



MAT1320-Linear Algebra Lecture Notes

Basis, Dimension and Coordinates

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4. $\mathcal{B} = \{\vec{e}_1 = (1, 0, \dots, 0), \dots, \vec{e}_n = (0, \dots, 0, 1)\} \subset V$
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Note: All vector spaces given above are of finite dimension.

Example

$$1. \mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subset V = \mathcal{M}_{2 \times 2}(\mathbb{R}) \Rightarrow \mathcal{M}_{2 \times 2}(\mathbb{R}) = \langle \mathcal{B} \rangle \text{ and } \dim(\mathcal{M}_{2 \times 2}(\mathbb{R})) = 4.$$

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2. $\mathcal{B} = \{1, x, x^2, \dots, x^{n-1}\} \subset V = \mathcal{P}_{n-1}(x) = \{a_0 + a_1x + \dots + a_{n-1}x^{n-1} \mid a_i \in \mathbb{R}\} \Rightarrow \mathcal{P}_{n-1}(x) = \langle \mathcal{B} \rangle$ and $\dim(\mathcal{P}_{n-1}(x)) = n$.

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3. $\mathcal{B} = \{1, x, x^2, \dots\} \subset V = \mathcal{P}(x) = \{\text{the set of all polynomials}\} \Rightarrow \mathcal{P}(x) = \langle \mathcal{B} \rangle$ and $\dim(\mathcal{P}(x)) = \infty$.

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4. Let $V = \left\{ \vec{\mathbf{0}} \right\}$ be the zero space, then $V = \left\langle \vec{\mathbf{0}} \right\rangle$ and $\dim(V) = 0$.

Example

Determine whether the set

$\mathcal{B} = \{\vec{v}_1 = (1, 2, 0), \vec{v}_2 = (2, 0, 1), \vec{v}_3 = (1, 2, 1)\}$ is a basis for \mathbb{R}^3 .

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$$\Rightarrow x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3 = (0, 0, 0)$$

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Thus $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

2. Span: For all $(x, y, z) \in \mathbb{R}^3$

$$(x, y, z) = x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3$$

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Hence, we have $x_1 = \frac{2x+y-4z}{4}$, $x_2 = \frac{2x-y}{4}$, $x_3 = \frac{-2x+y+4z}{4}$

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Consequently, $\mathbb{R}^3 = \langle \mathcal{B} \rangle$.

Theorem

Let V be a vector space of dimension n and

$\mathcal{B} = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n\} \subset V$. Then the following statements hold:

1. *If \mathcal{B} is a linearly independent set, then $\langle \mathcal{B} \rangle = V$.*

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Note: By the above Theorem, in order to check that $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subset V$ is a basis of the finite dimensional vector space V , it is just sufficient to check linearly independence.

Example

Determine whether the set $\mathcal{B} = \{\vec{\mathbf{v}}_1 = (1, 2), \vec{\mathbf{v}}_2 = (2, 0)\}$ is a basis for \mathbb{R}^2 .

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$$\Rightarrow x_1 \vec{v}_1 + x_2 \vec{v}_2 = (0, 0)$$

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Therefore, the vectors \vec{v}_1, \vec{v}_2 are linearly independent. Since $\dim(\mathbb{R}^2) = 2$ we have $\langle \mathcal{B} \rangle = \mathbb{R}^2$ and we conclude that \mathcal{B} is a basis for \mathbb{R}^2 .

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$$\begin{aligned}(x, y) &= x_1 \vec{v}_1 + x_2 \vec{v}_2 \\(x, y) &= x_1(1, 2) + x_2(2, 0) \\ \begin{cases} x_1 + 2x_2 = x \\ 2x_1 = y \end{cases} &\Rightarrow \left(\begin{array}{cc|c} 1 & 2 & x \\ 2 & 0 & y \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{y}{2} \\ 0 & 1 & \frac{2x-y}{4} \end{array} \right)\end{aligned}$$

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Since, $\mathbb{R}^2 = \langle \mathcal{B} \rangle$ the vectors \vec{v}_1, \vec{v}_2 are linearly independent. As a result, \mathcal{B} is a basis for \mathbb{R}^2 .

Example

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$$x = -2z$$

1. Show that \mathcal{U} is a subspace of \mathbb{R}^3 .

$$\begin{aligned} \begin{matrix} x \\ y \\ z \end{matrix} \quad \begin{bmatrix} -2z \\ y \\ z \end{bmatrix} &= \begin{bmatrix} -2z \\ 0 \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} \\ &= z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

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2. Find a basis for \mathcal{U} .
3. $\dim(\mathcal{U}) = ?$

1. **Subspace:** For all $(x, y, z), (a, b, c) \in \mathcal{U}$ and $\alpha, \beta \in \mathbb{R}$

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This says that \mathcal{U} is a subspace of \mathbb{R}^3 .

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It can be easily seen that the vectors $(-2, 0, 1)$ and $(0, 1, 0)$ are linearly independent and spans \mathcal{U} . Thus, $\mathcal{B} = \{(-2, 0, 1), (0, 1, 0)\}$ is a basis for \mathcal{U} .

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3. Dimension: $\dim(\mathcal{U}) = 2$.

Example

$$\text{Let } \mathcal{U} = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & 0 & a+b \end{pmatrix} \mid a, b \in \mathbb{R} \right\} \subset \mathcal{M}_{2 \times 3}(\mathbb{R}).$$

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Coordinates

Theorem

Let V be vector space of dimension n and with the ordered basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Then any vector $\vec{w} \in V$ can be expressed uniquely as a linear combination of basis vectors in \mathcal{B} , say

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Note: These n scalars x_1, x_2, \dots, x_n are called the **coordinates** of \vec{w} relative to the basis \mathcal{B} , and they form a vector (x_1, x_2, \dots, x_n) in \mathbb{R}^n called the **coordinate vector** of \vec{w} relative to \mathcal{B} . We denote this vector by $[\vec{w}]_{\mathcal{B}}$, or simply $[\vec{w}]$, when \mathcal{B} is understood. Thus,

$$[\vec{w}]_{\mathcal{B}} = (x_1, x_2, \dots, x_n).$$

Proof.

Let $\mathcal{B} = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n\}$ be an ordered basis of n -space V and $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in \mathbb{R}$.

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Let $\mathcal{B} = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n\}$ be an ordered basis of n -space V and $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in \mathbb{R}$. Assume that $\vec{\mathbf{w}}$ has two expressions as follows:

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Since basis elements are linearly independent we have

$$\Rightarrow (x_1 - y_1) \vec{v}_1 + (x_2 - y_2) \vec{v}_2 + \dots + (x_n - y_n) \vec{v}_n = \vec{0}$$

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Example

Find coordinate vector $[\vec{\mathbf{w}}]_{\mathcal{B}}$ of $\vec{\mathbf{w}} = (1, 2, 3) \in \mathbb{R}^3$ relative to ordered basis $\mathcal{B} = \{(1, 2, 0), (2, 0, 1), (1, 2, 1)\}$.

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$$(x, y, z) = x_1(1, 2, 0) + x_2(2, 0, 1) + x_3(1, 2, 1)$$

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$$(x, y, z) = x_1(1, 2, 0) + x_2(2, 0, 1) + x_3(1, 2, 1)$$

$$x_1 + 2x_2 + x_3 = x$$

$$2x_1 + 2x_3 = y$$

$$x_2 + x_3 = z$$

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Then we have $x_1 = \frac{2x+y-4z}{4}$, $x_2 = \frac{2x-y}{4}$, $x_3 = \frac{-2x+y+4z}{4}$

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Then we have $x_1 = \frac{2x+y-4z}{4}$, $x_2 = \frac{2x-y}{4}$, $x_3 = \frac{-2x+y+4z}{4}$ and hence,

$$(1, 2, 3) = -2 \cdot (1, 2, 0) + 0 \cdot (2, 0, 1) + 3(1, 2, 1).$$

?