BLM2502 Theory of Computation

Spring 2016

BLM2502 Theory of Computation

- **Course Outline**

Week	Content
1	Introduction to Course
2	Computability Theory, Complexity Theory, Automata Theory, Set Theory, Relations, Proofs, Pigeonhole Principle
3	Regular Expressions
4	Finite Automata
5	Deterministic and Nondeterministic Finite Automata
6	Epsilon Transition, Equivalence of Automata
7	Pumping Theorem
8	April 10 - 14 week is the first midterm week
9	Context Free Grammars
10	Parse Tree, Ambiguity,
11	**Pumping Theorem**
12	Turing Machines, Recognition and Computation, Church-Turing Hypothesis
13	Turing Machines, Recognition and Computation, Church-Turing Hypothesis
14	May 22 – 27 week is the second midterm week
15	Review
16	Final Exam date will be announced

The Pumping Lemma for CFL's

Pumping Lemma

- Recall the pumping lemma for regular languages.
- It told us that if there was a string long enough to cause a cycle in the DFA for the language, then we could "pump" the cycle and discover an infinite sequence of strings that had to be in the language.
- For CFL's the situation is a little more complicated.
- We can always find two pieces of any sufficiently long string to "pump" in tandem.
  - That is: if we repeat each of the two pieces the same number of times, we get another string in the language.

Statement of the CFL Pumping Lemma

» For every context-free language L There is an integer n, such that For every string z in L of length > n There exists z = uvwxy such that:

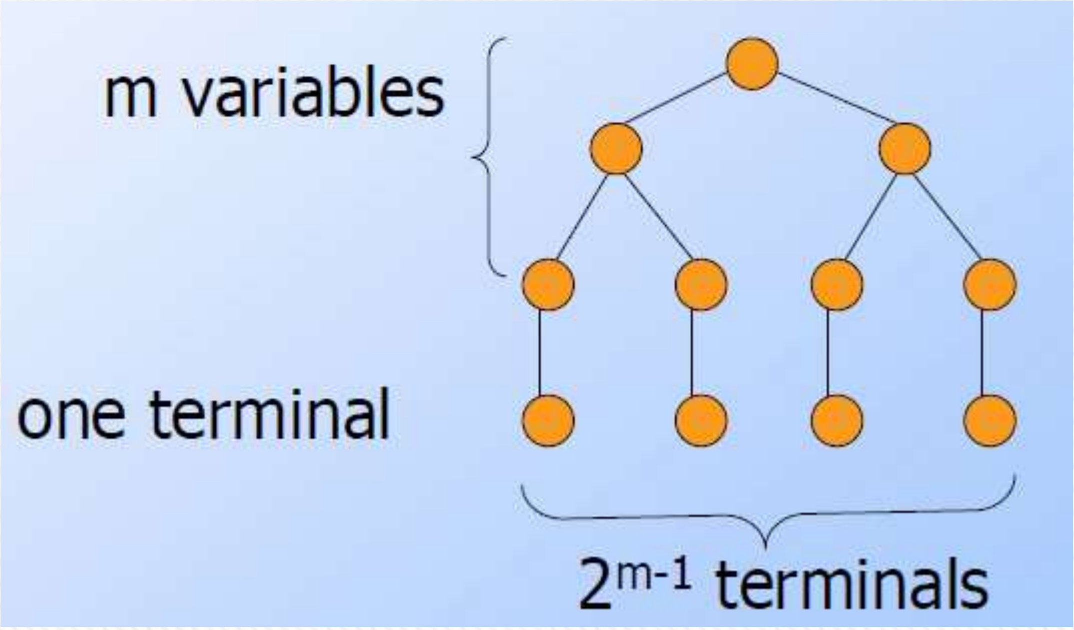
1. $|vwx| < n$.
2. $|vx| > 0$.
3. For all $i > 0$, uv^iwx^iy is in L.

Proof of the Pumping Lemma

- Start with a CNF grammar for L - {ε}.
- Let the grammar have m variables.
  - Pick $n = 2^m$.
  - Let $|z| > n$.
- We claim ("Lemma 1") that a parse tree with yield z must have a path of length m+2 or more.

Proof of Lemma 1

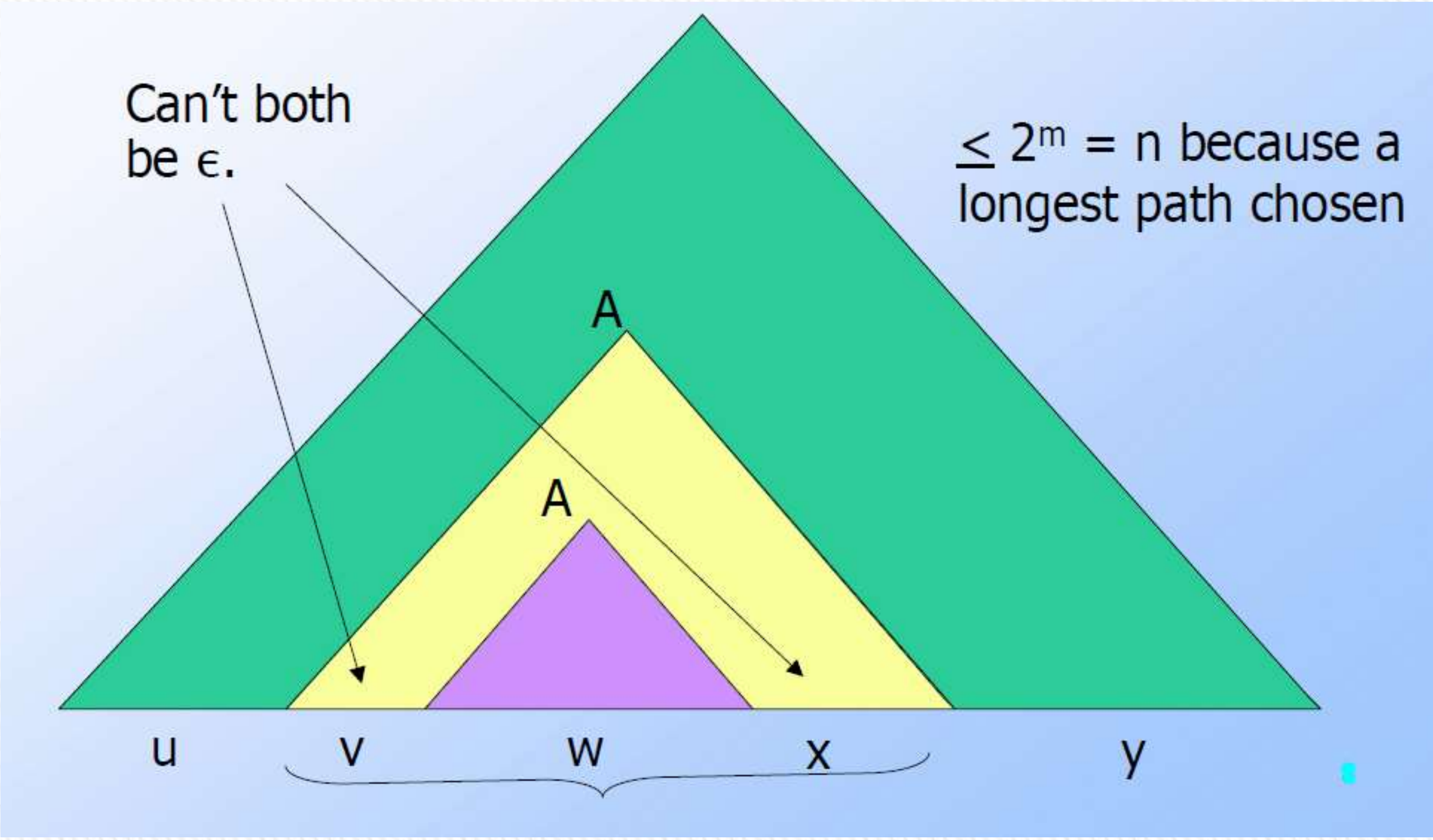
» If all paths in the parse tree of a CNF grammar are of length < m+1, then the longest yield has length 2^{m-1}, as in figure:



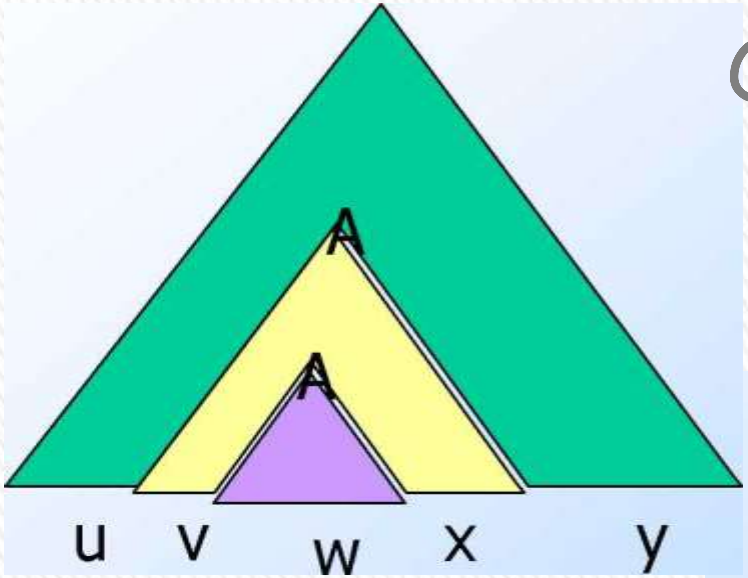
Proof of the Pumping Lemma

- Now we know that the parse tree for z has a path with at least m+1 variables.
- Consider some longest path.
- There are only m different variables, so among the lowest m+1 we can find two nodes with the same label, say A.
- The parse tree thus looks like:

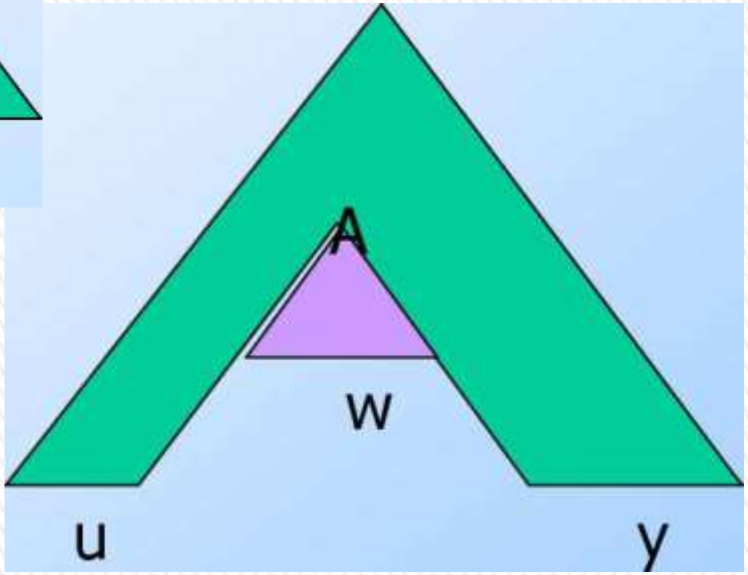
Parse Tree in the Pumping-Lemma



Pumping



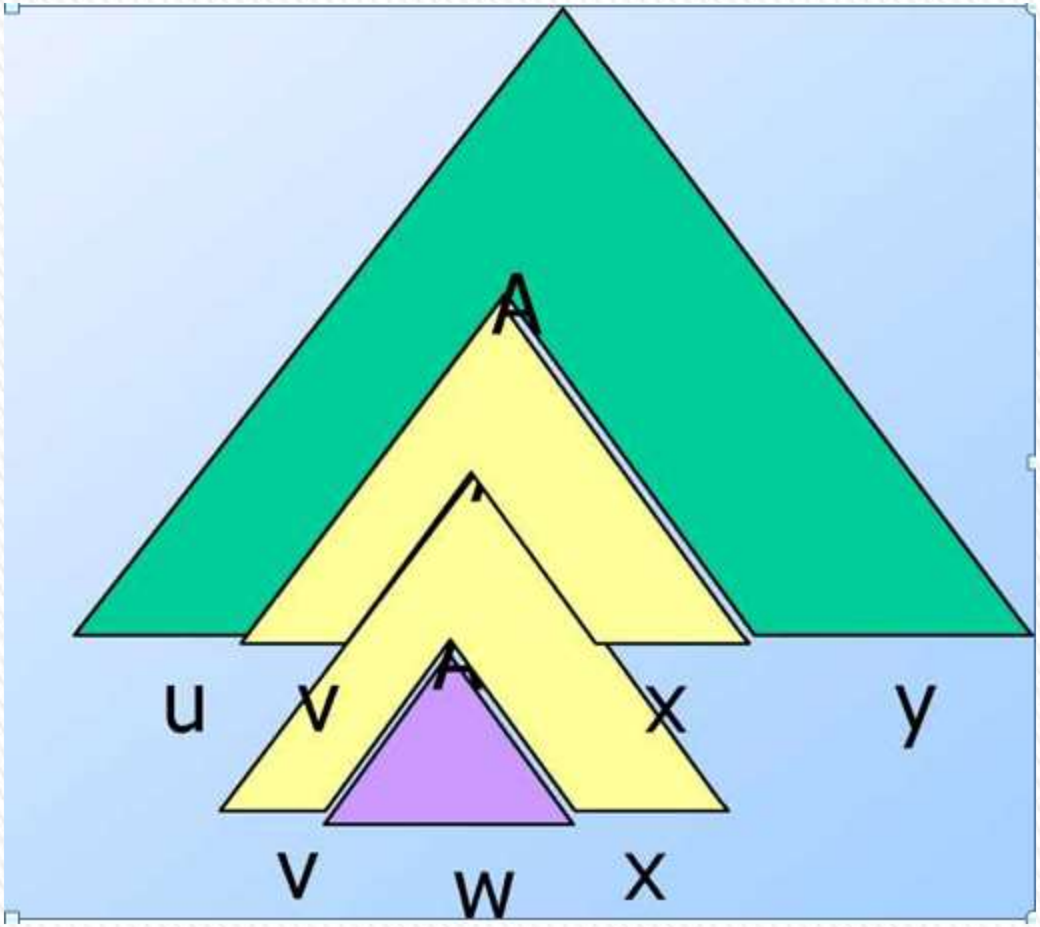
Once



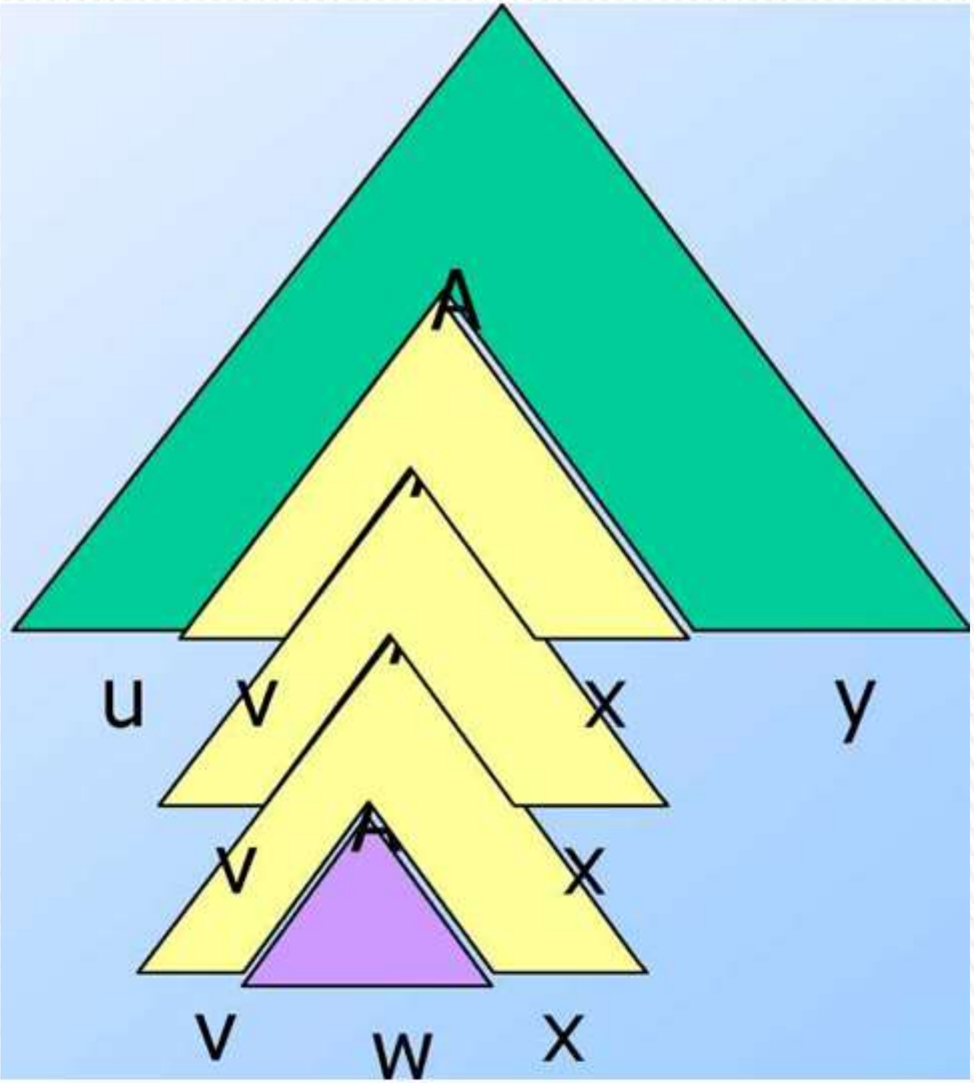
Zero times

Pumping

Twice



Pumping



Thrice, …

Using the Pumping Lemma

- Non-CFL's typically involve trying to match two pairs of counts or match two strings.
- Example: pumping lemma can be used to show that L = {ww | w in (0+1)*} is not a CFL.
- {0'10' | i >1} is a CFL.
  - We can match one pair of counts.

Using the Pumping Lemma

- $L = \{0^i10^i10^i \mid i > 1\}$ is not a CFL
  - We can't match two pairs, or three counts as a group.
  - Proof using the pumping lemma.
  - Suppose L were a CFL.
  - Let n be L's pumping-lemma constant.
  - Consider $z = 0^n10^n10^n$.
  - We can write $z = uvwxy$, where $|vwx| < n$, and $|vx| > 1$.
  - Case 1: vx has no 0's.
  - Then at least one of them is a 1, and uwy has at most one 1, which no string in L does.

Using the Pumping Lemma

- Still considering $z = 0^n10^n10^n$.
- Case 2: vx has at least one 0.
- vwx is too short (length < n) to extend to all three blocks of 0's in $0^n10^n10^n$.
- Thus, uwy has at least one block of n 0's, and at least one block with fewer than n 0's.
- Thus, uwy is not in L.

Simplifications of Context-Free Grammars

A Substitution Rule

$$S \to aB$$
$$A \to aaA$$
$$A \to abBc$$
$$B \to aA$$
$$B \to b$$

Substitute

$$B \to b$$

Equivalent grammar

$$S \to aB \mid ab$$
$$A \to aaA$$
$$A \to abBc \mid abbc$$
$$B \to aA$$

$$S \to aB \mid ab$$

$$A \to aaA$$

$$A \to abBc \mid abbc$$

$$B \to aA$$

Substitute

$$B \to aA$$

$$S \to \cancel{aB} \mid ab \mid aaA$$

$$A \to aaA$$

$$A \to \cancel{abBc} \mid abbc \mid abaAc$$

Equivalent grammar

In general: $A \to xBz$

$$B \to y_1$$

Substitute

$B \to y_1$

$$A \to xBz \mid xy_1z$$

equivalent grammar

Nullable Variables

$\varepsilon-\text{production}:$ $X \rightarrow \varepsilon$

Nullable Variable: $Y \Rightarrow \ldots \Rightarrow \varepsilon$

Example:

$$S \rightarrow aMb$$

$$M \rightarrow aMb$$

$$M \rightarrow \varepsilon$$

Nullable variable

$\varepsilon-\text{production}$

Removing ε – productions

$$S \to aMb$$

$$M \to aMb$$

$$\cancel{M \to \varepsilon}$$

Substitute $M \to \varepsilon$

$$S \to aMb \mid ab$$

$$M \to aMb \mid ab$$

After we remove all the ε - productions all the nullable variables disappear (except for the start variable)

Unit-Productions

Unit Production: $X \to Y$

(a single variable in both sides)

Example:

$$S \to aA$$

$$A \to a$$

$$\boxed{A \to B}$$

$$\boxed{B \to A}$$

Unit Productions

$$B \to bb$$

Removal of unit productions:

$$S \rightarrow aA$$
$$A \rightarrow a$$
$$\cancel{A \rightarrow B}$$
$$B \rightarrow A$$
$$B \rightarrow bb$$

Substitute

$$A \rightarrow B$$

$$S \rightarrow aA \mid aB$$
$$A \rightarrow a$$
$$B \rightarrow A \mid B$$
$$B \rightarrow bb$$

Unit productions of form $X \rightarrow X$

can be removed immediately

$$S \rightarrow aA \mid aB$$
$$A \rightarrow a$$
$$B \rightarrow A \mid \cancel{B}$$
$$B \rightarrow bb$$

Remove

$B \rightarrow B$

$$S \rightarrow aA \mid aB$$
$$A \rightarrow a$$
$$B \rightarrow A$$
$$B \rightarrow bb$$

$$S \rightarrow aA \mid aB$$

$$A \rightarrow a$$

$$\cancel{B \rightarrow A}$$

$$B \rightarrow bb$$

Substitute

$$B \rightarrow A$$

$$S \rightarrow aA \mid aB \mid aA$$

$$A \rightarrow a$$

$$B \rightarrow bb$$

Remove repeated productions

$$S \rightarrow aA \mid aB \mid \cancel{aA}$$

$$A \rightarrow a$$

$$B \rightarrow bb$$

Final grammar

$$S \rightarrow aA \mid aB$$

$$A \rightarrow a$$

$$B \rightarrow bb$$

Useless Productions

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

$$S \rightarrow A$$

$$A \rightarrow aA$$

Useless Production

Some derivations never terminate...

$$S \Rightarrow A \Rightarrow aA \Rightarrow aaA \Rightarrow \ldots \Rightarrow aa\ldots aA \Rightarrow \ldots$$

Another grammar:

$$S \to A$$

$$A \to aA$$

$$A \to \lambda$$

$$B \to bA$$

Useless Production

Not reachable from S

In general:

If there is a derivation

$$S \Rightarrow \ldots \Rightarrow xAy \Rightarrow \ldots \Rightarrow w \quad \in L(G)$$

consists of terminals

Then variable A is useful

Otherwise, variable A is useless

A production $A \rightarrow x$ is useless

if any of its variables is useless

$$S \rightarrow aSb$$

$$S \rightarrow \varepsilon$$

Productions

Variables $S \rightarrow A$ useless

useless $A \rightarrow aA$ useless

useless $B \rightarrow C$ useless

useless $C \rightarrow D$ useless

Removing Useless Variables and Productions

Example Grammar:

$$S \rightarrow aS \mid A \mid C$$

$$A \rightarrow a$$

$$B \rightarrow aa$$

$$C \rightarrow aCb$$

First: find all variables that can produce strings with only terminals or ε (possible useful variables)

$$S \to aS \mid A \mid C$$

$$A \to a$$

$$B \to aa$$

$$C \to aCb$$

Round 1: $\{A, B\}$

(the right hand side of production that has only terminals)

Round 2: $\{A, B, S\}$

(the right hand side of a production has terminals and variables of previous round)

This process can be generalized

Then, remove productions that use variables other than $\{A, B, S\}$

$$S \to aS \mid A \mid \cancel{C}$$

$$A \to a$$

$$B \to aa$$

$$\cancel{C \to aCb}$$

$\Rightarrow$

$$S \to aS \mid A$$

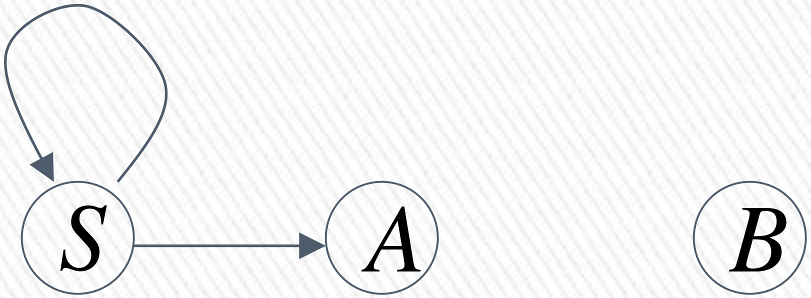
$$A \to a$$

$$B \to aa$$

Second: Find all variables reachable from S

Use a Dependency Graph where nodes are variables

$$S \to aS \mid A$$
$$A \to a$$
$$B \to aa$$



Keep only the variables reachable from S

$$S \rightarrow aS \mid A$$

$$A \rightarrow a$$

$$\cancel{B \rightarrow aa}$$

Final Grammar

$$S \rightarrow aS \mid A$$

$$A \rightarrow a$$

Contains only useful variables

Removing All

- **Step 1:** Remove Nullable Variables

- **Step 2:** Remove Unit-Productions

- **Step 3:** Remove Useless Variables

This sequence guarantees that unwanted variables and productions are removed

Normal Forms
for
Context-free Grammars

Chomsky Normal Form

Each productions has form:

$$A \to BC \qquad \text{or} \qquad A \to a$$

variable variable terminal

Examples:

$$S \to AS$$
$$S \to a$$
$$A \to SA$$
$$A \to b$$

Chomsky Normal Form

$$S \to AS$$
$$S \to AAS$$
$$A \to SA$$
$$A \to aa$$

Not Chomsky Normal Form

Conversion to Chomsky Normal Form

- Example:

$$S \to ABa$$
$$A \to aab$$
$$B \to Ac$$

Not in Chomsky Normal Form

We will convert it to Chomsky Normal Form

Introduce new variables for the terminals:

$$T_a, T_b, T_c$$

$$S \to ABa$$
$$A \to aab$$
$$B \to Ac$$

$$S \to ABT_a$$
$$A \to T_aT_aT_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

Introduce new intermediate variable V_1 to break first production:

$$\boxed{S \to ABT_a}$$
$$A \to T_a T_a T_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

$$\Rightarrow$$

$$\boxed{\begin{array}{l} S \to AV_1 \\ V_1 \to BT_a \end{array}}$$
$$A \to T_a T_a T_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

Introduce intermediate variable: V_2

$$S \to AV_1$$
$$V_1 \to BT_a$$
$$\boxed{A \to T_aT_aT_b}$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

$$\Rightarrow$$

$$S \to AV_1$$
$$V_1 \to BT_a$$
$$\boxed{\begin{aligned} A &\to T_aV_2 \\ V_2 &\to T_aT_b \end{aligned}}$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

Final grammar in Chomsky Normal Form:

$$S \to AV_1$$
$$V_1 \to BT_a$$
$$A \to T_a V_2$$
$$V_2 \to T_a T_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

Initial grammar

$$S \to ABa$$
$$A \to aab$$
$$B \to Ac$$

In general:

From any context-free grammar
(which doesn't produce ε)
not in Chomsky Normal Form

we can obtain:
an equivalent grammar
in Chomsky Normal Form

The Procedure

First remove:

- Nullable variables
- Unit productions
- (Useless variables optional)

Then, for every symbol a:

New variable: T_a

Add production $T_a \to a$

In productions with length at least 2 replace a with T_a

Productions of form $A \to a$ do not need to change!

Replace any production $A \to C_1 C_2 \cdots C_n$

with
$$A \to C_1 V_1$$
$$V_1 \to C_2 V_2$$
$$\cdots$$
$$V_{n-2} \to C_{n-1} C_n$$

New intermediate variables: $V_1, V_2, \ldots, V_{n-2}$

Observations

- Chomsky normal forms are good for parsing and proving theorems

- It is easy to find the Chomsky normal form for any context-free grammar

Greinbach Normal Form

All productions have form:

$$A \to a\, V_1 V_2 \cdots V_k \qquad k \ge 0$$

symbol

variables

Examples:

$$S \to cAB$$
$$A \to aA \mid bB \mid b$$
$$B \to b$$

Greinbach Normal Form

$$S \to abSb$$
$$S \to aa$$

Not Greinbach Normal Form

Conversion to Greinbach Normal Form:

$$S \to abSb$$

$$S \to aa$$

$$S \to aT_bST_b$$

$$S \to aT_a$$

$$T_a \to a$$

$$T_b \to b$$

Greinbach

Normal Form

Observations

- Greinbach normal forms are very good for parsing strings (better than Chomsky Normal Forms)
- However, it is difficult to find the Greinbach normal of a grammar

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