

Taylor's Series of $\sin x$

In order to use Taylor's formula to find the power series expansion of $\sin x$ we have to compute the derivatives of $\sin(x)$:

$$\begin{aligned}\sin'(x) &= \cos(x) \\ \sin''(x) &= -\sin(x) \\ \sin'''(x) &= -\cos(x) \\ \sin^{(4)}(x) &= \sin(x).\end{aligned}$$

Since $\sin^{(4)}(x) = \sin(x)$, this pattern will repeat.

Next we need to evaluate the function and its derivatives at 0:

$$\begin{aligned}\sin(0) &= 0 \\ \sin'(0) &= 1 \\ \sin''(0) &= 0 \\ \sin'''(0) &= -1 \\ \sin^{(4)}(0) &= 0.\end{aligned}$$

Again, the pattern repeats.

Taylor's formula now tells us that:

$$\begin{aligned}\sin(x) &= 0 + 1x + 0x^2 + \frac{-1}{3!}x^3 + 0x^4 + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\end{aligned}$$

Notice that the signs alternate and the denominators get very big; factorials grow very fast.

The radius of convergence R is infinity; let's see why. The terms in this sum look like:

$$\frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1} \cdot \frac{x}{2} \cdot \frac{x}{3} \cdots \frac{x}{(2n+1)}.$$

Suppose x is some fixed number. Then as n goes to infinity, the terms on the right in the product above will be very, very small numbers and there will be more and more of them as n increases.

In other words, the terms in the series will get smaller as n gets bigger; that's an indication that x may be inside the radius of convergence. But this would be true for any fixed value of x , so the radius of convergence is infinity.

Why do we care what the power series expansion of $\sin(x)$ is? If we use enough terms of the series we can get a good estimate of the value of $\sin(x)$ for any value of x .

This is very useful information about the function $\sin(x)$ but it doesn't tell the whole story. For example, it's hard to tell from the formula that $\sin(x)$ is periodic. The period of $\sin(x)$ is 2π ; how is this series related to the number π ?

Power series are very good for some things but can also hide some properties of functions.

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