



# BLM2502

# Theory of Computation

# BLM2502 Theory of Computation

## » Course Outline

Week	Content
1.	Introduction to Course
2.	Computability Theory, Complexity Theory, Automata Theory, Set Theory, Relations, Proofs, Pigeonhole Principle
3.	Regular Expressions
4.	Finite Automata
5.	<b>Deterministic and Nondeterministic Finite Automata</b>
6.	<b>Epsilon Transition, Equivalence of Automata</b>
7.	Pumping Theorem
8.	Context Free Grammars
9.	Parse Tree, Ambiguity,
10.	Pumping Theorem
11.	Turing Machines, Recognition and Computation, Church-Turing Hypothesis
12.	Turing Machines, Recognition and Computation, Church-Turing Hypothesis
13.	Review

NFA

Non-Deterministic

Finite Automata



# Formal Definition of NFA

$$M = (Q, \Sigma, \delta, q_0, F)$$

$Q$ : Set of states, i.e.  $\{q_0, q_1, q_2\}$

$\Sigma$ : Input alphabet, i.e.  $\{a, b\}$        $\varepsilon \notin \Sigma$

$\delta$ : Transition function  $Q \times \Sigma \rightarrow 2^Q$

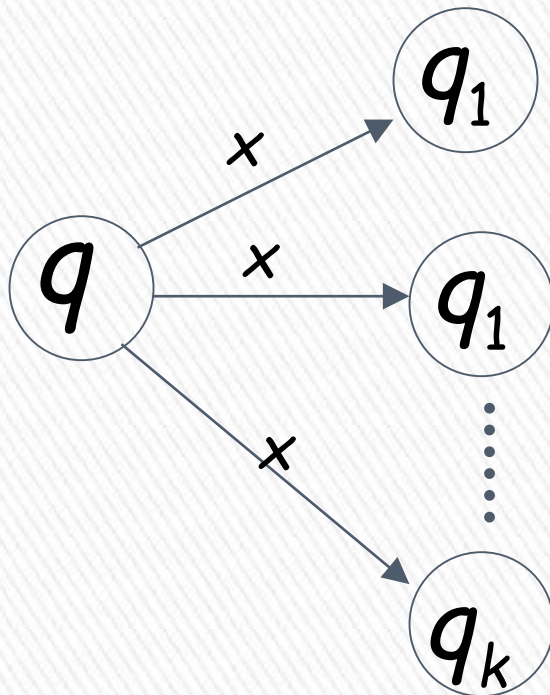
$q_0$ : Initial state

$F$ : Accepting states



# Transition Function $\delta$

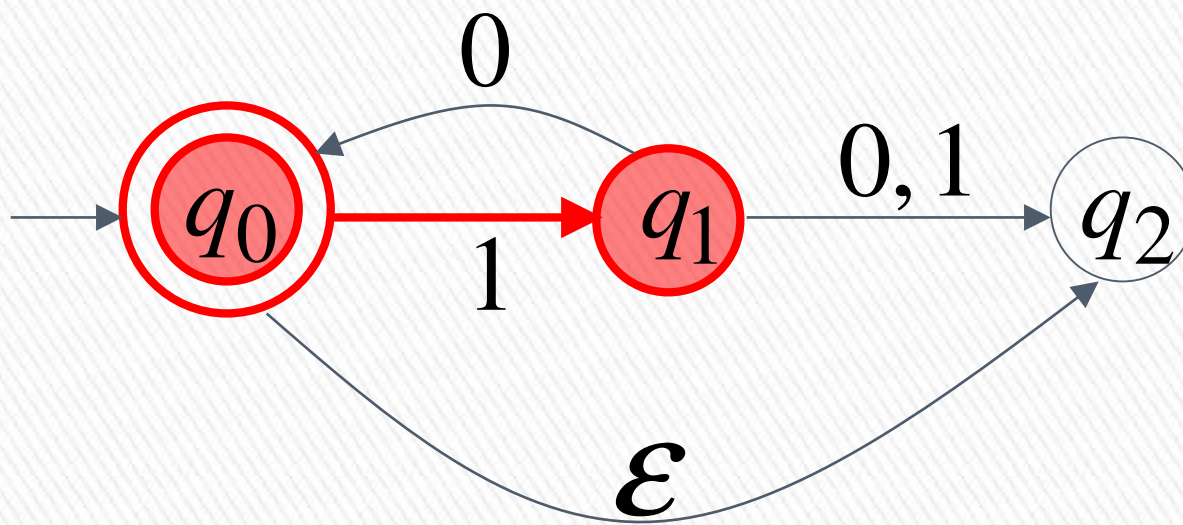
$$\delta(q, x) = \{q_1, q_2, \dots, q_k\}$$



resulting states with  
following **one** transition  
with symbol  $x$

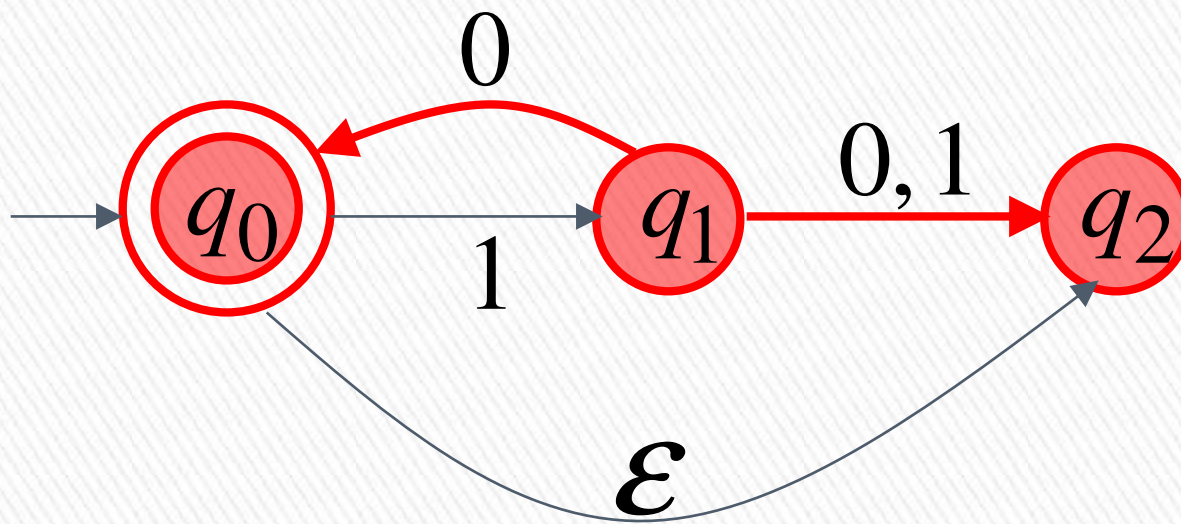


$$\delta(q_0, 1) = \{q_1\}$$

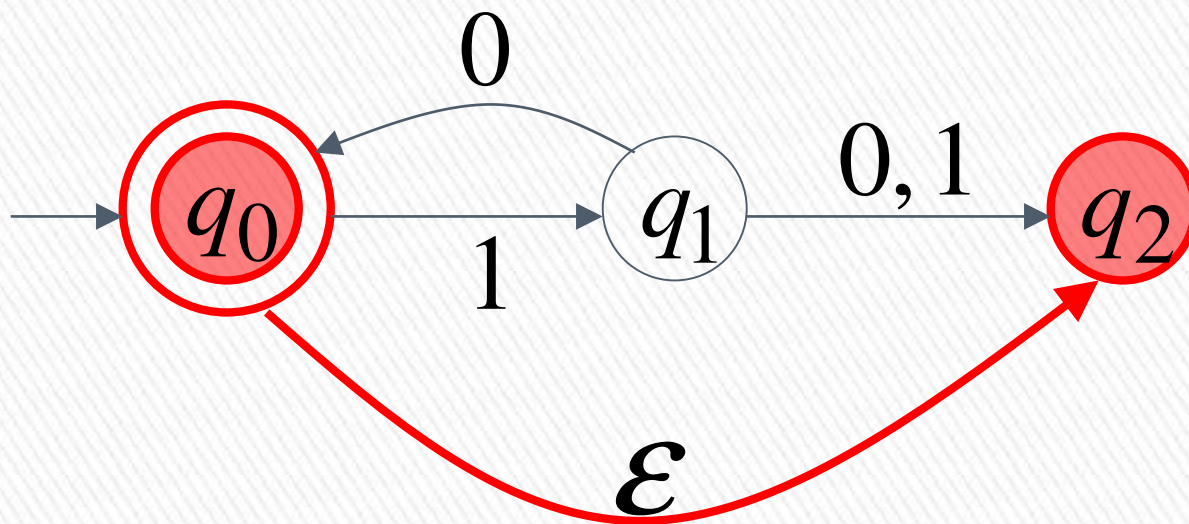




$$\delta(q_1, 0) = \{q_0, q_2\}$$

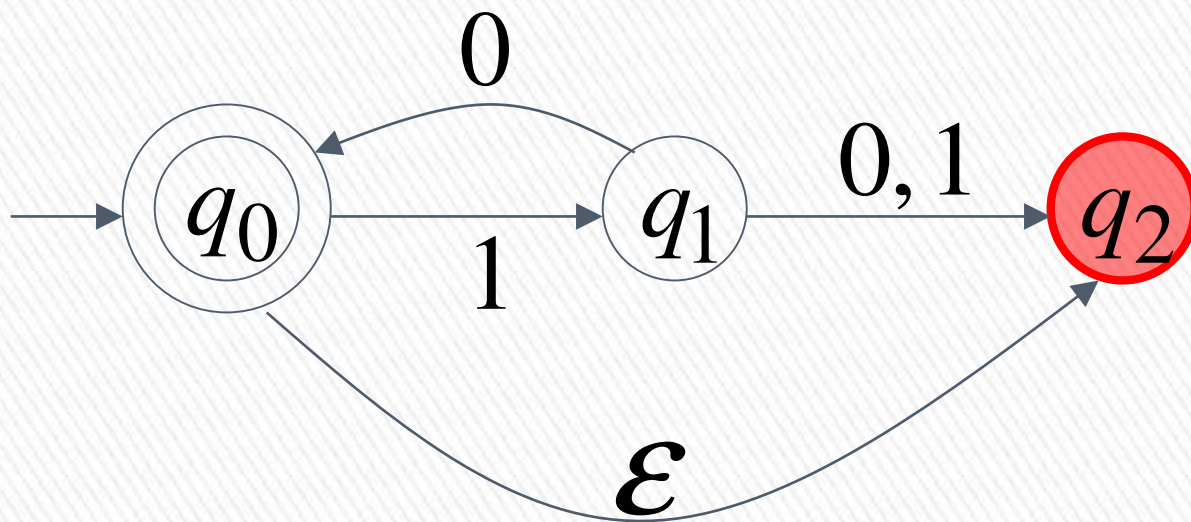


$$\delta(q_0, \varepsilon) = \{q_2\}$$



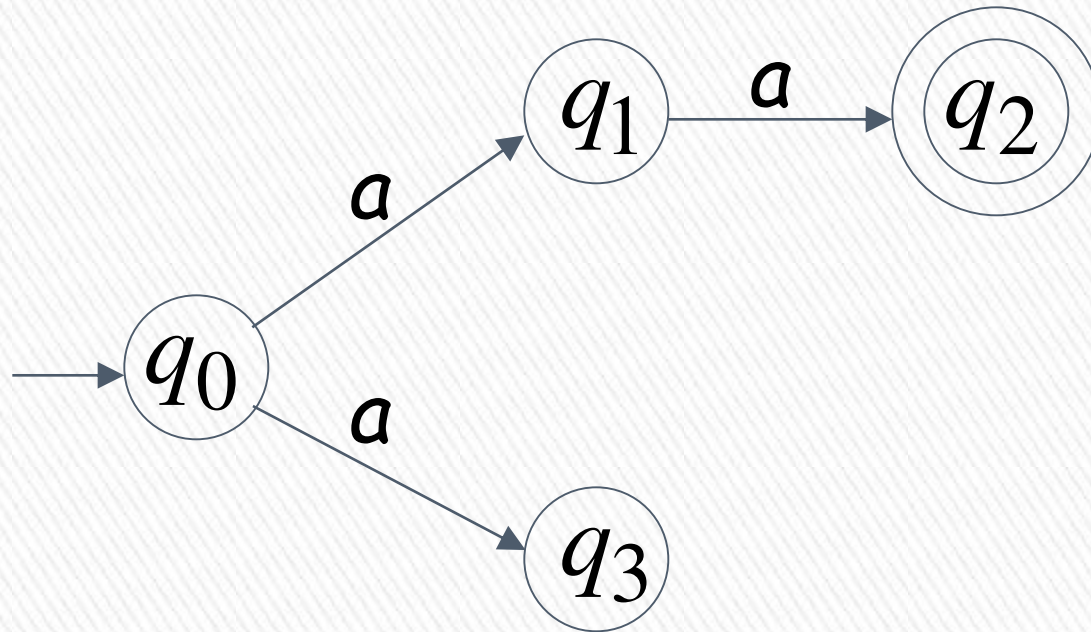


$$\delta(q_2, 1) = \emptyset$$

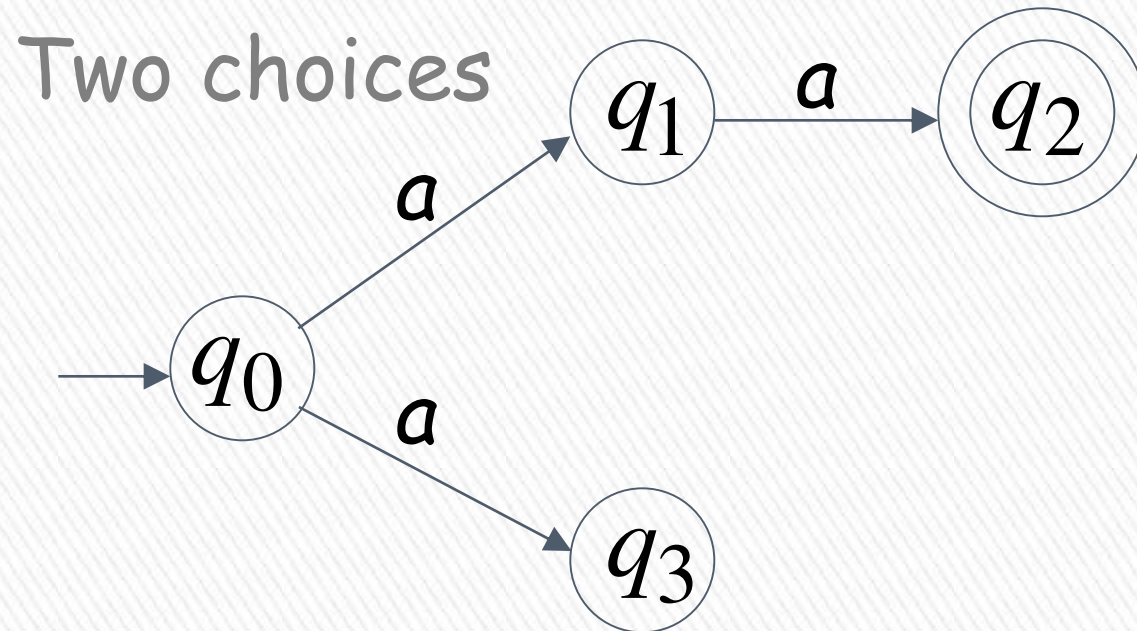


# Nondeterministic Finite Automaton (NFA)

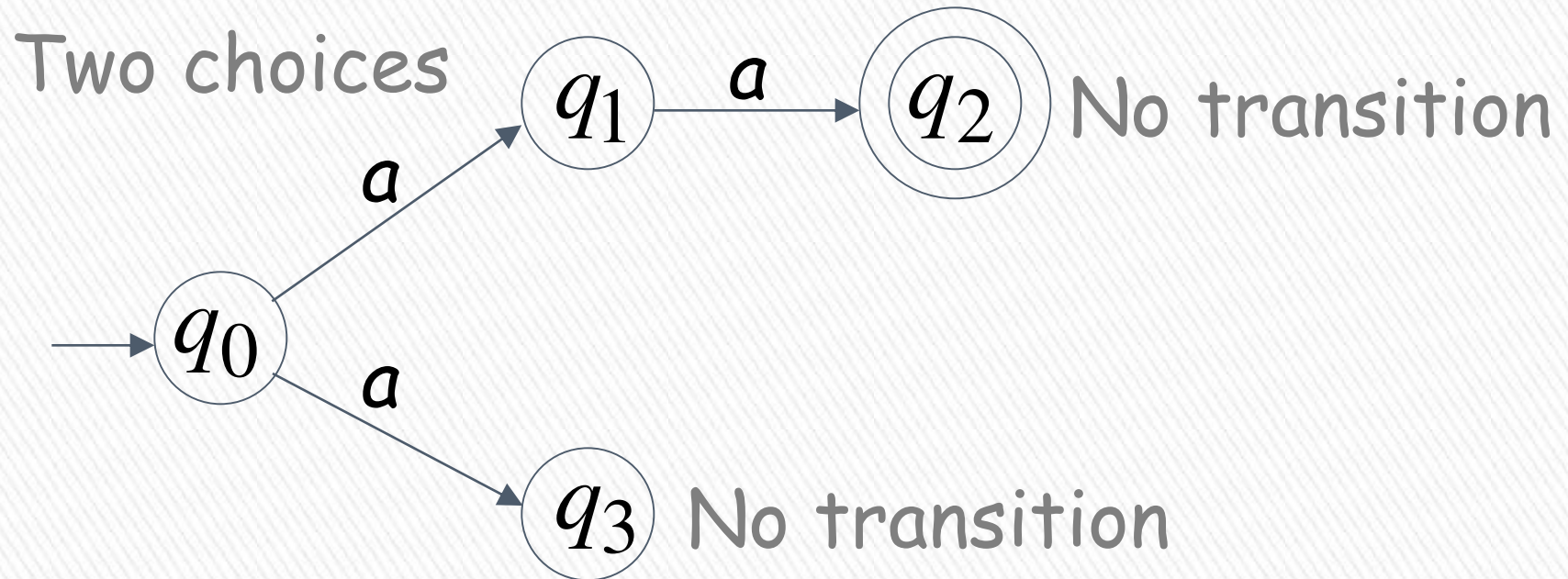
Alphabet =  $\{a\}$



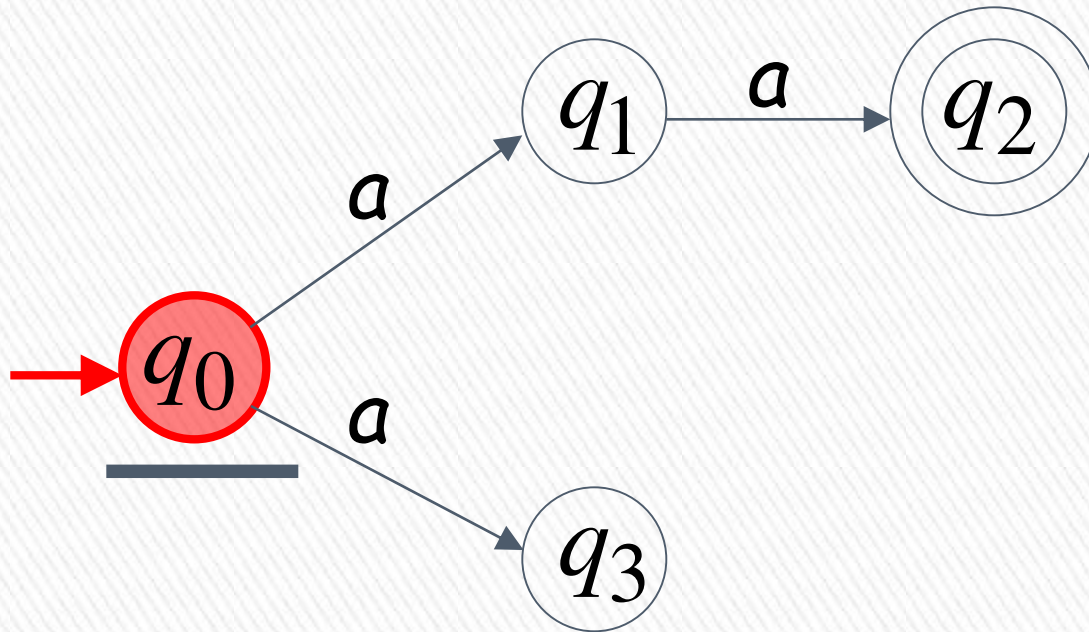
Alphabet =  $\{a\}$



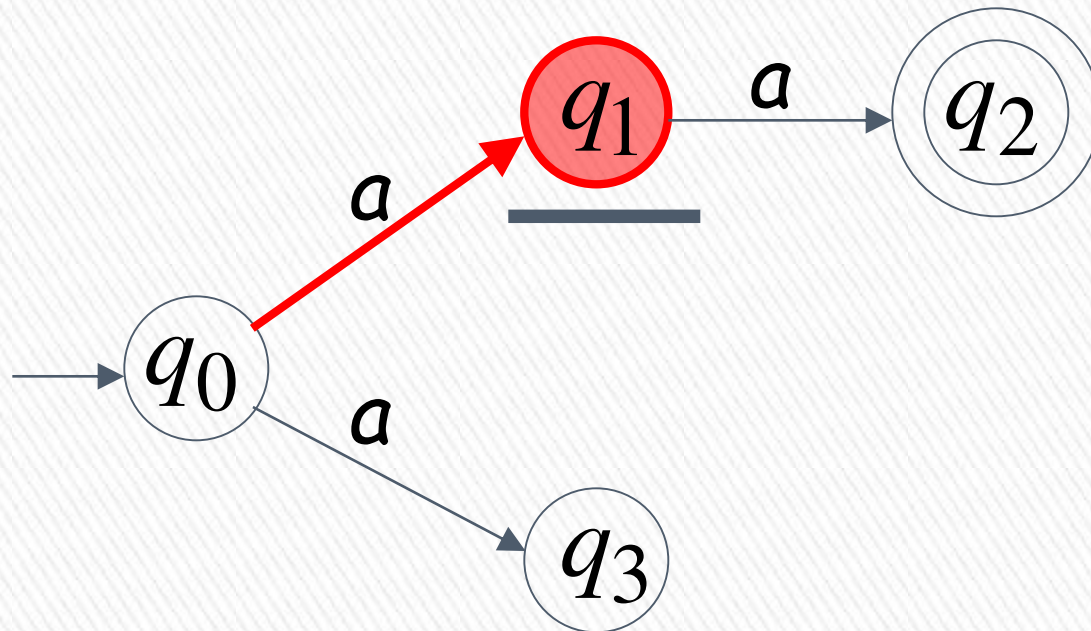
Alphabet =  $\{a\}$



# First Choice



# First Choice

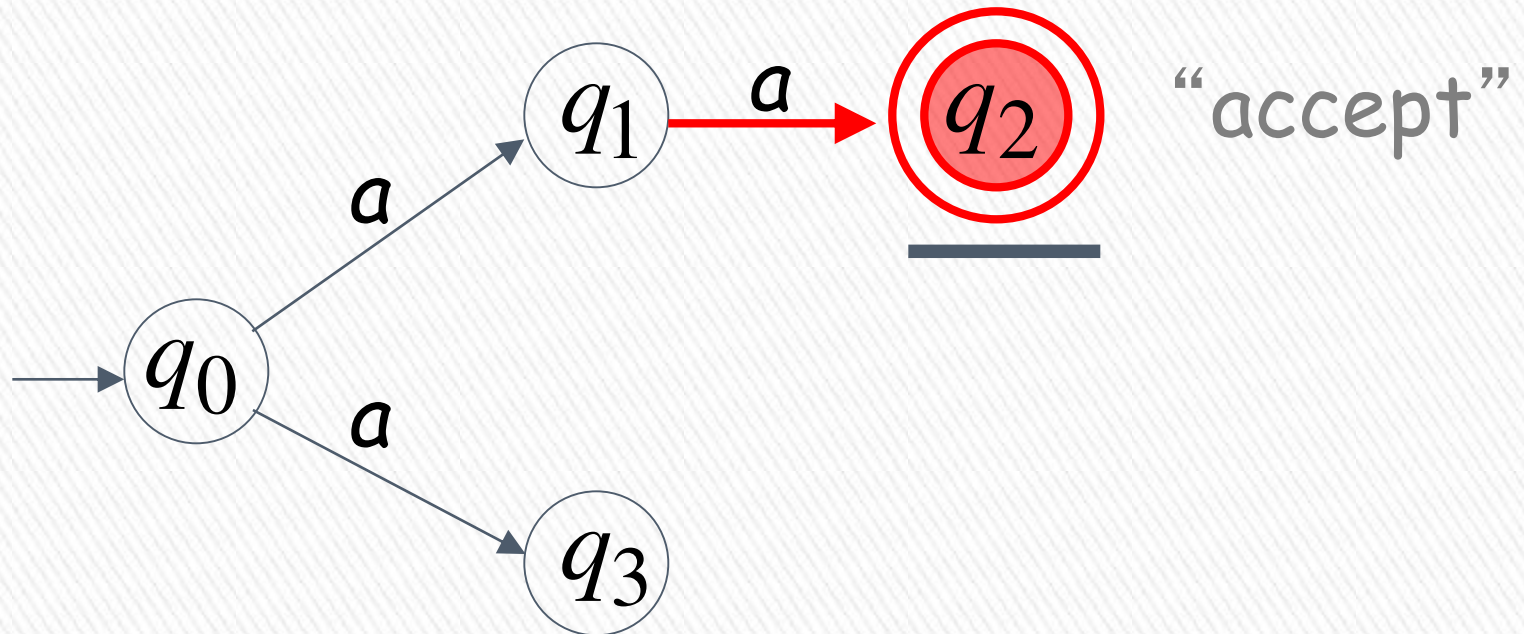




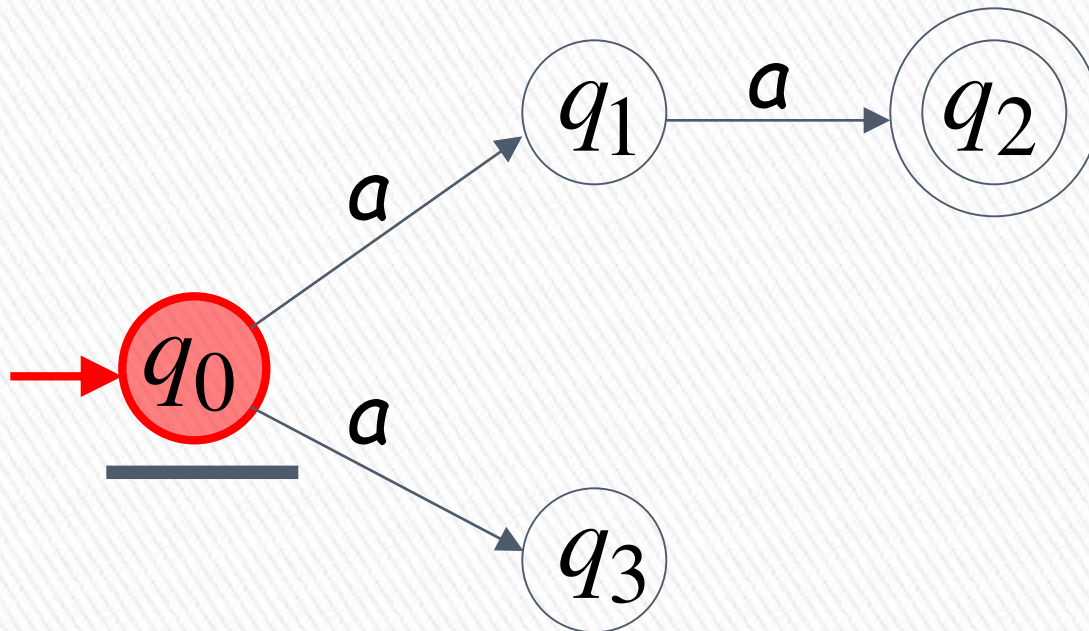
# First Choice



All input is consumed



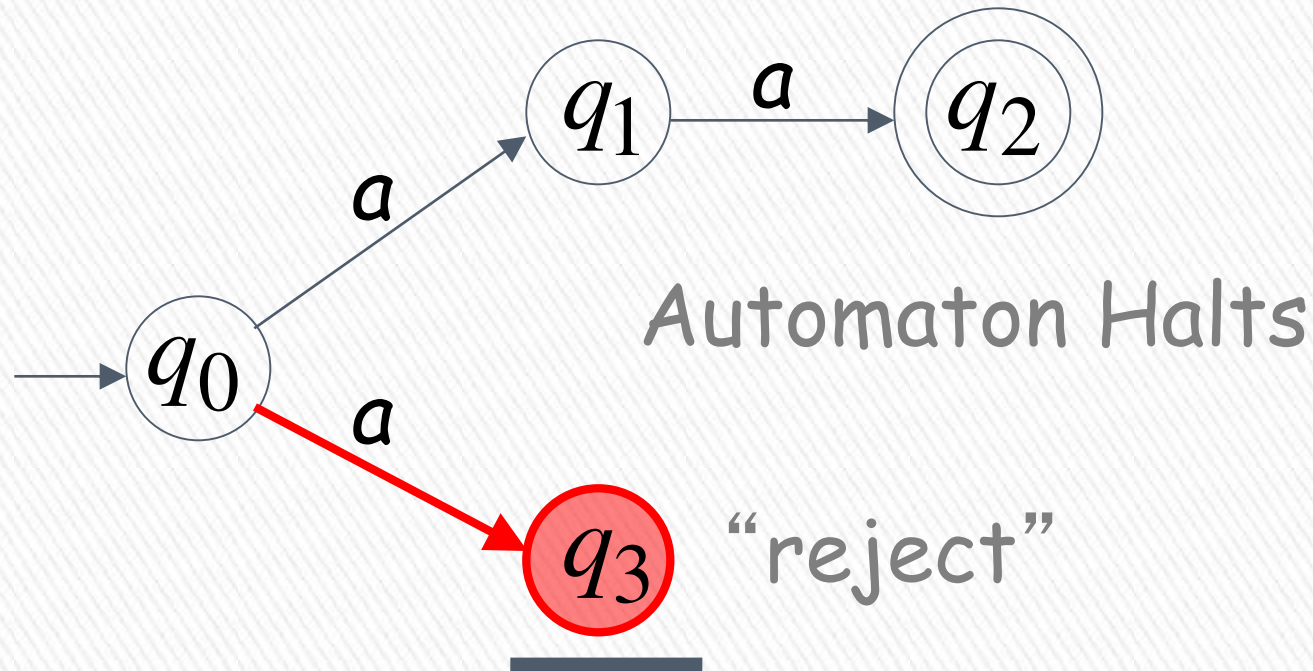
# Second Choice



## Second Choice



Input cannot be consumed



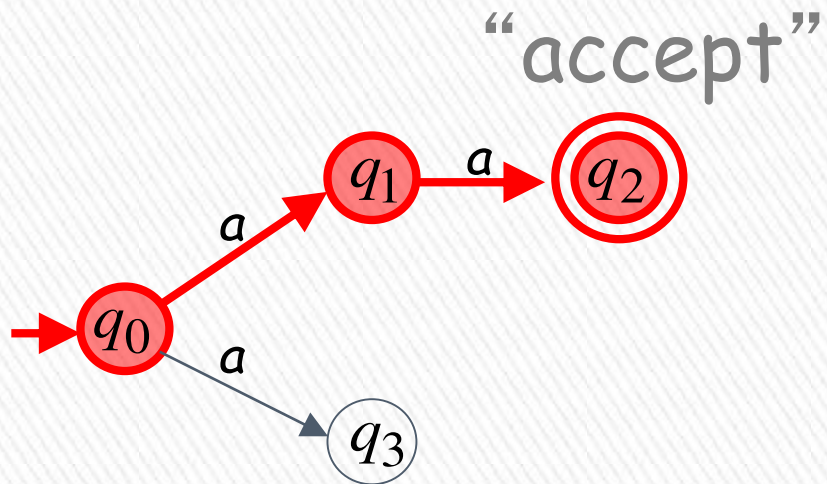
## An NFA accepts a string:

if there exists a computation of the NFA that accepts the string

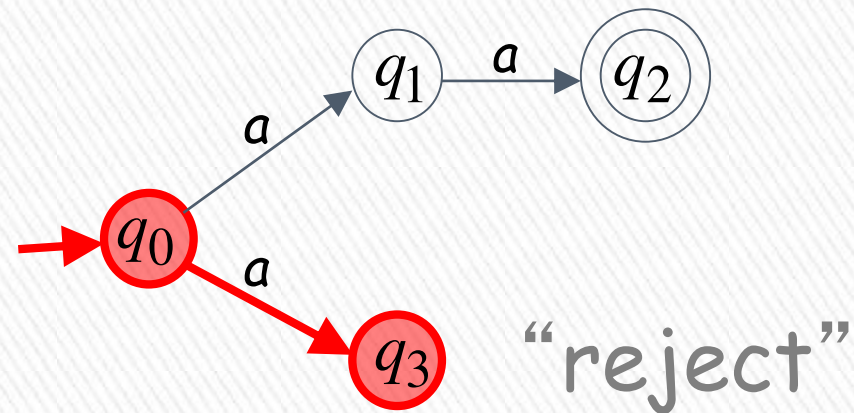
i.e., all the input string is processed and the automaton is in an accepting state



*aa* is accepted by the NFA:

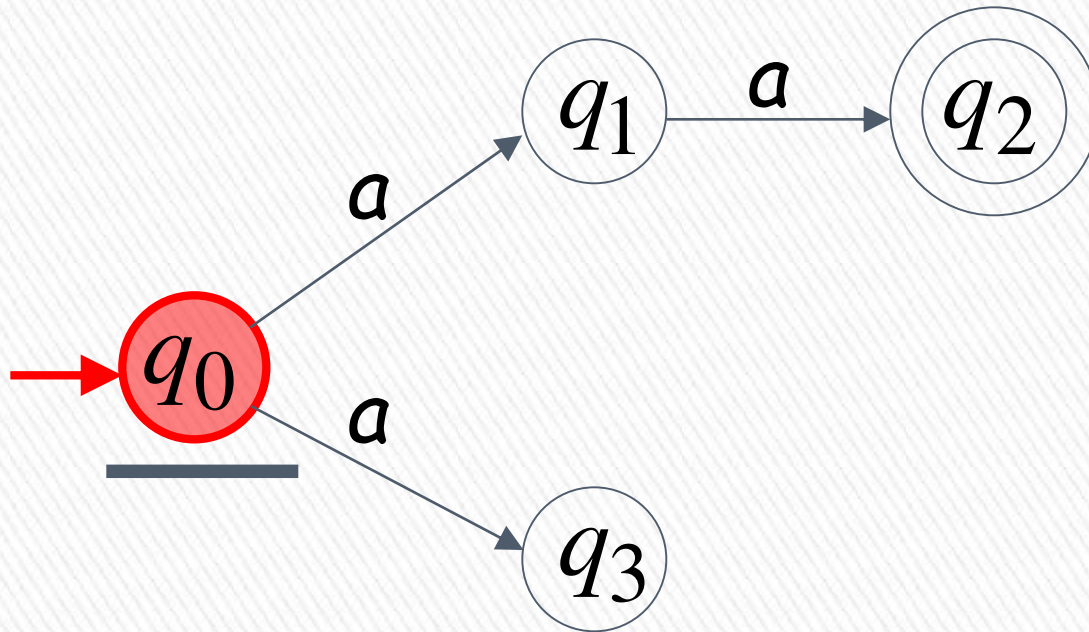
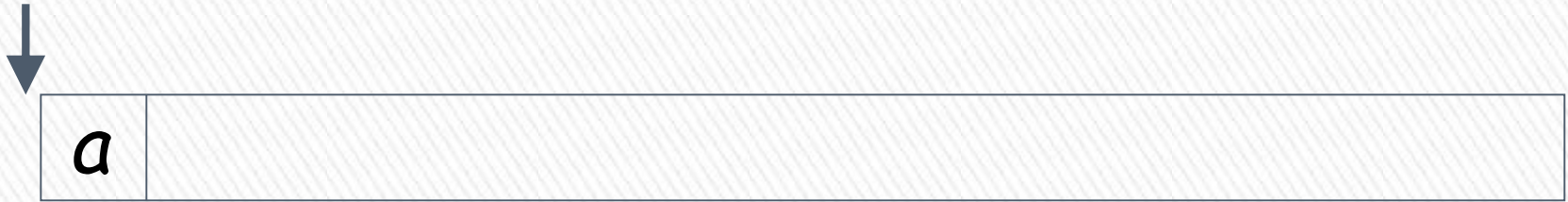


because this  
computation  
accepts *aa*



this computation  
is ignored

# Rejection example

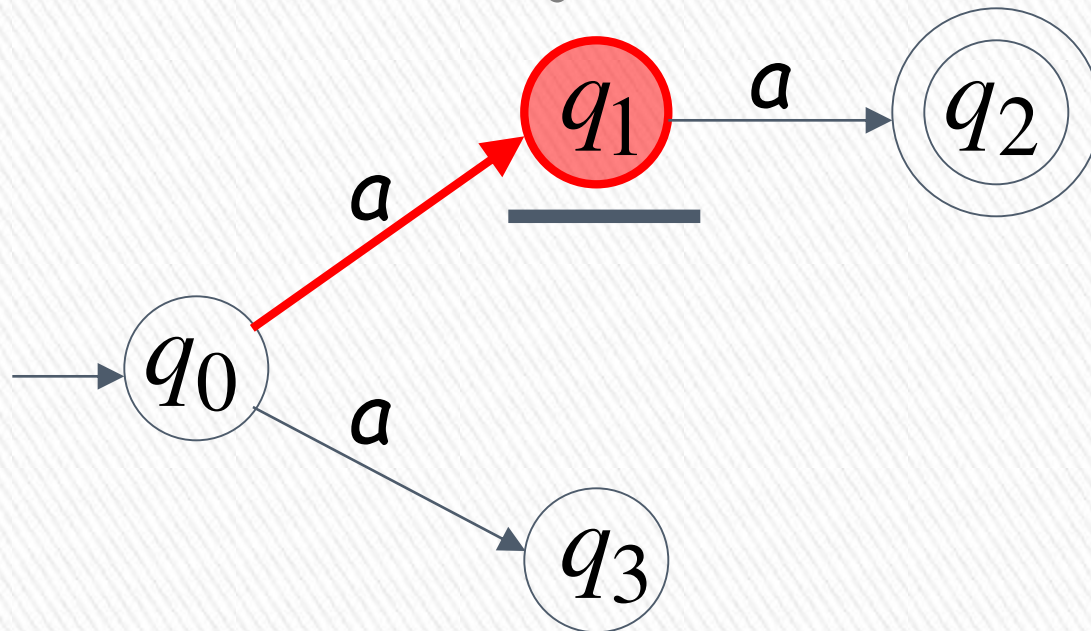




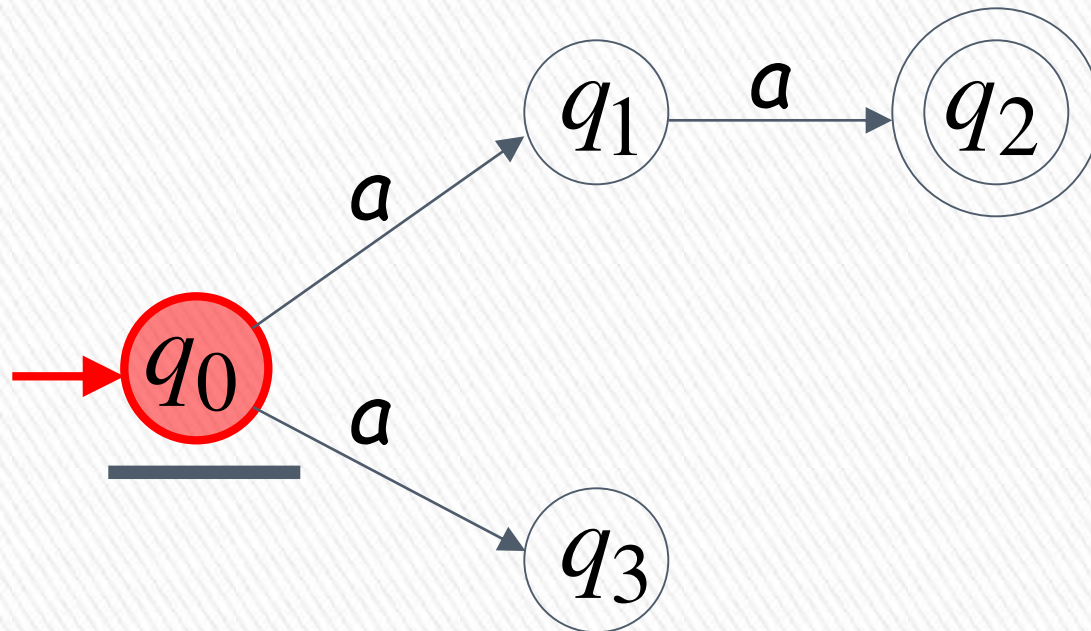
# First Choice



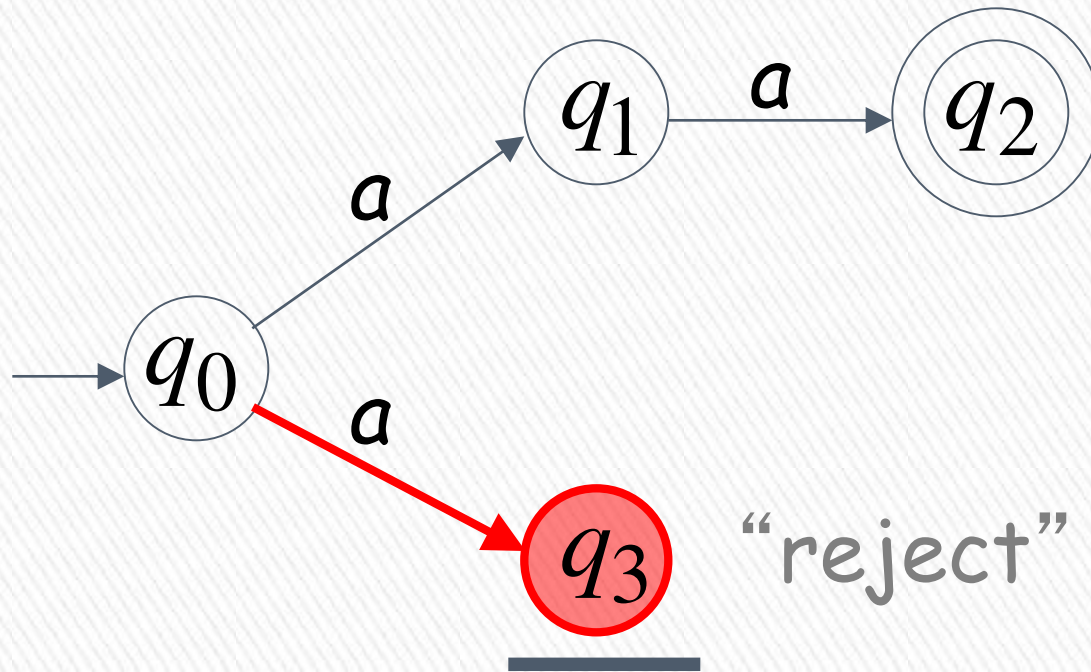
“reject”



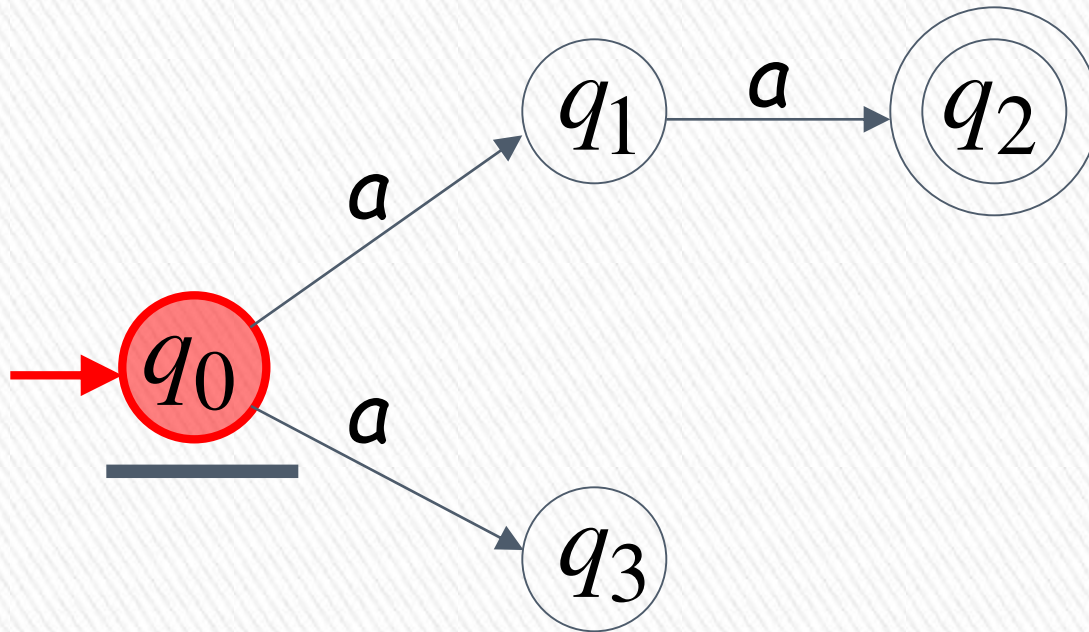
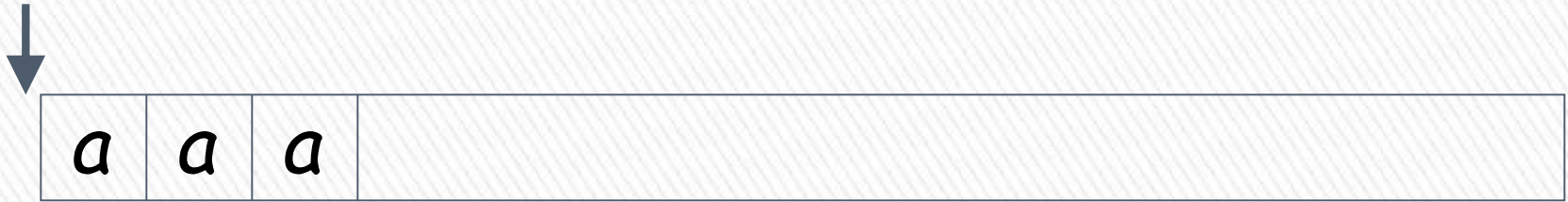
# Second Choice



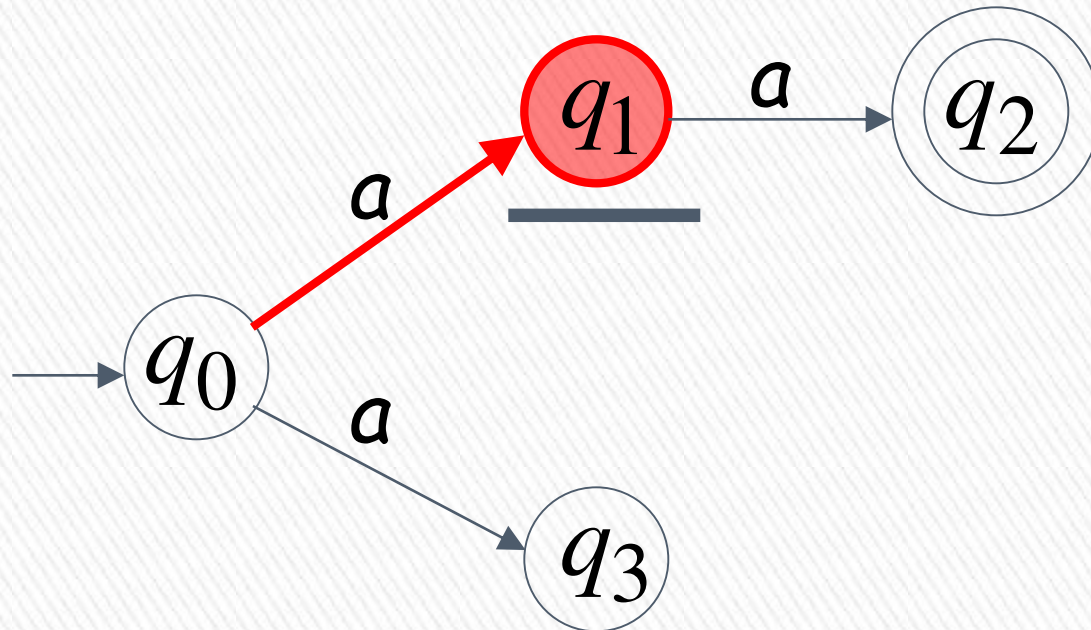
# Second Choice



# Another Rejection example



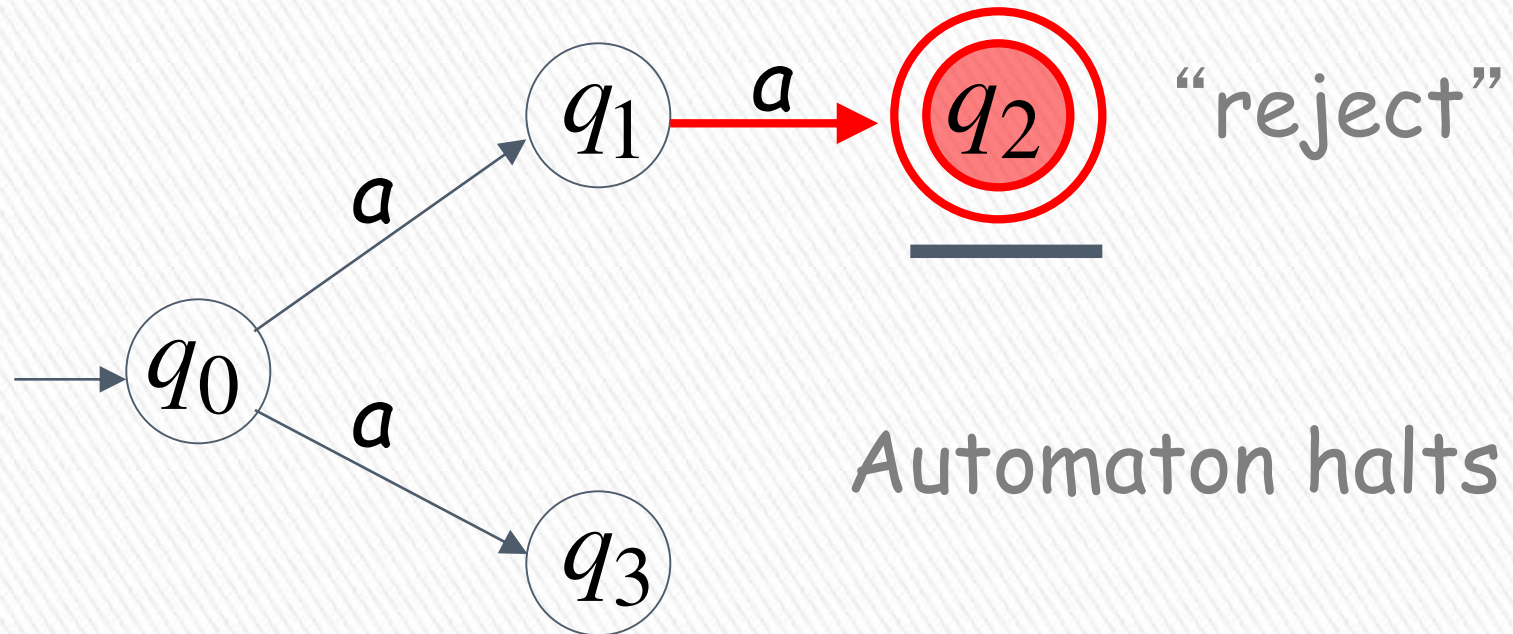
# First Choice



# First Choice

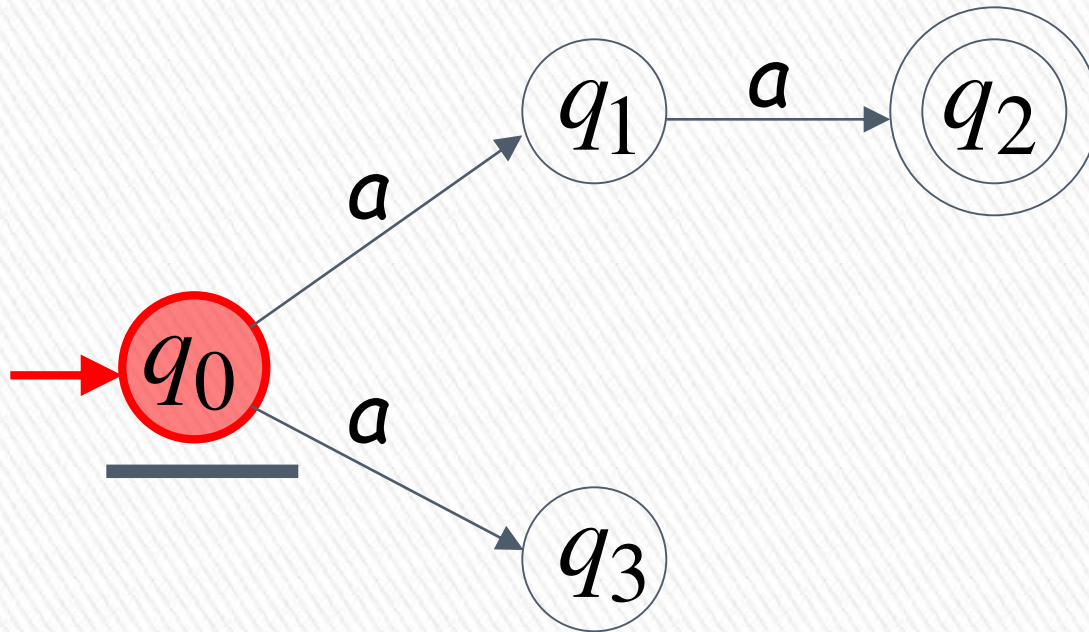


Input cannot be consumed





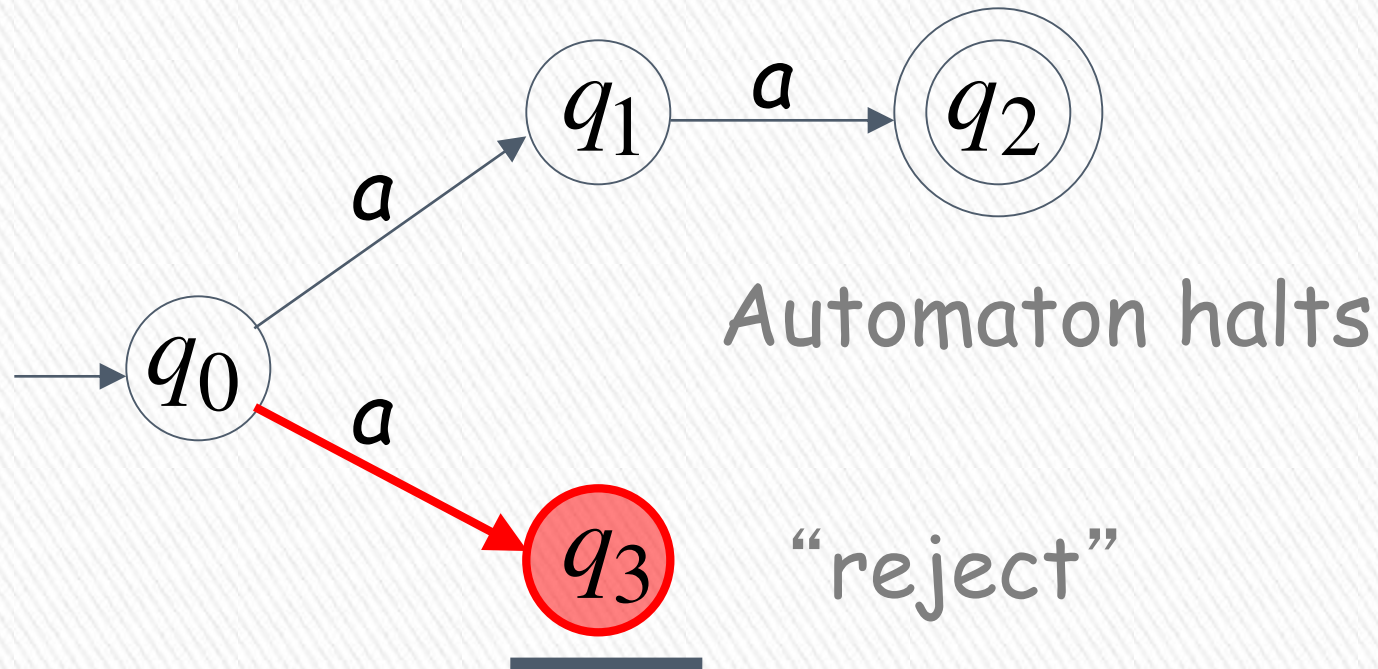
# Second Choice



# Second Choice



Input cannot be consumed



## An NFA rejects a string:

if there is no computation of the NFA that accepts the string.

For each computation:

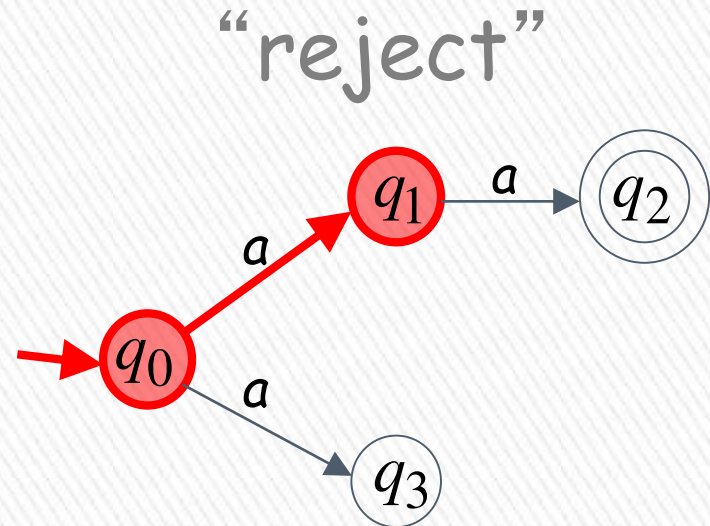
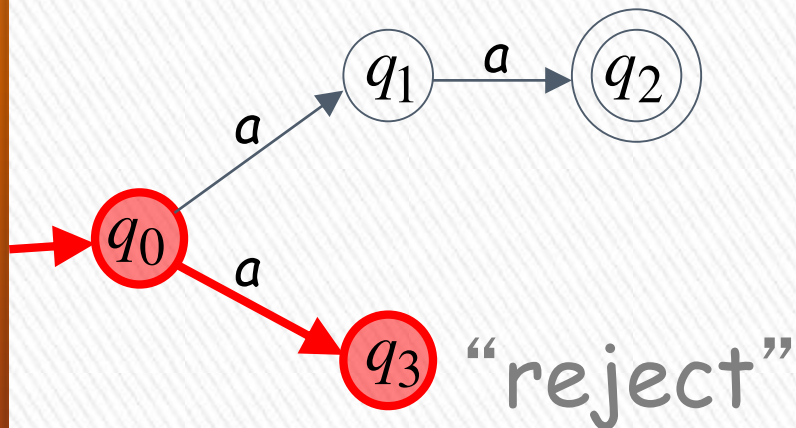
- All the input is consumed and the automaton is in a non final state

OR

- The input cannot be consumed

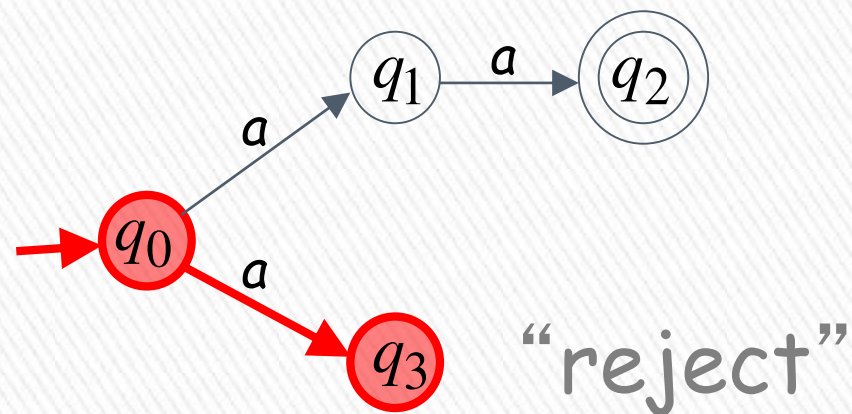
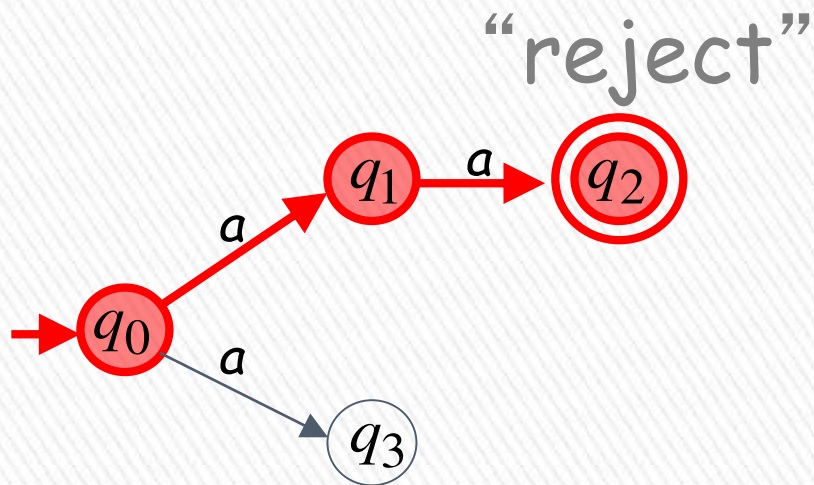


$a$  is rejected by the NFA:



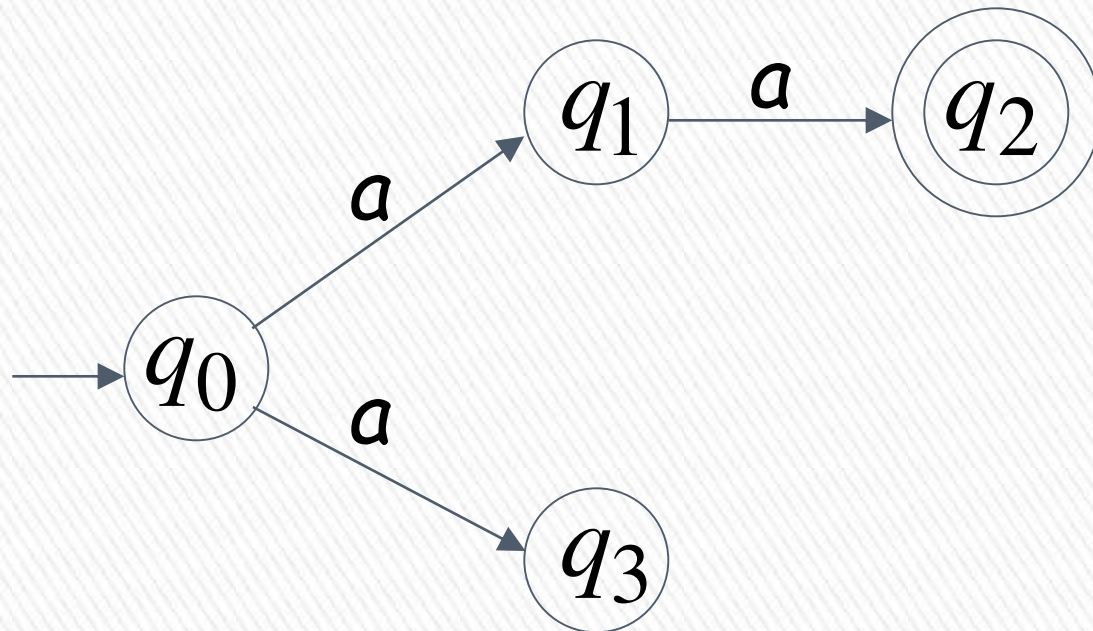
All possible computations lead to rejection >

aaa is rejected by the NFA:



All possible computations lead to rejection

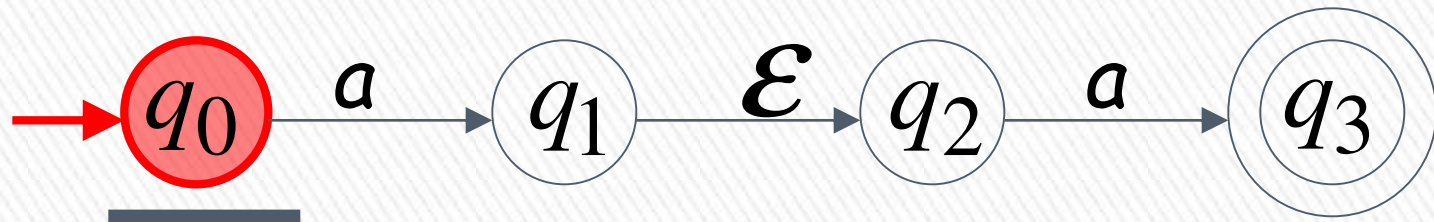
Language accepted:  $L = \{aa\}$





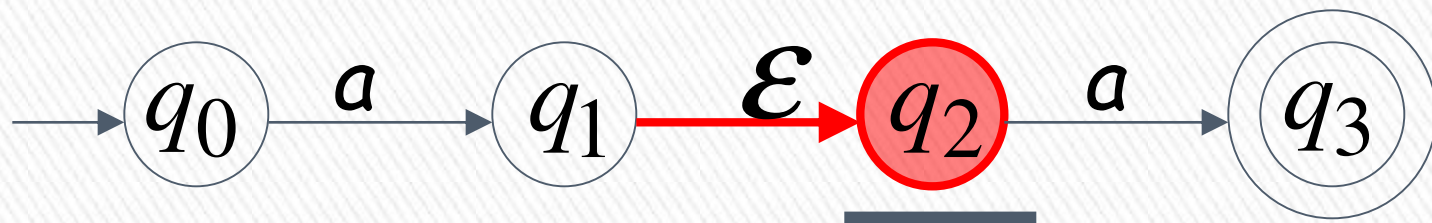
# Epsilon Transition



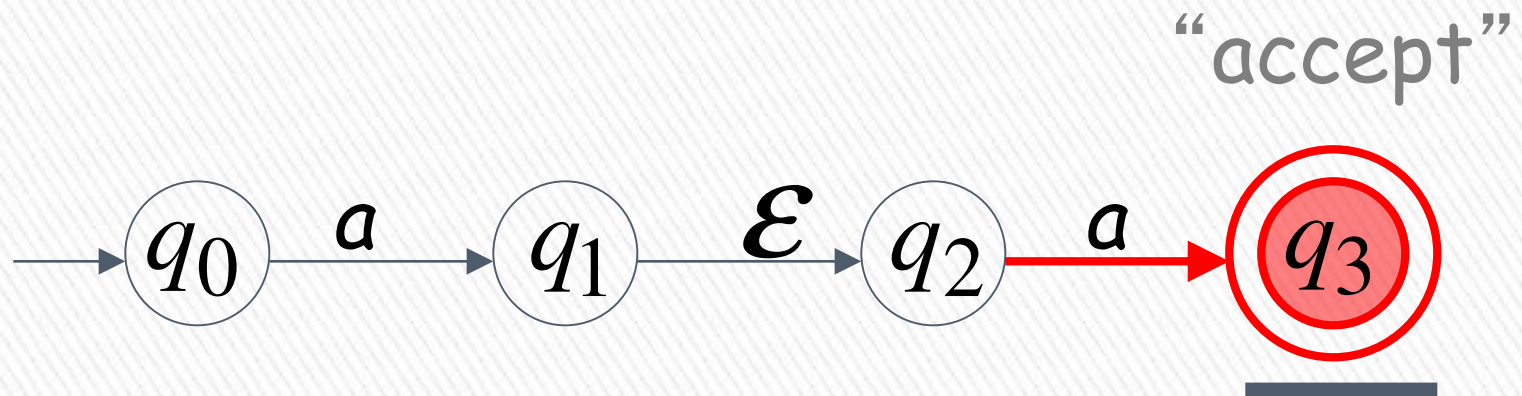




input tape head does not move



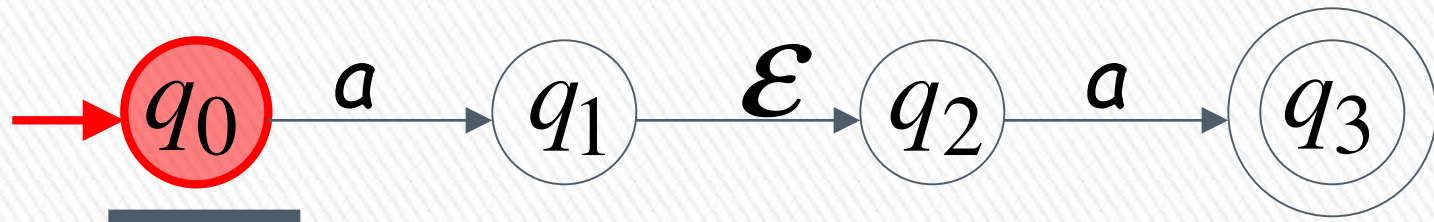
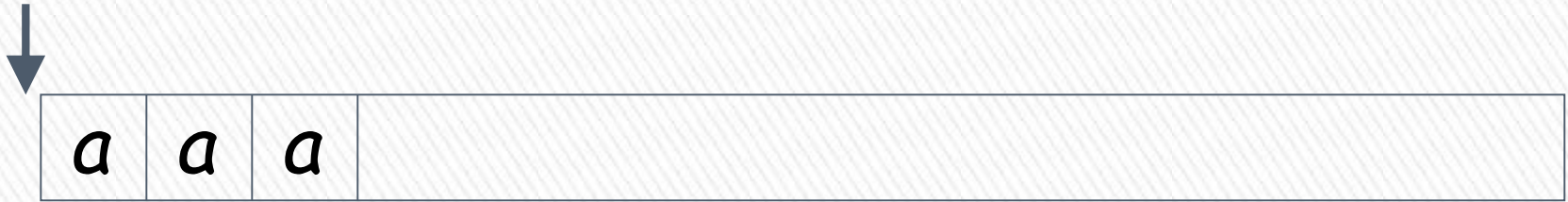
all input is consumed



String  $aa$  is accepted



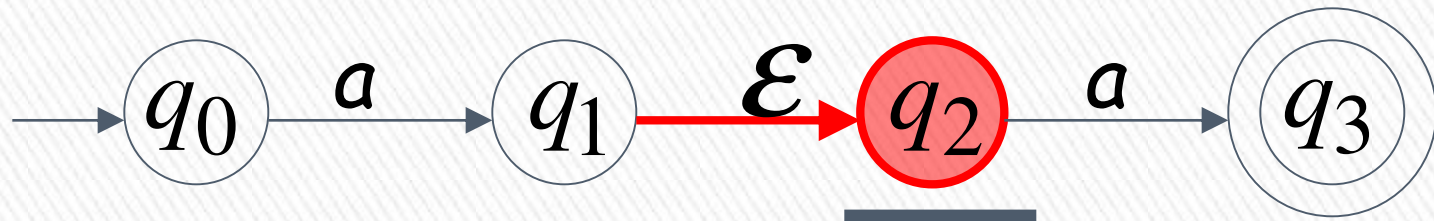
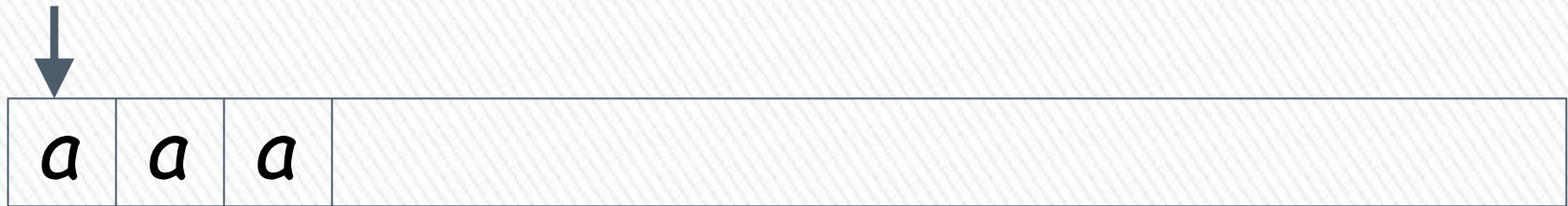
# Rejection Example



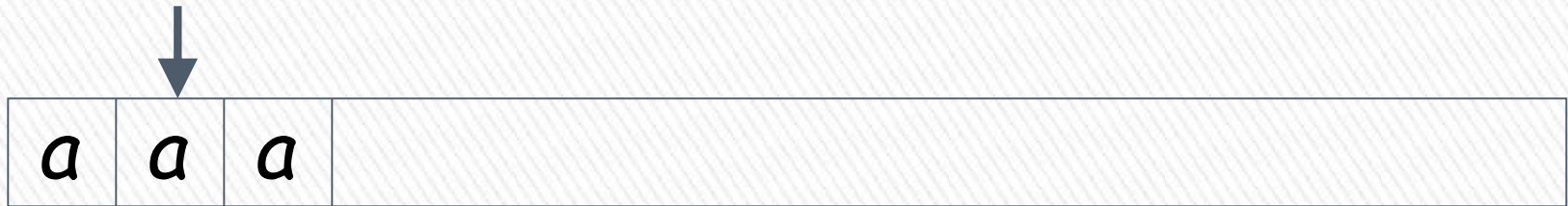




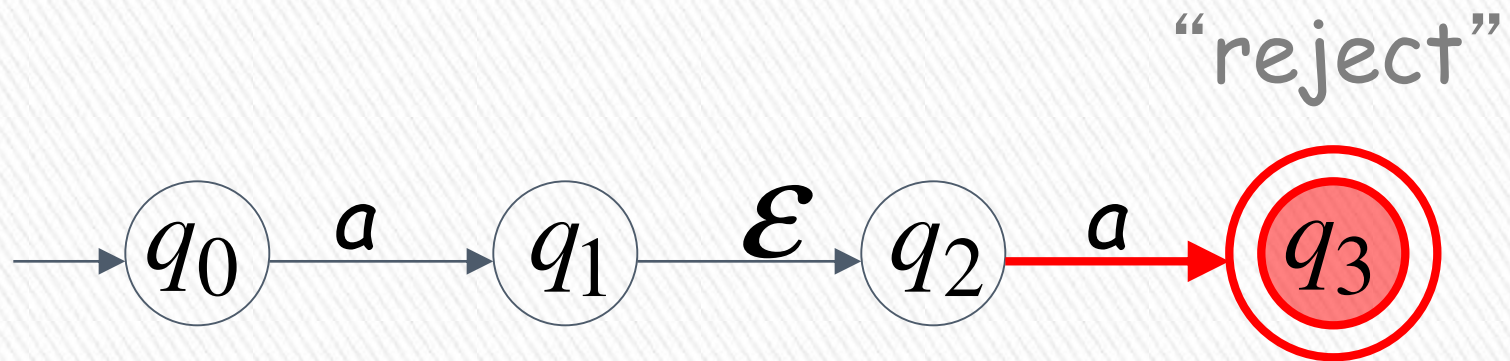
(read head doesn't move)



Input cannot be consumed



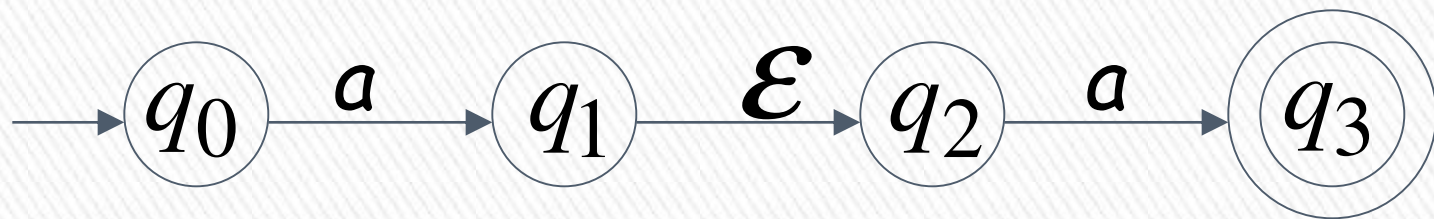
Automaton halts



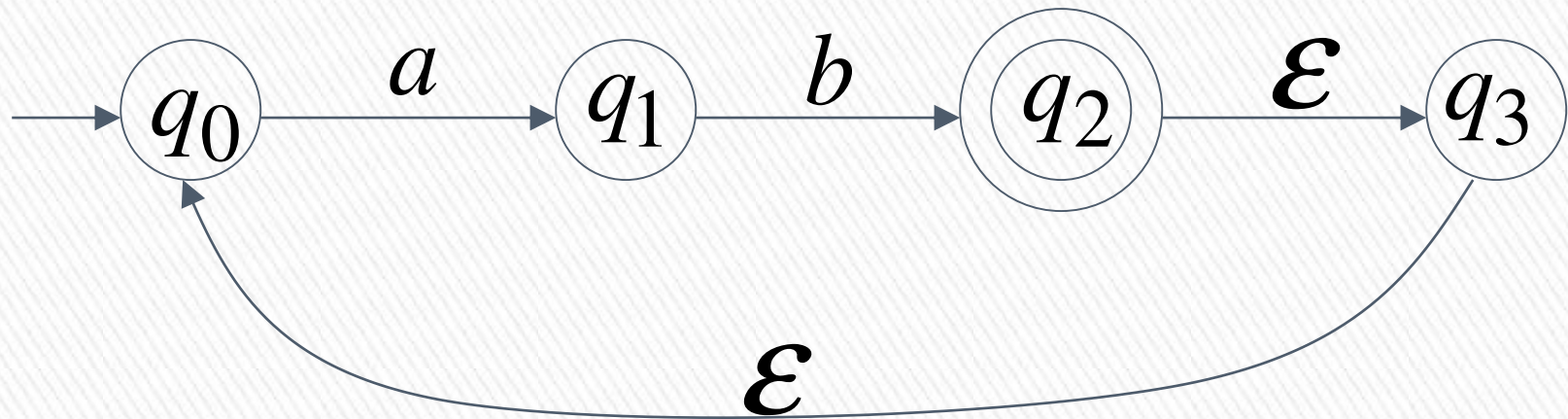
String **aaa** is rejected

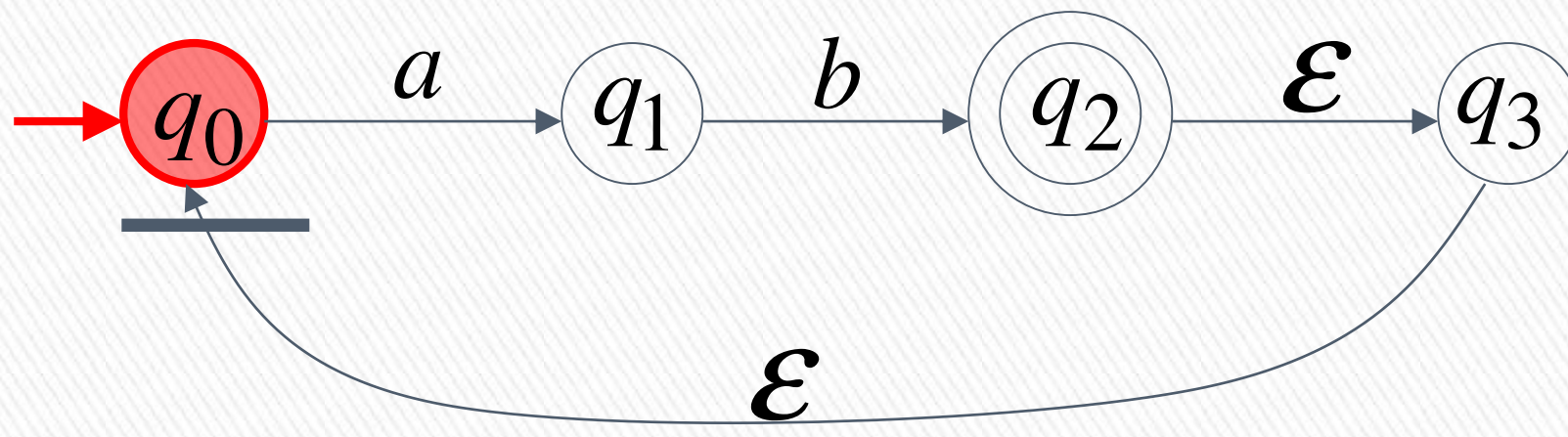


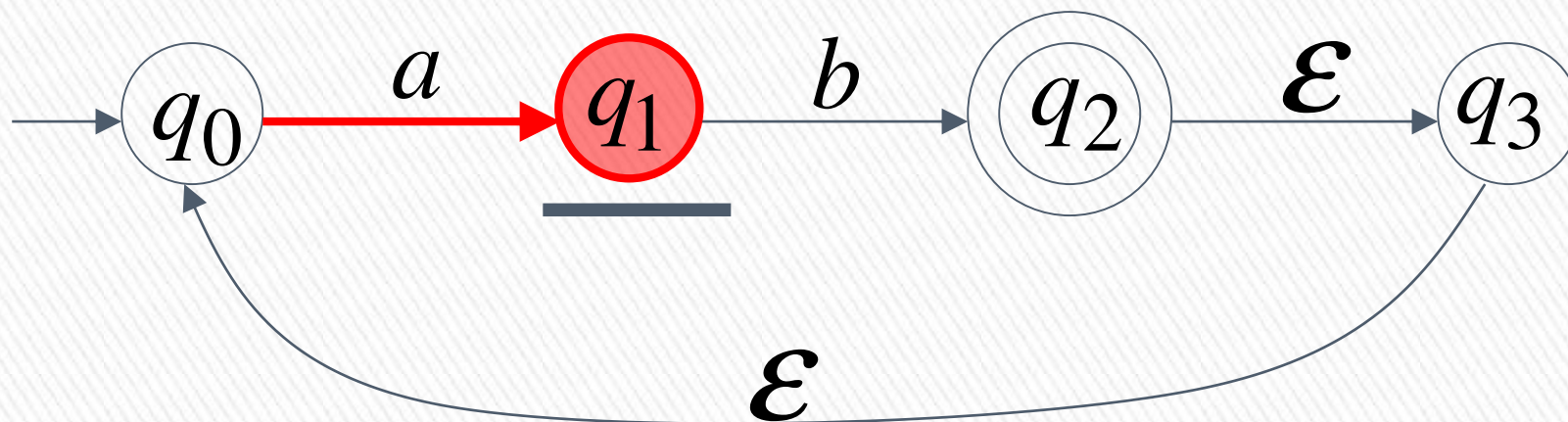
Language accepted:  $L = \{aa\}$

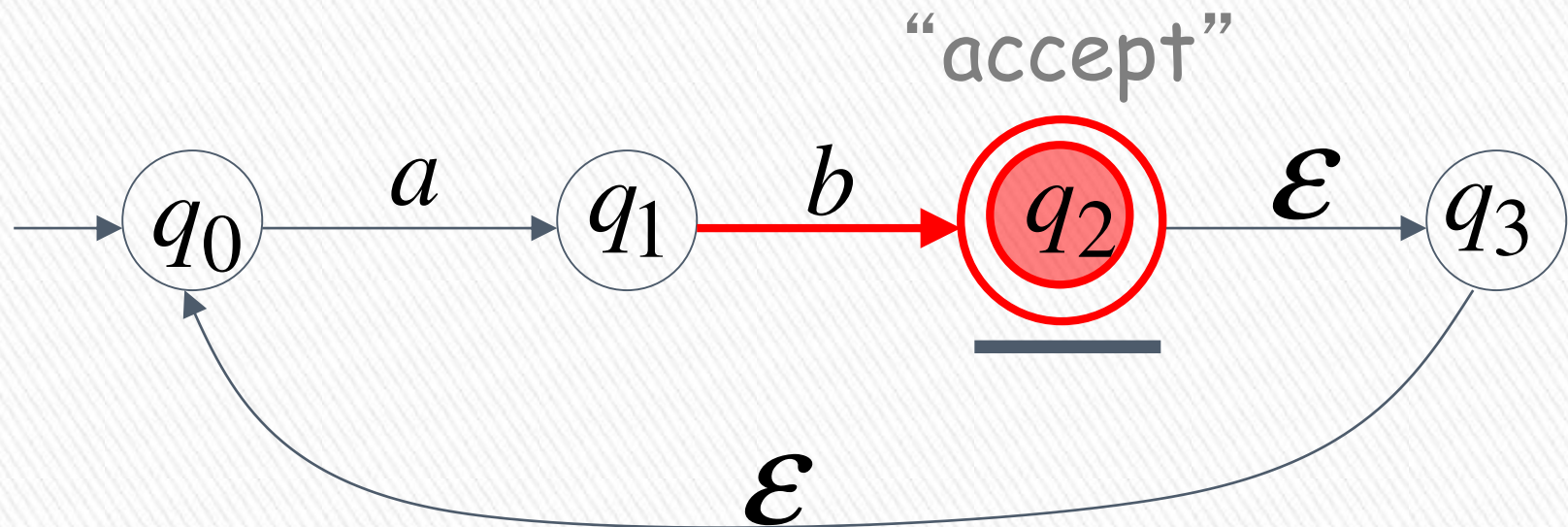


# Another NFA Example



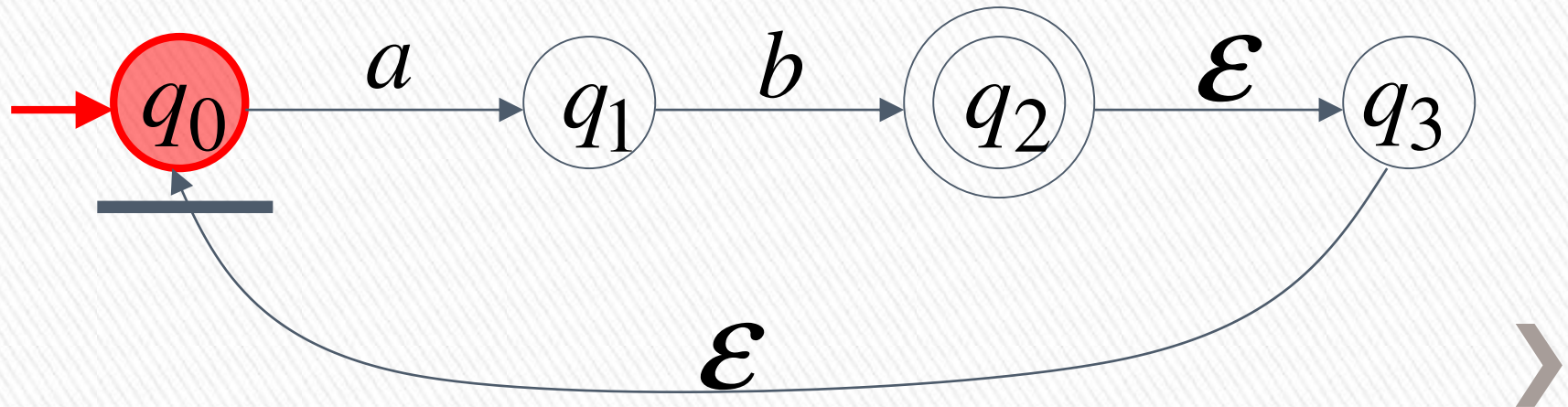


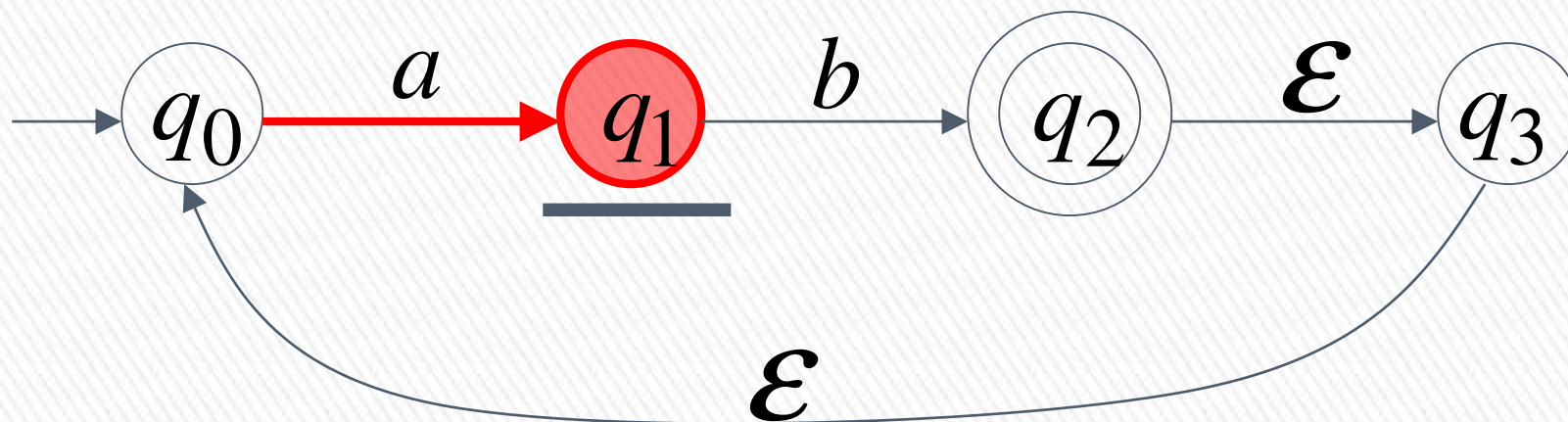


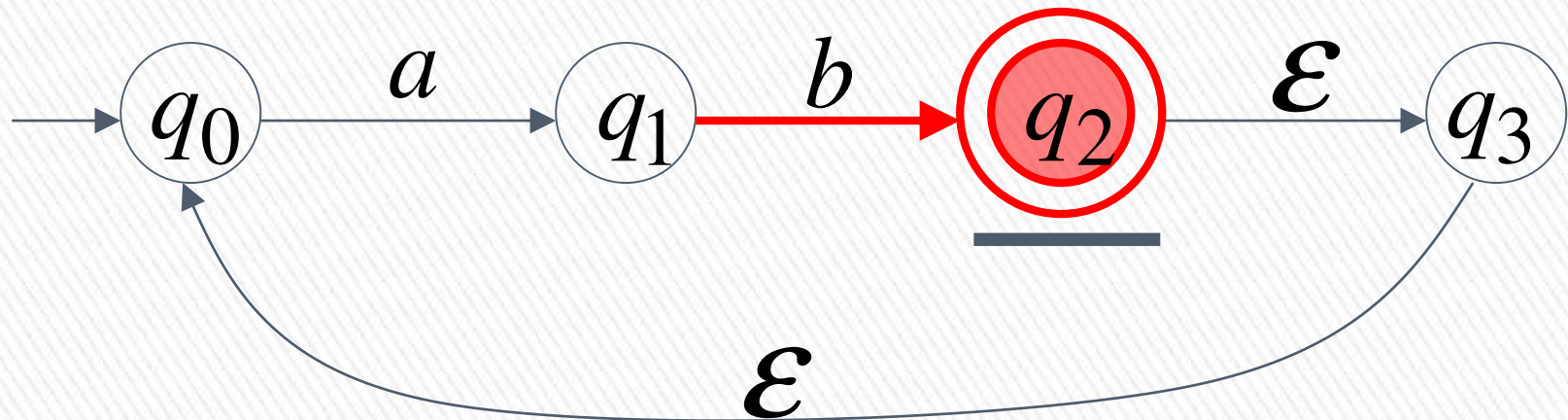


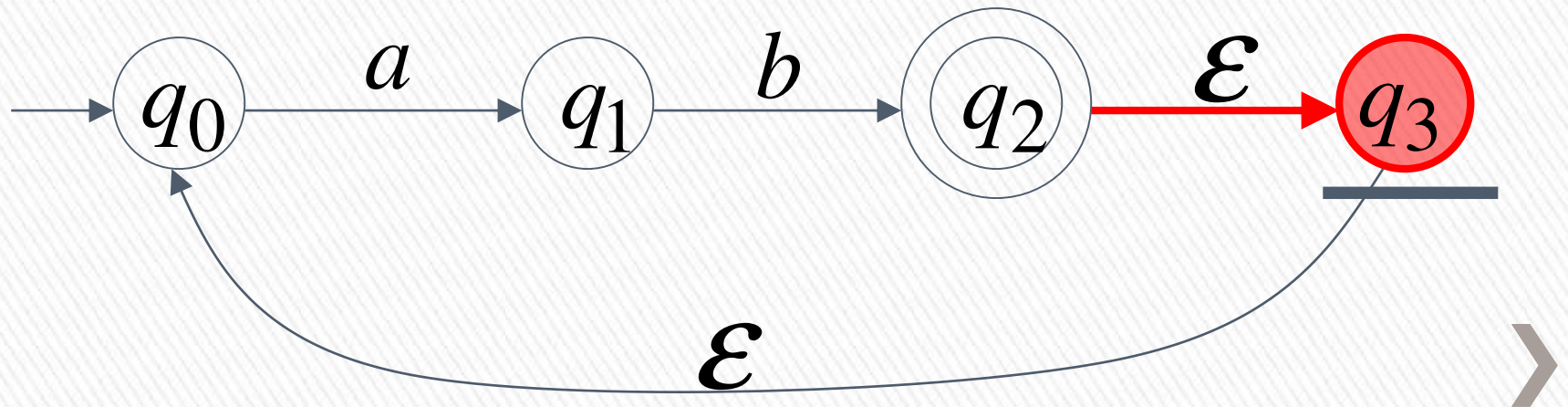


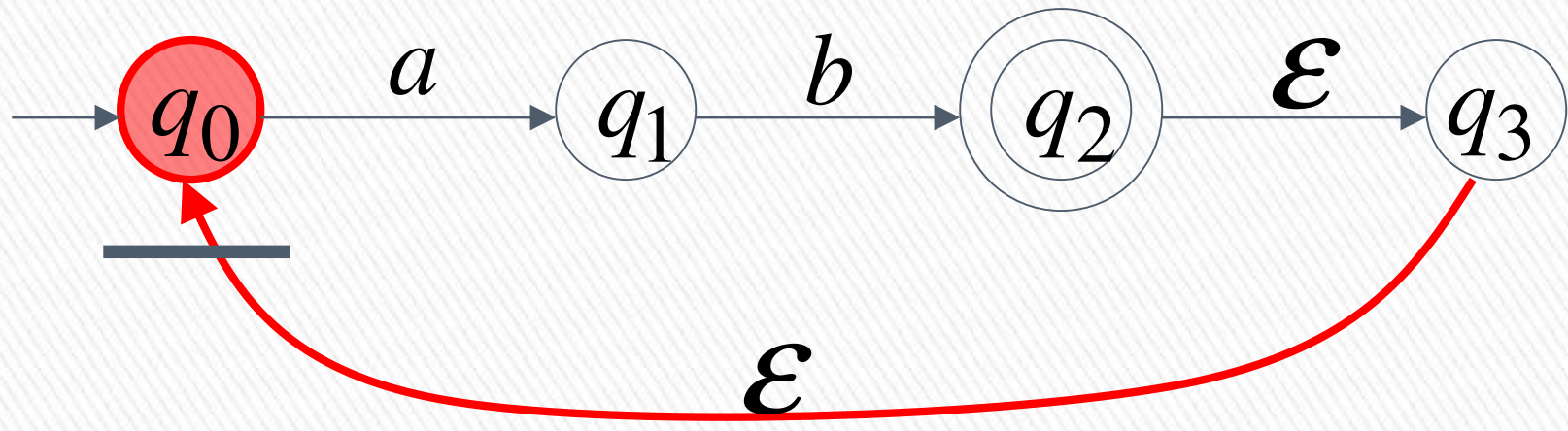
## Another String

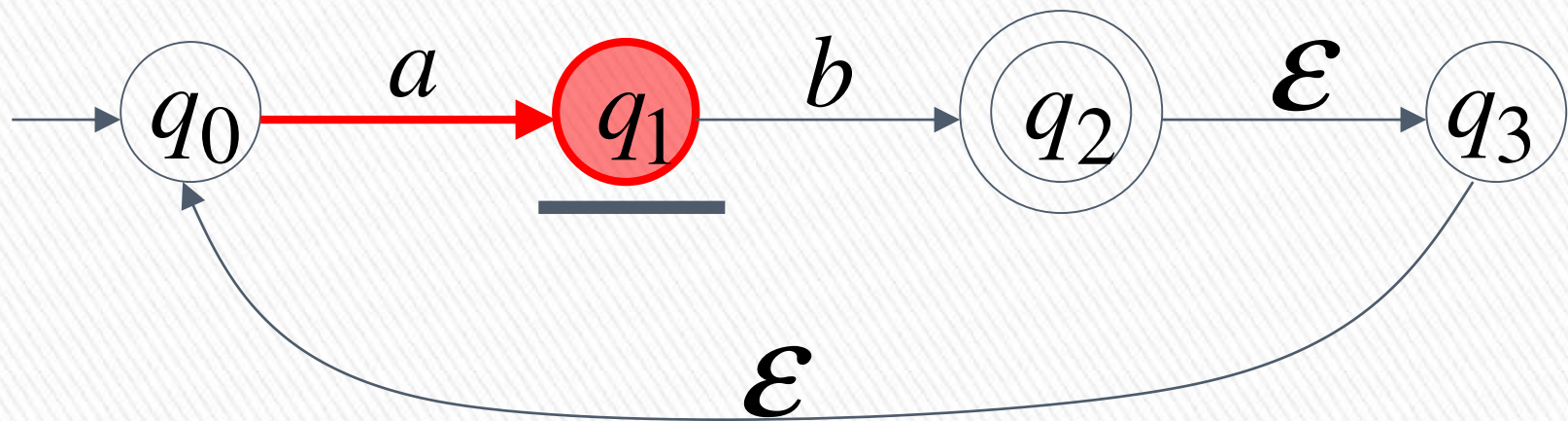


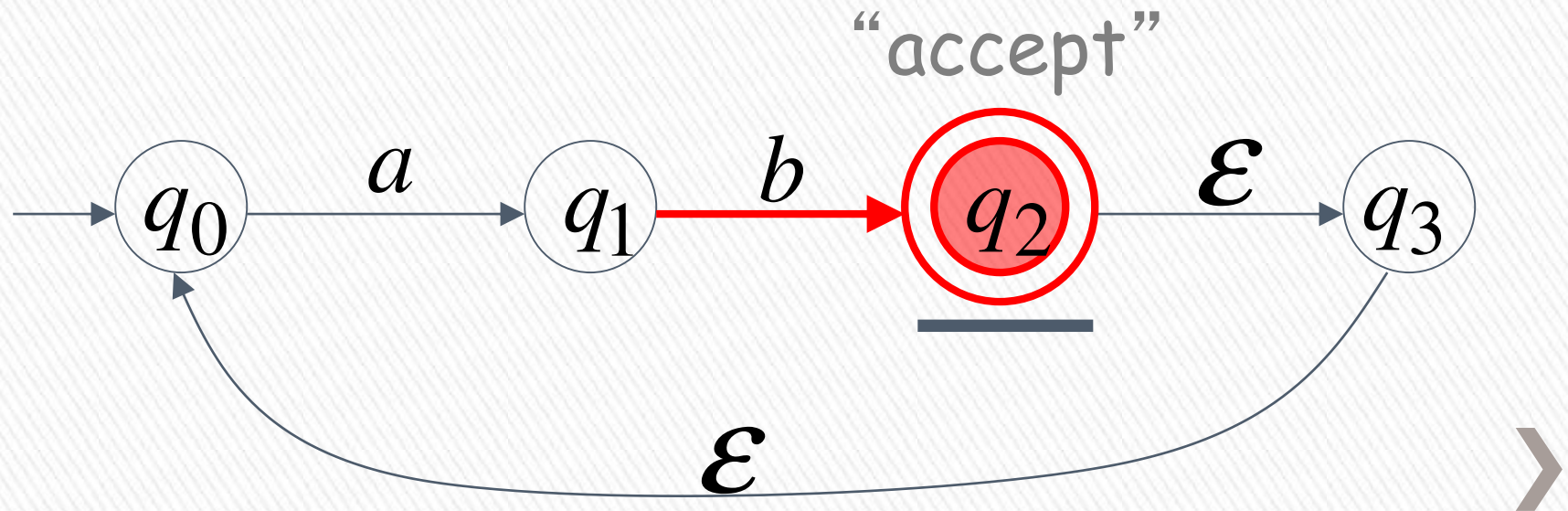








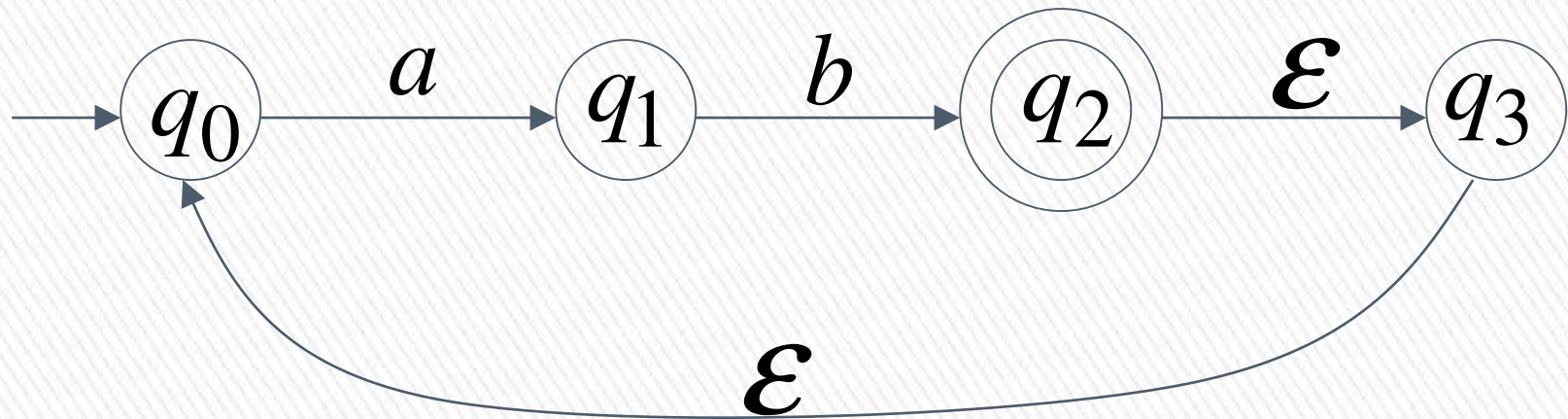




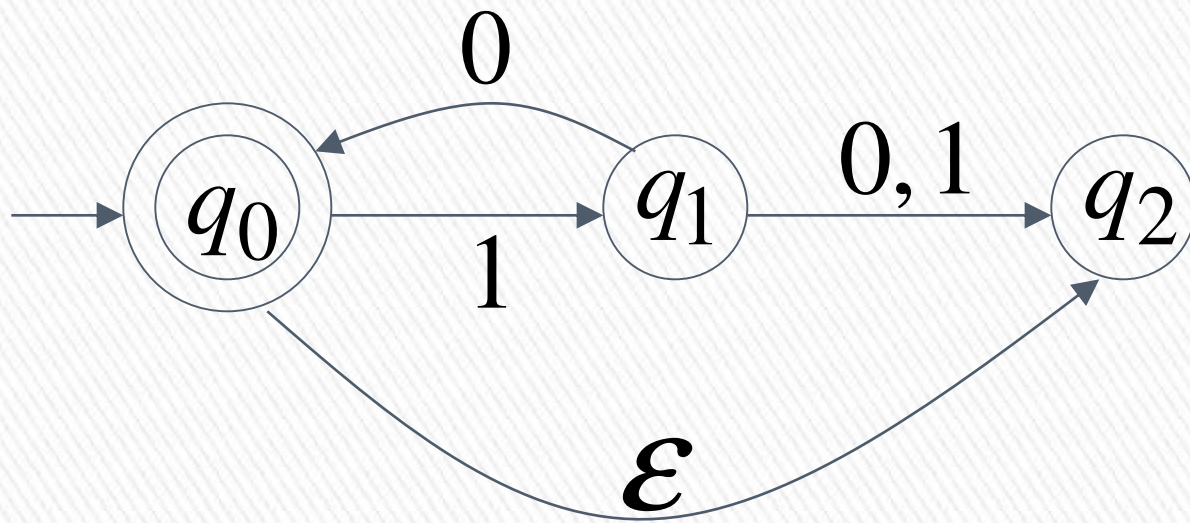


Language accepted

$$L = \{ab, abab, ababab, \dots\}$$
$$= \{ab\}^+$$

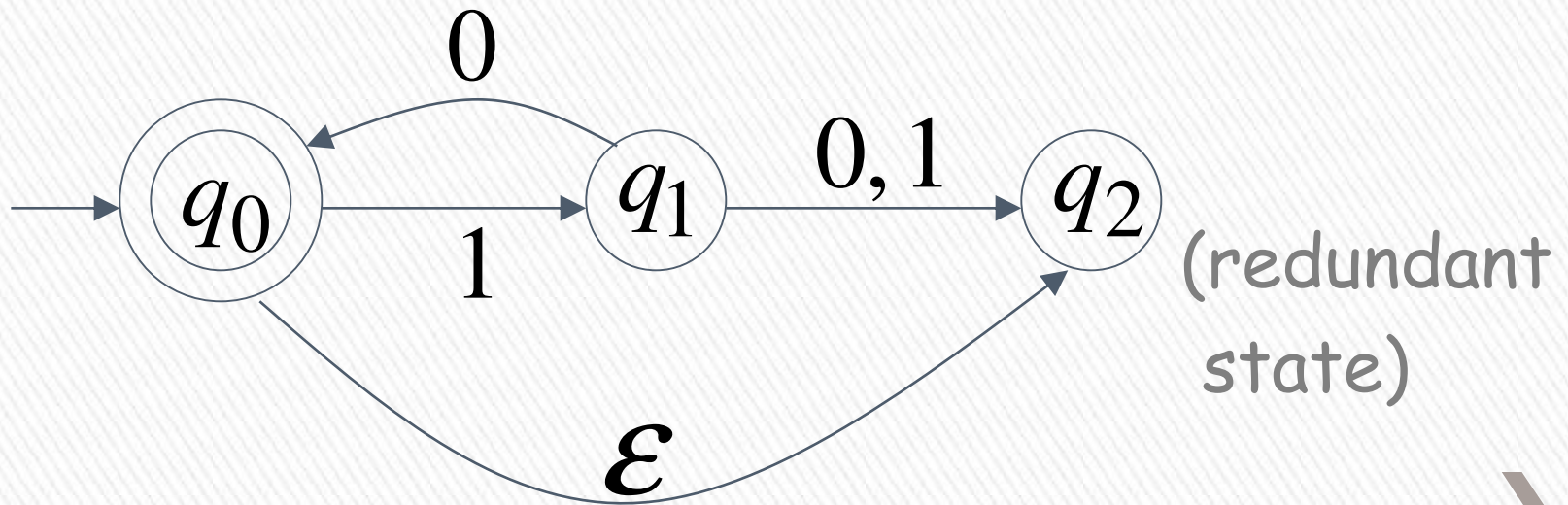


# Another NFA Example



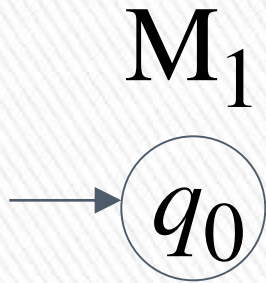
# Language accepted

$$\begin{aligned} L(M) &= \{\varepsilon, 10, 1010, 101010, \dots\} \\ &= \{10\}^* \end{aligned}$$

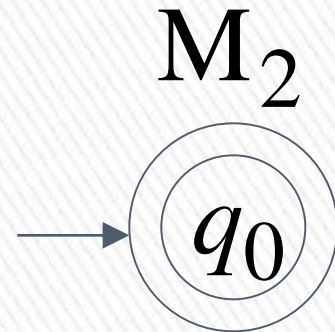


## Remarks:

- The  $\varepsilon$  symbol never appears on the input tape
- Simple automata:



$$L(M_1) = \{ \}$$

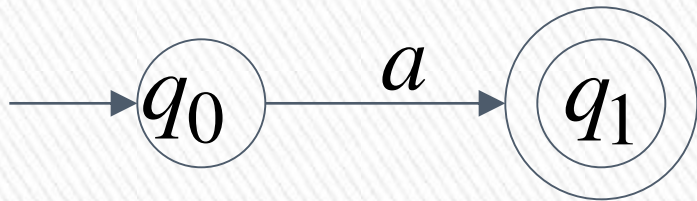


$$L(M_2) = \{ \varepsilon \}$$

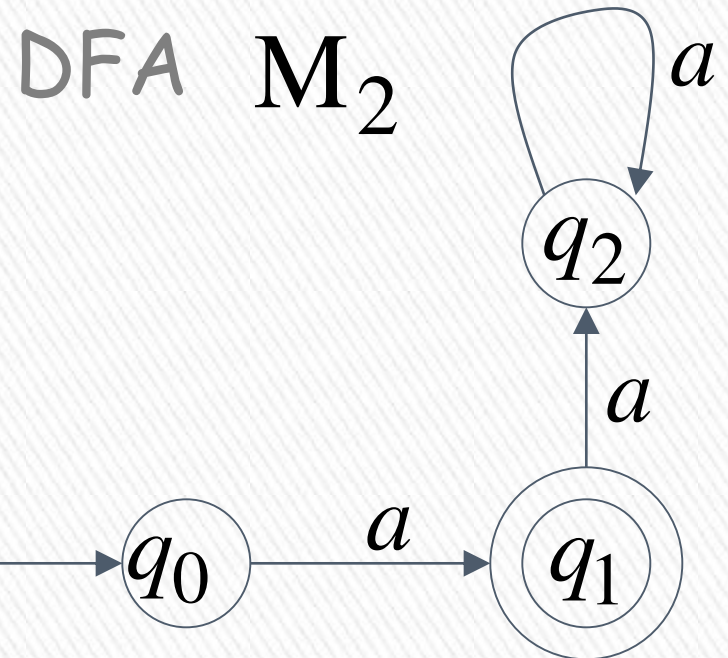


- NFAs are interesting because we can express languages easier than DFAs

NFA  $M_1$



$$L(M_1) = \{a\}$$

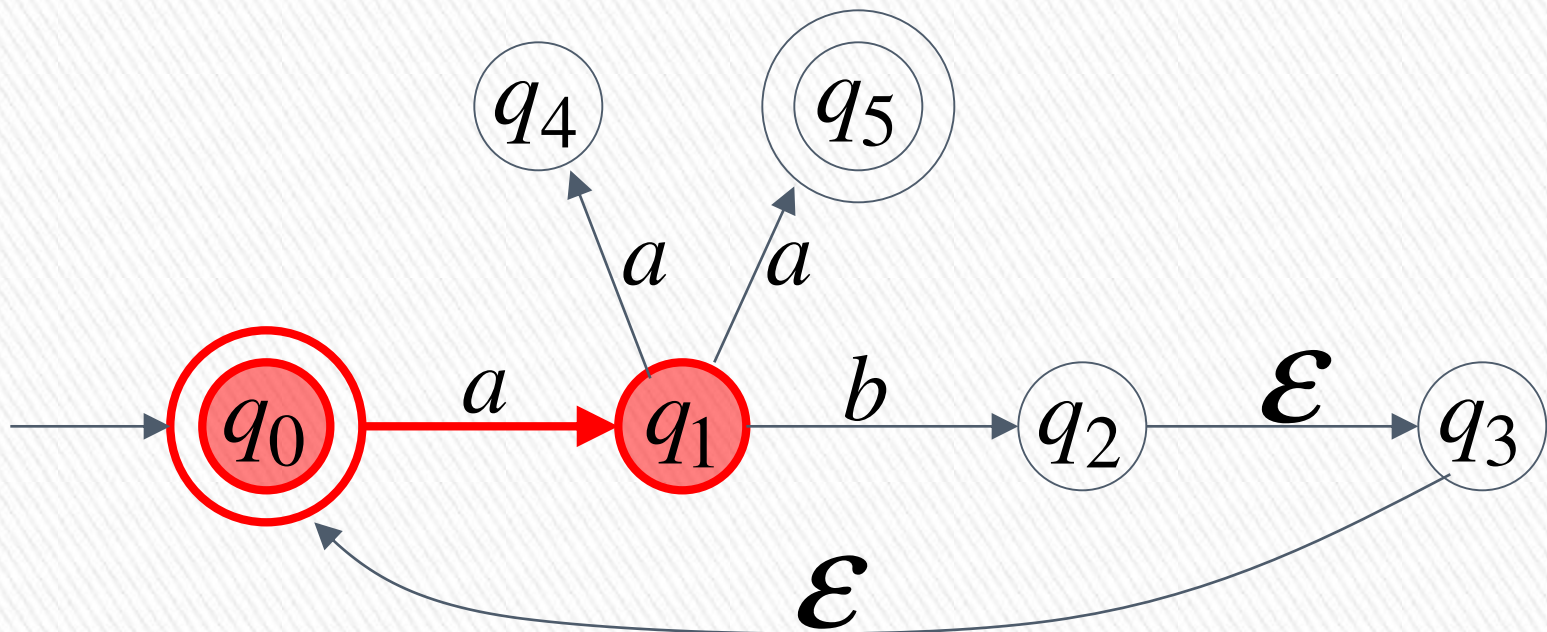


$$L(M_2) = \{a\}$$

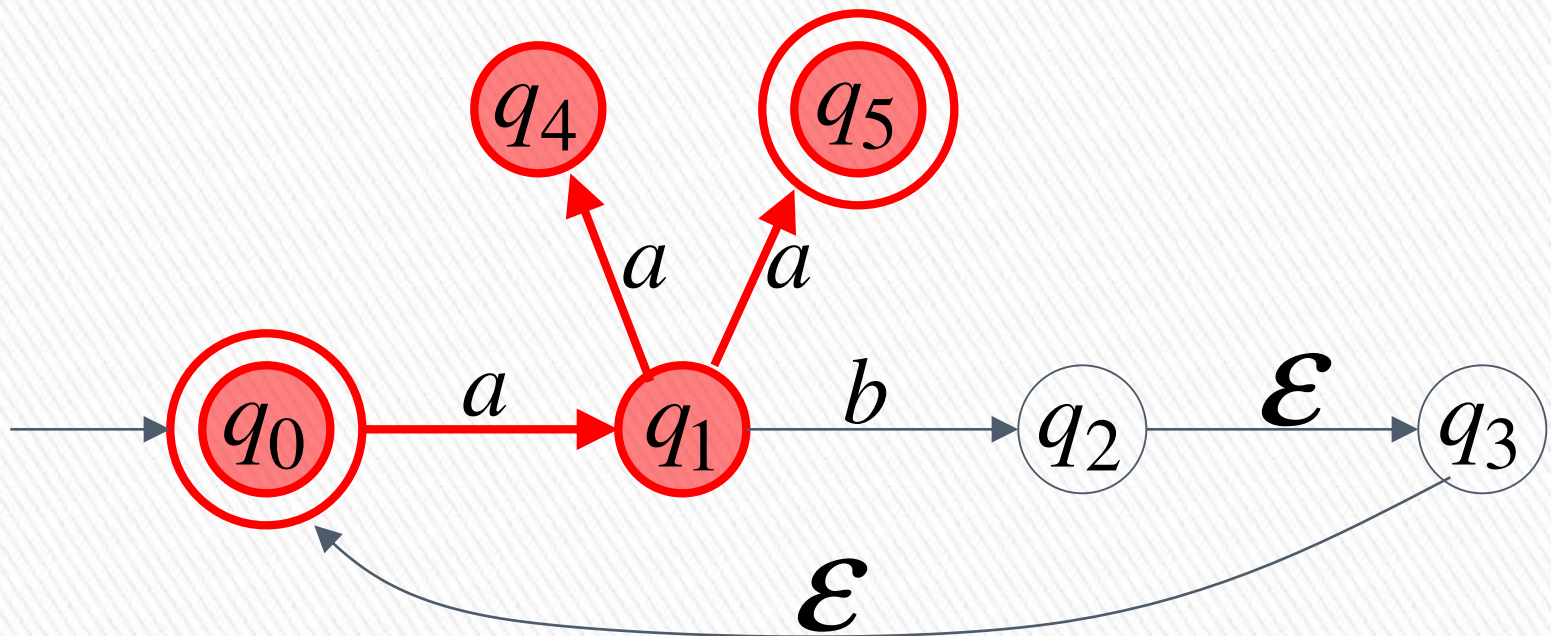
# Extended Transition Function $\delta^*$

Same with  $\delta$  but applied on strings

$$\delta^*(q_0, a) = \{q_1\}$$

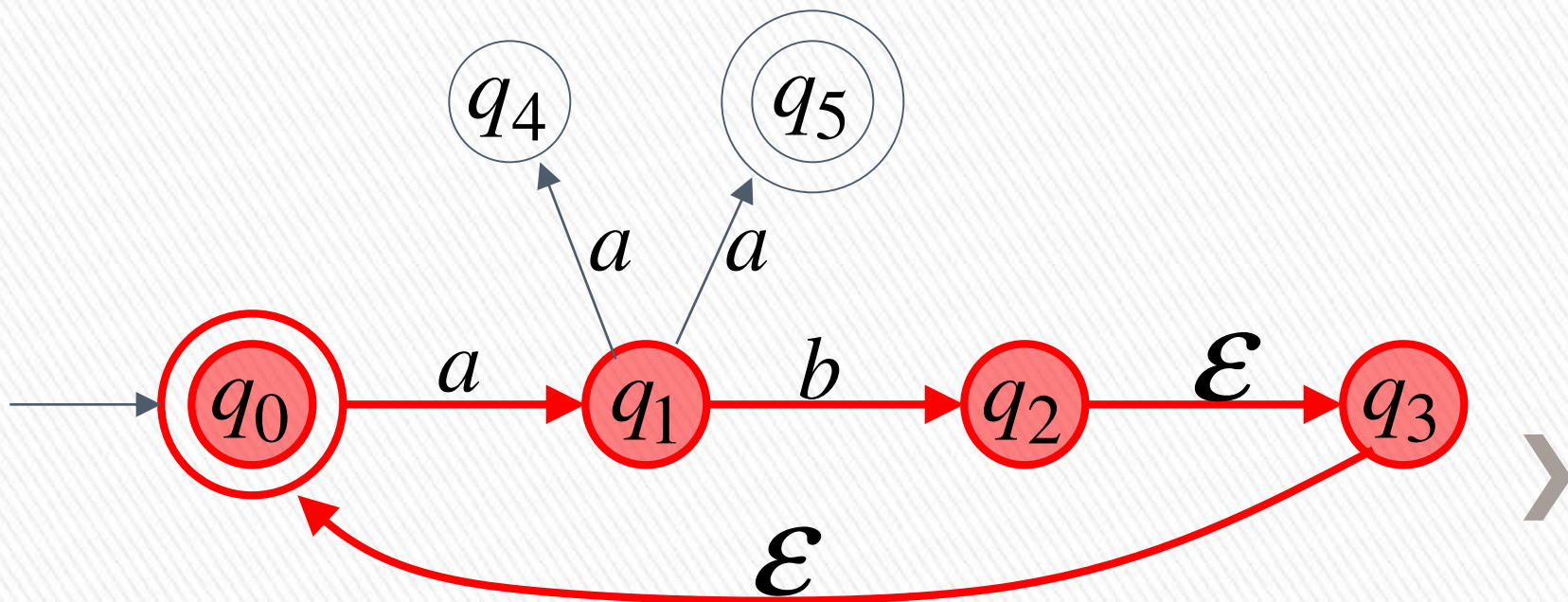


$$\delta^*(q_0, aa) = \{q_4, q_5\}$$





$$\delta^*(q_0, ab) = \{q_2, q_3, q_0\}$$



In general

$q_j \in \delta^*(q_i, w)$  : there is a walk from  $q_i$  to  $q_j$   
with label  $w$



$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$



# The Language of an NFA

» The language accepted by  $M$  is:

$$L(M) = \{w_1, w_2, \dots, w_n\}$$

» where

$$\delta^*(q_0, w_m) = \{q_i, \dots, q_k, \dots, q_j\}$$

» and there is some  $q_k \in F$

(accepting state)

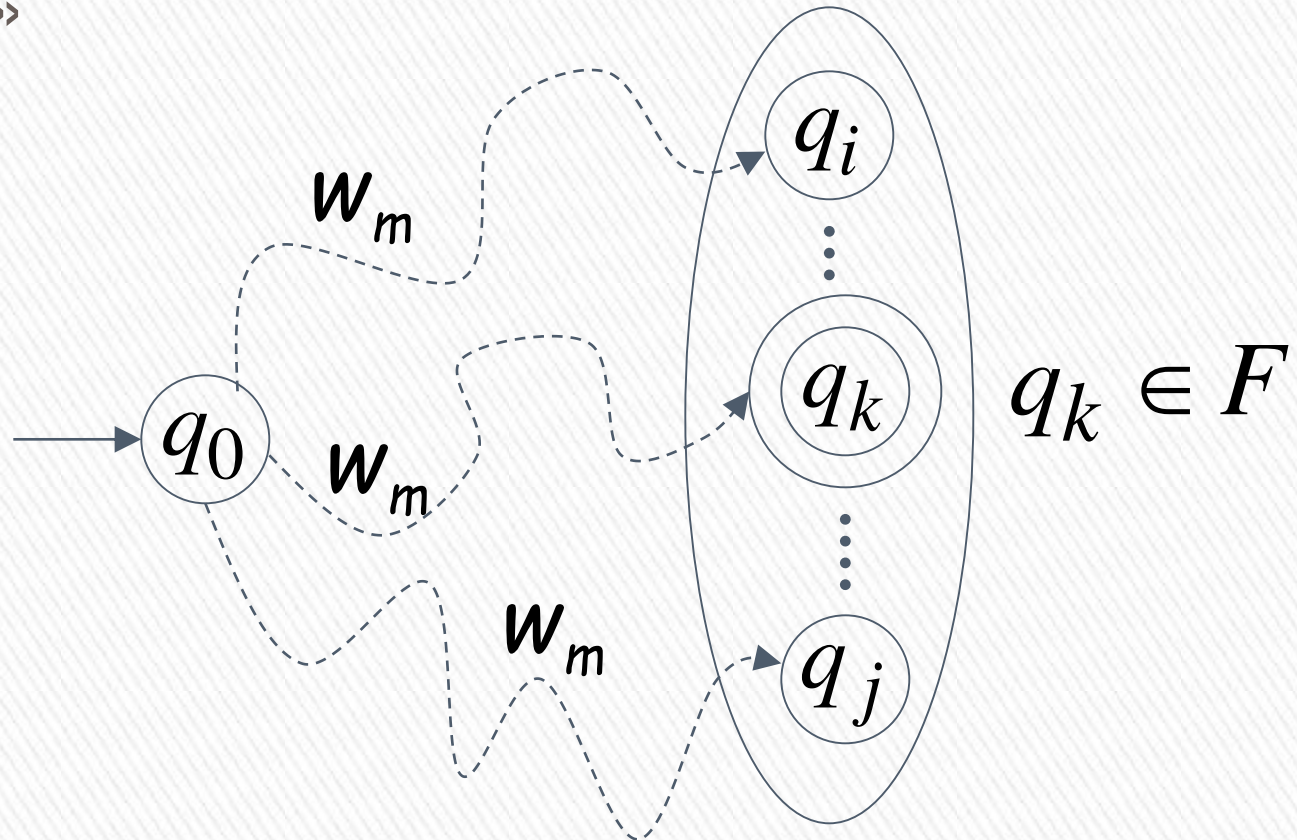


$$w_m \in L(M)$$

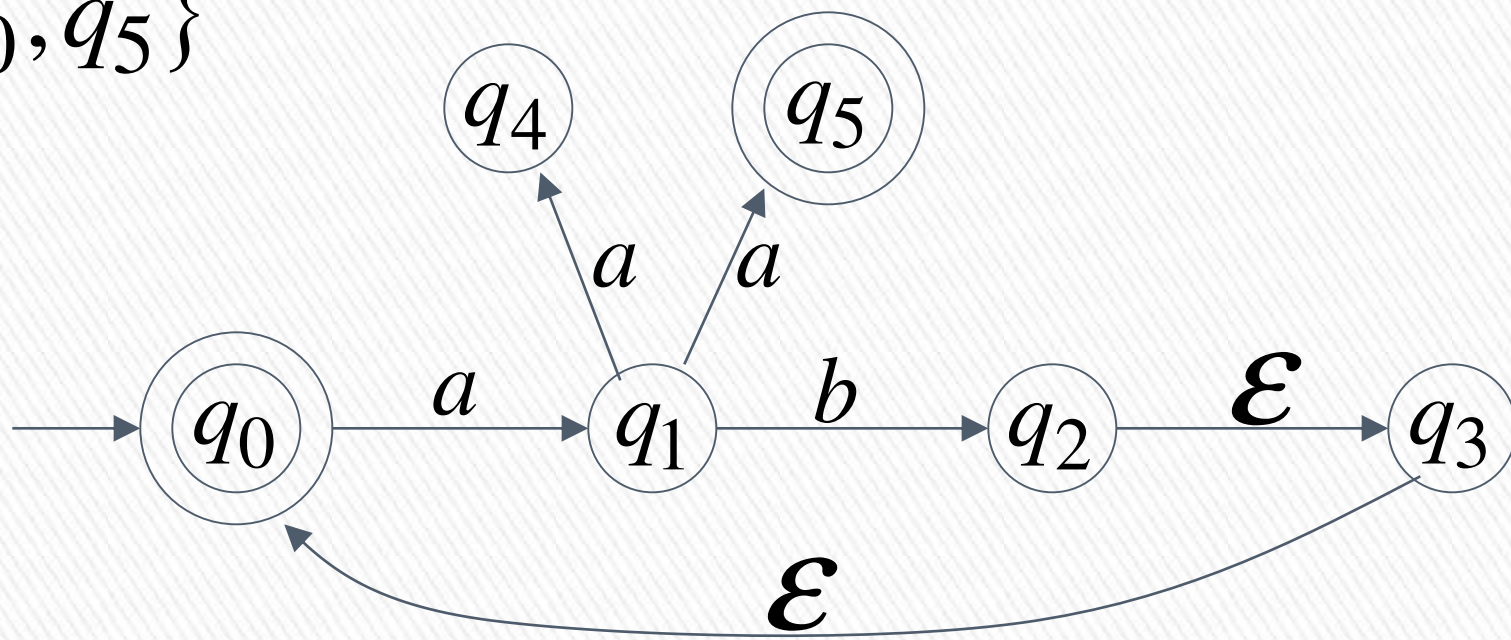
>>

>>

$$\delta^*(q_0, w_m)$$



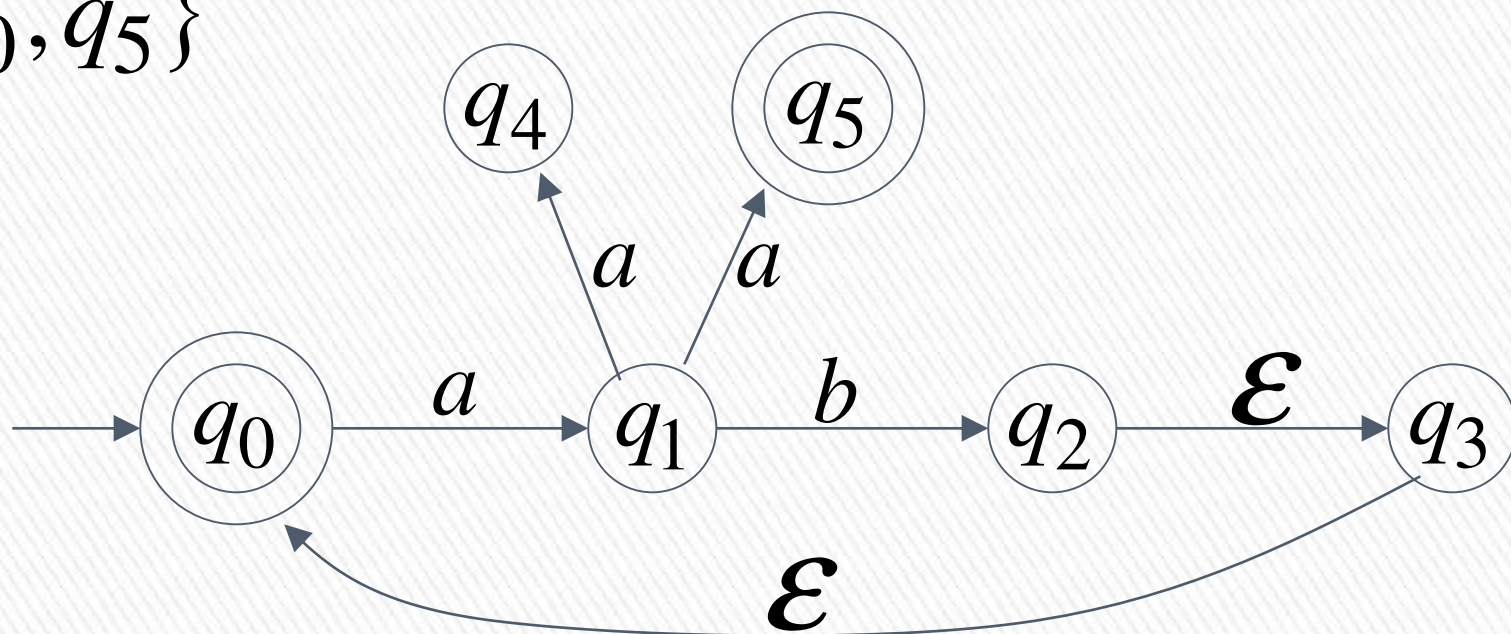
$$F = \{q_0, q_5\}$$



$$\delta^*(q_0, aa) = \{q_4, \underline{q_5}\} \xrightarrow{\quad} aa \in L(M)$$

$\swarrow \in F$

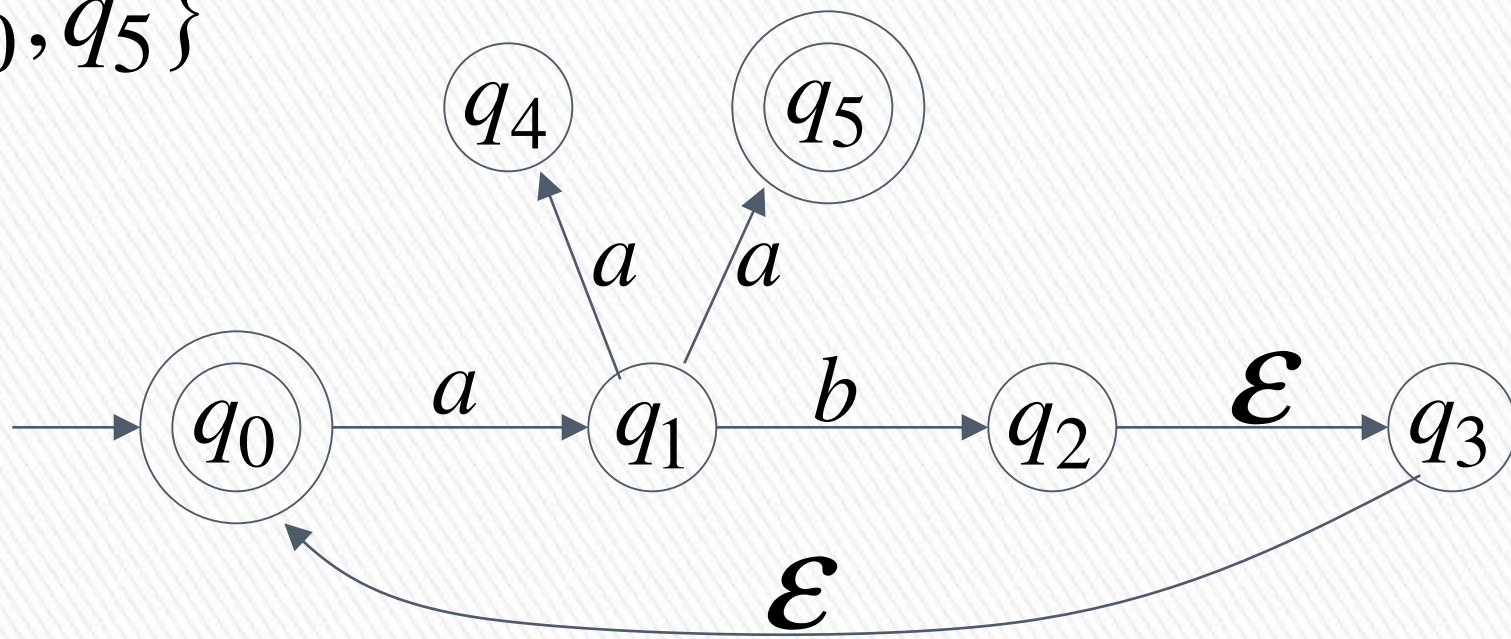
$$F = \{q_0, q_5\}$$



$$\delta^*(q_0, ab) = \{q_2, q_3, \underline{q_0}\} \xrightarrow{\quad} ab \in L(M) \gg$$

$\swarrow$   
 $\in F$

$$F = \{q_0, q_5\}$$

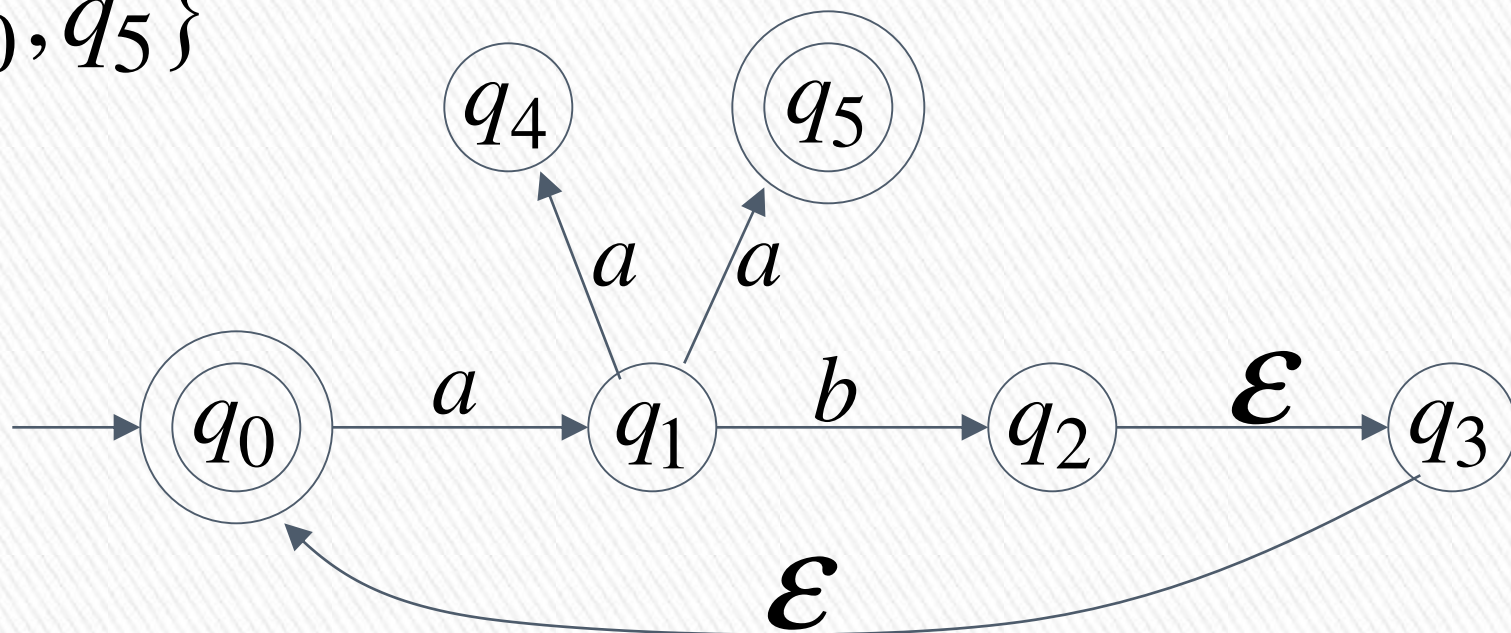


$$\delta^*(q_0, abaa) = \{q_4, \underline{q_5}\} \quad \xrightarrow{\text{yellow arrow}} \quad abaa \in L(M)$$

$\nwarrow \in F$ 
}

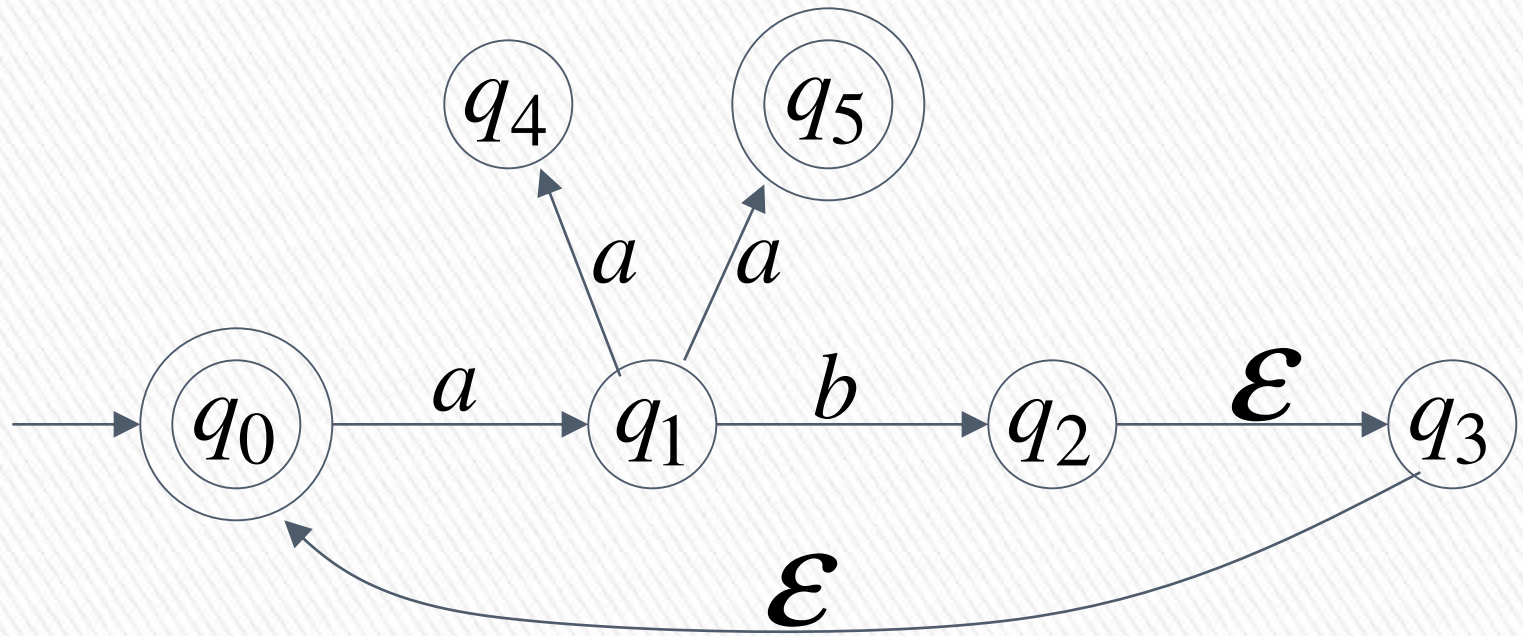


$$F = \{q_0, q_5\}$$



$$\delta^*(q_0, aba) = \{q_1\} \xrightarrow{\text{yellow arrow}} aba \notin L(M) \succ$$

$\nwarrow$   
 $\notin F$



$$L(M) = \{ab\}^* \cup \{ab\}^* \{aa\}$$



# Equivalence of Machines

» Definition:

» Machine  $M_1$  is equivalent to machine  $M_2$

if  $L(M_1) = L(M_2)$

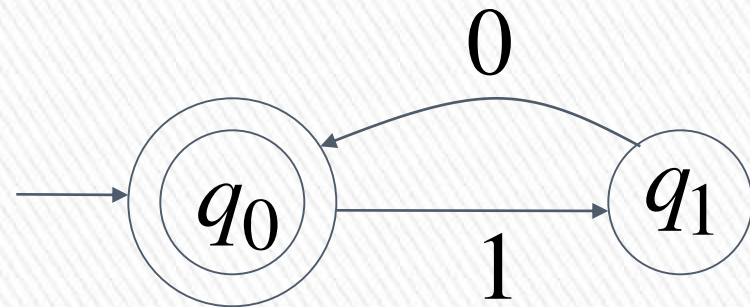


# Example of equivalent machines

»

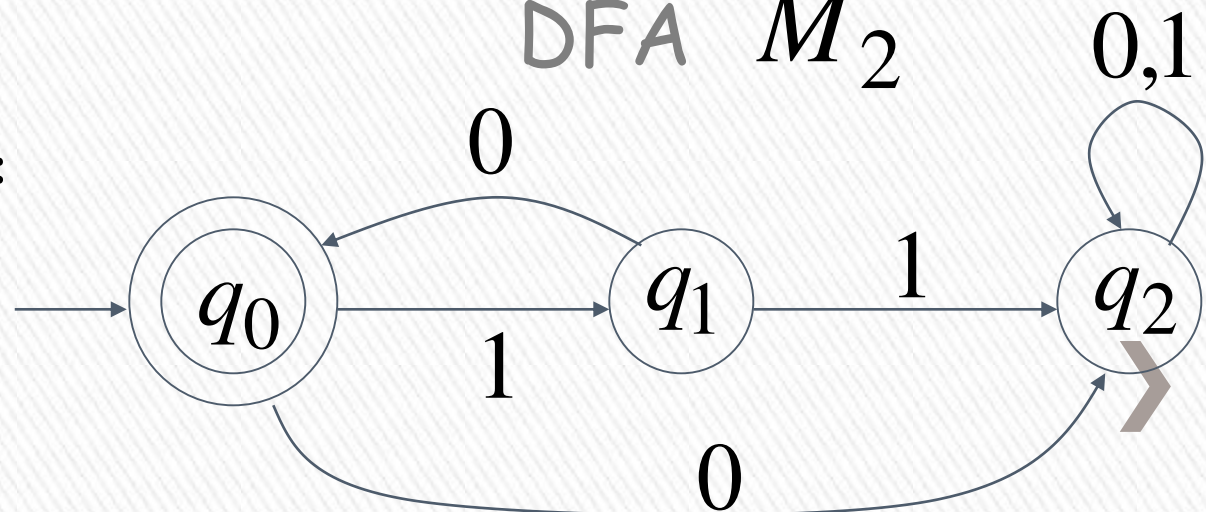
$$L(M_1) = \{10\}^*$$

NFA  $M_1$



$$L(M_2) = \{10\}^*$$

DFA  $M_2$



# NFAs accept Regular Languages

## Theorem:

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} = \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Languages Accepted  
by DFAs

NFAs and DFAs have the same computation power,  
accept the same set of languages

**Proof:** we only need to show

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

AND

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$





## Proof-Step 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Every DFA is trivially an NFA



Any language  $L$  accepted by a DFA  
is also accepted by an NFA





## Proof-Step 2

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Any NFA can be converted to an equivalent DFA



Any language  $L$  accepted by an NFA is also accepted by a DFA 

## Lemma:

If we convert NFA  $M$  to DFA  $M'$   
then the two automata are equivalent:

$$L(M) = L(M')$$

## Proof:

We only need to show:  $L(M) \subseteq L(M')$

AND

$$L(M) \supseteq L(M') \quad \rangle$$

First we show:  $L(M) \subseteq L(M')$

We only need to prove:

$$w \in L(M) \quad \longrightarrow \quad w \in L(M')$$



# NFA

Consider  $w \in L(M)$



symbols

$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$



symbol



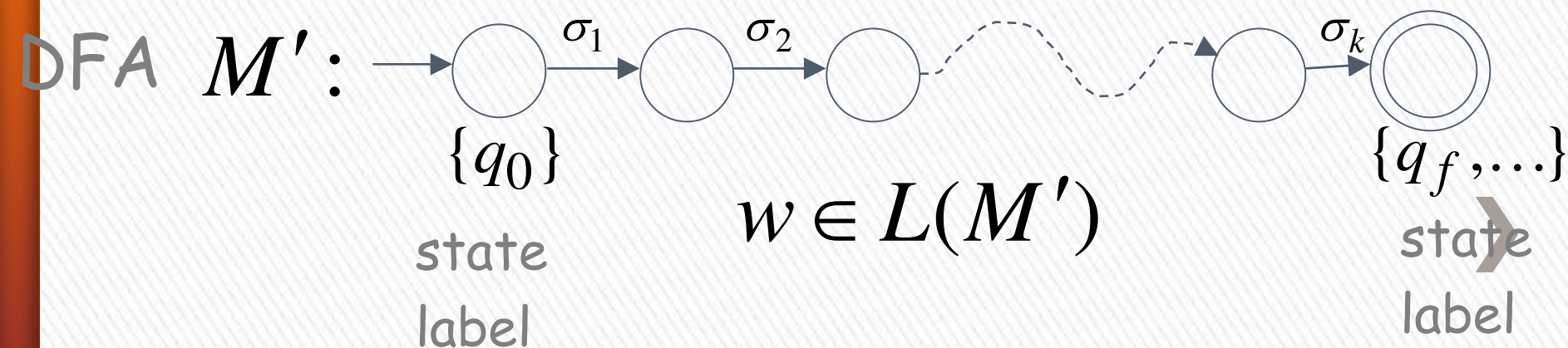
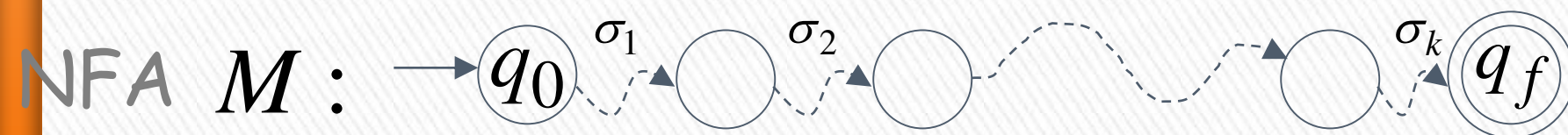
denotes a possible sub-path like

symbol



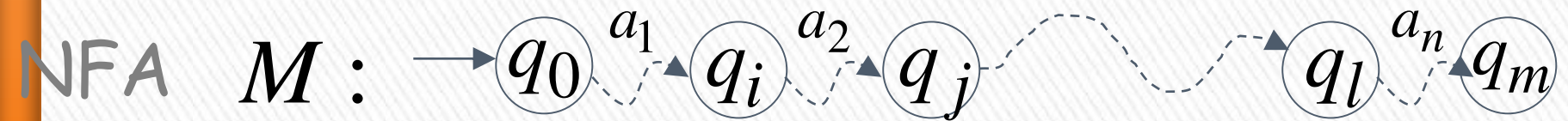
We will show that if  $w \in L(M)$

$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$

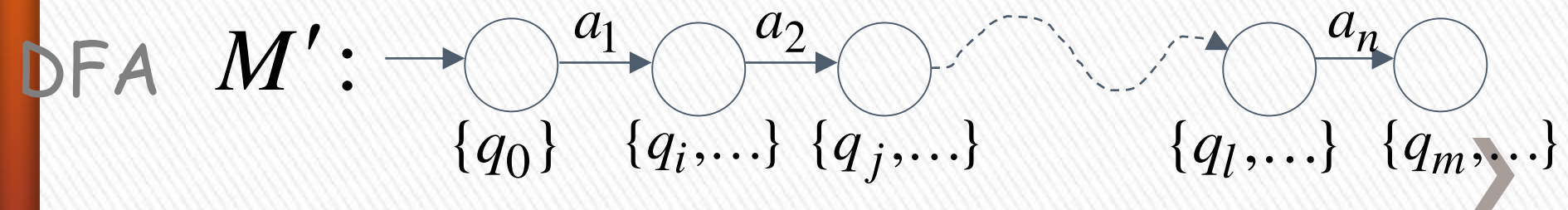


More generally, we will show that if in  $M$ :

(arbitrary string)  $v = a_1 a_2 \cdots a_n$



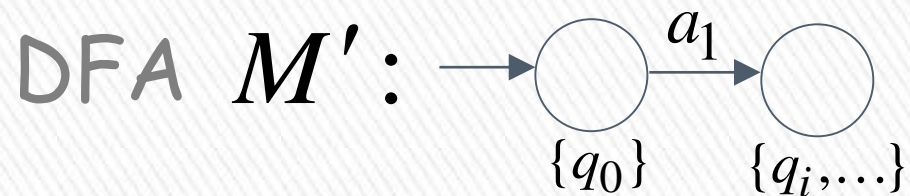
then





# Proof by induction on $|v|$

Induction Basis:  $|v| = 1$        $v = a_1$



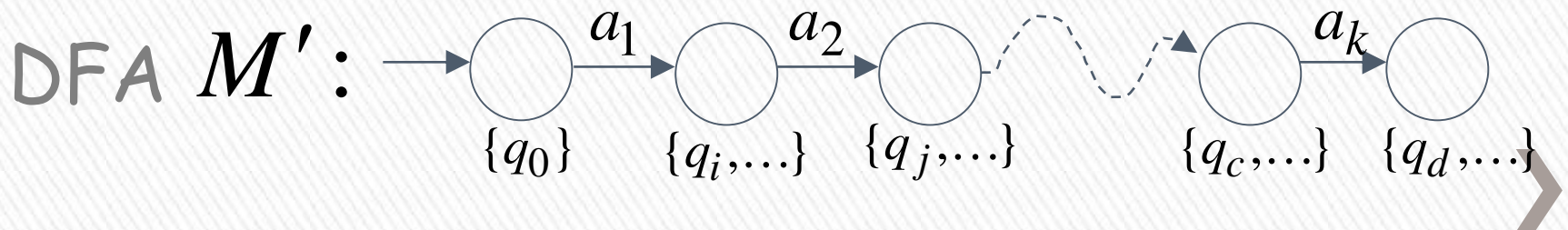
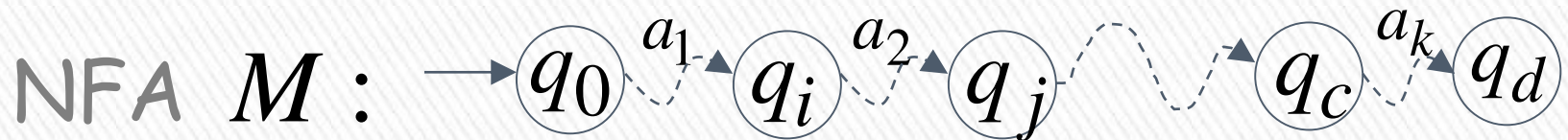
is true by construction of  $M'$



Induction hypothesis:  $1 \leq |v| \leq k$

$$v = a_1 a_2 \cdots a_k$$

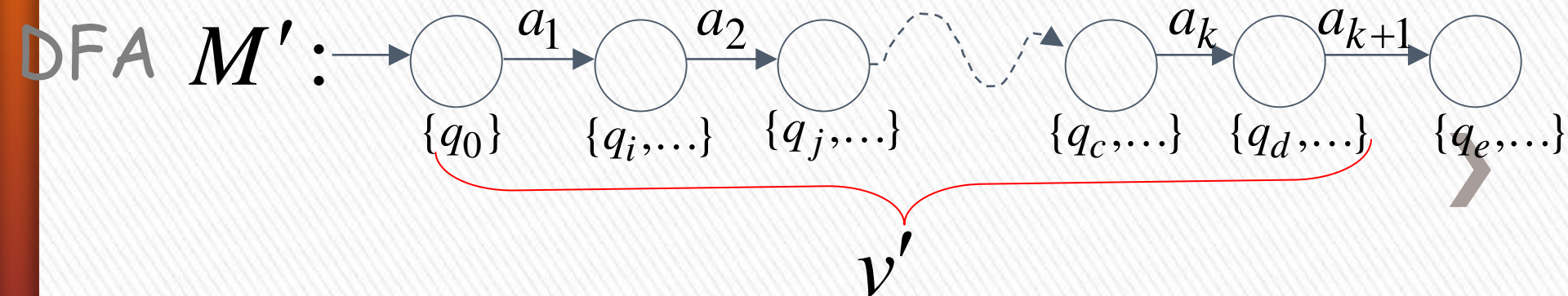
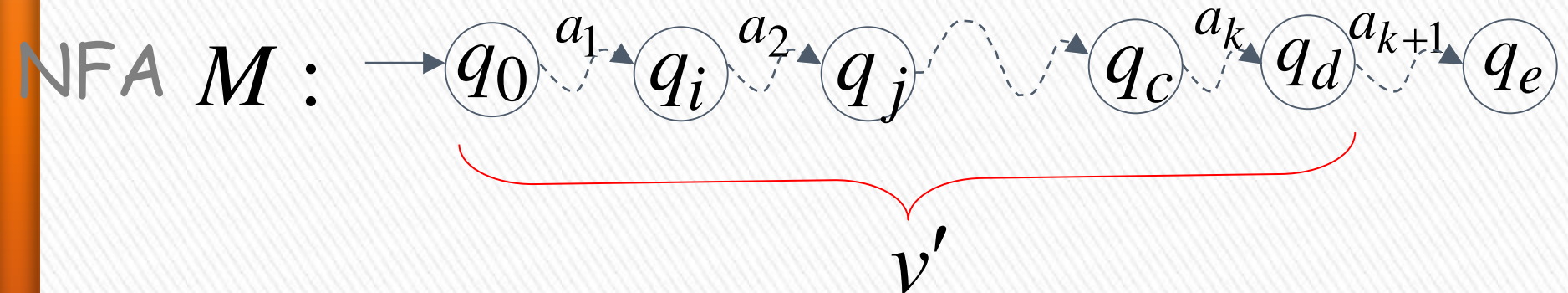
Suppose that the following hold



Induction Step:  $|v| = k + 1$

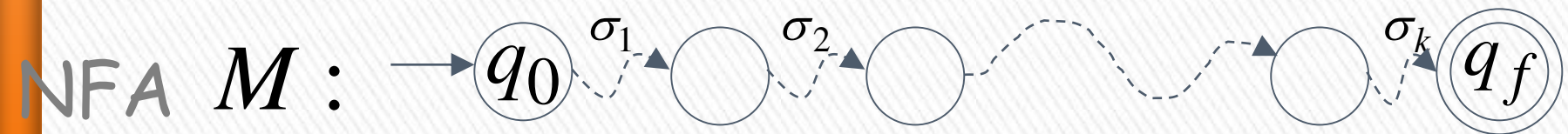
$$v = \underbrace{a_1 a_2 \cdots a_k}_{v'} a_{k+1} = v' a_{k+1}$$

Then this is true by construction of  $M'$

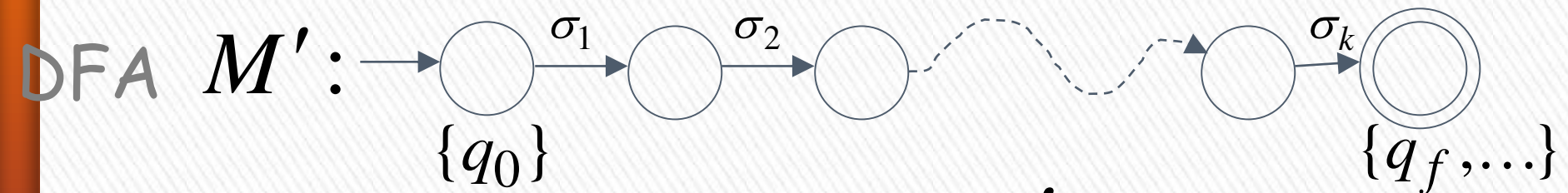


Therefore if  $w \in L(M)$

$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$



then



$$w \in L(M')$$



We have shown:  $L(M) \subseteq L(M')$

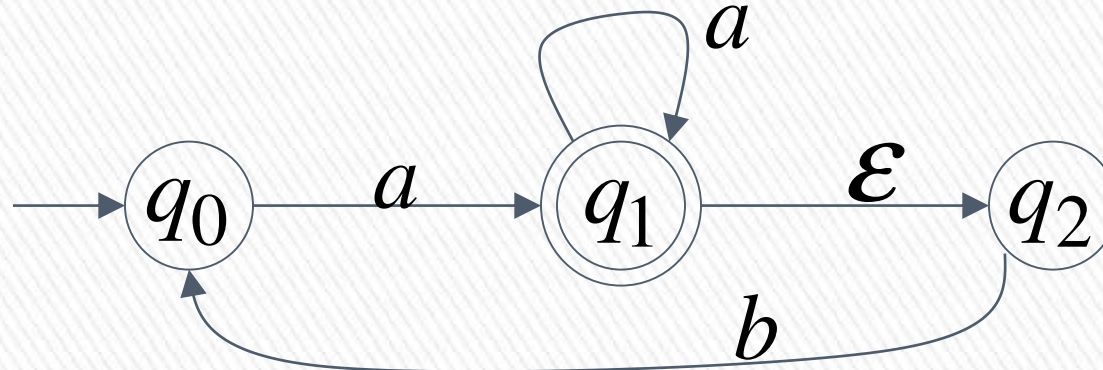
With a similar proof  
we can show:  $L(M) \supseteq L(M')$

Therefore:  $L(M) = L(M')$

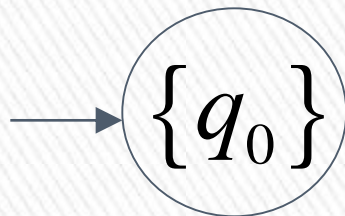


END OF LEMMA PROOF

# Conversion NFA to DFA



NFA  $M$

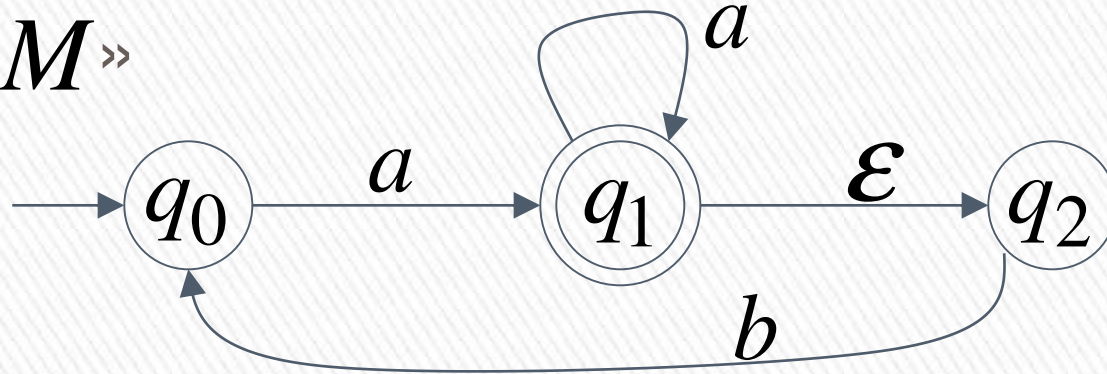


DFA  $M'$



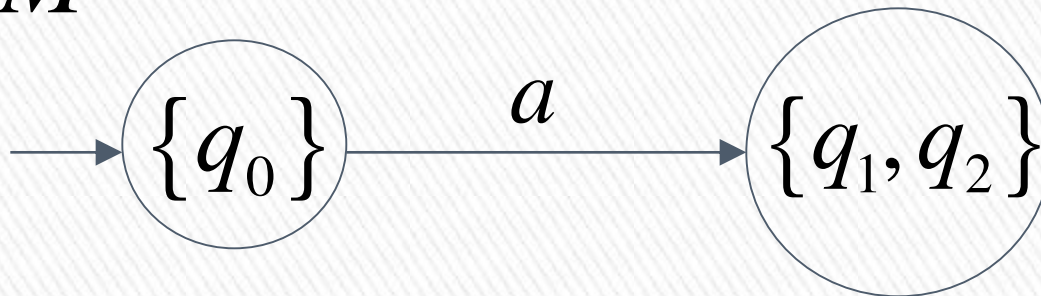


NFA  $M \gg$



$$\delta^*(q_0, a) = \{q_1, q_2\}$$

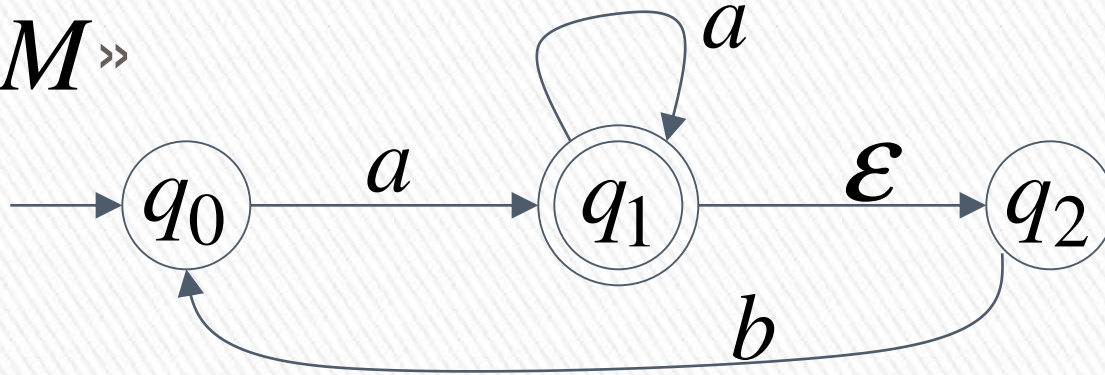
DFA  $M'$



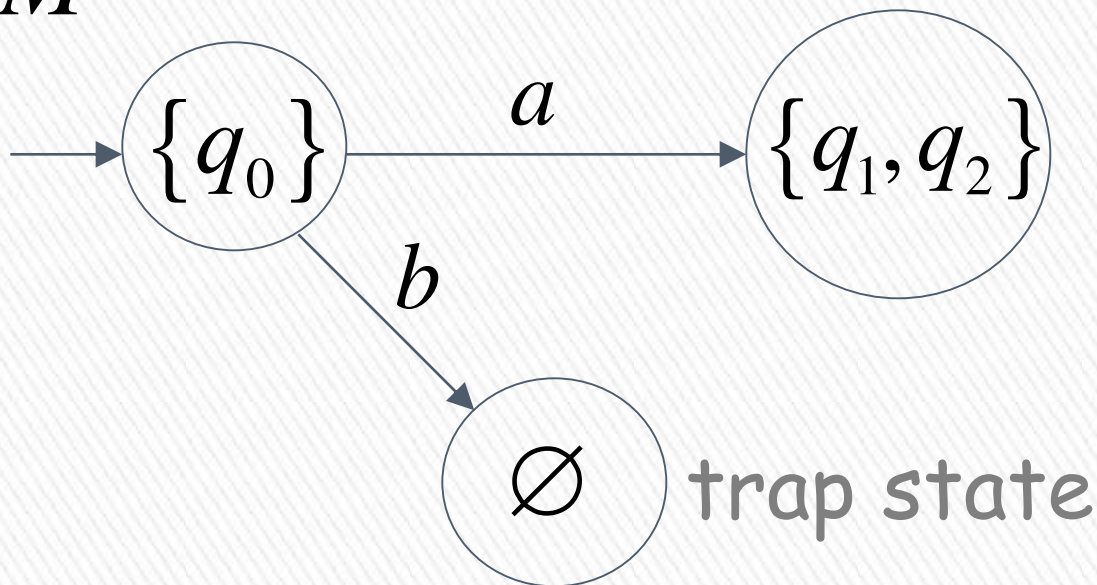


$$\delta^*(q_0, b) = \emptyset \quad \text{empty set}$$

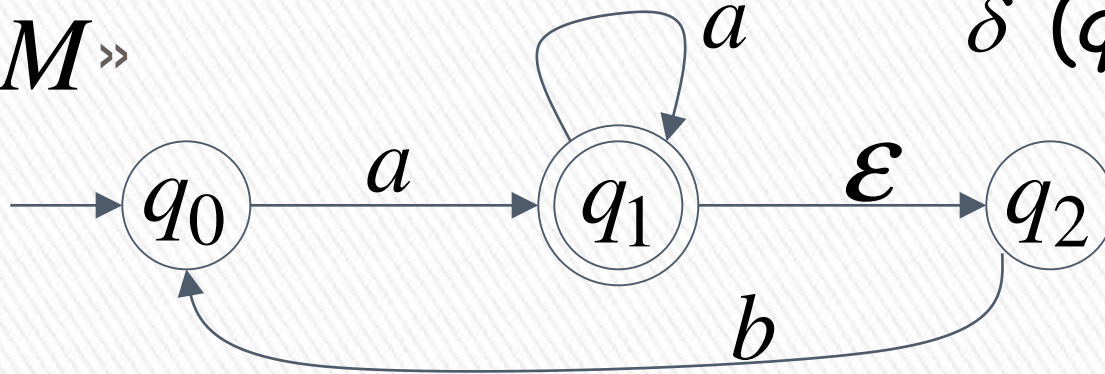
NFA  $M$



DFA  $M'$



NFA  $M \gg$



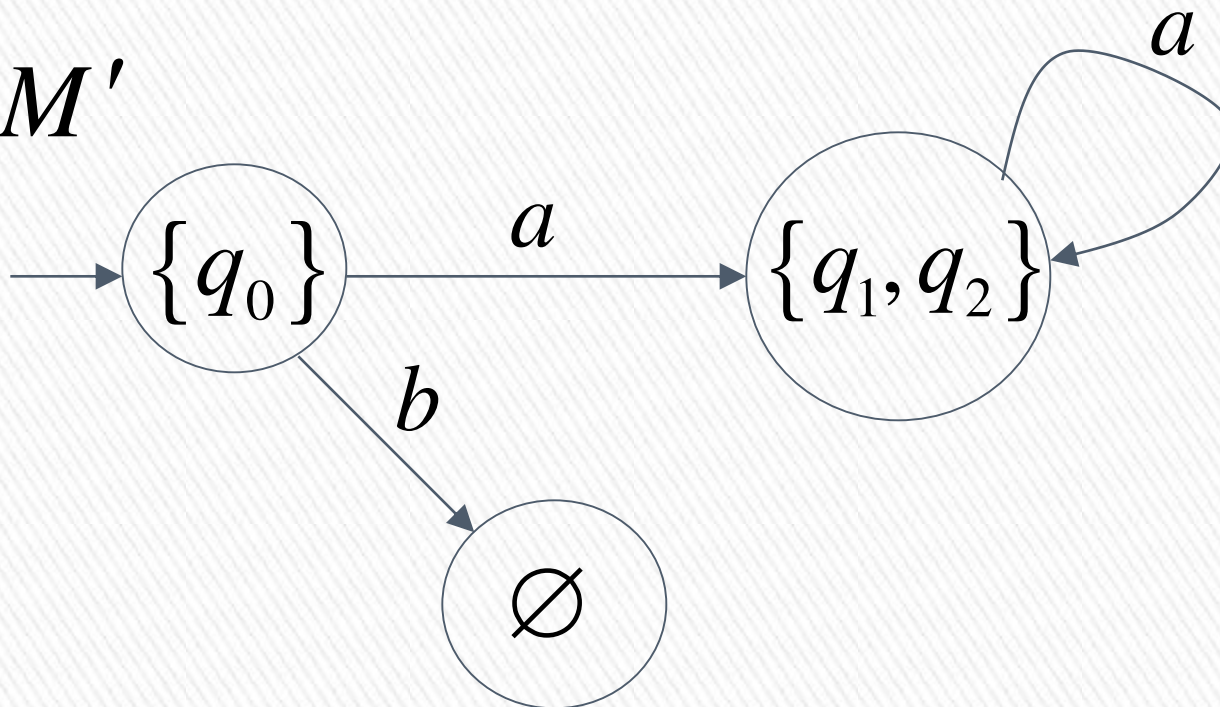
$$\delta^*(q_1, a) = \{q_1, q_2\}$$

$$\delta^*(q_2, a) = \emptyset$$

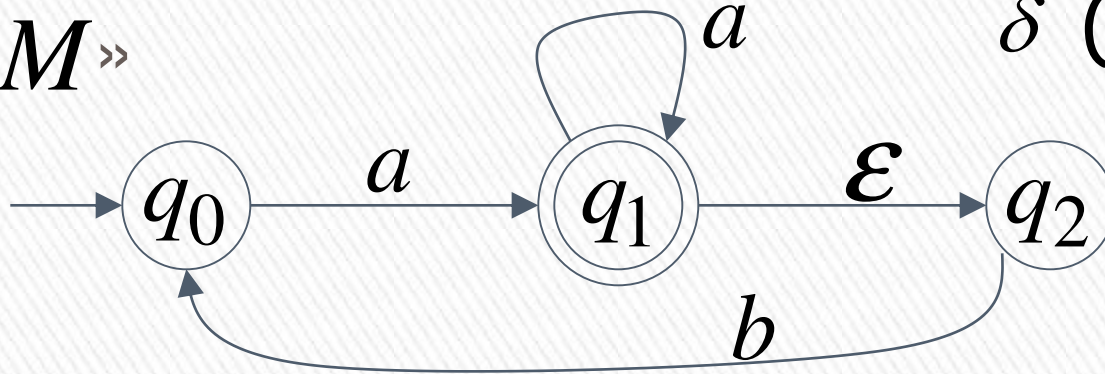
union

$$\{q_1, q_2\}$$

DFA  $M'$



NFA  $M \gg$



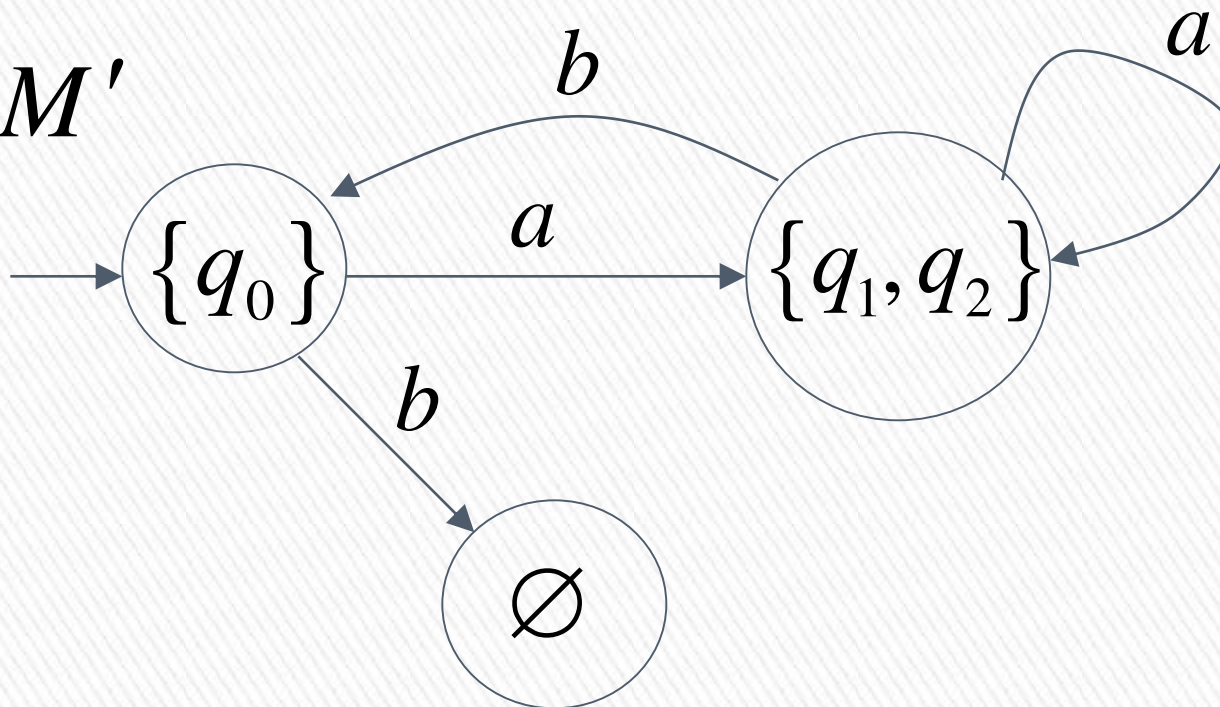
$$\delta^*(q_1, b) = \{q_0\}$$

$$\delta^*(q_2, b) = \{q_0\}$$

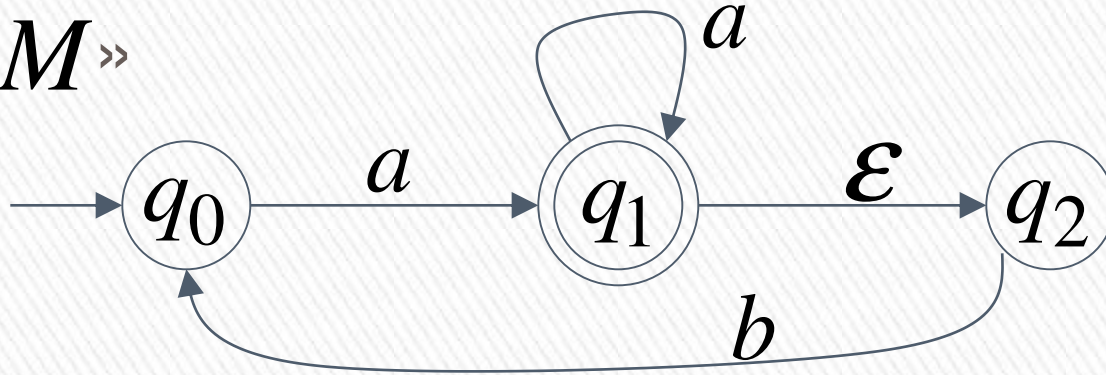
union

$\{q_0\}$

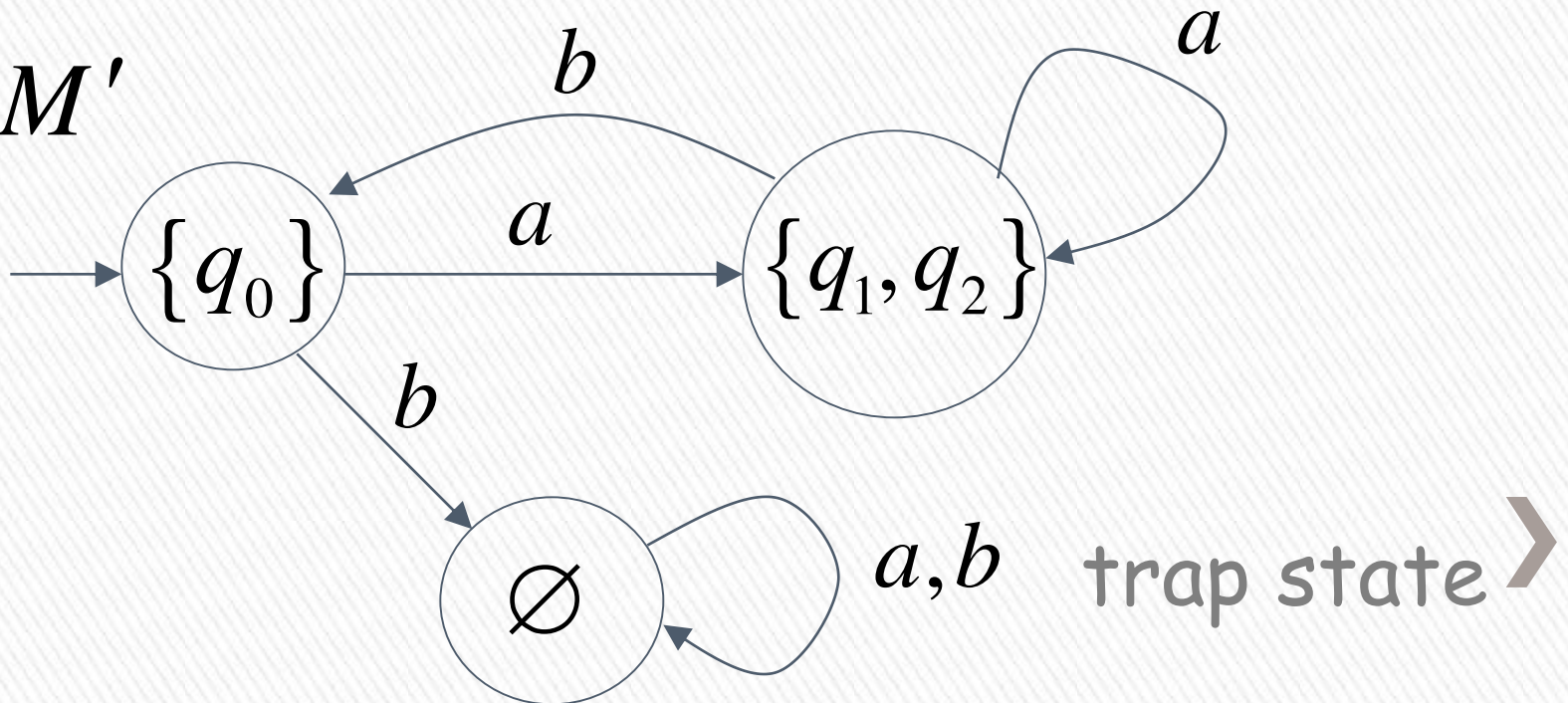
DFA  $M'$



NFA  $M_{\gg}$

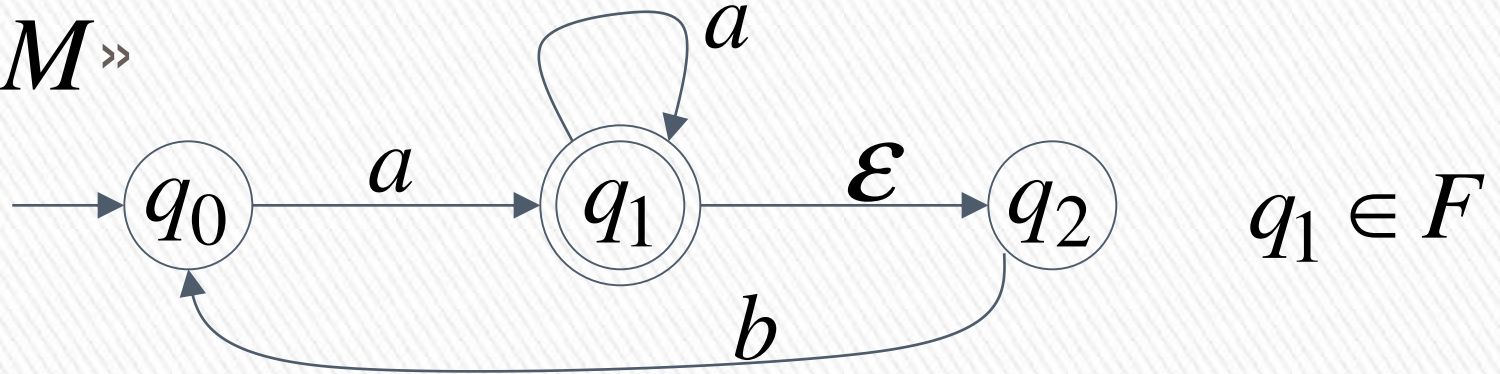


DFA  $M'$

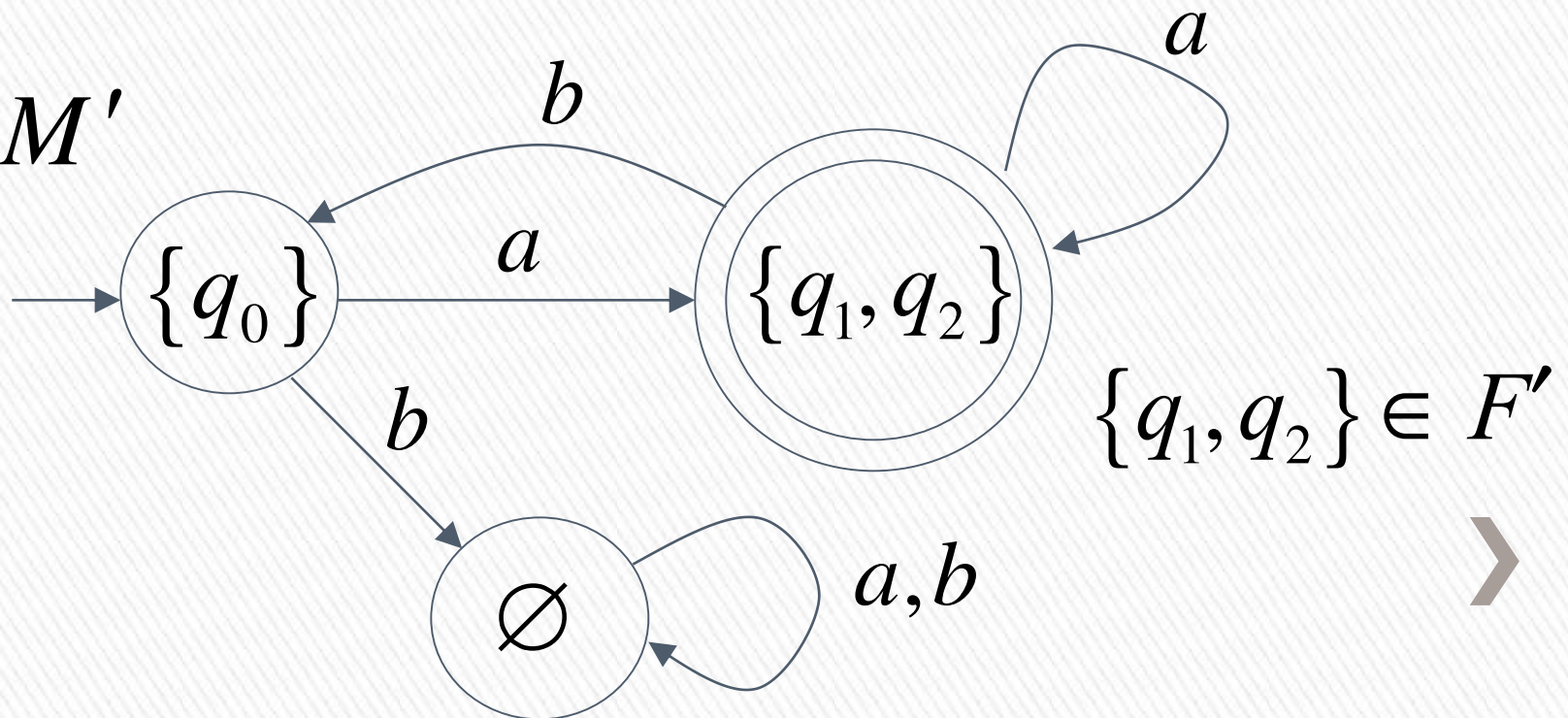


# END OF CONSTRUCTION

NFA  $M_{\gg}$



DFA  $M'$



# General Conversion Procedure

» Input: an NFA  $M$

» Output: an equivalent DFA  $M'$   
with  $L(M) = L(M')$





» The NFA has states

$$q_0, q_1, q_2, \dots$$

» The DFA has states from the power set

$$\emptyset, \{q_0\}, \{q_1\}, \{q_0, q_1\}, \{q_1, q_2, q_3\}, \dots$$





# Conversion Procedure Steps

»

## Step 1

» Initial state of NFA:  $q_0$

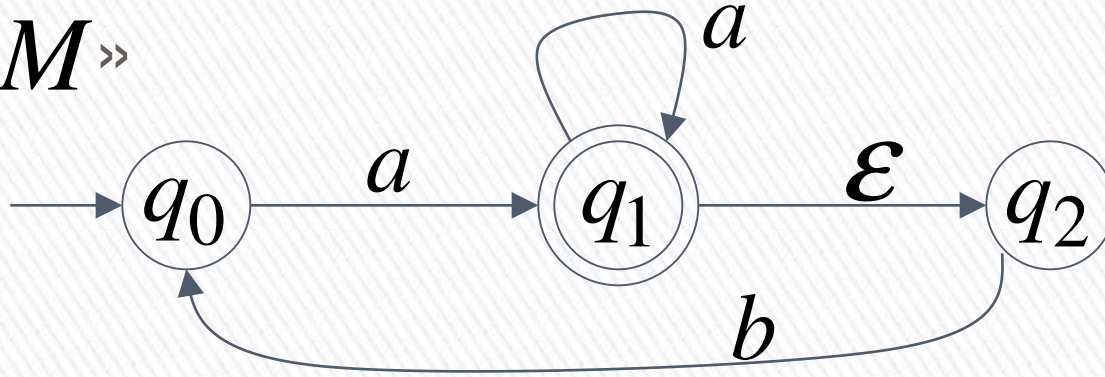


» Initial state of DFA:  $\{q_0\}$

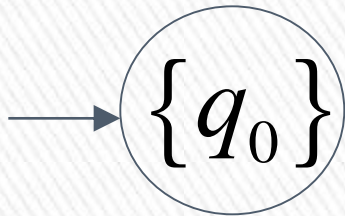


# Example

NFA  $M$



DFA  $M'$



## Step 2

For every DFA's state  $\{q_i, q_j, \dots, q_m\}$

compute in the NFA

$$\left. \begin{array}{l} \delta^*(q_i, a) \\ \cup \delta^*(q_j, a) \\ \dots \\ \cup \delta^*(q_m, a) \end{array} \right\} = \text{Union } \{q'_k, q'_l, \dots, q'_n\}$$

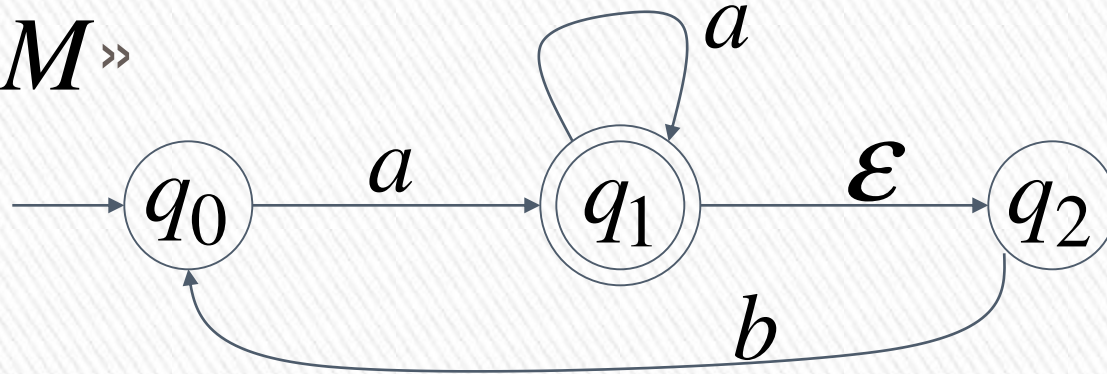
add transition to DFA

$$\delta(\{q_i, q_j, \dots, q_m\}, a) = \{q'_k, q'_l, \dots, q'_n\}$$

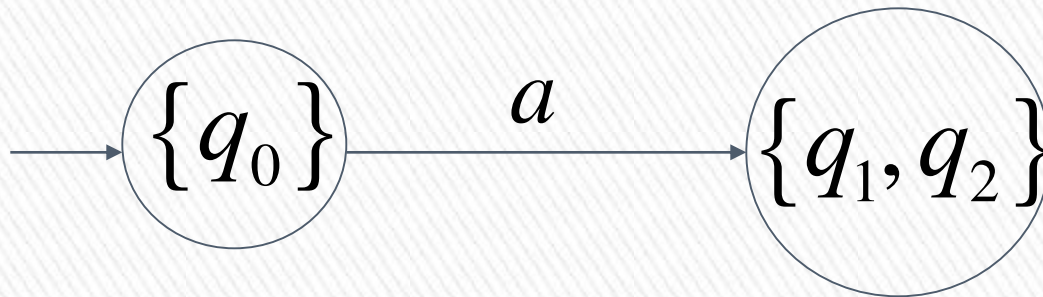


**Example**  $\delta^*(q_0, a) = \{q_1, q_2\}$

**NFA**  $M \gg$



**DFA**  $M'$   $\delta(\{q_0\}, a) = \{q_1, q_2\}$



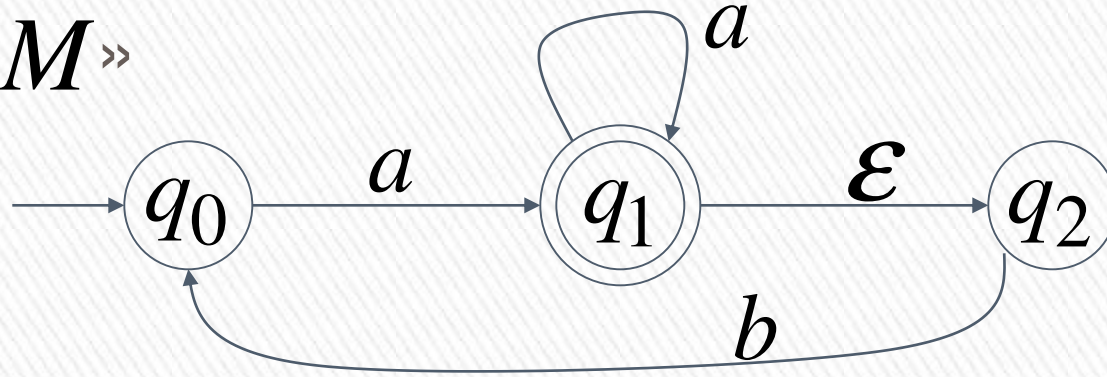
## Step 3

- » Repeat Step 2 for every state in DFA and symbols in alphabet until no more states can be added in the DFA

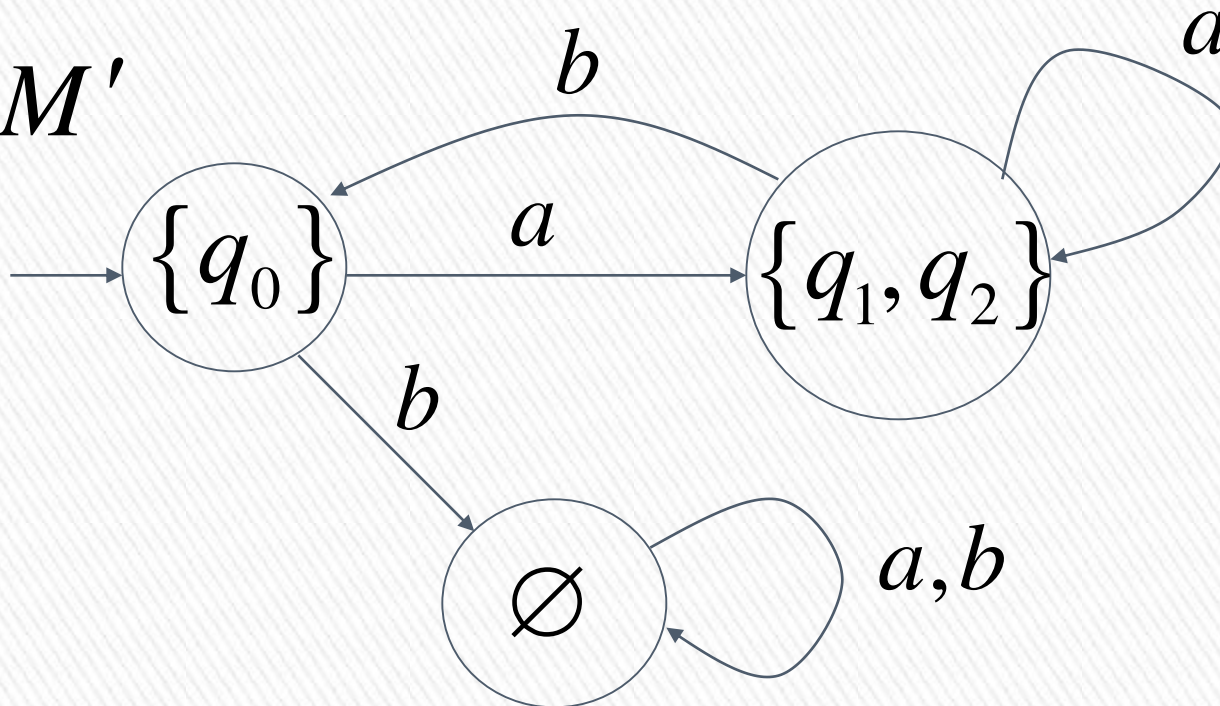


# Example

NFA  $M$



DFA  $M'$



## Step 4

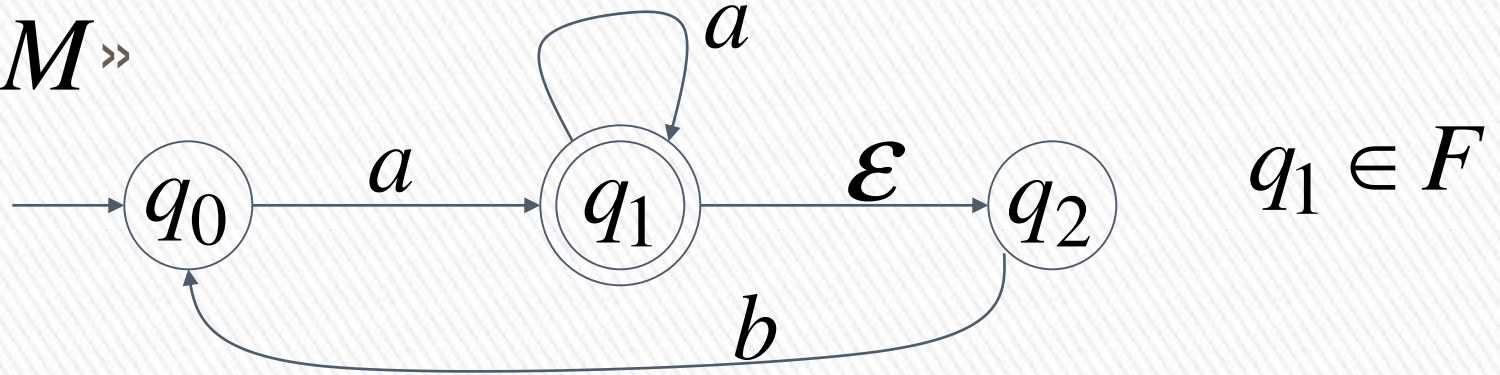
- » For any DFA state  $\{q_i, q_j, \dots, q_m\}$
- » if some  $q_j$  is accepting state in NFA
- » Then,  $\{q_i, q_j, \dots, q_m\}$   
is accepting state in DFA





# Example

NFA  $M \gg$



DFA  $M'$

