarrow{\;\sigma,\, a \to \varepsilon\;} \bigcirc \xrightarrow{\;\varepsilon,\, \varepsilon \to b\;} q_j$

PDA M_3

$$q_i \xrightarrow{\sigma, \; \varepsilon \rightarrow \varepsilon} q_j$$

PDA M_4

$$q_i \xrightarrow{\sigma, \; \varepsilon \rightarrow \delta} \bigcirc \xrightarrow{\varepsilon, \; \delta \rightarrow \varepsilon} q_j$$
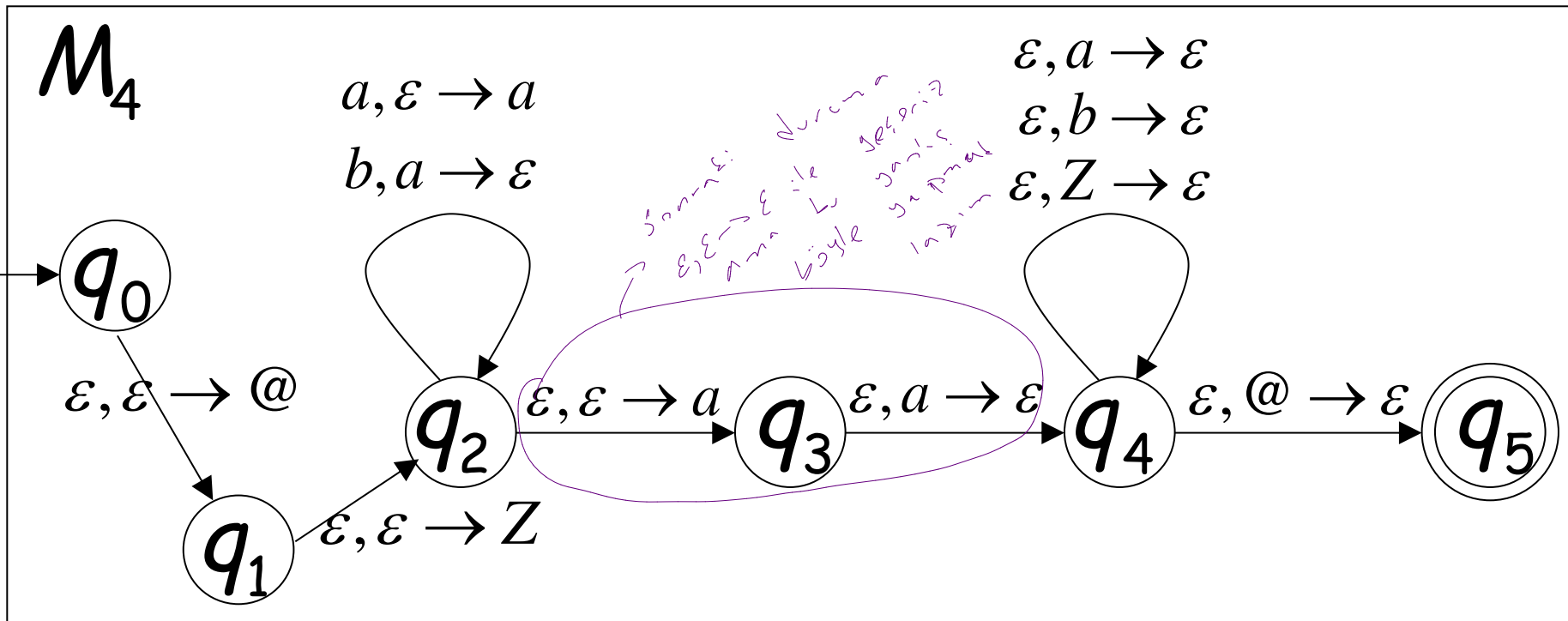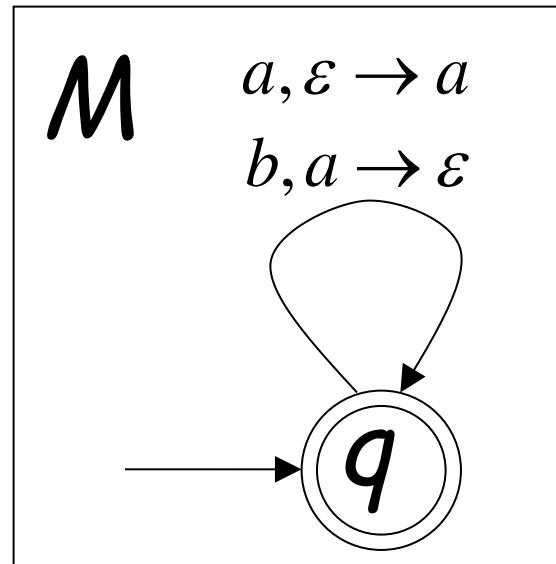
Where δ is a symbol of the stack alphabet

PDA M_4 is the final modified PDA

Note that the new initial stack symbol # is never used in any transition

Example:

M

$$a, \varepsilon \to a$$
$$b, a \to \varepsilon$$

M_4

$$a, \varepsilon \to a$$
$$b, a \to \varepsilon$$

$$\varepsilon, a \to \varepsilon$$
$$\varepsilon, b \to \varepsilon$$
$$\varepsilon, Z \to \varepsilon$$

q_0

$$\varepsilon, \varepsilon \to @$$

q_1

$$\varepsilon, \varepsilon \to Z$$

q_2

$$\varepsilon, \varepsilon \to a$$

q_3

$$\varepsilon, a \to \varepsilon$$

q_4

$$\varepsilon, @ \to \varepsilon$$

q_5

Grammar Construction

Variables: A_{q_i, q_j}

States of PDA

PDA

Kind 1: for each state

$$q$$

Grammar

$$A_{qq} \rightarrow \varepsilon$$

PDA

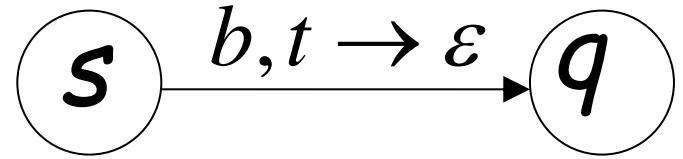## Kind 2: for every three states

$$p \qquad q \qquad r$$

Grammar

$$A_{pq} \rightarrow A_{pr} A_{rq}$$

PDA

Kind 3: for every pair of such transitions

$$p \xrightarrow{a,\varepsilon \to t} r \qquad s \xrightarrow{b,t \to \varepsilon} q$$

Grammar

$$A_{pq} \to a A_{rs} b$$

PDA

Initial state

Accept state

q_0

q_f

Grammar

Start variable

$A_{q_0 q_f}$

Example: PDA

M_4

$a, \varepsilon \to a$
$b, a \to \varepsilon$

$\varepsilon, a \to \varepsilon$
$\varepsilon, b \to \varepsilon$
$\varepsilon, Z \to \varepsilon$

q_0

$\varepsilon, \varepsilon \to @$

$\varepsilon, \varepsilon \to a \qquad q_3 \qquad \varepsilon, a \to \varepsilon \qquad q_4 \qquad \varepsilon, @ \to \varepsilon \qquad q_5$

q_2

$\varepsilon, \varepsilon \to Z$

q_1

Grammar

Kind 1: from single states

$$A_{q_0 q_0} \rightarrow \varepsilon$$

$$A_{q_1 q_1} \rightarrow \varepsilon$$

$$A_{q_2 q_2} \rightarrow \varepsilon$$

$$A_{q_3 q_3} \rightarrow \varepsilon$$

$$A_{q_4 q_4} \rightarrow \varepsilon$$

$$A_{q_5 q_5} \rightarrow \varepsilon$$

Kind 2: from triplets of states

$$A_{q_0 q_0} \rightarrow A_{q_0 q_0} A_{q_0 q_0} \mid A_{q_0 q_1} A_{q_1 q_0} \mid A_{q_0 q_2} A_{q_2 q_0} \mid A_{q_0 q_3} A_{q_3 q_0} \mid A_{q_0 q_4} A_{q_4 q_0} \mid A_{q_0 q_5} A_{q_5 q_0}$$

$$A_{q_0 q_1} \rightarrow A_{q_0 q_0} A_{q_0 q_1} \mid A_{q_0 q_1} A_{q_1 q_1} \mid A_{q_0 q_2} A_{q_2 q_1} \mid A_{q_0 q_3} A_{q_3 q_1} \mid A_{q_0 q_4} A_{q_4 q_1} \mid A_{q_0 q_5} A_{q_5 q_1}$$
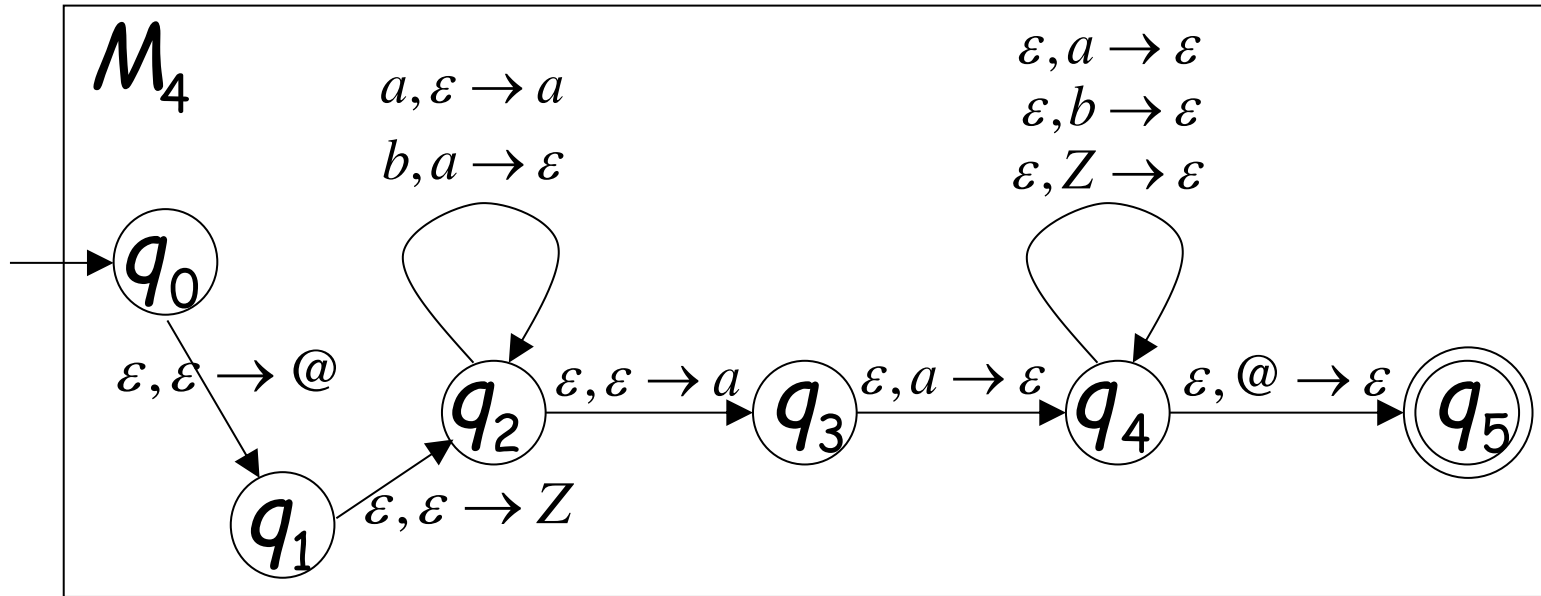
$\vdots$

$$A_{q_0 q_5} \rightarrow A_{q_0 q_0} A_{q_0 q_5} \mid A_{q_0 q_1} A_{q_1 q_5} \mid A_{q_0 q_2} A_{q_2 q_5} \mid A_{q_0 q_3} A_{q_3 q_5} \mid A_{q_0 q_4} A_{q_4 q_5} \mid A_{q_0 q_5} A_{q_5 q_5}$$

$\vdots$

$$A_{q_5 q_5} \rightarrow A_{q_5 q_0} A_{q_0 q_5} \mid A_{q_5 q_1} A_{q_1 q_5} \mid A_{q_5 q_2} A_{q_2 q_5} \mid A_{q_5 q_3} A_{q_3 q_5} \mid A_{q_5 q_4} A_{q_4 q_5} \mid A_{q_5 q_5} A_{q_5 q_5}$$

Start variable $A_{q_0 q_5}$

Kind 3: from pairs of transitions

$$A_{q_0 q_5} \rightarrow A_{q_1 q_4}$$

$$A_{q_2 q_4} \rightarrow a A_{q_2 q_4}$$

$$A_{q_2 q_2} \rightarrow A_{q_2 q_2} b$$

$$A_{q_1 q_4} \rightarrow A_{q_2 q_4}$$

$$A_{q_2 q_2} \rightarrow a A_{q_2 q_2} b$$

$$A_{q_2 q_4} \rightarrow A_{q_3 q_3}$$

$$A_{q_2 q_4} \rightarrow a A_{q_2 q_3}$$

$$A_{q_2 q_4} \rightarrow A_{q_3 q_4}$$

Suppose that a PDA M is converted to a context-free grammar G

We need to prove that $L(G) = L(M)$

or equivalently

$$L(G) \subseteq L(M)$$

$$L(G) \supseteq L(M)$$

$L(G) \subseteq L(M)$

We need to show that if G has derivation:

$$A_{q_0 q_f} \overset{*}{\Rightarrow} w \qquad \text{(string of terminals)}$$

Then there is an accepting computation in M :

$$(q_0, w, \#) \overset{*}{\succ} (q_f, \varepsilon, \#)$$

with input string w

We will actually show that if G has derivation:

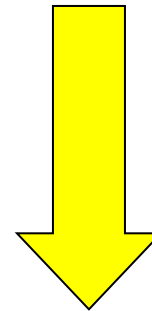$$A_{pq} \overset{*}{\Rightarrow} w$$

Then there is a computation in M :

$$(p, w, \varepsilon) \overset{*}{\succ} (q, \varepsilon, \varepsilon)$$

Therefore:

$$A_{q_0 q_f} \overset{*}{\Longrightarrow} w$$

$$(q_0, w, \varepsilon) \overset{*}{\succ} (q_f, \varepsilon, \varepsilon)$$

Since there is no transition
with the # symbol

$$(q_0, w, \#) \overset{*}{\succ} (q_f, \varepsilon, \#)$$

Lemma:

If $\quad A_{pq} \overset{*}{\Longrightarrow} w$ (string of terminals)

then there is a computation from state p to state q on string w which leaves the stack empty:

$$(p, w, \varepsilon) \overset{*}{\succ} (q, \varepsilon, \varepsilon)$$

Proof Intuition:

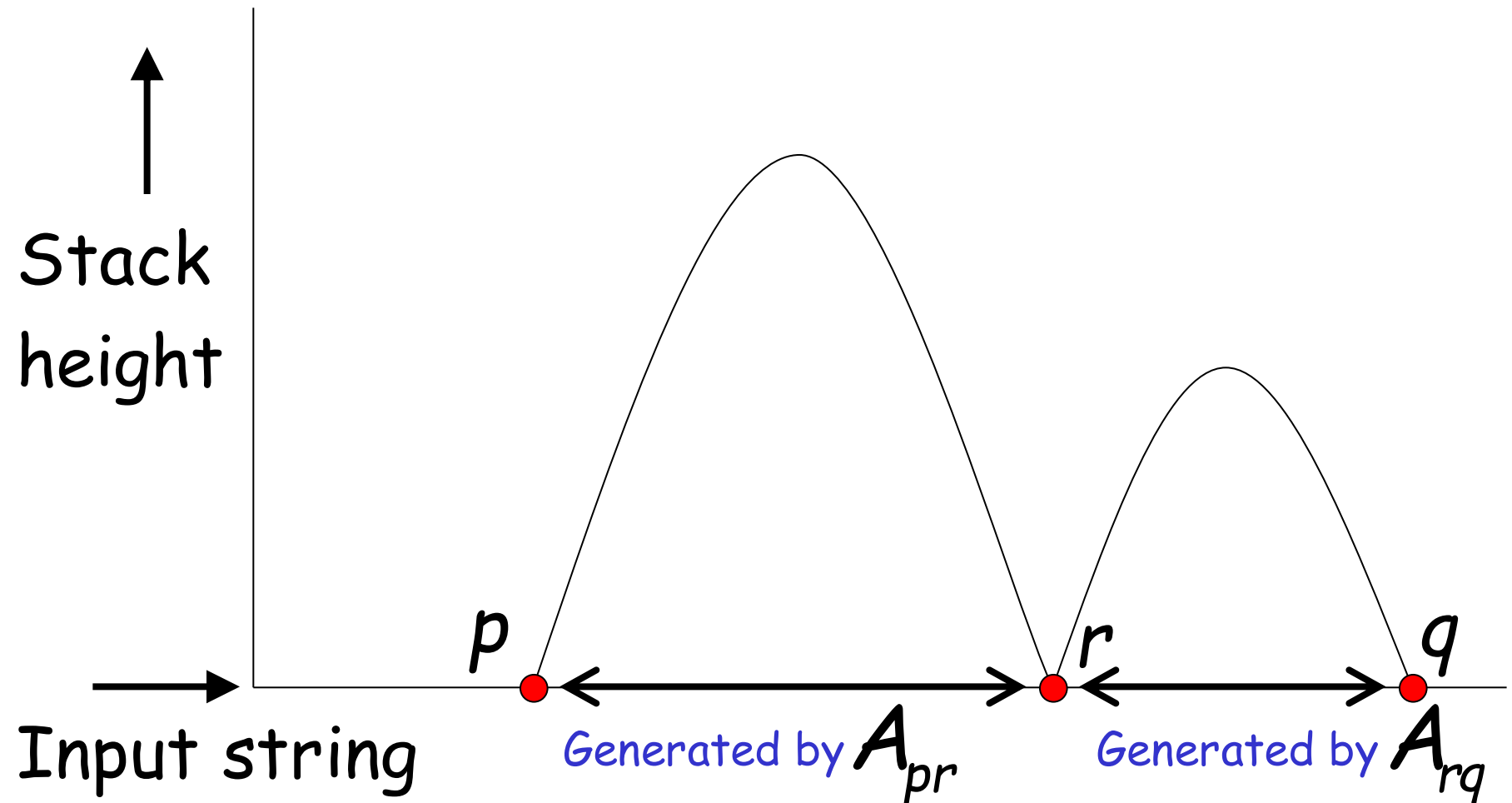$$A_{pq} \Rightarrow \cdots \Rightarrow w$$

Type 2

Case 1: $A_{pq} \Rightarrow A_{pr} A_{rq} \Rightarrow \cdots \Rightarrow w$

Type 3

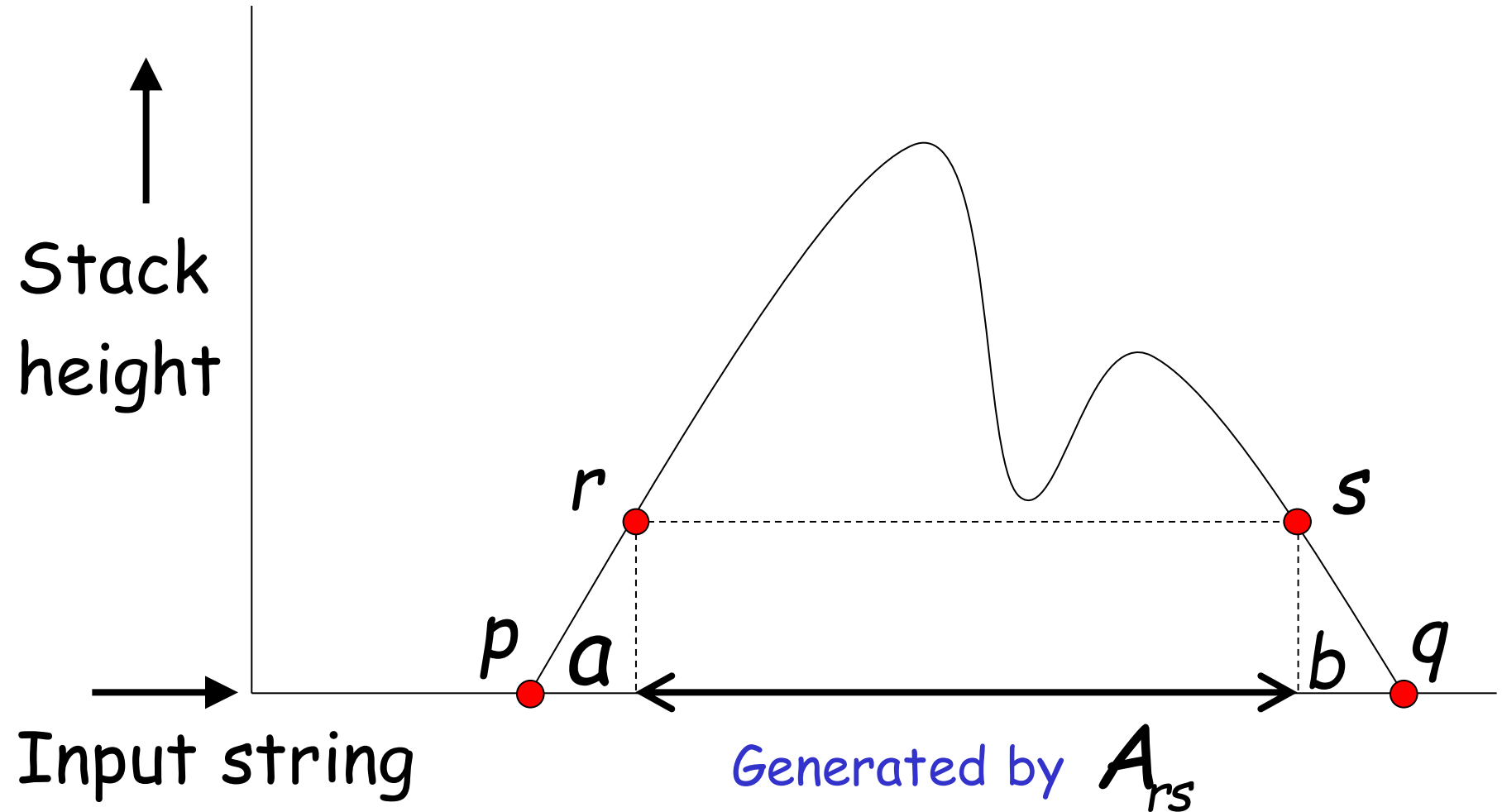Case 2: $A_{pq} \Rightarrow a A_{rs} b \Rightarrow \cdots \Rightarrow w$

Type 2

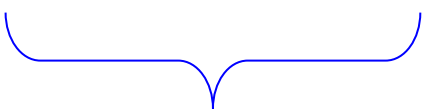Case 1: $A_{pq} \Rightarrow A_{pr} A_{rq} \Rightarrow \cdots \Rightarrow w$

Stack height

Input string

p Generated by A_{pr} r Generated by A_{rq} q

Type 3

Case 2: $A_{pq} \Rightarrow aA_{rs}b \Rightarrow \cdots \Rightarrow w$

Stack height

Input string

Generated by A_{rs}

Formal Proof:

We formally prove this claim
by induction on the number
of steps in derivation:

$$A_{pq} \Rightarrow \cdots \Rightarrow w$$

number of steps

Induction Basis: $A_{pq} \Rightarrow w$

(one derivation step)

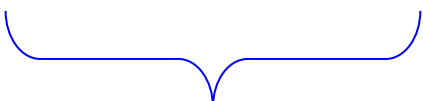A Kind 1 production must have been used:

$$A_{pp} \rightarrow \varepsilon$$

Therefore, $p = q$ and $w = \varepsilon$

This computation of PDA trivially exists:

$$(p, \varepsilon, \varepsilon) \overset{*}{\succ} (p, \varepsilon, \varepsilon)$$

Induction Hypothesis:
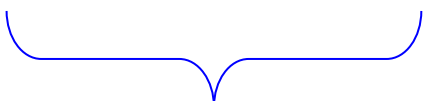
$$A_{pq} \Rightarrow \cdots \Rightarrow w$$

k derivation steps

suppose it holds:

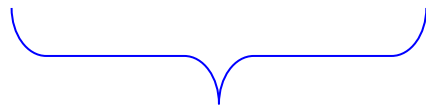$$(p, w, \varepsilon) \overset{*}{\succ} (q, \varepsilon, \varepsilon)$$

Induction Step:

$$A_{pq} \Rightarrow \cdots \Rightarrow w$$

$$k + 1 \text{ derivation steps}$$

We have to show:

$$(p, w, \varepsilon) \overset{*}{\succ} (q, \varepsilon, \varepsilon)$$

$$A_{pq} \Rightarrow \cdots \Rightarrow w$$

$$k + 1 \text{ derivation steps}$$

Type 2

Case 1: $\quad A_{pq} \Rightarrow A_{pr} A_{rq} \Rightarrow \cdots \Rightarrow w$

Type 3

Case 2: $\quad A_{pq} \Rightarrow a A_{rs} b \Rightarrow \cdots \Rightarrow w$

Type 2

Case 1: $A_{pq} \Rightarrow A_{pr} A_{rq} \Rightarrow \cdots \Rightarrow w$

$\underbrace{\qquad\qquad\qquad\qquad}_{k+1 \text{ steps}}$

We can write $w = yz$

$A_{pr} \Rightarrow \cdots \Rightarrow y$

$\underbrace{\quad\quad}_{\text{At most } k \text{ steps}}$

$A_{rq} \Rightarrow \cdots \Rightarrow z$

$\underbrace{\quad\quad}_{\text{At most } k \text{ steps}}$

$$A_{pr} \Rightarrow \cdots \Rightarrow y$$

At most k steps

$$A_{rq} \Rightarrow \cdots \Rightarrow z$$

At most k steps

From induction hypothesis, in PDA:

$$(p, y, \varepsilon) \overset{*}{\succ} (r, \varepsilon, \varepsilon)$$

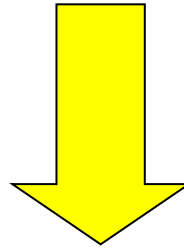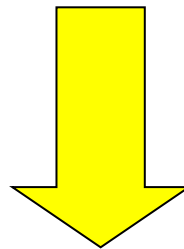From induction hypothesis, in PDA:

$$(r, z, \varepsilon) \overset{*}{\succ} (q, \varepsilon, \varepsilon)$$

$$(p, y, \varepsilon) \overset{*}{\succ} (r, \varepsilon, \varepsilon) \qquad (r, z, \varepsilon) \overset{*}{\succ} (q, \varepsilon, \varepsilon)$$

$$\Downarrow$$

$$(p, yz, \varepsilon) \overset{*}{\succ} (r, z, \varepsilon) \overset{*}{\succ} (q, \varepsilon, \varepsilon)$$

since $w = yz$

$$\Downarrow$$

$$(p, w, \varepsilon) \overset{*}{\succ} (q, \varepsilon, \varepsilon)$$

$$\text{Type 3}$$

Case 2: $A_{pq} \Rightarrow aA_{rs}b \Rightarrow \cdots \Rightarrow w$

$\underbrace{\qquad\qquad\qquad\qquad}_{k+1 \text{ steps}}$

We can write $w = ayb$

$A_{rs} \Rightarrow \cdots \Rightarrow y$

$\underbrace{\qquad}_{\text{At most } k \text{ steps}}$
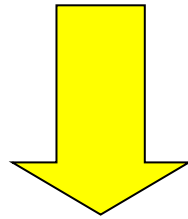
$$A_{rs} \Rightarrow \cdots \Rightarrow y$$

At most k steps

From induction hypothesis, the PDA has computation:

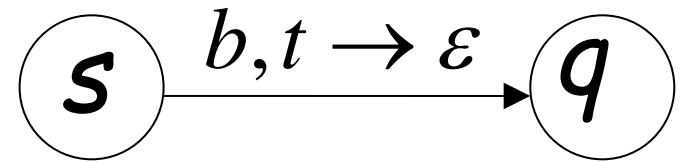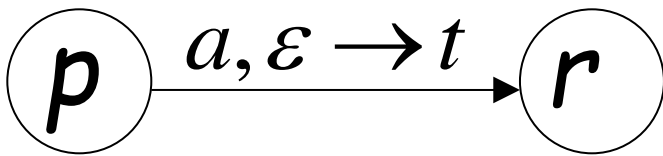$$(r, y, \varepsilon) \overset{*}{\succ} (s, \varepsilon, \varepsilon)$$

$$A_{pq} \Rightarrow aA_{rs}\,b \Rightarrow \cdots \Rightarrow w$$
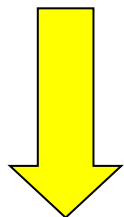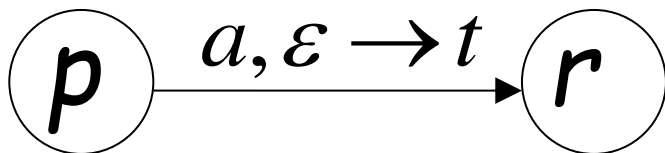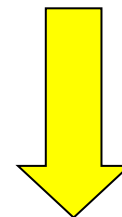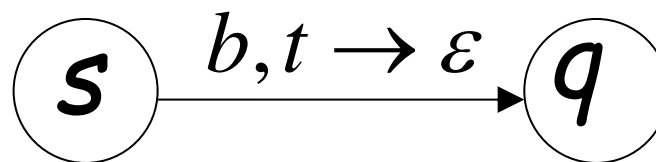
Grammar contains production

$$A_{pq} \rightarrow aA_{rs}\,b$$

And PDA Contains transitions

$$p \xrightarrow{\; a,\varepsilon \rightarrow t \;} r \qquad\qquad s \xrightarrow{\; b,t \rightarrow \varepsilon \;} q$$

$$p \xrightarrow{a, \varepsilon \to t} r$$

$$s \xrightarrow{b, t \to \varepsilon} q$$

$$(p, ayb, \varepsilon) \succ (r, yb, t)$$

$$(s, b, t) \succ (q, \varepsilon, \varepsilon)$$

We know

$$(r, y, \varepsilon) \overset{*}{\succ} (s, \varepsilon, \varepsilon) \quad \Longrightarrow \quad (r, yb, t) \overset{*}{\succ} (s, b, t)$$

We also know

$$(p, ayb, \varepsilon) \succ (r, yb, t)$$

$$(s, b, t) \succ (q, \varepsilon, \varepsilon)$$

Therefore:

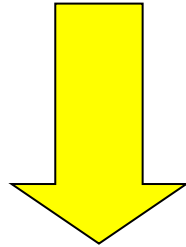$$(p, ayb, \varepsilon) \succ (r, yb, t) \overset{*}{\succ} (s, b, t) \succ (q, \varepsilon, \varepsilon)$$

$$(p, ayb, \varepsilon) \succ (r, yb, t) \overset{*}{\succ} (s, b, t) \succ (q, \varepsilon, \varepsilon)$$

since $w = ayb$

$$(p, w, \varepsilon) \overset{*}{\succ} (q, \varepsilon, \varepsilon)$$

END OF PROOF

So far we have shown:
$$L(G) \subseteq L(M)$$

With a similar proof we can show

$$L(G) \supseteq L(M)$$
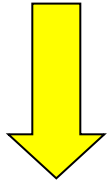
Therefore: $L(G) = L(M)$