BLM2502 Theory of Computation

Spring 2017

BLM2502 Theory of Computation

- **Course Outline**
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- 1 Introduction to Course
- 2 Computability Theory, Complexity Theory, Automata Theory, Set Theory, Relations, Proofs, Pigeonhole Principle
- **3 Regular Languages**
- 4 **Finite Automata**
- 5 Deterministic and Nondeterministic Finite Automata
- 6 Epsilon Transition, Equivalence of Automata
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- 9 Context Free Grammars
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- 14 Turing Machines, Recognition and Computation, Church-Turing Hypothesis
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Regular Expressions

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Regular Languages

» Keywords / Definitions:

- *Alphabet*: A finite nonempty set of symbols. The members of the alphabet are the *symbols* of the alphabet. We generally use capital Greek letters Σ and Γ to designate alphabets. The following are a few examples of alphabets.

  $\Sigma_1 = \{0,1\}$

  $\Sigma_2 = \{a,b,c,d,e,f,g.h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z\}$

  $\Gamma = \{0, 1, x, y, z\}$

- A string over an alphabet is a finite sequence of symbols from that alphabet, usually written next to one another and not separated by commas. If $\Sigma_1 = \{0,1\}$, then 01101 is a string over Σ_1. If $\Sigma_2 = \{a, b, c, \ldots, z\}$, then abracadabra is a string over Σ_2.

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Regular Expressions

» Keywords / Definitions:

- If w is a string over Σ, the *length* of w, written $|w|$, is the number of symbols that it contains. The string of length zero is called the *empty string* and is written as ε (or λ). The empty string plays the role similiar to 0 in a number system.
- If w has length n, we can write

  $w = w_1w_2 \ldots w_n$ where each $w_i \in \Sigma$.

  The reverse of w, written as w^R, is the string obtained by writing w in the opposite order (i.e., $w_nw_{n-1} \ldots w_1$).

  String z is a *substring* of w if z appears consecutively within w. For example, *cad* is a substring of *abracadabra*

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Regular Expressions

» Keywords / Definitions:

> If we have string x of length m and string y of length n, the *concatenation* of x and y, written xy, is the string obtained by appending y to the end of x, as in $x_1 \ldots x_m y_1 \ldots y_n$. To concatenate a string with itself many times we use the superscript notation:
>
> $x^3 = xxx$,
>
> $x^n = xx....x$ (n times)
>
> To indicate all possible recurrencies, a special superscript (in fact a special operator) is used:
>
> * (kleene star)

> *Language* : A set of strings (finite or infinite?) over an alphabet. Languages are used to describe computation problems.

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Regular Expressions

» Keywords / Definitions:

- *Regular Languages*: A subset of all languages. There is no way to define regular set. However, it is possible to say whether a set is regular or not.
- Best way is using *Finite Automata*. If some finite automata recognizes the language, then the language is regular.
- Regular (set) operations are used to build up regular sets:
- Union, concatenation, and kleene star operations are as follows.
  - *Union*: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
  - *Concatenation*: $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$.
  - *Star*: $A^* = \{x_1x_2 \ldots x_k \mid k > 0 \text{ and each } x_i \in A\}$.

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Regular Expressions

- Keywords / Definitions:
  - Class of regular languages is closed under **union**
  - Class of regular languages is closed under **intersection**
  - Class of regular languages is closed under **complement**
  - Class of regular languages is closed under **concenation**
  - Class of regular languages is closed under (kleene) **star**

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Alphabet:

Strings:

Decimal numbers

Alphabet:

Strings:

Binary numbers

Alphabet:

Strings:

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Unary numbers

Alphabet:

Strings:

Decimal equivalent:

» Examples on String Operations

$w = a_1a_2\ldots a_n$, $v = b_1b_2\ldots b_m$ $a,b \in \{x, y\}$

eg, w = xyyxy, v = xxxyyy

Concenation:

$wv = a_1a_2\ldots a_nb_1b_2\ldots b_n$

eg, xyyxyxxxyyy

Reverse:

$w^R = a_na_{n-1}\ldots a_1$ c

eg, yxyyx

» Examples on String Operations

$w = a_1 a_2 \ldots a_n$, $v = b_1 b_2 \ldots b_m$ $a, b \in \{x, y\}$

Length:

$|w| = n$, $|v| = m$

eg, $|yxyyx| = 5$; $|xxxyyy| = 6$

$|wv| = |w| + |v|$

eg, $|yxyyxxxxyyy| = 11$ (recall example above)

Empty String:

$|\varepsilon| = 0$

» Examples on String Operations

$w = a_1a_2\ldots a_n$, $v = b_1b_2\ldots b_m$ $a,b \in \{x, y\}$

Substring:

$a_1, a_2, \ldots, a_n$ (each symbol of the string are its substrings)

$a_1, a_1a_2, a_1a_2a_3\ldots$ (these are prefix substrings)

$a_2, a_2a_3, a_2a_3a_4, \ldots$

$a_n, a_{n-1}a_n, a_{n-2}a_{n-1}a_n, \ldots$ (these are suffix substrings)

special cases:

ε is prefix and suffix of each string

any string is prefix and suffix of itself

» Examples on String Operations

Special cases:

ε is prefix and suffix of each string

any string is prefix and suffix of itself

String: abba

Prefix	Suffix
ε	abba
a	bba
ab	ba
abb	a
abba	ε

Observe that:

$\varepsilon w = w\varepsilon = w$

» Examples on String Operations

$w = a_1a_2\ldots a_n$, $v = b_1b_2\ldots b_m$ $a,b \in \{x, y\}$

Self Concenation:

$ww = w^2$, $www = w^3$, etc

$w^0 = \varepsilon$

eg, w = abba;

$(abba)^0 = \varepsilon$

$(abba)^1 = abba$

$(abba)^2 = abbaabba$

$(abba)^3 = abbaabbaabba$

Kleene Star:

$w^* = \{w^0, w^1, w^2, w^3, \ldots\}$

- The * operation:

  The set of all possible strings from the alphabet

  $\Sigma = \{a, b\}$

  $\Sigma^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$

- The + operation:

  The set of all possible strings from the alphabet excluding ε

  $\Sigma = \{a, b\}$

  $\Sigma^+ = \Sigma^* - \{\varepsilon\} = \{a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$

» Language:

A Language over an alphabet $\Sigma = \{a, b\}$

is a subset of $\Sigma^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$

Examples:

$L_1 = \{\varepsilon\}$

$L_2 = \{a, b, aa, ab, ba, bb\}$

$L_3 = \{\varepsilon, a, aa, aaa, aaaa, \ldots\}$

$L_4 = \{a^n b^n, n \geq 0\} = \{\varepsilon, ab, aabb, aaabbbb, \ldots\}$

$\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

PRIME_NUMBERS = $\{x \mid x \in \Sigma^*$ and x is prime $\}$

$= \{2, 3, 5, 7, 11, \ldots\}$

EVEN_NUMBERS = $\{x \mid x \in \Sigma^*$ and x is even$\}$ $= \{0, 2, 4, \ldots\}$

ODD_NUMBERS = $\{x \mid x \in \Sigma^*$ and x is odd$\}$ $= \{1, 3, 5, \ldots\}$

» Unary Addition

Alphabet $\Sigma = \{1, +, =\}$

ADDITION = $\{x + y = z : x = 1^m, y=1^n, z = 1^k; k = m + n\}$

$111 + 11 = 11111 \in$ ADDITION

$111 + 111 = 1111 \notin$ ADDITION

» Squaring

Alphabet $\Sigma = \{1, \#\}$

SQUARES= $\{x \# y : x = 1^m, y=1^n, n = m^2\}$

$111 \# 111111111 \in$ SQUARES

$1111 \# 11111111 \notin$ SQUARES

» Operations On Languages

Set Operations: Regular sets are closed under union, intersection negation and complement.

$A = \{a, ab, aaaa\}$

$B = \{ab, bb\}$

$A \cup B = \{a, ab, bb, aaaa\}$

$A \cap B = \{ab\}$

$A - B = \{a, bb, aaaa\}$

$\bar{A} = \Sigma^* - A = \overline{\{a, ab, aaaa\}} = \{\varepsilon, aa, ba, bb, aba, \ldots\}$

Note that :

$\emptyset = \{\} \neq \{\varepsilon\}$

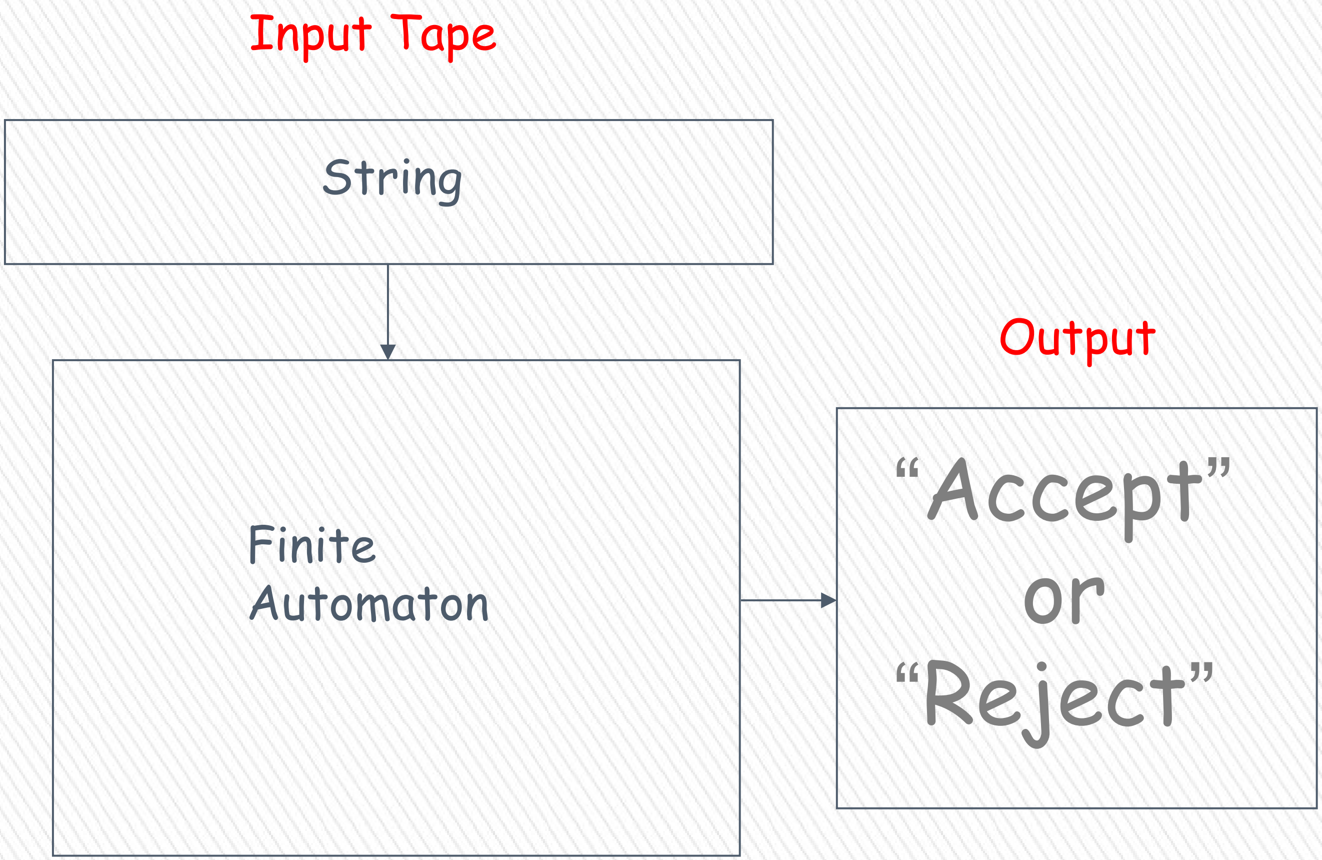
$|\emptyset| = |\{\}| = 0 \neq |\{\varepsilon\}|$

$|\{\varepsilon\}| = 1$ * This is set size

$|\varepsilon| = 0$ * This is string length

Finite Automata

»


Input Tape
String
Finite
Automaton
Output
"Accept"
or
"Reject"

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Finite Automata

» A finite automaton is a 5-tuple:

$(Q, \Sigma, \delta, q_0, F)$ where;

1. Q is a finite set called the ***states,***
2. Σ is a finite set called the ***alphabet,***
3. $\delta: Q \times \Sigma \rightarrow Q$ is the ***transition function,***
4. $q_0 \in Q$ is the ***start state,*** and
5. $F \subseteq Q$ is the ***set of accept states.***

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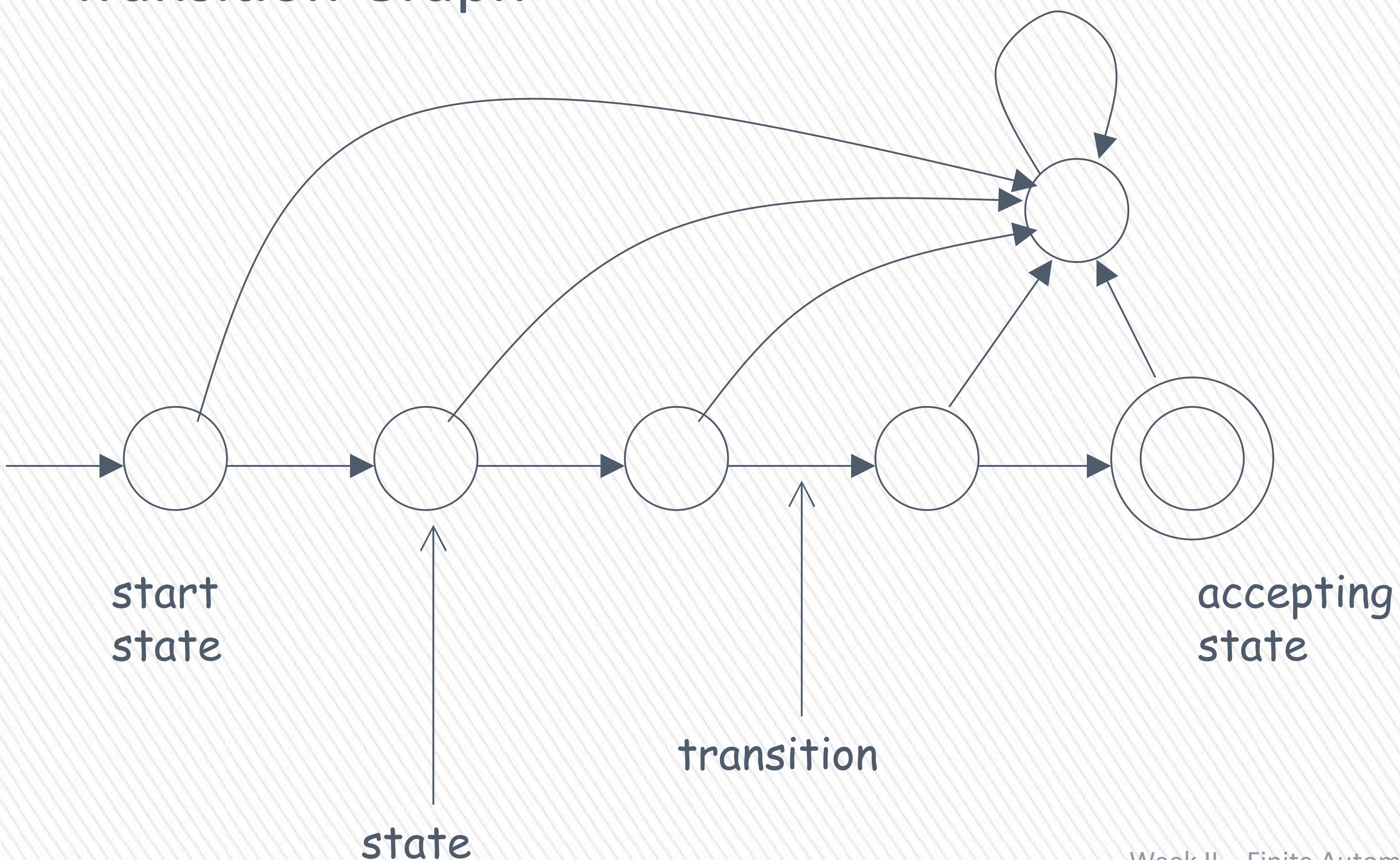
Transition Graph



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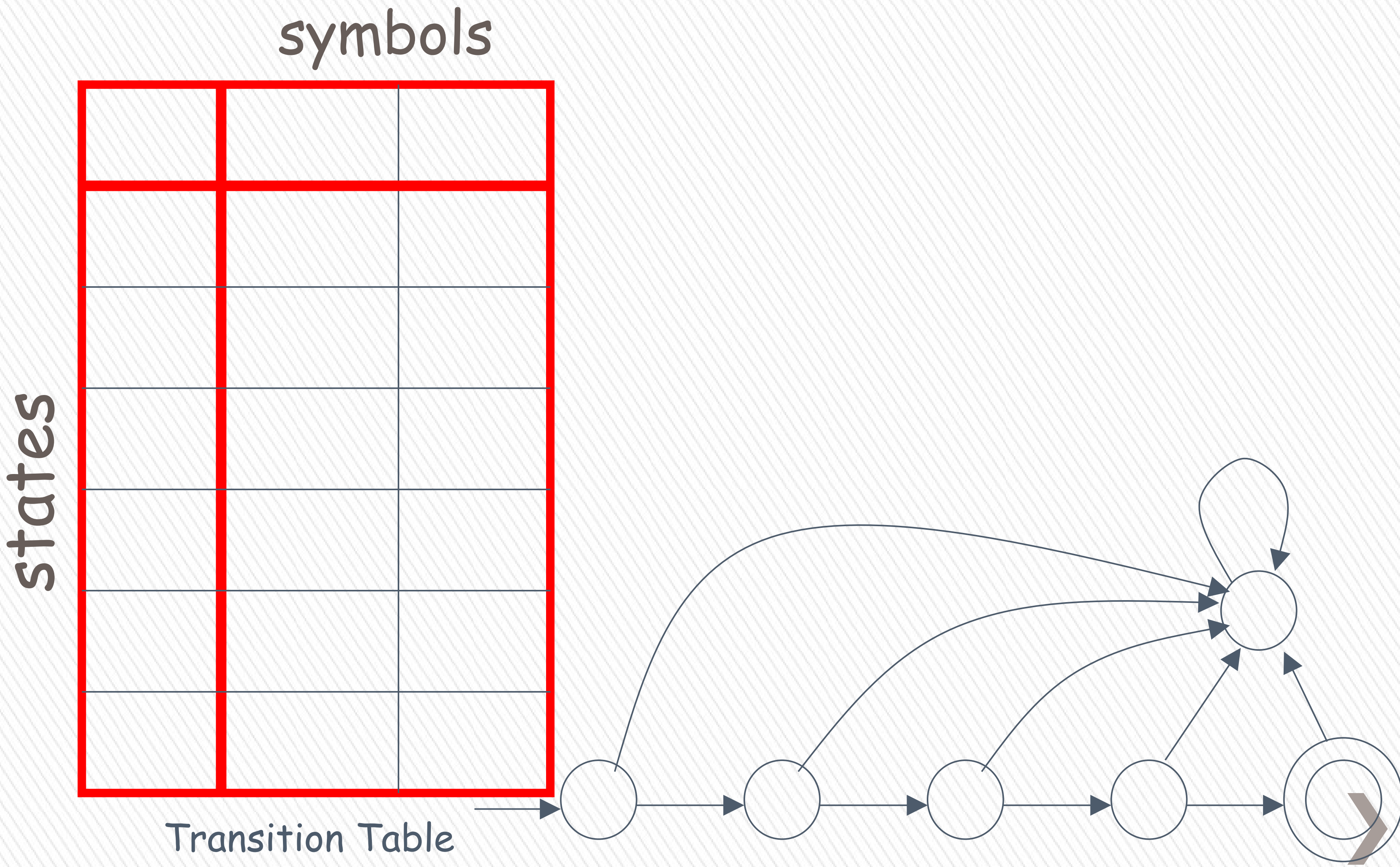

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- You can think of the transition function as being the "program" of the finite automaton M. This function tells us what M can do in "one step":
  - Let r be a state of Q and let a be a symbol of the alphabet Σ. If the finite automaton M is in state r and reads the symbol a, then it switches from state r to state $\delta(r, a)$. (In fact, $\delta(r, a)$ may be equal to r.)
- Example:
  - A = {w : w is a binary string containing an odd number of 1s}.

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Design:

- The finite automaton reads the input string w from left to right and keeps track of the number of 1s it has seen. After having read the entire string w, it checks whether this number is odd (in which case w is accepted) or even (in which case w is rejected).
- Using this approach, the finite automaton needs a state for every integer $i \geq 0$, indicating that the number of 1s read so far is equal to i.

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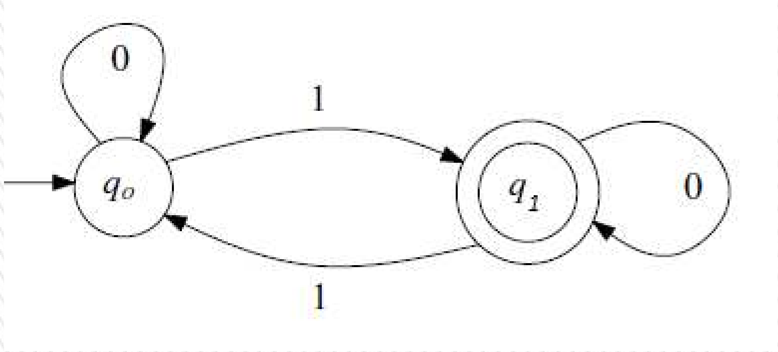
- Design – Continued
- However, this design is not feasible since FA have finite number of states.
- ???
- A better, and correct approach, is to keep track of whether the number of 1s read so far is even or odd.



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$M = (Q, \Sigma, \delta, q_0, F)$

$Q = \{q_0, q_1\}$

$\Sigma = \{0, 1\}$ (trivial from the problem)

$q_0 \in Q$

$F = \{q_1\}$

δ:

$(q_0, 0) \rightarrow q_0$

$(q_0, 1) \rightarrow q_1$

$(q_1, 0) \rightarrow q_1$

$(q_1, 1) \rightarrow q_0$

δ:	0	1
q_0	q_0	q_1
q_1	q_1	q_0

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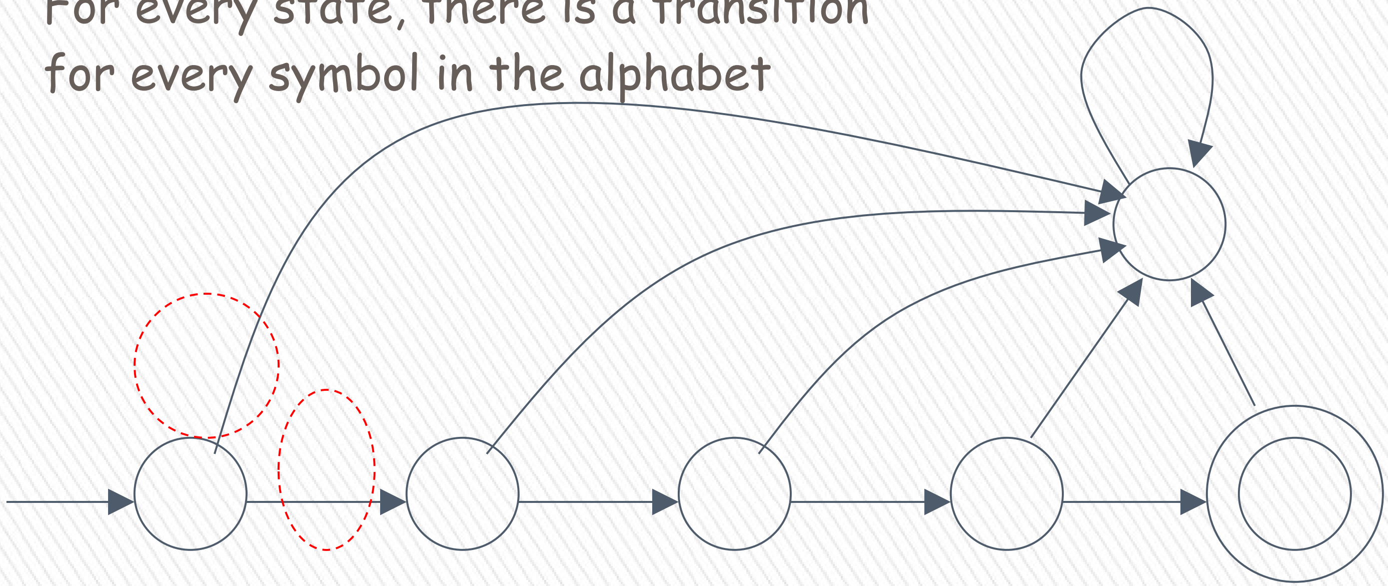
» DFA - Deterministic Finite Automata

For every state, there is a transition
for every symbol in the alphabet



Alphabet $\Sigma = \{a, b\}$

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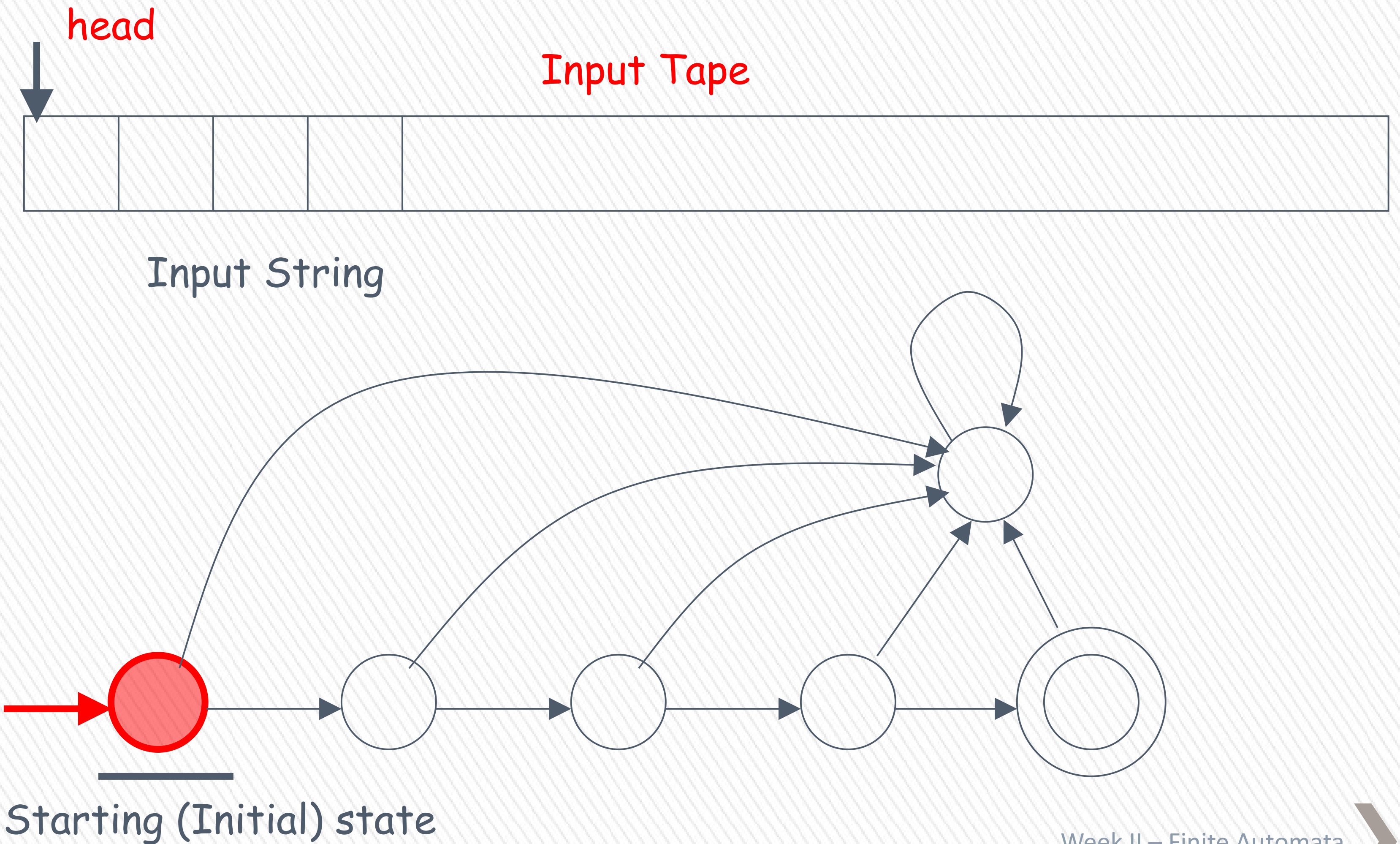

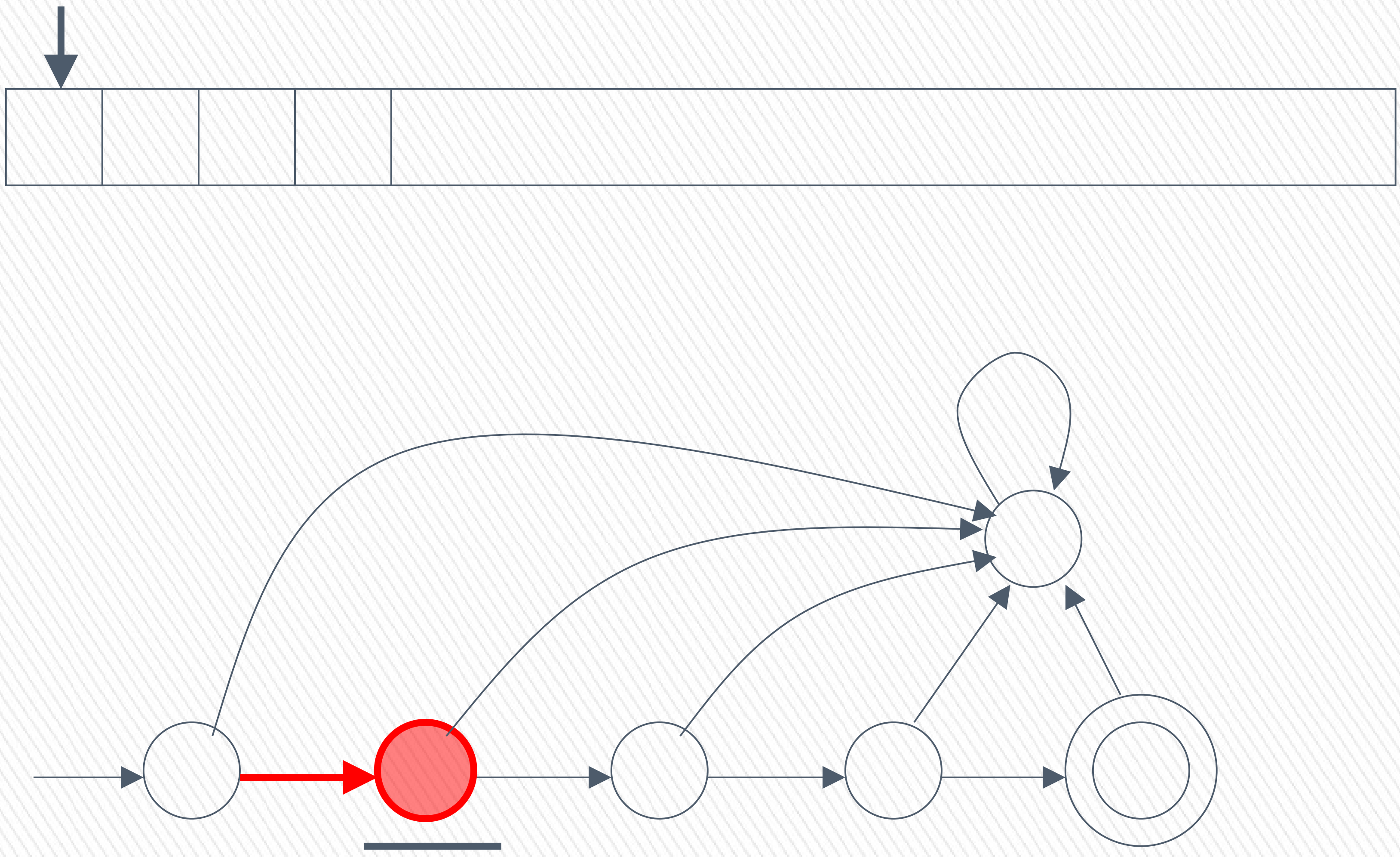

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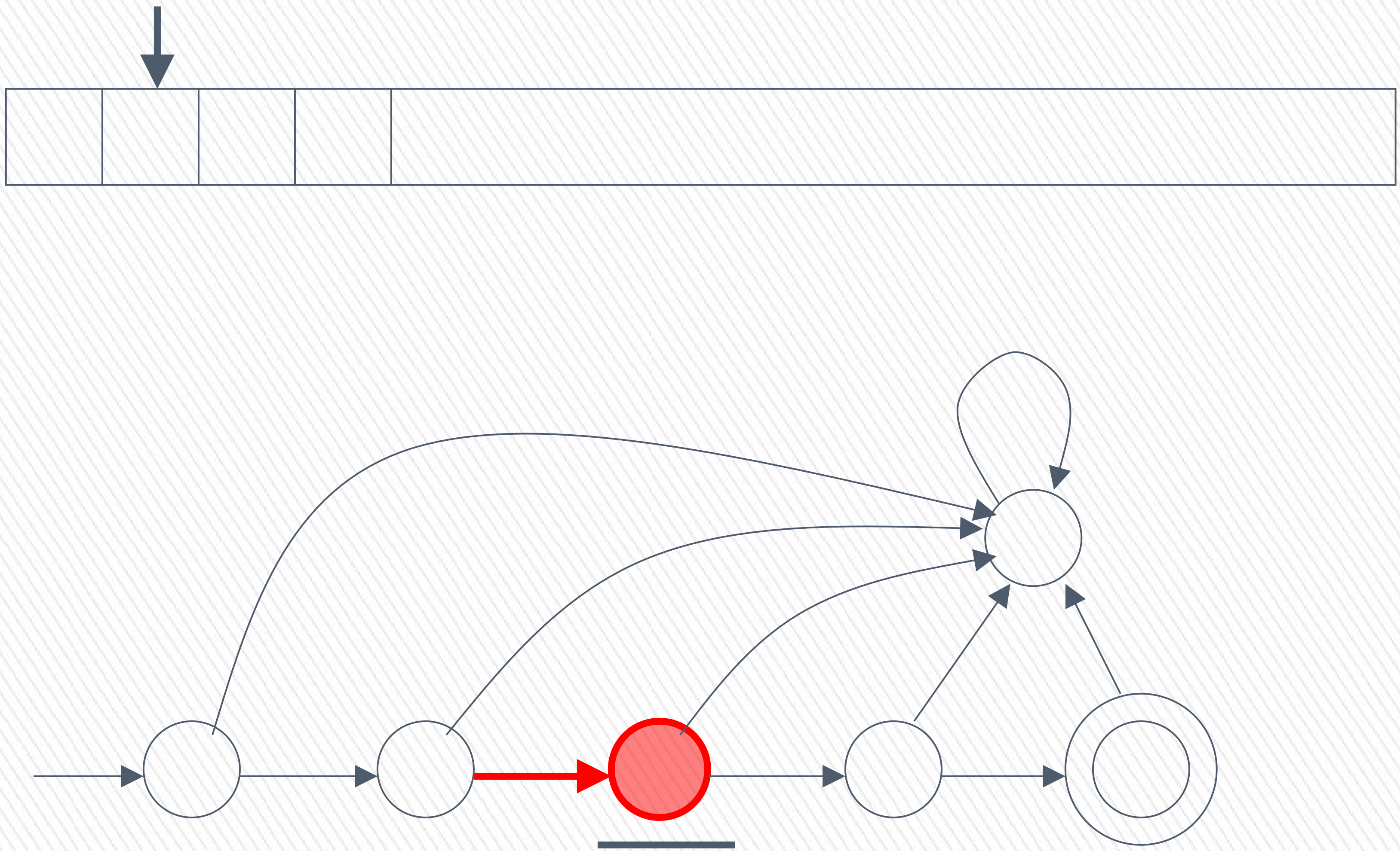

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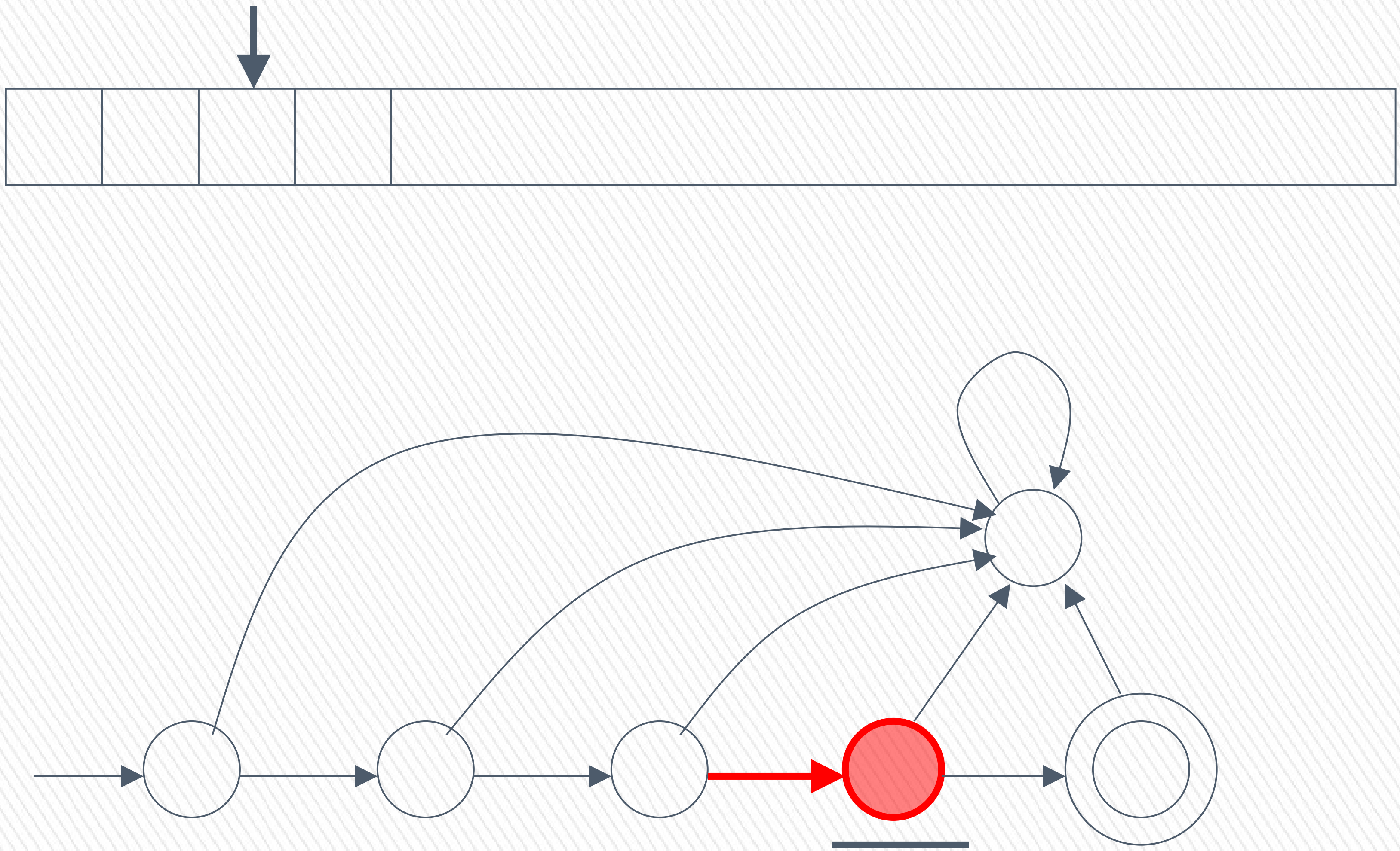

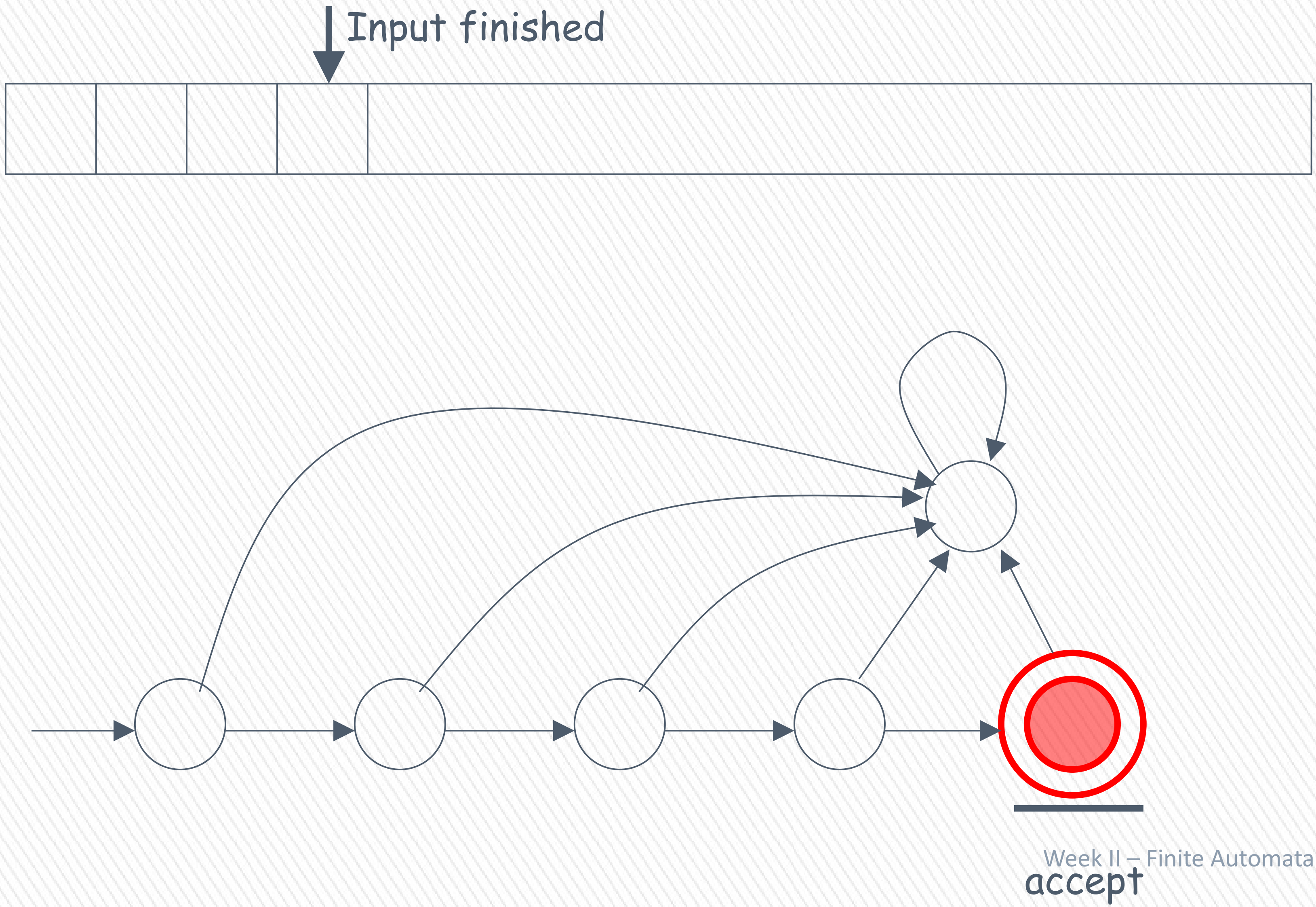
Input finished
accept

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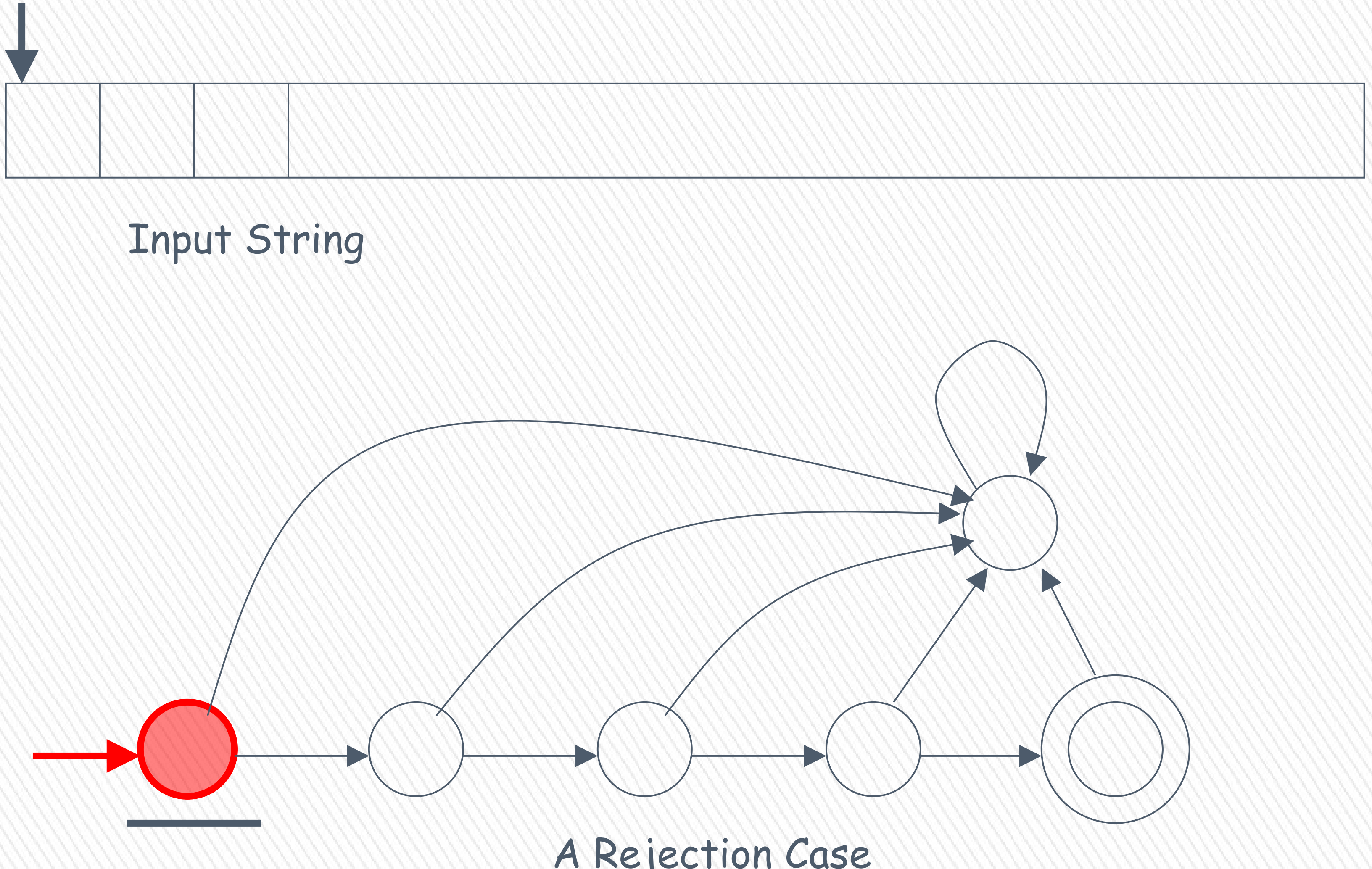

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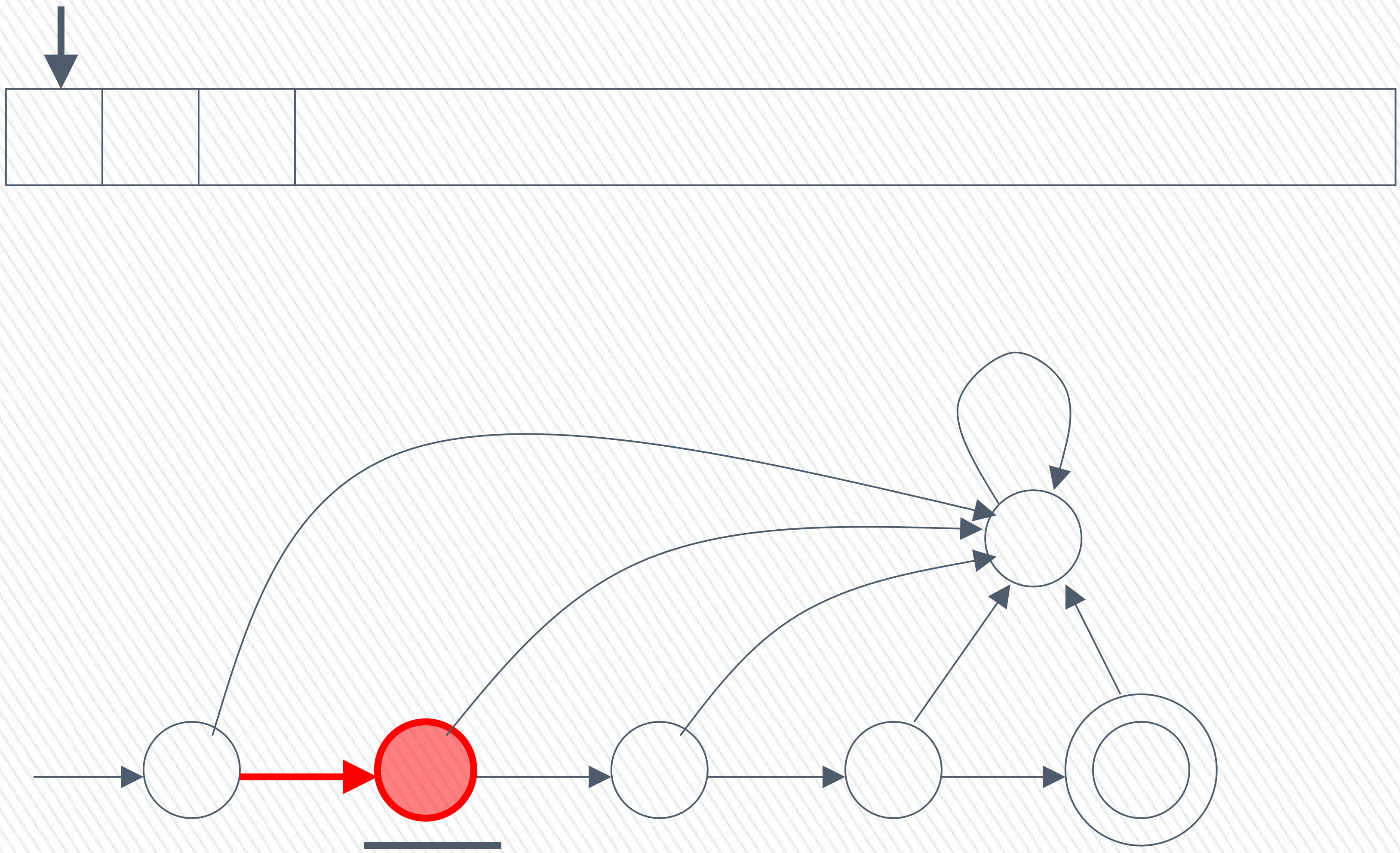

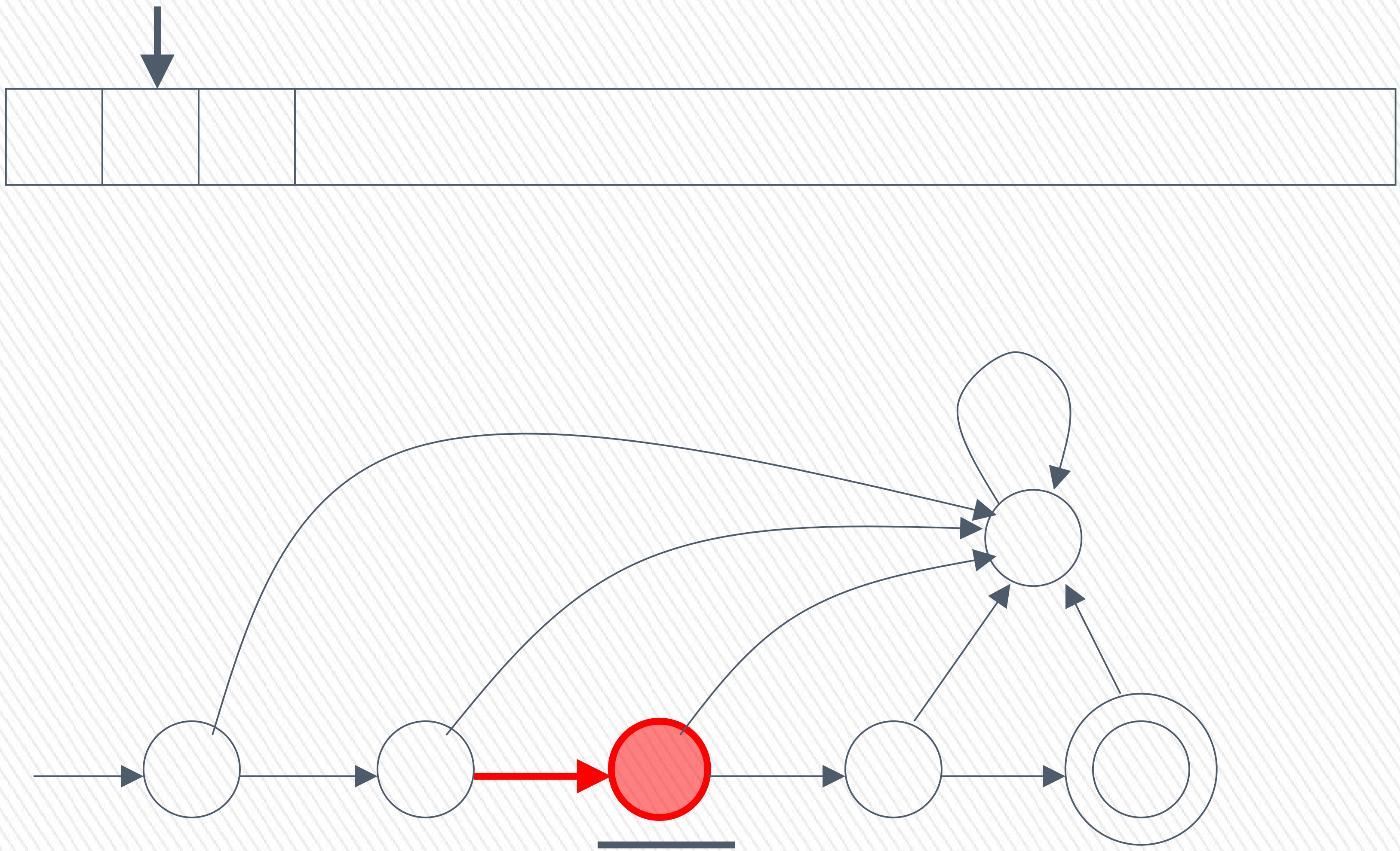

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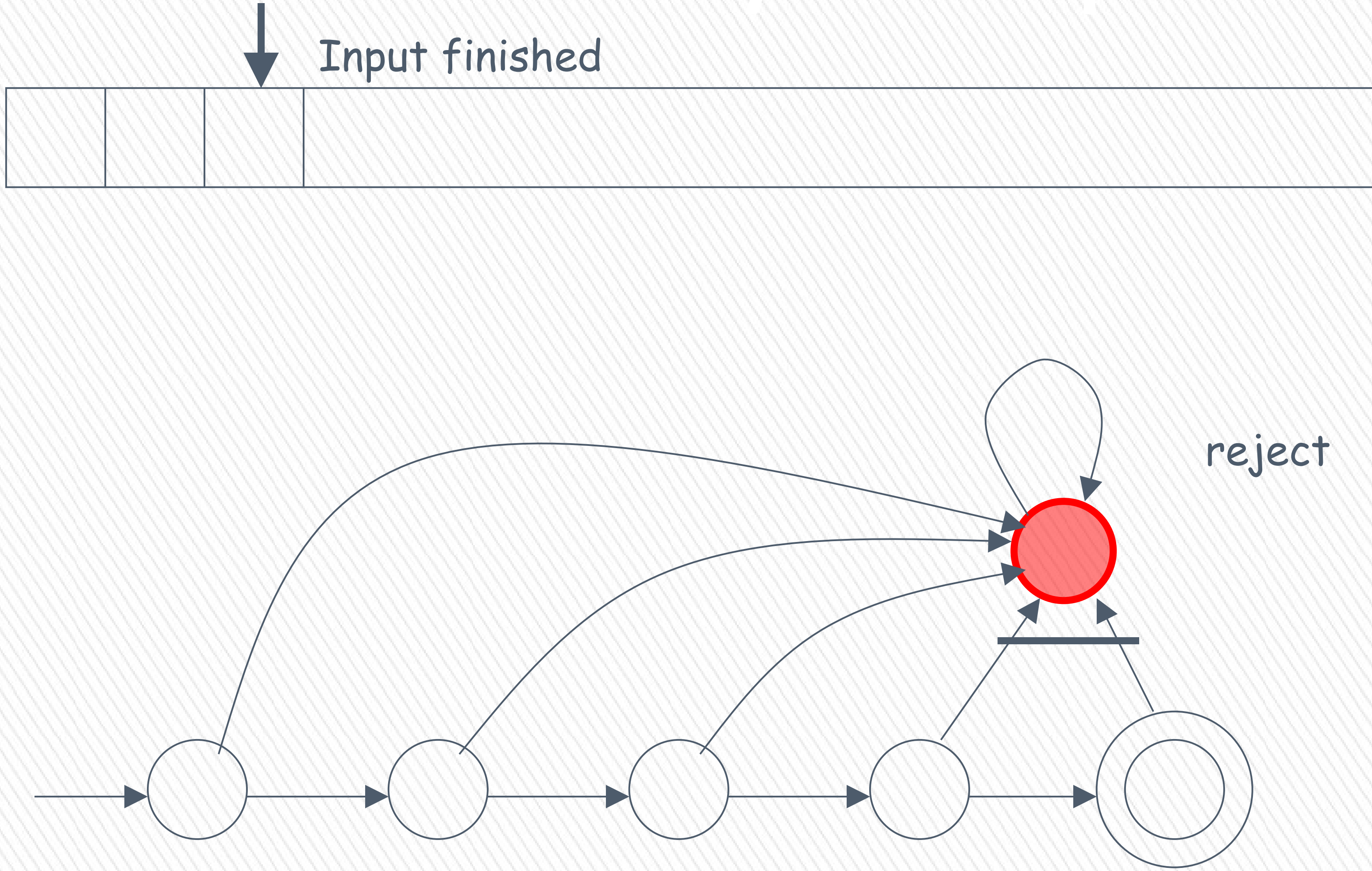

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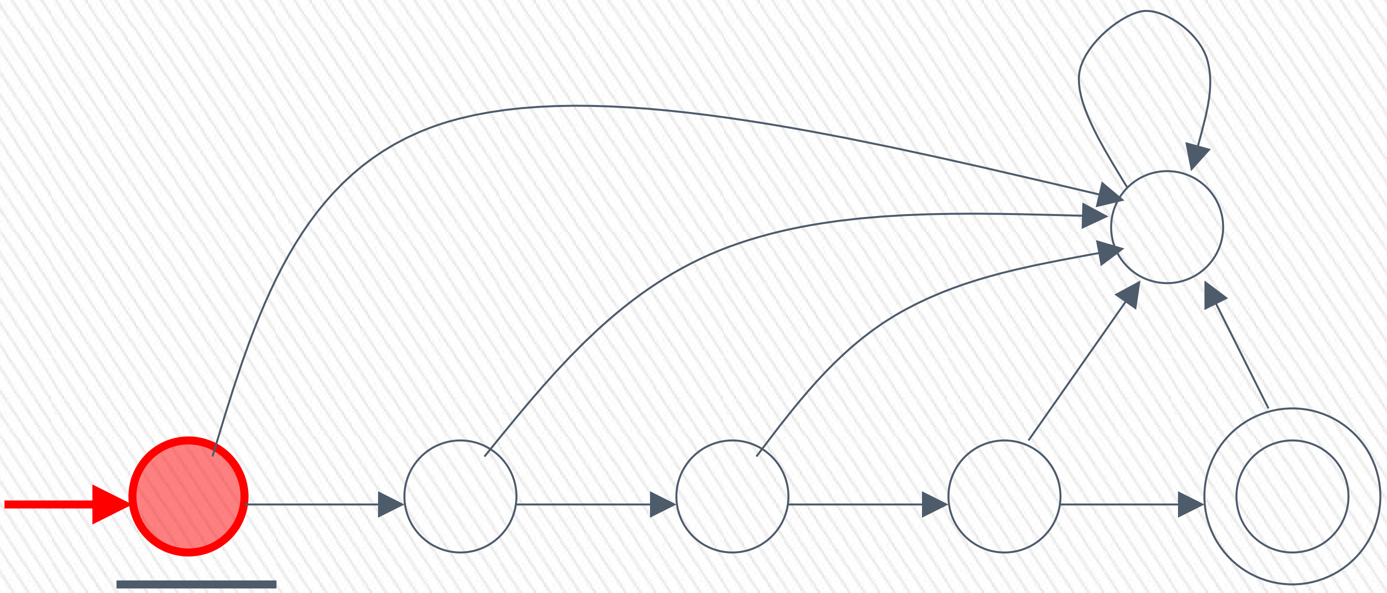
Tape is empty

Input Finished



reject

Another Rejection Case

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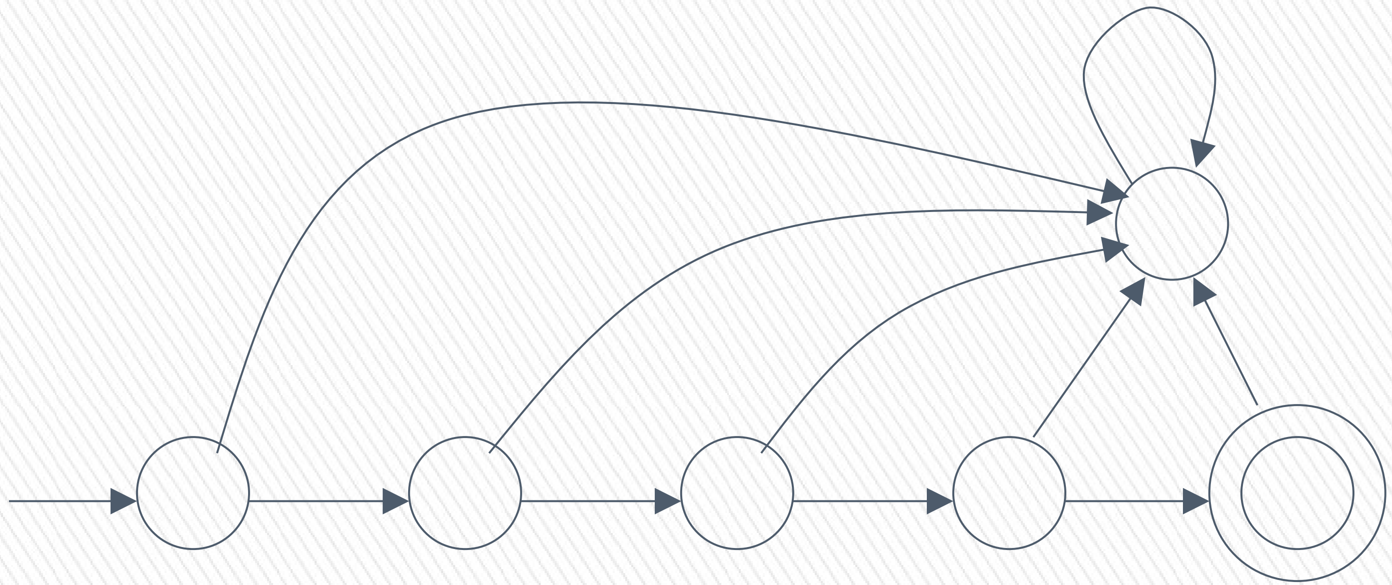
Language Accepted:



To accept a string:

- all the input string is scanned

and

- the last state is accepting

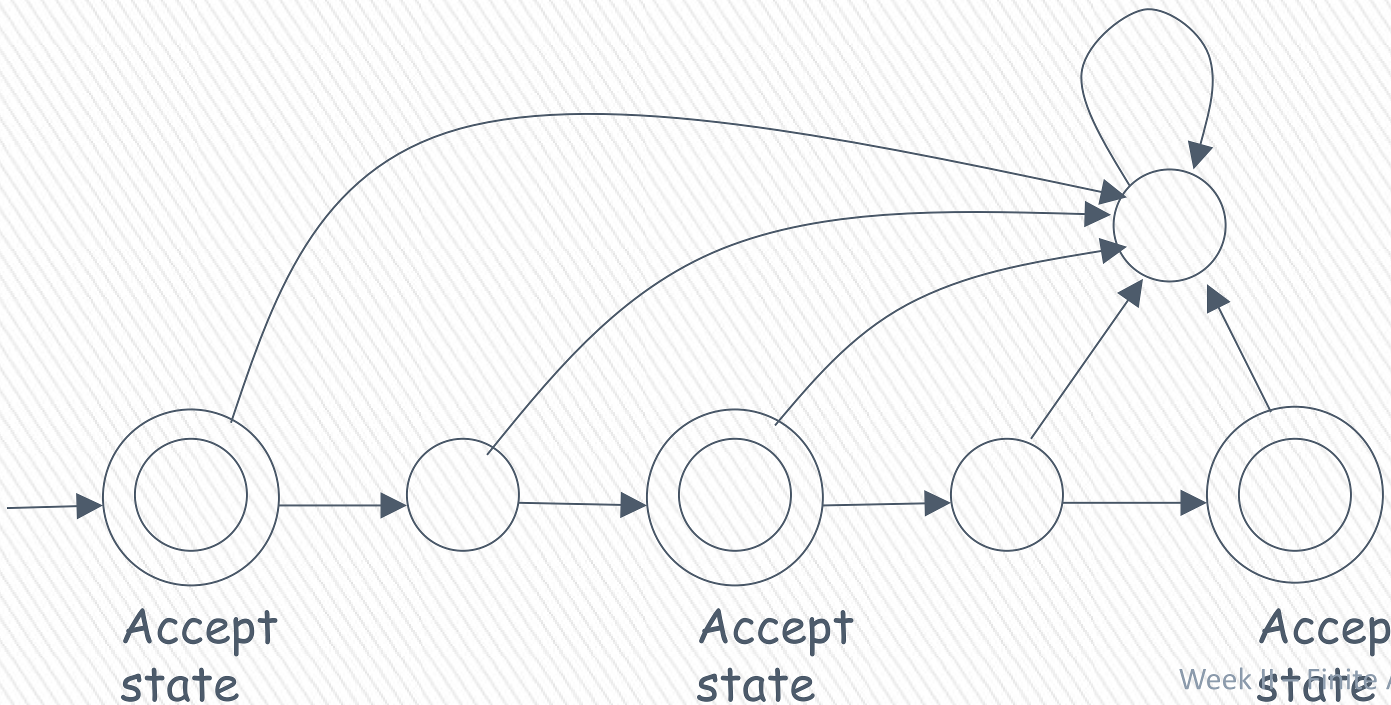
To reject a string:

- all the input string is scanned

and

- the last state is non-accepting

L = {ε, ab, abba}



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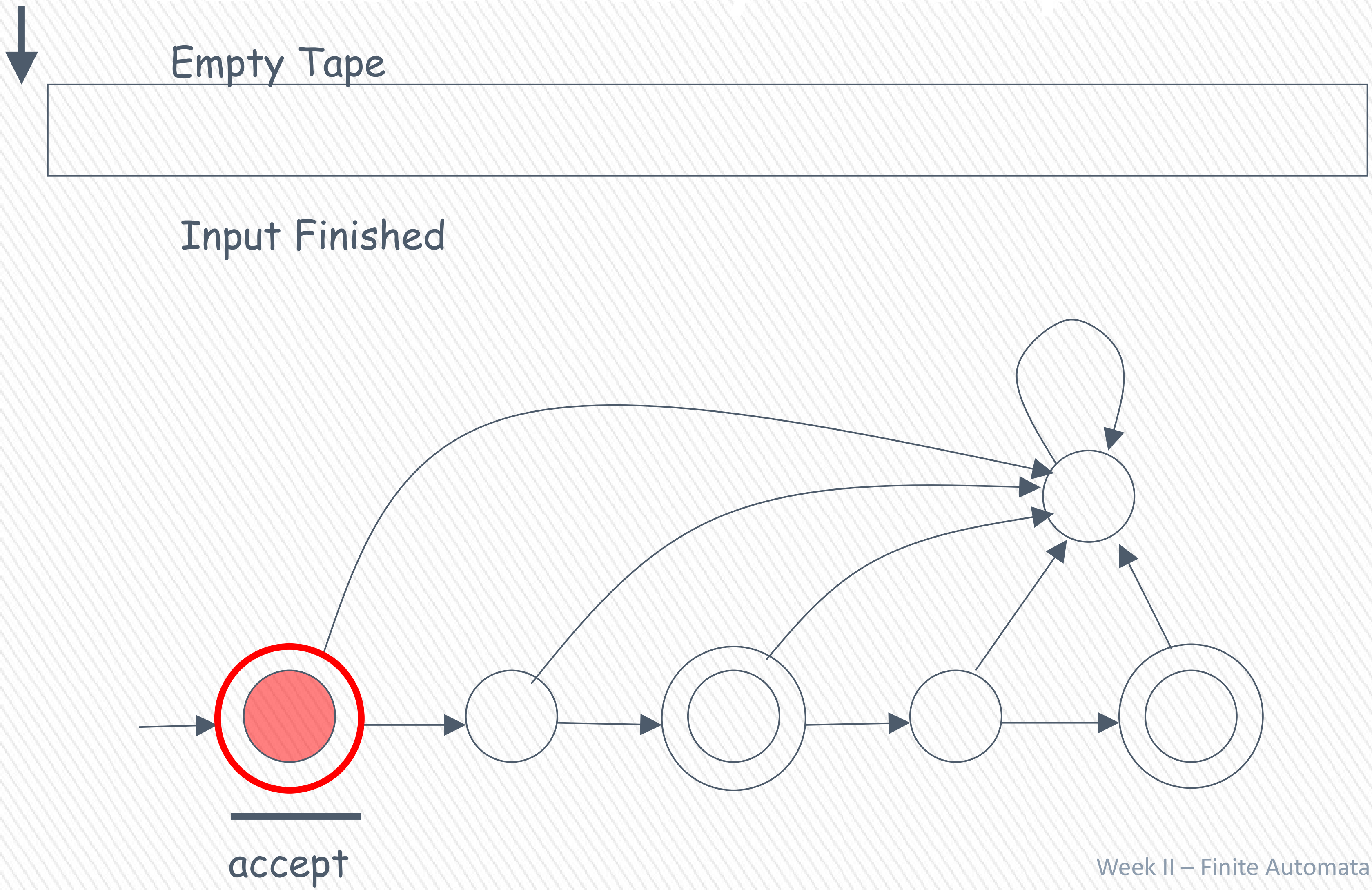

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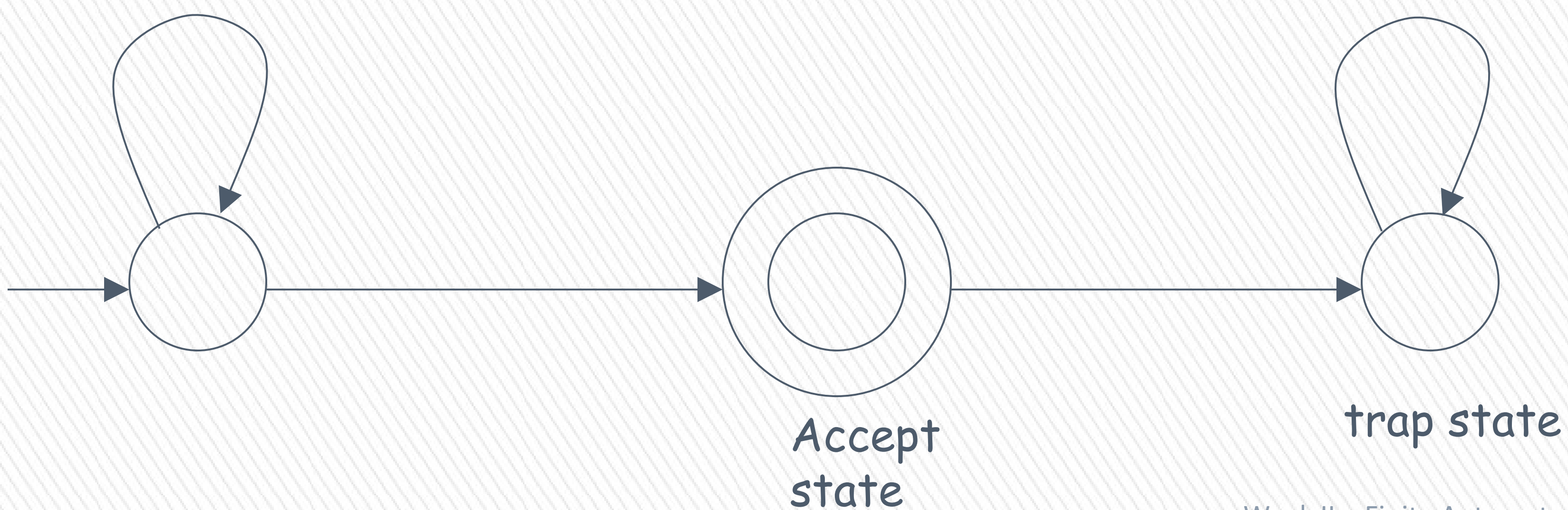


Input String

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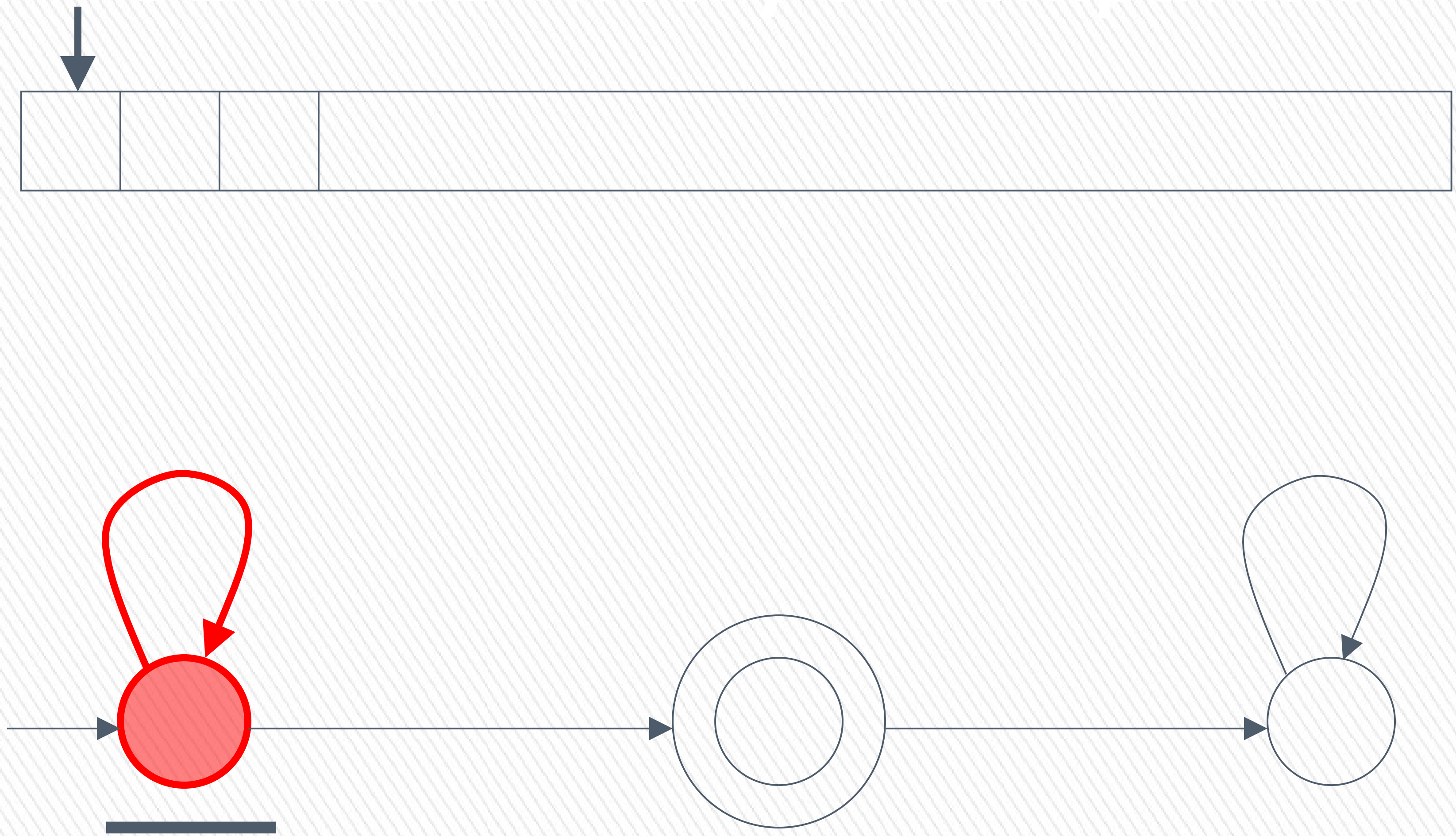

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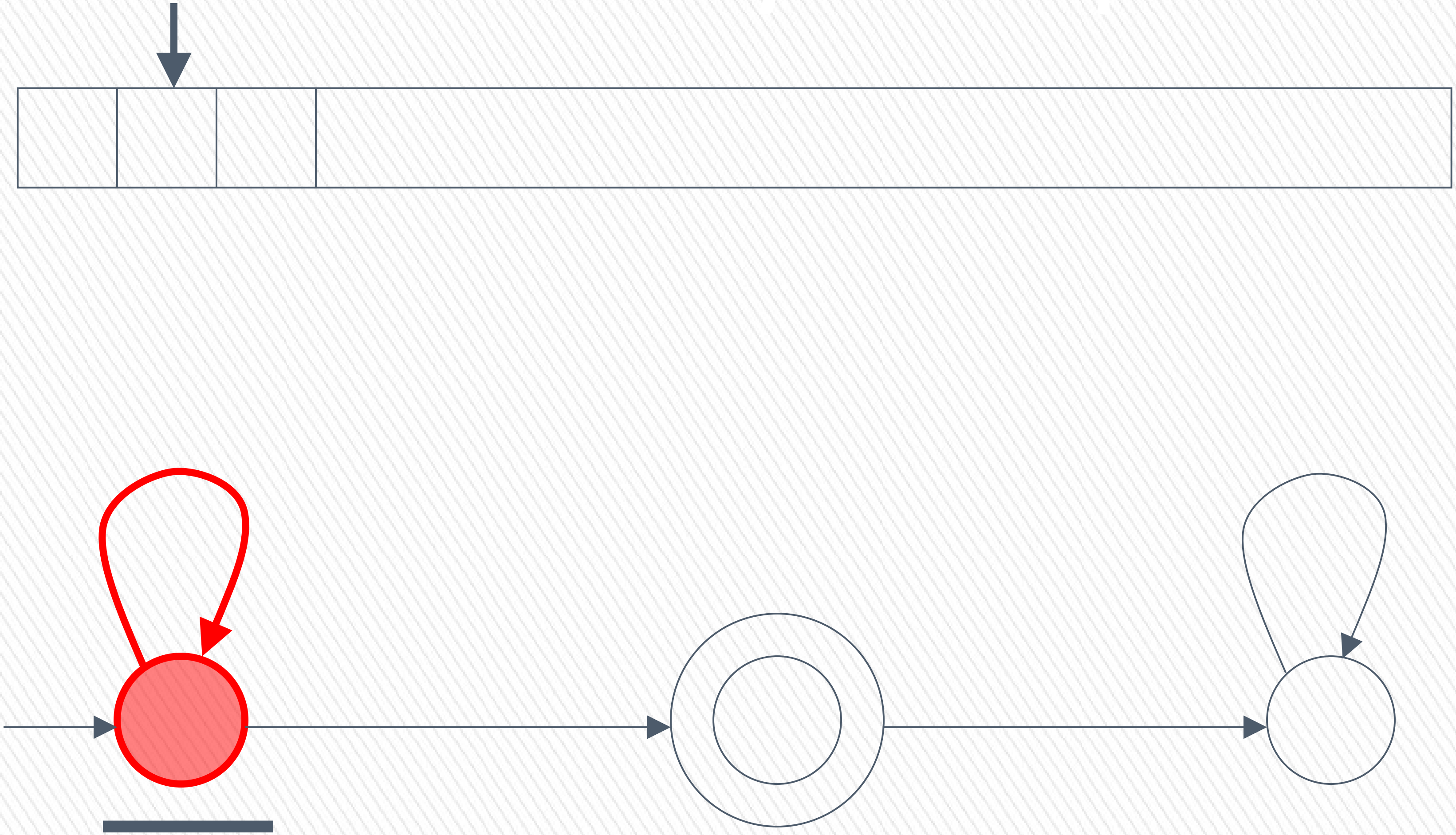

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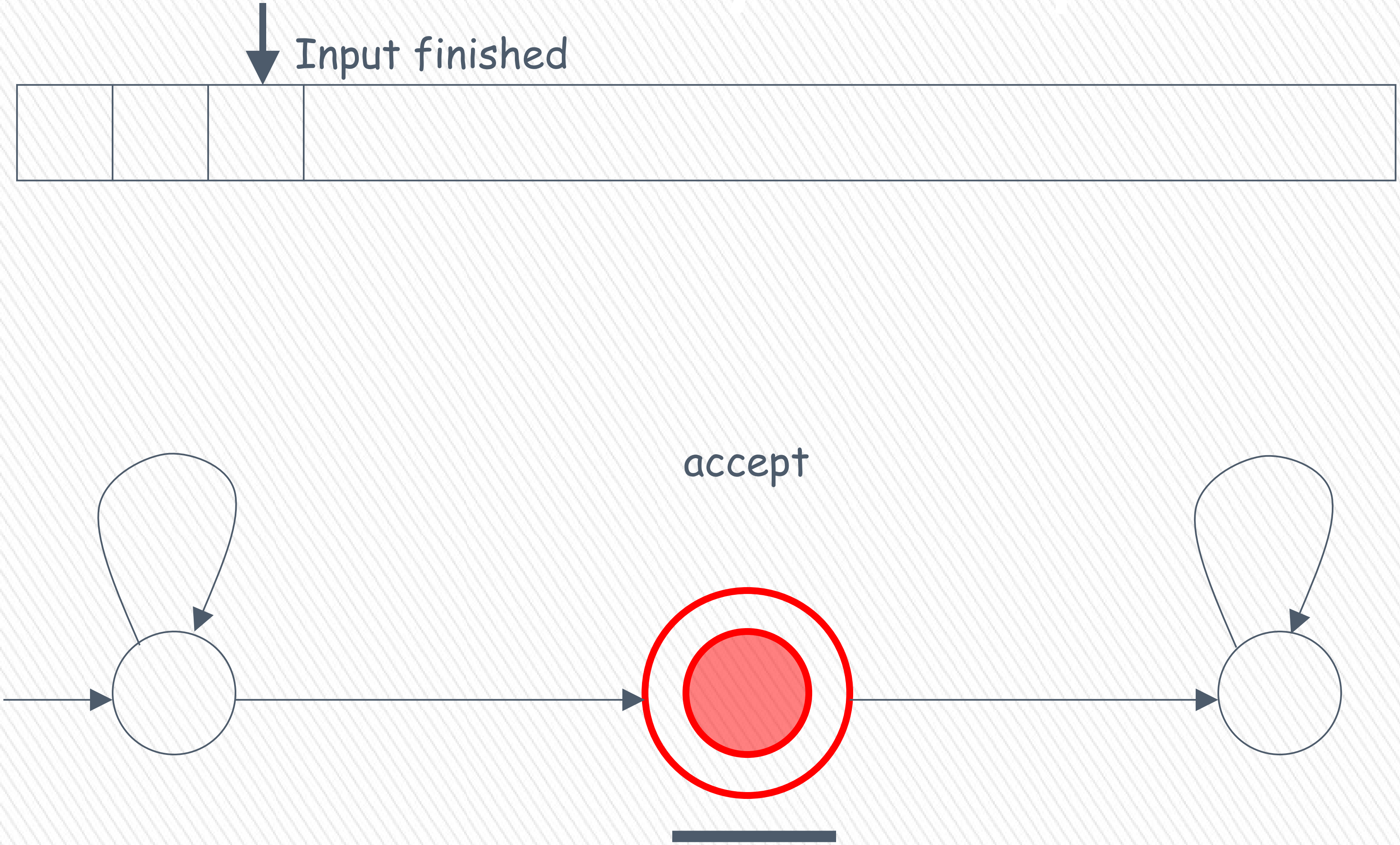

A rejection case

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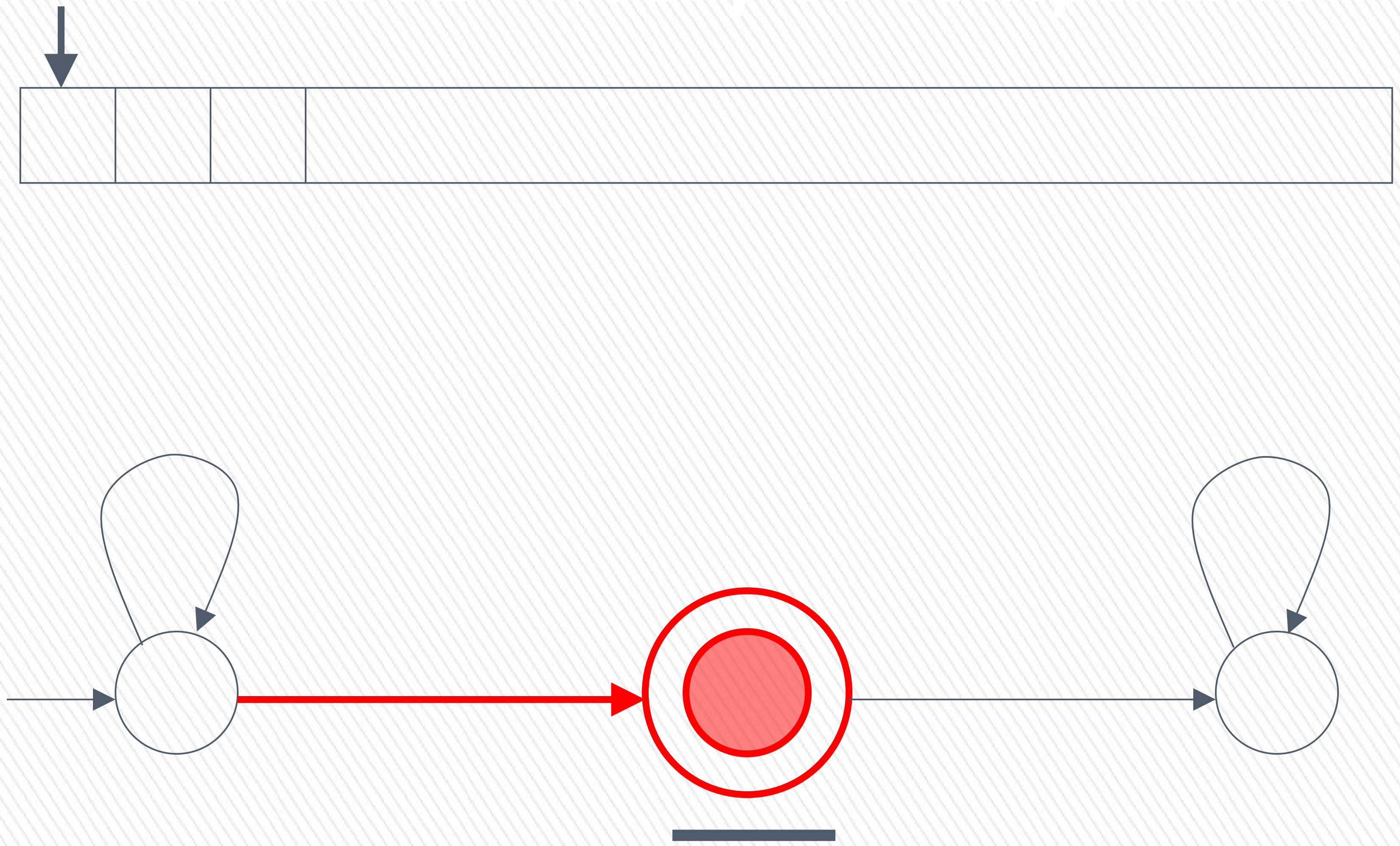

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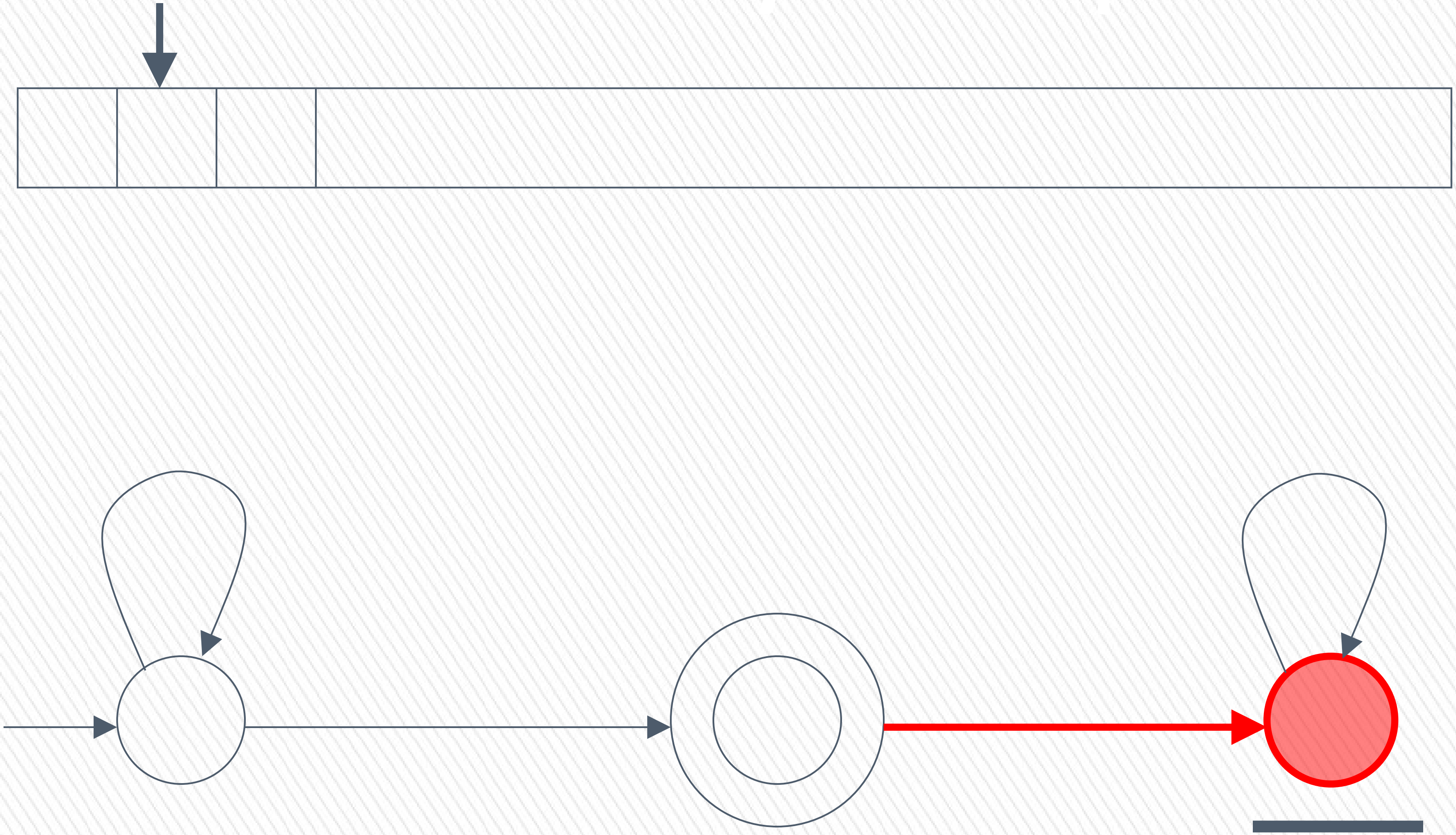

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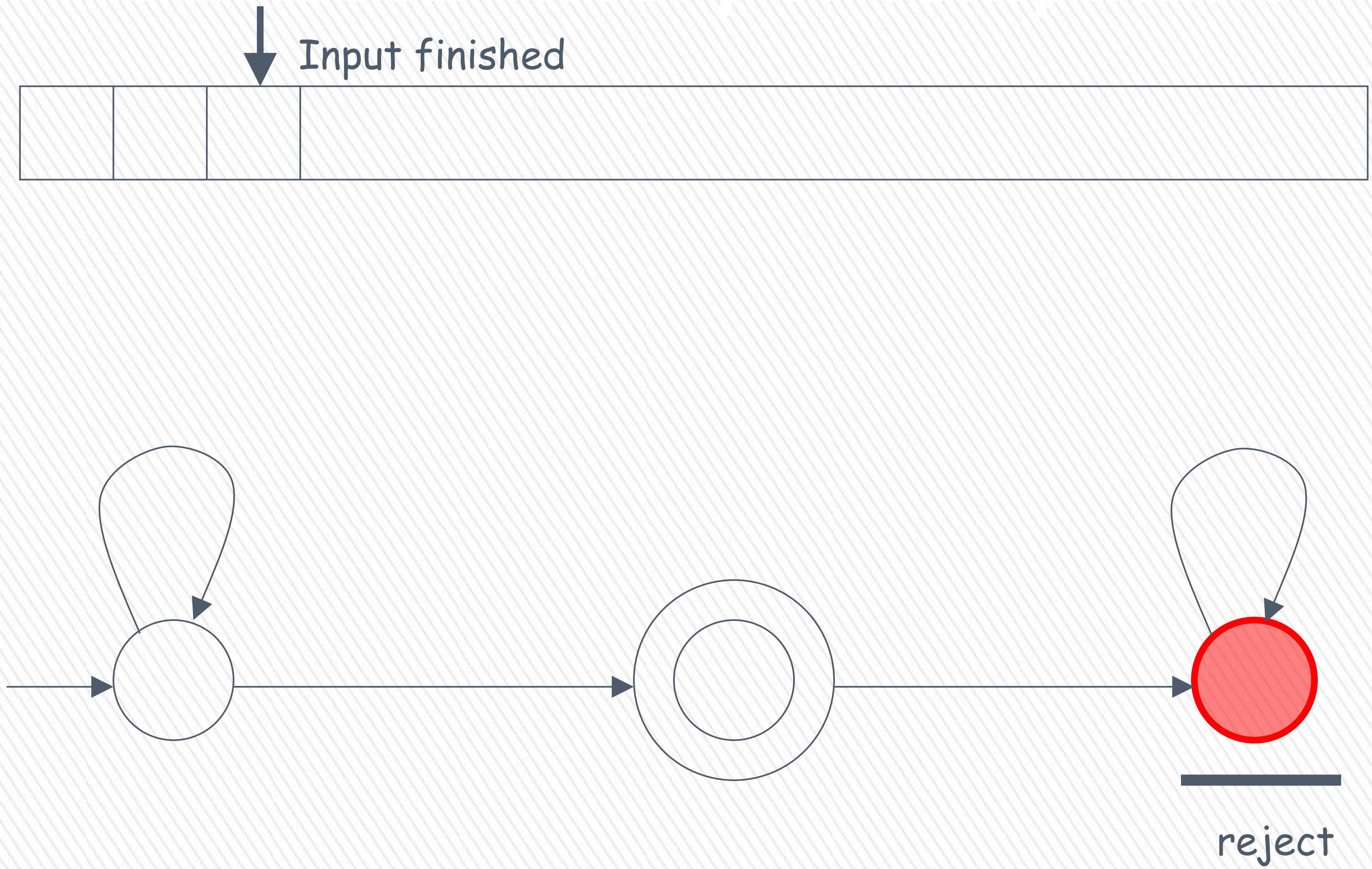
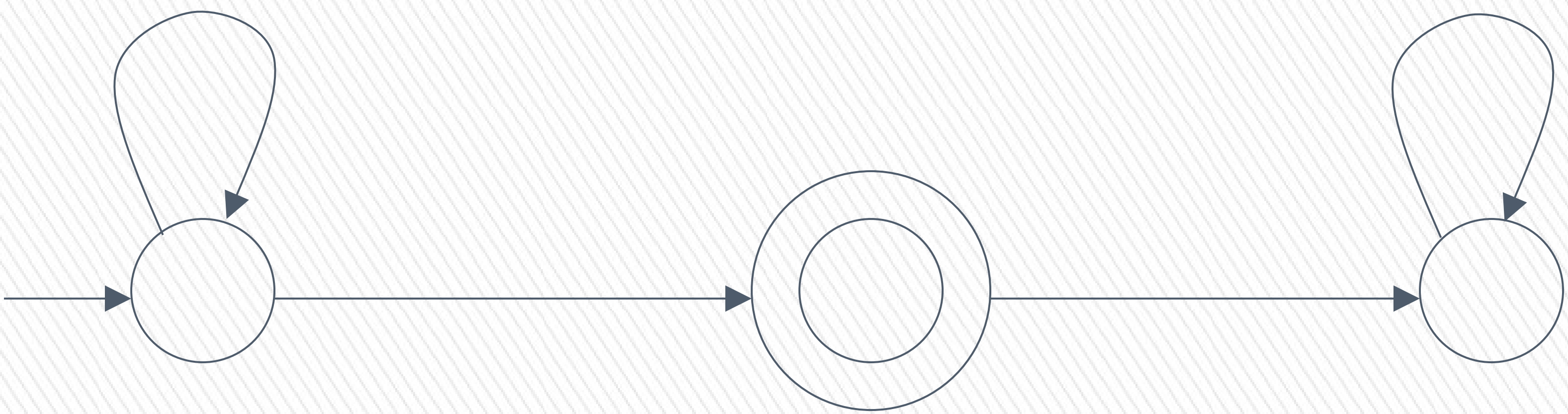
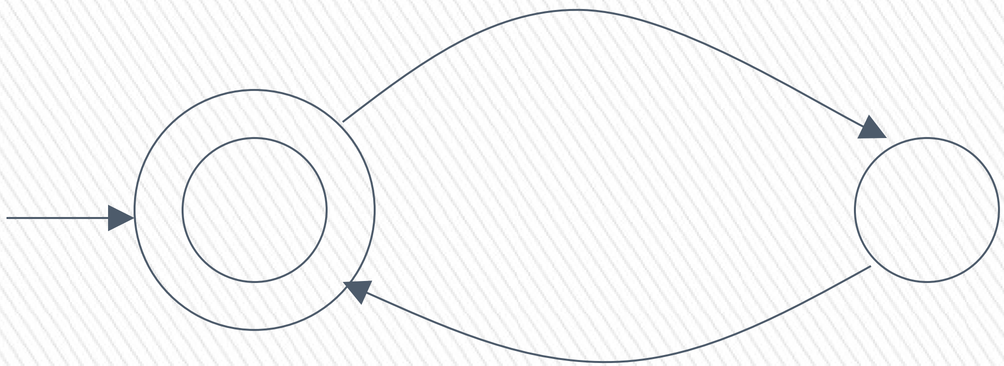


Language Accepted:



Alphabet: $\Sigma = \{1\}$



Language Accepted:

EVEN $= \{x\text{: } x \text{ is in } \Sigma^* \text{ and } x \text{ is even}\}$

$= \{\varepsilon, 11, 1111, 111111, \ldots\}$

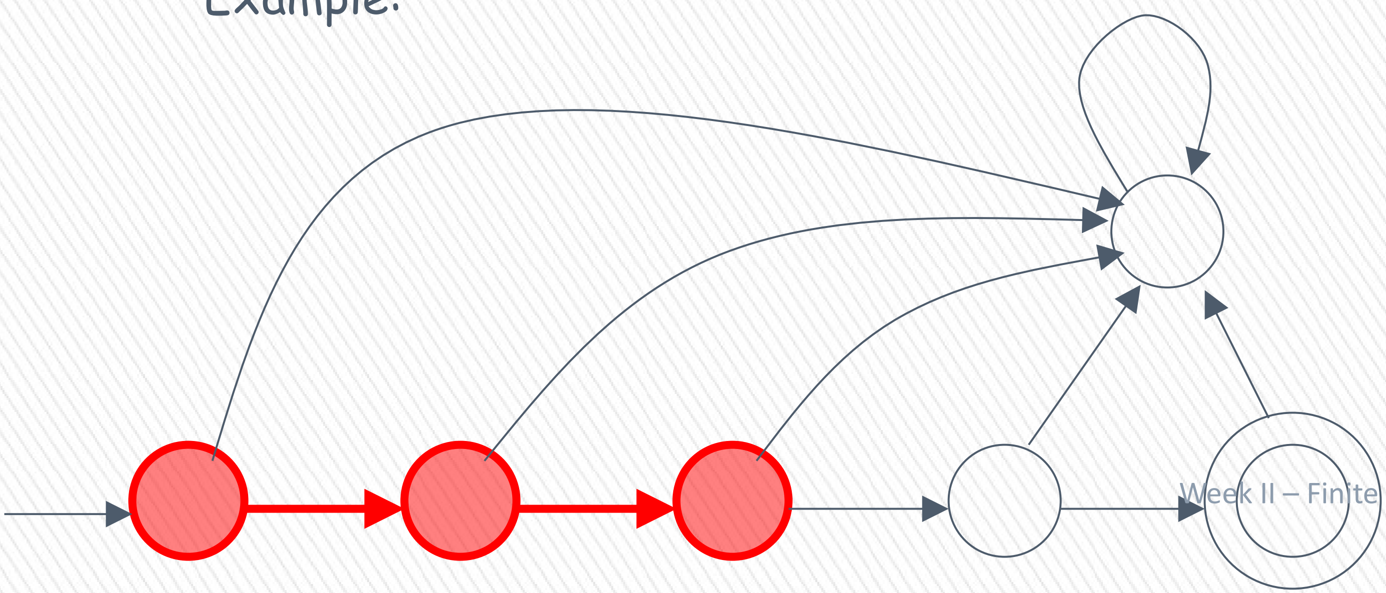
» Extended Transition Function

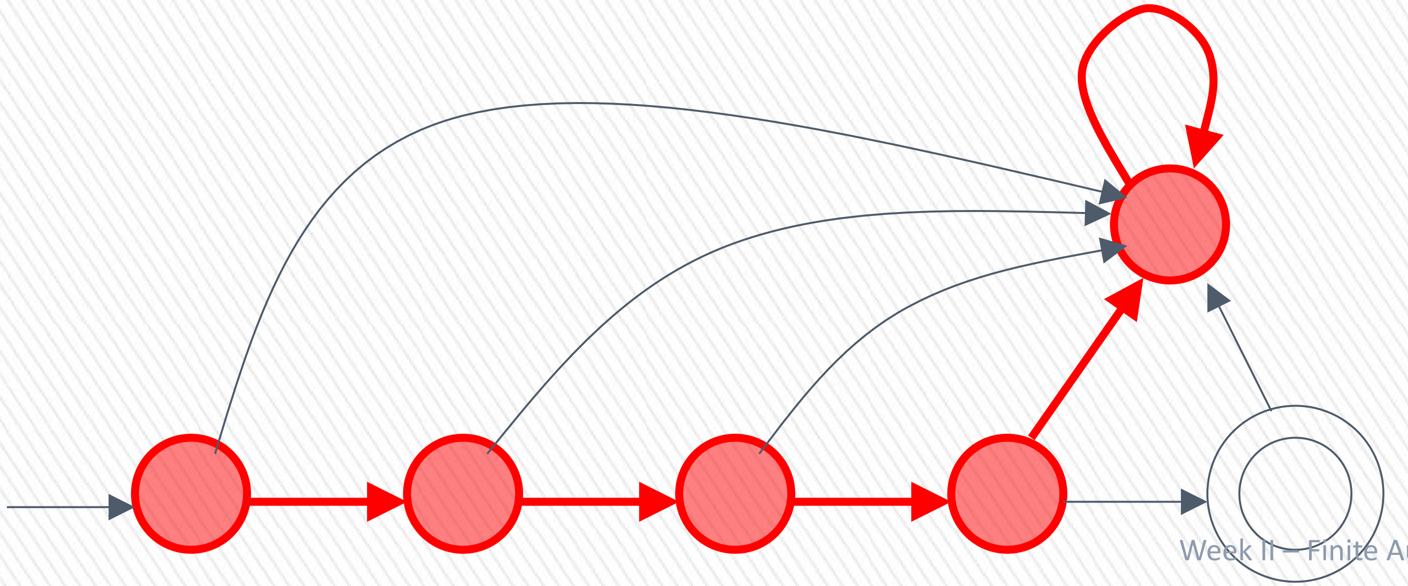
$\delta^* : Q \times \Sigma^* \rightarrow Q$

$\delta\,(q, w) \rightarrow q'$

> Describes the resulting state after scanning string **w** from state **q**

Example:





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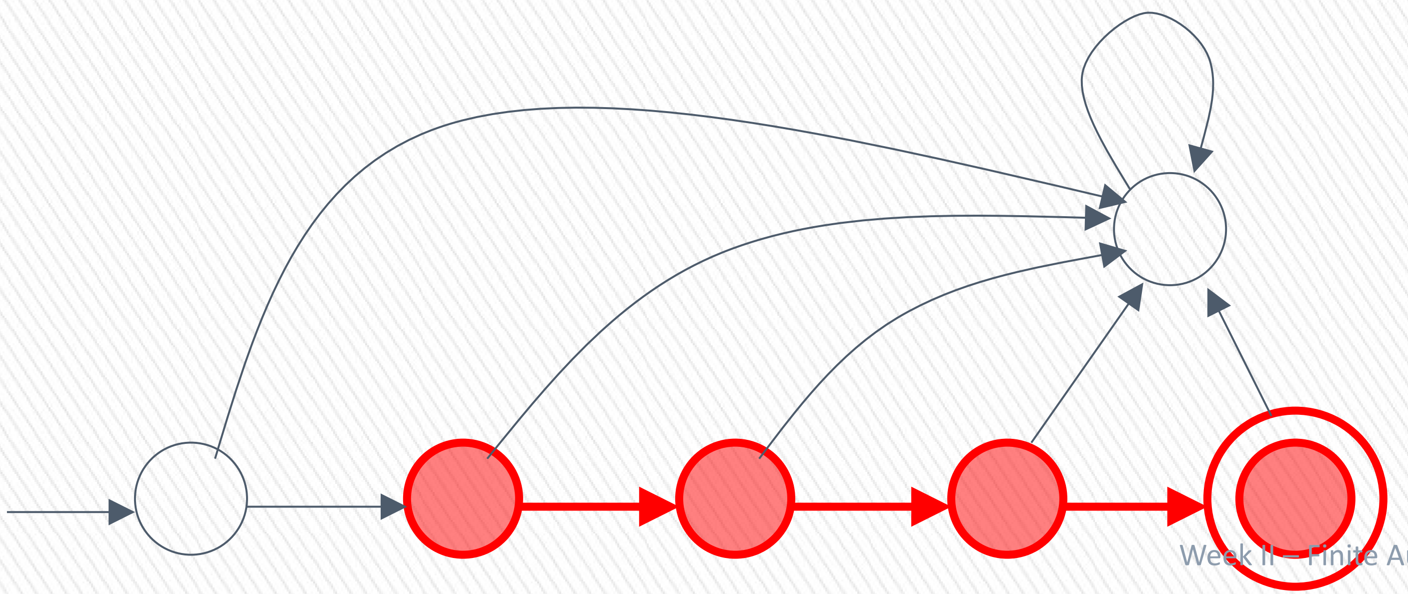
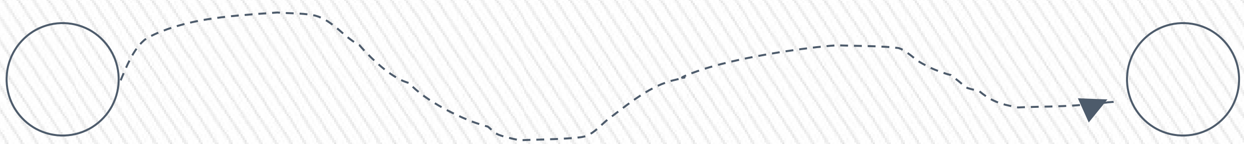


In general:

$\delta^*(q, w) \rightarrow q'$ implies that there is a walk of transitions

states may be repeated



Language Accepted By DFA

The language accepted by an automaton M, is denoted as L(M) and contains all the strings accepted by M

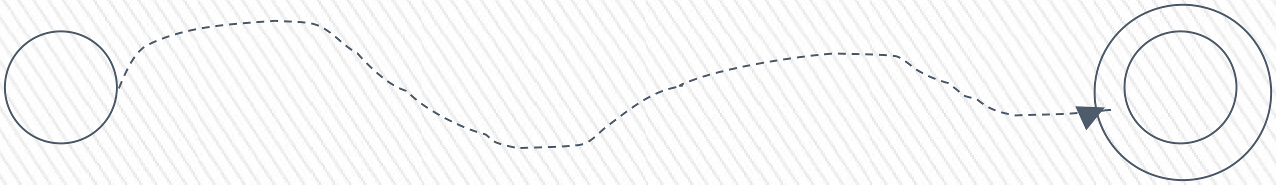
We say that a language L' is accepted (or recognized) by the DFA M if

$$L(M) = L'$$

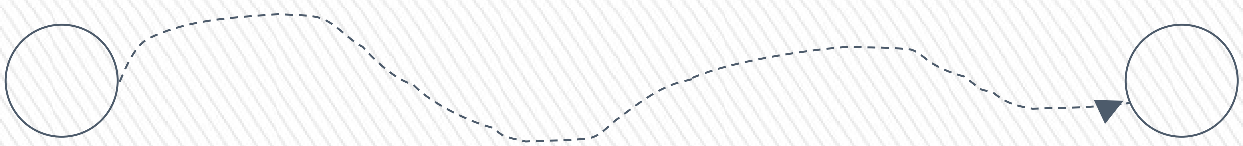
An automaton accepts one and only one language.
A language can be accepted by a number of automata

- For a DFA **M** = $(Q, \Sigma, \delta, q_0, F)$
- Language accepted by **M**:

  $L(M) = \{w \in \Sigma^* : \delta^*(q_0,w) \in F\}$



» Language rejected by **M** :



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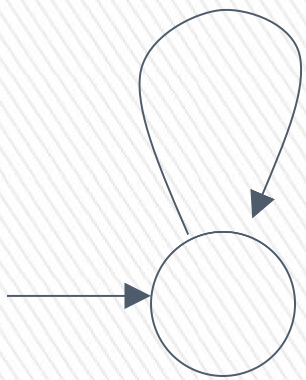
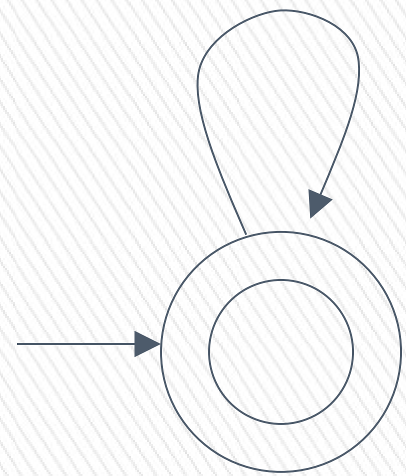
More DFA Examples



Empty language



All strings

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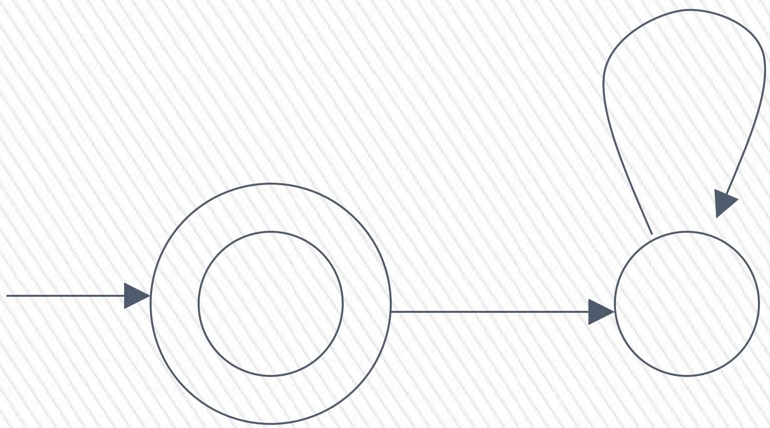
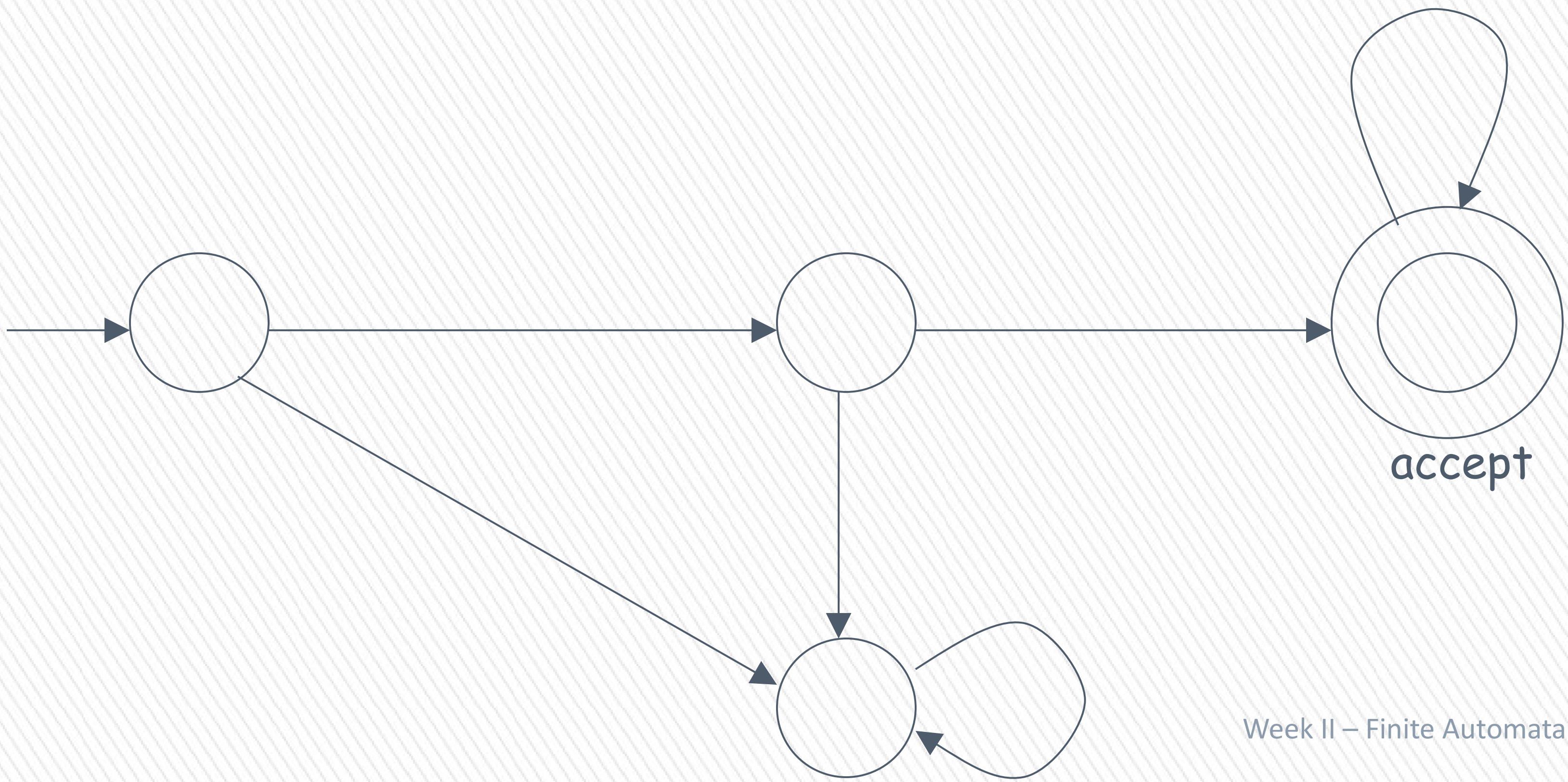
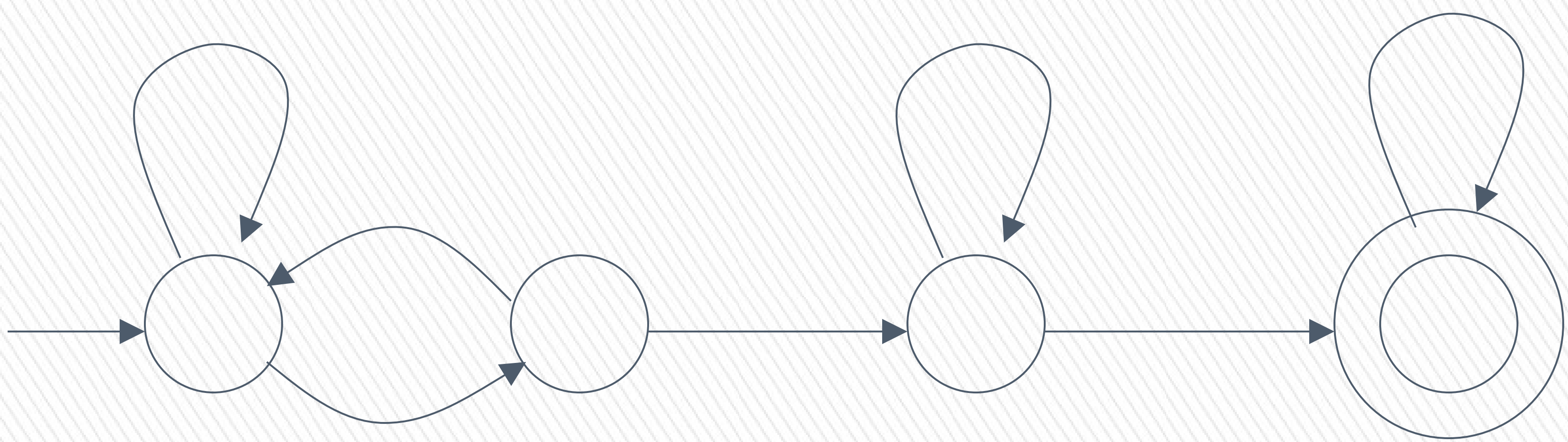


Language of the empty string

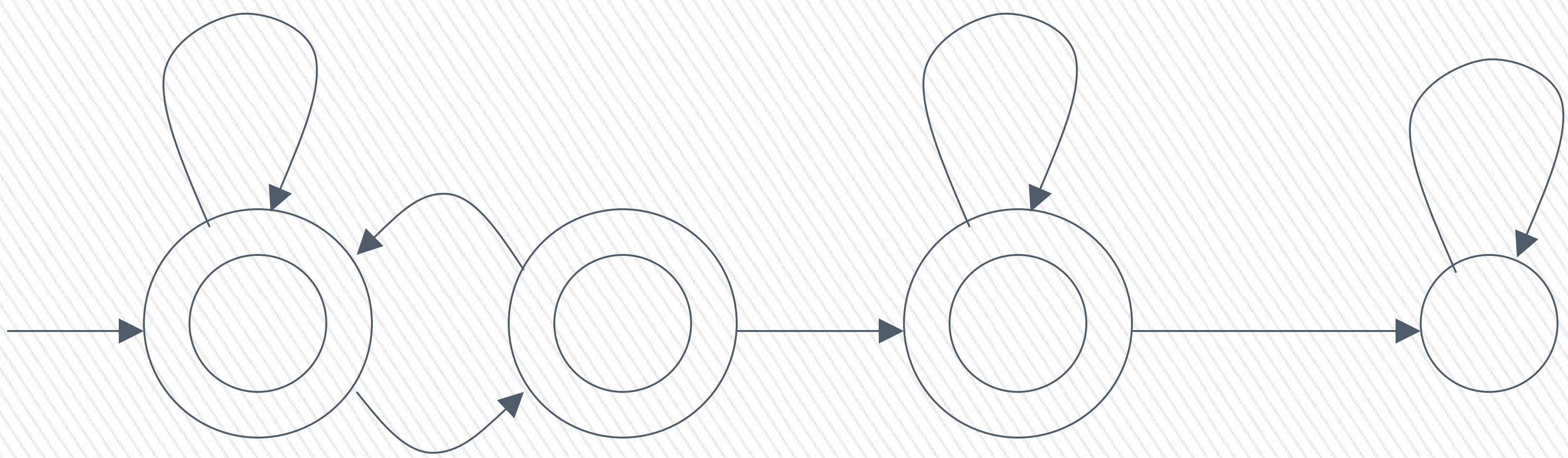
L(M) = { all strings with prefix ab }



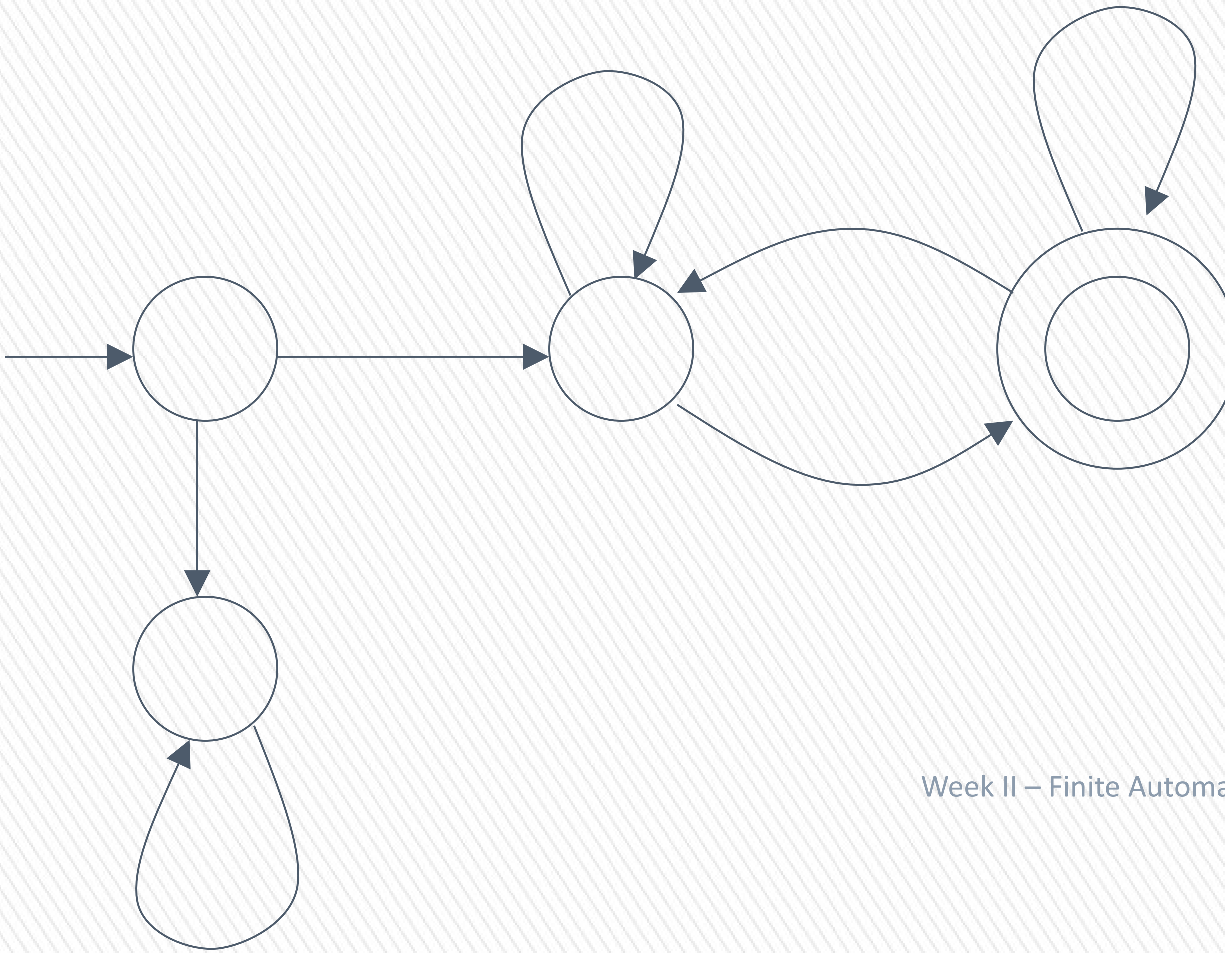
L(M) = { all binary strings containing substring 001 }



L(M) = { all binary strings without substring 001}



BIM202 Theory of Computation



{ all binary strings without substring 001 }

{ all strings in $\{a,b\}^*$ with prefix ab }

There exist automata that accept these languages (see previous slides).

There exist languages which are <u>not</u> Regular:

There is no DFA that accepts these languages