BLM1612 - Circuit Theory

The Instructors:

Dr. Öğretim Üyesi Erkan Uslu

euslu@yildiz.edu.tr

Dr. Öğretim Üyesi Hamza Osman İlhan

hoilhan@yildiz.edu.tr

RC and RL Circuits

First Order Circuits

Objectives of Lecture

- Explain the operation of a RC circuit in dc circuits
  - As the capacitor stores energy when voltage is first applied to the circuit or the voltage applied across the capacitor is increased during the circuit operation.
  - As the capacitor releases energy when voltage is removed from the circuit or the voltage applied across the capacitor is decreased during the circuit operation.
- Explain the operation of a RL circuit in dc circuit
  - As the inductor stores energy when current begins to flow in the circuit or the current flowing through the inductor is increased during the circuit operation.
  - As the inductor releases energy when current stops flowing in the circuit or the current flowing through the inductor is decreased during the circuit operation.

Natural Response

- The behavior of the circuit with no external sources of excitation.
  - There is stored energy in the capacitor or inductor at $time = 0$ s.
  - For $t > 0$ s, the stored energy is released
    - Current flows through the circuit and voltages exist across components in the circuit as the stored energy is released.
    - The stored energy will decays to zero as time approaches infinite, at which point the currents and voltages in the circuit become zero.

RC Circuit

- Suppose there is some charge on a capacitor at time $t = 0$ s.

  - This charge could have been stored because a voltage or current source had been in the circuit at $t < 0$ s, but was switched off at $t = 0$ s.

- We can use the equations relating voltage and current to determine how the charge on the capacitor is removed as a function of time.

  - The charge flows from one plate of the capacitor through the resistor R to the other plate to neutralize the charge on the opposite plate of the capacitor.

Equations for RC Circuit
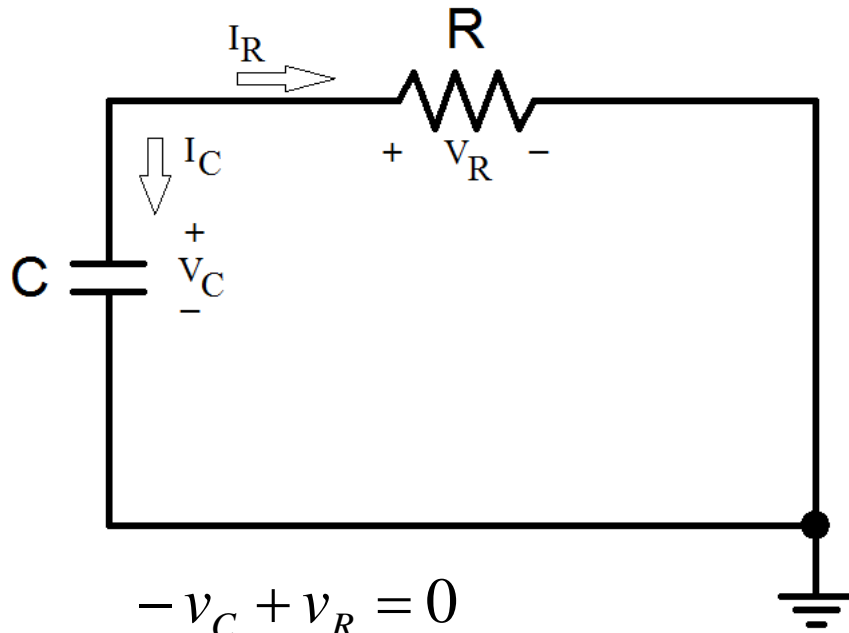
$$-v_C + v_R = 0$$

$$i_C = -i_R$$

$$i_C = C\frac{dv_C}{dt}$$

$$i_R = \frac{v_R}{R}$$
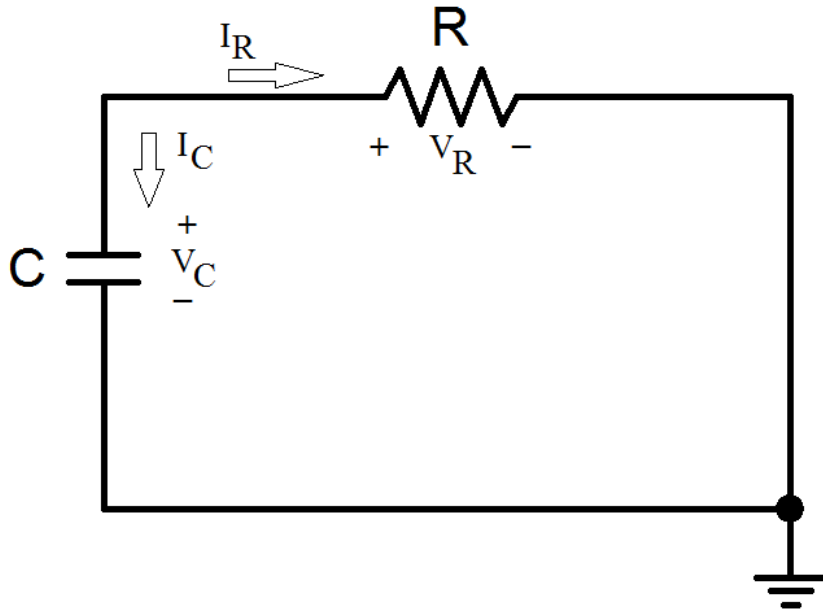
$$C\frac{dV_C}{dt} + \frac{V_R}{R} = 0$$

$$V_R = V_C$$

$$\frac{dV_C}{dt} + \frac{V_C}{RC} = 0$$

$$\frac{1}{V_C}\frac{dV_C}{dt} + \frac{1}{RC} = 0$$

$$\frac{dV_C}{V_C} = -\frac{1}{RC}dt$$

$$\ln(V_C) = -\frac{t}{RC} + \ln\left(V_C\big|_{t=t_o}\right)$$

Equations for RC Circuit

If $V_o = V_C\big|_{t=0s}$ and $\tau = RC$

$$V_C(t) = V_o e^{-\frac{t}{\tau}} \quad \text{when } t \geq 0s$$

$$I_R(t) = -I_C(t) = \frac{V_o}{R} e^{-\frac{t}{\tau}}$$

Since the voltages are equal and the currents have the opposite sign, the power that is dissipated by the resistor is the power that is being released by the capacitor.

$$p_R(t) = V_R I_R = \frac{V_o^2}{R} e^{-\frac{2t}{\tau}}$$

$$w(t) = \int_{0s}^{t} p_R(t)dt = \frac{CV_o^2}{2}\left[1 - e^{-\frac{2t}{\tau}}\right]$$

The Key to Working with a Source-Free RC Circuit Is Finding:

- The initial voltage $v(0) = V_0$ across the capacitor.
  - Can be obtained by inserting a d.c. source to the circuit for a time much longer than τ (at least t = -5τ) and then removing it at $t = 0$.
    - Capacitor
      - Open Circuit Voltage

- The time constant τ.
  - In finding the time constant $\tau = RC$, R is often the Thevenin equivalent resistance at the terminals of the capacitor;
    - that is, we take out the capacitor C and find $R = R_{Th}$ at its terminals

Time constant

- The natural response of a capacitive circuit refers to the behavior (in terms of voltages) of the circuit itself, with no external sources of excitation.

  - The natural response depends on the nature of the circuit alone, with no external sources.

    - In fact, the circuit has a response only because of the energy initially stored in the capacitor.

- The voltage response of the RC circuit
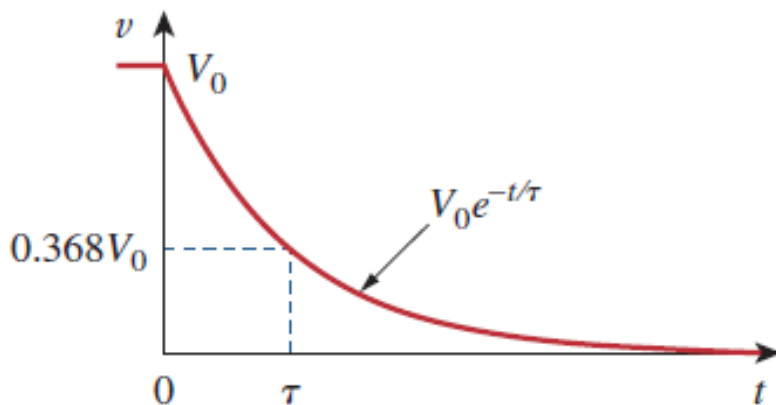
  - Time constant, $\tau = RC$

    - The time required for the voltage across the capacitor to decay by a factor of 1/e or 36.8% of its initial value.

9

Equations for RL Circuits
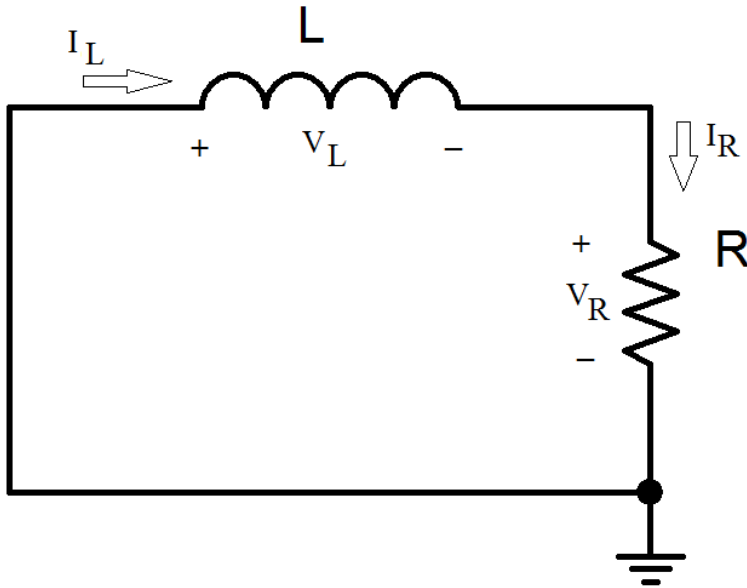
$$V_L + V_R = 0$$

$$I_L = I_R$$

$$V_L = L\frac{dI_L}{dt}$$

$$I_R = V_R/R$$

$$L\frac{dI_L}{dt} + RI_R = 0$$

$$\frac{dI_L}{dt} + \frac{RI_L}{L} = 0$$

$$\frac{1}{I_L}\frac{dI_L}{dt} + \frac{R}{L} = 0$$

$$\frac{dI_L}{I_L} = -\frac{R}{L}dt$$

$$\ln\left(I_L\right) = -\frac{R}{L}t + \ln\left(I_L\big|_{t=0s}\right)$$

Equations for RL Circuit
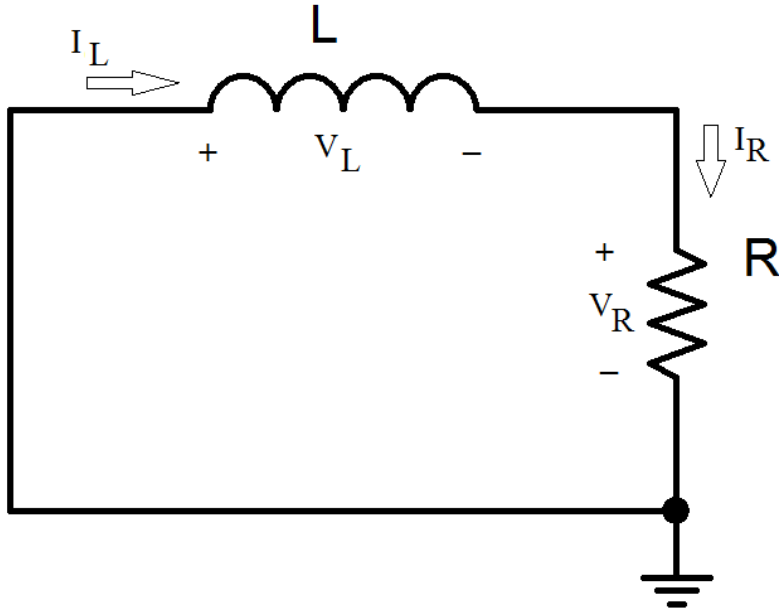
If $I_o = I_L\big|_{t=0s}$ and $\tau = \dfrac{L}{R}$

$I_L(t) = I_o e^{-\frac{t}{\tau}}$ when $t \geq 0s$

$V_R(t) = -V_L(t) = RI_o e^{-\frac{t}{\tau}}$

Since the voltages are equal and the currents have the opposite sign, the power that is dissipated by the resistor is the power that is being released by the inductor.

$p_R(t) = V_R I_R = RI_o^2 e^{-\frac{2t}{\tau}}$

$w(t) = \displaystyle\int_{0s}^{t} p_R(t)dt = \frac{LI_o^2}{2}\left[1 - e^{-\frac{2t}{\tau}}\right]$

The Key to Working with a Source-Free RL Circuit Is Finding:

- The initial current $i(0) = I_0$ through the inductor.
  - Can be obtained by inserting a d.c. source to the circuit for a time much longer than τ (at least t = -5τ) and then removing it at $t = 0$.
    - Inductor
      - Short Circuit Current

- The time constant τ.
  - In finding the time constant $\tau = $ L/R, R is often the Thevenin equivalent resistance at the terminals of the inductor;
    - that is, we take out the inductor L and find $R = R_{\text{Th}}$ at its terminals

Time constant

- The natural response of an inductive circuit refers to the behavior (in terms of currents) of the circuit itself, with no external sources of excitation.
  - The natural response depends on the nature of the circuit alone, with no external sources.
    - In fact, the circuit has a response only because of the energy initially stored in the inductor.

- The current response of the RL circuit
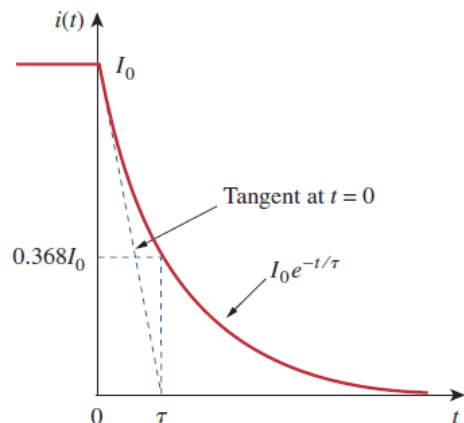
  - Time constant, $\tau = L/R$
    - The time required for the current in the inductor to decay by a factor of 1/e or 36.8% of its initial value.

13

Singularity Functions

- Singularity functions (also called switching functions) are very useful in circuit analysis.
- They serve as good approximations to the switching signals that arise in circuits with switching operations.
- They are helpful in the neat, compact description of some circuit phenomena,
  - especially the step response of *RC* or *RL* circuits
- Singularity functions are functions that either are discontinuous or have discontinuous derivatives.

Unit Step Function

- The unit step function ($u(t)$) is 0 for negative values of t and 1 for positive values of t.
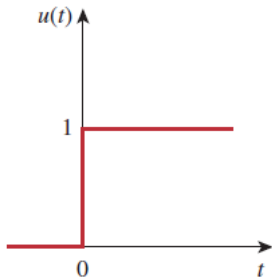
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$
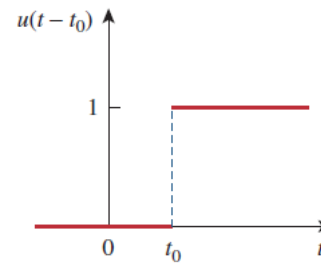
$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$
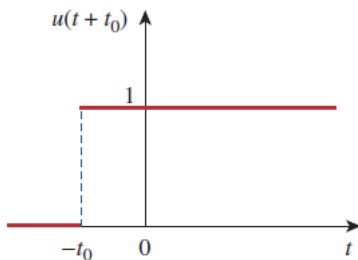
$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases}$$
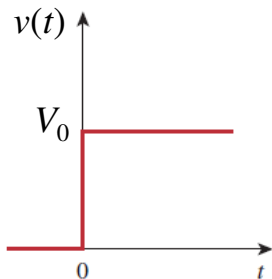
$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases} \qquad \Longrightarrow \qquad v(t) = V_0 u(t - t_0)$$

15

Unit Step Function

- Voltage source of $V_0 u(t)$ and its equivalent circuit.
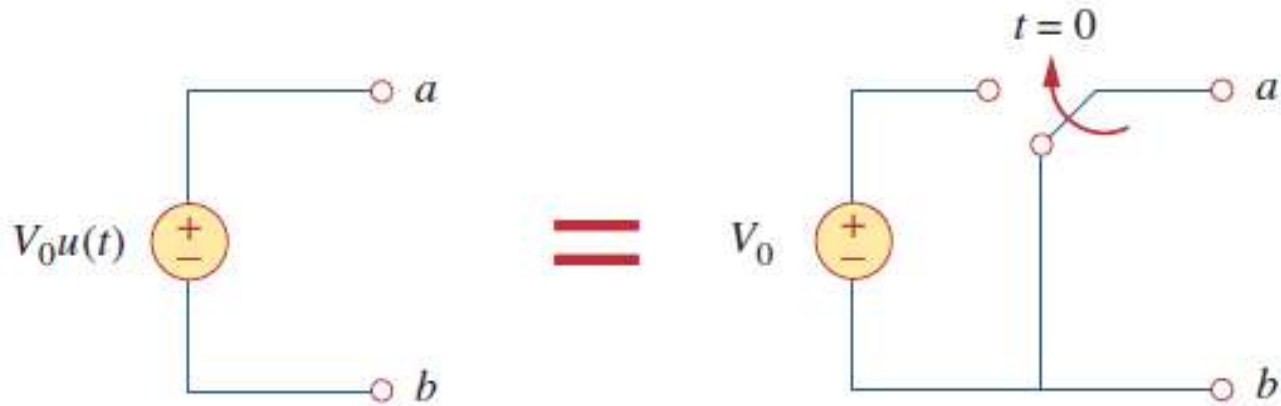
- Current source of $I_0 u(t)$ and its equivalent circuit.
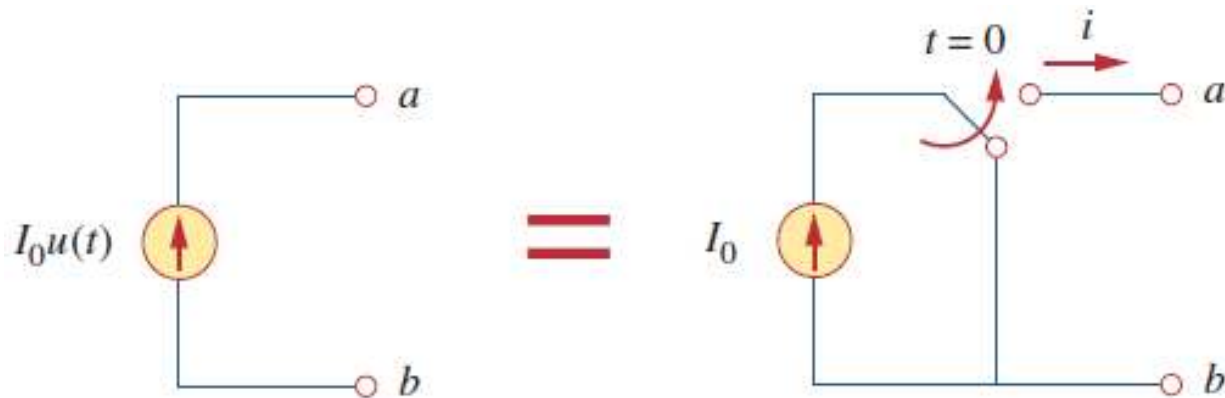
Unit Impulse Function

- The derivative of the unit step function $u(t)$ is the unit impulse function ($\delta(t)$)
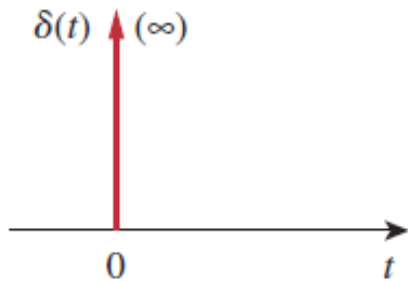
$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$

$$\int_{0^-}^{0^+} \delta(t)\, dt = 1$$
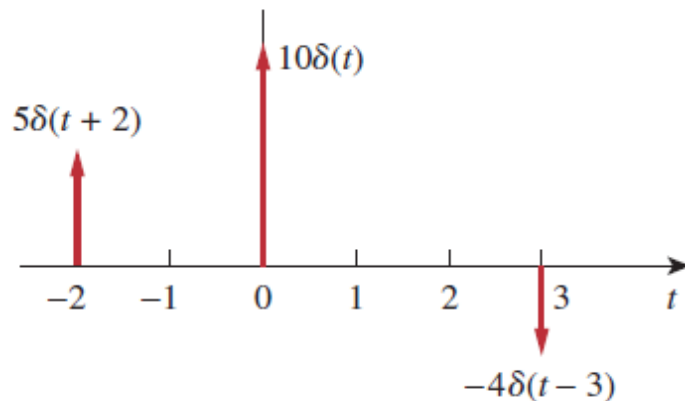
$x(t) = 5\delta(t+2) + 10\delta(t) - 4\delta(t\text{-}3)$

Integration of Unit Functions

- To illustrate how the impulse function affects other functions, let us evaluate the integral

$$\int_a^b f(t)\delta(t - t_0)\,dt$$

$$\int_a^b f(t)\delta(t - t_0)\,dt = \int_a^b f(t_0)\delta(t - t_0)\,dt$$

$$= f(t_0)\int_a^b \delta(t - t_0)\,dt = f(t_0)$$

- This is a highly useful property of the impulse function known as the sampling or sifting property.

Unit Ramp Function
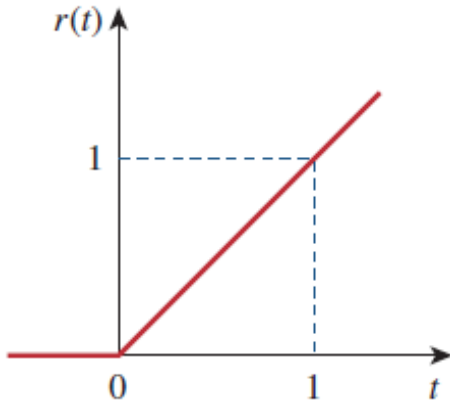
$$r(t) = \int_{-\infty}^{t} u(\lambda)d\lambda = tu(t)$$

$$r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases}$$
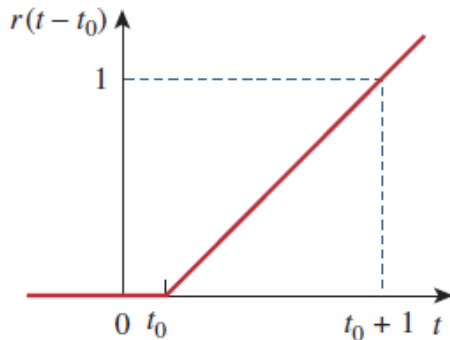
$$r(t - t_0) = \begin{cases} 0, & t \leq t_0 \\ t - t_0, & t \geq t_0 \end{cases}$$
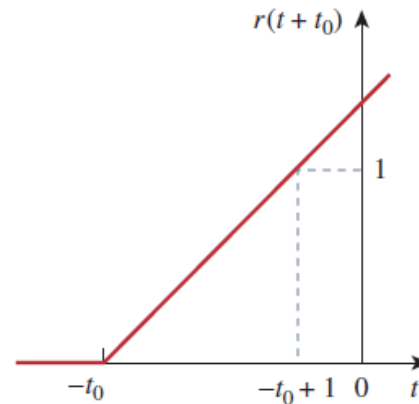
$$r(t + t_0) = \begin{cases} 0, & t \leq -t_0 \\ t + t_0, & t \geq -t_0 \end{cases}$$

Relationships of singularity functions

- The three singularity functions (impulse, step, and ramp) are related by differentiation as

$$\delta(t) = \frac{du(t)}{dt}, \qquad u(t) = \frac{dr(t)}{dt}$$

- or by integration as

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)\,d\lambda, \qquad r(t) = \int_{-\infty}^{t} u(\lambda)\,d\lambda$$

Transient responses of RC and RL circuits

- AKA a forced response to an independent source

- Capacitor and inductor store energy when there is:

  – a transition in a unit step function source, u(t-to)

  – a voltage or current source is switched into the circuit.

RC Circuit

$I_C = 0A$ when $t < t_o$

$V_C = 0V$ when $t < t_o$
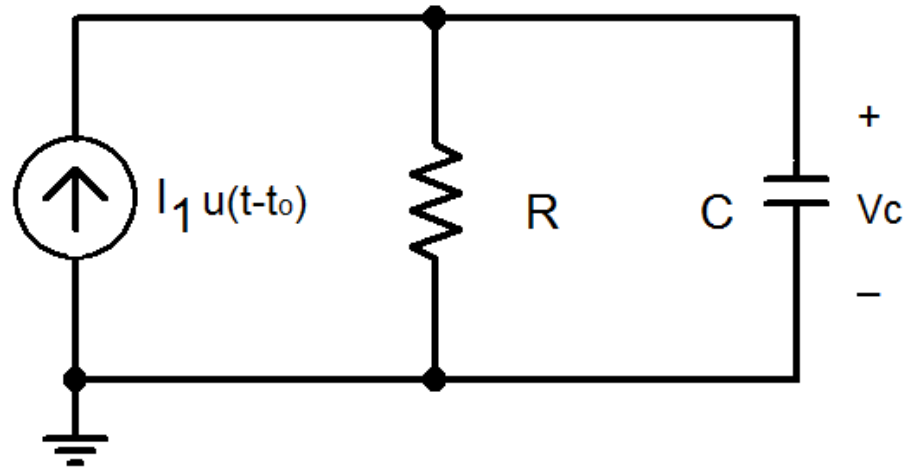
Because $I_1 = 0A$ (replace it with an open circuit).

RC Circuit

- Find the final condition of the voltage across the capacitor.
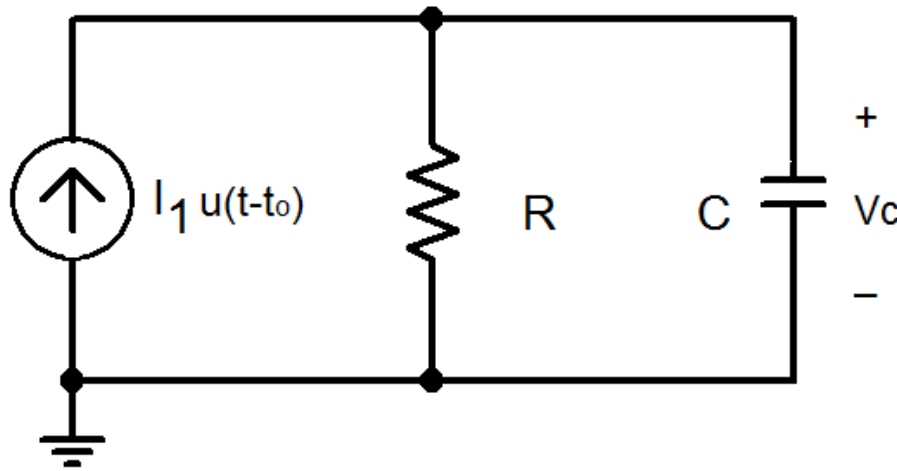
– Replace C with an open circuit and determine the voltage across the terminal.

$I_C = 0A$ when $t \sim \infty$ s

$V_C = V_R = I_1 R$ when $t \sim \infty$ s

RC Circuit

- In the time between t_o and $t = \infty$ s, the capacitor stores energy and currents flow through R and C.

$$V_C = V_R$$

$$I_C = C\frac{dV_C}{dt}$$

$$I_R = \frac{V_R}{R}$$

$$I_R + I_C - I_1 = 0$$

$$\frac{V_C}{R} + C\frac{dV_C}{dt} - I_1 = 0$$

$$V_C(t) = RI_1\left[1 - e^{-\frac{t-t_0}{\tau}}\right] \qquad \tau = RC$$
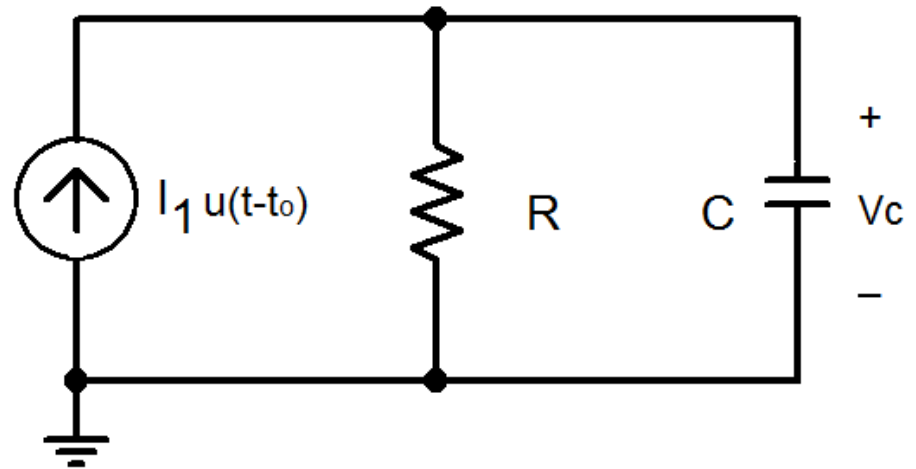
RL Circuit
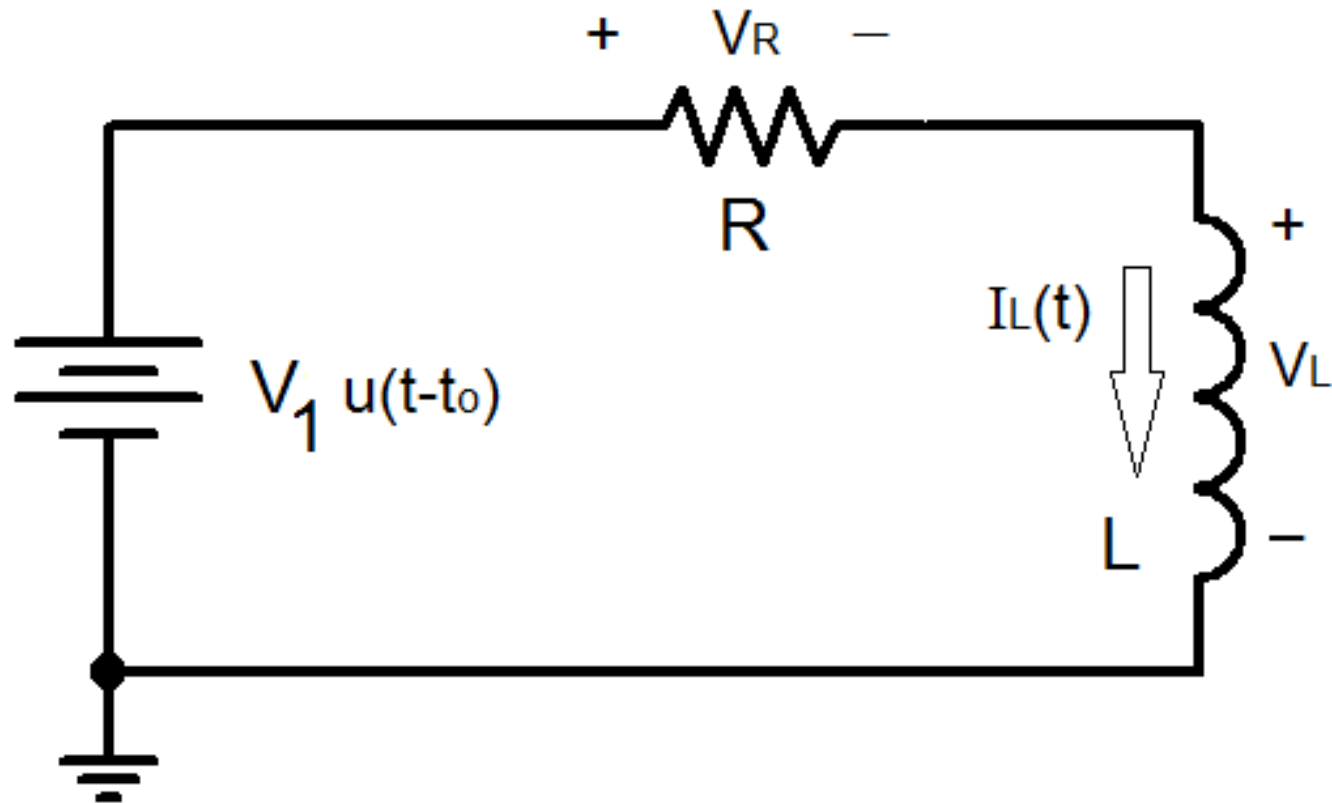
RL Circuit

- Initial condition is not important as the magnitude of the voltage source in the circuit is equal to 0V when $t \leq t_o$.
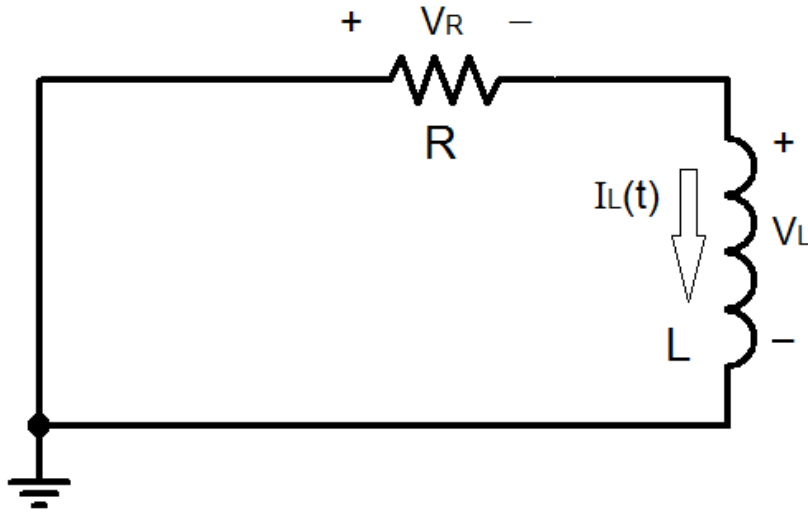
- Since the voltage source has only been turned on at $t = t_o$, the circuit at $t \leq t_o$ is as shown on the left.

- As the inductor has not stored any energy because no power source has been connected to the circuit as of yet, all voltages and currents are equal to zero.

RL Circuit

- So, the final condition of the inductor current needs to be calculated after the voltage source has switched on.

  - Replace L with a short circuit and calculate $I_L(\infty)$.
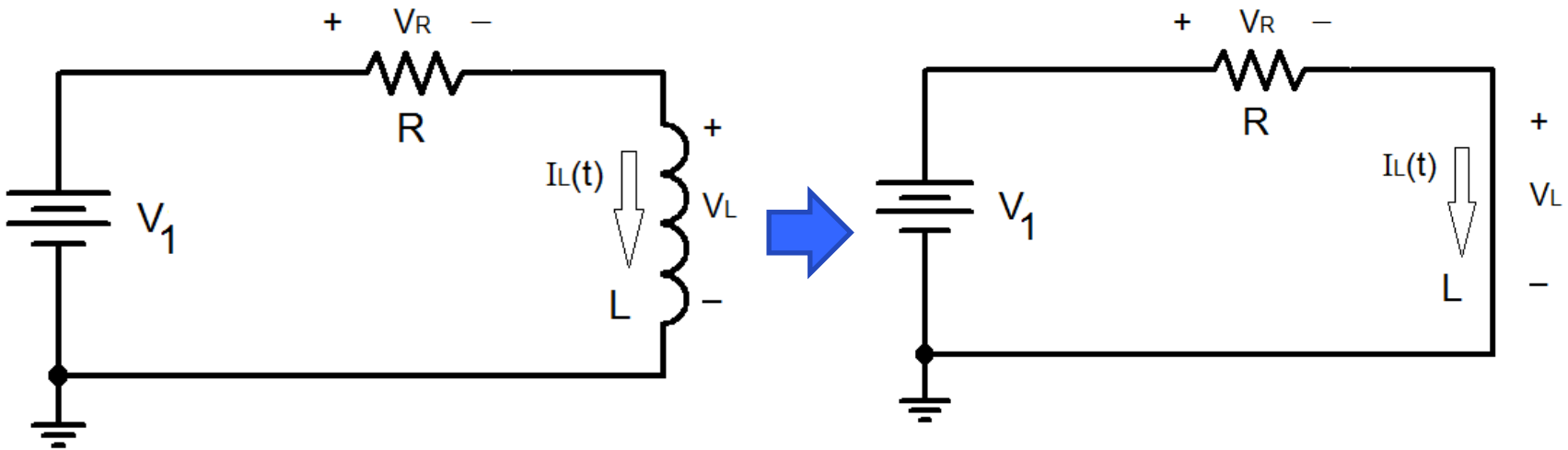
Final Condition
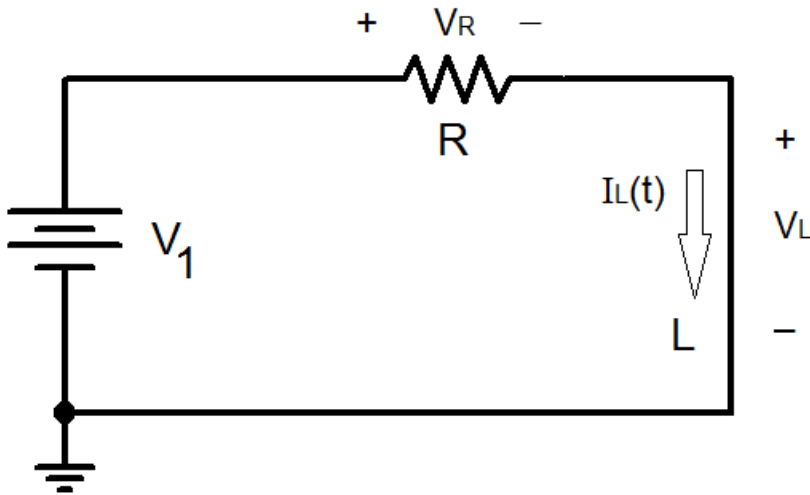
$$V_L(\infty) = 0V$$

$$I_L(\infty) = I_R$$

$$I_R = \frac{V_1}{R}$$

RL Circuit
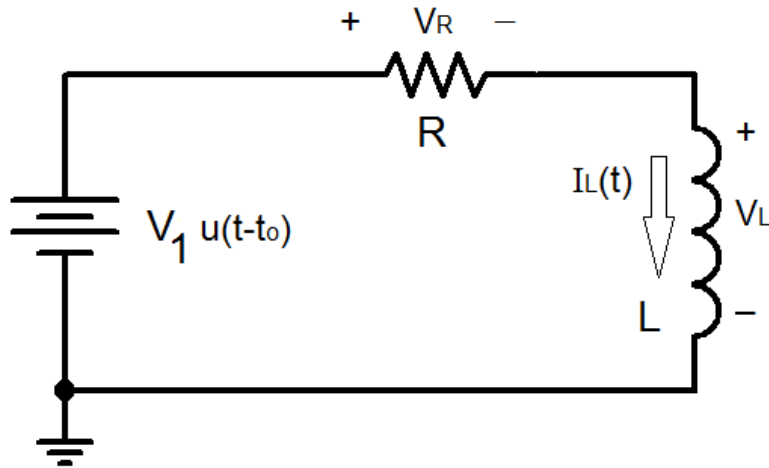
$$-V_1 + V_L + V_R = 0$$

$$I_L = I_R = V_R / R$$

$$V_L = L\frac{dI_L}{dt}$$

$$\frac{dI_L}{dt} + RI_R - V_1 = 0$$

$$\frac{dI_L}{dt} + \frac{R}{L}I_L - \frac{V_1}{L} = 0$$

$$I_L(t) = \frac{V_1}{R}\left[1 - e^{-(t-t_o)/\tau}\right]$$

$$\tau = \frac{L}{R}$$

Complete Response

- Is equal to the natural response of the circuit plus the forced response
  - Use superposition to determine the final equations for voltage across components and the currents flowing through them.

- Typically, it is assumed that the currents and voltages in a circuit have reached steady-state once 5τ have passed after a change has been made to the value of a current or voltage source in the circuit.
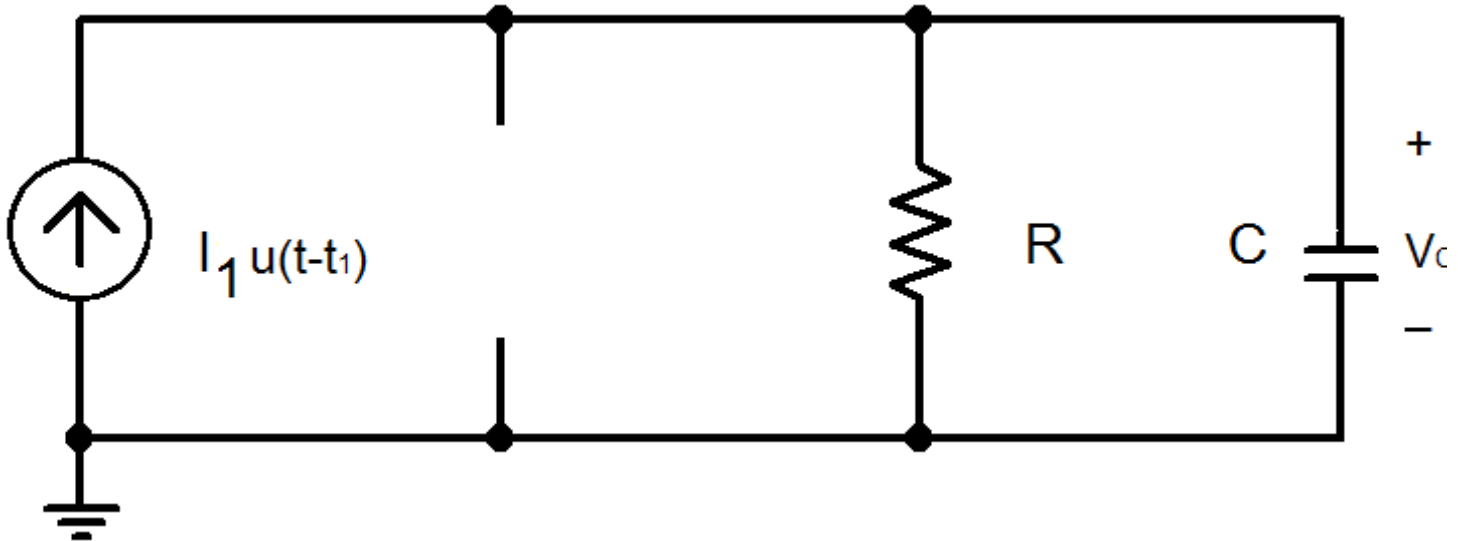
Example 01…

- Suppose there were two unit step function sources in the circuit.
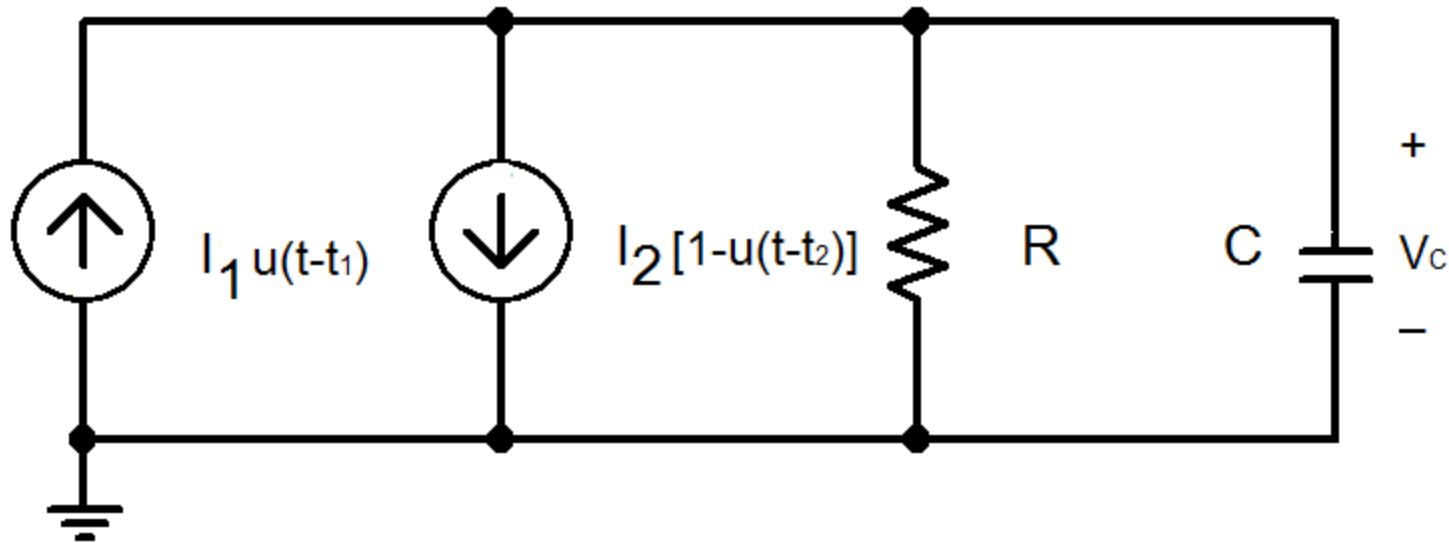
…Example 01…

- The solution for Vc would be the result of superposition where:
  - $I_2 = 0A$, I_1 is left on
    - The solution is a forced response since I_1 turns on at $t = t_1$
  - $I_1 = 0A$, I_2 is left on
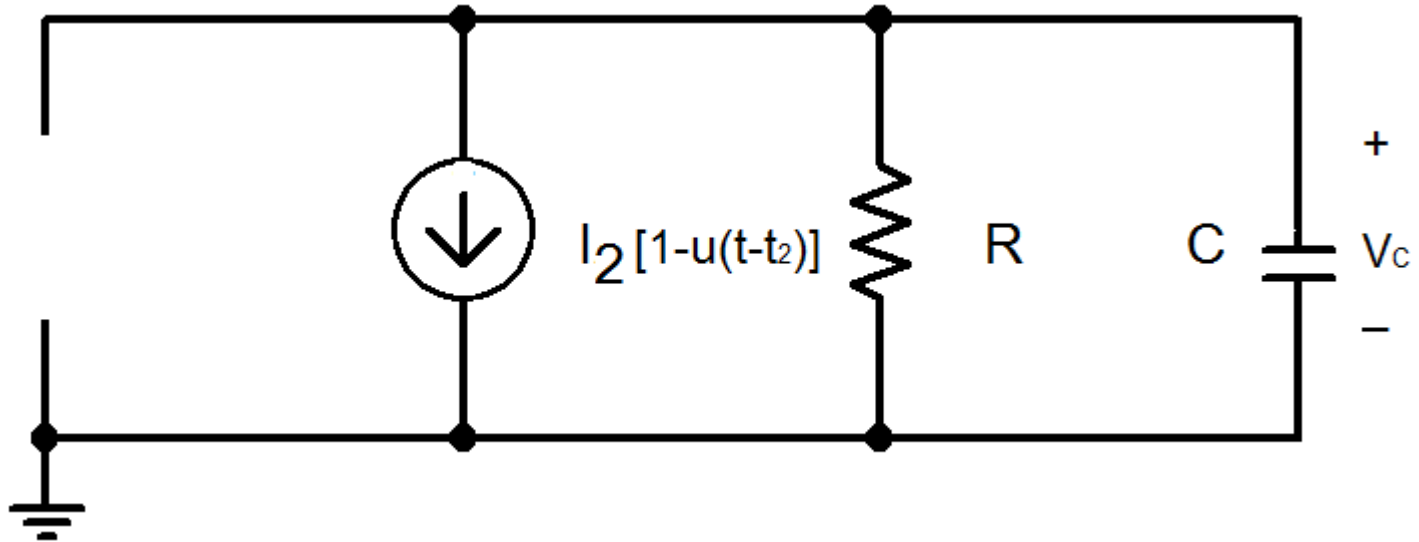   - The solution is a natural response since I_2 turns off at $t = t_2$

…Example 01…

$$V_C(t) = 0V \quad \text{when } t < t_1$$

$$V_C(t) = RI_1 \left[1 - e^{-\frac{(t-t_1)}{RC}} \right] \quad \text{when } t > t_1$$

…Example 01…

$$V_C(t) = -RI_2 \quad \text{when } t < t_2$$

$$V_C(t) = -RI_2 e^{-\frac{(t-t_2)}{RC}} \quad \text{when } t > t_2$$

…Example 01…

- If $t_1 < t_2$

$$V_C(t) = 0V - RI_2 \qquad\qquad \text{when } t < t_1$$

$$V_C(t) = RI_1\left[1 - e^{-\frac{(t-t_1)}{RC}}\right] - RI_2 \qquad \text{when } t_1 < t < t_2$$

$$V_C(t) = RI_1\left[1 - e^{-\frac{(t-t_1)}{RC}}\right] - RI_2 e^{-\frac{(t-t_2)}{RC}} \qquad \text{when } t > t_2$$

General Equations

- When a voltage or current source changes its magnitude at t = 0s in a simple RC or RL circuit.
  - Equations for a simple RC circuit

$$V_C(t) = V_C(\infty) + \left[V_C(0) - V_C(\infty)\right]e^{-t/\tau}$$

$$I_C(t) = \frac{C}{\tau}\left[V_C(\infty) - V_C(0)\right]e^{-t/\tau}$$

$$\tau = RC$$

  - Equations for a simple RL circuit

$$I_L(t) = I_L(\infty) + \left[I_L(0) - I_L(\infty)\right]e^{-t/\tau}$$

$$V_L(t) = \frac{L}{\tau}\left[I_L(\infty) - I_L(0)\right]e^{-t/\tau}$$

$$\tau = L/R$$

Example 02…
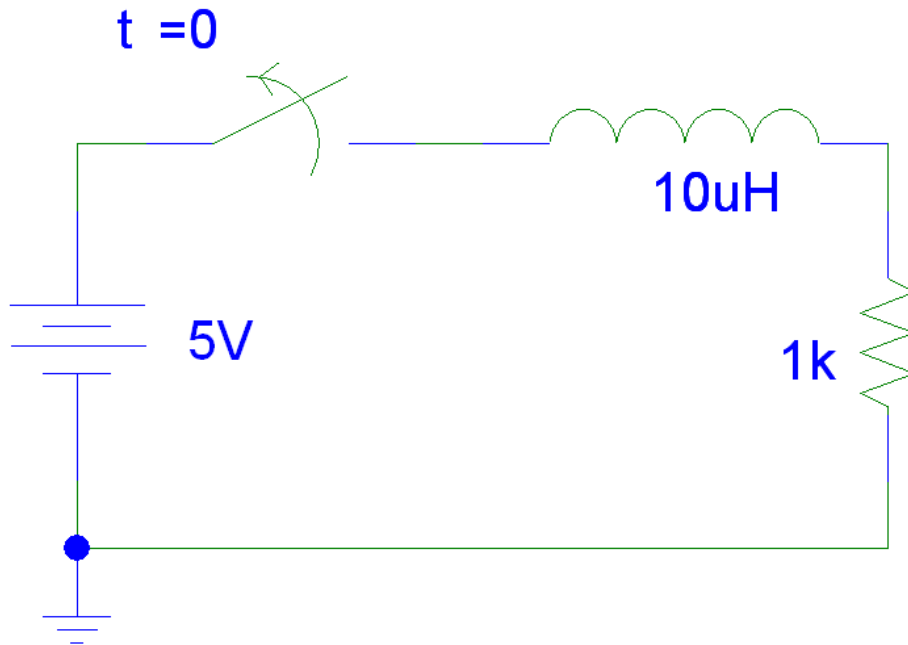
t < 0

$$V_L = 0V$$
$$V_R = 5V$$
$$I_L = I_R = 5mA$$

$$V(t) = 5V\ [1 - u(t)]$$

…Example 02

t = 0

10uH

5V

1k
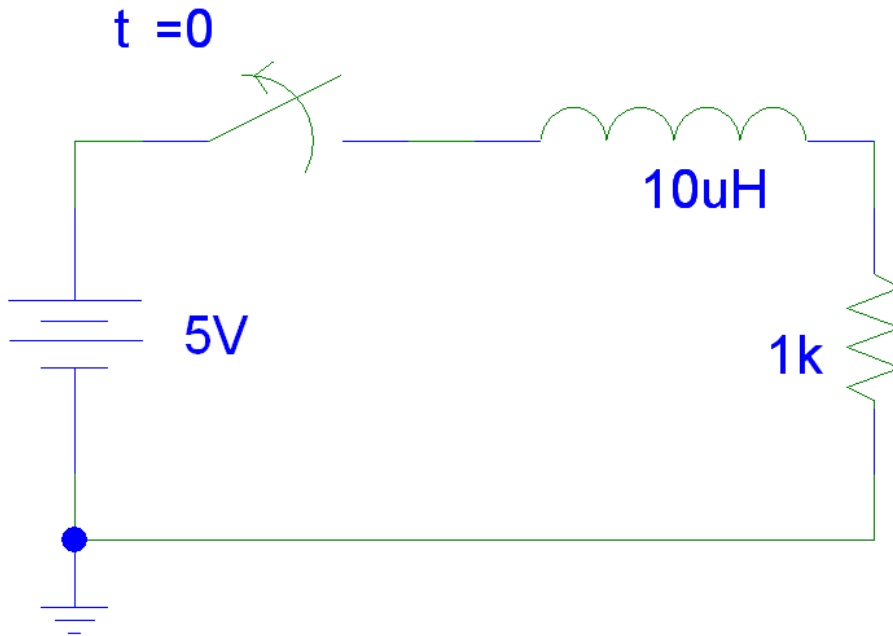
$V(t) = 5V [1 - u(t)]$

t > 0

$\tau = L/R = 10mH/1k\Omega = 10$ ns

$I_L = I_R = i(0)e^{-t/\tau} = 5mA\ e^{-t/10ns}$

$V_R = 1k\Omega\ I_R = 5V\ e^{-t/10ns}$

$V_L = L(dI_L/dt) = -5V\ e^{-t/10ns}$

Note $V_R + V_L = 0$ V

Example 03…
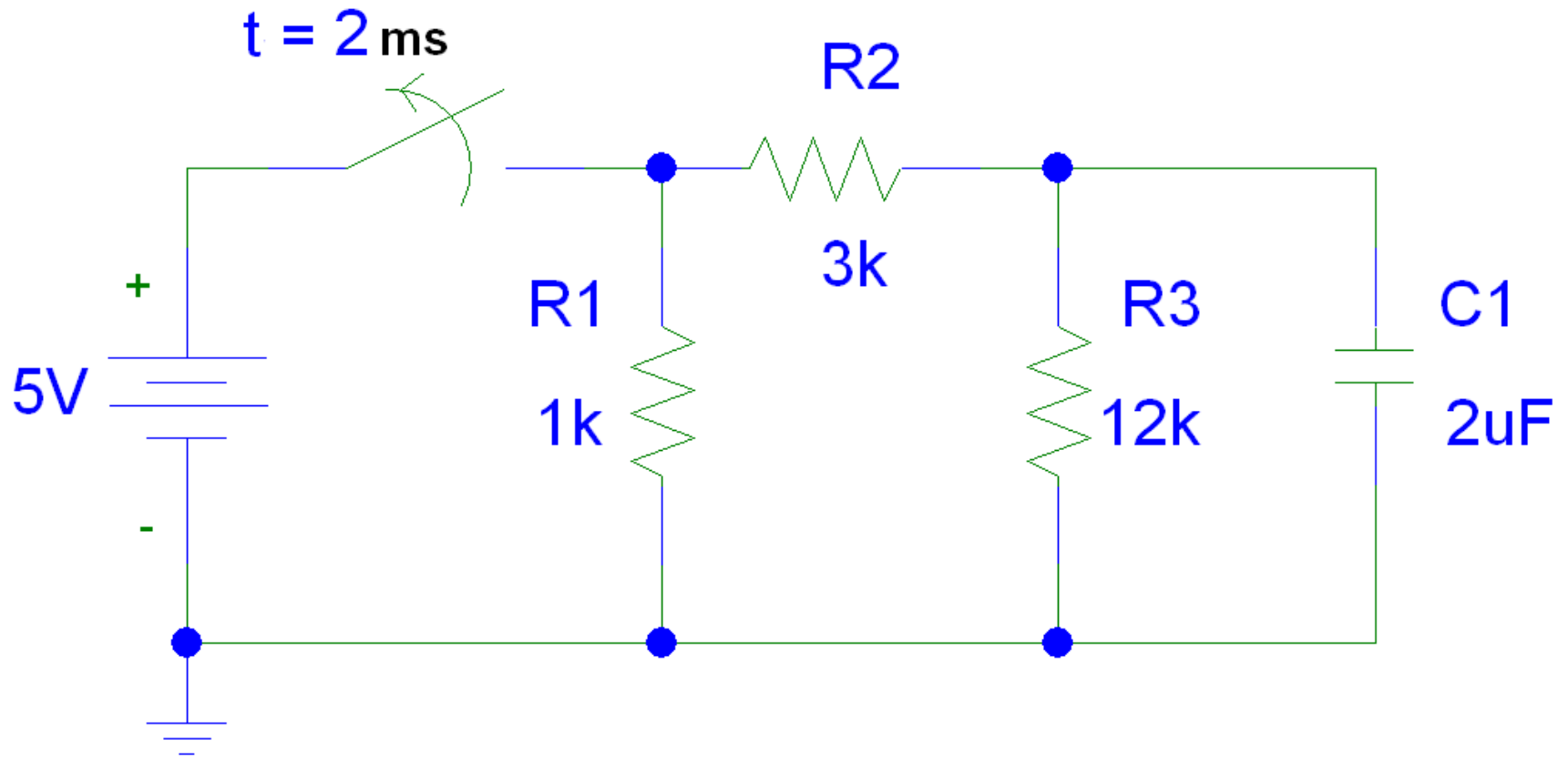
$$V(t) = 5V\ [1 - u(t - 2ms)]$$

…Example 03…

t < 2ms
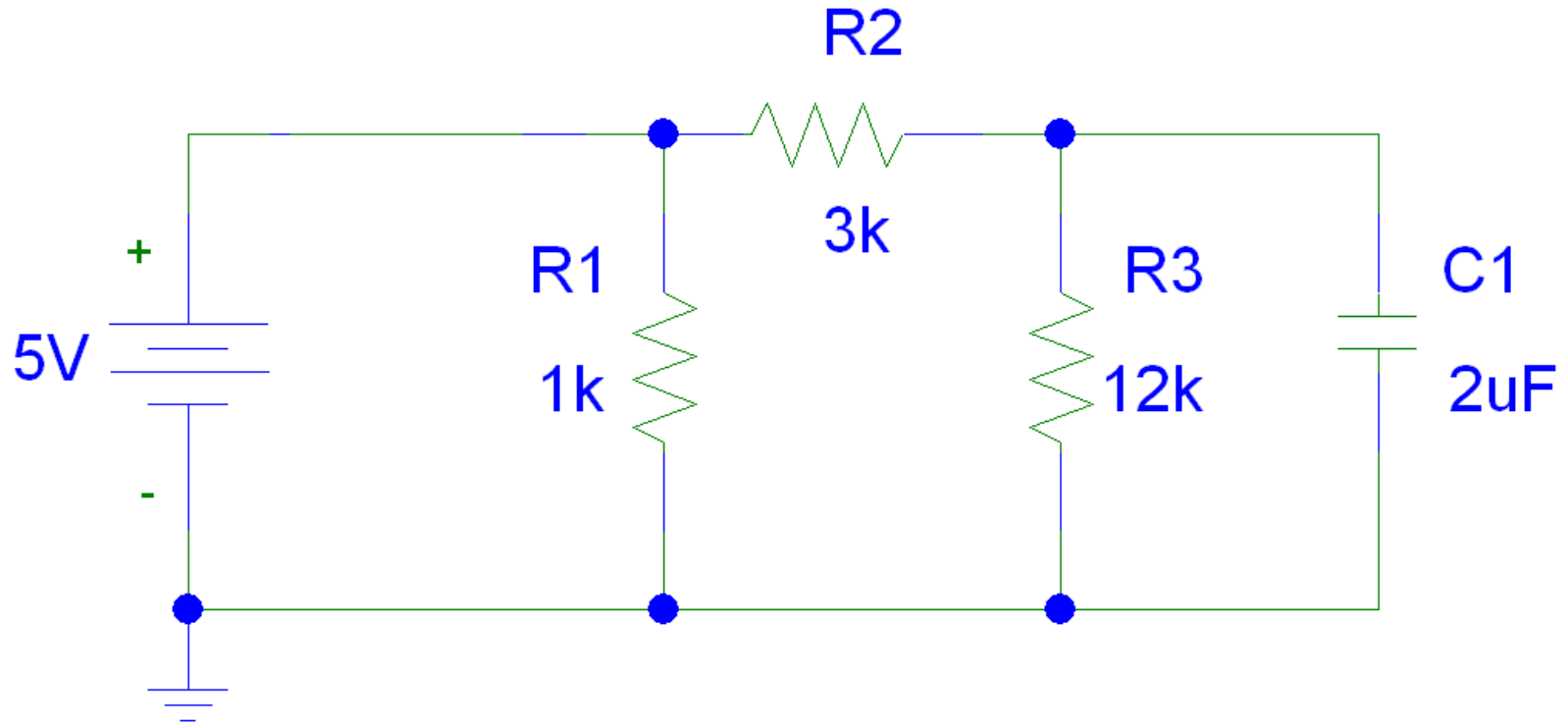
…Example 03…

- **t < 2ms**
  - C1 is an open.
    - The voltage across the capacitor is equal to the voltage across the 12kΩ resistor.

$$V_C = [12k\Omega / 15k\Omega]\ 5V = 4V$$
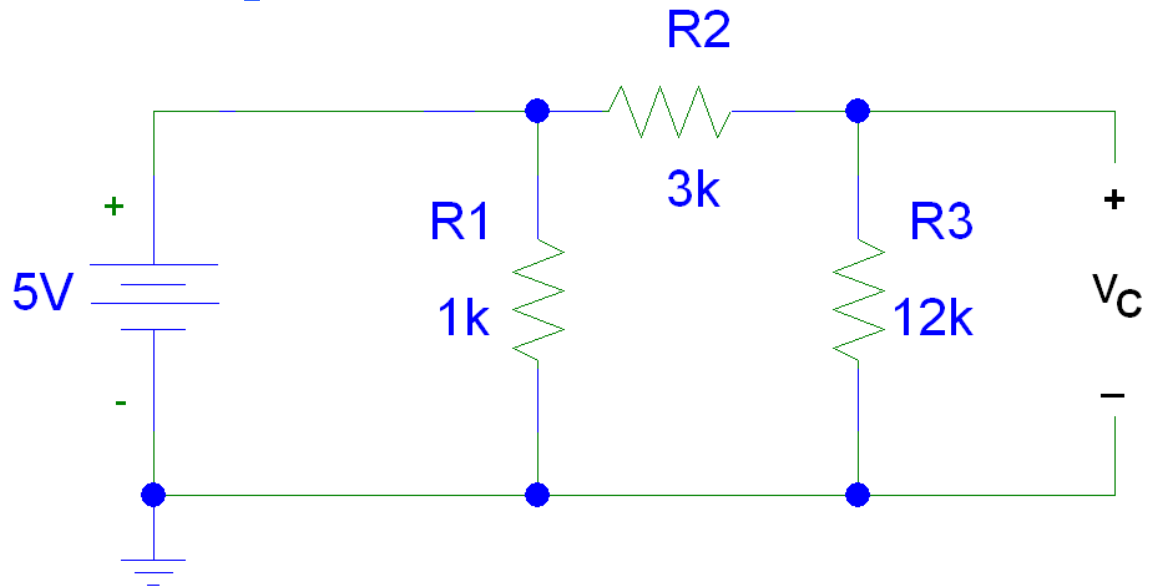
…Example 03…

t > 2ms
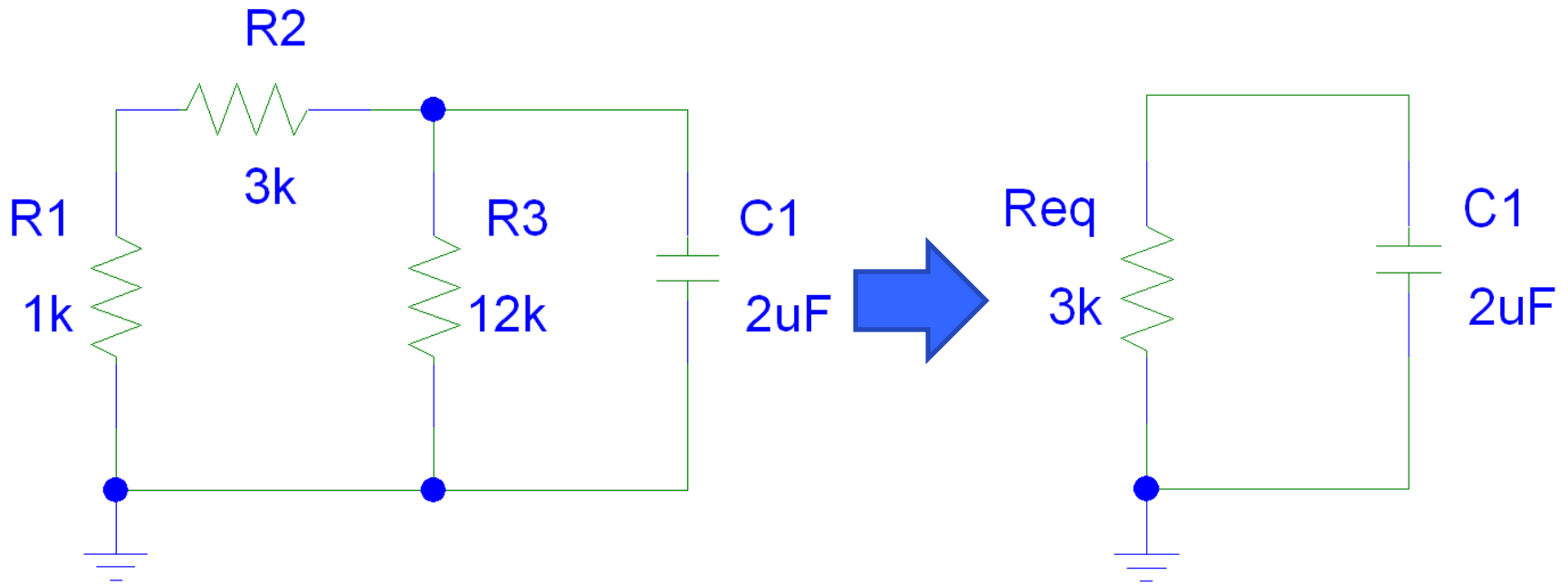
…Example 03

t > 2ms

$$\tau = R_{eq}C = 3k\Omega(2\mu F) = 6 \text{ ms}$$
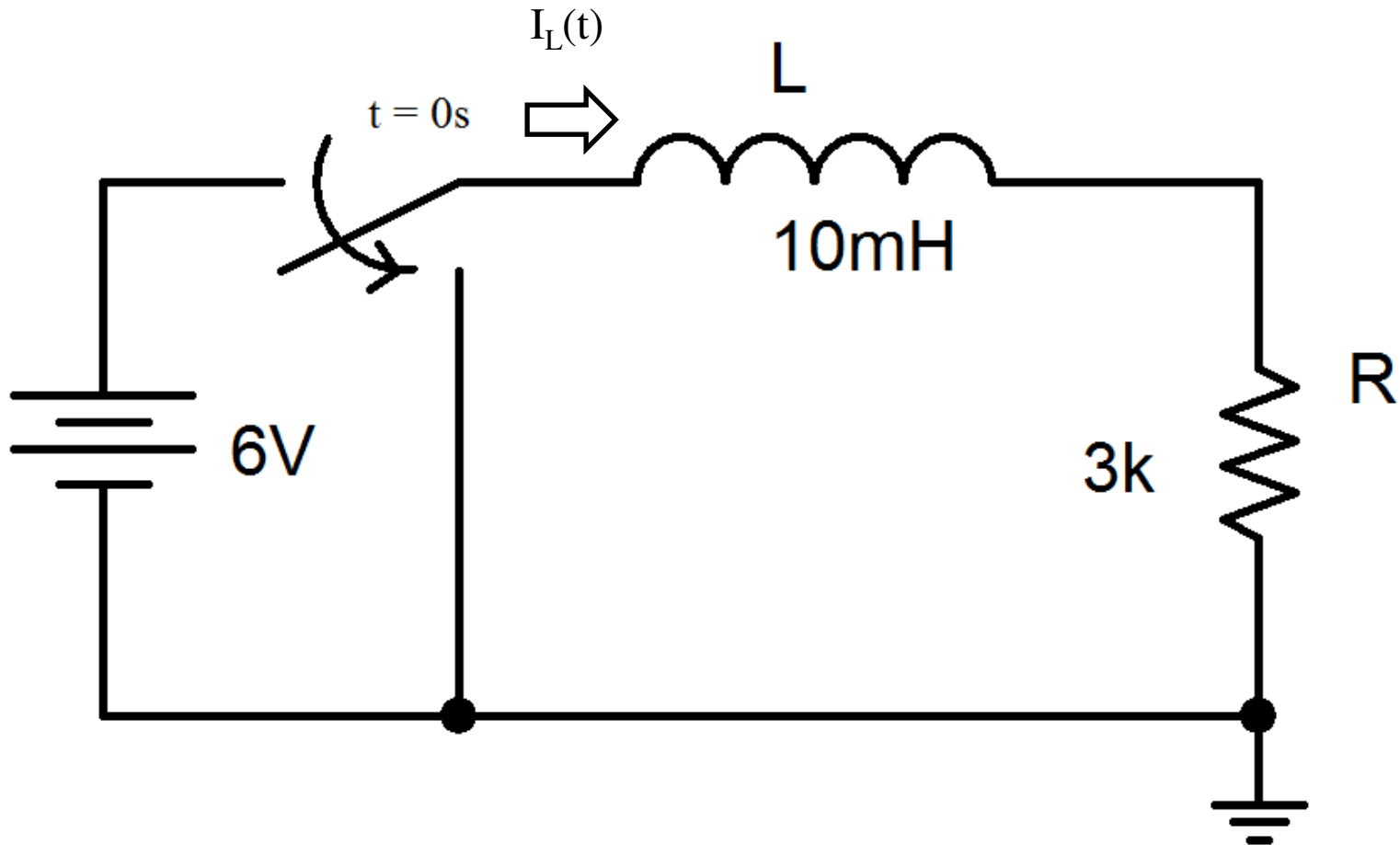
$$V_C = V_C(2ms)e^{-(t-2ms)/\tau} = 4V\ e^{-(t-2ms)/6ms}$$

$$V_R = V_C$$

$$I_C = C\ dV_c/dt = 2\mu F(-4V/6ms)\ e^{-(t-2ms)/6ms}$$
$$= -1.33\ e^{-(t-2ms)/6ms}\ mA$$

$$I_R = -I_C = 1.33\ e^{-(t-2ms)/6ms}\ mA$$

$$\text{Note } I_R + I_L = 0 \text{ mA}$$

Example 04…

…Example 04…
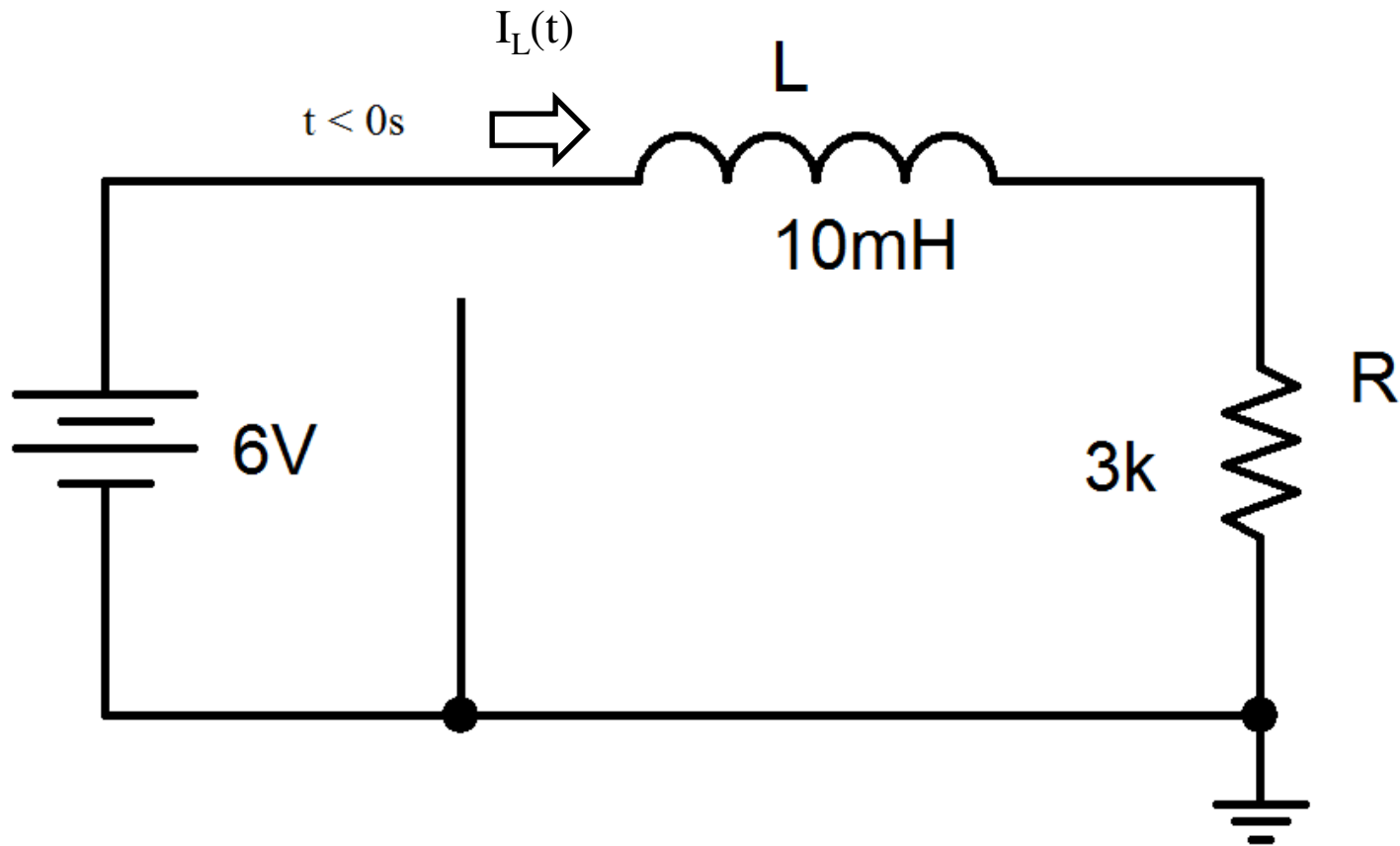
…Example 04…

Find the initial condition.
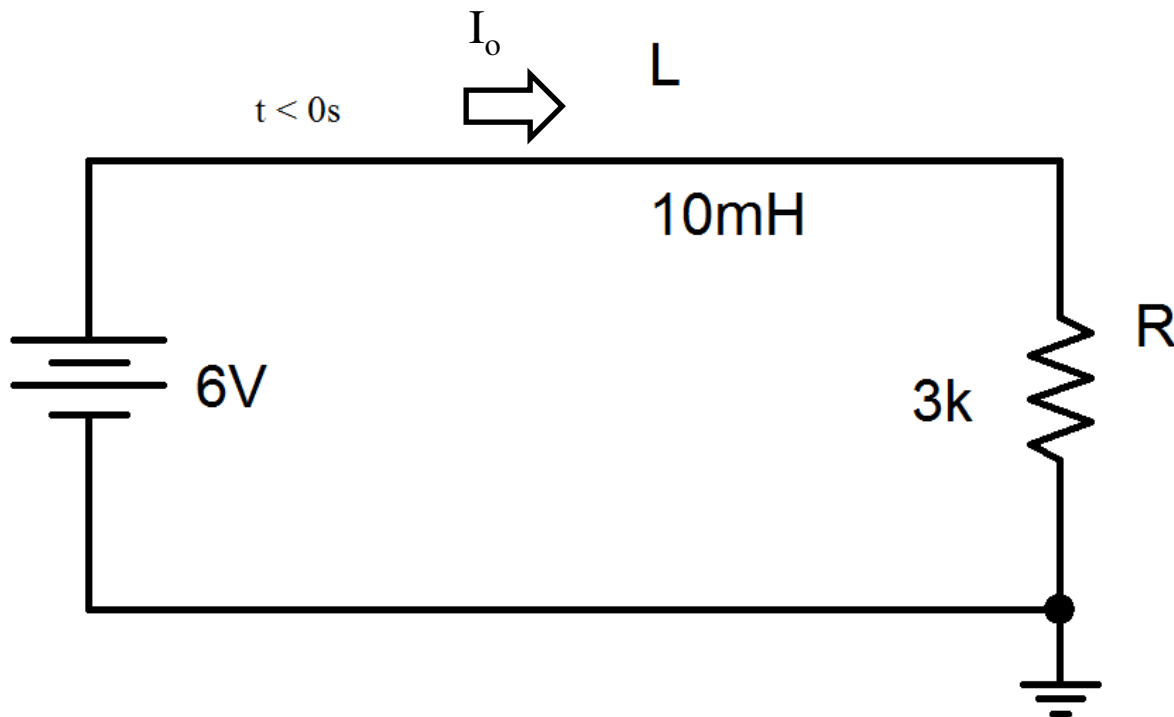
t < 0s

$V_L = 0V$

$V_R = 6V$

$I_L = I_R = 2mA$

Therefore,

$I_o = 2mA$

…Example 04…
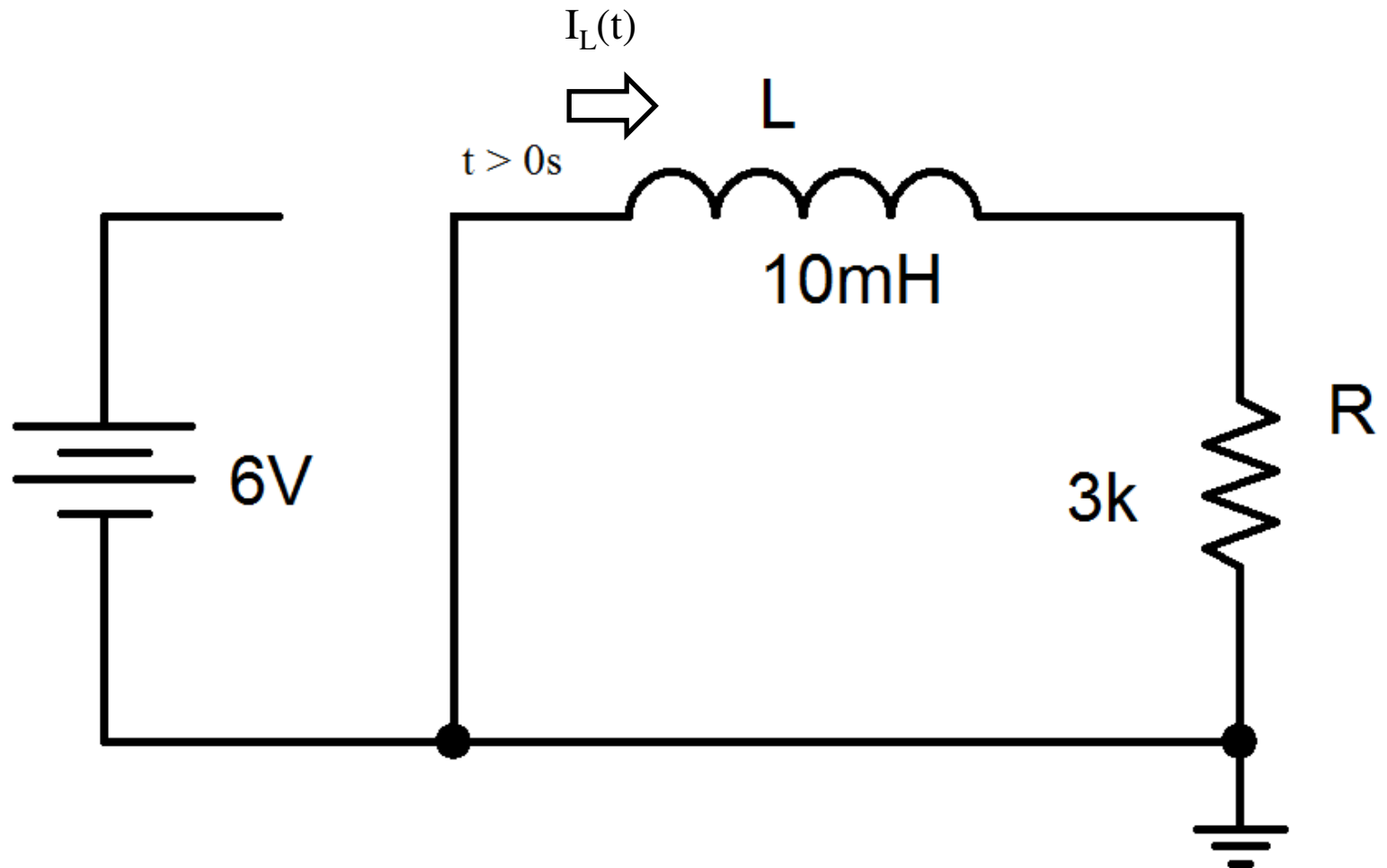
…Example 04
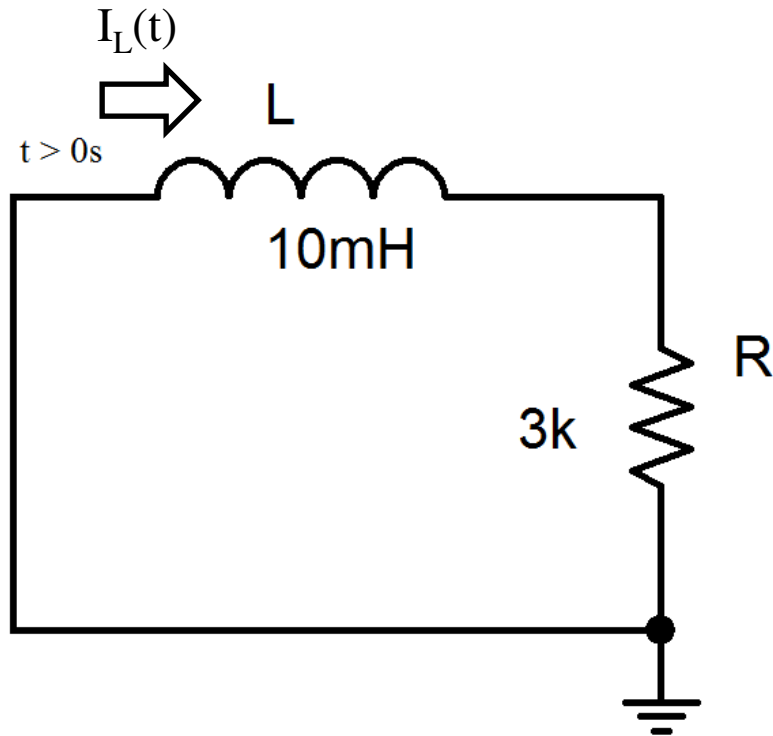
$t > 0s$

$t = L/R = 10mH/3k\Omega/ = 3.33\mu s$

$I_L = I_R = I_o e^{-t/\tau} = 2mA \; e^{-(t/3.33\mu s)}$

$V_R = 3k\Omega \; I_R = 6V \; e^{-(t/3.33\mu s)}$

$V_L = L \; dI_L/dt = -6V \; e^{-(t/3.33\mu s)}$

Note $V_R + V_L = 0 \; V$