

# Reductions

Computable function  $f$  :

There is a deterministic Turing machine  $M$   
which for any input string  $w$  computes  $f(w)$   
and writes it on the tape

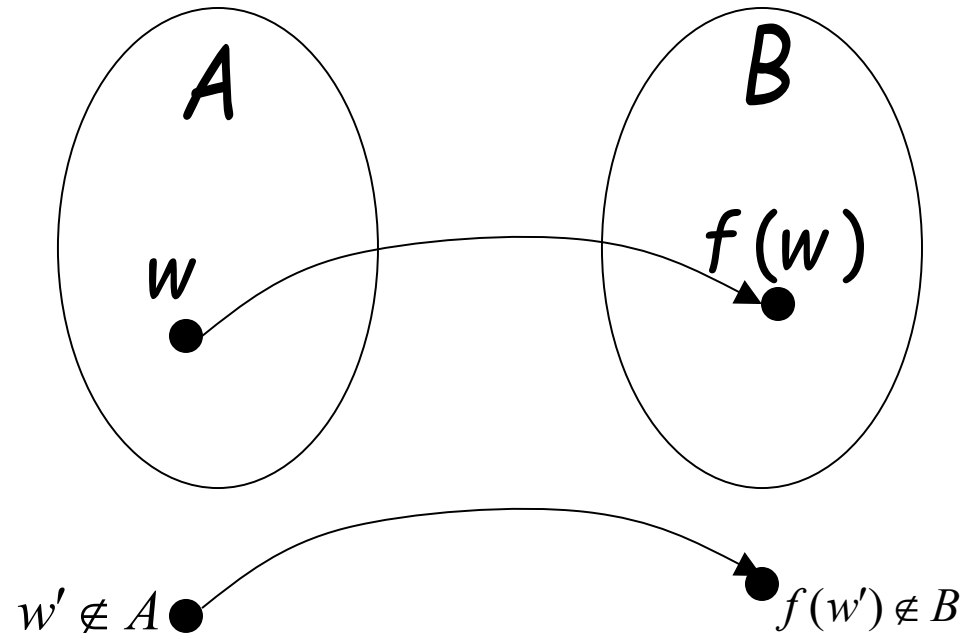
Problem  $X$  is reduced to problem  $Y$



If we can solve problem  $Y$   
then we can solve problem  $X$

## Definition:

Language  $A$   
is reduced to  
language  $B$



There is a computable  
function  $f$  (*reduction*) such that:

$$w \in A \iff f(w) \in B$$

# Theorem 1:

If: Language  $A$  is reduced to  $B$   
and language  $B$  is decidable

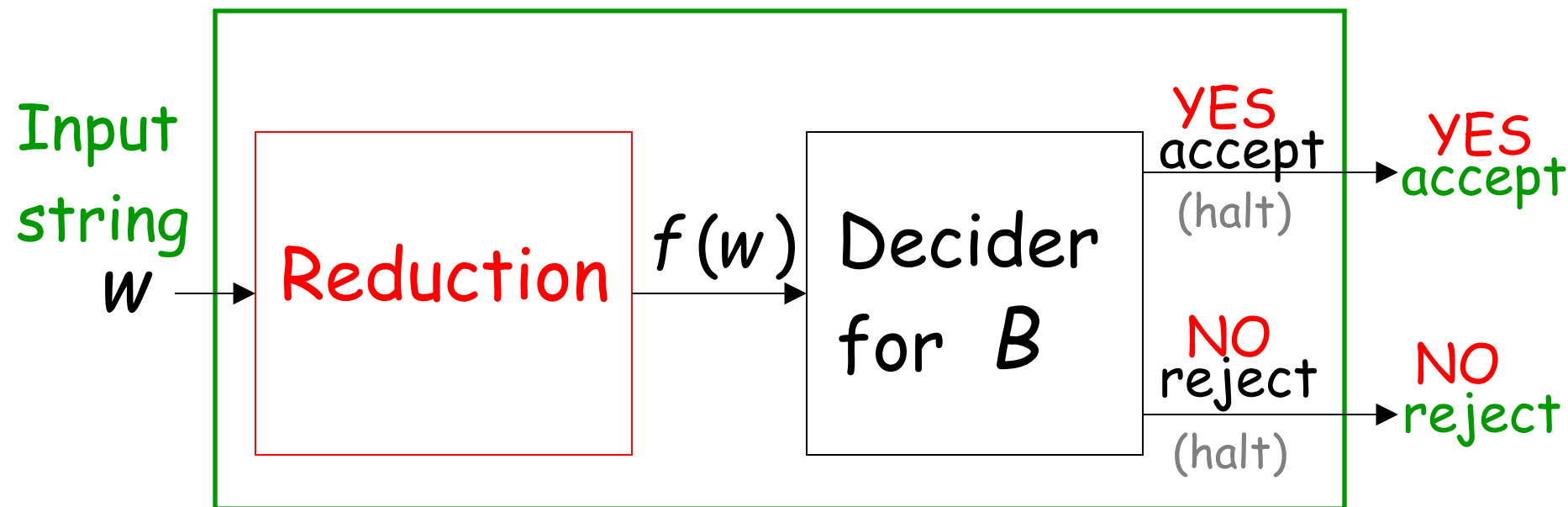
Then:  $A$  is decidable

## Proof:

Basic idea:

Build the decider for  $A$   
using the decider for  $B$

## Decider for $A$



From reduction:  $w \in A \iff f(w) \in B$

END OF PROOF

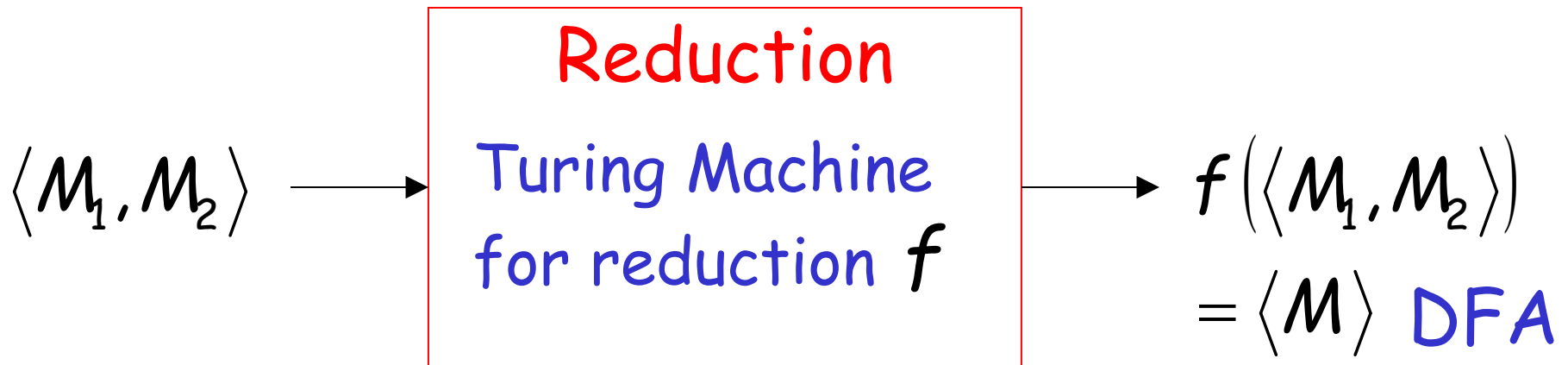
## Example:

$$EQUAL_{DFA} = \{\langle M_1, M_2 \rangle : M_1 \text{ and } M_2 \text{ are DFAs} \\ \text{that accept the same languages}\}$$

is reduced to:

$$EMPTY_{DFA} = \{\langle M \rangle : M \text{ is a DFA that accepts} \\ \text{the empty language } \emptyset\}$$

We only need to construct:

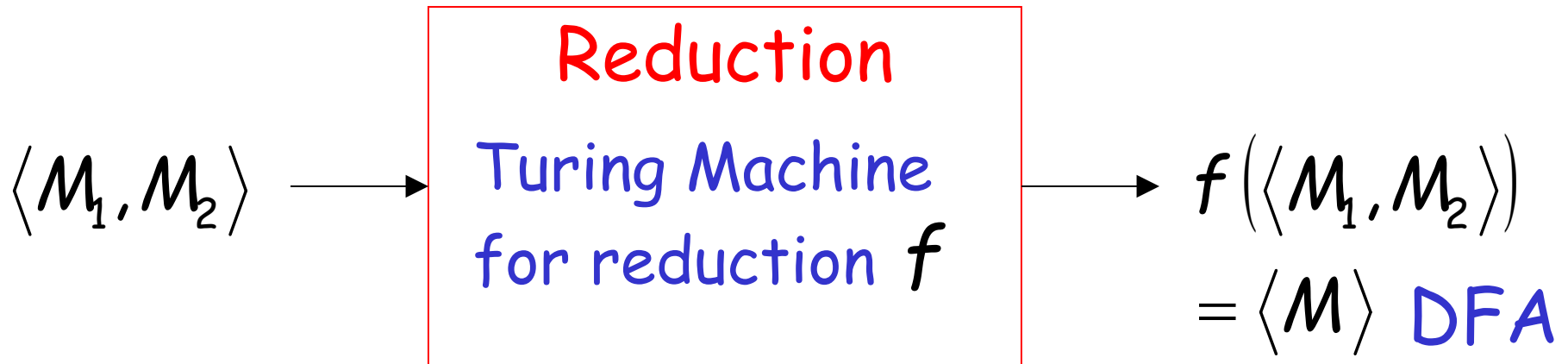


$$\langle M_1, M_2 \rangle \in EQUAL_{DFA} \iff \langle M \rangle \in EMPTY_{DFA}$$



Let  $L_1$  be the language of DFA  $M_1$

Let  $L_2$  be the language of DFA  $M_2$

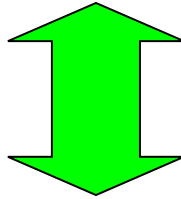


construct DFA  $M$

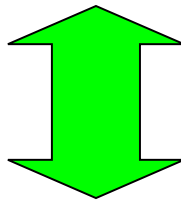
by combining  $M_1$  and  $M_2$  so that:

$$L(M) = (L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2)$$

$$L(M) = (L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2)$$

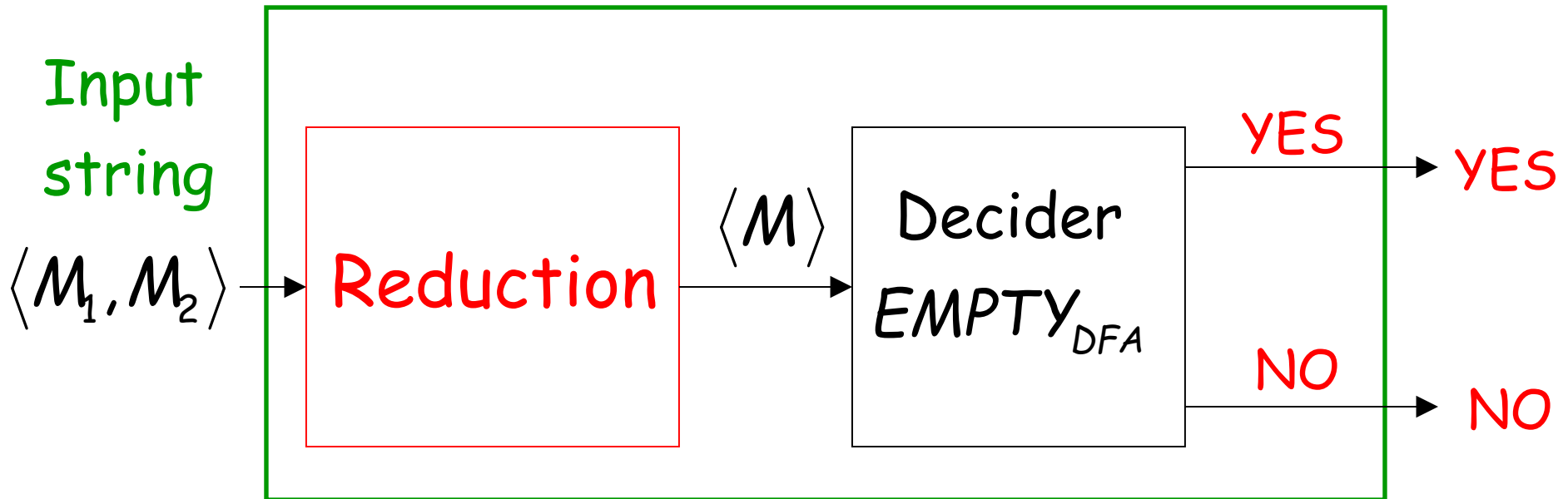


$$L_1 = L_2 \quad \Leftrightarrow \quad L(M) = \emptyset$$



$$\langle M_1, M_2 \rangle \in EQUAL_{DFA} \quad \Leftrightarrow \quad \langle M \rangle \in EMPTY_{DFA}$$

## Decider for $EQUAL_{DFA}$



## Theorem 2:

If: Language  $A$  is reduced to  $B$   
and language  $A$  is undecidable

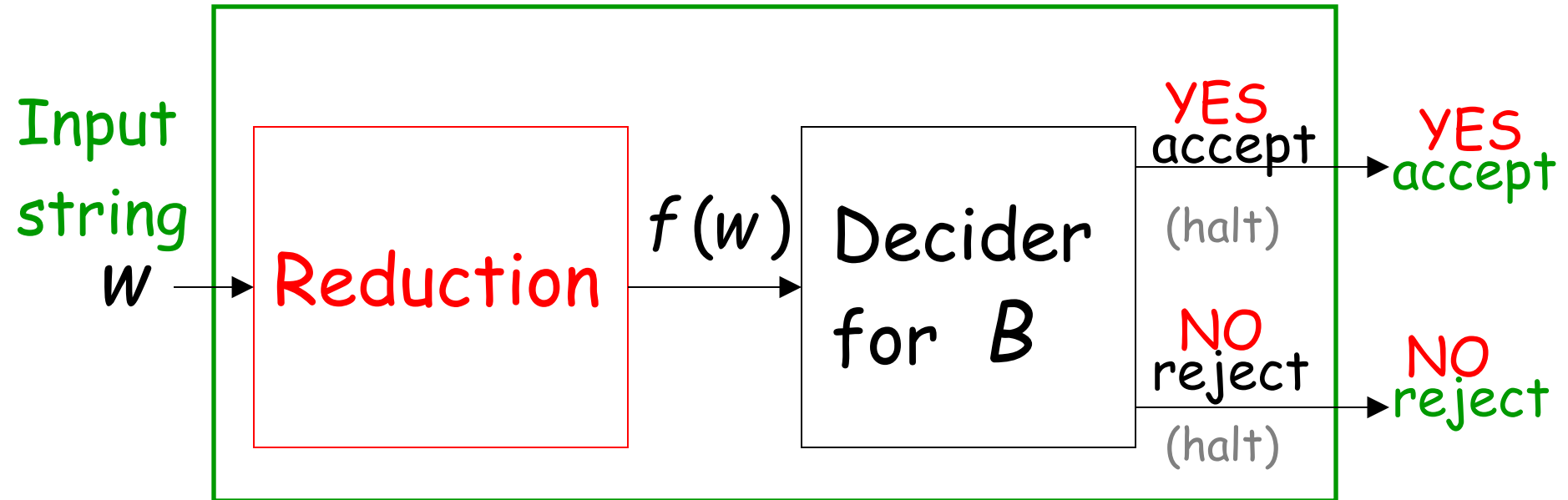
Then:  $B$  is undecidable

**Proof:** Suppose  $B$  is decidable  
Using the decider for  $B$   
build the decider for  $A$

Contradiction!

If  $B$  is decidable then we can build:

Decider for  $A$



$$w \in A \iff f(w) \in B$$

CONTRADICTION!

END OF PROOF

## Observation:

To prove that language  $B$  is undecidable  
we only need to reduce  
a known undecidable language  $A$  to  $B$

# State-entry problem

Input:

- Turing Machine  $M$
- State  $q$
- String  $w$

Question: Does  $M$  enter state  $q$   
while processing input string  $w$  ?

Corresponding language:

$STATE_{TM} = \{ \langle M, w, q \rangle : M \text{ is a Turing machine that} \\ \text{enters state } q \text{ on input string } w \}$   
(while processing)

**Theorem:**  $STATE_{TM}$  is undecidable

(state-entry problem is unsolvable)

**Proof:**

Reduce

$HALT_{TM}$  (halting problem)

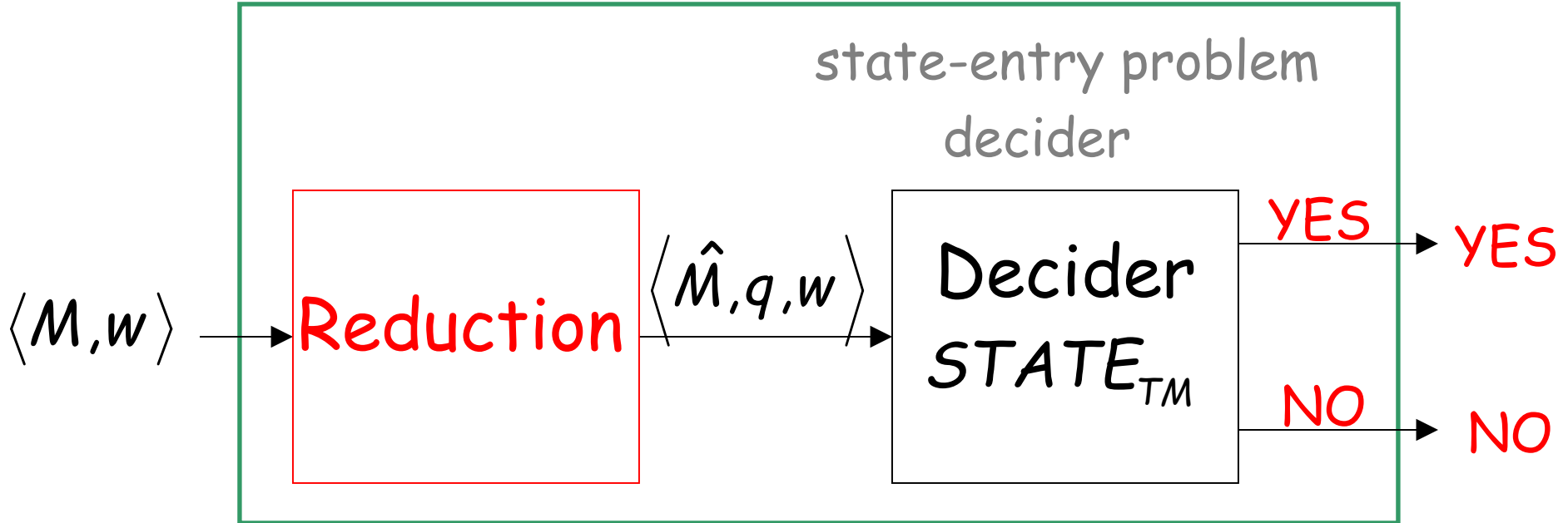
to

$STATE_{TM}$  (state-entry problem)



# Halting Problem Decider

## Decider for $HALT_{TM}$



Given the reduction,  
if  $STATE_{TM}$  is decidable,  
then  $HALT_{TM}$  is decidable

A contradiction!  
since  $HALT_{TM}$   
is undecidable

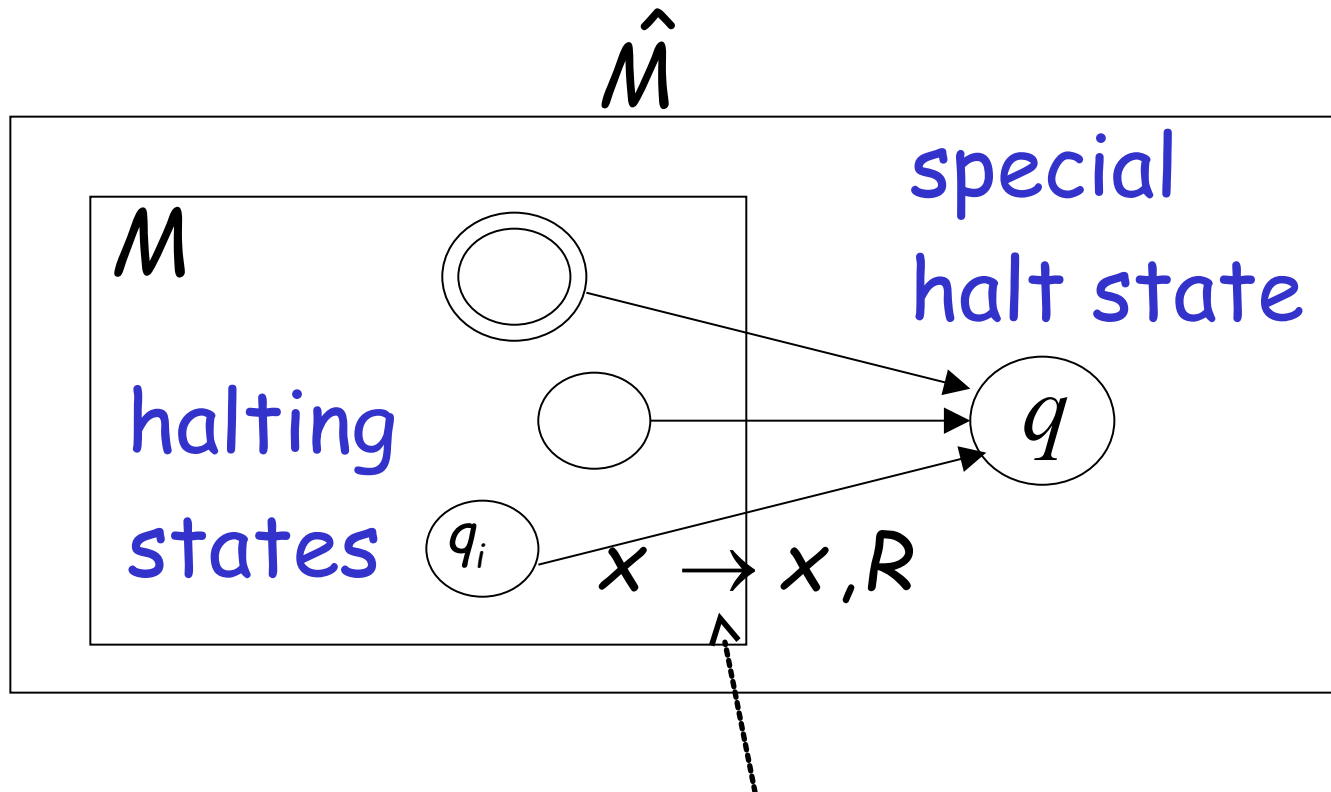
We only need to build the reduction:



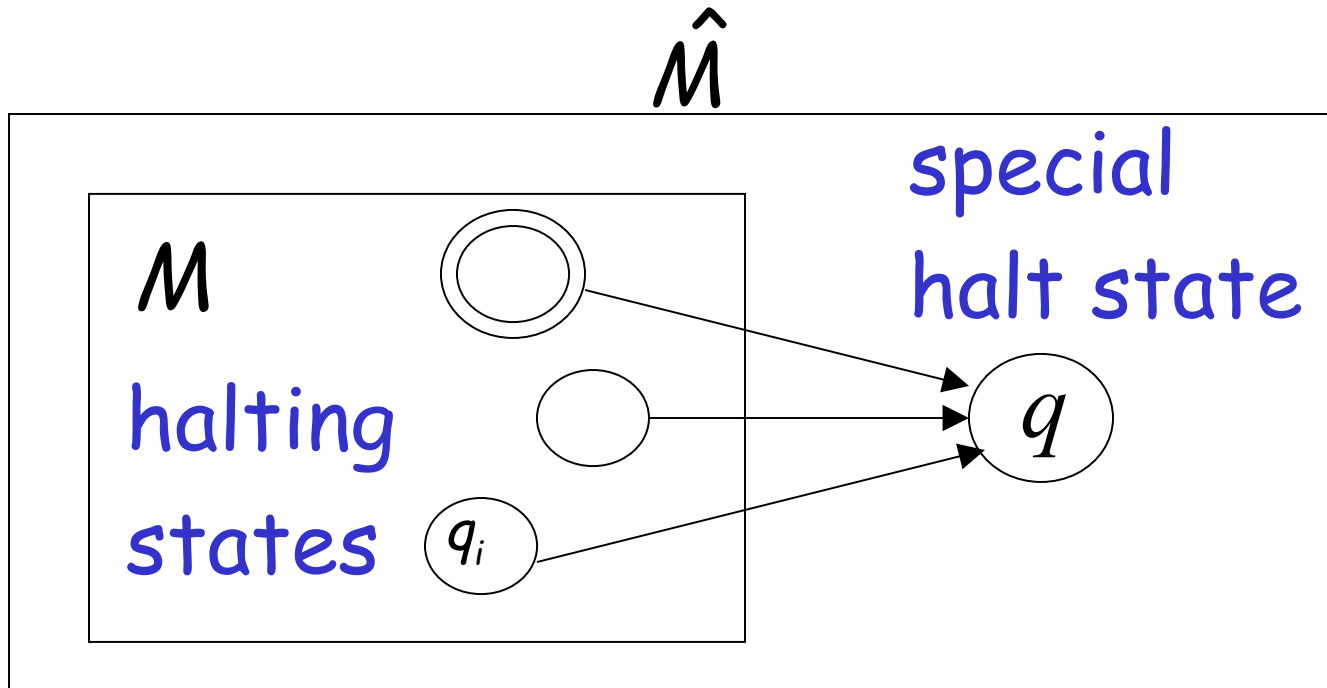
So that:

$$\langle M, w \rangle \in HALT_{TM} \iff \langle \hat{M}, w, q \rangle \in STATE_{TM}$$

For the reduction, construct  $\hat{M}$  from  $M$  :

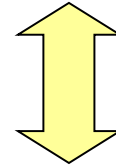


A transition for every unused  
tape symbol  $x$  of  $q_i$



$M$  halts  $\longleftrightarrow \hat{M}$  halts on state  $q$

Therefore:  $M$  halts on input  $w$



$\hat{M}$  halts on state  $q$  on input  $w$

Equivalently:

$$\langle M, w \rangle \in \text{HALT}_{TM} \iff \langle \hat{M}, w, q \rangle \in \text{STATE}_{TM}$$

END OF PROOF

# Blank-tape halting problem

Input: Turing Machine  $M$

Question: Does  $M$  halt when started with a blank tape?

Corresponding language:

$BLANK_{TM} = \{\langle M \rangle : M \text{ is a Turing machine that halts when started on blank tape}\}$

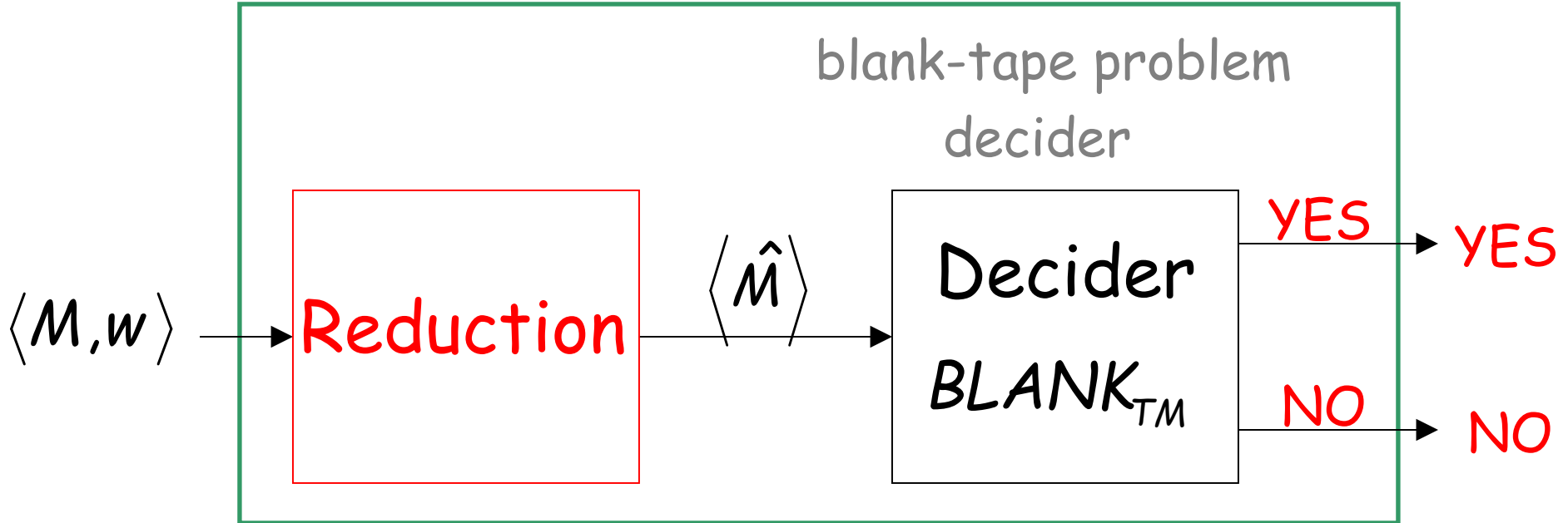
**Theorem:**  $BLANK_{TM}$  is undecidable

(blank-tape halting problem is unsolvable)

**Proof:** Reduce  
 $HALT_{TM}$  (halting problem)  
to  
 $BLANK_{TM}$  (blank-tape problem)

# Halting Problem Decider

## Decider for $HALT_{TM}$

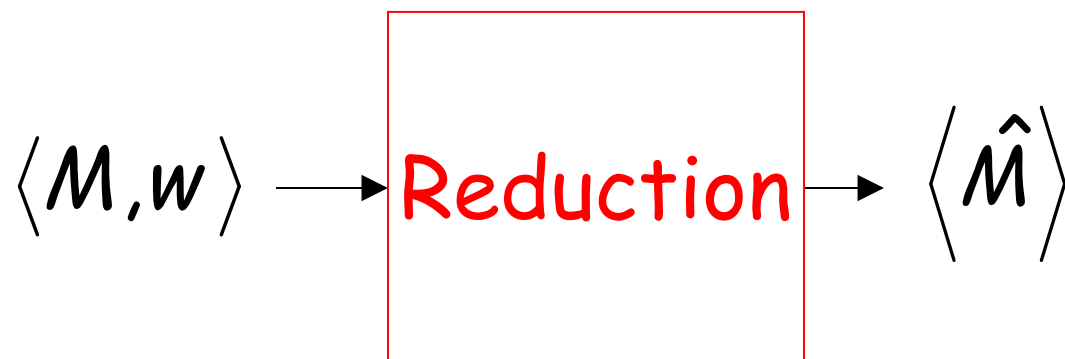


Given the reduction,  
If  $BLANK_{TM}$  is decidable,  
then  $HALT_{TM}$  is decidable

A contradiction!  
since  $HALT_{TM}$   
is undecidable



We only need to build the reduction:

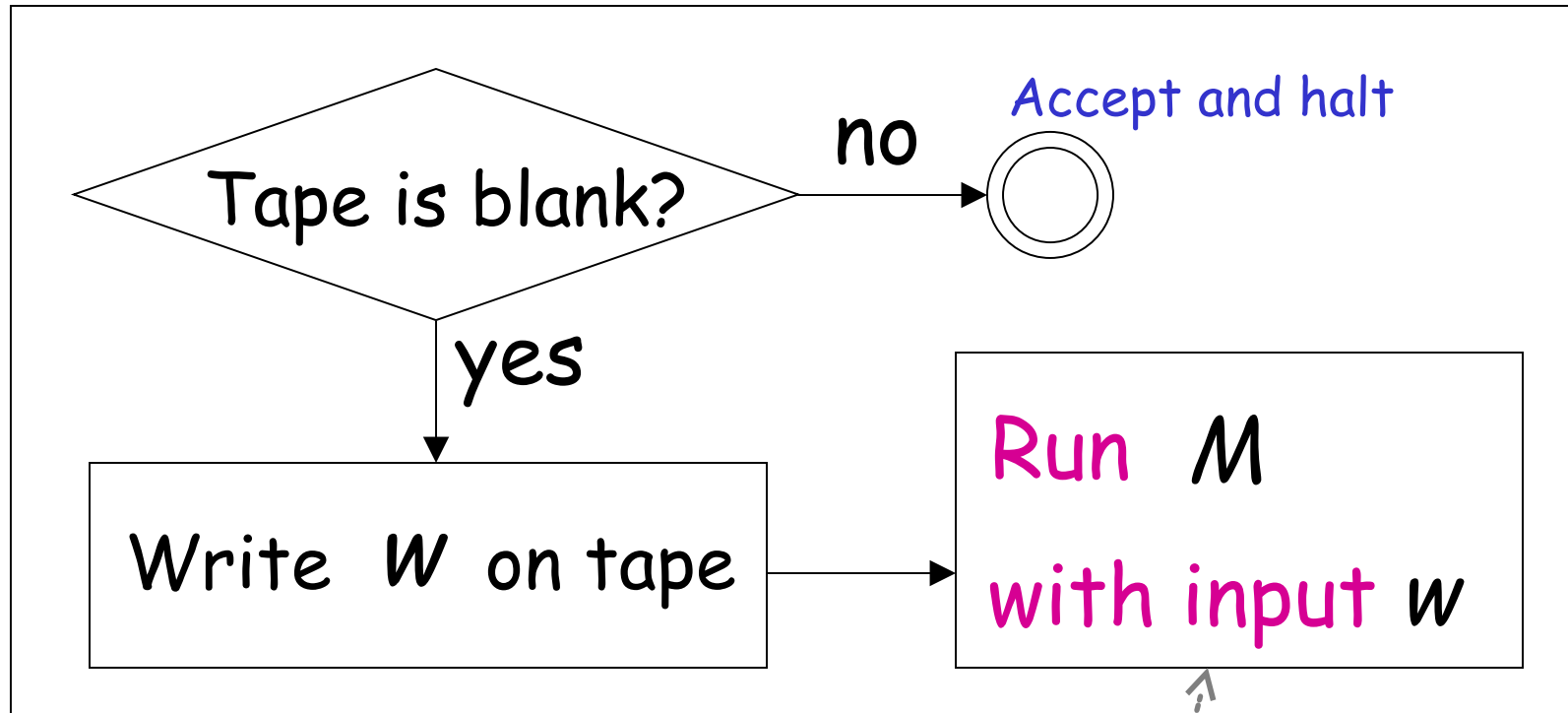


So that:

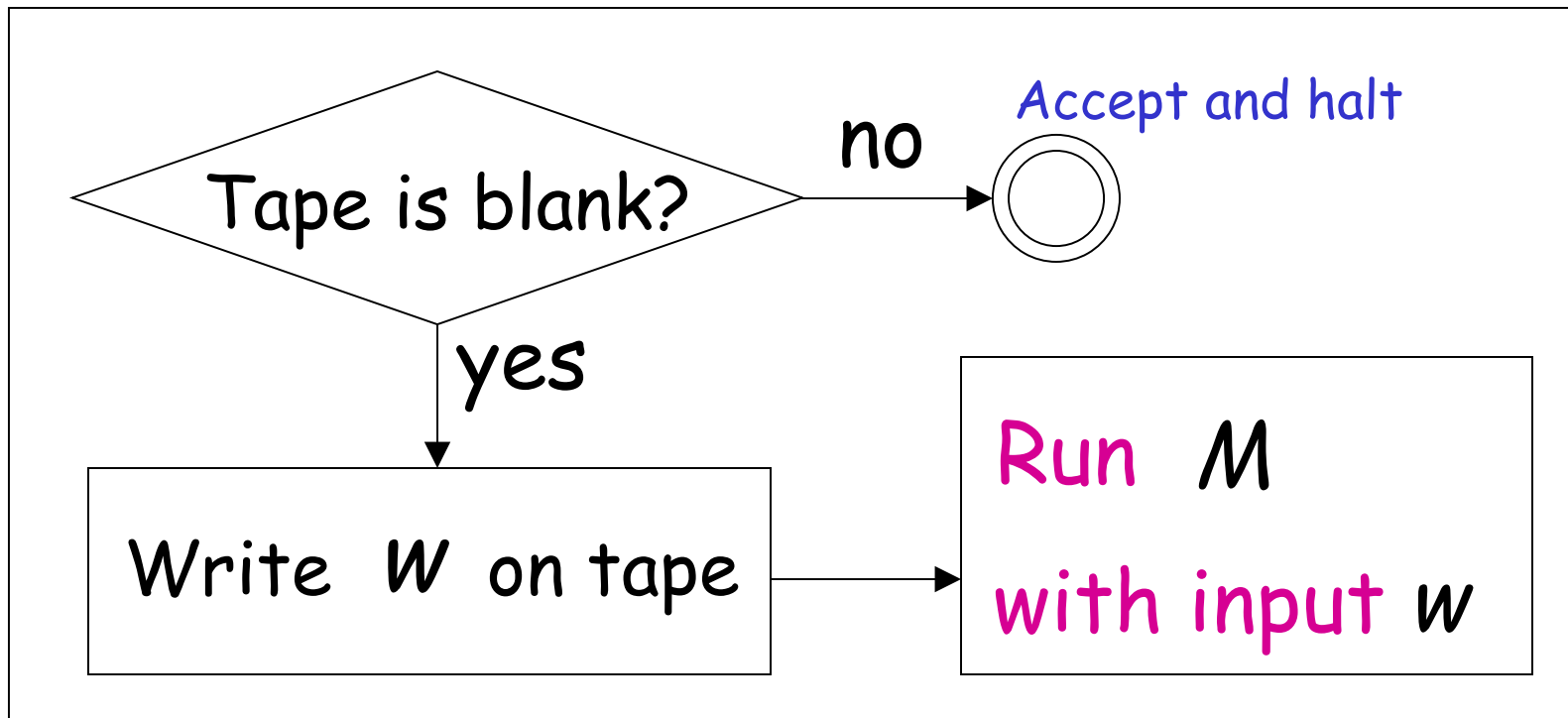
$$\langle M, w \rangle \in HALT_{TM} \iff \langle \hat{M} \rangle \in BLANK_{TM}$$

Construct  $\langle \hat{M} \rangle$  from  $\langle M, w \rangle$ :

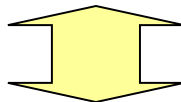
$\hat{M}$



If  $M$  halts then  $\hat{M}$  halts too

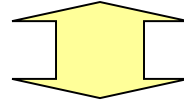
$\hat{M}$ 

$M$  halts on input  $w$



$\hat{M}$  halts when started on blank tape

$M$  halts on input  $w$



$\hat{M}$  halts when started on blank tape

Equivalently:

$$\langle M, w \rangle \in \text{HALT}_{TM} \longleftrightarrow \langle \hat{M} \rangle \in \text{BLANK}_{TM}$$

END OF PROOF

## Theorem 3:

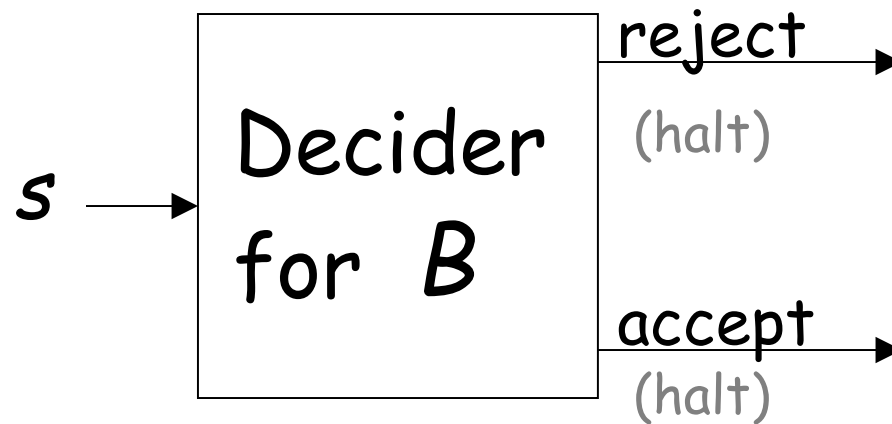
If: Language  $A$  is reduced to  $\overline{B}$   
and language  $A$  is undecidable

Then:  $B$  is undecidable

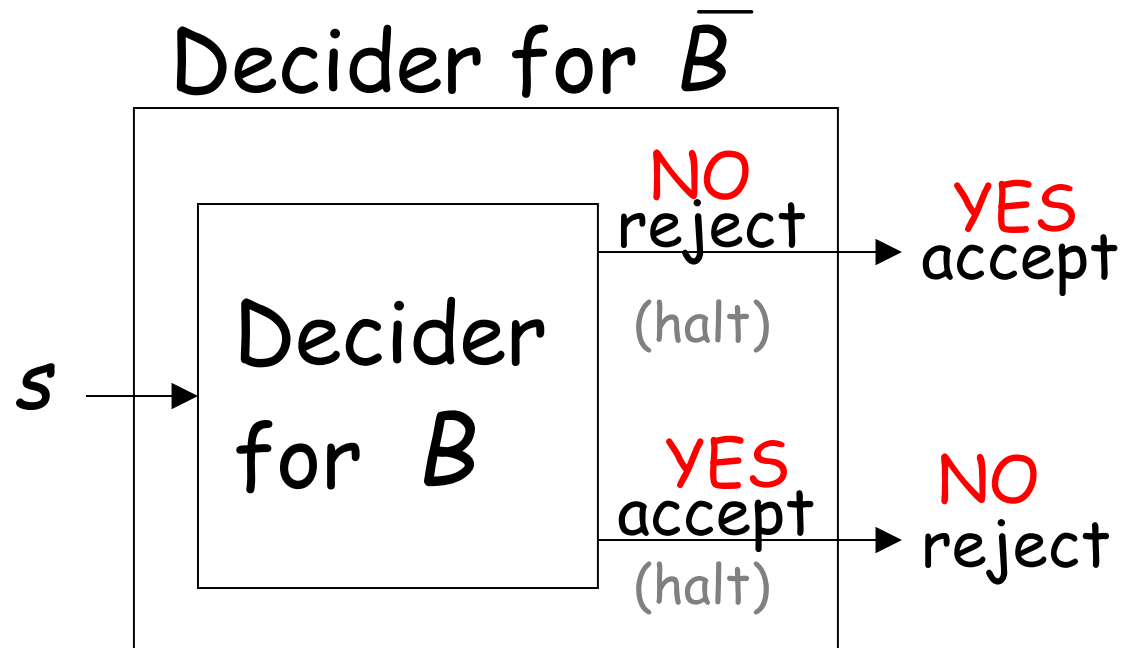
**Proof:** Suppose  $B$  is decidable  
Then  $\overline{B}$  is decidable  
Using the decider for  $\overline{B}$   
build the decider for  $A$

Contradiction!

Suppose  $B$  is decidable

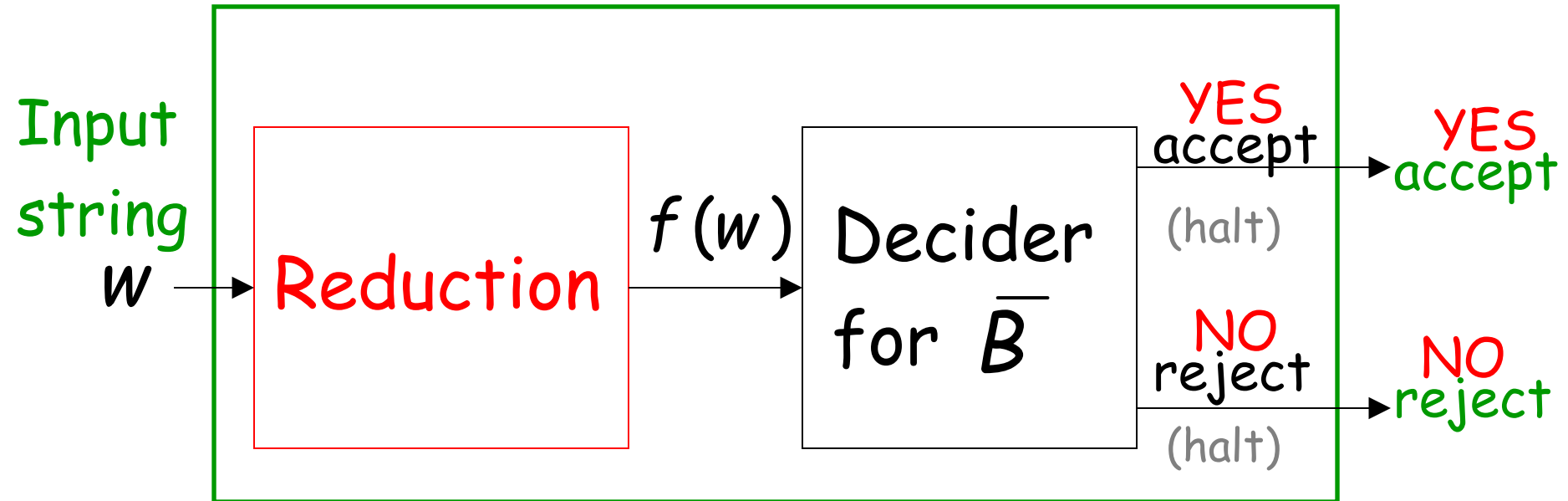


Suppose  $B$  is decidable  
Then  $\bar{B}$  is decidable



If  $\bar{B}$  is decidable then we can build:

Decider for  $A$



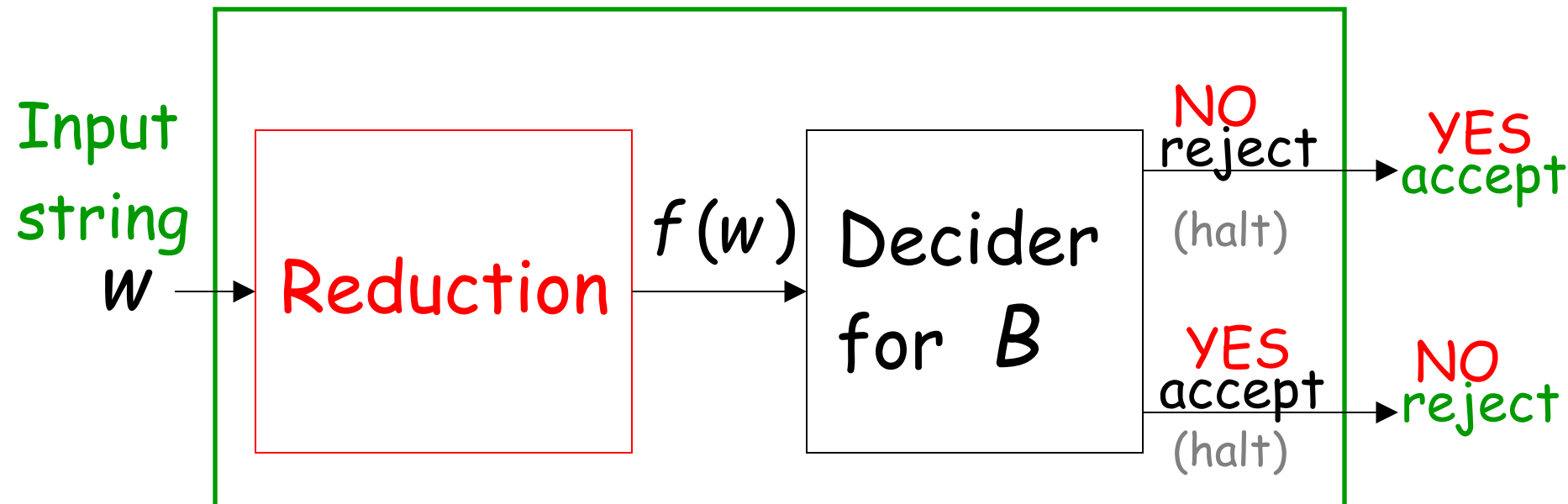
$$w \in A \iff f(w) \in \bar{B}$$

CONTRADICTION!



Alternatively:

Decider for  $A$



$$w \in A \iff f(w) \notin B$$

CONTRADICTION!

END OF PROOF

## Observation:

To prove that language  $B$  is undecidable  
we only need to reduce  
a known undecidable language  $A$   
to  $B$  (Theorem 2)  
or  $\overline{B}$  (Theorem 3)

# Undecidable Problems for Turing Recognizable languages

Let  $L$  be a Turing-acceptable language

- $L$  is empty?
- $L$  is regular?
- $L$  has size 2?

All these are undecidable problems

Let  $L$  be a Turing-acceptable language

- $L$  is empty?
- $L$  is regular?
- $L$  has size 2?

# Empty language problem

Input: Turing Machine  $M$

Question: Is  $L(M)$  empty?  $L(M) = \emptyset$ ?

Corresponding language:

$EMPTY_{TM} = \{\langle M \rangle : M \text{ is a Turing machine that accepts the empty language } \emptyset\}$

**Theorem:**  $EMPTY_{TM}$  is undecidable

(empty-language problem is unsolvable)

**Proof:**

Reduce

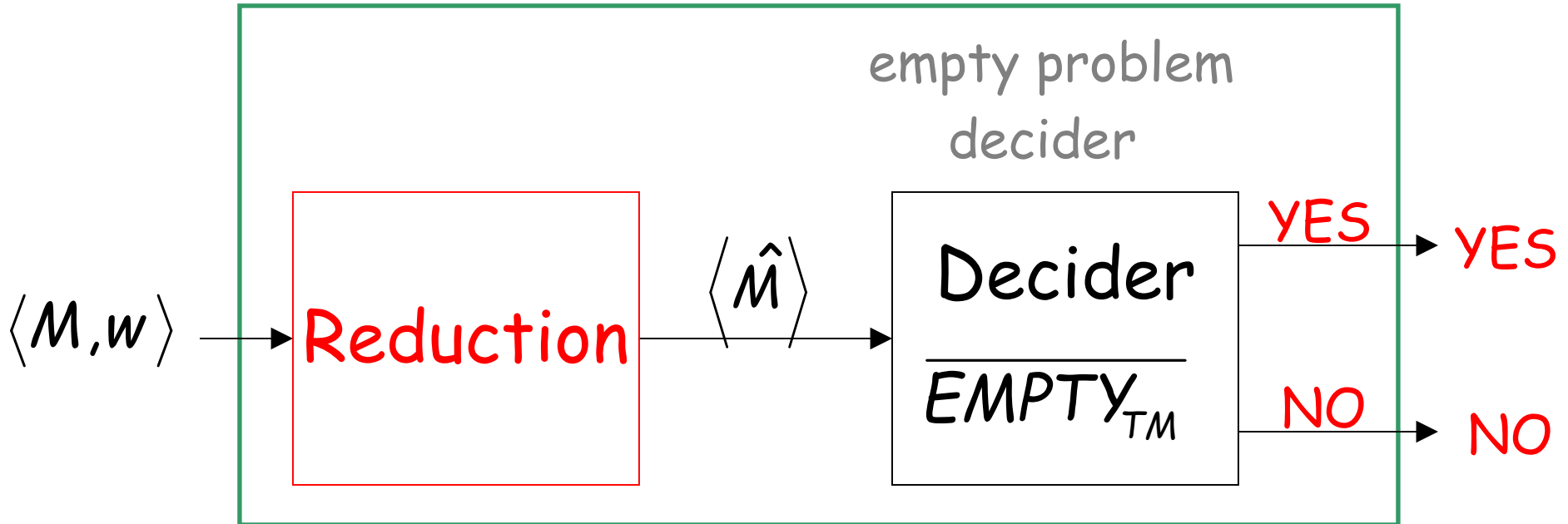
$A_{TM}$  (membership problem)

to

$\overline{EMPTY_{TM}}$  (empty language problem)

membership problem decider

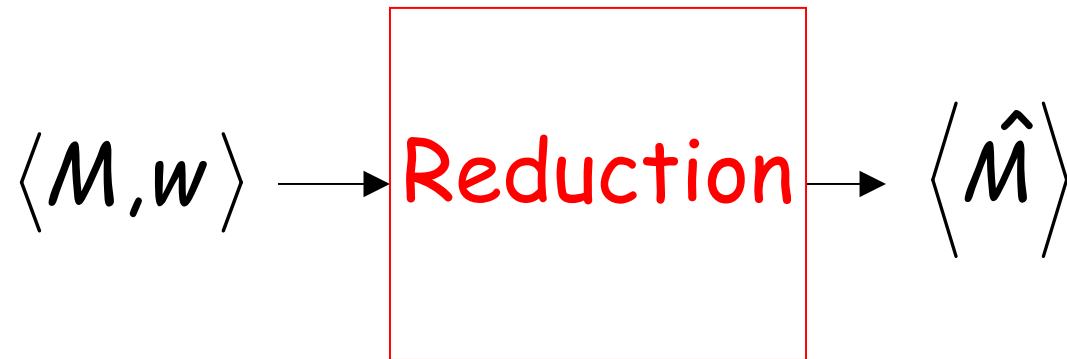
## Decider for $A_{TM}$



Given the reduction,  
if  $\overline{EMPTY}_{TM}$  is decidable,  
then  $A_{TM}$  is decidable

A contradiction!  
since  $A_{TM}$   
is undecidable

We only need to build the reduction:



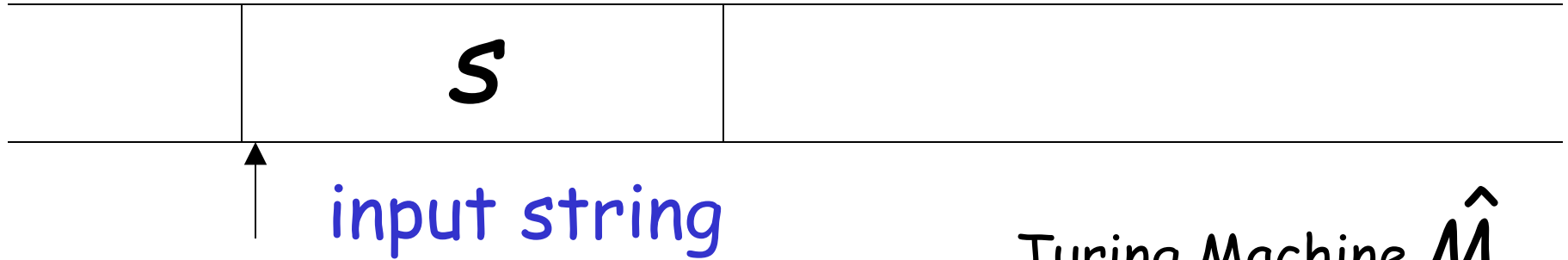
So that:

$$\langle M, w \rangle \in A_{TM} \iff \langle \hat{M} \rangle \in \overline{EMPTY_{TM}}$$



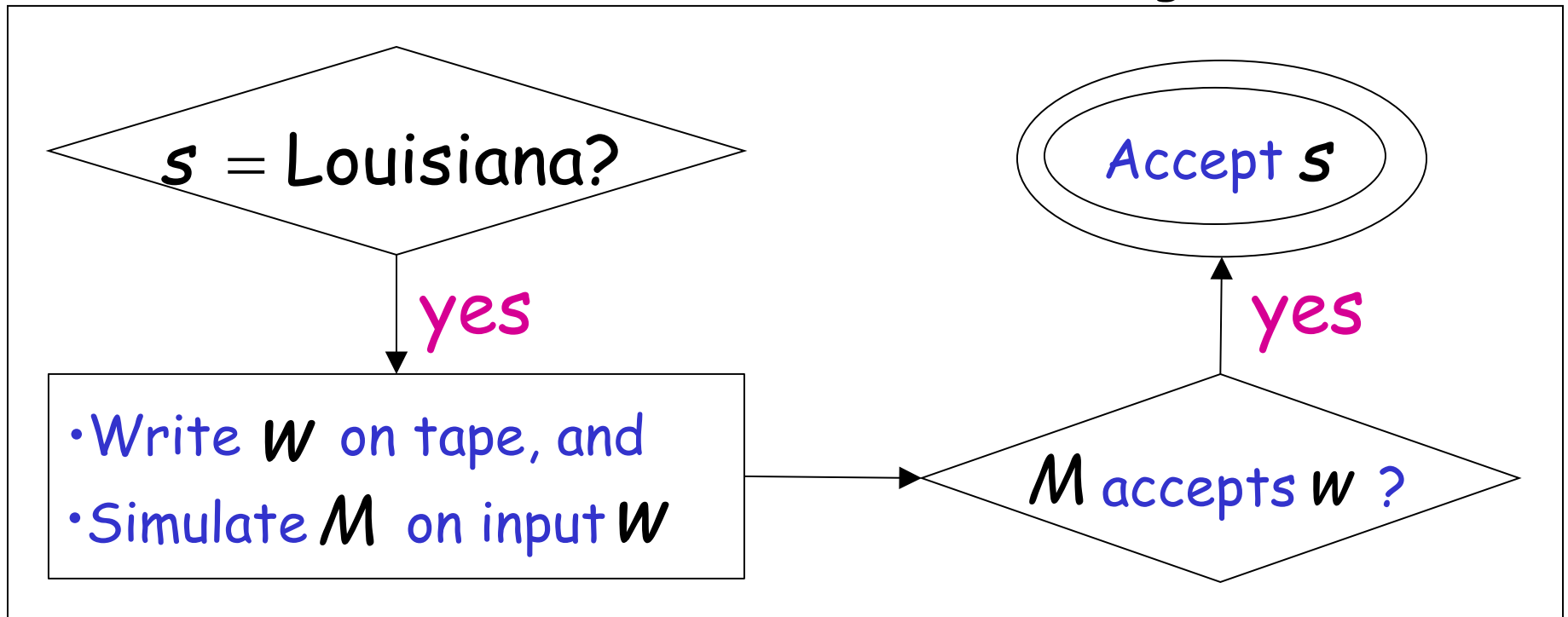
Construct  $\langle \hat{M} \rangle$  from  $\langle M, w \rangle$ :

Tape of  $\hat{M}$



input string

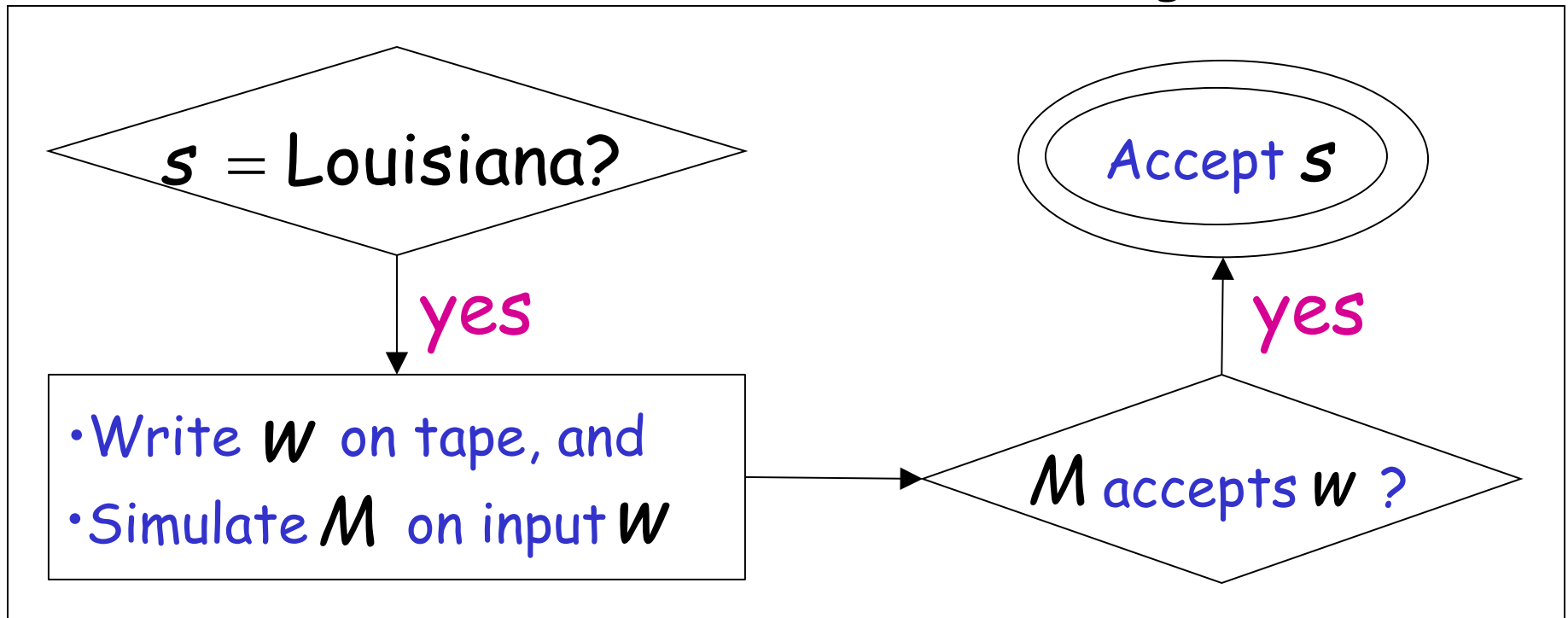
Turing Machine  $\hat{M}$



The only possible accepted string  $s$

Louisiana

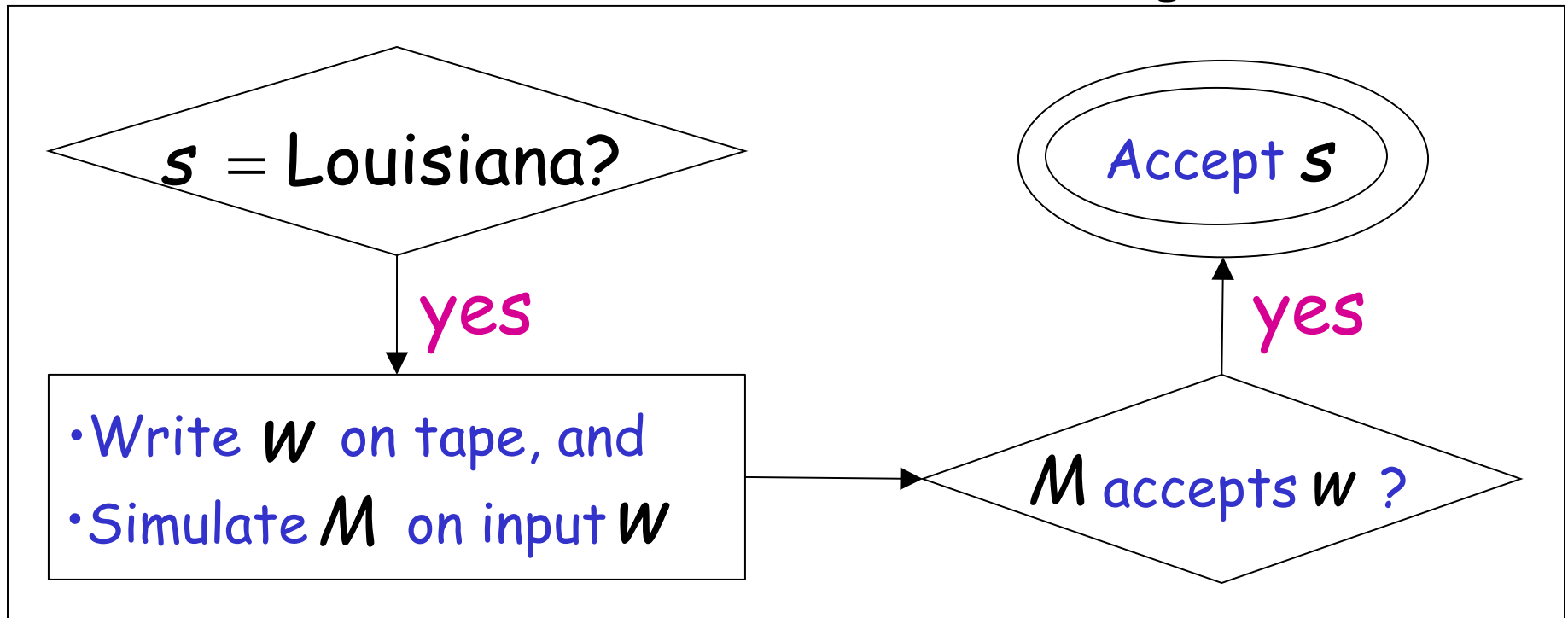
Turing Machine  $\hat{M}$



$M$  accepts  $w \implies L(\hat{M}) = \{\text{Louisiana}\} \neq \emptyset$

$M$  does not accept  $w \implies L(\hat{M}) = \emptyset$

Turing Machine  $\hat{M}$



Therefore:

$$M \text{ accepts } w \iff L(\hat{M}) \neq \emptyset$$

Equivalently:

$$\langle M, w \rangle \in A_{TM} \iff \langle \hat{M} \rangle \in \overline{EMPTY_{TM}}$$

END OF PROOF

Let  $L$  be a Turing-acceptable language

- $L$  is empty?
- $L$  is regular?
- $L$  has size 2?

# Regular language problem

Input: Turing Machine  $M$

Question: Is  $L(M)$  a regular language?

Corresponding language:

$$REGULAR_{TM} = \{\langle M \rangle : M \text{ is a Turing machine that accepts a regular language}\}$$

**Theorem:**  $REGULAR_{TM}$  is undecidable

(regular language problem is unsolvable)

**Proof:**

Reduce

$A_{TM}$

(membership problem)

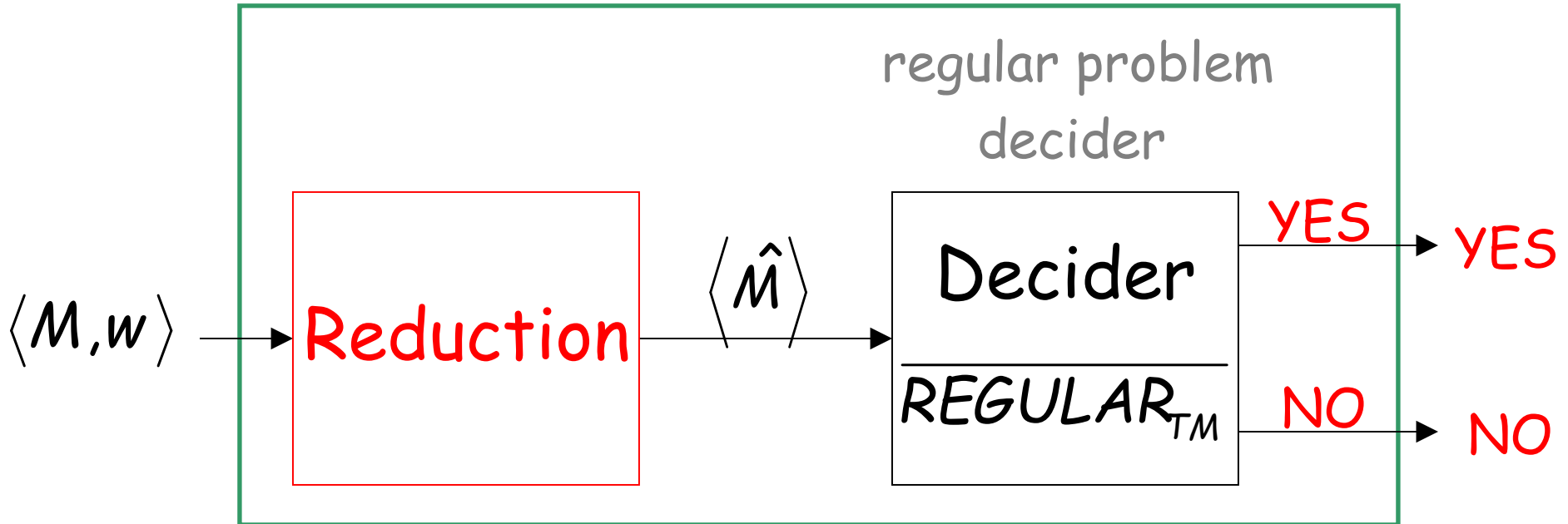
to

$REGULAR_{TM}$

(regular language problem)

membership problem decider

## Decider for $A_{TM}$

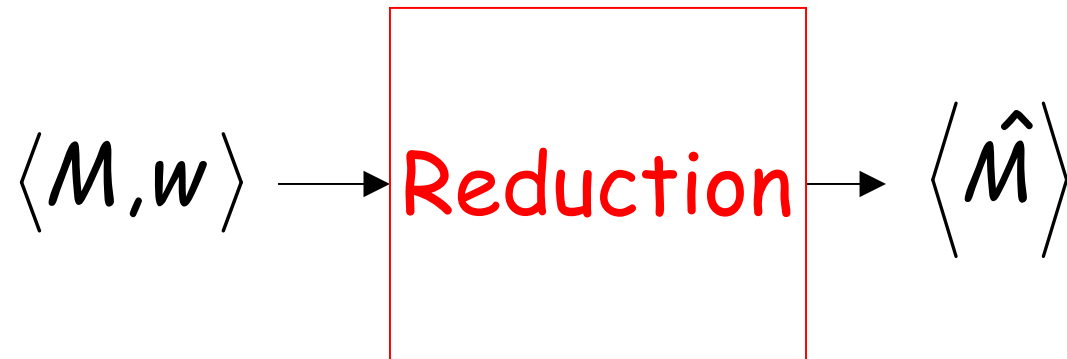


Given the reduction,  
If  $\overline{REGULAR}_{TM}$  is decidable,  
then  $A_{TM}$  is decidable

A contradiction!  
since  $A_{TM}$   
is undecidable



We only need to build the reduction:

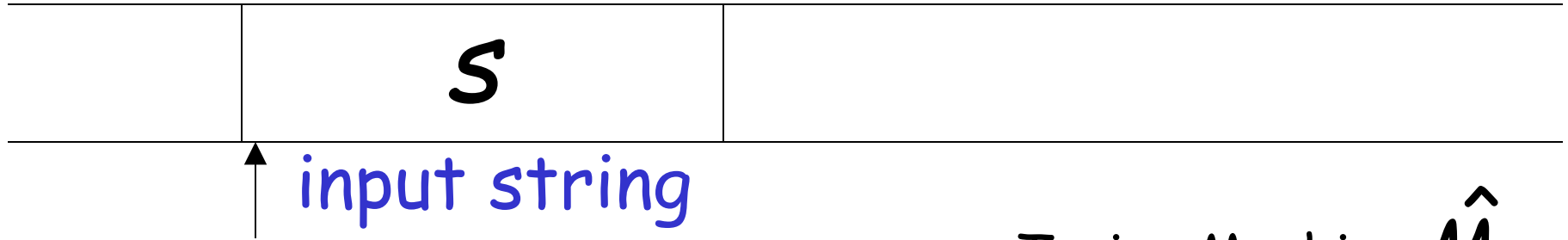


So that:

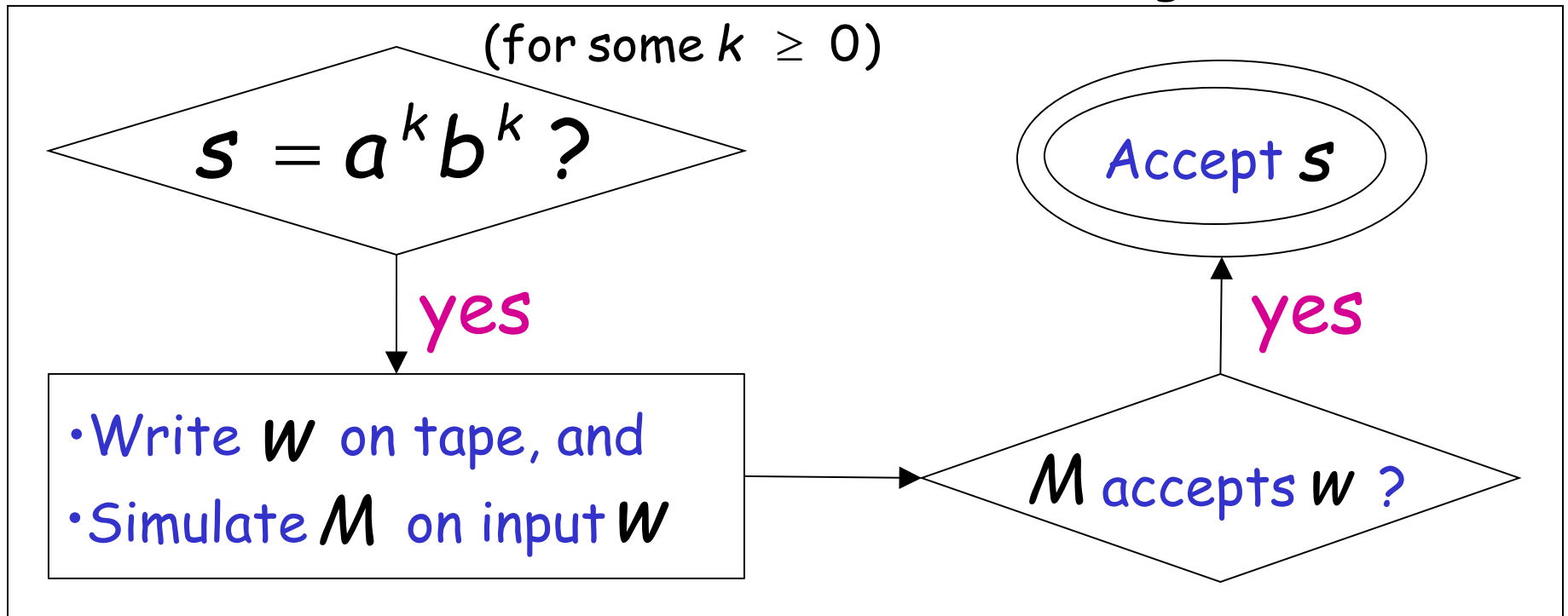
$$\langle M, w \rangle \in A_{TM} \iff \langle \hat{M} \rangle \in \overline{REGULAR_{TM}}$$

Construct  $\langle \hat{M} \rangle$  from  $\langle M, w \rangle$ :

Tape of  $\hat{M}$



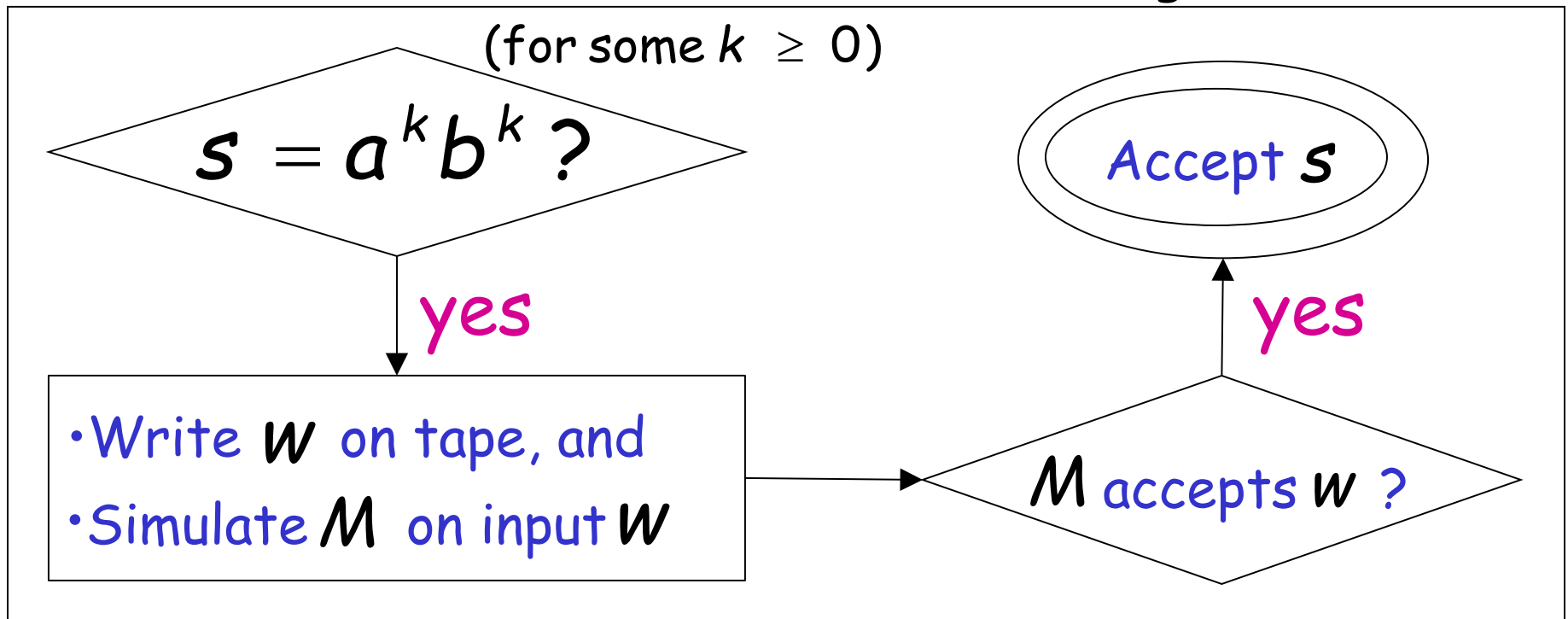
Turing Machine  $\hat{M}$



$M$  accepts  $w \longrightarrow L(\hat{M}) = \{a^n b^n : n \geq 0\}$  not regular

$M$  does not accept  $w \longrightarrow L(\hat{M}) = \emptyset$  regular

Turing Machine  $\hat{M}$



Therefore:

$M$  accepts  $w$   $\iff L(\hat{M})$  is not regular

Equivalently:

$\langle M, w \rangle \in A_{TM} \iff \langle \hat{M} \rangle \in \overline{REGULAR_{TM}}$

END OF PROOF

Let  $L$  be a Turing-acceptable language

- $L$  is empty?
- $L$  is regular?
- $L$  has size 2?

## Size2 language problem

Input: Turing Machine  $M$

Question: Does  $L(M)$  have size 2 (two strings)?  
 $|L(M)| = 2?$

Corresponding language:

$SIZE2_{TM} = \{\langle M \rangle : M \text{ is a Turing machine that accepts exactly two strings}\}$

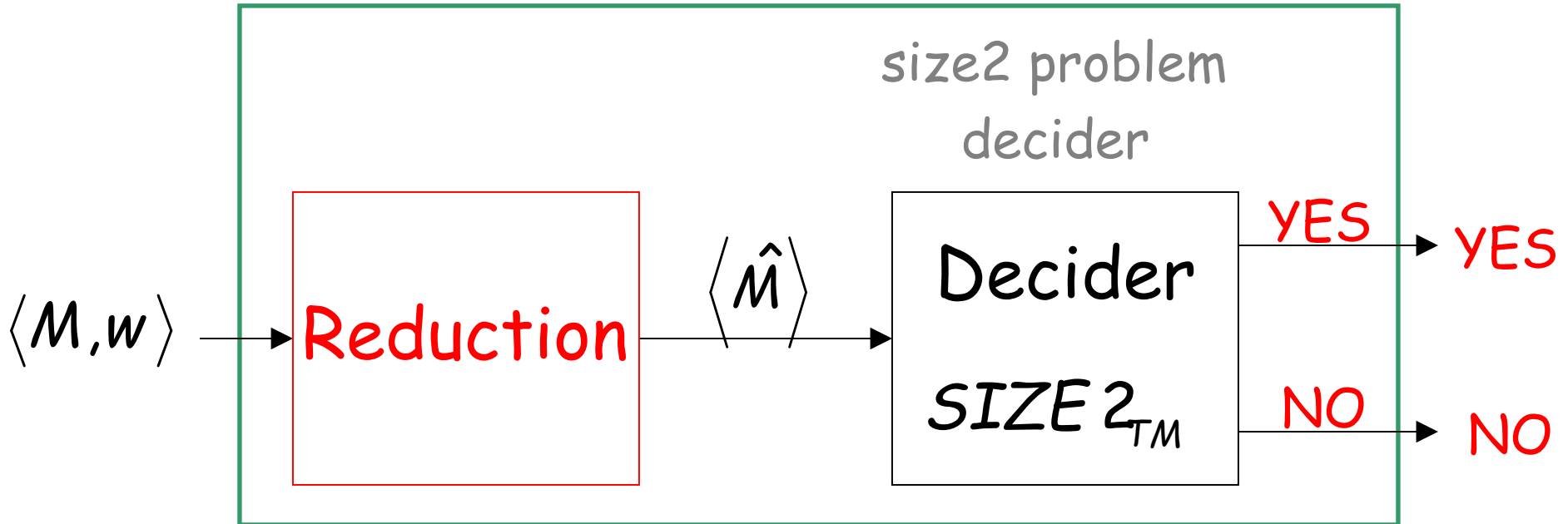
**Theorem:**  $SIZE2_{TM}$  is undecidable

(size2 language problem is unsolvable)

**Proof:** Reduce  
 $A_{TM}$  (membership problem)  
to  
 $SIZE2_{TM}$  (size 2 language problem)

membership problem decider

## Decider for $A_{TM}$

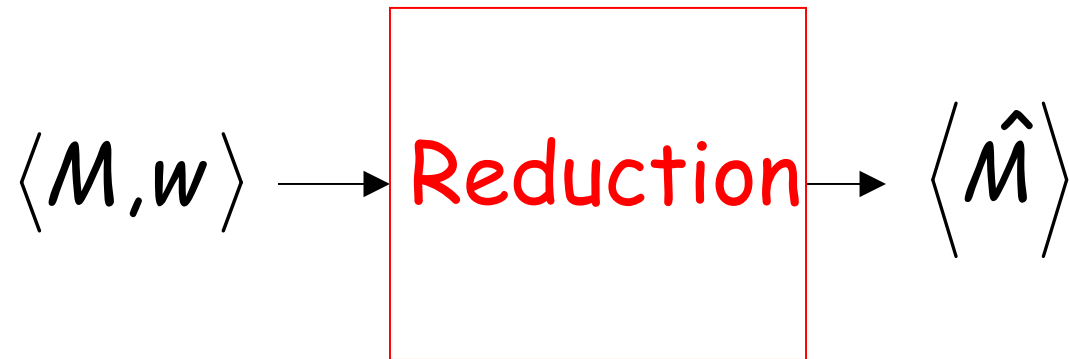


Given the reduction,  
If  $SIZE 2_{TM}$  is decidable,  
then  $A_{TM}$  is decidable

A contradiction!  
since  $A_{TM}$   
is undecidable



We only need to build the reduction:

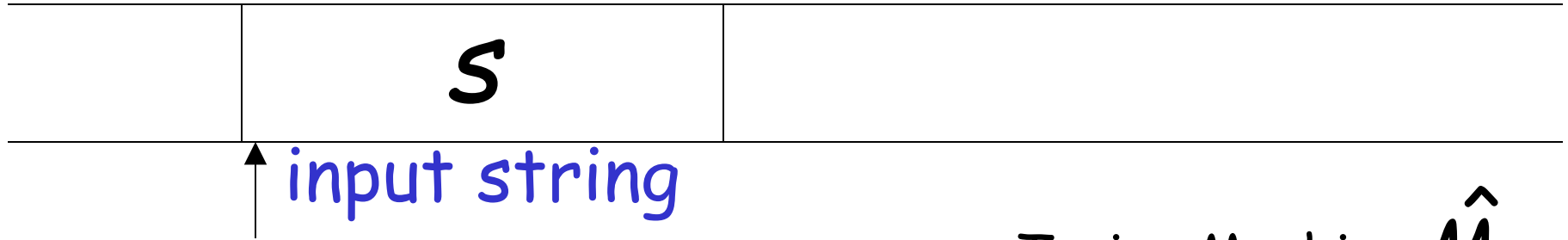


So that:

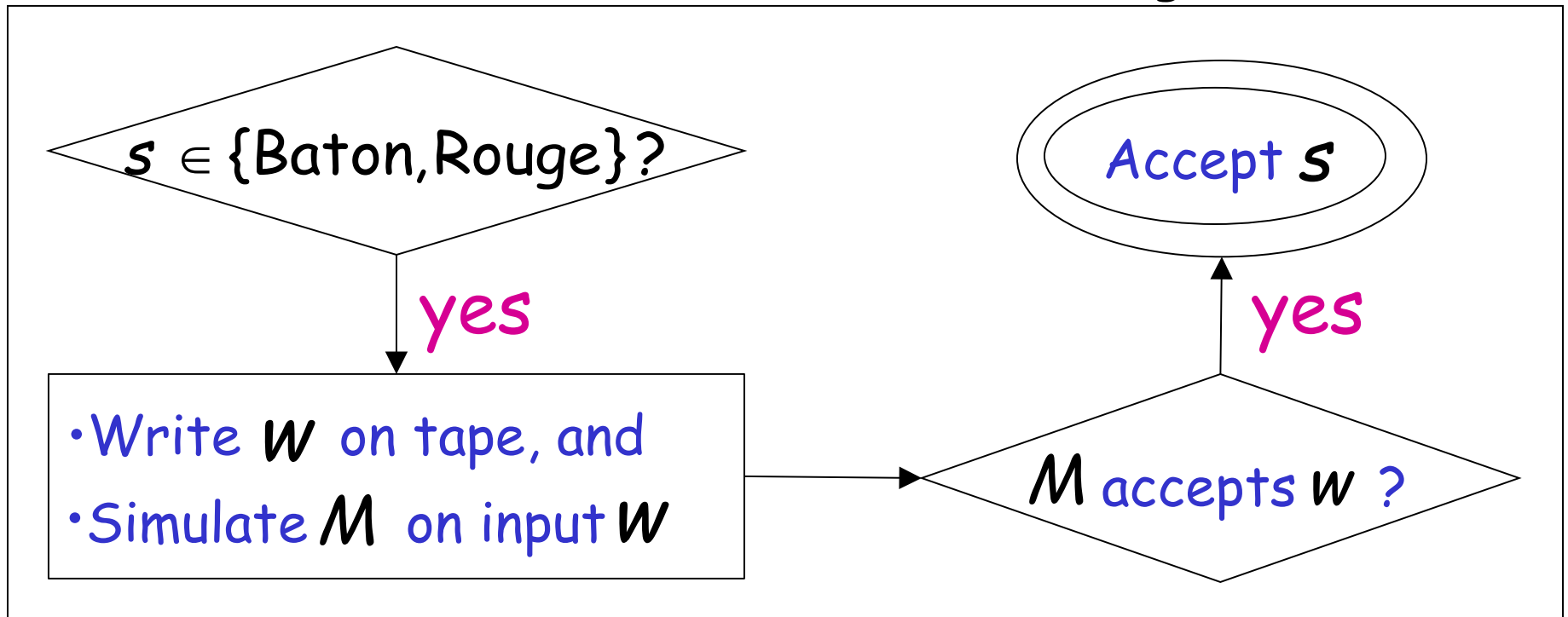
$$\langle M, w \rangle \in A_{TM} \iff \langle \hat{M} \rangle \in SIZE2_{TM}$$

Construct  $\langle \hat{M} \rangle$  from  $\langle M, w \rangle$ :

Tape of  $\hat{M}$



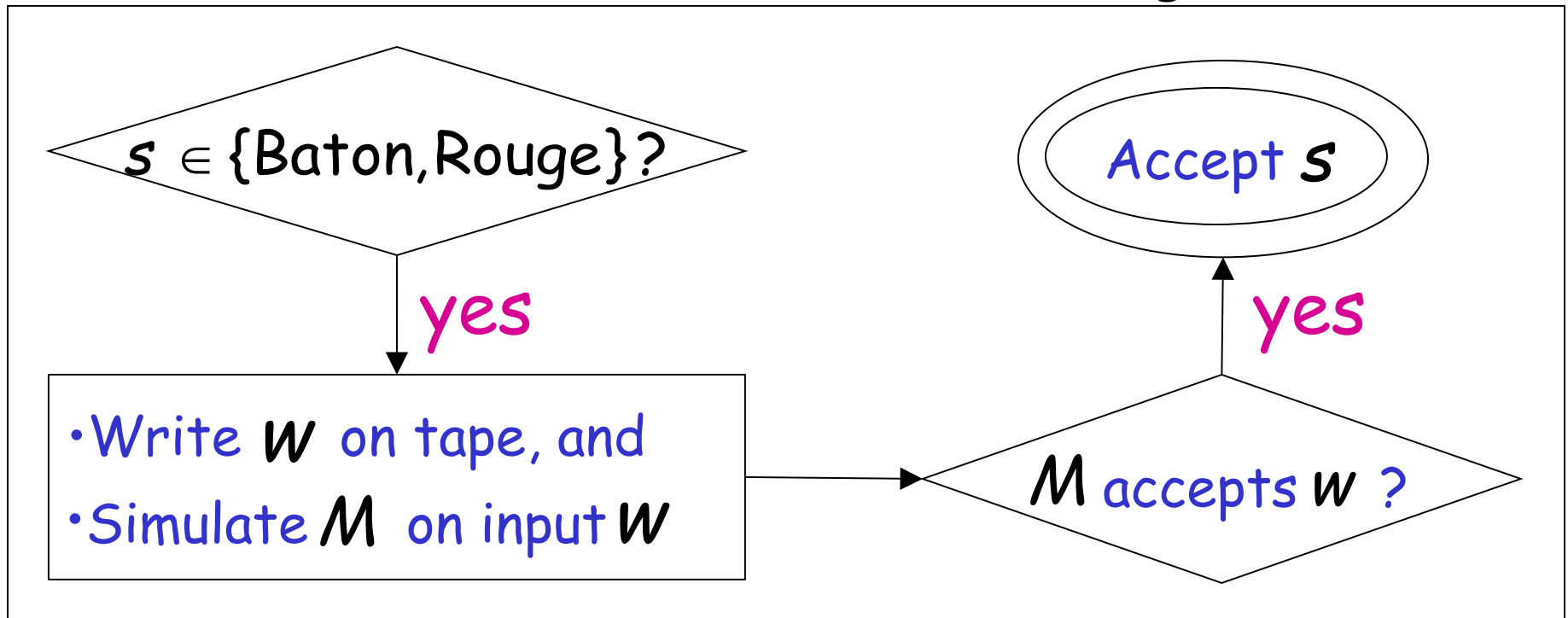
Turing Machine  $\hat{M}$



$M$  accepts  $w$   $\longrightarrow$   $L(\hat{M}) = \{\text{Baton, Rouge}\}$  2 strings

$M$  does not accept  $w$   $\longrightarrow$   $L(\hat{M}) = \emptyset$  0 strings

Turing Machine  $\hat{M}$



Therefore:

$M$  accepts  $w$   $\iff L(\hat{M})$  has size 2

Equivalently:

$\langle M, w \rangle \in A_{TM} \iff \langle \hat{M} \rangle \in SIZE2_{TM}$

END OF PROOF