Turing's Thesis

Turing's thesis (1930):

Any computation carried out
by mechanical means
can be performed by a Turing Machine

Algorithm:

An algorithm for a problem is a
Turing Machine which solves the problem

The algorithm describes the steps of
the mechanical means

This is easily translated to computation steps
of a Turing machine

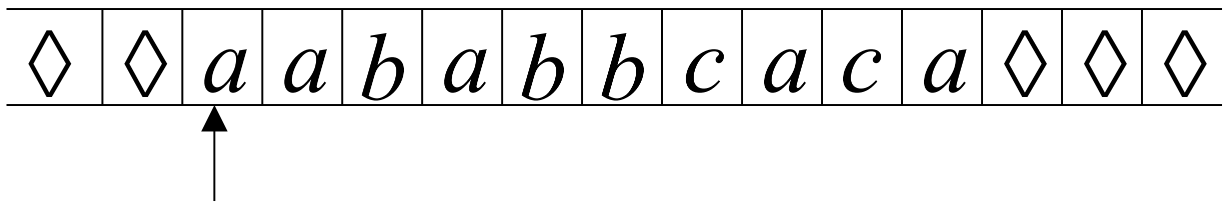
When we say: There exists an algorithm

We mean: There exists a Turing Machine that executes the algorithm

Variations of the Turing Machine

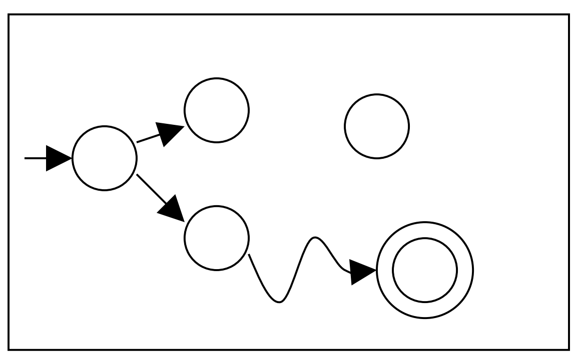
The Standard Model

Infinite Tape



Read-Write Head (Left or Right)

Control Unit



Deterministic

Variations of the Standard Model

Turing machines with:

- Stay-Option
- Semi-Infinite Tape
- Multitape
- Multidimensional
- Nondeterministic

Different Turing Machine **Classes**

Same Power of two machine classes:

both classes accept the same set of languages

We will prove:

each new class has the same power with Standard Turing Machine

(accept Turing-Recognizable Languages)

Same Power of two classes means:

for any machine M_1 of first class

there is a machine M_2 of second class

such that: $L(M_1) = L(M_2)$

and vice-versa

Simulation: A technique to prove same power.

Simulate the machine of one class with a machine of the other class

First Class
Original Machine

M_1

Second Class
Simulation Machine

M_2

M_1

simulates M_1

Configurations in the Original Machine M_1 have corresponding configurations in the Simulation Machine M_2

Original Machine: $d_0 \succ d_1 \succ \cdots \succ d_n$ (moves of M_1)

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

Simulation Machine: $d'_0 \overset{*}{\succ} d'_1 \overset{*}{\succ} \cdots \overset{*}{\succ} d'_n$ (moves of M_2)

Accepting Configuration

Original Machine: d_f

$$\updownarrow$$

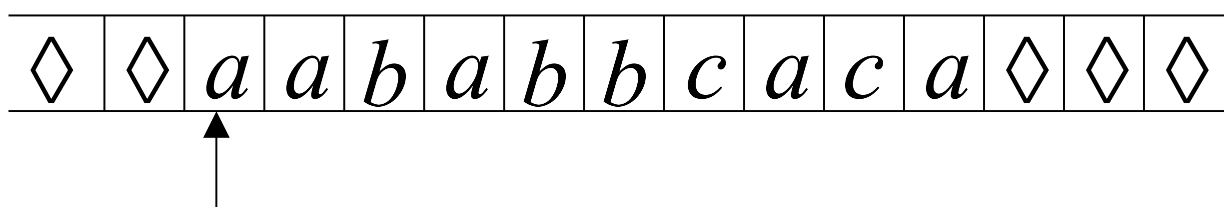
Simulation Machine: d'_f

the Simulation Machine
and the Original Machine
accept the same strings

$$L(M_1) = L(M_2)$$

Turing Machines with Stay-Option

The head can stay in the same position



Left, Right, Stay

L,R,S: possible head moves

Example:

Time 1

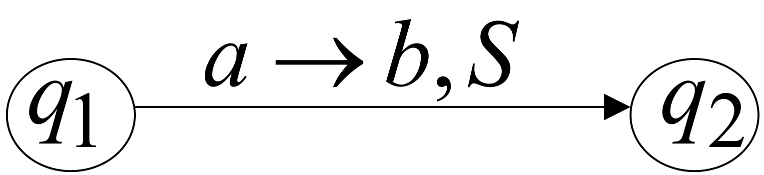
◊	◊	a	a	b	a	b	b	c	a	c	a	◊	◊	◊

q_1

Time 2

◊	◊	b	a	b	a	b	b	c	a	c	a	◊	◊	◊

q_2



Theorem: Stay-Option machines have the same power with Standard Turing machines

Proof: 1. Stay-Option Machines simulate Standard Turing machines

2. Standard Turing machines simulate Stay-Option machines

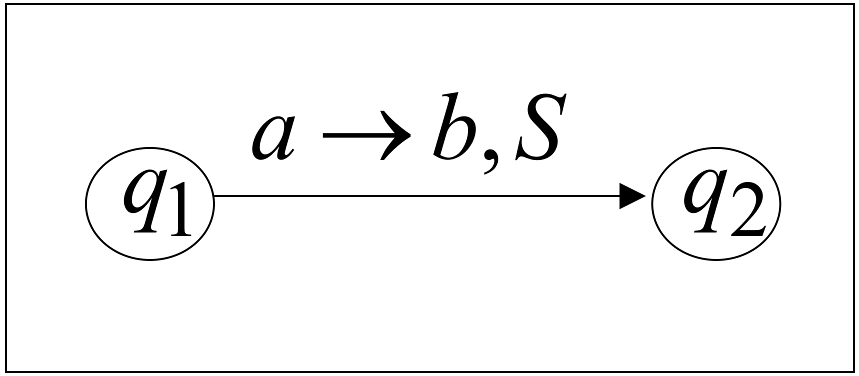
1. Stay-Option Machines simulate Standard Turing machines

Trivial: any standard Turing machine is also a Stay-Option machine

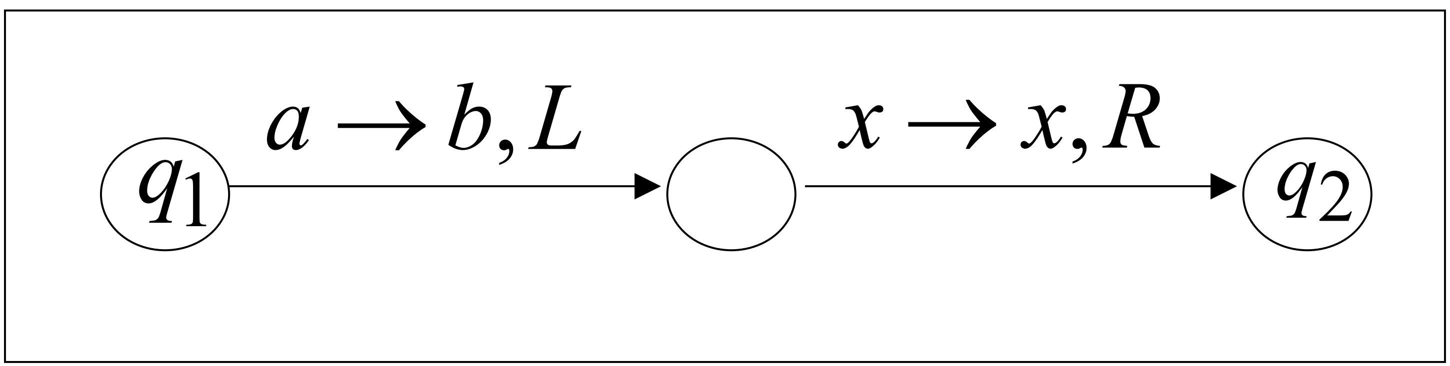
2. Standard Turing machines simulate Stay-Option machines

We need to simulate the stay head option with two head moves, one left and one right

Stay-Option Machine



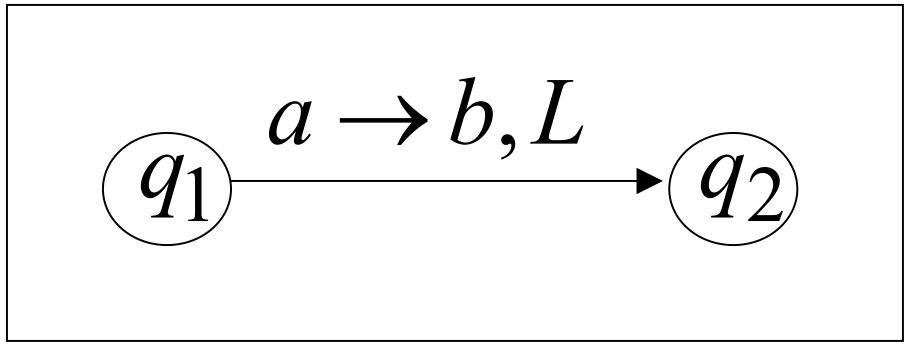
Simulation in Standard Machine



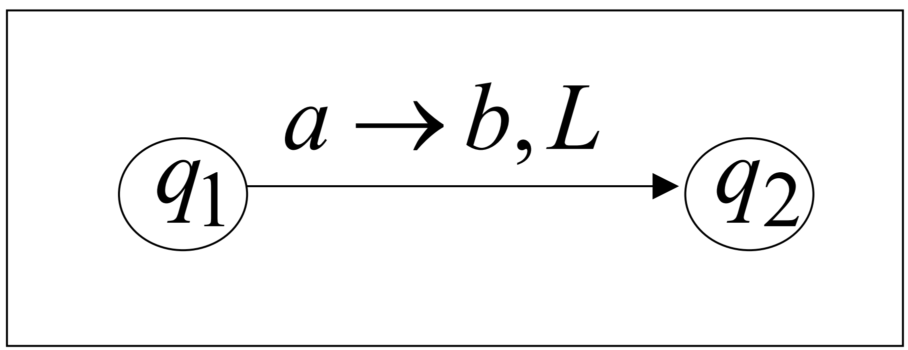
For every possible tape symbol x

For other transitions nothing changes

Stay-Option Machine



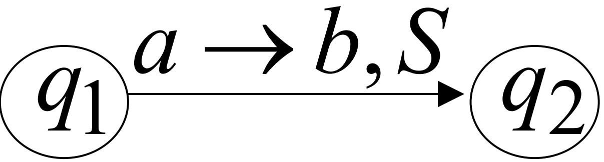
Simulation in Standard Machine

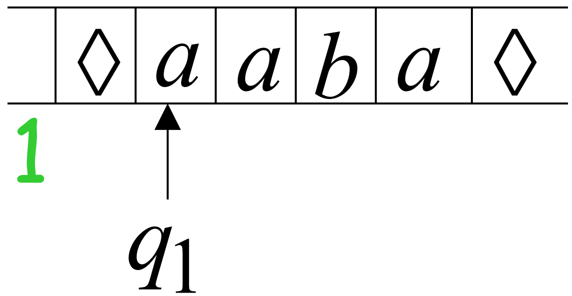


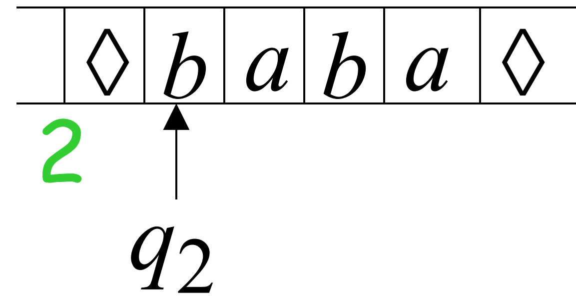
Similar for Right moves

example of simulation

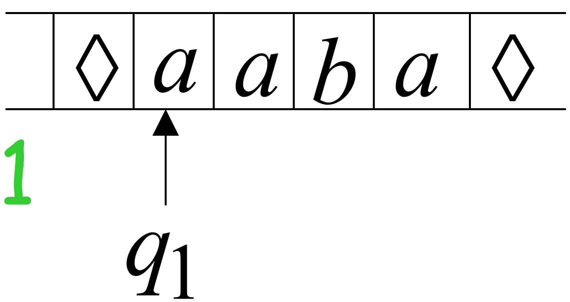
Stay-Option Machine:

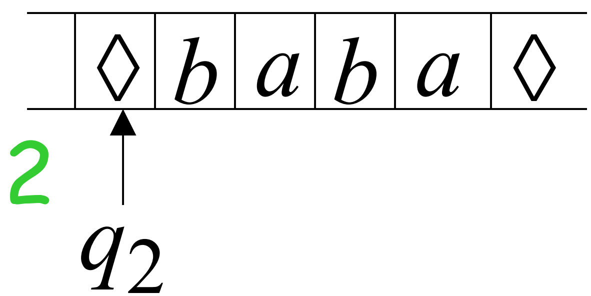


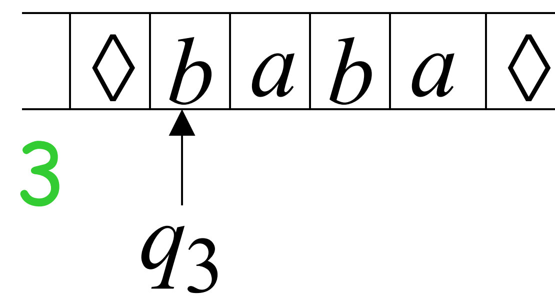




Simulation in Standard Machine:





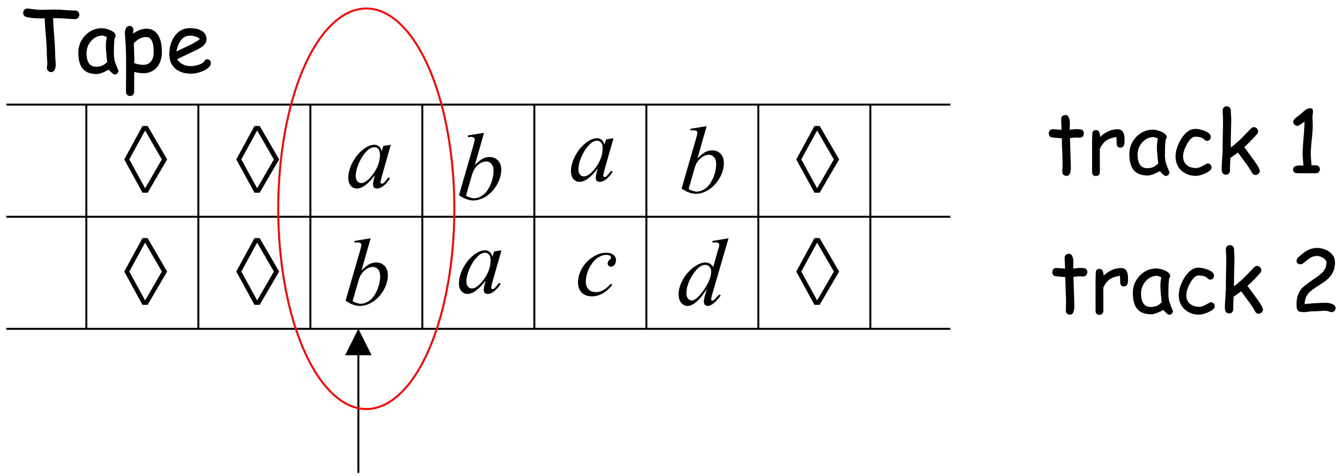


END OF PROOF

A useful trick: Multiple Track Tape

helps for more complicated simulations

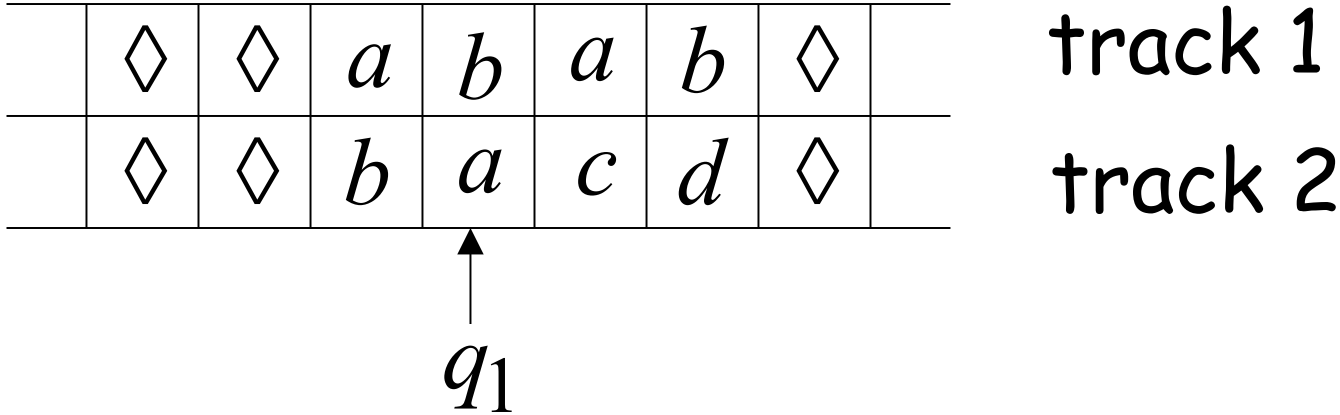
One Tape

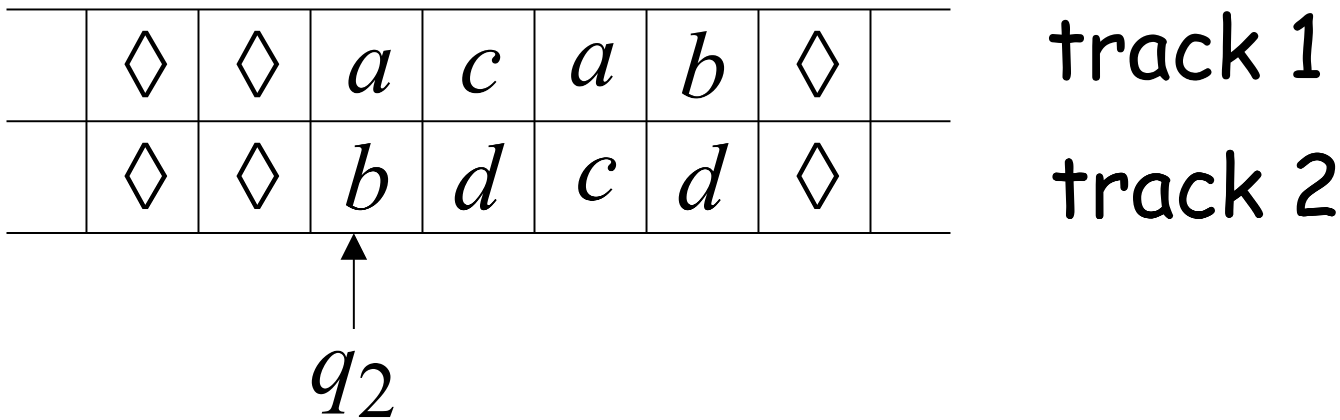


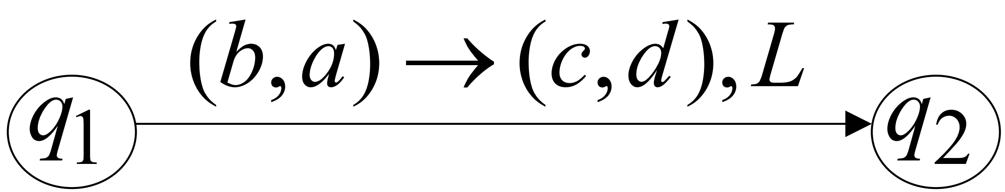
One head

One symbol (a, b)

It is a standard Turing machine, but each tape alphabet symbol describes a pair of actual useful symbols

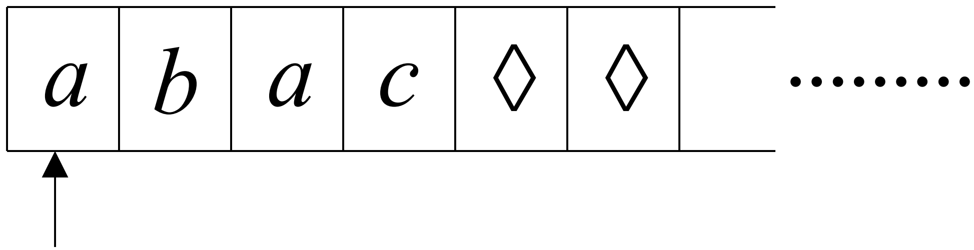

track 1
track 2
q_1


track 1
track 2
q_2


$(b,a) \rightarrow (c,d), L$
q_1
q_2

Semi-Infinite Tape

The head extends infinitely only to the right



- Initial position is the leftmost cell

- When the head moves left from the border, it returns back to leftmost position

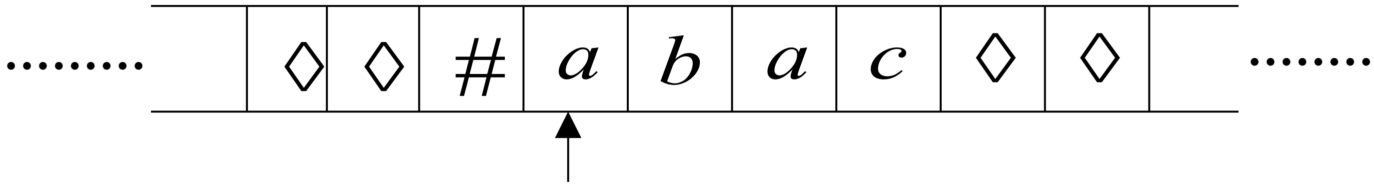
Theorem: Semi-Infinite machines have the same power with Standard Turing machines

Proof:

1. Standard Turing machines simulate Semi-Infinite machines

2. Semi-Infinite Machines simulate Standard Turing machines

1. Standard Turing machines simulate Semi-Infinite machines:

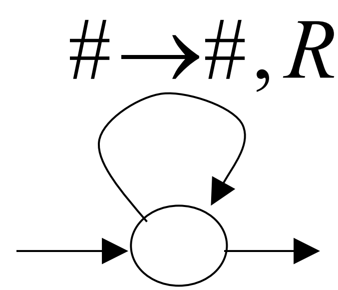


Standard Turing Machine

Semi-Infinite machine modifications

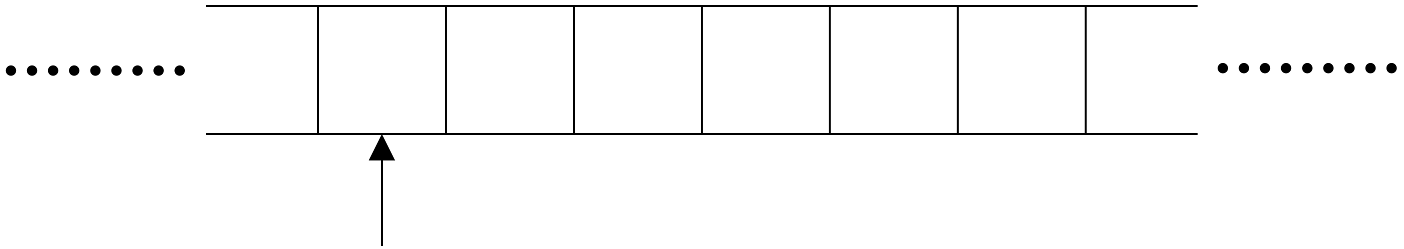
a. insert special symbol $\#$
at left of input string

b. Add a self-loop
to every state
(except states with no
outgoing transitions)

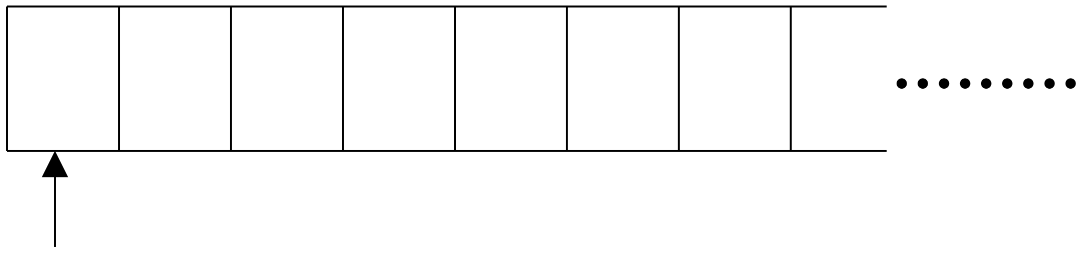


2. Semi-Infinite tape machines simulate Standard Turing machines:

Standard machine

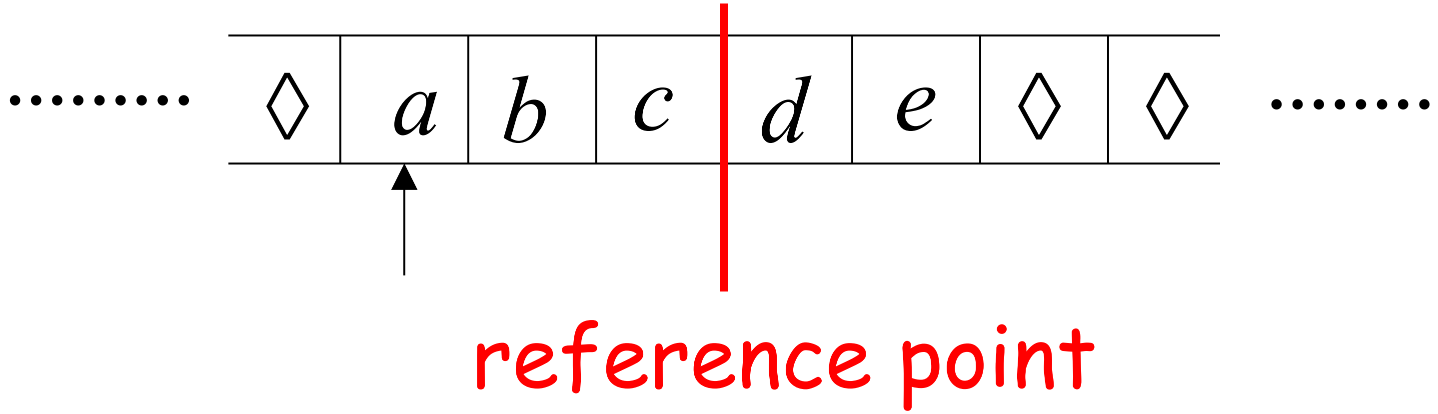


Semi-Infinite tape machine

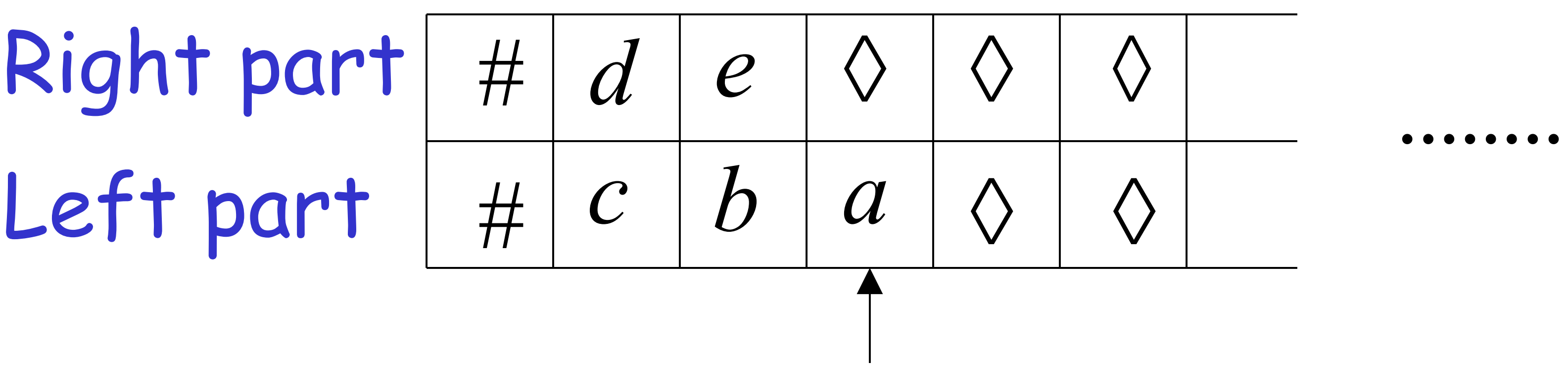


Squeeze infinity of both directions to one direction

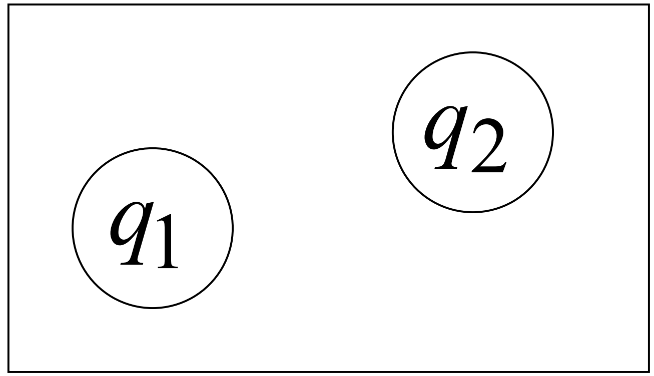
Standard machine



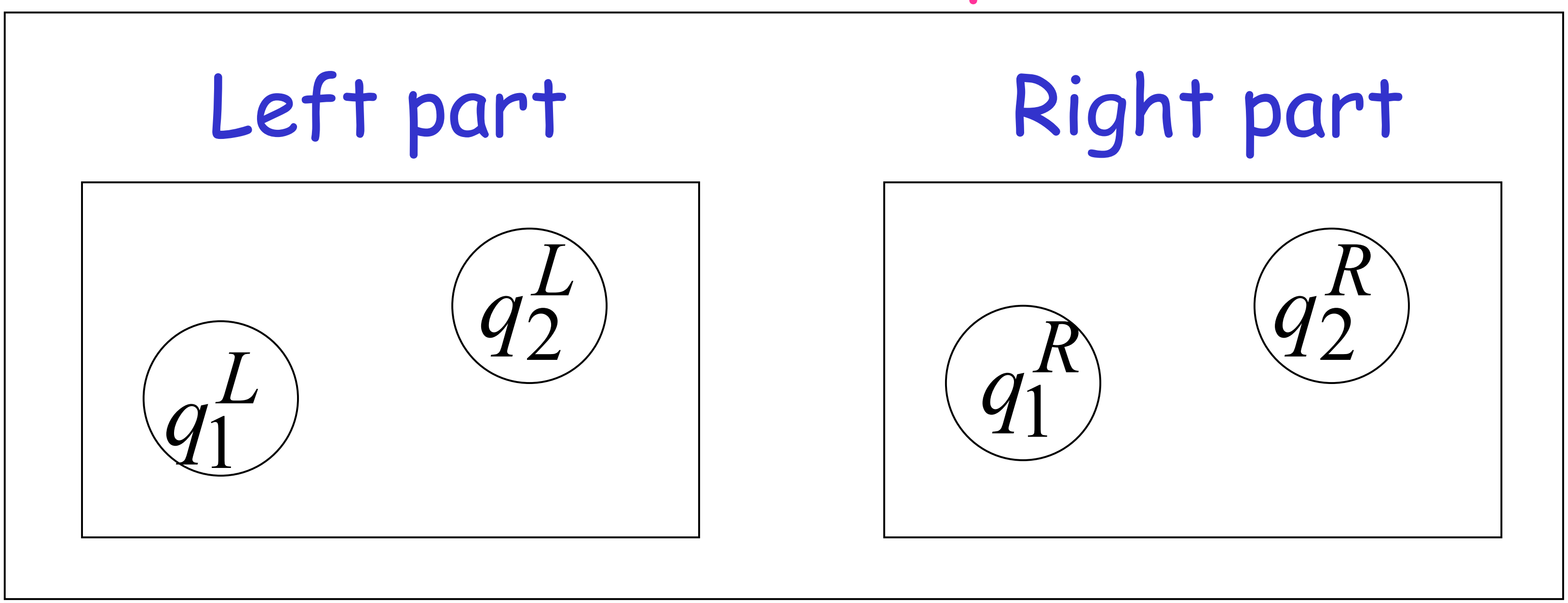
Semi-Infinite tape machine with two tracks



Standard machine



Semi-Infinite tape machine



Standard machine

$$q_1 \xrightarrow{a \rightarrow g, R} q_2$$

Semi-Infinite tape machine

Right part

$$q_1^R \xrightarrow{(a, x) \rightarrow (g, x), R} q_2^R$$

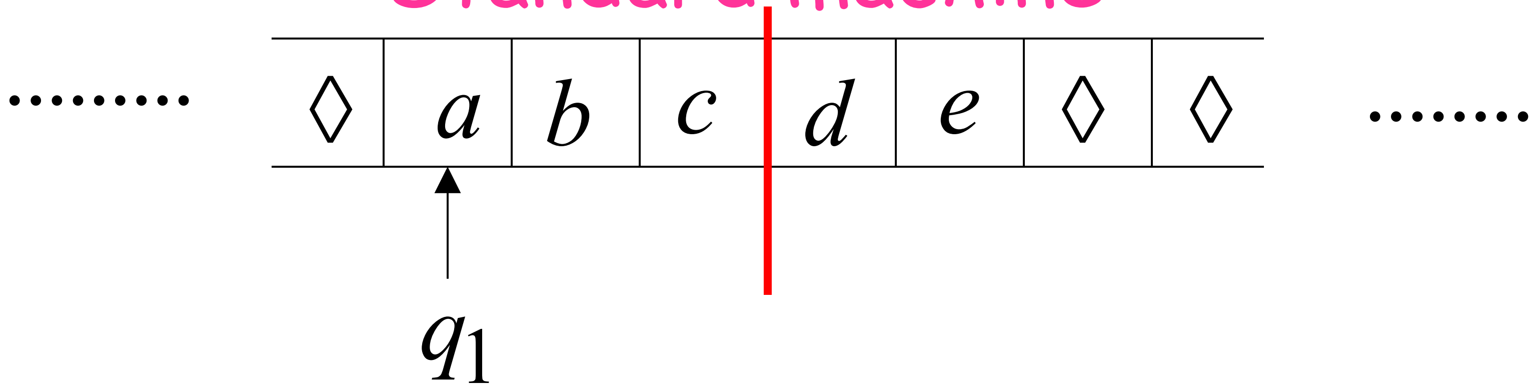
Left part

$$q_1^L \xrightarrow{(x, a) \rightarrow (x, g), L} q_2^L$$

For all tape symbols x

Time 1

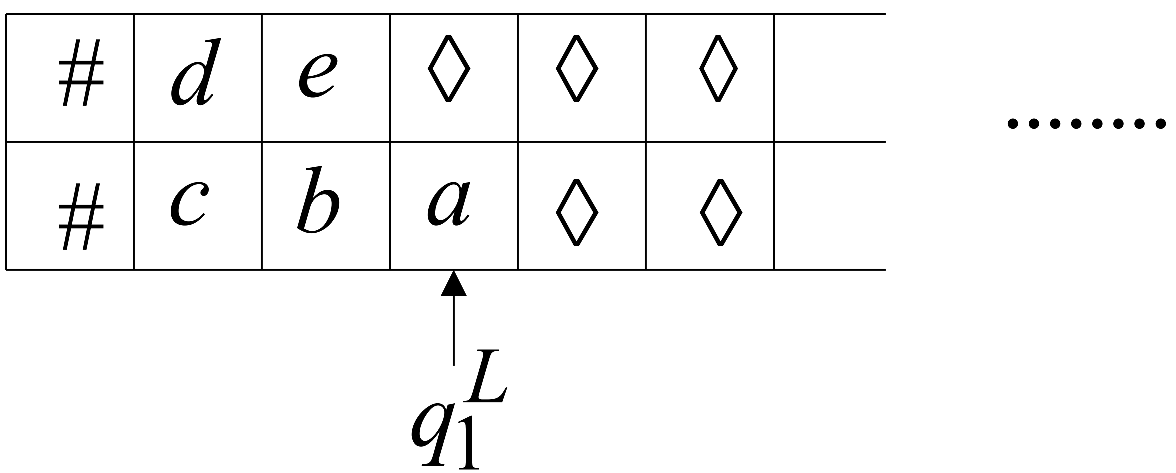
Standard machine



Semi-Infinite tape machine

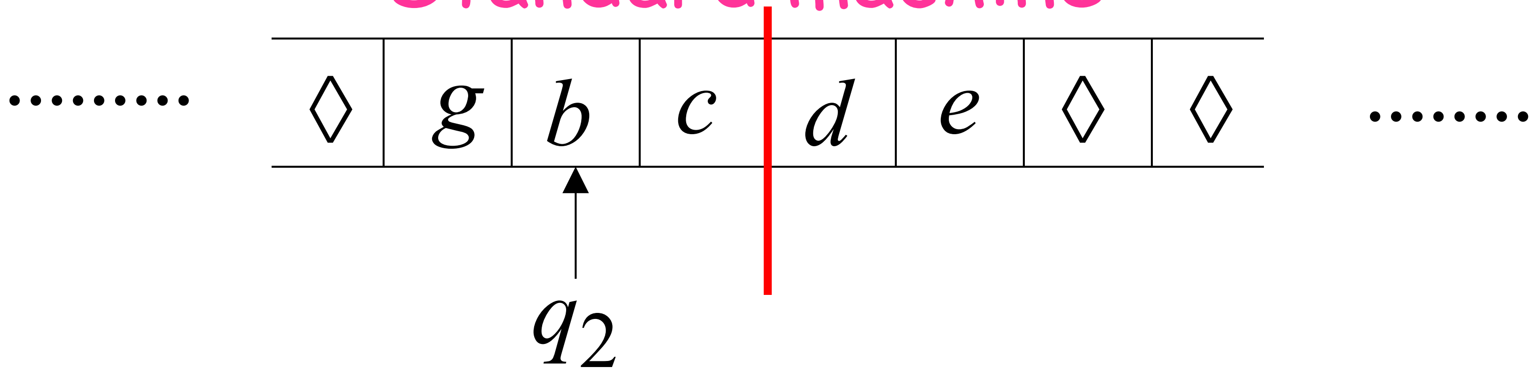
Right part

Left part

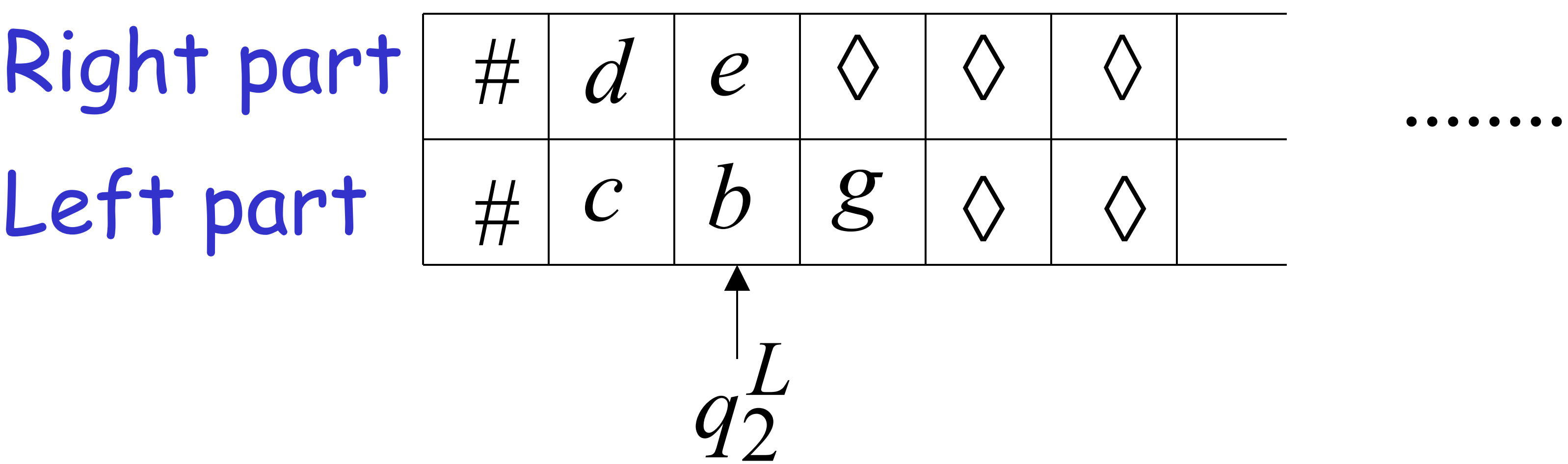


Time 2

Standard machine



Semi-Infinite tape machine



At the border:

Semi-Infinite tape machine

Right part

$$q_1^R \xrightarrow{(\#,\#) \rightarrow (\#,\#), R} q_1^L$$

Left part

$$q_1^L \xrightarrow{(\#,\#) \rightarrow (\#,\#), R} q_1^R$$

Semi-Infinite tape machine

Time 1

Right part	#	d	e	$\Diamond$	$\Diamond$	$\Diamond$	....
Left part	#	c	b	g	$\Diamond$	$\Diamond$	

q_1^L

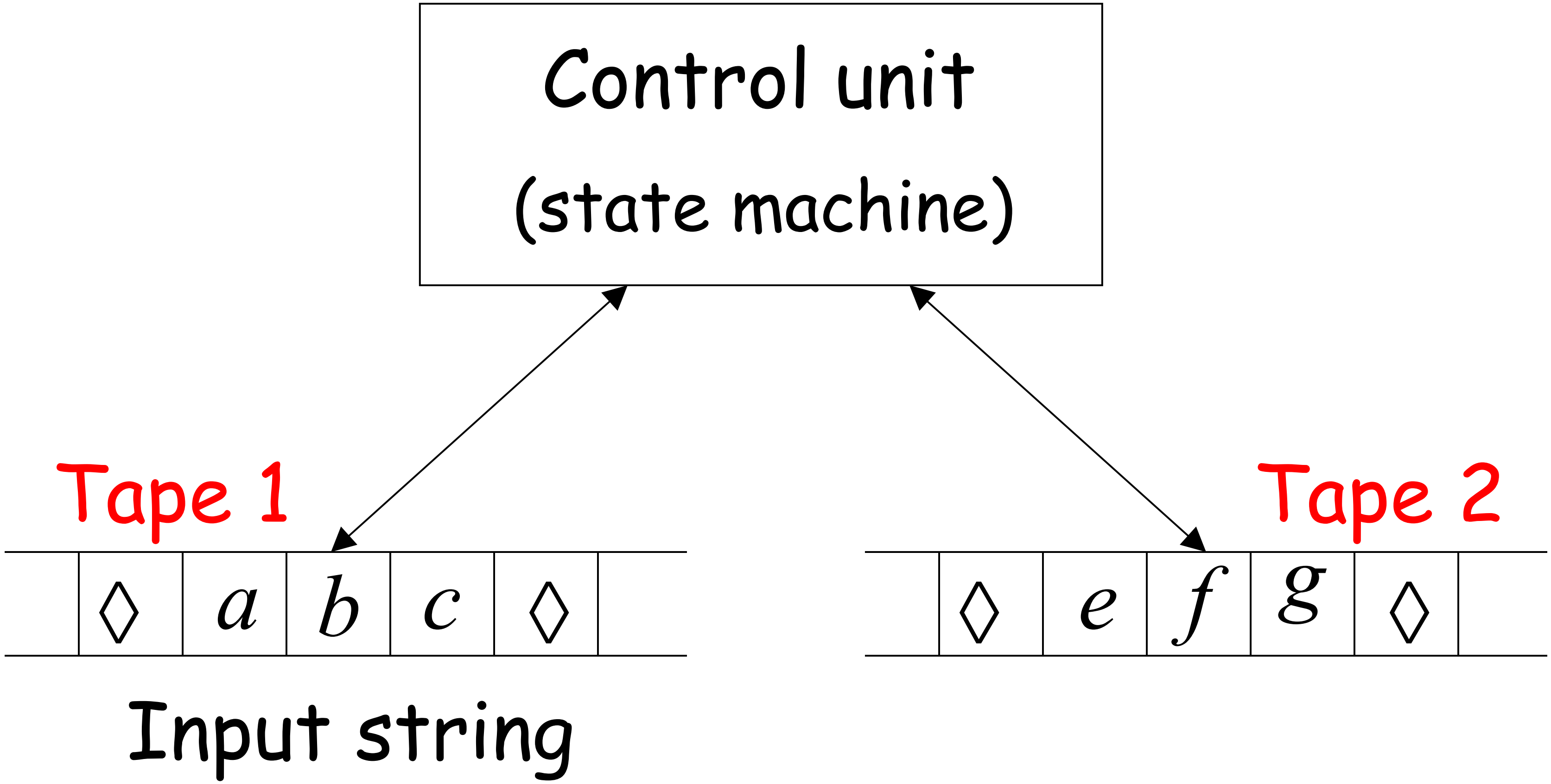
Time 2

Right part	#	d	e	$\Diamond$	$\Diamond$	$\Diamond$	....
Left part	#	c	b	g	$\Diamond$	$\Diamond$	

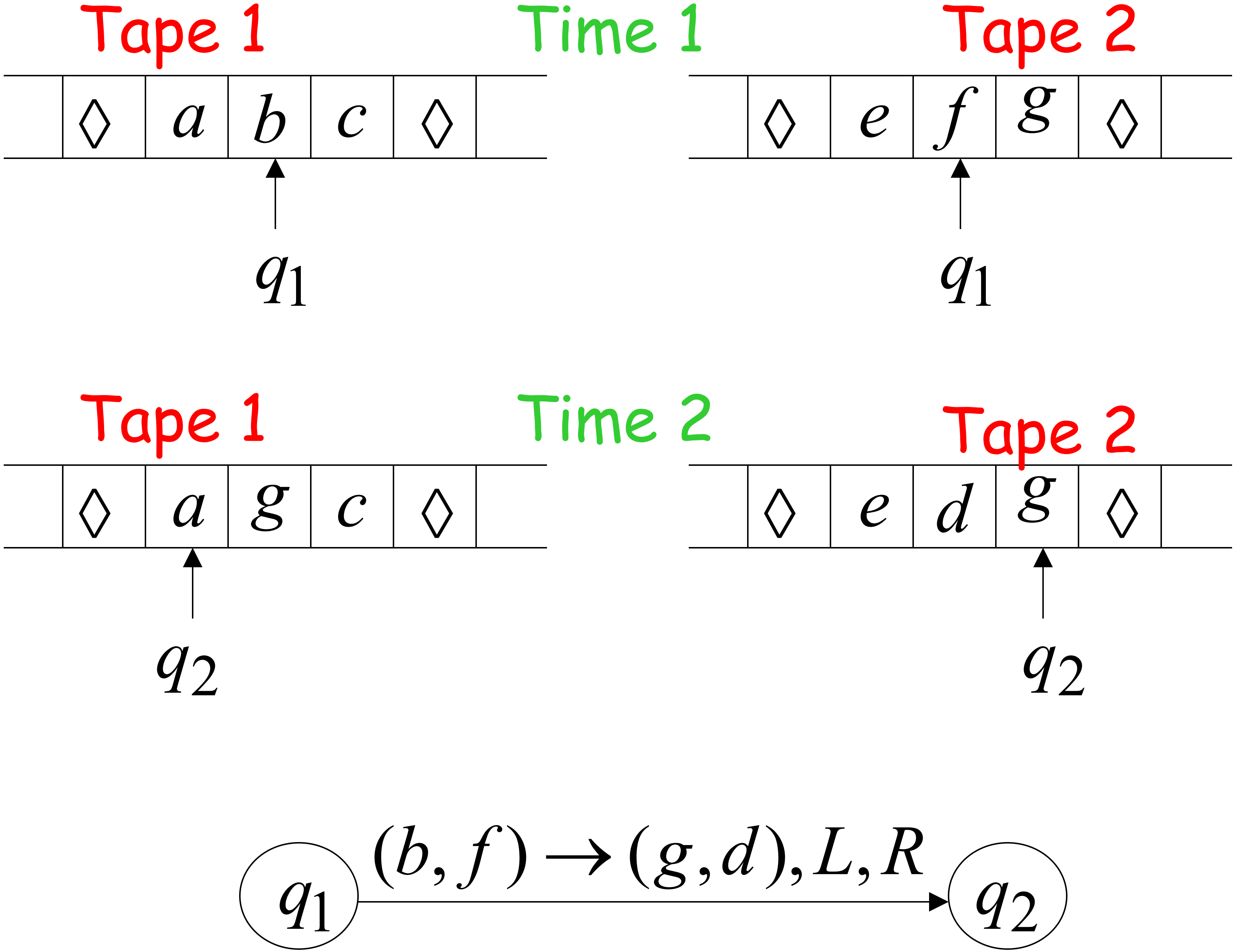
q_1^R

END OF PROOF

Multi-tape Turing Machines



Input string appears on Tape 1


Tape 1
Time 1
Tape 2
◊ a b c ◊
q_1
◊ e f g ◊
q_1
Tape 1
Time 2
Tape 2
◊ a g c ◊
q_2
◊ e d g ◊
q_2
q_1
$(b, f) \rightarrow (g, d), L, R$
q_2

Theorem: Multi-tape machines have the same power with Standard Turing machines

Proof: 1. Multi-tape machines simulate Standard Turing machines

2. Standard Turing machines simulate Multi-tape machines

1. Multi-tape machines simulate Standard Turing Machines:

Trivial: Use one tape

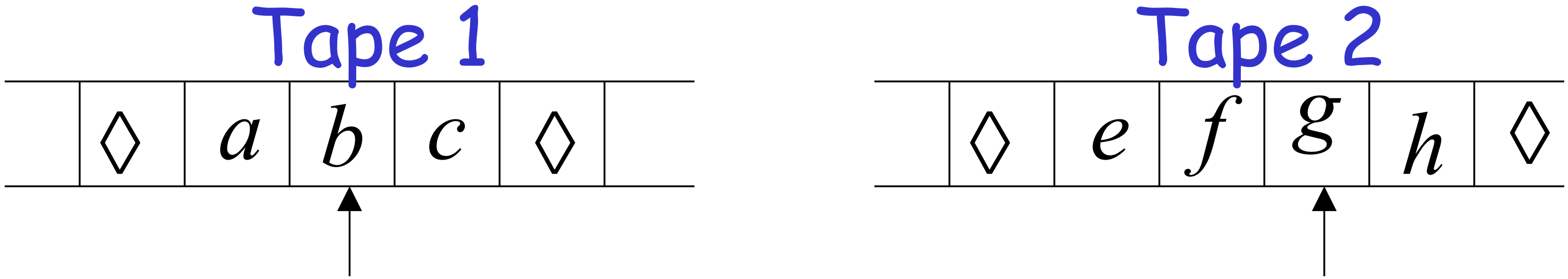
2. Standard Turing machines simulate Multi-tape machines:

Standard machine:

- Uses a multi-track tape to simulate the multiple tapes

- A tape of the Multi-tape machine corresponds to a pair of tracks

Multi-tape Machine



Standard machine with four track tape

		a	b	c				Tape 1
		0	1	0				head position
		e	f	g	h			Tape 2
		0	0	1	0			head position

Reference point

	#	a	b	c				Tape 1
	#	0	1	0				head position
	#	e	f	g	h			Tape 2
	#	0	0	1	0			head position

Repeat for each Multi-tape state transition:

1. Return to reference point
2. Find current symbol in Track 1 and update
3. Return to reference point
4. Find current symbol in Tape 2 and update

END OF PROOF

Same power doesn't imply same speed:

$$L = \{a^n b^n\}$$

Standard Turing machine: $O(n^2)$ time

Go back and forth $O(n^2)$ times to match the a's with the b's

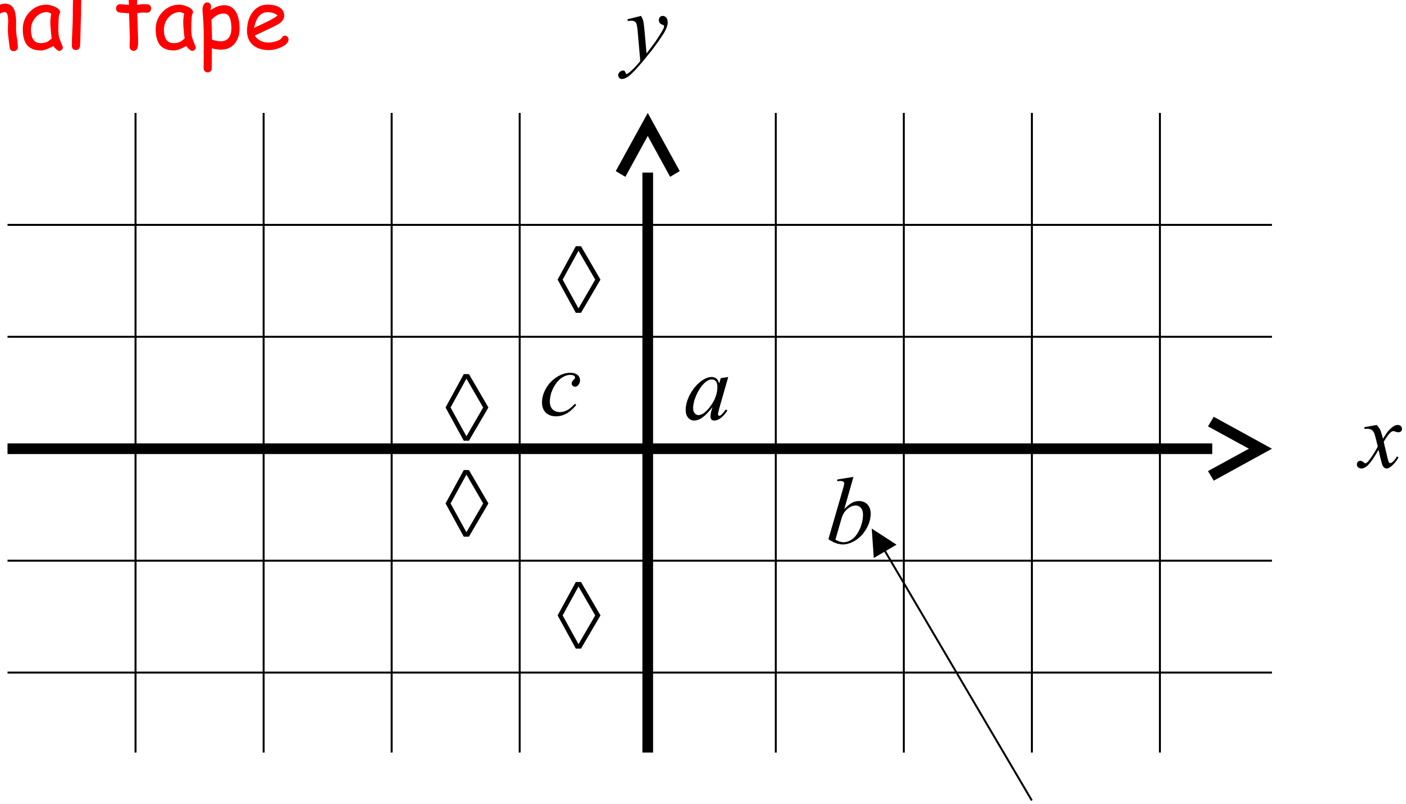
2-tape machine: $O(n)$ time

1. Copy b^n to tape 2 ($O(n)$ steps)

2. Compare a^n on tape 1 and b^n on tape 2 ($O(n)$ steps)

Multidimensional Turing Machines

2-dimensional tape



MOVES: L,R,U,D

U: up D: down

HEAD

Position: +2, -1

Theorem: Multidimensional machines have the same power with Standard Turing machines

Proof: 1. Multidimensional machines simulate Standard Turing machines

2. Standard Turing machines simulate Multi-Dimensional machines

1. Multidimensional machines simulate Standard Turing machines

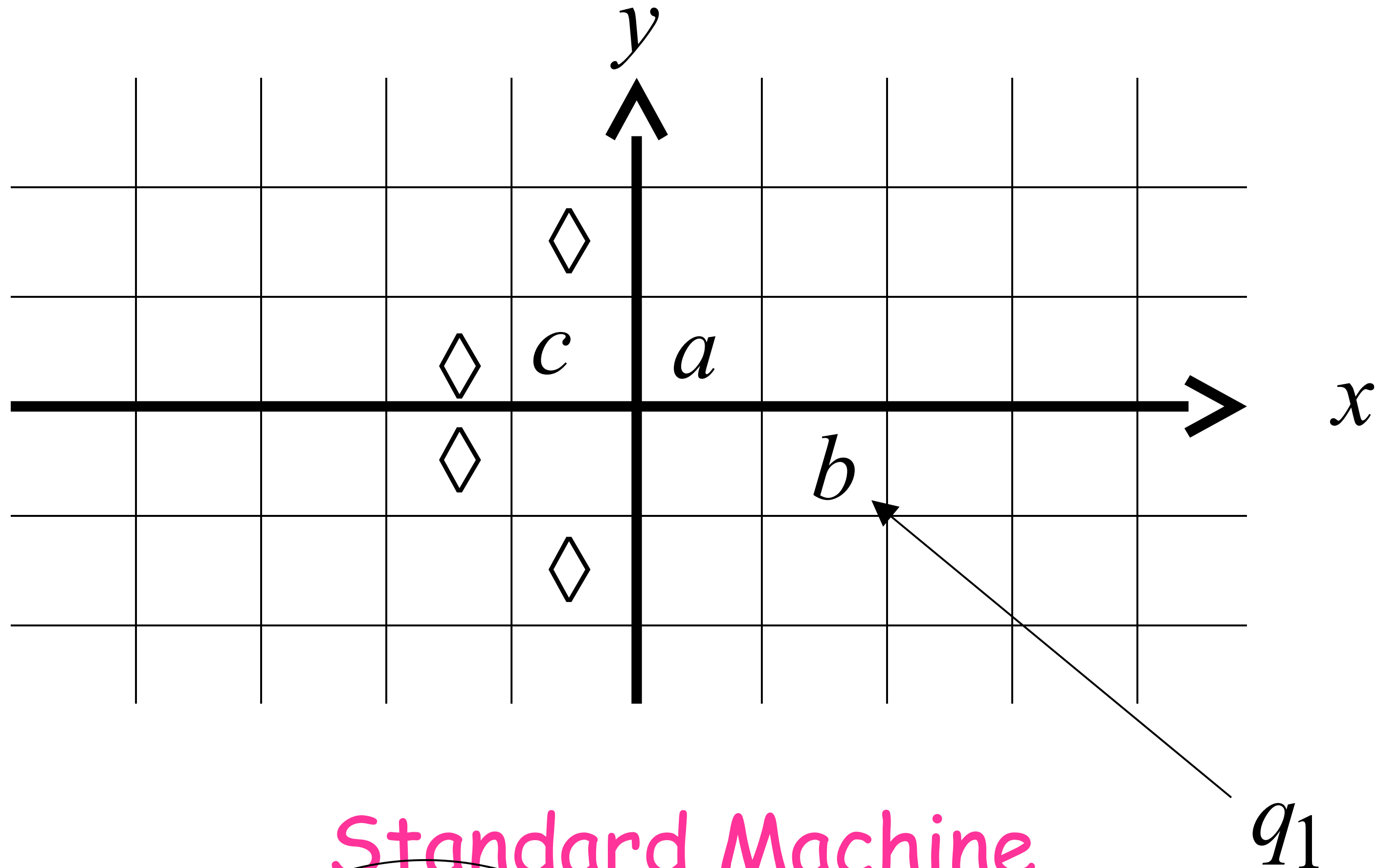
Trivial: Use one dimension

2. Standard Turing machines simulate Multidimensional machines

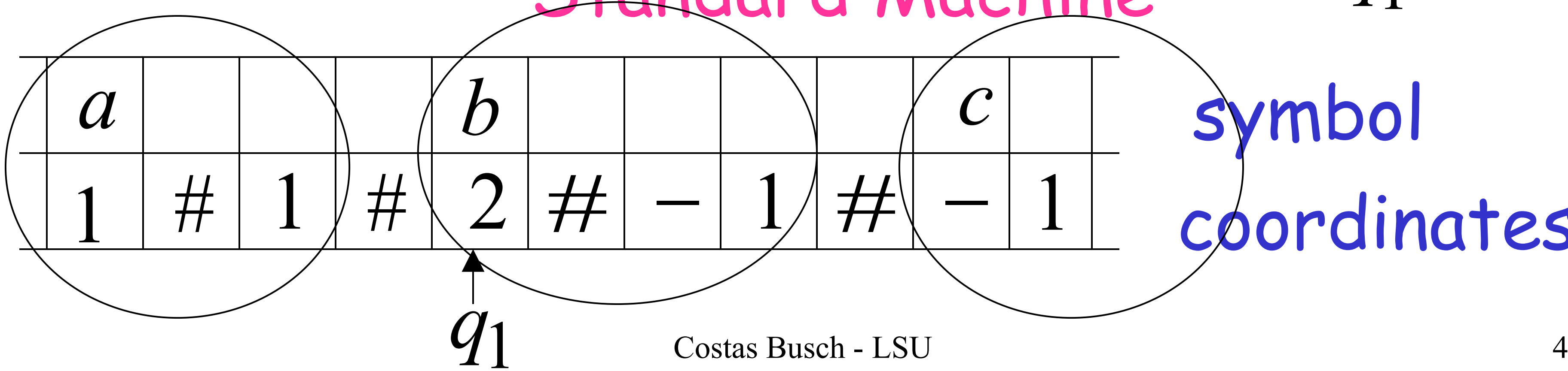
Standard machine:

- Use a two track tape
- Store symbols in track 1
- Store coordinates in track 2

2-dimensional machine



Standard Machine



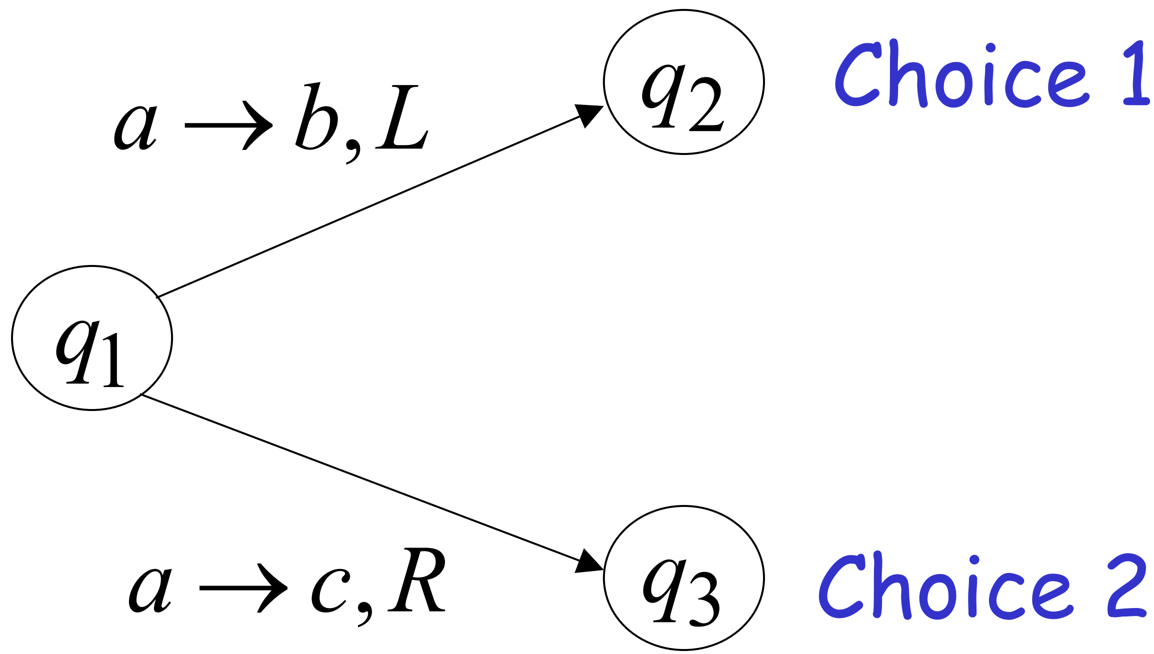
Standard machine:

Repeat for each transition followed in the 2-dimensional machine:

1. Update current symbol
2. Compute coordinates of next position
3. Find next position on tape

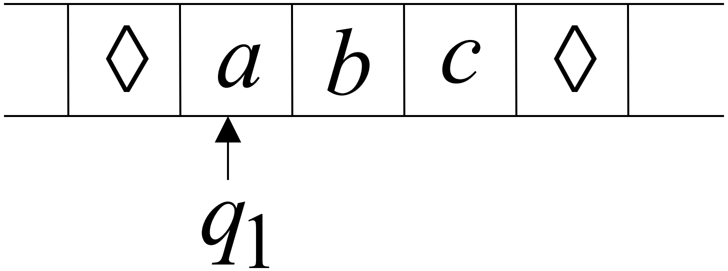
END OF PROOF

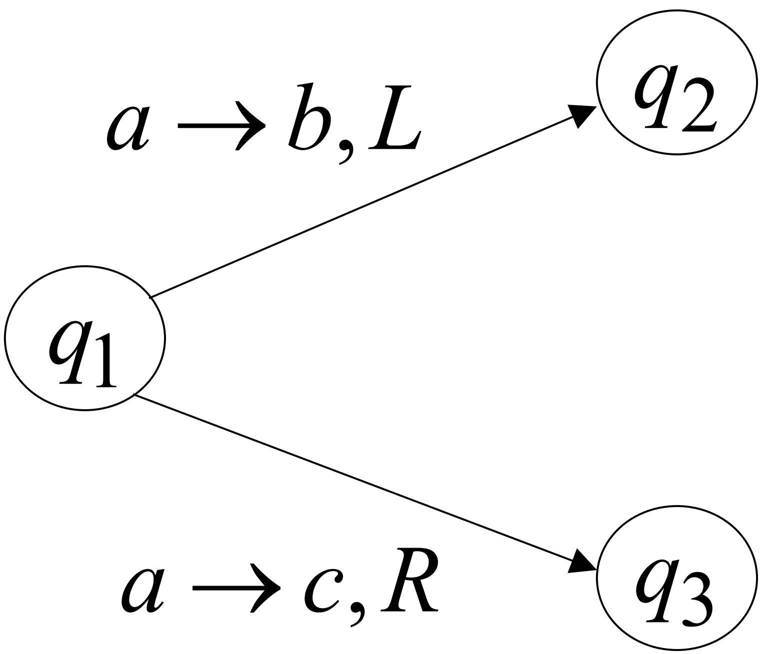
Nondeterministic Turing Machines



Allows Non Deterministic Choices

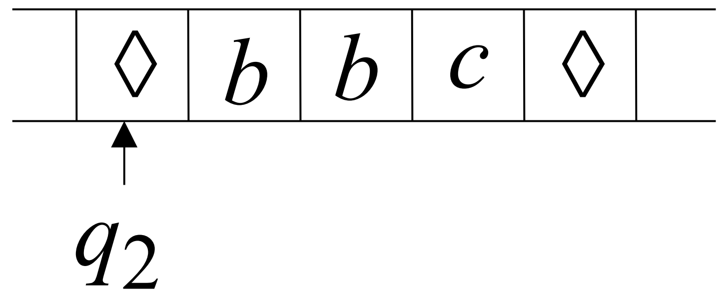
Time 0



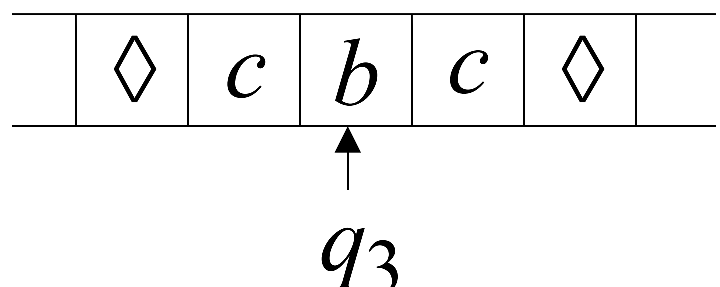


Time 1

Choice 1



Choice 2



Input string w is accepted if there is a computation:

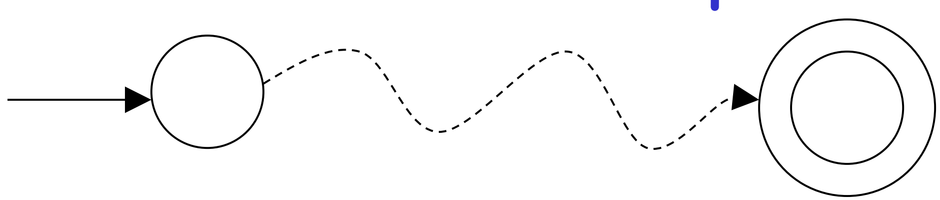
$$q_0 w \overset{*}{\succ} x \, q_f \, y$$

Initial configuration

Final Configuration

Any accept state

There is a computation:



Theorem: Nondeterministic machines have the same power with Standard Turing machines

Proof: 1. Nondeterministic machines simulate Standard Turing machines

2. Standard Turing machines simulate Nondeterministic machines

1. Nondeterministic Machines simulate Standard (deterministic) Turing Machines

Trivial: every deterministic machine is also nondeterministic

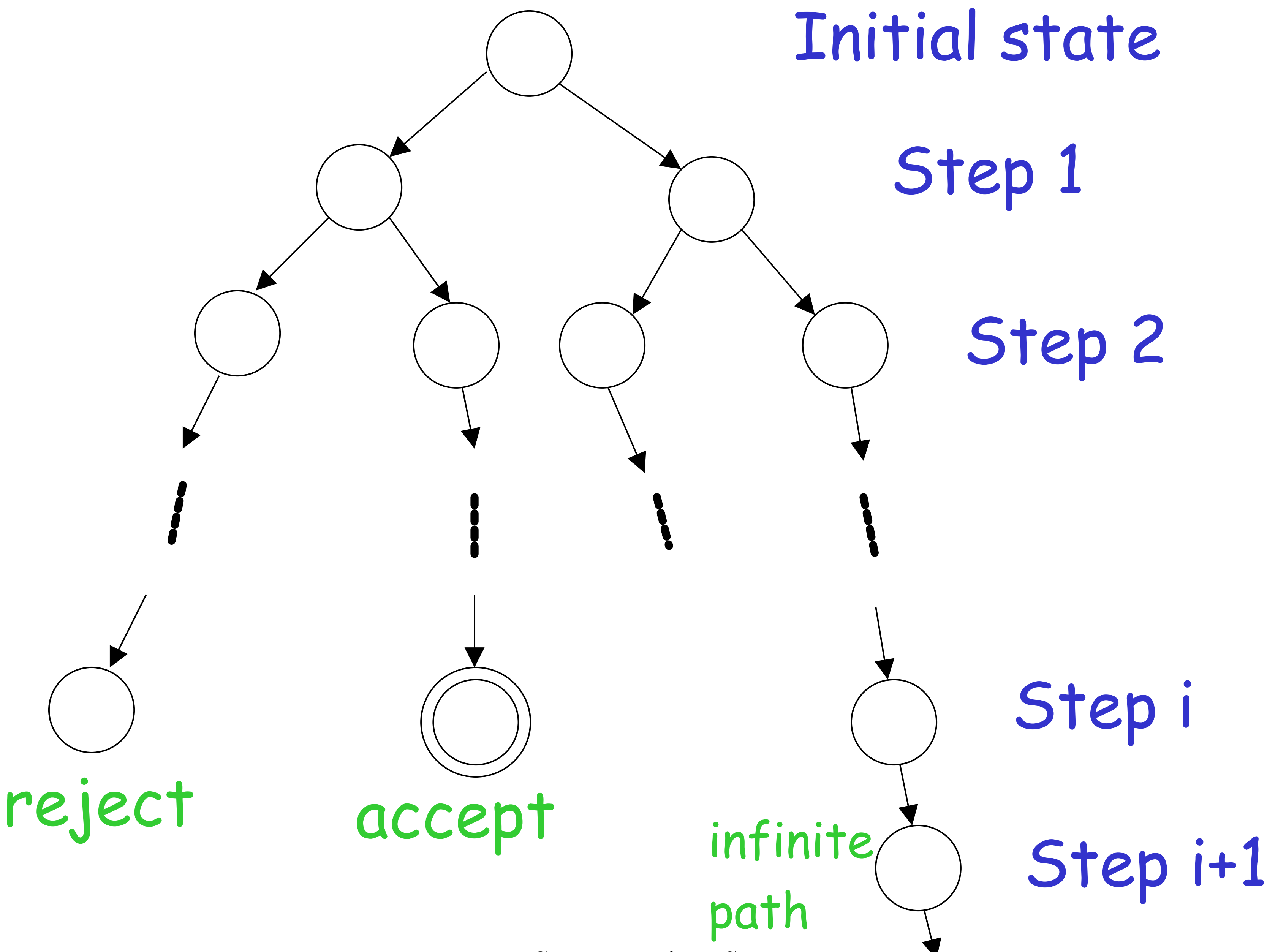
2. Standard (deterministic) Turing machines simulate Nondeterministic machines:

Deterministic machine:

- Uses a 2-dimensional tape
  (equivalent to standard Turing machine with one tape)

- Stores all possible computations of the non-deterministic machine on the 2-dimensional tape

All possible computation paths

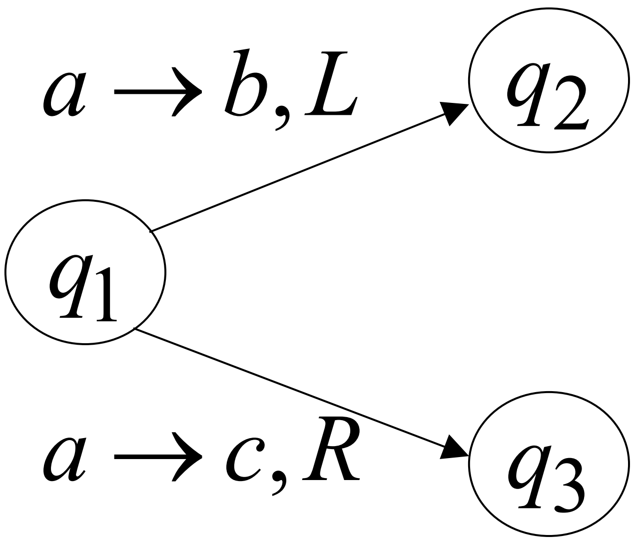


The Deterministic Turing machine simulates all possible computation paths:

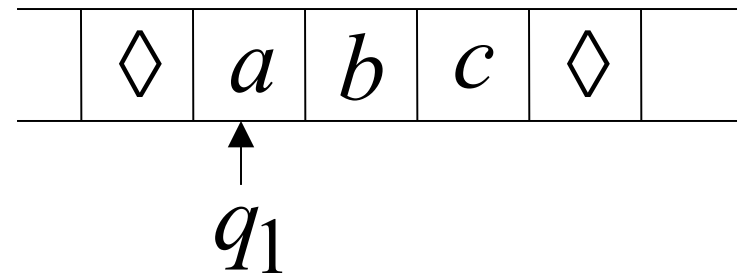
- simultaneously
- step-by-step
- with breadth-first search

depth-first may result getting stuck at exploring an infinite path before discovering the accepting path

NonDeterministic machine



Time 0

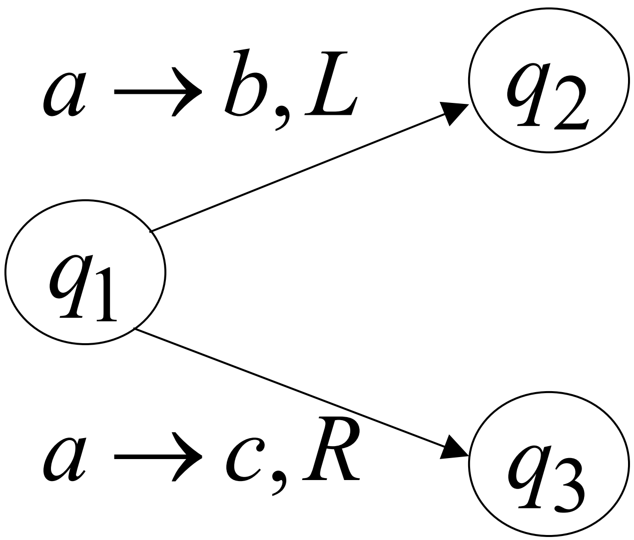


Deterministic machine

	#	#	#	#	#	#
	#	a	b	c	#	
	#	q_1			#	
	#	#	#	#	#	

current configuration

NonDeterministic machine



Time 1

Choice 1

	◊	b	b	c	◊	
	$\uparrow q_2$					

Choice 2

	◊	c	b	c	◊	
			$\uparrow q_3$			

Deterministic machine

	#	#	#	#	#	#	
#		b	b	c	#		
#	q_2				#		
#		c	b	c	#		
#			q_3		#		

Computation 1

Computation 2

Deterministic Turing machine

Repeat

For each configuration in current step
of non-deterministic machine,
if there are two or more choices:

1. Replicate configuration
2. Change the state in the replicas

Until either the input string is accepted
or rejected in all configurations

If the non-deterministic machine accepts the input string:

The deterministic machine accepts and halts too

The simulation takes in the worst case exponential time compared to the shortest length of an accepting path

If the non-deterministic machine does not accept the input string:

1. The simulation halts if all paths reach a halting state

OR

2. The simulation never terminates if there is a never-ending path (infinite loop)

In either case the deterministic machine rejects too (1. by halting or 2. by simulating the infinite loop)

END OF PROOF