Optimization Techniques
Section 2

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Steepest Descent

- Exact step size
- x_new = x_old - eps * df;
- Eps is calculated as:
  - z(eps)=x – eps*df
  - Find eps point where f'(z(eps))=0

Steepest Descent

- Find the minimum point of f(x)=x^2
- z(eps)=x-eps*2*x=x(1-2*eps) = search direction
- f(z(eps))= the value of f at a point on the search direction
- Find eps value which is minimum of f(z(eps)), f'(z(eps))=0
- f(z(eps))=(x(1-2*eps))^2=x^2*(1-2*eps)^2
- f(z(eps))=x^2*(1-4*eps+4*eps^2)
- f'(z(eps))=x^2*(-4+8*eps)=0
- eps=1/2
- X_{n+1}=X_n-eps*df=X_n-eps*2*X_n=X_n-X_n=0
- Wherever you start, the minimum point is found at one iteration !

Steepest Descent

- Find the minimum point of f(x)=x^4
- z(eps)=x-eps*4*x^3= search direction
- If we know that the minimum point of f(z(eps)) is 0 (But, we do not know, in reality)
- eps*4*x^3=x than,
- eps=1/(4*x^2)
- X_{n+1}=X_n-epx*4*x^3=X_n-*(1/(4*X_n^2))*4*X_n^3=
- X_n-X_n=0
- Wherever you start, the minimum point is found at one iteration !

Steepest Descent in 2 dims.

- Find the iteration equation to find the minimum of f(x1,x2)=x1^2+3*x2^2
- df=[2*x1 ; 6*x2]
- t=eps
- z(t) have 2 dims as x
- z(t)=[x1 ; x2]-t*df
- z(t)=[x1;x2]-[2*x1*t; 6*x2*t]
- z(t)=[x1*(1-2*t) ; x2*(1-6*t)]
- f(z(t))= (x1^2)*(1-2*t) ^2+3*(x2^2)(1-6*t)^2

Steepest Descent in 2 dims.

- f(z(t))= (x1^2)*(1-2*t) ^2+3*(x2^2)(1-6*t)^2
- df(z(t))/dt=f'(z(t))=
- =(x1^2)*2*(1-2*t)*(-2)+3*(x2^2)*2*(1-6*t)*(-6)
- =(x1^2)*(-4)*(1-2*t)-36*(x2^2)*(1-6*t)
- =(x1^2)*(-4+8*t)- (x2^2)*(36-216*t)
- =-4*(x1^2)+(x1^2)*8*t-36*(x2^2)+216*t*(x2^2)
- =0 because f'(z(t))=0
- (x1^2)*8*t+216*t*(x2^2)=4*(x1^2)+36*(x2^2)
- t= (4*(x1^2)+36*(x2^2)) / ((x1^2)*8+216*(x2^2))
- t= ((x1^2)+9*(x2^2)) / (2*(x1^2)+54*(x2^2))

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Steepest Descent in 2 dims.

- So the iteration equation is
- $X_{n+1}=X_n$-t*[2*x1; 6*x2] where

t= ((x1^2)+9*(x2^2)) / (2*(x1^2)+54*(x2^2))

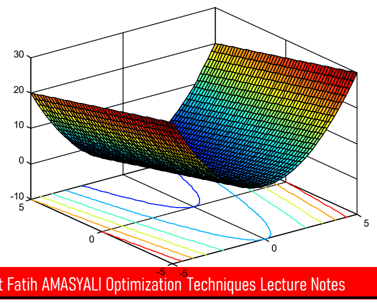
Steepest Descent Examples

- f(x)=x1^2+x2^2
- df=[2*x1 ; 2*x2]
- z(t)=[x1 ; x2]-t*df
- z(t)=[x1;x2]-[2*x1*t; 2*x2*t]
- z(t)=[x1*(1-2*t) ; x2*(1-2*t)]
- f(z(t))= (x1^2)*(1-2*t) ^2+(x2^2)(1-2*t)^2
- f(z(t))=((1-2*t) ^2)(x1^2+ x2^2)
- f'(z(t))=(x1^2 + x2^2)*(8*t - 4)=0
- t=1/2
- z(t)=[x1 ; x2]-t* [2*x1 ; 2*x2]
- z(t)=[x1 ; x2]-[x1 ; x2]= [0 0]
- Wherever you start, the minimum point is found at one iteration !

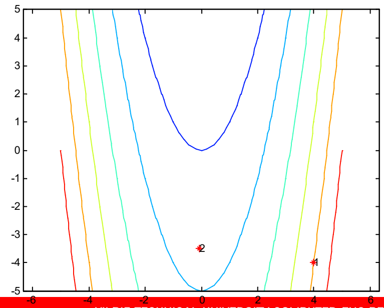
Steepest Descent Examples

- f(x)=x1^2+x2^2, t=1/2
- f(x) =x1^2-x2, t= 0.5+1/(8*x1^2)

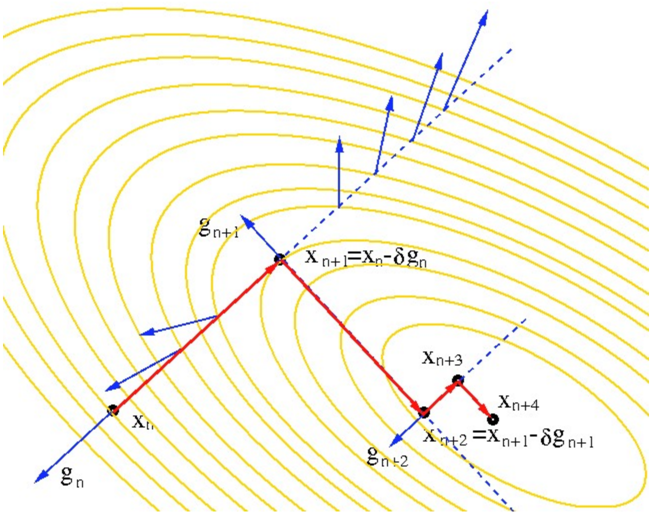
steepest_desc_2dim_2.m



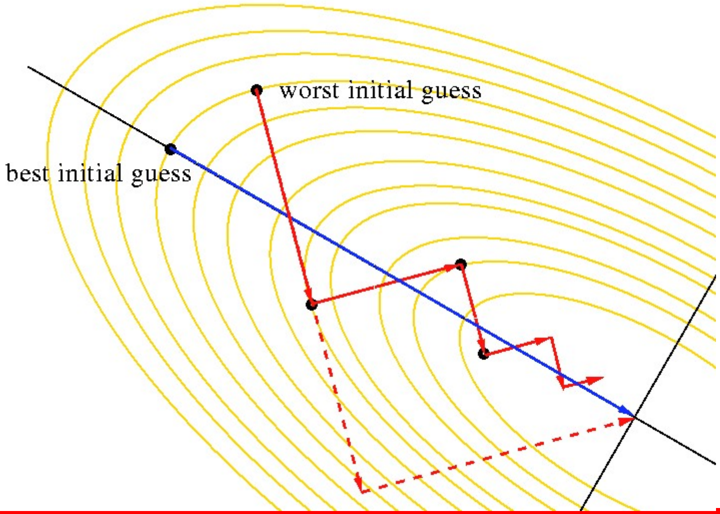
where is the 3rd,4th, iterations…?



the new search direction will always be perpendicular to the previous direction.

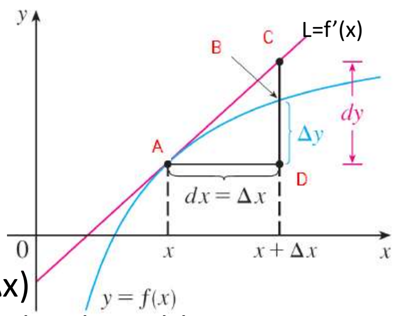


the new search direction will always be perpendicular to the previous direction.



Linear Approximation

- Assuming the function is linear at a point.



$f(x+\Delta x) \approx L(x+\Delta x)$

$\lim_{(\Delta x \to 0)} (f(x+\Delta x) - L(x+\Delta x)) = 0$

$L(x+\Delta x) = f(x) + dy = f(x) + \Delta x f'(x)$ since $f'(x)=dy/dx$

New point: $domx=x+\Delta x$,

$\Delta x=domx-x$, $f(domx) \approx f(x)+(domx-x)f'(x)$

Linear Approximation Example 1

- $(1.0002)^{50} \approx ?$
- $f(x)=x^{50}$
- $f(x+\Delta x) \approx f(x) + \Delta x\, f'(x)$
- domx=1.0002, x=1 , $\Delta x=0.0002$
- $f(1+0.0002) \approx f(1) + 0.0002\, f'(1)$
- $f(1+0.0002) \approx f(1) + 0.0002 * 50 * 1^{49}$
- $f(1+0.0002) \approx 1 + 0.0002 * 50 * 1 = 1.01$

Linear Approximation Example 2

- Find the linear approximation for x tends to 1 where $f(x) = \ln x$.
- $f(x+\Delta x) \approx f(x) + \Delta x\, f'(x)$
- domx=x+Δx , Δx = domx-x , x=1, $f'(x)=1/x$
- f(domx) $\approx \ln 1 + f'(1)$ (domx - 1) = domx -1
- $\ln x \approx x-1$, for x close to 1

- For x tends to 2
- Δx = domx-x, x=2
- f(domx) $\approx \ln 2 + f'(2)$ (domx - 2) = ln 2 + (domx-2)/2
- $\ln x \approx \ln 2 + (x-2)/2$, for x close to 2

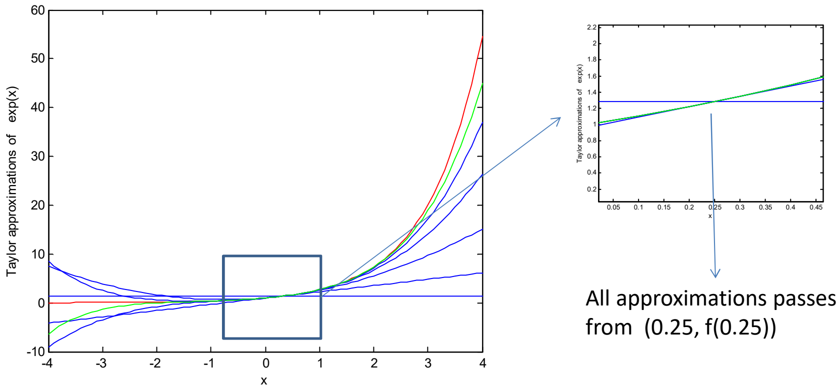
For a better approximation

- 1^{st} order Taylor: (linear approx.)

  $f(x+\Delta x) \approx f(x) + \Delta x\, f'(x)$
- 2^{nd} order Taylor: (non-linear approx.)

  $f(x+\Delta x) \approx f(x) + \Delta x\, f'(x) + ½\, f''(x)\, \Delta x^2$
- …
- N^{th} order Taylor: (non-linear approx.)
- $f(x+\Delta x) \approx \sum \ (f^{(i)'}(x)\, \Delta x^i\,)/\, i! \quad i=0 \dots N$

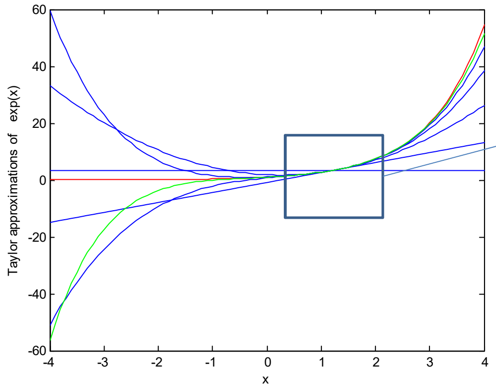
Function approx. with Taylor

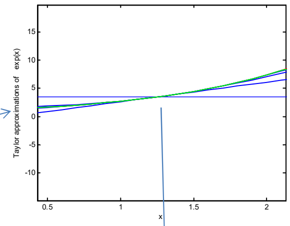


- f(x)=exp(x), N=0:5, X=0.25
- Red: real f, Blues and green: approximations
- Green: The last approx.

approx_taylor.m

Function approx. with Taylor



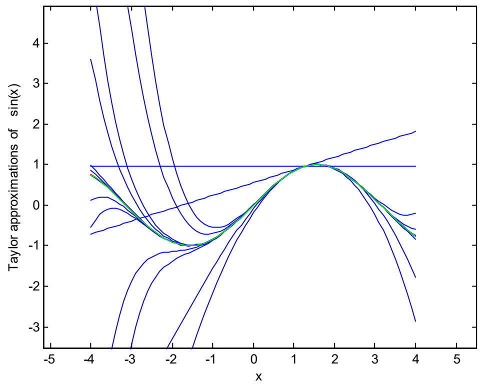


All approximations passes from (1.25, f(1.25))

- f(x)=exp(x), N=0:5, X=1.25
- Red: real f, Blues and green: approximations
- Green: The last approx.

approx_taylor.m

Function approx. with Taylor

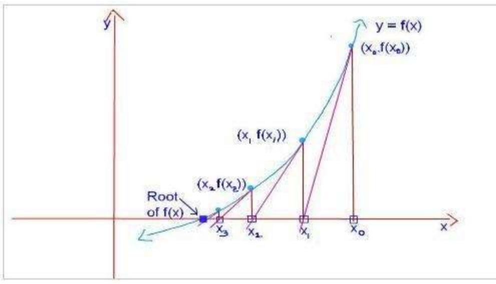


- f(X)=sin(x)
- X=1.25
- N=0:15
- Red: real f
- Blues and green: approximations
- Green: The last approx.

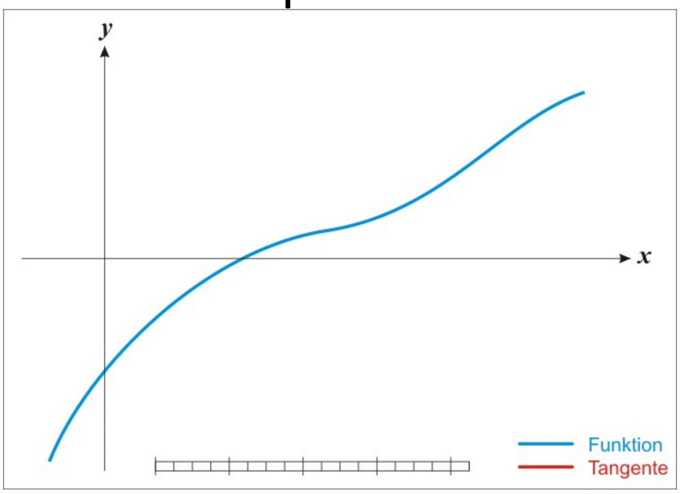
approx_taylor.m

Finding a root of f(x) iteratively (find a point x where f(x)=0) Newton Raphson 1st order

- $f'(x_n) = f(x_n) / (x_n - x_{n+1})$
- $x_n - x_{n+1} = f(x_n) / f'(x_n)$
- $x_{n+1} = x_n - f(x_n) / f'(x_n)$
  n = 0,1,2,3....
- If we require the root correct up to 6 decimal places, we stop when the digits in x_{n+1} and x_n agree till the 6th decimal place.



Newton Raphson- 1st order



Example

- Find sqrt(2)
- Means find the root of $x^2-2=0$
- $x_0=1$
- $x_{n+1}= x_n-f(x_n)/f'(x_n)$
- $x_{n+1}= x_n - (x_n*x_n-2)/(2*x_n)$
- $x_0=1$
  $x_1=1.5$
  $x_2=1.41667$
  $x_3=1.41422$
  $x_4=1.41421$
  if the current improvement (0.00001) is insignificant, we can say sqrt(2) =1.41421, if not go on the iterations.

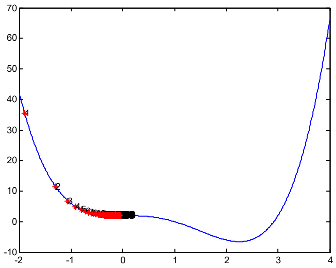
Taylor Series - Newton Raphson 2nd order

- 1st order Taylor:
  $f(x+\Delta x) \approx f(x) + \Delta x\, f'(x)$ (1)
- According to 1st order Taylor, to find $f(x+\Delta x)=0$,
  $\Delta x=-f(x)/f'(x)$
- To find $f(x+\Delta x)'=0$, take derivative of (1)
  $f'(x+\Delta x) \approx f'(x) + \Delta x\, f''(x)$
  $\Delta x=-f'(x)/f''(x) \leftarrow$ Newton Raphson 2nd order
  $x_{n+1} = x_n - f(x_n) / f'(x_n)$
  $\Delta x = x_{n+1} - x_n$

Newton Raphson

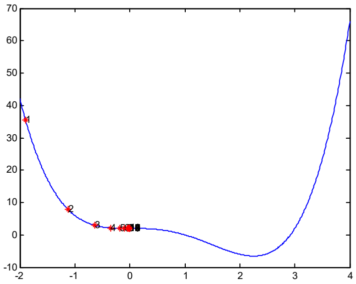
- Generally faster converge, because it uses more information (2nd derivative)
- No explicit step size selection
- 1st order : x_new = x_old - f/df; (f(x)=0)
- 2nd order : x_new = x_old - df/ddf; (f’(x)=0)
- instead of x_new = x_old - eps * df; (gradient descent)

Gradient Descent Step size=0.01 Starting point=-1.9



Does not converge with 200 iterations

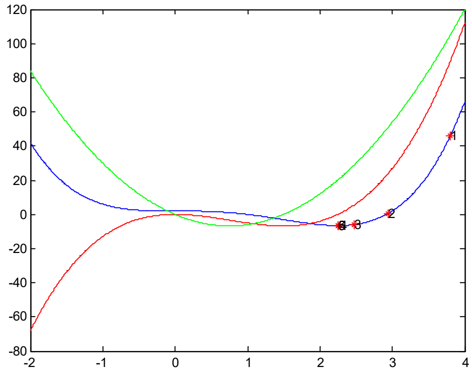
Newton Raphson 2nd order



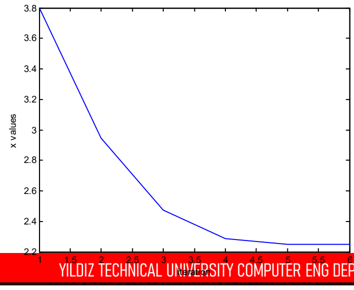
Converged at 20 iterations

newton_raphson_1.m

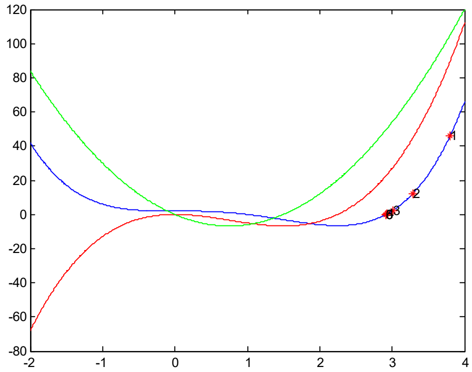
newton_raphson_1.m
2nd order finds f'(x)=0



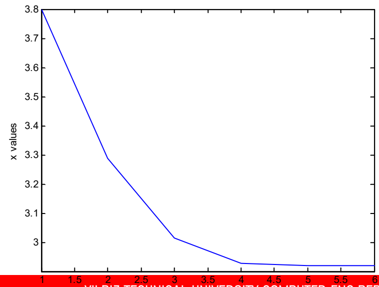
- Blue f
- Red f'
- Green f''



newton_raphson_1.m
1nd order finds f(x)=0

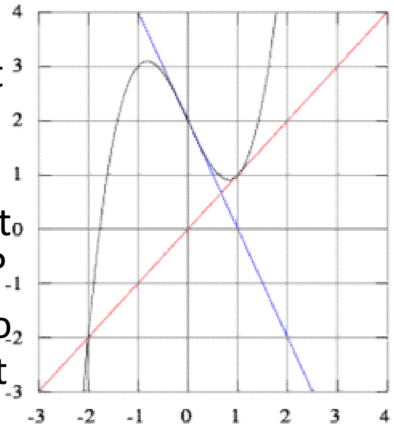


- Blue f
- Red f'
- Green f''



Newton Raphson 1st order -Cycle problem

- The tangent lines of $x^3 - 2x + 2$ at 0 and 1 intersect the x-axis at 1 and 0 respectively.
- What happened if we start at a point in (0,1) interval?
- If we start at 0.1, it goes to 1, then it goes to 0, then it goes to 1 …



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Newton Raphson

- Faster convergence (lower iteration number)
- But, more calculation for each iteration

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