BLM2041 Signals and Systems

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BLM2041 Signals and Systems

Spectrum Representation

Problem Solving Skills

- Math Formula
  - Sum of Cosines
  - Amp, Freq, Phase
- Recorded Signals
  - Speech
  - Music
  - No simple formula
- Plot & Sketches
  - *S*(*t*) versus *t*
  - Spectrum
- MATLAB
  - Numerical
  - Computation
  - Plotting list of numbers

LECTURE OBJECTIVES

- Sinusoids with DIFFERENT Frequencies
  - SYNTHESIZE by Adding Sinusoids

$$x(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

- SPECTRUM Representation
  - Graphical Form shows **DIFFERENT** Freqs

LECTURE OBJECTIVES

- Signals with **HARMONIC** Frequencies
  - Add Sinusoids with $f_k = kf_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

FREQUENCY can change vs. TIME

Chirps: $x(t) = \cos(\alpha t^2)$

Introduce Spectrogram Visualization (`specgram.m`) (`plotspec.m`)

LECTURE OBJECTIVES

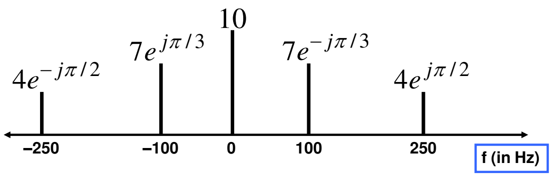
- Work with the Fourier Series Integral

$$\boxed{a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi k/T_0)t} dt}$$

- **ANALYSIS** via Fourier Series
  - For **PERIODIC** signals: $x(t+T_0) = x(t)$
- **SPECTRUM** from Fourier Series
  - a_k is Complex Amplitude for k-th Harmonic

FREQUENCY DIAGRAM

- Plot Complex Amplitude vs. Freq



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Another FREQ. Diagram

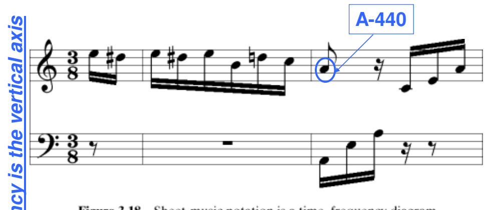

Figure 3.18 Sheet-music notation is a time–frequency diagram.

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MOTIVATION

- Synthesize Complicated Signals
  - Musical Notes
    - Piano uses 3 strings for many notes
    - Chords: play several notes simultaneously
  - Human Speech
    - Vowels have dominant frequencies
    - Application: computer generated speech
  - Can all signals be generated this way?
    - Sum of sinusoids?

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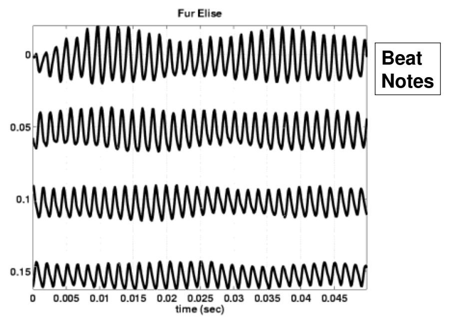
Fur Elise WAVEFORM



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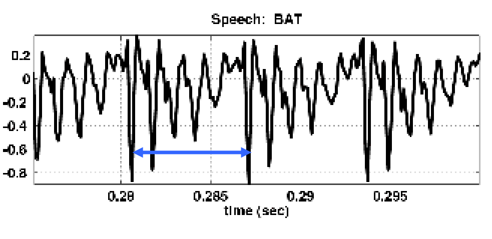
Speech Signal: BAT

- Nearly **Periodic** in Vowel Region
  - Period is (Approximately) T = 0.0065 sec



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Euler's Formula Reversed

- Solve for cosine (or sine)

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

$$\boxed{\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})}$$

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INVERSE Euler's Formula

- Solve for cosine (or sine)

$$\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \tfrac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

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SPECTRUM Interpretation

- Cosine = sum of 2 complex exponentials:

$$A\cos(7t) = \tfrac{A}{2} e^{j7t} + \tfrac{A}{2} e^{-j7t}$$

One has a positive frequency
The other has negative freq.
Amplitude of each is half as big

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NEGATIVE FREQUENCY

- Is negative frequency real?
- Doppler Radar provides an example
  - Police radar measures speed by using the Doppler shift principle
  - Let's assume 400Hz ←→60 mph
  - +400Hz means towards the radar
  - -400Hz means away (opposite direction)
  - Think of a train whistle

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SPECTRUM of SINE

- Sine = sum of 2 complex exponentials:

$$A\sin(7t) = \tfrac{A}{2j} e^{j7t} - \tfrac{A}{2j} e^{-j7t}$$

$$= \tfrac{1}{2} A e^{-j0.5\pi} e^{j7t} + \tfrac{1}{2} A e^{j0.5\pi} e^{-j7t}$$

$$\boxed{\frac{-1}{j} = j = e^{j0.5\pi}}$$

- Positive freq. has phase = -0.5π
- Negative freq. has phase = +0.5π

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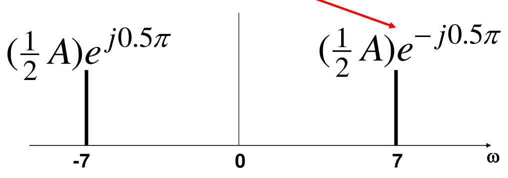
GRAPHICAL SPECTRUM

EXAMPLE of SINE

$$\boxed{A\sin(7t) = \tfrac{1}{2} A e^{-j0.5\pi} e^{j7t} + \tfrac{1}{2} A e^{j0.5\pi} e^{-j7t}}$$



AMPLITUDE, PHASE & FREQUENCY are shown

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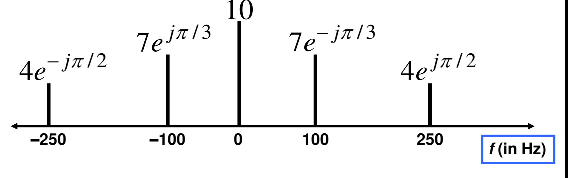
SPECTRUM ---> SINUSOID

- Add the spectrum components:



What is the formula for the signal *x*(*t*)?

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Gather (A,ω,ϕ) information

- Frequencies:
  - -250 Hz
  - -100 Hz
  - **0** Hz
  - 100 Hz
  - 250 Hz
- Amplitude & Phase
  - 4 -π/2
  - 7 +π/3
  - 10 **0**
  - 7 -π/3
  - 4 +π/2

Note the **conjugate phase**

DC is another name for zero-freq component

DC component always has ϕ=0 or π (for real ***x(t)***)

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Add Spectrum Components-1

- **Frequencies:**
  - **-250 Hz**
  - **-100 Hz**
  - **0 Hz**
  - **100 Hz**
  - **250 Hz**
- **Amplitude & Phase**
  - **4** **-π/2**
  - **7** **+π/3**
  - **10** **0**
  - **7** **-π/3**
  - **4** **+π/2**

$$x(t) = 10 + \\ 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} \\ 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

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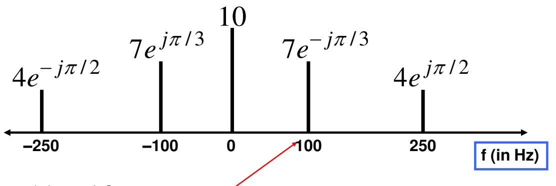
Add Spectrum Components-2



$$x(t) = 10 + \\ 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} \\ 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

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Simplify Components

$$x(t) = 10 + \\ 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} \\ 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

<u>Use Euler's Formula to get REAL sinusoids:</u>

$$A\cos(\omega t + \varphi) = \tfrac{1}{2}Ae^{-j\varphi}e^{j\omega t} + \tfrac{1}{2}Ae^{-j\varphi}e^{-j\omega t}$$

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FINAL ANSWER

$$\boxed{x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) \\ + 8\cos(2\pi(250)t + \pi/2)}$$

<u>So, we get the general form:</u>

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

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Summary: GENERAL FORM

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

$$x(t) = X_0 + \sum_{k=1}^{N} \Re e\left\{X_k e^{j2\pi f_k t}\right\}$$

$$X_k = A_k e^{j\varphi_k}$$

$$\text{Frequency} = f_k$$

$$\Re e\{z\} = \tfrac{1}{2}z + \tfrac{1}{2}z^*$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{\tfrac{1}{2}X_k e^{j2\pi f_k t} + \tfrac{1}{2}X_k^* e^{-j2\pi f_k t}\right\}$$

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Example: Synthetic Vowel

- Sum of 5 Frequency Components
  - Complex amplitudes for harmonic signal that approximates the vowel sound «ah»

f_k (Hz)	X_k	Mag	Phase (rad)
200	$(771 + j12202)$	12,226	1.508
400	$(-8865 + j28048)$	29,416	1.876
500	$(48001 - j8995)$	48,836	-0.185
1600	$(1657 - j13520)$	13,621	-1.449
1700	$4723 + j0$	4723	0

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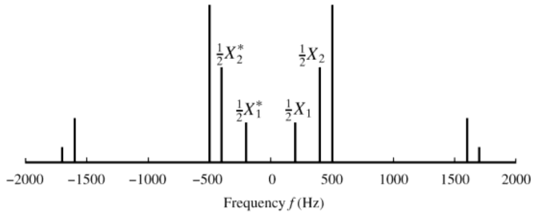
SPECTRUM of VOWEL



- Note: Spectrum has $0.5X_k$ (except X_{DC})
- Conjugates in negative frequency

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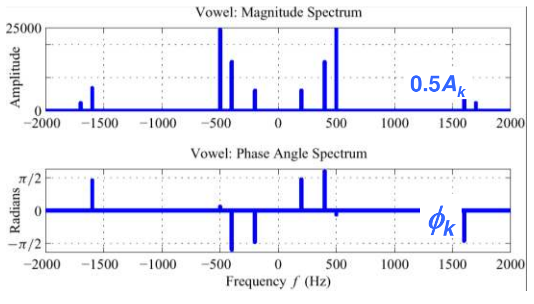
SPECTRUM of VOWEL (Polar Format)



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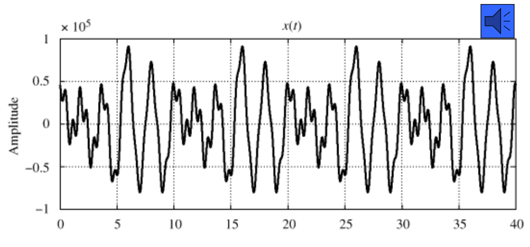
Vowel Waveform (sum of all 5 components)

- Sum of all of the signals in the prevous slides



- Note that the period is 10 ms, which equals $1/f_0$

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Periodic Signals, Harmonics & Time-Varying Sinusoids

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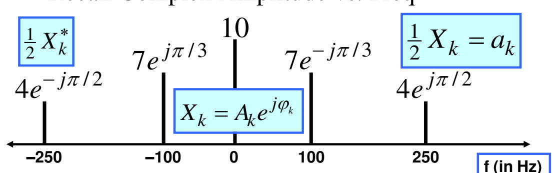
SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$\boxed{\begin{aligned} x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) \\ + 8\cos(2\pi(250)t + \pi/2) \end{aligned}}$$

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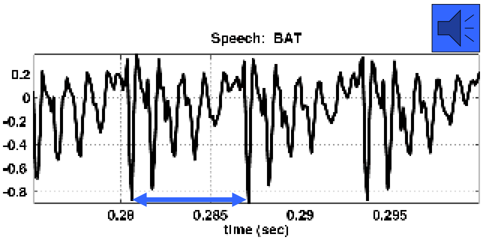
SPECTRUM for PERIODIC ?

- Nearly Periodic in the Vowel Region
  - Period is (Approximately) T = 0.0065 sec



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PERIODIC SIGNALS

- Repeat every T secs
  - Definition

    $$x(t) = x(t+T)$$
  - Example:

    $$x(t) = \cos^2(3t) \qquad T = ?$$
  - Speech can be "quasi-periodic"

    $$T = \frac{2\pi}{3} \qquad T = \frac{\pi}{3}$$

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Period of Complex Exponential

$$x(t) = e^{j\omega t}$$

$$x(t+T) = x(t) \; ?$$

Definition: Period is T

$$e^{j\omega(\cancel{t}+T)} = e^{j\omega \cancel{t}}$$

$$\boxed{e^{j2\pi k} = 1}$$

$$\Rightarrow e^{j\omega T} = 1 \quad \Rightarrow \omega T = 2\pi k$$

$$\omega = \frac{2\pi k}{T} = \left(\frac{2\pi}{T}\right) k = \omega_0 k$$

k = integer

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Harmonic Signal Spectrum

$$\boxed{\text{Periodic signal can only have: } f_k = k f_0}$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$f_0 = \frac{1}{T}$$

$$\boxed{X_k = A_k e^{j\varphi_k}}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t} \right\}$$

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Define FUNDAMENTAL FREQ

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$f_k = k f_0 \qquad (\omega_0 = 2\pi f_0)$$

$$f_0 = \frac{1}{T_0}$$

f_0 = fundamental Frequency (largest)

T_0 = fundamental Period (shortest)

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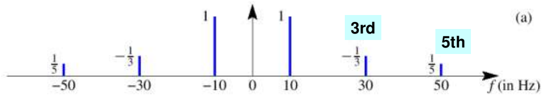
Harmonic Signal (3 Freqs)



What is the fundamental frequency? **10 Hz**

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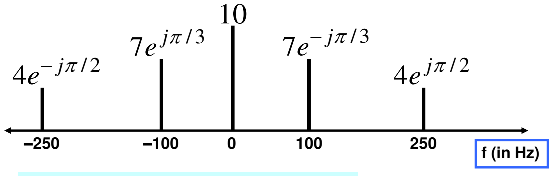
POP QUIZ: FUNDAMENTAL

- Here's another spectrum:



What is the fundamental frequency?

100 Hz ?   50 Hz ?

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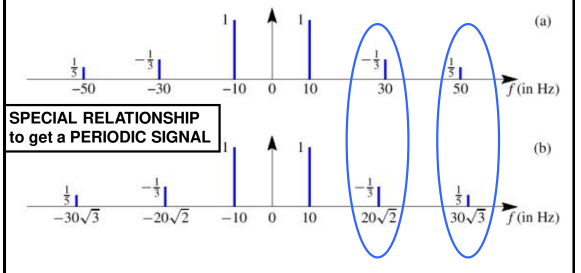
IRRATIONAL SPECTRUM



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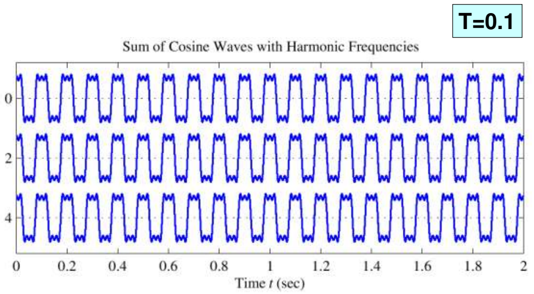
Harmonic Signal (3 Freqs)



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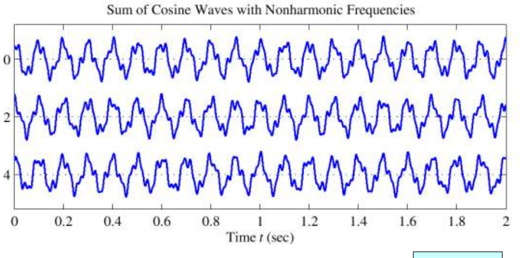
NON-Harmonic Signal



NOT PERIODIC

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FREQUENCY ANALYSIS

- **Now, a much HARDER problem**
- Given a recording of a song, have the computer write the music
- Can a machine extract frequencies?
  - Yes, if we COMPUTE the spectrum for $x(t)$
    - During short intervals

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Time-Varying FREQUENCIES Diagram

Figure 3.18 Sheet-music notation is a time–frequency diagram.

Time is the horizontal axis

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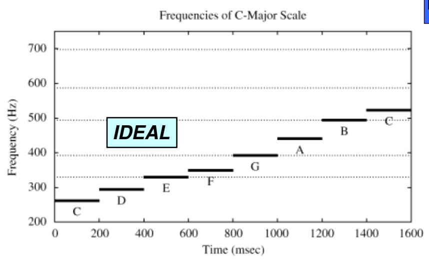
SIMPLE TEST SIGNAL

- C-major SCALE: stepped frequencies



  – Frequency is constant for each note

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SPECTROGRAM

- SPECTROGRAM Tool
  – MATLAB function is `specgram.m`
  – SP-First has `plotspec.m` & `spectgr.m`
- ANALYSIS program
  – Takes x(t) as input &
  – Produces spectrum values X_k
  – Breaks x(t) into SHORT TIME SEGMENTS
    - Then uses the FFT (Fast Fourier Transform)

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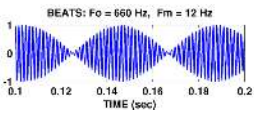
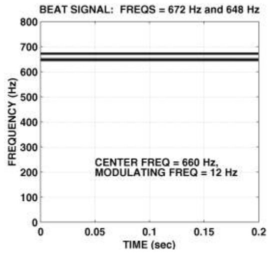
SPECTROGRAM EXAMPLE

- Two **Constant** Frequencies: Beats





$\cos(2\pi(660)t)\sin(2\pi(12)t)$

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AM Radio Signal

- Same as BEAT Notes

$$\cos(2\pi(660)t)\sin(2\pi(12)t)$$

$$\frac{1}{2}\left(e^{j2\pi(660)t}+e^{-j2\pi(660)t}\right)\frac{1}{2j}\left(e^{j2\pi(12)t}-e^{-j2\pi(12)t}\right)$$

$$\frac{1}{4j}\left(e^{j2\pi(672)t}-e^{-j2\pi(672)t}-e^{j2\pi(648)t}+e^{-j2\pi(648)t}\right)$$

$$\tfrac{1}{2}\cos(2\pi(672)t-\tfrac{\pi}{2})+\tfrac{1}{2}\cos(2\pi(648)t+\tfrac{\pi}{2})$$

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SPECTRUM of AM (Beat)

- 4 complex exponentials in AM:

What is the fundamental frequency?

648 Hz ? 24 Hz ?

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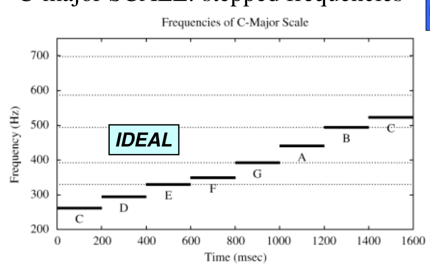
STEPPED FREQUENCIES

- C-major SCALE: stepped frequencies



  – Frequency is constant for each note

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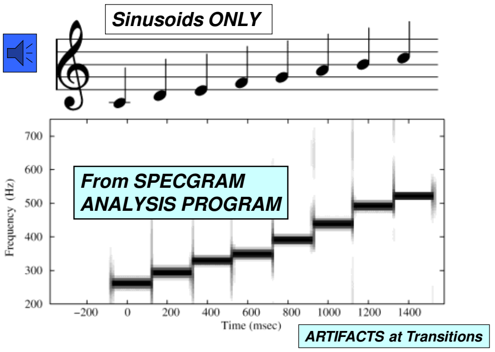
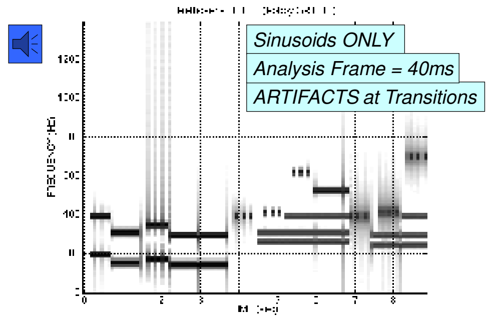
SPECTROGRAM of C-Scale



Spectrogram of LAB SONG



Time-Varying Frequency

- Frequency can change vs. time
  - Continuously, not stepped
- **FREQUENCY MODULATION (FM)**

$$x(t) = \cos(2\pi f_c t + v(t))$$

VOICE

- CHIRP SIGNALS
  - Linear Frequency Modulation (LFM)

New Signal: Linear FM

- Called Chirp Signals (LFM)
  - Quadratic phase

QUADRATIC

$$x(t) = A\cos(\alpha t^2 + 2\pi f_0 t + \varphi)$$

- Freq will change LINEARLY vs. time
  - Example of Frequency Modulation (FM)
  - Define "instantaneous frequency"

INSTANTANEOUS FREQ

- Definition

$$x(t) = A\cos(\psi(t))$$
$$\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t)$$

Derivative of the "Angle"

- For Sinusoid:

$$x(t) = A\cos(2\pi f_0 t + \varphi)$$
$$\psi(t) = 2\pi f_0 t + \varphi$$
$$\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t) = 2\pi f_0$$

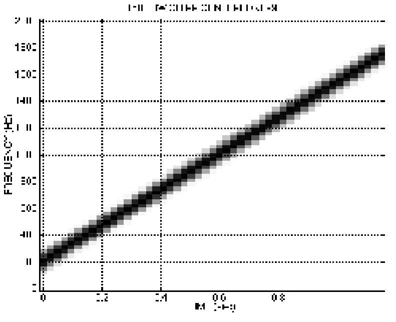
Makes sense

INSTANTANEOUS FREQ of the Chirp

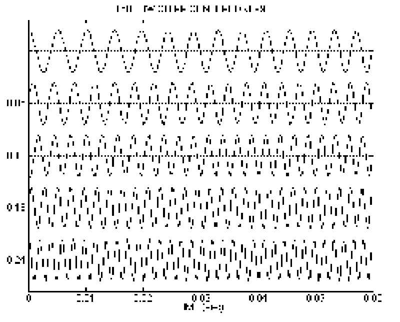
- Chirp Signals have Quadratic phase
- Freq will change LINEARLY vs. time

$$x(t) = A\cos(\alpha t^2 + \beta t + \varphi)$$
$$\Rightarrow \psi(t) = \alpha t^2 + \beta t + \varphi$$
$$\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t) = 2\alpha t + \beta$$

CHIRP SPECTROGRAM



CHIRP WAVEFORM



OTHER CHIRPS

$\psi(t)$ can be anything:

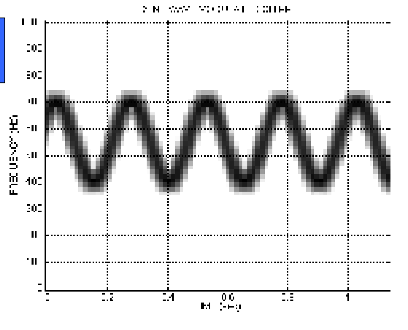
$$x(t) = A\cos(\alpha\cos(\beta t) + \varphi)$$

$$\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t) = -\alpha\beta\sin(\beta t)$$

$\psi(t)$ could be speech or music:

- FM radio broadcast

SINE-WAVE FREQUENCY MODULATION (FM)



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Fourier Series Coefficients

HISTORY

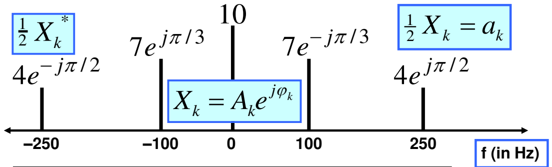
- Jean Baptiste Joseph Fourier (1768-1830)

  - Napoleonic era
  - Studied the mathematical theory of heat conduction
  - Established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric funcions.

- http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = \boxed{a_0} + \sum_{k=1}^{N} \left\{ \boxed{a_k} e^{j2\pi f_k t} + \boxed{a_k^*} e^{-j2\pi f_k t} \right\}$$

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Harmonic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

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Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

$$a_k = \tfrac{1}{2} X_k = \tfrac{1}{2} A_k e^{j\varphi_k}$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$X_k = A_k e^{j\varphi_k}$$

COMPLEX AMPLITUDE

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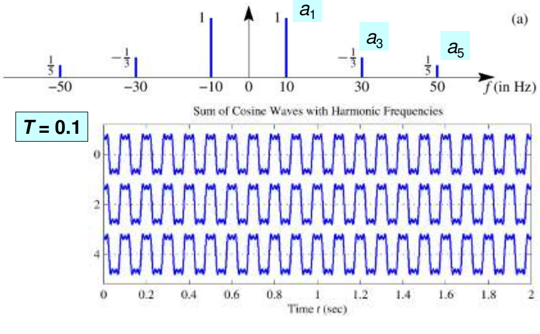
Harmonic Signal (3 Freqs)



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SYNTHESIS vs. ANALYSIS

- SYNTHESIS
  - **Easy**
  - Given (ω_k, A_k, ϕ_k) create $x(t)$
- Synthesis can be HARD
  - Synthesize Speech so that it sounds good
- ANALYSIS
  - **Hard**
  - Given $x(t)$, extract (ω_k, A_k, ϕ_k)
  - How many?
  - Need algorithm for computer

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STRATEGY: $x(t) \rightarrow a_k$

- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} \, dt$$

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INTEGRAL Property of *exp(j)*

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}dt = \frac{T_0}{-j2\pi m}e^{-j(2\pi/T_0)mt}\Big|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m}(e^{-j2\pi m}-1)$$

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}dt = 0$$

$$m \neq 0$$

$$\omega_0 = \frac{2\pi}{T_0}$$

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ORTHOGONALITY of *exp(j)*

- PRODUCT of *exp*(+*j*) and *exp*(-*j*)

$$\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)\ell t}e^{-j(2\pi/T_0)kt}dt = \begin{cases}0 & k \neq \ell \\ 1 & k = \ell\end{cases}$$

$$\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)(\ell-k)t}dt$$

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Isolate One FS Coefficient

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}$$

$$\tfrac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)\ell t}dt = \tfrac{1}{T_0}\int_0^{T_0}\left(\sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}\right)e^{-j(2\pi/T_0)\ell t}dt$$

$$\tfrac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)\ell t}dt = \sum_{k=-\infty}^{\infty} a_k\left(\tfrac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)kt}e^{-j(2\pi/T_0)\ell t}dt\right) = a_\ell$$

$$\Rightarrow a_k = \tfrac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

Integral is zero except for $k = \ell$

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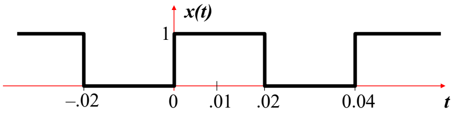
SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases}1 & 0 \le t < \tfrac{1}{2}T_0 \\ 0 & \tfrac{1}{2}T_0 \le t < T_0\end{cases}$$

for $T_0 = 0.04$ sec.



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FS for a SQUARE WAVE $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k \neq 0)$$

$$a_k = \frac{1}{.04}\int_0^{.02} 1e^{-j(2\pi/.04)kt}dt = \frac{1}{.04(-j2\pi k/.04)}e^{-j(2\pi/.04)kt}\Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)}(e^{-j(\pi)k}-1) = \frac{1-(-1)^k}{j2\pi k}$$

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DC Coefficient: a_0

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k = 0)$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt = \frac{1}{T_0}(\text{Area})$$

$$a_0 = \frac{1}{.04}\int_0^{.02} 1dt = \frac{1}{.04}(.02-0) = \tfrac{1}{2}$$

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Fourier Coefficients a_k

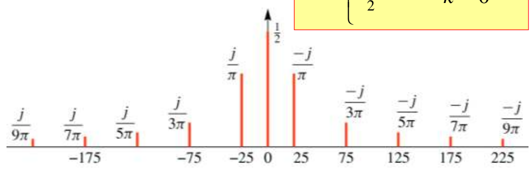
- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

$$a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

$\omega_0 = 2\pi/(0.04) = 2\pi(25)$

$$a_k = \begin{cases} \frac{-j}{\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$



Fourier Series Integral

- HOW do you determine a_k from $x(t)$?

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

Fundamental Frequency $f_0 = 1/T_0$

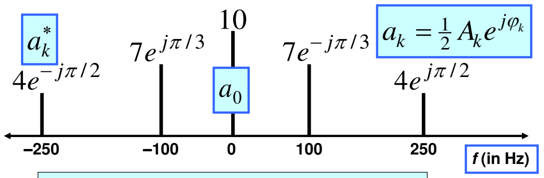
$a_{-k} = a_k^*$ when $x(t)$ is real

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt \qquad \text{(DC component)}$$

Fourier Series & Spectrum

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = a_0 + \sum_{k=1}^{N}\left\{ a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t}\right\}$$

Harmonic Signal

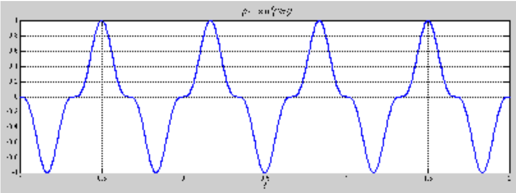
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

Example

$$x(t) = \sin^3(3\pi t)$$



$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

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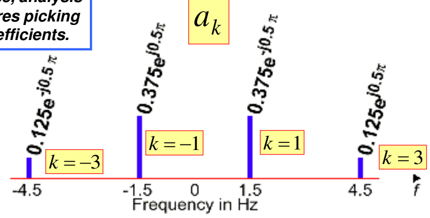
Example

$$x(t) = \sin^3(3\pi t)$$

$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

In this case, analysis just requires picking off the coefficients.

a_k



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STRATEGY: $x(t) \rightarrow a_k$

- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 k t}dt$$

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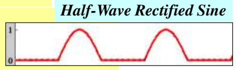
FS: Rectified Sine Wave $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k \neq \pm 1)$$

Half-Wave Rectified Sine



$$\begin{aligned} a_k &= \frac{1}{T_0}\int_0^{T_0/2} \sin(\tfrac{2\pi}{T_0}t)\, e^{-j(2\pi/T_0)kt}dt \\ &= \frac{1}{T_0}\int_0^{T_0/2} \frac{e^{j(2\pi/T_0)t} - e^{-j(2\pi/T_0)t}}{2j}\, e^{-j(2\pi/T_0)kt}dt \\ &= \frac{1}{j2T_0}\int_0^{T_0/2} e^{-j(2\pi/T_0)(k-1)t}dt - \frac{1}{j2T_0}\int_0^{T_0/2} e^{-j(2\pi/T_0)(k+1)t}dt \\ &= \left.\frac{e^{-j(2\pi/T_0)(k-1)t}}{j2T_0(-j(2\pi/T_0)(k-1))}\right|_0^{T_0/2} - \left.\frac{e^{-j(2\pi/T_0)(k+1)t}}{j2T_0(-j(2\pi/T_0)(k+1))}\right|_0^{T_0/2} \end{aligned}$$

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FS: Rectified Sine Wave $\{a_k\}$

$$\begin{aligned} a_k &= \left.\frac{e^{-j(2\pi/T_0)(k-1)t}}{j2T_0(-j(2\pi/T_0)(k-1))}\right|_0^{T_0/2} - \left.\frac{e^{-j(2\pi/T_0)(k+1)t}}{j2T_0(-j(2\pi/T_0)(k+1))}\right|_0^{T_0/2} \\ &= \frac{1}{4\pi(k-1)}\left(e^{-j(2\pi/T_0)(k-1)T_0/2} - 1\right) - \frac{1}{4\pi(k+1)}\left(e^{-j(2\pi/T_0)(k+1)T_0/2} - 1\right) \\ &= \frac{1}{4\pi(k-1)}\left(e^{-j\pi(k-1)} - 1\right) - \frac{1}{4\pi(k+1)}\left(e^{-j\pi(k+1)} - 1\right) \\ &= \left(\frac{k+1-(k-1)}{4\pi(k^2-1)}\right)\left(-(-1)^k - 1\right) = \begin{cases} 0 & k \text{ odd} \\ \pm\frac{1}{j4} & k = \pm 1 \\ \frac{-1}{\pi(k^2-1)} & k \text{ even} \end{cases} \end{aligned}$$

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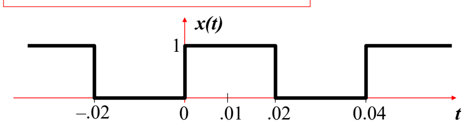
SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2}T_0 \\ 0 & \frac{1}{2}T_0 \le t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



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Fourier Coefficients a_k

- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

$$a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \dfrac{1}{j\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

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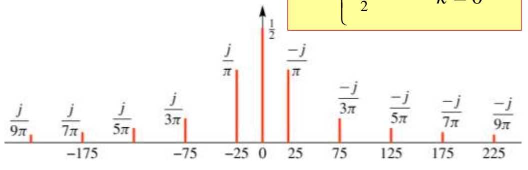
Spectrum from Fourier Series

$\omega_0 = 2\pi/(0.04) = 2\pi(25)$

$$a_k = \begin{cases} \dfrac{-j}{\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$



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Fourier Series Synthesis

- HOW do you **APPROXIMATE** $x(t)$?

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}\,dt$$

- Use FINITE number of coefficients

$$\boxed{x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k f_0 t}}$$

$a_{-k} = a_k^*$ when $x(t)$ is real

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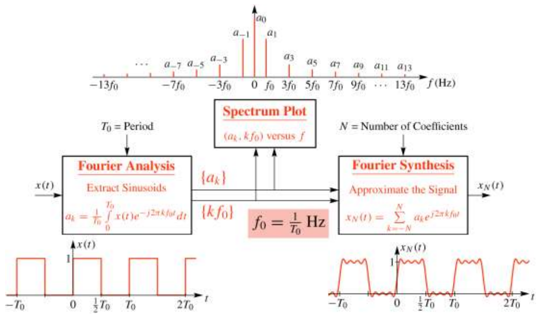
Fourier Series Synthesis



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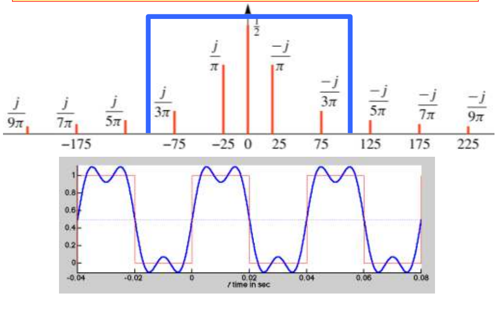
Synthesis: 1st & 3rd Harmonics

$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi(25)t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\cos(2\pi(75)t - \tfrac{\pi}{2})$$



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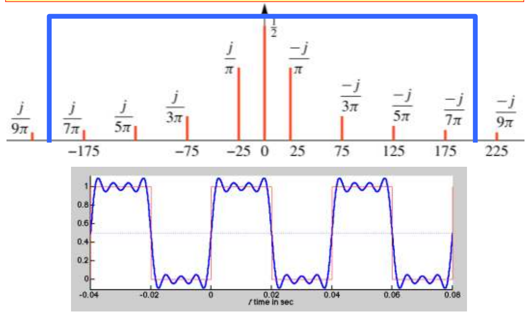
Synthesis: up to 7th Harmonic

$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(50\pi t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\sin(150\pi t) + \frac{2}{5\pi}\sin(250\pi t) + \frac{2}{7\pi}\sin(350\pi t)$$



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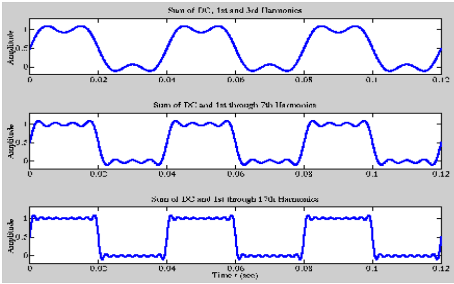
Fourier Synthesis

$$x_N(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \ldots$$



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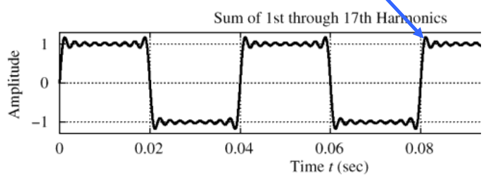
Gibbs' Phenomenon

- Convergence at DISCONTINUITY of $x(t)$
  - There is always an overshoot
  - 9% for the Square Wave case



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