MAT1072 / Matematik 2

Parametrik Denklemler / Kutupsal Koordinatlar

1) $x=4\sin t$, $y=2\cos t$ eğrisinin $t=\frac{\pi}{4}$ deki teğetinin denklemini bulunuz.

$$m=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\left.\frac{-2\sin t}{4\cos t}\right|_{t=\frac{\pi}{4}}=-\frac{1}{2}$$

$$t=\frac{\pi}{4} \Rightarrow \begin{array}{l} x_0=2\sqrt{2} \\ y_0=\sqrt{2} \end{array}$$

Denklem: $y-y_0=m(x-x_0) \Rightarrow y-\sqrt{2}=-\frac{1}{2}(x-2\sqrt{2}) \Rightarrow y=-\frac{1}{2}x+2\sqrt{2}$

2) $x=x(t)$, $y=y(t)$ olarak tanımlandıklarını kabul ederek,

$\left.\begin{array}{l} t^2\sin x+x^3=e^t \\ y=t\sin t-2t \end{array}\right\}$ parametrik denklemleri ile verilen eğrinin $t=0$ daki teğet doğrusunun eğimini bulunuz.

$t=0 \Rightarrow (x,y)=(1,0)$

$$t^2\sin x+x^3=e^t \Rightarrow 2t\sin x+t^2\cos x\frac{dx}{dt}+3x^2\frac{dx}{dt}=e^t$$

$$\left.\begin{array}{l} t=0 \\ x=1 \end{array}\right\} 0+0+3\frac{dx}{dt}=1 \Rightarrow \left.\frac{dx}{dt}\right|_{t=0}=\frac{1}{3}$$

$$y=t\sin t-2t \Rightarrow \frac{dy}{dt}=\sin t+t\cos t-2 \Rightarrow \left.\frac{dy}{dt}\right|_{t=0}=-2$$

$$\left.\frac{dy}{dx}\right|_{t=0}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{-2}{\frac{1}{3}}=-6$$

3) $\left.\begin{array}{l} x=2t^2+3 \\ y=t^4 \end{array}\right\}$ parametrik denklemleri ile verilen eğrinin $t=-1$ noktasındaki teğet doğrusunun denklemini yazınız.

$$t=-1 \Rightarrow \left.\begin{array}{l} x=5 \\ y=1 \end{array}\right\} y-1=m.(x-5)$$

$$m=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{4t^3}{4t}=t^2 \Rightarrow m|_{t=-1}=1 \Rightarrow y-1=x-5$$

$$\Rightarrow y=x-4$$

4) $x = 8\cos t + 8t\sin t$
$y = 8\sin t - 8t\cos t$
$0 \leq t \leq \frac{\pi}{2}$
parametrizasyonu ile verilen eğrinin uzunluğunu bulunuz

$$L = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = -\cancel{8\sin t} + \cancel{8\sin t} + 8t\cos t \Rightarrow \left(\frac{dx}{dt}\right)^2 = 64t^2\cos^2 t$$

$$\frac{dy}{dt} = \cancel{8\cos t} - \cancel{8\cos t} + 8t\sin t \Rightarrow \left(\frac{dy}{dt}\right)^2 = 64t^2\sin^2 t$$

$$L = \int_0^{\pi/2} \sqrt{64t^2} = \int_0^{\pi/2} 8t\,dt = 4t^2 \Big|_0^{\pi/2} = \pi^2$$

5) $-1 \leq t \leq 0$ olmak üzere $x(t) = t^2$, $y(t) = 1 - t^2$ ile çizilmiş yolun uzunluğunu bulunuz.

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = -2t$$

$$L = \int_{-1}^{0} \sqrt{(2t)^2 + (-2t)^2}\, dt = 2\sqrt{2}\int_{-1}^{0} |t|\,dt = -2\sqrt{2}\int_{-1}^{0} t\,dt = \sqrt{2}$$

6) $x = 2t^2 + 3$
$y = t^4$
parametrik denklemi ile verilen eğrinin $t = -1$ deki normal doğrusunun denklemini bulunuz.

$t = -1$ $\begin{cases} x = 5 \\ y = 1 \end{cases}$ $\Rightarrow (5, 1)$ noktasından geçen normal doğrunun denklemi:

$$y - 1 = m_N(x - 5)$$

$$m_T = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t^3}{4t} = t^2 \Rightarrow m_T \Big|_{t=-1} = (-1)^2 = 1$$

$$m_T \cdot m_N = -1 \Rightarrow m_N = -1$$

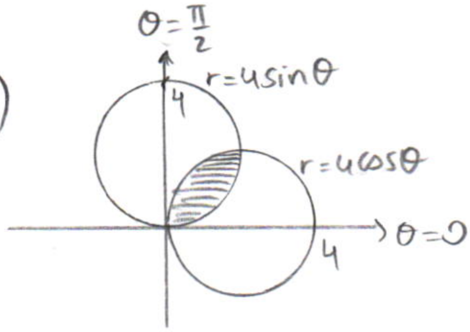
$$y - 1 = -1(x - 5) \Rightarrow y = 6 - x$$

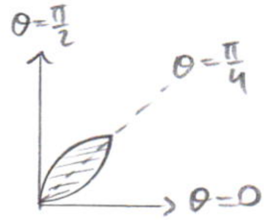
(7) a) $r=4\cos\theta$, $r=4\sin\theta$ } eğrilerinin sınırladığı ortak alanı

b) $r=4\cos\theta$ içinde, $r=4\sin\theta$ dışında } kalan alanı

c) $r=4\cos\theta$ dışında, $r=4\sin\theta$ içinde } kalan alanı bulunuz. (integralleri hesaplamayın)

a)



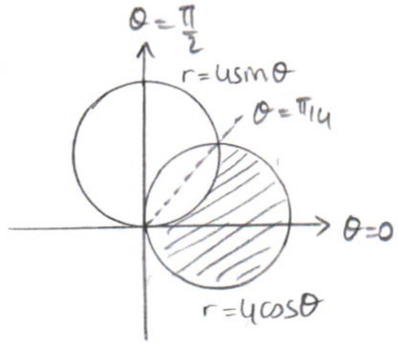


$4\cos\theta=4\sin\theta$

$\theta=\frac{\pi}{4}$

$$A=\frac{1}{2}\int_0^{\pi/4}(4\sin\theta)^2\,d\theta+\frac{1}{2}\int_{\pi/4}^{\pi/2}(4\cos\theta)^2\,d\theta$$

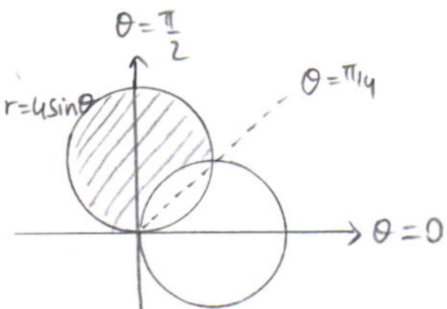
b)



1. yol: $$A=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(4\cos\theta)^2\,d\theta-\frac{1}{2}\left[\int_0^{\pi/4}(4\sin\theta)^2\,d\theta+\int_{\pi/4}^{\pi/2}(4\cos\theta)^2\,d\theta\right]$$

2. yol: $$A=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\pi/4}(4\cos\theta)^2\,d\theta-\frac{1}{2}\int_0^{\pi/4}(4\sin\theta)^2\,d\theta$$

c)



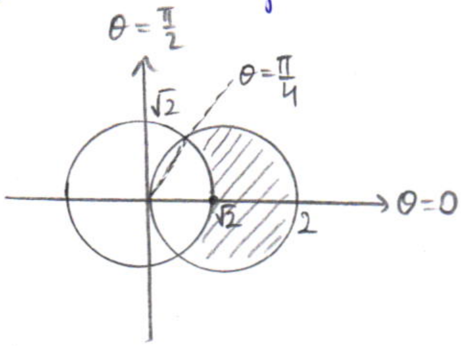
1. yol:

$$A=\frac{1}{2}\int_0^{\pi}(4\sin\theta)^2\,d\theta-\frac{1}{2}\left[\int_0^{\pi/4}(4\sin\theta)^2\,d\theta+\int_{\pi/4}^{\pi/2}(4\cos\theta)^2\,d\theta\right]$$

2. yol:

$$A=\frac{1}{2}\int_{\pi/4}^{\pi}(4\sin\theta)^2\,d\theta-\frac{1}{2}\int_{\pi/4}^{\pi/2}(4\cos\theta)^2\,d\theta$$

⑧ $r=2\cos\theta$ eğrisinin içinde, $r=\sqrt{2}$ eğrisinin dışında kalan alanı bulunuz.



$$2\cos\theta=\sqrt{2} \Rightarrow \cos\theta=\frac{\sqrt{2}}{2} \Rightarrow \theta=\frac{\pi}{4}$$

$$\frac{A}{2}=\frac{1}{2}\int_0^{\pi/4}(2\cos\theta)^2d\theta-\frac{1}{2}\int_0^{\pi/4}(\sqrt{2})^2d\theta$$

$$=\frac{1}{2}\left(\int_0^{\pi/4}(4\cos^2\theta-2)\,d\theta\right)=\frac{1}{2}\int_0^{\pi/4}2\cos2\theta\,d\theta$$

$$=\frac{\sin2\theta}{2}\Big|_0^{\pi/4}=\frac{1}{2} \Rightarrow A=1$$

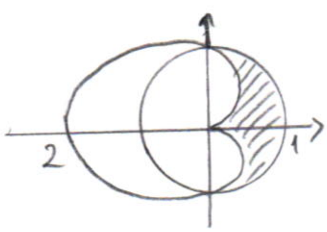
⑨ a) $r=1$ eğrisinin içinde $r=1-\cos\theta$ eğrisinin dışında kalan

b) $r=1$ 〃 dışında $r=1-\cos\theta$ 〃 içinde 〃

c) $r=1$ ve $r=1-\cos\theta$ eğrilerinin sınırladığı ortak alanı bulunuz.

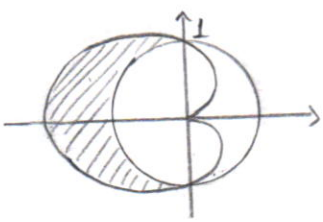
(İnt. hesaplamayın)

a)



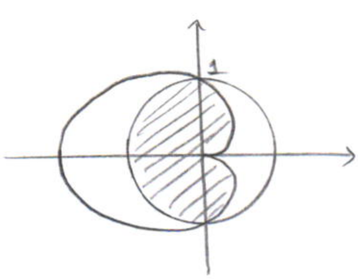
$$\frac{A}{2}=\frac{1}{2}\int_0^{\pi/2}1^2d\theta-\frac{1}{2}\int_0^{\pi/2}(1-\cos\theta)^2d\theta$$

b)



$$\frac{A}{2}=\frac{1}{2}\int_{\pi/2}^{\pi}[(1-\cos\theta)^2-1^2]\,d\theta$$

c)



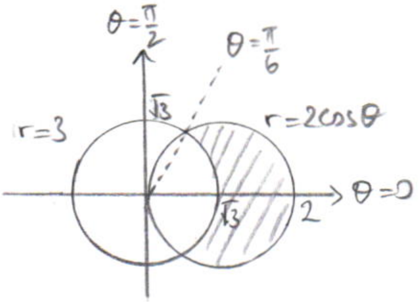
$$\frac{A}{2}=\frac{1}{2}\int_0^{\pi/2}(1-\cos\theta)^2d\theta+\frac{1}{2}\int_{\pi/2}^{\pi}1^2d\theta$$

10) a) $r=2\cos\theta$ içinde, $r=\sqrt{3}$ dışında kalan,

b) $r=2\cos\theta$ dışında, $r=\sqrt{3}$ içinde kalan,

c) $r=2\cos\theta$ ve $r=\sqrt{3}$ sınırladığı ortak alanı bulunuz.

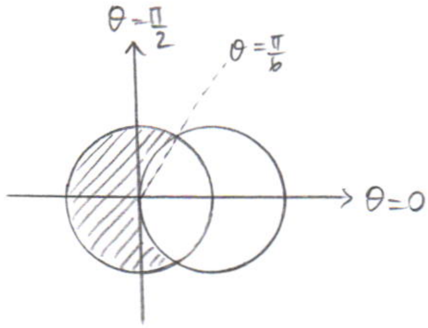
a)



$$2\cos\theta=\sqrt{3} \Rightarrow \theta=\frac{\pi}{6}$$

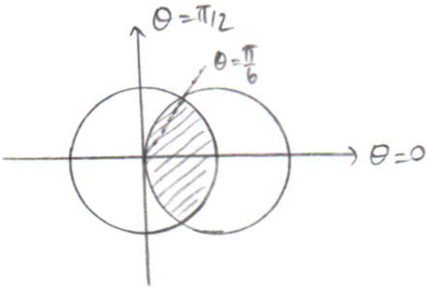
$$\frac{A}{2}=\frac{1}{2}\int_0^{\pi/6}\left((2\cos\theta)^2-3\right)d\theta$$

b)



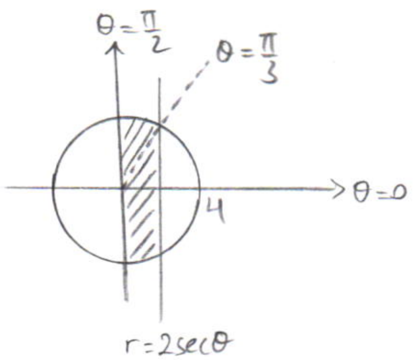
$$\frac{A}{2}=\frac{1}{2}\int_{\pi/6}^{\pi}3\,d\theta-\int_{\pi/6}^{\pi/2}(2\cos\theta)^2\,d\theta$$

c)



$$\frac{A}{2}=\frac{1}{2}\int_0^{\pi/6}3\,d\theta+\frac{1}{2}\int_{\pi/6}^{\pi/2}(2\cos\theta)^2\,d\theta$$

11) $r=4$, $\theta=\frac{\pi}{2}$, $r=2\sec\theta$ arasında kalan alanı hesaplayınız.

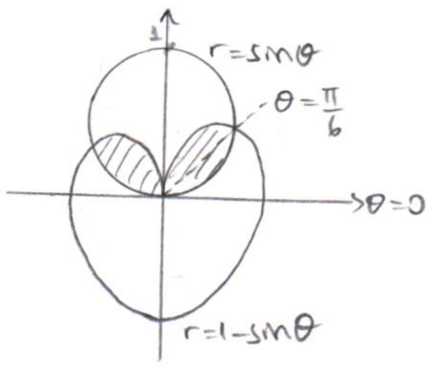


$$r=2\sec\theta=\frac{2}{\cos\theta} \Rightarrow r\cos\theta=2 \Rightarrow x=2 \text{ doğrusu}$$

$$\left.\begin{array}{l} r=4 \\ r=\frac{2}{\cos\theta} \end{array}\right\} \cos\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}$$

$$\frac{A}{2}=\frac{1}{2}\int_0^{\pi/3}(2\sec\theta)^2\,d\theta+\frac{1}{2}\int_{\pi/3}^{\pi/2}4^2\,d\theta \Rightarrow A=4\tan\theta\Big|_0^{\pi/3}+16\theta\Big|_{\pi/3}^{\pi/2}=4\sqrt{3}+\frac{8\pi}{3}$$

12) $r = 1-\sin\theta$ ve $r=\sin\theta$ eğrilerinin sınırladığı bölgenin alanını hesaplayın.



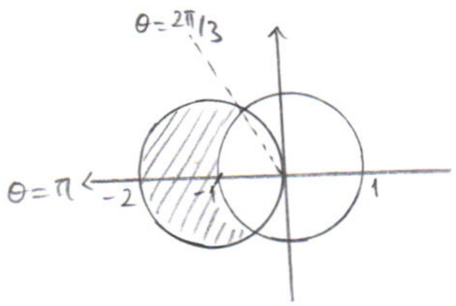
$$1-\sin\theta = \sin\theta \Rightarrow \theta = \frac{\pi}{6} \quad \left(\text{ve } \frac{5\pi}{6}\right)$$

$$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/6} (\sin\theta)^2\, d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} (1-\sin\theta)^2 d\theta$$

$$A = \int_0^{\pi/6} \frac{1-\cos 2\theta}{2}\, d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{2}\left(3-4\sin\theta-\cos 2\theta\right) d\theta$$

$$= \left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right) + \left(\frac{\pi}{2} - \frac{7\sqrt{3}}{8}\right) = \frac{7\pi}{12} - \sqrt{3}$$

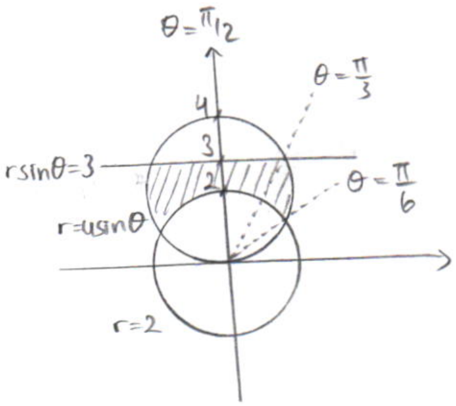
13) $r=-2\cos\theta$ çemberinin içinde, $r=1$ çemberinin dışında kalan alanı bulunuz.



$$-2\cos\theta = 1 \Rightarrow \theta = \frac{2\pi}{3}$$

$$\frac{A}{2} = \frac{1}{2}\int_{2\pi/3}^{\pi} (-2\cos\theta)^2 d\theta - \frac{1}{2}\int_{2\pi/3}^{\pi} 1^2\, d\theta$$

14) $r=2$, $r=4\sin\theta$ ve $r\sin\theta=3$ ile sınırlı bölgenin alanını veren belirli integrali yazınız.



$$r\sin\theta = 3 \Rightarrow y=3 \text{ doğrusu}$$

$$4\sin\theta = \frac{3}{\sin\theta} \Rightarrow \sin^2\theta = \frac{3}{4} \Rightarrow \sin\theta = \mp\frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$4\sin\theta = 2 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

$$\frac{A}{2} = \frac{1}{2}\int_{\pi/6}^{\pi/3} \left[(4\sin\theta)^2 - 2^2\right] d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} \left[\left(3/\sin\theta\right)^2 - 2^2\right] d\theta$$

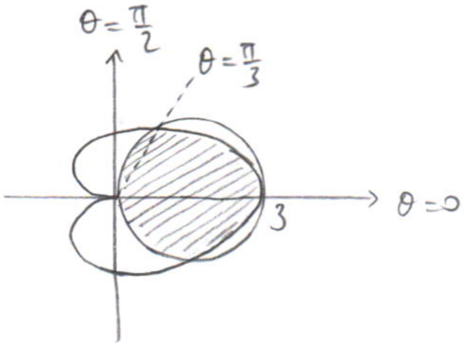
15) a) $r=3\cos\theta$ ve $r=1+\cos\theta$ eğrilerinin sınırladığı ortak alanı

b) $r=3\cos\theta$ nın içinde $r=1+\cos\theta$ nın dışında kalan alanı

c) $r=3\cos\theta$ nın dışında $r=1+\cos\theta$ nın içinde kalan alanı

veren integralleri yazın.

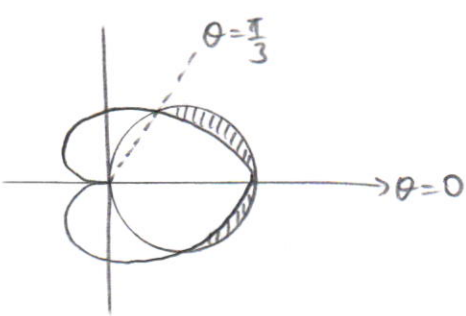
a)



$$1+\cos\theta = 3\cos\theta \Rightarrow \theta=\frac{\pi}{3}$$

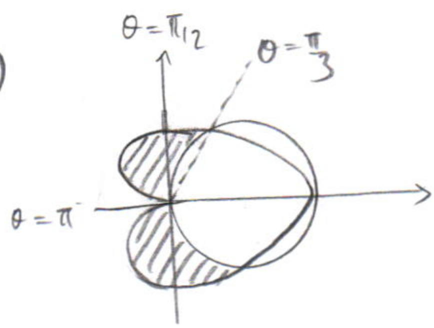
$$\frac{A}{2}=\frac{1}{2}\int_0^{\pi/3}(1+\cos\theta)^2 d\theta+\frac{1}{2}\int_{\pi/3}^{\pi/2}(3\cos\theta)^2 d\theta$$

b)



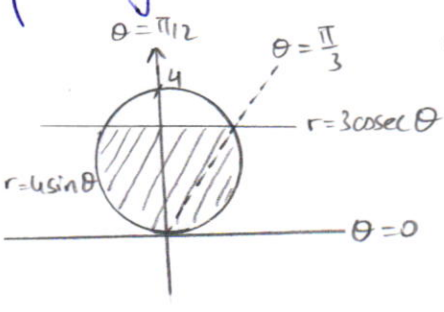
$$\frac{A}{2}=\frac{1}{2}\int_0^{\pi/3}\left[(3\cos\theta)^2-(1+\cos\theta)^2\right]d\theta$$

c)



$$\frac{A}{2}=\frac{1}{2}\int_{\pi/3}^{\pi}(1+\cos\theta)^2 d\theta-\frac{1}{2}\int_{\pi/3}^{\pi/2}(3\cos\theta)^2 d\theta$$

16) $r=4\sin\theta$, $r=3\operatorname{cosec}\theta$ ve $\theta=0$ ile sınırlı bölgenin alanını veren integrali yazınız.

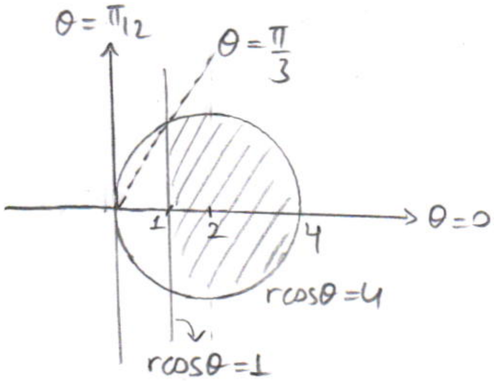


$$r=3\operatorname{cosec}\theta=\frac{3}{\sin\theta} \Rightarrow y=3 \text{ doğrusu}$$

$$4\sin\theta=\frac{3}{\sin\theta} \Rightarrow \theta=\frac{\pi}{3}$$

$$\frac{A}{2}=\frac{1}{2}\int_0^{\pi/3}(3\sin\theta)^2 d\theta+\frac{1}{2}\int_{\pi/3}^{\pi/2}(3\operatorname{cosec}\theta)^2 d\theta$$

(17) $r = 4\cos\theta$ ile $r\cos\theta \geq 1$ sınırladığı ortak alanı bulunuz.



$r\cos\theta \geq 1 \Rightarrow x \geq 1$

$$\left.\begin{array}{l} r = 4\cos\theta \\ r = \dfrac{1}{\cos\theta} \end{array}\right\} \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

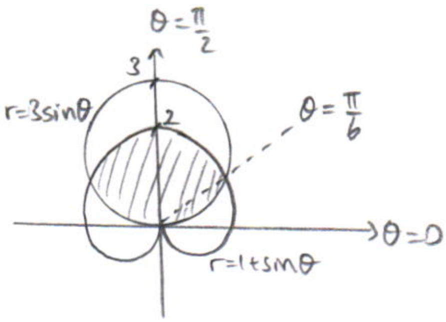
$$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/3} (4\cos\theta)^2 d\theta - \frac{1}{2}\int_0^{\pi/3} \sec^2\theta \, d\theta$$

(18) a) $r = 3\sin\theta$ ve $r = 1+\sin\theta$ eğrilerinin sınırladığı ortak alanı

b) $r = 3\sin\theta$ dışında, $r = 1+\sin\theta$ içinde kalan alanı

c) $r = 3\sin\theta$ içinde, $r = 1+\sin\theta$ dışında kalan alanı bulunuz.

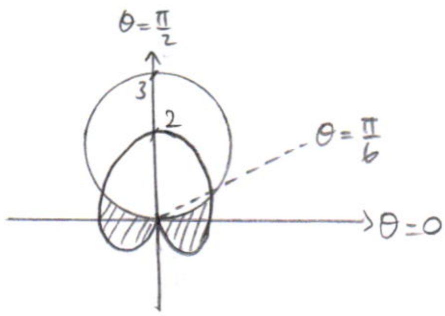
a)



$1+\sin\theta = 3\sin\theta \Rightarrow \theta = \frac{\pi}{6}$

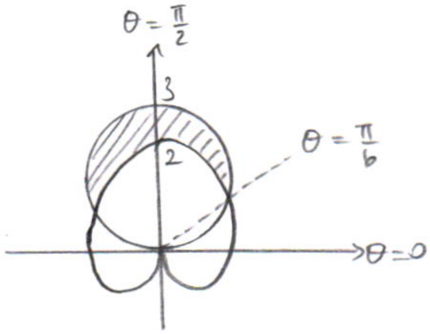
$$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/6} (3\sin\theta)^2 + \frac{1}{2}\int_{\pi/6}^{\pi/2} (1+\sin\theta)^2 d\theta$$

b)



$$\frac{A}{2} = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\pi/6} (1+\sin\theta)^2 d\theta - \frac{1}{2}\int_0^{\pi/6} (3\sin\theta)^2 d\theta$$

c)



$$\frac{A}{2} = \frac{1}{2}\int_{\pi/6}^{\pi/2} \left[(3\sin\theta)^2 - (1+\sin\theta)^2\right] d\theta$$