

MAT 1320 LINEAR ALGEBRA

FURTHER EXERCISES

Note: $\text{rank } A$: the number of nonzero rows in an echelon form of A .

1. If $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ -1 & 1 & -2 \end{pmatrix}$, then find $\text{rank}(A)$.

Answer: $\text{rank } A = 2$

2. If $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 2 & -1 & 2 & 5 \\ 5 & 6 & 3 & 2 \\ 1 & 3 & -1 & -3 \end{pmatrix}$, then find $\text{rank}(A)$.

$$\begin{array}{l}
 \begin{array}{l}
 r_2 \rightarrow r_2 - 3r_1 \\
 r_3 \rightarrow r_3 - 2r_1 \\
 r_4 \rightarrow r_4 - 5r_1 \\
 r_5 \rightarrow r_5 - r_1
 \end{array}
 \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -4 & -2 & 2 \\ 0 & -5 & 0 & 5 \\ 0 & -4 & -2 & 2 \\ 0 & 1 & -2 & -3 \end{pmatrix}
 \xrightarrow{\substack{r_4 \rightarrow r_4 - r_2 \\ r_3 \cdot \frac{-1}{5}}}
 \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -4 & -2 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 \end{pmatrix}
 \xrightarrow{\substack{r_2 \leftrightarrow r_3 \\ r_4 \leftrightarrow r_3}}
 \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -4 & -2 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
 \\
 \begin{array}{l}
 r_3 \rightarrow r_3 + 4r_2 \\
 r_4 \rightarrow r_4 - r_2
 \end{array}
 \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}
 \xrightarrow{r_4 \rightarrow r_4 - r_3}
 \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
 \xrightarrow{r_3 \cdot \frac{-1}{2}}
 \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
 \end{array}$$

$\Rightarrow \text{rank } A = 3$.

Note: $[A : I_n] \longrightarrow \dots \longrightarrow [I_n : A^{-1}]$ (if A^{-1} exists)
(via elementary row operations)

3. Find the inverse of $A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$ (if exists).

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array} \right) \xrightarrow{\substack{r_2 \rightarrow r_2 - r_1 \\ r_3 \rightarrow r_3 - r_1}} \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right)$$

$$\xrightarrow{r_1 \rightarrow r_1 - 3r_2 - 3r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right) \Rightarrow A^{-1} = \begin{pmatrix} 4 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$\underbrace{\hspace{10em}}_{I_3} \quad \underbrace{\hspace{10em}}_{A^{-1}}$

4. Calculate $\begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix}$.

$$\begin{array}{l} r_2 \rightarrow r_2 - r_1 \\ r_3 \rightarrow r_3 - r_1 \end{array} \begin{vmatrix} 1 & x & y+z \\ 0 & y-x & x-y \\ 0 & z-x & x-z \end{vmatrix} \xrightarrow{\substack{\text{Laplace for} \\ \text{first column}}} 1 \cdot (-1)^{1+1} \begin{vmatrix} y-x & x-y \\ z-x & x-z \end{vmatrix}$$

$$\begin{aligned} &= (y-x)(x-z) - (x-y)(z-x) \\ &= (y-x)(x-z) + (y-x)(z-x) = 0 \end{aligned}$$

5. Calculate $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$.

$$\begin{array}{l} r_2 \rightarrow r_2 - r_1 \\ r_3 \rightarrow r_3 - r_1 \end{array} = \begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & b^3-a^3 \\ 0 & c-a & c^3-a^3 \end{vmatrix} = 1 \cdot (-1)^{1+1} \begin{vmatrix} b-a & b^3-a^3 \\ c-a & c^3-a^3 \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & b^2+ab+a^2 \\ 1 & c^2+ca+a^2 \end{vmatrix}$$

$$\begin{aligned} &= (b-a)(c-a) (c^2+ca+a^2 - b^2-ab-a^2) \\ &= (b-a)(c-a) (c^2-b^2 - a(c-b)) \\ &= (b-a)(c-a)(c-b)(a+b+c) \end{aligned}$$

6. Calculate $\begin{vmatrix} a & a^3 & bc \\ b & b^3 & ca \\ c & c^3 & ab \end{vmatrix} = \frac{1}{abc} \begin{vmatrix} a^2 & a^4 & abc \\ b^2 & b^4 & abc \\ c^2 & c^4 & abc \end{vmatrix} = \frac{abc}{abc} \begin{vmatrix} a^2 & a^4 & 1 \\ b^2 & b^4 & 1 \\ c^2 & c^4 & 1 \end{vmatrix}$

$$\begin{aligned} & \begin{vmatrix} a^2 & a^4 & 1 \\ b^2 & b^4 & 1 \\ c^2 & c^4 & 1 \end{vmatrix} \xrightarrow{r_2 \rightarrow r_2 - r_1} \begin{vmatrix} a^2 & a^4 & 1 \\ b^2 - a^2 & b^4 - a^4 & 0 \\ c^2 - a^2 & c^4 - a^4 & 0 \end{vmatrix} \\ & \xrightarrow{r_3 \rightarrow r_3 - r_1} \begin{vmatrix} a^2 & a^4 & 1 \\ b^2 - a^2 & b^4 - a^4 & 0 \\ c^2 - a^2 & c^4 - a^4 & 0 \end{vmatrix} = 1 \cdot (-1)^{1+3} \begin{vmatrix} b^2 - a^2 & b^4 - a^4 \\ c^2 - a^2 & c^4 - a^4 \end{vmatrix} = (b^2 - a^2)(c^2 - a^2) \begin{vmatrix} 1 & b^2 + a^2 \\ 1 & c^2 + a^2 \end{vmatrix} \\ & = (b^2 - a^2)(c^2 - a^2)(c^2 - b^2) \end{aligned}$$

7. If $\begin{vmatrix} r & t & 1 \\ p & -1 & w \\ 2 & s & u \end{vmatrix} = 1$, then $\begin{vmatrix} r-2 & t-s & 1-u \\ -p+2u & 1+su & -w+u^2 \\ 4 & 2s & 2u \end{vmatrix} = ?$

$$\begin{aligned} & = 2 \cdot \begin{vmatrix} r-2 & t-s & 1-u \\ -p+2u & 1+su & -w+u^2 \\ 2 & s & u \end{vmatrix} \xrightarrow{\substack{r_2 \rightarrow r_2 - u r_3 \\ r_1 \rightarrow r_1 + r_3}} \begin{vmatrix} r & t & 1 \\ -p & 1 & -w \\ 2 & s & u \end{vmatrix} = -2 \begin{vmatrix} r & t & 1 \\ p & -1 & -w \\ 2 & s & u \end{vmatrix} \\ & = -2 \cdot 1 = -2. \end{aligned}$$

8. Show that $\begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ n+1 & n+2 & n+3 & \dots & 2n-1 & 2n \\ 2n+1 & 2n+2 & 2n+3 & \dots & 3n-1 & 3n \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ n^2-n+1 & n^2-n+2 & n^2-n+3 & \dots & n^2-1 & n^2 \end{vmatrix} = 0$.

$$\begin{aligned} & \begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ n & n & n & \dots & n & n \\ 2n & 2n & 2n & \dots & 2n & 2n \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ n^2-n & n^2-n & n^2-n & \dots & n^2-n & n^2-n \end{vmatrix} \xrightarrow{r_3 \rightarrow r_3 - n r_2} \begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ n & n & n & \dots & n & n \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ n^2-n & n^2-n & n^2-n & \dots & n^2-n & n^2-n \end{vmatrix} \end{aligned}$$

There is zero row. Then,

3 = 0.

9. Show that
$$\begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix}_{n \times n} = (-1)^{n-1} (n-1).$$

See that the sum of entries in each row is $n-1$. Then,

$$C_1 \rightarrow C_1 + C_2 + \cdots + C_n = \begin{vmatrix} n-1 & 1 & 1 & \cdots & 1 \\ n-1 & 0 & 1 & \cdots & 1 \\ n-1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & 1 & 1 & \cdots & 0 \end{vmatrix} = (n-1) \cdot \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{vmatrix} = (n-1) \cdot \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix}$$

10. Calculate
$$\begin{vmatrix} x+y & x & \cdots & x \\ x & x+y & \cdots & x \\ \vdots & \vdots & \ddots & \vdots \\ x & x & \cdots & x+y \end{vmatrix}_{n \times n}.$$

$$= (n-1)! \cdot 1 \cdot (-1)^{1+n-1} \cdot \begin{vmatrix} -1 & & & \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{vmatrix}_{(n-1) \times (n-1)} = (n-1)! \cdot (-1)^{n+1}$$

$$= \begin{vmatrix} nx+y & x & \cdots & x \\ nx+y & x+y & \cdots & x \\ \vdots & \vdots & \ddots & \vdots \\ nx+y & x & \cdots & x+y \end{vmatrix} = (nx+y) \cdot \begin{vmatrix} 1 & x & \cdots & x \\ 1 & x+y & \cdots & x \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x & \cdots & x+y \end{vmatrix} = (nx+y) \cdot \begin{vmatrix} 1 & x & \cdots & x \\ 0 & y & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y \end{vmatrix}$$

$$= (nx+y) \cdot 1 \cdot (-1)^{n+1} \cdot \begin{vmatrix} y & & & 0 \\ & \ddots & & \\ 0 & & & y \end{vmatrix}_{(n-1) \times (n-1)} = (nx+y)! \cdot y^{n-1}$$

11. Calculate
$$\begin{vmatrix} 2a & 2b & b-c \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix}.$$

$$r_1 + r_2 = \begin{vmatrix} 2a+2b & 2b+2a & b+a \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix} = (a+b) \cdot \begin{vmatrix} 2 & 2 & 1 \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ 2b-2a-2c & c & a+c \\ a-b & a-b & b \end{vmatrix} = 1 \cdot (-1)^{1+3} \cdot \begin{vmatrix} 2b-2a-2c & c \\ a-b & a-b \end{vmatrix} = (a-b) \cdot \begin{vmatrix} 2b-2a-2c & c \\ 1 & 1 \end{vmatrix}$$

$$= (a-b)(2b-2a-3c)$$

12. Calculate
$$\begin{vmatrix} 1+x & 1 & 1 & \vdots & 1 \\ 1 & 1+x & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \ddots & 1+x & 1 \\ 1 & \cdots & \cdots & 1 & 1+x \end{vmatrix}_{n \times n}.$$

(Similar to exercise 10.)
(Replace x and y by 1 and x respectively.)

13. Calculate
$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}.$$

14. Calculate
$$\begin{vmatrix} x^2 & x^2+2x+1 & x^2+4x+4 \\ x^2+2x+1 & x^2+4x+4 & x^2+6x+9 \\ x^2+4x+4 & x^2+6x+9 & x^2+8x+16 \end{vmatrix}.$$

$$= \begin{vmatrix} x^2 & 2x+1 & 4x+4 \\ x^2+2x+1 & 2x+3 & 4x+8 \\ x^2+4x+4 & 2x+5 & 4x+12 \end{vmatrix} \xrightarrow{r_2-r_1, r_3-r_1} \begin{vmatrix} x^2 & 2x+1 & 4x+4 \\ 2x+1 & 2 & 4 \\ 4x+4 & 4 & 8 \end{vmatrix} \xrightarrow{c_3 \rightarrow 2c_2} \begin{vmatrix} x^2 & 2x+1 & 2x+2 \\ 2x+1 & 2 & 0 \\ 4x+4 & 4 & 0 \end{vmatrix}$$

Laplace expansion for 3rd column.

$$= (2x+2) \cdot (-1)^{1+3} \begin{vmatrix} 2x+1 & 2 \\ 4x+4 & 4 \end{vmatrix} = -(2x+2) \cdot 4 = -8x-8.$$

15. Find the solution of the system of linear equations

$$3x_1 - 2x_2 - x_4 = 7$$

$$2x_2 + 2x_3 + x_4 = 5$$

$$x_1 - 2x_2 - 3x_3 - 2x_4 = 1.$$

16. For which values of a , the system of linear equations

$$x_1 + x_2 + x_3 = 2$$

$$2x_1 + 3x_2 + (a^2 - 1)x_3 = a + 1$$

$$2x_1 + 3x_2 + 2x_3 = 5$$

The number of the unknowns $n = 3$.

a) is inconsistent? $\text{rank } A \neq \text{rank}(A:b)$

b) has unique solution? $\text{rank } A = \text{rank}(A:b) = n = 3$

c) has infinitely many solutions? $\text{rank } A = \text{rank}(A:b) < n = 3$.

$$\begin{bmatrix} 1 & 1 & 1 & : & 2 \\ 2 & 3 & a^2-1 & : & a+1 \\ 2 & 3 & 2 & : & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & : & 2 \\ 0 & 1 & a^2-3 & : & a-3 \\ 0 & 1 & 0 & : & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & : & 2 \\ 0 & 1 & a^2-3 & : & a-3 \\ 0 & 0 & 3-a^2 & : & 4-a \end{bmatrix}$$

c) In the case there are infinite solutions, the last row must be zero.

This is not possible because there is no real number a such that $3-a^2=0$ and $4-a$ at the same time.

a) In the case there is no solution, $3-a^2$ must be 0 and $4-a$ must be nonzero. $\Rightarrow 3-a^2=0 \Rightarrow a=\pm\sqrt{3}$ (Then, clearly, $4-a=4-\sqrt{3} \neq 0$)

b) In the case there is unique solution, $3-a^2 \neq 0$. This guarantees that $\text{rank } A = \text{rank}(A:b) = 3$. $\Rightarrow 3 \neq a^2 \Rightarrow a \neq \pm\sqrt{3}$.

17. Determine the solution of the system of linear equations $\begin{matrix} x + 2y + \lambda z = 1 \\ 2x + \lambda y + 8z = 3 \end{matrix}$ with respect to λ .

$$\begin{pmatrix} 1 & 2 & \lambda & : & 1 \\ 2 & \lambda & 8 & : & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & \lambda & : & 1 \\ 0 & \lambda - 4 & 8 - 2\lambda & : & 1 \end{pmatrix}$$

$n=3$

If $\lambda - 4 = 0$ and $8 - 2\lambda = 0$ at the same time, then $\text{rank} A = 2$ and $\text{rank}(A:b) = 3$ and so there is no solution. That is, if $\lambda = 4$, there is no solution.

If $\lambda \neq 4$, then $\text{rank} A = 2 = \text{rank}(A:b)$ and there are infinite solutions depending on $n - \text{rank} A = 3 - 2 = 1$ parameter.

18. Find the solution of $\begin{matrix} x + 3y - 2z + t = 4 \\ x + 2y - 4z - 3t = 6 \\ 2x + 5y - 5z - 2t = 10 \end{matrix}$?

19. For which values of a and b , the system of linear equations $\begin{matrix} y - 2z = b \\ x - y + z = 2 \\ x + ay = 3 \end{matrix} \Rightarrow n=3.$

- is inconsistent? $a = -\frac{1}{2}, b \neq 2$
- has unique solution? $a \neq -\frac{1}{2}, b \in \mathbb{R}$
- has infinitely many solutions? $a = -\frac{1}{2}, b = 2$

$$\begin{pmatrix} 0 & 1 & -2 & : & b \\ 1 & -1 & 1 & : & 2 \\ 1 & a & 0 & : & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 1 & : & 2 \\ 0 & 1 & -2 & : & b \\ 1 & a & 0 & : & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 1 & : & 2 \\ 0 & 1 & -2 & : & b \\ 0 & a+1 & -1 & : & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & -1 & 1 & : & 2 \\ 0 & 1 & -2 & : & b \\ 0 & 0 & 2a+1 & : & -ab-b+1 \end{pmatrix}$$

$r_3 \rightarrow r_3 - (a+1)r_2$

• If $2a+1 = 0$ and $-ab-b+1 \neq 0$, then there is no solution. $\Rightarrow a = -\frac{1}{2}, b \neq 2$.

• If $2a+1 = 0$ and $-ab-b+1 = 0$, then there is infinite solutions. $\Rightarrow a = -\frac{1}{2}, b = 2$

• If $2a+1 \neq 0$, then there is unique solution. $\Rightarrow a \neq -\frac{1}{2}, b \in \mathbb{R}$.

20. For which values of a and b , the system $\begin{matrix} 2x - y + 2az + t = b \\ 2x - y + (2a+1)z + (a+1)t = 0 \\ -2x + y + (1-2a)z - 2t = -2b-2 \end{matrix} \longrightarrow n=4.$

- is inconsistent? $-1-a=0 \Rightarrow a=-1, b \in \mathbb{R}$
- has unique solution? No possible because no possible for $\text{rank } A \neq 4$.
- has infinitely many solutions? $-1-a \neq -1 \Rightarrow a \neq -1, b \in \mathbb{R}$.

$$\begin{pmatrix} 2 & -1 & 2a & 1 & : & b \\ 2 & -1 & 2a+1 & a+1 & : & 0 \\ -2 & 1 & 1-2a & -2 & : & -2b-2 \end{pmatrix} \xrightarrow[r_3 \rightarrow r_3 + r_1]{r_2 \rightarrow r_2 - r_1} \begin{pmatrix} 2 & -1 & 2a & 1 & : & b \\ 0 & 0 & -1 & -a & : & b \\ 0 & 0 & 1 & -1 & : & -b-2 \end{pmatrix}$$

Firstly $r_2 \leftrightarrow r_1$,
and then $r_2 \leftrightarrow r_3$

$$\longrightarrow \begin{pmatrix} 2 & -1 & 2a+1 & a+1 & : & 0 \\ 0 & 0 & -1 & -a & : & b \\ 0 & 0 & 0 & -1-a & : & -2 \end{pmatrix}$$

Observe that $\text{rank}(A:b) = 3$

• If $-1-a=0$, then $\text{rank } A = 2 \neq \text{rank}(A:b) = 3$ and so there is no solution

• If $-1-a \neq 0$, then $\text{rank } A = 3 = \text{rank}(A:b) < n=4$ and so infinite solutions with $4-3=1$ parameter.