BLM3620 Digital Signal Processing

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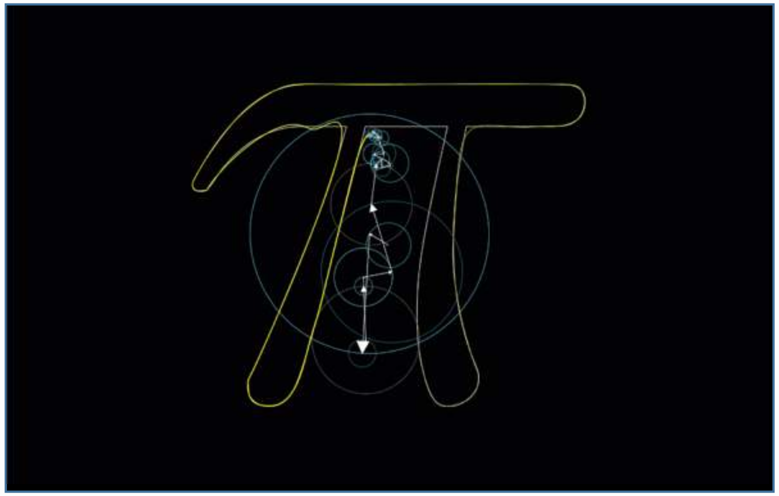
Lecture #3 – Spectrum Representation (for continuous-time signals)

- Spectrum of a Sum of Sinusoids
- Fourier Series – Analysis and Synthesis
- Example: Amplitude Modulation
- Spectrogram
- MATLAB Applications



Course Materials

Important Materials:

- James H. McClellan, R. W. Schafer, M. A. Yoder, *DSP First Second Edition*, Pearson, 2015.

- Lizhe Tan, Jean Jiang, *Digital Signal Processing: Fundamentals and Applications*, Third Edition, Academic Press, 2019.

Auxilary Materials:

- Prof. Sarp Ertürk, *Sayısal İşaret İşleme*, Birsen Yayınevi.

- Prof. Nizamettin Aydin, DSP Lecture Notes.

- J. G. Proakis, D. K. Manolakis, *Digital Signal Processing Fourth Edition*, Peason, 2014.

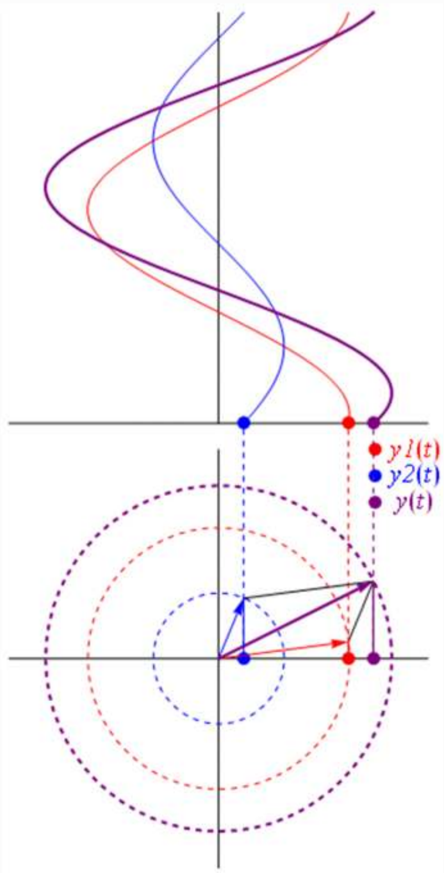
- J. K. Perin, *Digital Signal Processing*, *Lecture Notes,* Standford University, 2018.

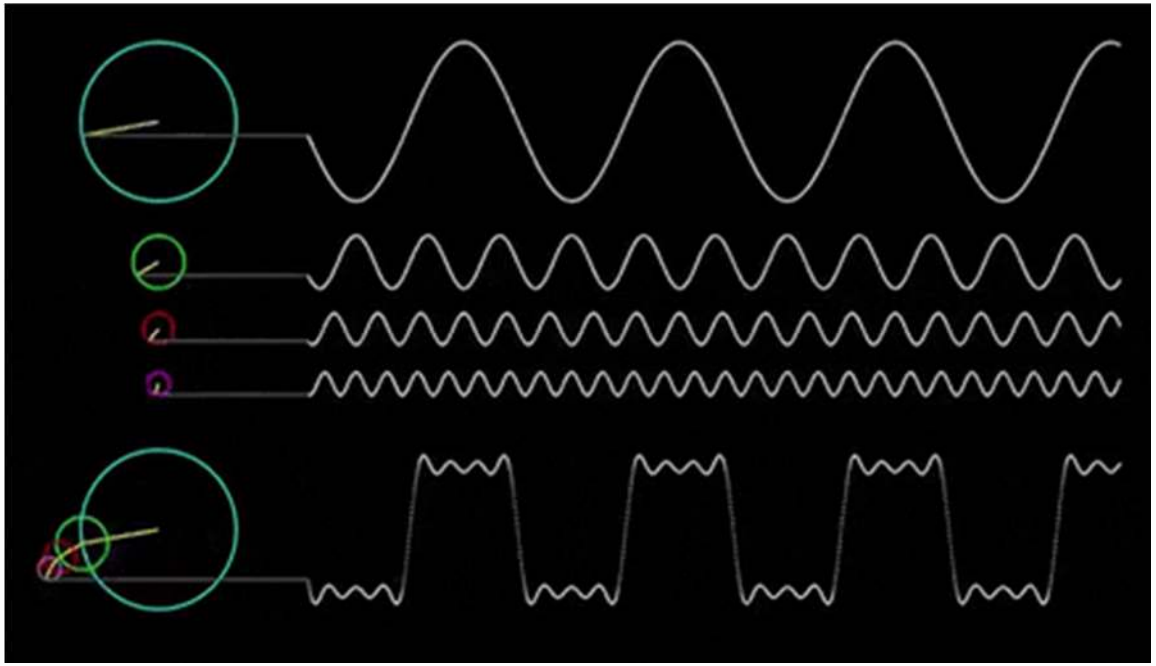
Syllabus

Week	Lectures
1	Introduction to DSP and MATLAB
2	Sinuzoids and Complex Exponentials
3	Spectrum Representation
4	Sampling and Aliasing
5	Discrete Time Signal Properties and Convolution
6	Convolution and FIR Filters
7	Frequency Response of FIR Filters
8	Midterm Exam
9	Discrete Time Fourier Transform and Properties
10	Discrete Fourier Transform and Properties
11	Fast Fourier Transform and Windowing
12	z- Transforms
13	FIR Filter Design and Applications
14	IIR Filter Design and Applications
15	Final Exam

For more details -> Bologna page: http://www.bologna.yildiz.edu.tr/index.php?r=course/view&id=5730&aid=3

Recall: Sum of Phasors and Fourier Series





$$x(t) = \sum_{k=-M}^{M} a_k e^{j2\pi f_k t}$$

Demo Link:https://dspfirst.gatech.edu/chapters/02sines/demos/phasors/index.html

Fourier Series

- Sinusoids with DIFFERENT Frequencies
  - SYNTHESIZE by Adding Sinusoids

$$\textbf{\textit{Harmonic}} \text{ freqs}: f_k = k f_0$$

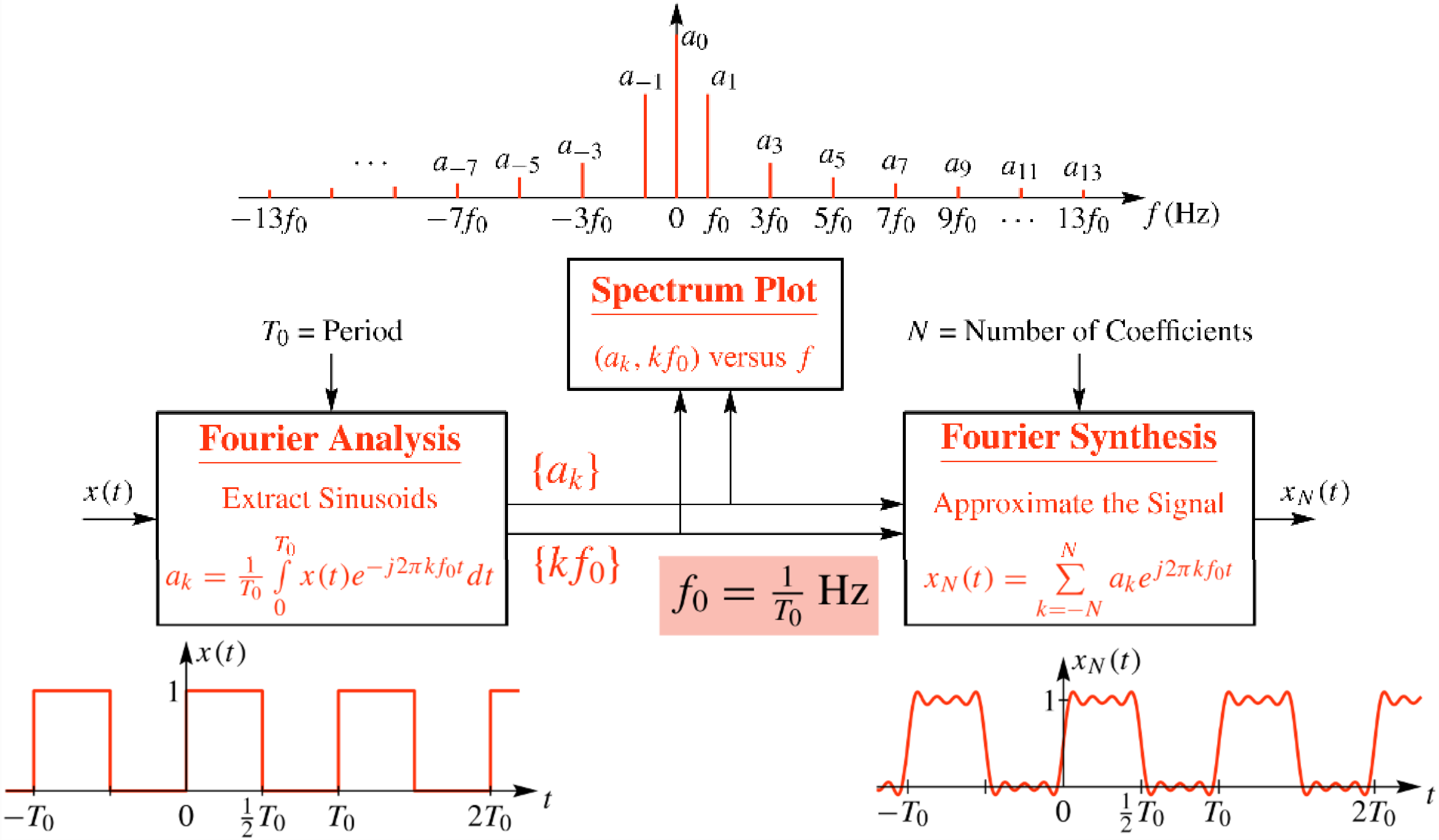
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k F_0 t}$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k F_0 t + \varphi_k)$$

- SPECTRUM Representation
  - Graphical Form shows **DIFFERENT** Freqs

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi k / T_0)t} dt$$

Fourier Series Summary



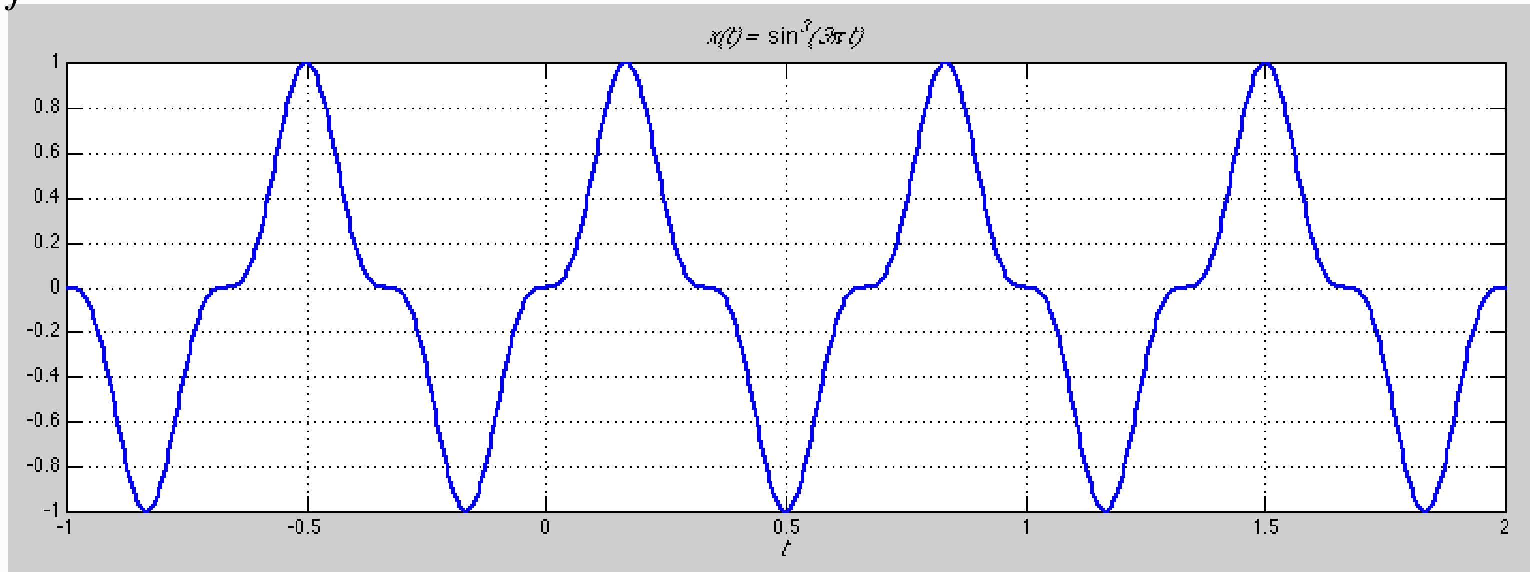
Strategies to Find Fourier Series Coefficients

- <u>Some thoughts:</u>
  - Starting from signal, x(t), which frequencies and complex amplitudes are required?
  - ONLY FOR PERIODIC SIGNALS!
  - Two possible analysis methods:
    - **1. Read off coefficients from inverse Euler's**
    - **2. Evaluate Fourier series integral**
  - Can plot the spectrum for the Fourier Series
    - **Equally spaced lines at kF_0**

STRATEGY 1:

$$x(t) = \sin^3(3\pi t)$$

$$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$



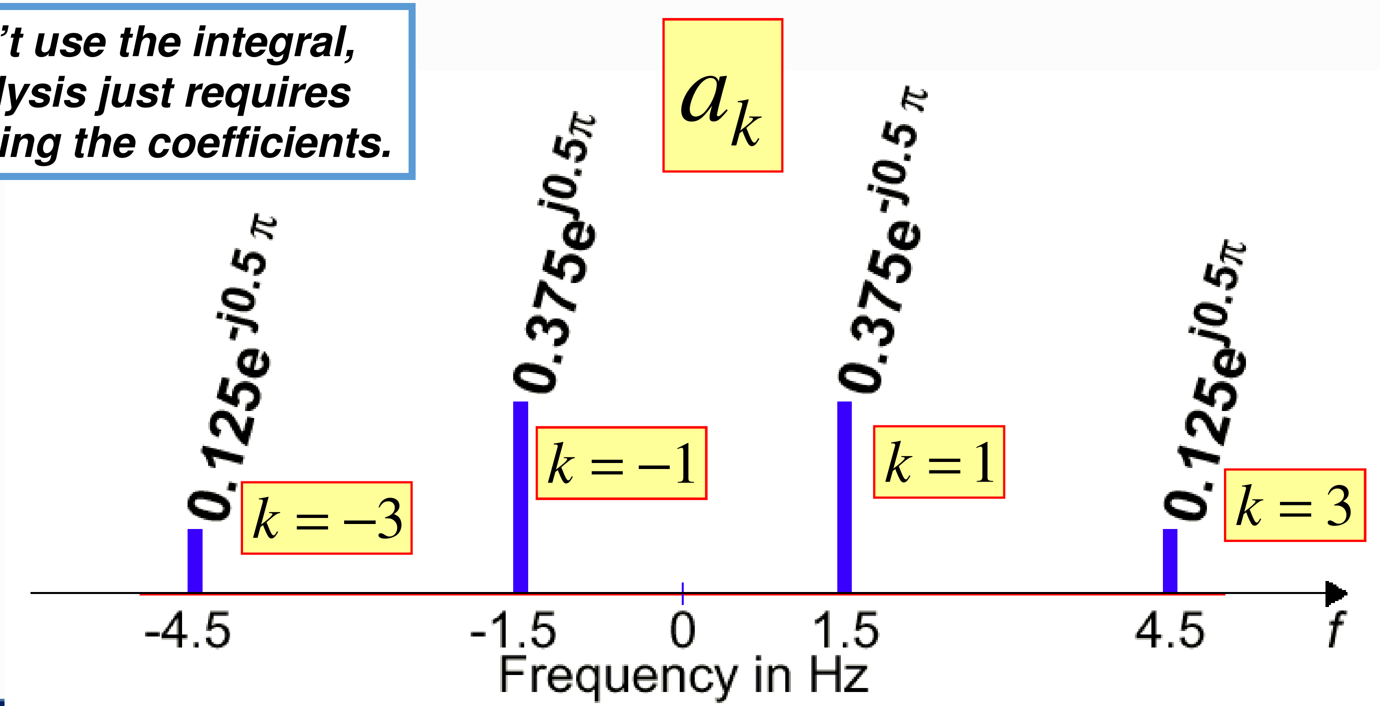
$$x(t) = \left(\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})\right)^3 = \frac{j}{8}\left(e^{j\omega t} - e^{-j\omega t}\right)^3$$

Example

$$x(t) = \sin^3(3\pi t)$$

$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

Don't use the integral, Analysis just requires picking the coefficients.



STRATEGY 2: x(t) → a_k

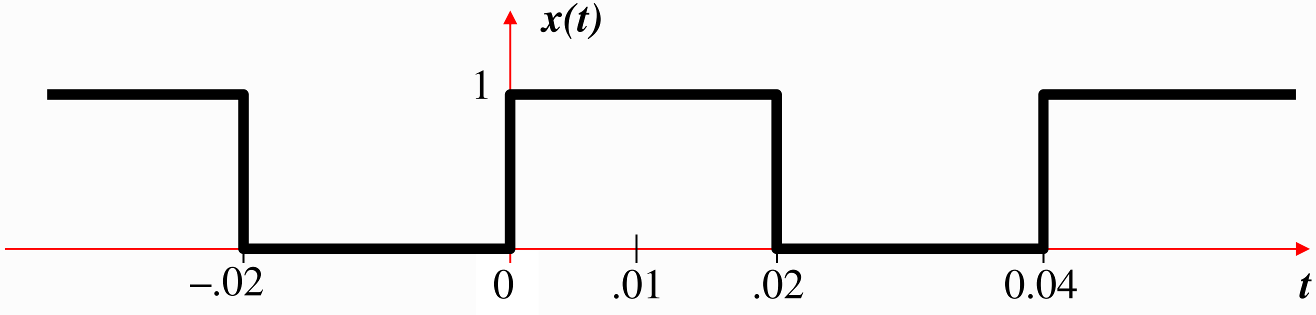
- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}dt$$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2}T_0 \\ 0 & \frac{1}{2}T_0 \le t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



FS for a SQUARE WAVE $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}\,dt \qquad (k \neq 0)$$

$$x(t) = \begin{cases} 1 & 0 \le t < .02 \\ 0 & .02 \le t < .04 \end{cases}$$

$$a_k = \frac{1}{0.04}\int_0^{.02} 1\,e^{-j(2\pi/.04)kt}\,dt = \frac{1}{.04(-j2\pi k/.04)}\,e^{-j(2\pi/.04)kt}\Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)}(e^{-j(\pi)k} - 1) = \frac{1-(-1)^k}{j2\pi k} \qquad (k \neq 0)$$

Square Wave Coeffs: $\{a_k\}$

- Complex Amplitude a_k for k-th Harmonic

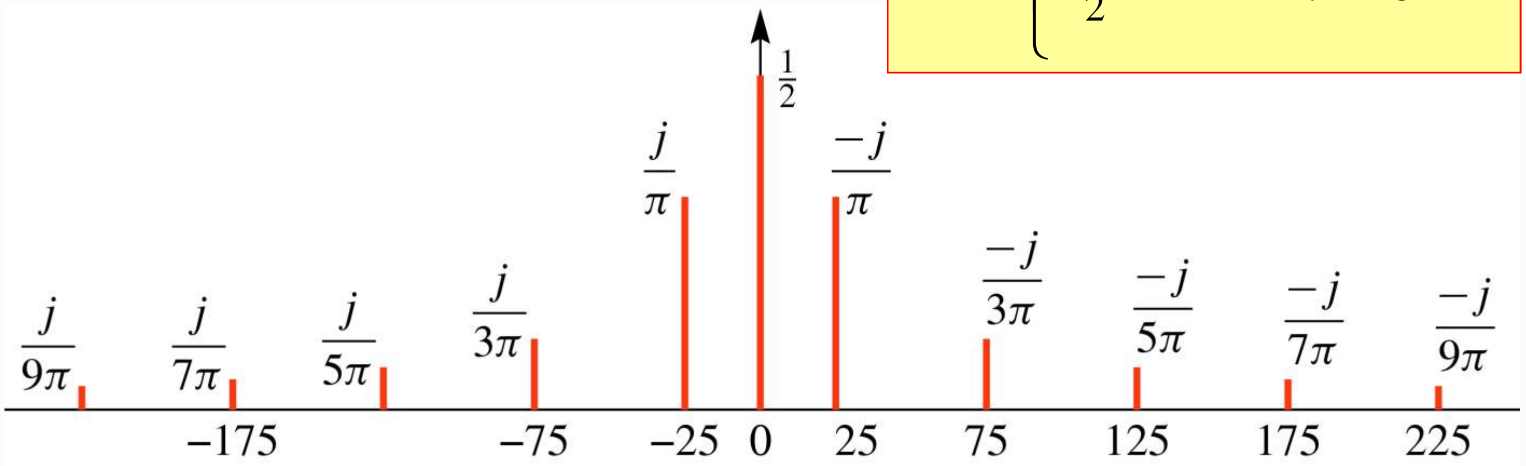
  - Does not depend on the period, T_0
  - DC value is 0.5

$$a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \dfrac{1}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

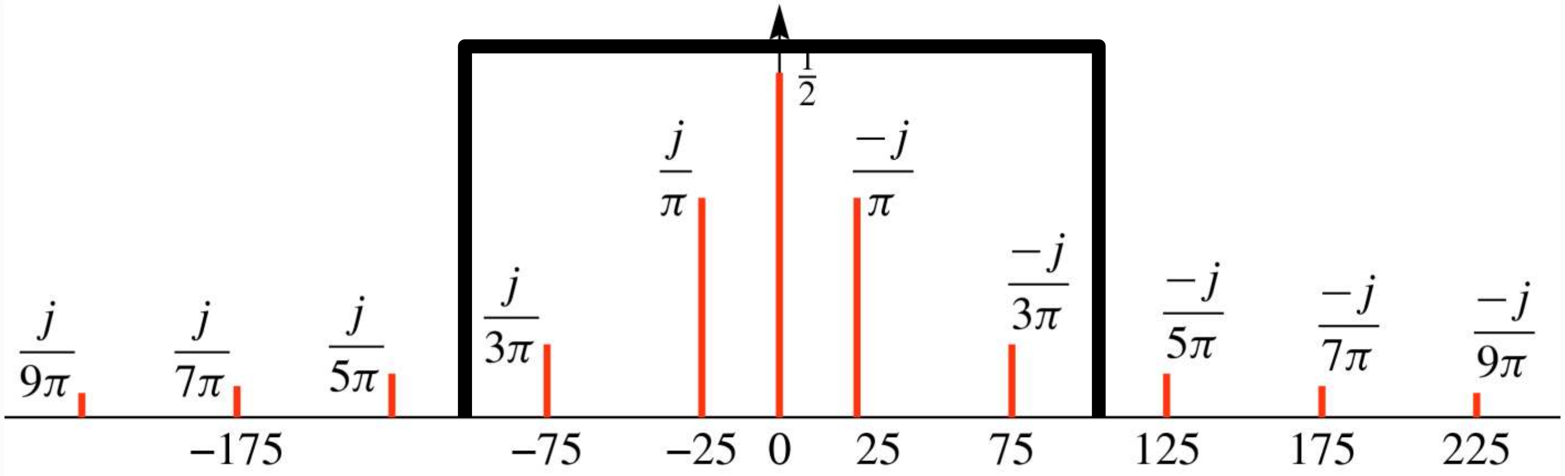
$$T_0 = 0.04 \;\Rightarrow$$
$$\omega_0 = 2\pi/(0.04) = 2\pi(25)$$

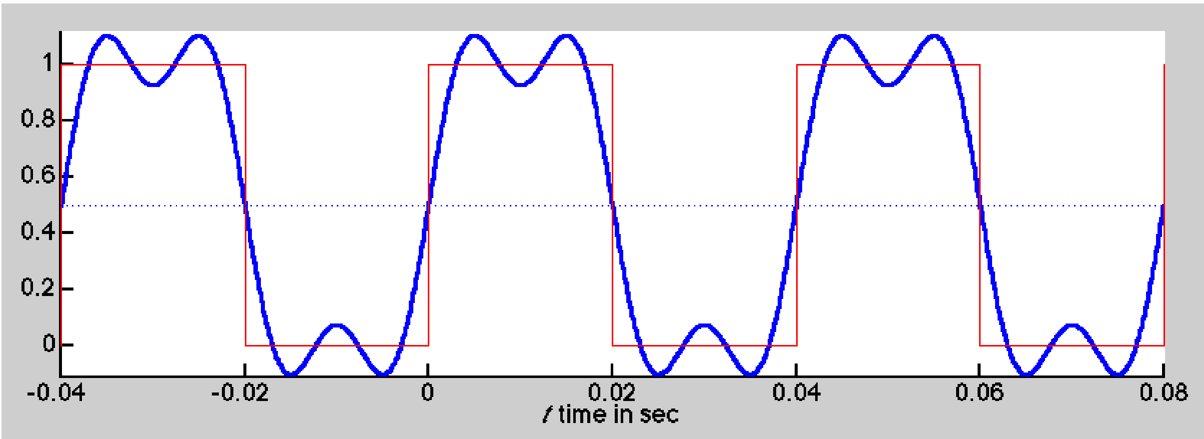
$$a_k = \begin{cases} \dfrac{-j}{\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$



Synthesis: 1st & 3rd Harmonics

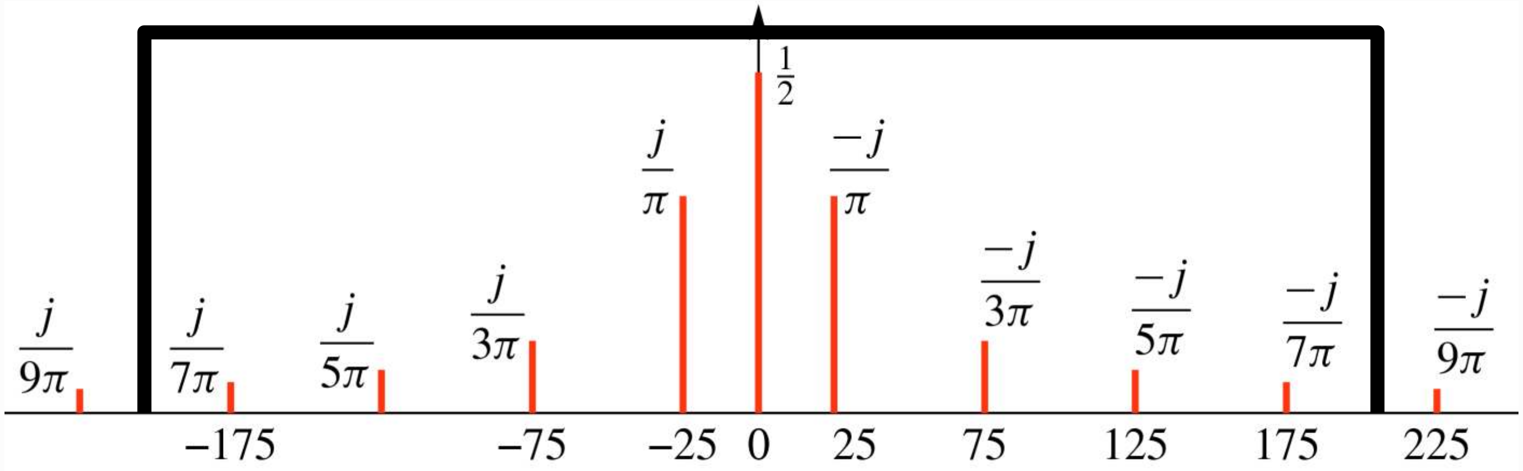
$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi(25)t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\cos(2\pi(75)t - \tfrac{\pi}{2})$$

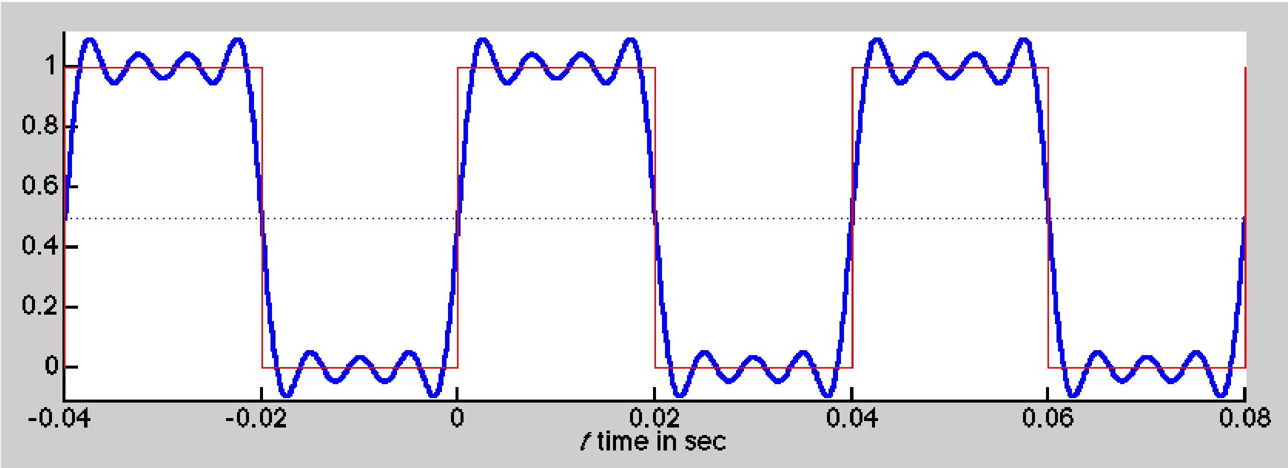




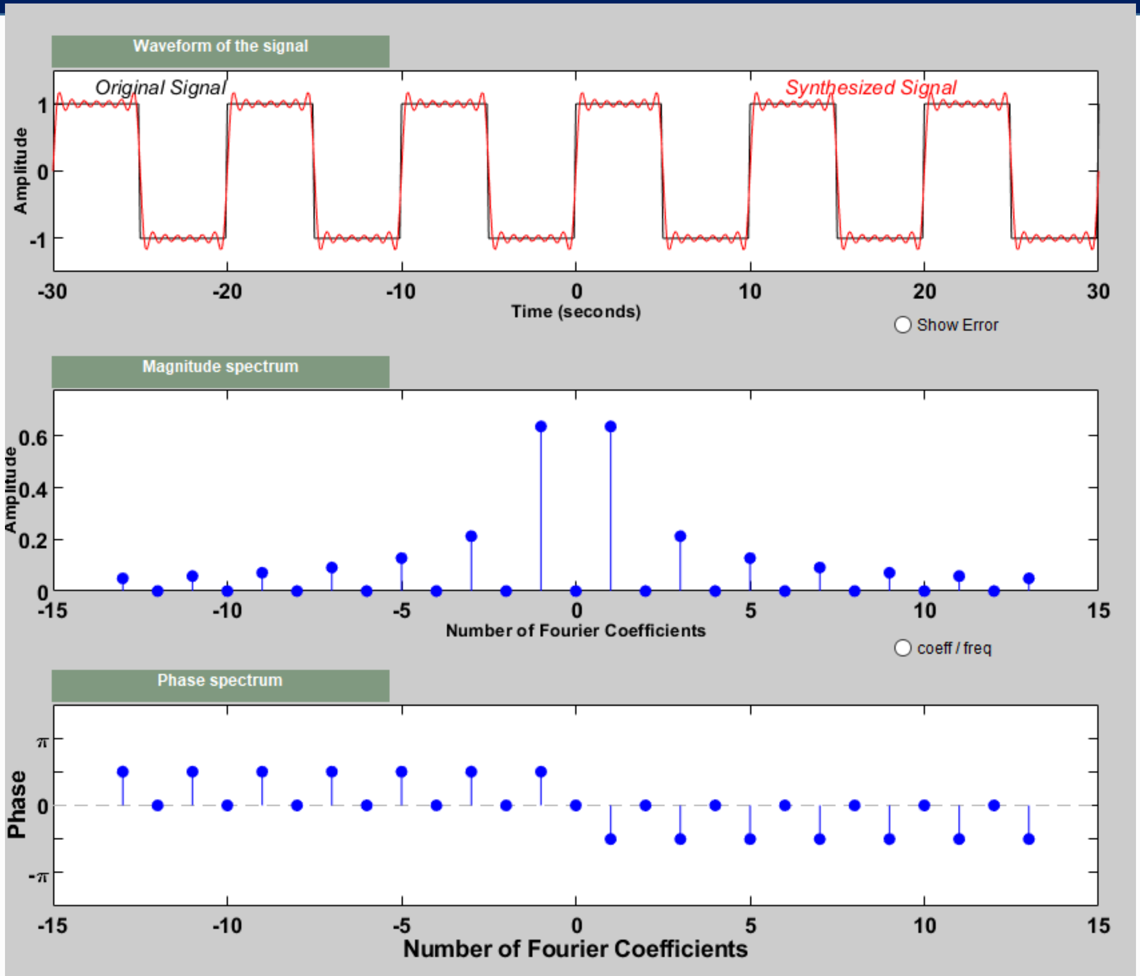
Synthesis: up to 7th Harmonic

$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(50\pi t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\sin(150\pi t) + \frac{2}{5\pi}\sin(250\pi t) + \frac{2}{7\pi}\sin(350\pi t)$$





Fourier Series Demo



https://dspfirst.gatech.edu/chapters/03spect/demosLV/fseries/index.html

More Examples for Strategy -1

$$x(t) = 10 + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2)$$

Find spectrum of signal x(t) .

Apply inverse Euler formula:

$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

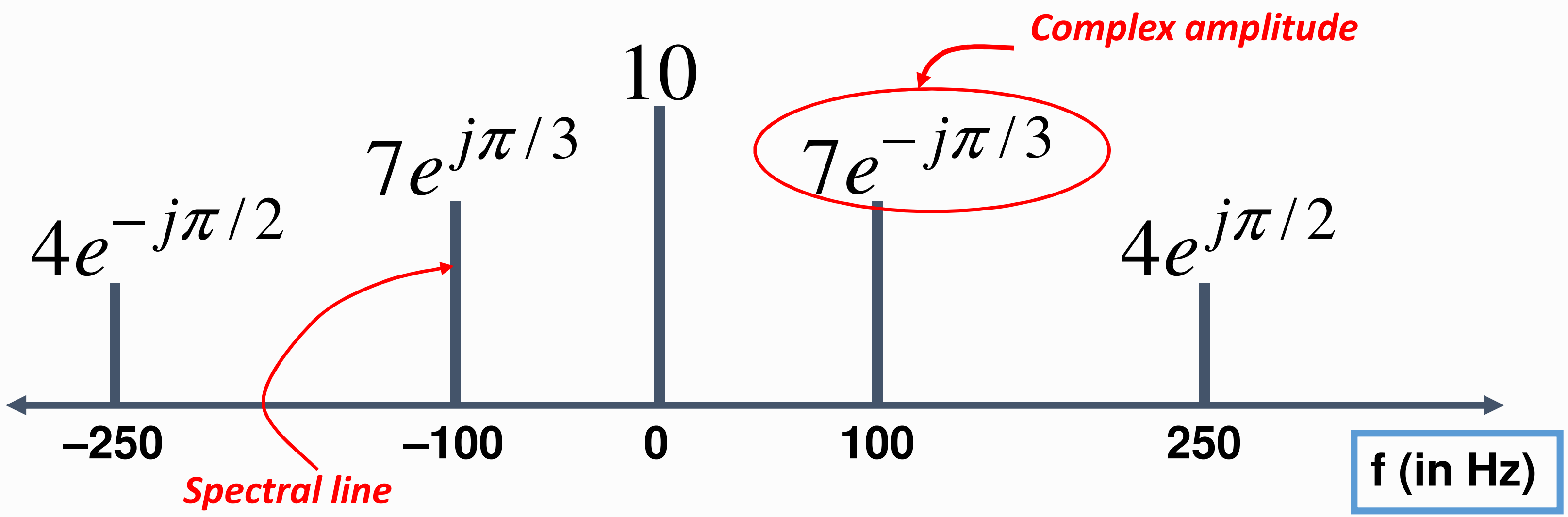
$$\sin(\omega t) = \tfrac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

$$\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

Find the complex amplitude and frequency of these phasors:

$$\{(\underbrace{0}_{f_0},\ \underbrace{10}_{a_0}),\ (\underbrace{100}_{f_2},\ \underbrace{7e^{-j\pi/3}}_{a_2}),\ (\underbrace{-100}_{f_{-2}},\ \underbrace{7e^{j\pi/3}}_{a_{-2}}),\ (250,\ 4e^{j\pi/2}),\ (-250,\ 4e^{-j\pi/2})\}$$

Spectrum Representation



Spectrum Interpretation

$$A\cos(7\pi t + 0.1) = \frac{A}{2} e^{j0.1} e^{j7\pi t} + \frac{A}{2} e^{-j0.1} e^{-j7\pi t}$$

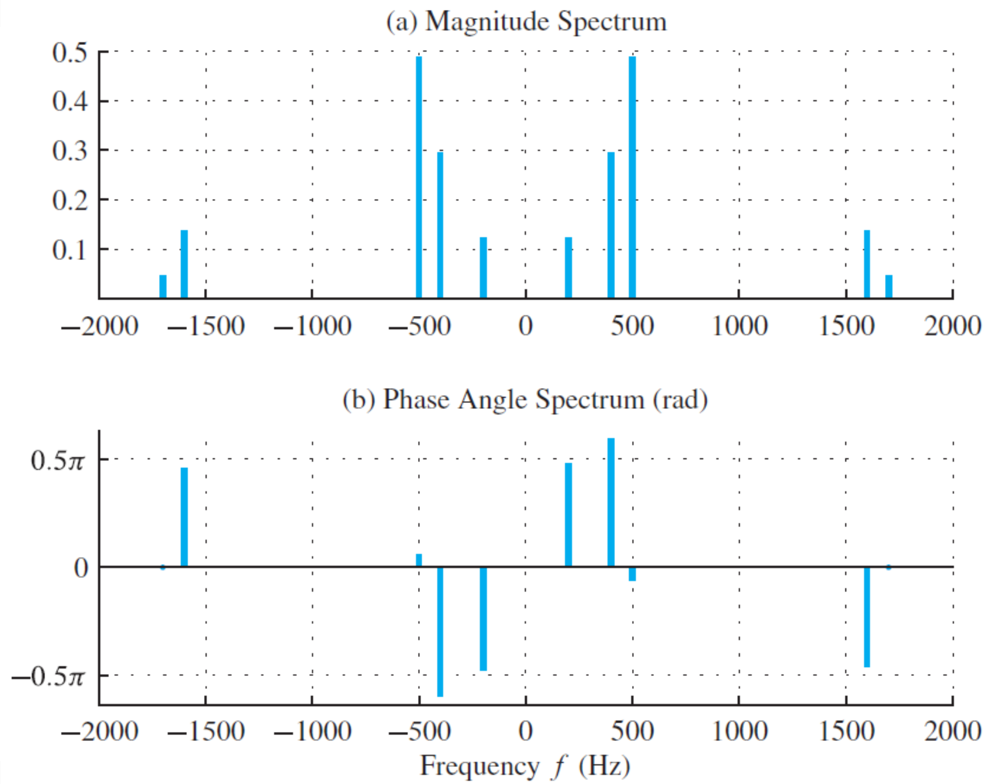
- One has a positive frequency
- The other has negative freq.
- Amplitude of each is half as big

$$\boxed{A\cos(\omega t + \varphi) = \tfrac{1}{2} A e^{j\varphi} e^{j\omega t} + \tfrac{1}{2} A e^{-j\varphi} e^{-j\omega t}}$$

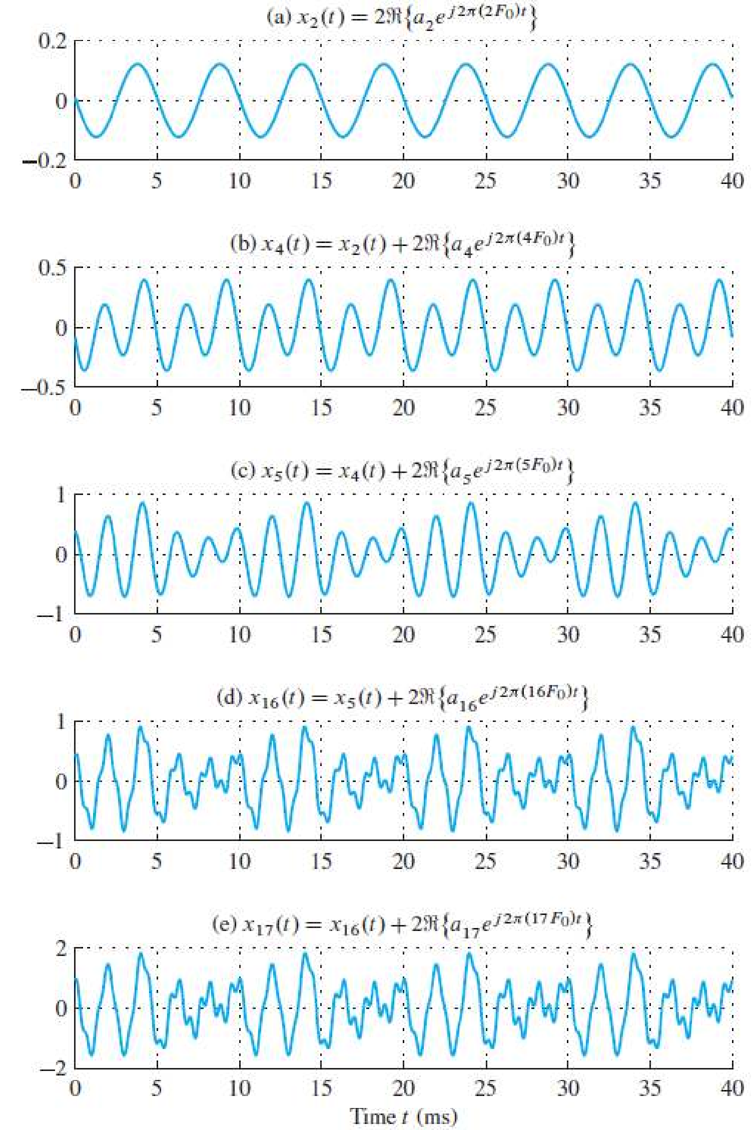
Example: Sythetic Vowel

Table 3-1 Complex amplitudes for the periodic signal that approximates a complicated waveform like a vowel, such as "ah." The a_k coefficients are given for positive indices k, but the values for negative k are the conjugates, $a_{-k} = a_k^*$.

k	f_k (Hz)	a_k	Mag	Phase
1	100	0	0	0
2	200	$0.00772 + j0.122$	0.1223	1.508
3	300	0	0	0
4	400	$-0.08866 + j0.2805$	0.2942	1.877
5	500	$0.48 - j0.08996$	0.4884	−0.185
6	600	0	0	0
⋮	⋮	⋮	⋮	⋮
15	1500	0	0	0
16	1600	$0.01656 - j0.1352$	0.1362	−1.449
17	1700	$0.04724 + j0$	0.04724	0



Vowel Waveform



(a) The 200-Hz term alone. (b) Sum of the 400-Hz and 200-Hz terms. Additional terms are added one at a time until the entire synthetic vowel signal is created in (e). (c) Adding the 500-Hz term, which changes the fundamental period, (d) adding the 1600-Hz term, and (e) adding the 1700-Hz term.

FREQUENCY ANALYSIS

- **Now, a much HARDER problem**

- Given a recording of a song, have the computer write the music

- Can a machine extract frequencies?
  - Yes, if we COMPUTE the spectrum for x(t)
    - During short intervals

Frequency can change with time ☹ What can we do?

Time-Varying FREQUENCIES Diagram

Frequency is the vertical axis

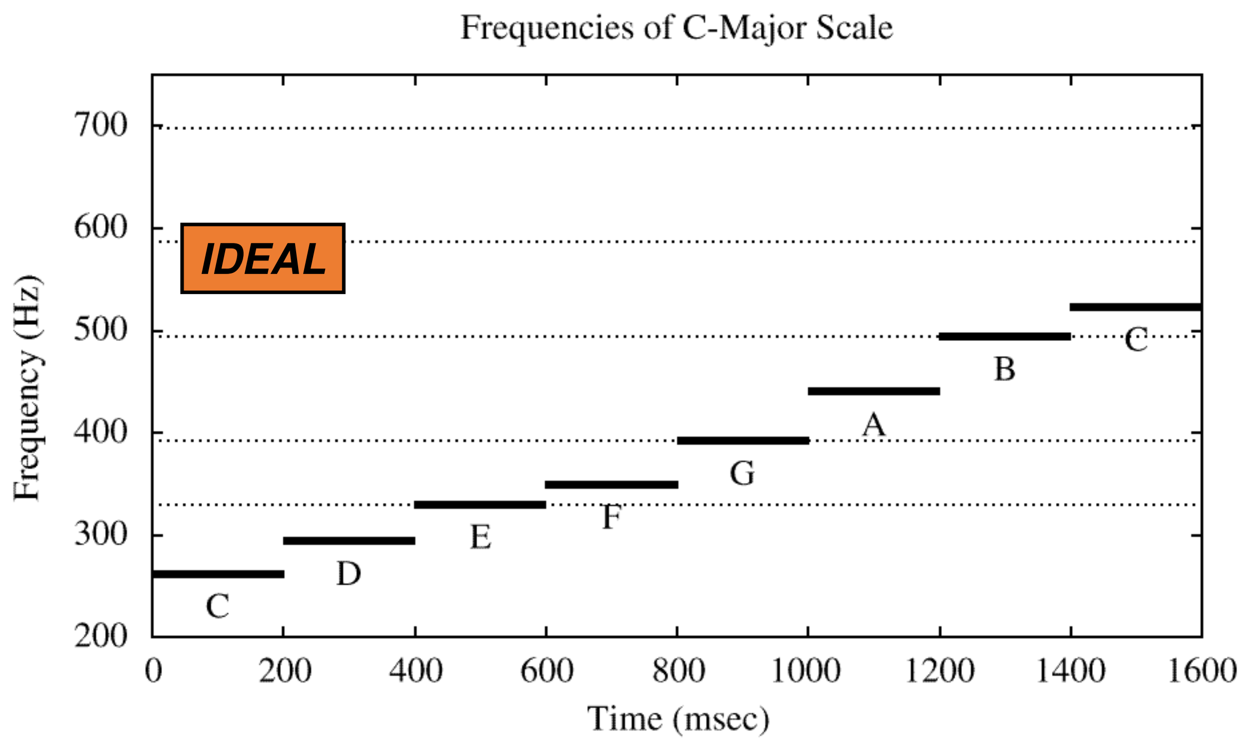
Figure 3.18 Sheet-music notation is a time–frequency diagram.

Time is the horizontal axis

SIMPLE TEST SIGNAL

- C-major SCALE: stepped frequencies
  - Frequency is constant for each note

Middle C	D_4	E_4	F_4	G_4	A_4	B_4	C_5
262 Hz	294	330	349	392	440	494	523

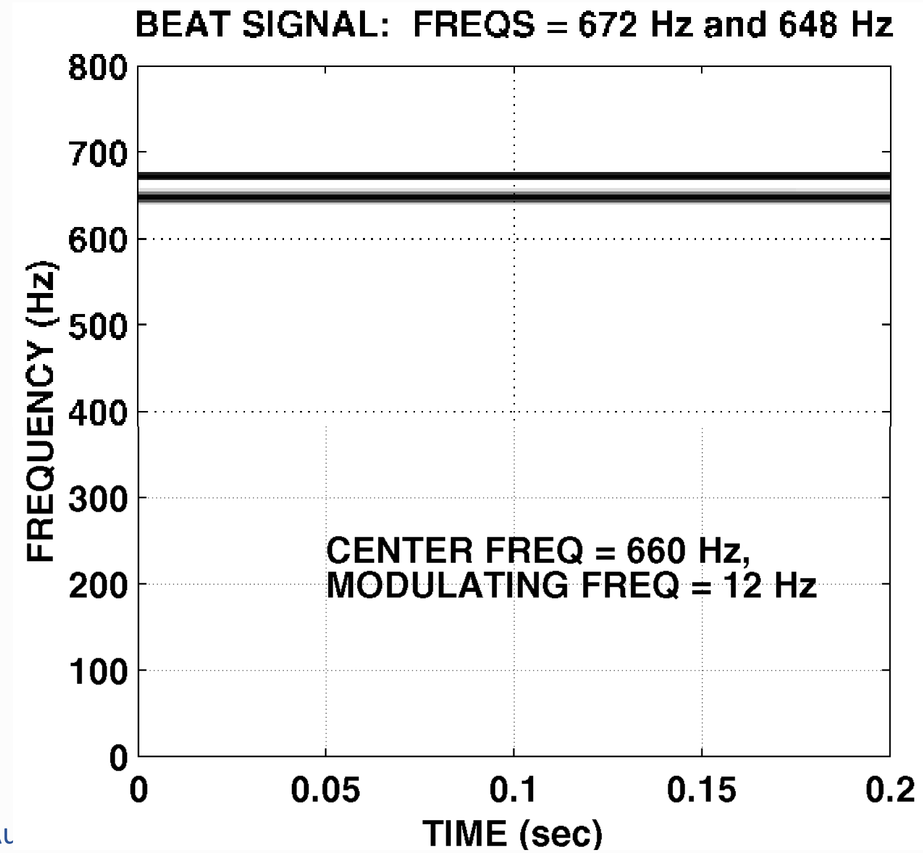


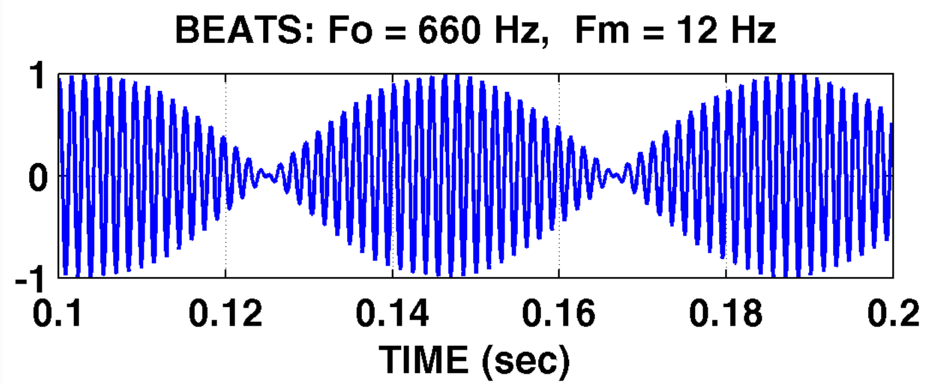
SPECTROGRAM

- SPECTROGRAM Tool
  - MATLAB function is `spectrogram.m`
  - SP-First has `plotspec.m` & `spectgr.m`
- <u>**ANALYSIS**</u> program
  - Takes x(t) as input
  - Produces spectrum values X_k
  - Breaks x(t) into SHORT TIME SEGMENTS
    - Then uses the FFT (<u>Fast Fourier Transform</u>)

SPECTROGRAM EXAMPLE

- Two **Constant** Frequencies: Beats





$$\cos(2\pi(672)t) + \cos(2\pi(648)t)$$
$$= 2\cos(2\pi(12)t)\cos(2\pi(660)t)$$

AM Radio Signal

- Same form as BEAT Notes, but higher in freq

$$\boxed{\cos(2\pi(\underline{660})t)\sin(2\pi(12)t)}$$

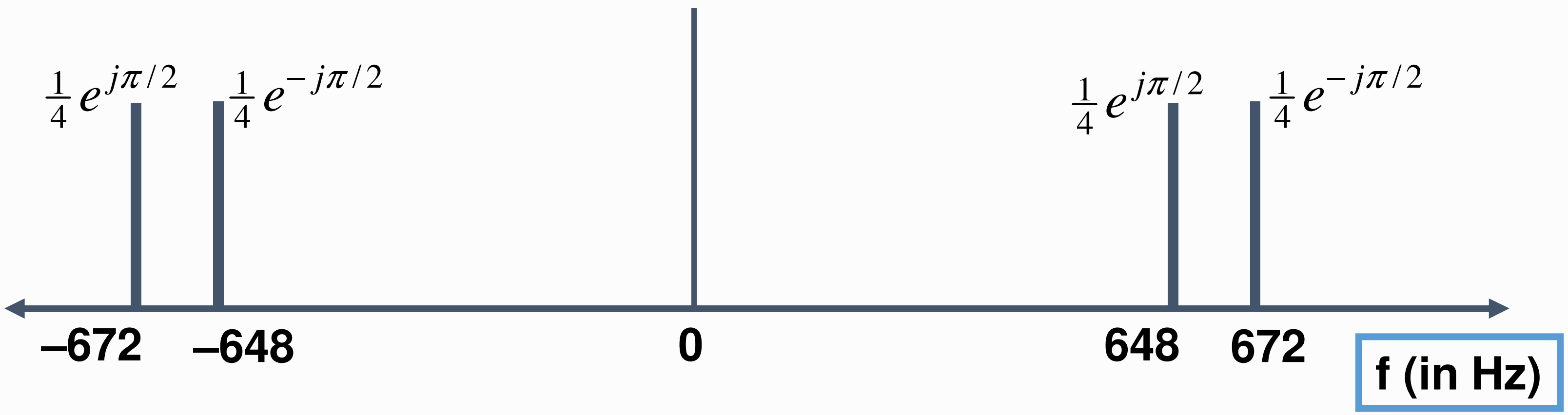
$$\frac{1}{2}\left(e^{j2\pi(660)t}+e^{-j2\pi(660)t}\right)\frac{1}{2j}\left(e^{j2\pi(12)t}-e^{-j2\pi(12)t}\right)$$

$$\frac{1}{4j}\left(e^{j2\pi(672)t}-e^{-j2\pi(672)t}-e^{j2\pi(648)t}+e^{-j2\pi(648)t}\right)$$

$$\boxed{\tfrac{1}{2}\cos(2\pi(672)t-\tfrac{\pi}{2})+\tfrac{1}{2}\cos(2\pi(648)t+\tfrac{\pi}{2})}$$

SPECTRUM of AM (Amplitude Modulation)

- **SUM** of 4 complex exponentials:

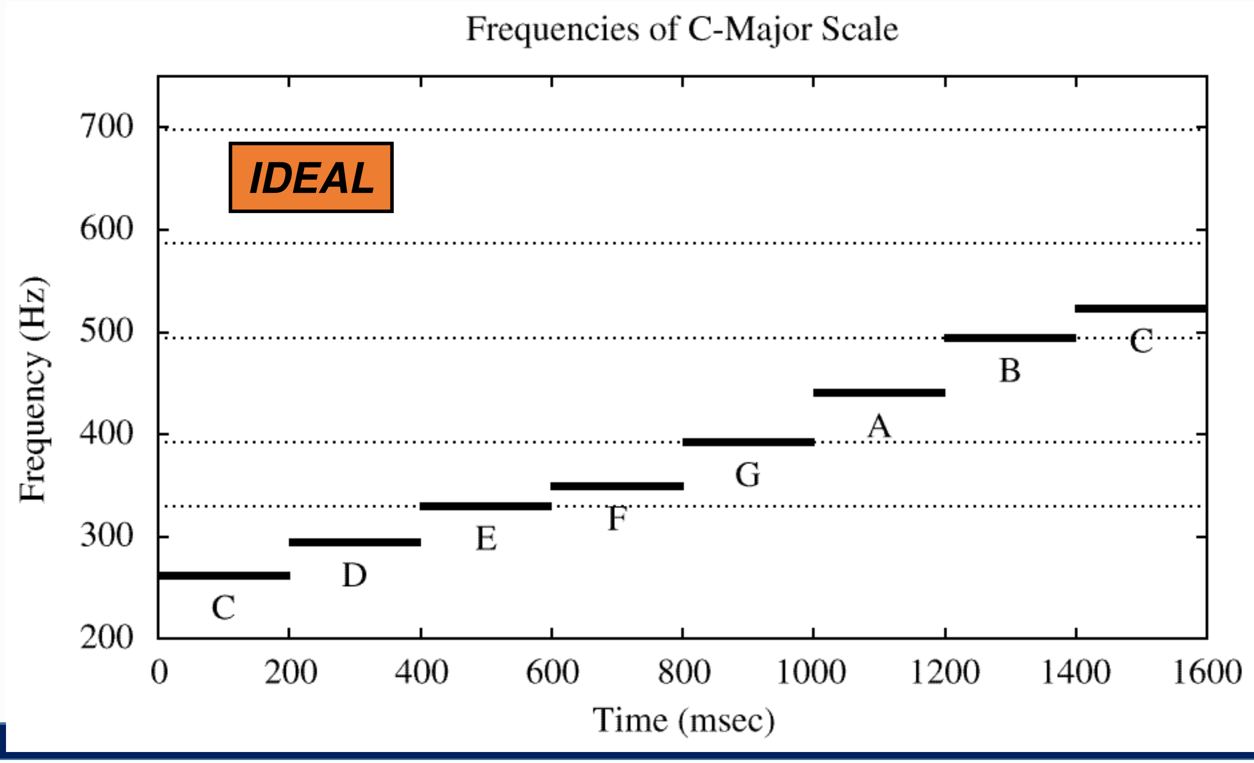


What is the fundamental frequency?

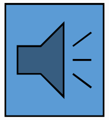
648 Hz ? **24 Hz ?**

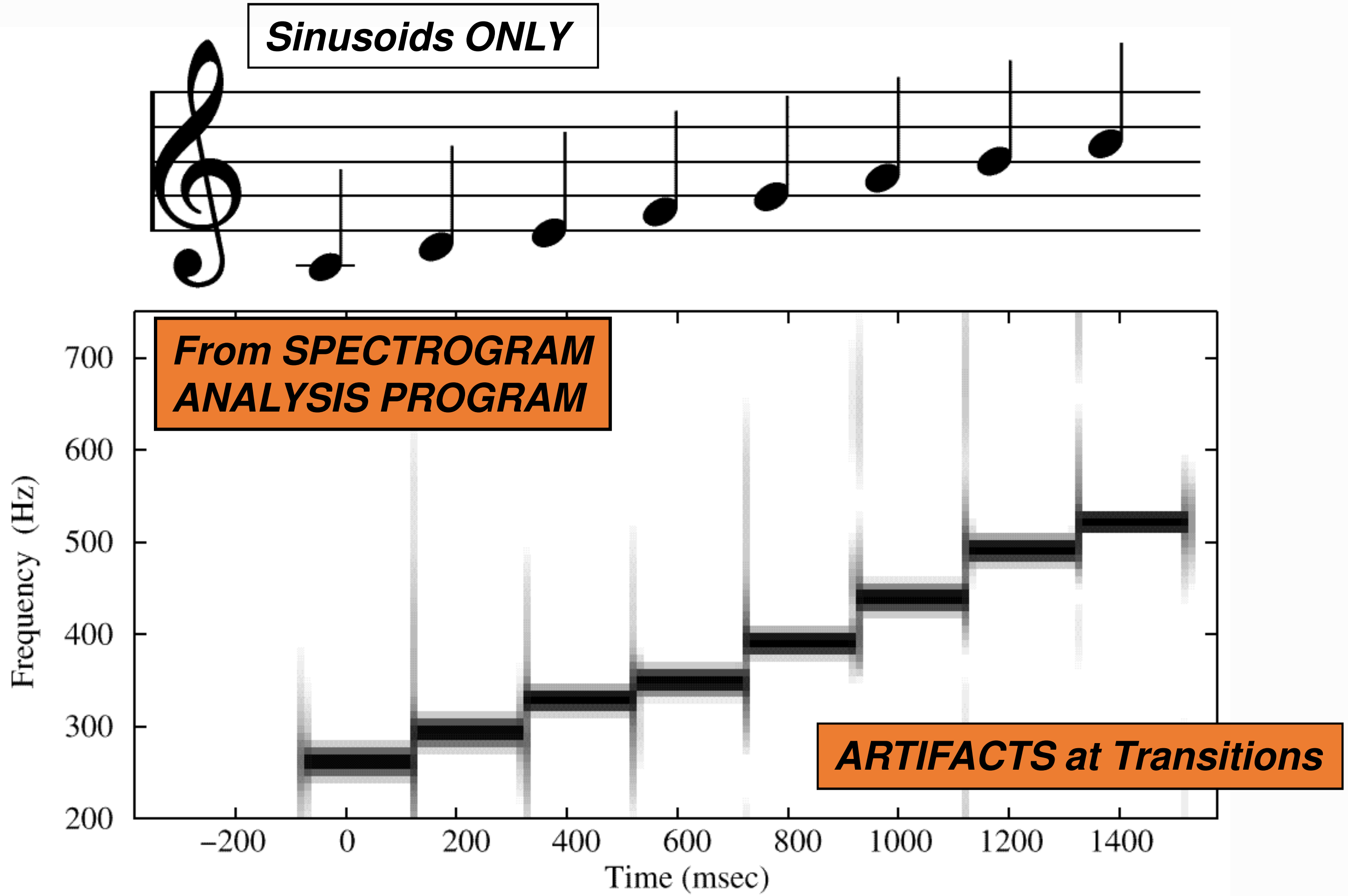
STEPPED FREQUENCIES

- C-major SCALE: successive sinusoids
  - Frequency is constant for each note



SPECTROGRAM of C-Scale

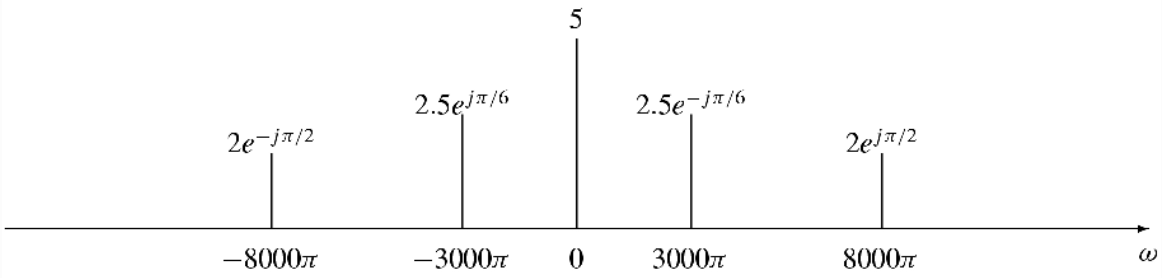




Example 1

PROBLEM:

A real signal $x(t)$ has the following two-sided spectrum:



(a) Write an equation for $x(t)$ as a sum of cosines.

(b) Plot the spectrum of the signal $y(t) = 2x(t) - 3\cos(5000\pi(t - 0.002))$.

a) $x(t) = 5 + (2.5 \times 2)\cos\left(3000\pi t - \frac{\pi}{6}\right) + (2 \times 2)\cos\left(8000\pi t + \frac{\pi}{2}\right)$

careful! do not forget this factor 2 !

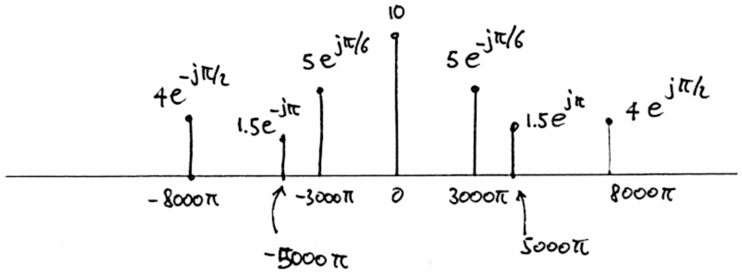
b) $y(t) = 2x(t) - 3\cos(5000\pi(t - 0.002))$

$= 2x(t) - 3\cos(5000\pi t - 10\pi)$

$= 2x(t) - 3\cos(5000\pi t)$

$= 2x(t) + 3\cos(5000\pi t + \pi)$

$\text{Spec}(y) = 2 \cdot \text{Spec}(x) + \text{Spec}(3\cos(5000\pi t + \pi))$



Example 2

A signal composed of sinusoids is given by the equation

$$x(t) = 3\cos(50\pi t - \pi/8) - 5\cos(150\pi t + \pi/6)$$

(a) Sketch the spectrum of this signal indicating the complex amplitude of each frequency component. You do not have to make separate plots for real/imaginary parts or magnitude/phase. Just indicate the complex amplitude value at the appropriate frequency.

(b) Is $x(t)$ periodic? If so, what is the period? Which harmonics are present?

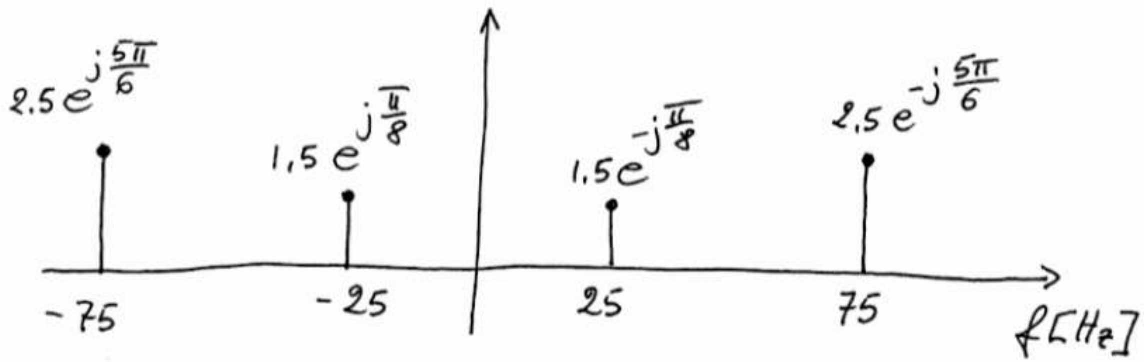
Answer

a) $$x(t) = \frac{3}{2} e^{-j\frac{\pi}{8}} e^{j50\pi t} + \frac{3}{2} e^{j\frac{\pi}{8}} e^{-j50\pi t}$$

$$- \frac{5}{2} e^{j\frac{\pi}{6}} e^{j150\pi t} - \frac{5}{2} e^{-j\frac{\pi}{6}} e^{-j150\pi t} =$$

$$= \frac{5}{2} e^{j\frac{5\pi}{6}} e^{-j150\pi t} + \frac{3}{2} e^{j\frac{\pi}{8}} e^{-j50\pi t}$$

$$+ \frac{3}{2} e^{-j\frac{\pi}{8}} e^{j50\pi t} + \frac{5}{2} e^{-j\frac{5\pi}{6}} e^{j150\pi t}$$



b) Yes, x(t) is periodic : $T = \frac{1}{25} = 40\ ms$
First and third harmonics are present.

Example 3

A periodic signal, $x(t)$, is given by

$$x(t) = 2 + \sin(300\pi t) + 3\cos(600\pi t + \pi/3)$$

(a) What is the period of $x(t)$?

FUNDAMENTAL FREQ.: $\omega_0 = 300\pi = \frac{2\pi}{T}$

$$\Rightarrow \boxed{T = \frac{1}{150} \text{ SEC.}}$$

(b) Find the Fourier series coefficients of $x(t)$ for $-6 \le k \le 6$.

USING EULER'S RELATION

$$x(t) = 2 + \frac{1}{2} e^{-j\pi/2} \cdot e^{j300\pi t} + \frac{1}{2} e^{j\pi/2} e^{-j300\pi t} + \frac{3}{2} e^{j\pi/3} \cdot e^{j2(300\pi)t} + \frac{3}{2} e^{-j\pi/3} e^{-j2(300\pi)t}$$

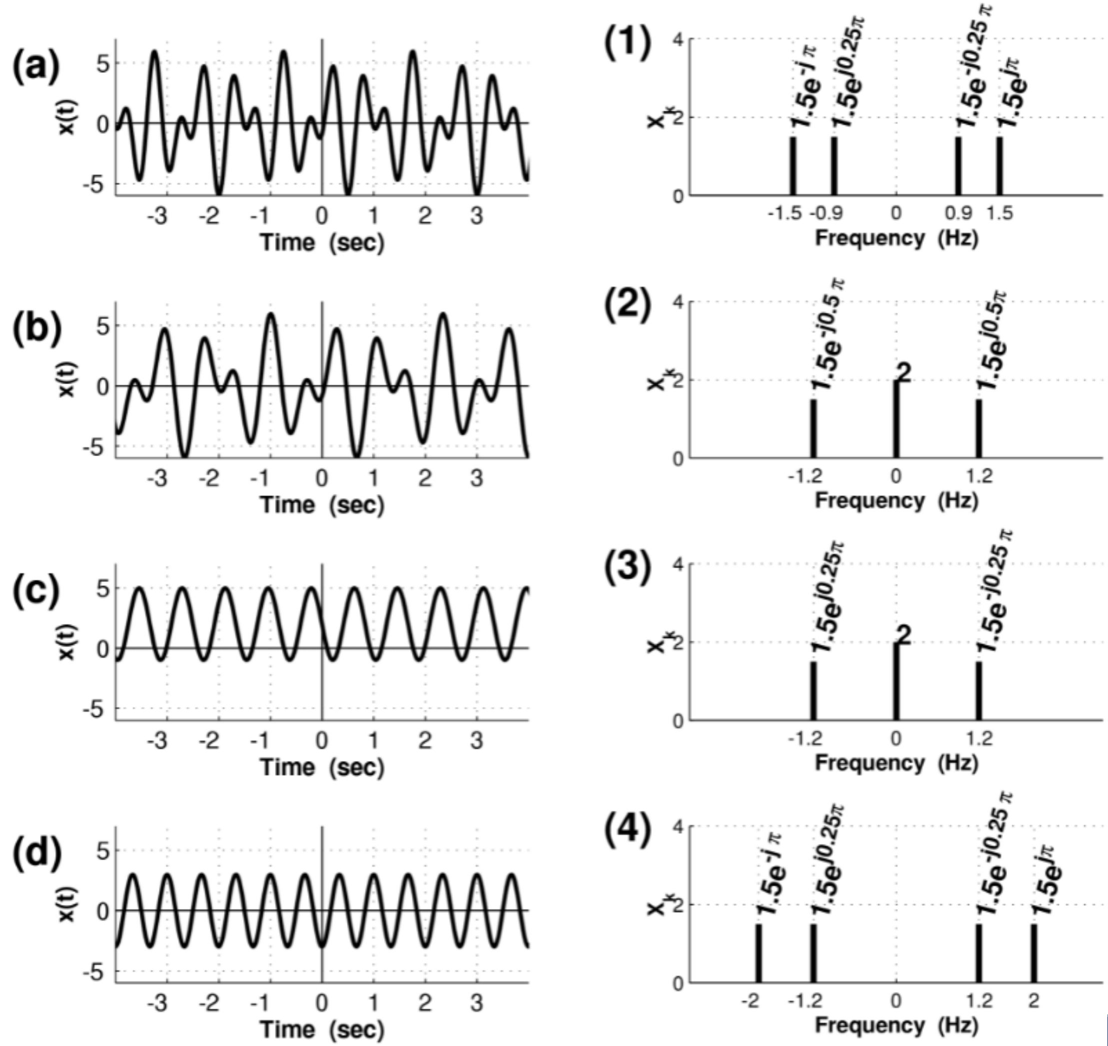
$$\boxed{\begin{array}{ll} a_0 = 2 & a_2 = \frac{3}{2} e^{j\pi/3} \\ a_1 = \frac{1}{2} e^{-j\pi/2} & a_{-2} = \frac{3}{2} e^{-j\pi/3} \\ a_{-1} = \frac{1}{2} e^{j\pi/2} & a_k = 0 \text{ FOR ALL OTHER } k \end{array}}$$

Example 4

Several signals are plotted below along with their corresponding spectra. However, they are in a random order. For each of the signals (a)–(e), determine the correct spectrum (1)–(5). Write your answers in the following table:

(a)	(b)	(c)	(d)	(e)

(a)	(b)	(c)	(d)	(e)
4	1	2	5	3



More Examples

Can be found here:

https://dspfirst.gatech.edu/database/?d=homework&chap=3

https://dspfirst.gatech.edu/chapters/03spect/demos/spectrog/index.html