BLM5106- Advanced Algorithm Analysis and Design

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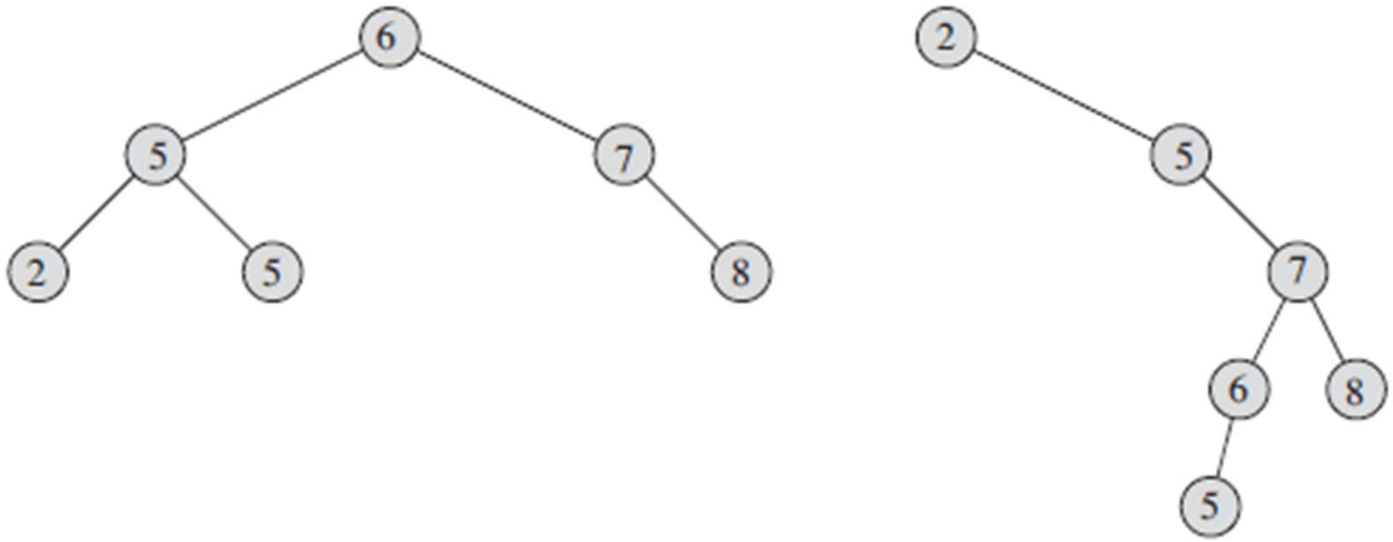
Binary Search Trees

The search tree data structure supports many dynamic-set operations, including SEARCH, MINIMUM, MAXIMUM, PREDECESSOR, SUCCESSOR, INSERT, and DELETE. Thus, we can use a search tree both as a dictionary and as a priority queue.

Basic operations on a binary search tree take time proportional to the height of the tree. For a complete binary tree with n nodes, such operations run in $\Theta(\lg n)$ worst-case time. If the tree is a linear chain of n nodes, however, the same operations take $\Theta(n)$ worst-case time.

Binary Search Trees



(a) (b)

Figure 12.1 Binary search trees. For any node x, the keys in the left subtree of x are at most $x.key$, and the keys in the right subtree of x are at least $x.key$. Different binary search trees can represent the same set of values. The worst-case running time for most search-tree operations is proportional to the height of the tree. **(a)** A binary search tree on 6 nodes with height 2. **(b)** A less efficient binary search tree with height 4 that contains the same keys.

Binary Search Trees

```
TREE-SEARCH(x, k)
if x == NIL or k == x.key
    return x
if k < x.key
    return TREE-SEARCH(x.left, k)
else return TREE-SEARCH(x.right, k)
```

Binary Search Trees

TREE-MINIMUM(x)

```
while x.left ≠ NIL
    x = x.left
return x
```

TREE-MAXIMUM(x)

```
while x.right ≠ NIL
    x = x.right
return x
```

Binary Search Trees

```
TREE-SUCCESSOR(x)
if x.right ≠ NIL
    return TREE-MINIMUM(x.right)
y = x.p
while y ≠ NIL and x == y.right
    x = y
    y = y.p
return y
```

Binary Search Trees

```
TREE-INSERT(T, z)
y = NIL
x = T.root
while x ≠ NIL
    y = x
    if z.key < x.key
        x = x.left
    else x = x.right
z.p = y
if y == NIL
    T.root = z          // tree T was empty
elseif z.key < y.key
    y.left = z
else y.right = z
```

Binary Search Trees

Deletion

The overall strategy for deleting a node z from a binary search tree T has three basic cases but, as we shall see, one of the cases is a bit tricky.

- If z has no children, then we simply remove it by modifying its parent to replace z with NIL as its child.
- If z has just one child, then we elevate that child to take z's position in the tree by modifying z's parent to replace z by z's child.
- If z has two children, then we find z's successor y—which must be in z's right subtree—and have y take z's position in the tree. The rest of z's original right subtree becomes y's new right subtree, and z's left subtree becomes y's new left subtree. This case is the tricky one because, as we shall see, it matters whether y is z's right child.

Binary Search Trees

```
TRANSPLANT(T, u, v)
if u.p == NIL
    T.root = v
elseif u == u.p.left
    u.p.left = v
else u.p.right = v
if v ≠ NIL
    v.p = u.p
```

Binary Search Trees

```
TREE-DELETE(T, z)
    if z.left == NIL
        TRANSPLANT(T, z, z.right)
    elseif z.right == NIL
        TRANSPLANT(T, z, z.left)
    else y = TREE-MINIMUM(z.right)
        if y.p ≠ z
            TRANSPLANT(T, y, y.right)
            y.right = z.right
            y.right.p = y
        TRANSPLANT(T, z, y)
        y.left = z.left
        y.left.p = y
```

Red-Black Trees

- A binary search tree of height h can support any of the basic dynamic-set operations—such as SEARCH, PREDECESSOR, SUCCESSOR, MINIMUM, MAXIMUM, INSERT, and DELETE—in $O(h)$ time.
- Thus, the set operations are fast if the height of the search tree is small. If its height is large, however, the set operations may run no faster than with a linked list.
- Red-black trees are one of many search-tree schemes that are “balanced” in order to guarantee that basic dynamic-set operations take $O(\lg n)$ time in the worst case.

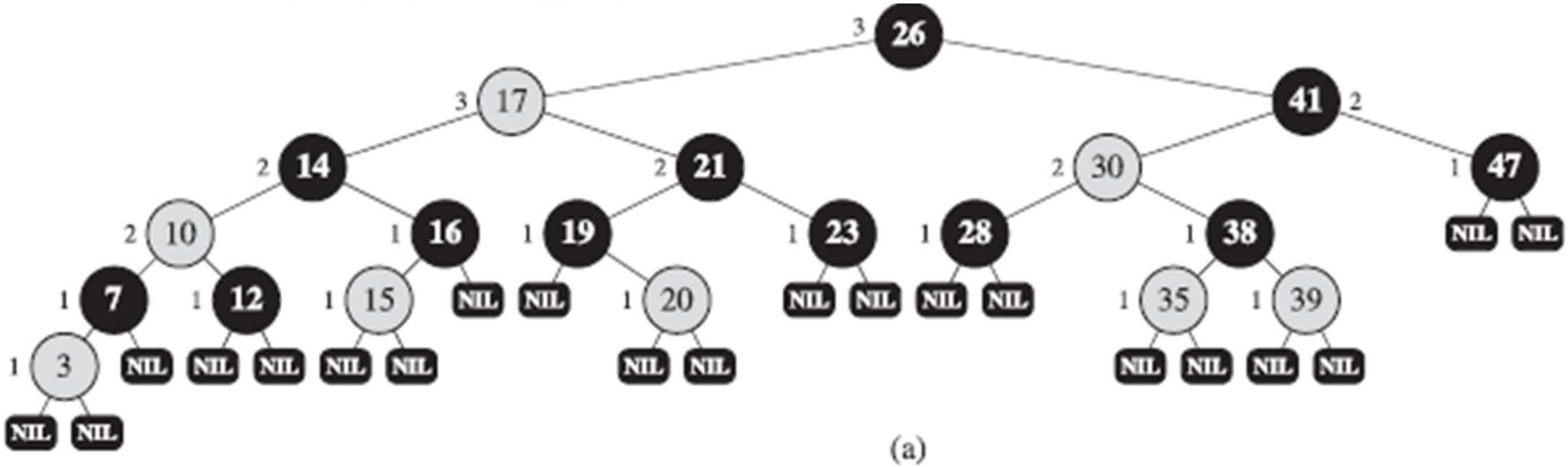
Red-Black Trees

- A ***red-black tree*** is a binary search tree with one extra bit of storage per node: its ***color***, which can be either RED or BLACK
- By constraining the node colors on any simple path from the root to a leaf, red-black trees ensure the tree is approximately balanced.
- Each node of the tree now contains the attributes color, key, left, right, and p. If a child or the parent of a node does not exist, the corresponding pointer attribute of the node contains the value NIL

Red-Black Trees

A red-black tree is a binary tree that satisfies the following ***red-black properties***:

1. Every node is either red or black.
2. The root is black.
3. Every leaf (NIL) is black.
4. If a node is red, then both its children are black.
5. For each node, all simple paths from the node to descendant leaves contain the same number of black nodes.



(a)

Red-Black Trees

- A red-black tree with n internal nodes has height at most 2*lg(n+1)

- Proof?

Rotations

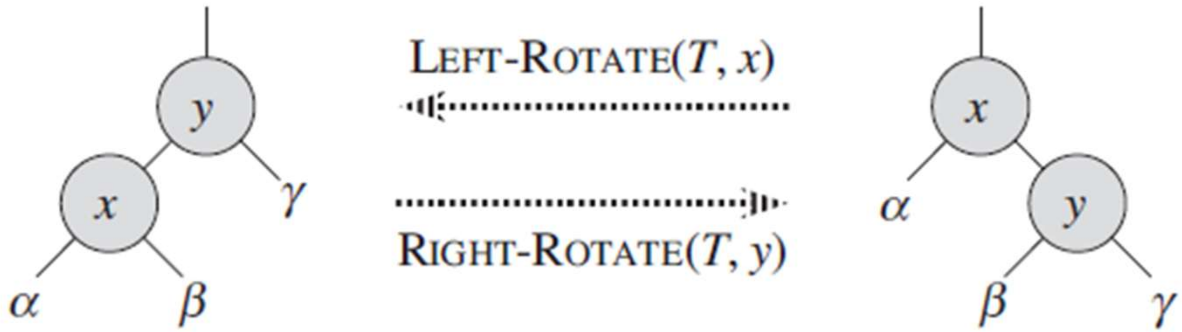


Figure 13.2 The rotation operations on a binary search tree. The operation LEFT-ROTATE(T, x) transforms the configuration of the two nodes on the right into the configuration on the left by changing a constant number of pointers. The inverse operation RIGHT-ROTATE(T, y) transforms the configuration on the left into the configuration on the right. The letters α, β, and γ represent arbitrary subtrees. A rotation operation preserves the binary-search-tree property: the keys in α precede $x.key$, which precedes the keys in β, which precede $y.key$, which precedes the keys in γ.

Rotations

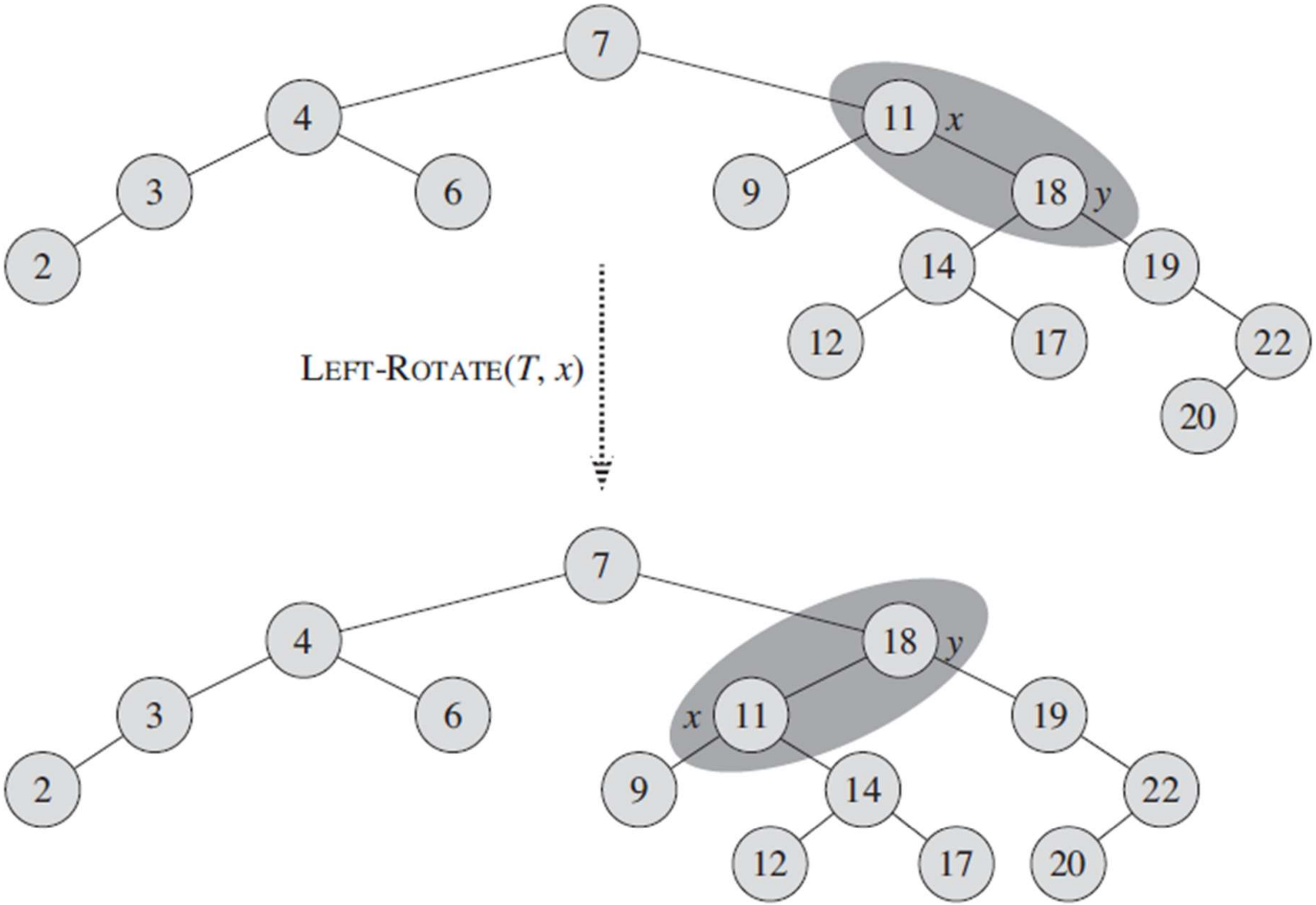


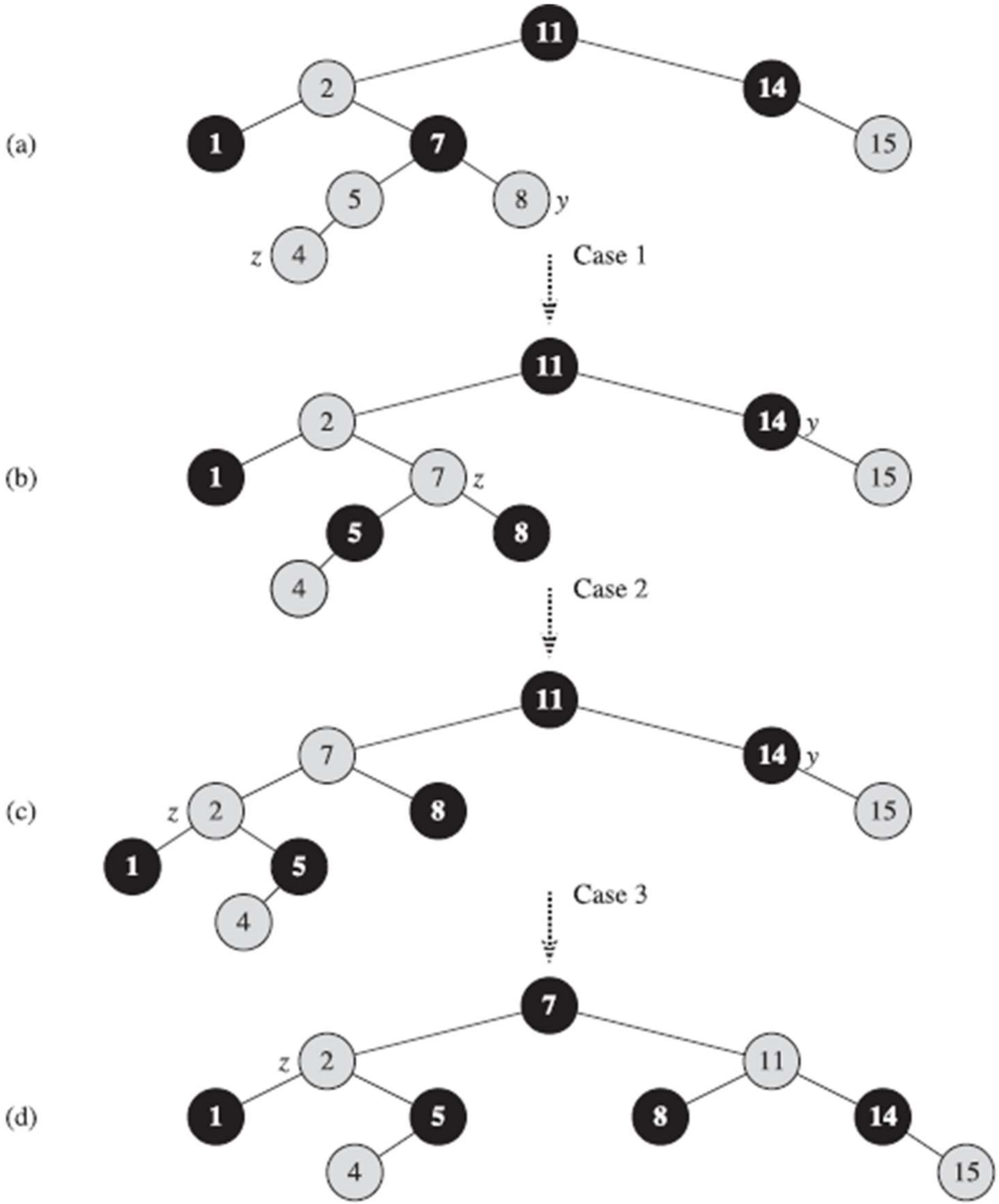
Figure 13.3 An example of how the procedure LEFT-ROTATE(T, x) modifies a binary search tree. Inorder tree walks of the input tree and the modified tree produce the same listing of key values.

RB-INSERT(T, z)

```
y = T.nil
x = T.root
while x ≠ T.nil
    y = x
    if z.key < x.key
        x = x.left
    else x = x.right
z.p = y
if y == T.nil
    T.root = z
elseif z.key < y.key
    y.left = z
else y.right = z
z.left = T.nil
z.right = T.nil
z.color = RED
RB-INSERT-FIXUP(T, z)
```

TREE-INSERT(T, z)

```
y = NIL
x = T.root
while x ≠ NIL
    y = x
    if z.key < x.key
        x = x.left
    else x = x.right
z.p = y
if y == NIL
    T.root = z          // tree T was empty
elseif z.key < y.key
    y.left = z
else y.right = z
```


(a)
11
2
14
1
7
15
5
8
y
z
4
Case 1
(b)
11
2
14
y
1
7
z
15
5
8
4
Case 2
(c)
11
7
14
y
z
2
8
15
1
5
4
Case 3
(d)
7
z
2
11
1
5
8
14
4
15

B-Trees

- B-trees are balanced search trees designed to work well on disks or other direct access secondary storage devices.
- B-trees are similar to red-black trees but they are better at minimizing disk I/O operations.
- Many database systems use B-trees, or variants of B-trees, to store information.
- B-trees differ from red-black trees in that B-tree nodes may have many children, from a few to thousands. That is, the "branching factor" of a B-tree can be quite large, although it usually depends on characteristics of the disk unit used.
- B-trees are similar to red-black trees in that every n-node B-tree has height $O(\lg n)$
- We can also use B-trees to implement many dynamic-set operations in time $O(\lg n)$

B-Tree

If an internal B-tree node x contains $x.n$ keys, then x has $x.n + 1$ children. The keys in node x serve as dividing points separating the range of keys handled by x into $x.n + 1$ subranges, each handled by one child of x.

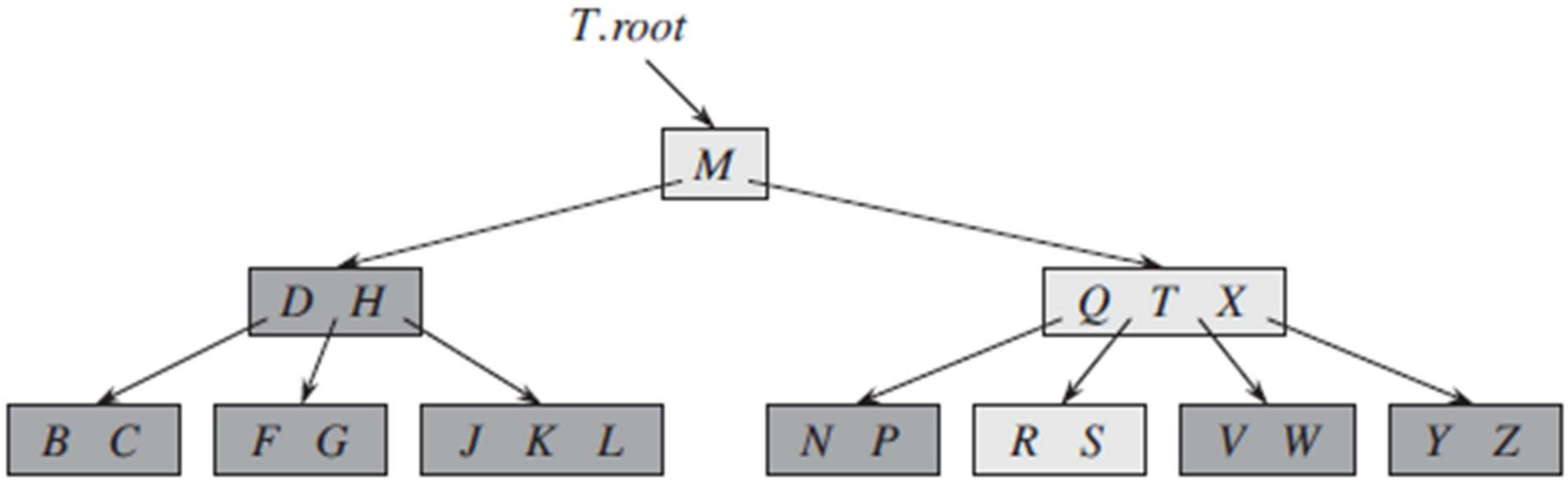


Figure 18.1 A B-tree whose keys are the consonants of English. An internal node x containing $x.n$ keys has $x.n + 1$ children. All leaves are at the same depth in the tree. The lightly shaded nodes are examined in a search for the letter R.

B-Tree

- In a typical B-tree application, the amount of data handled is so large that all the data do not fit into main memory at once.
- The B-tree algorithms copy selected pages from disk into main memory as needed and write back onto disk the pages that have changed.
- B-tree algorithms keep only a constant number of pages in main memory at any time; thus, the size of main memory does not limit the size of B-trees that can be handled.
- A B-tree node is usually as large as a whole disk page, and this size limits the number of children a B-tree node can have.
- For a large B-tree stored on a disk, we often see branching factors between 50 and 2000, depending on the size of a key relative to the size of a page.
- A large branching factor dramatically reduces both the height of the tree and the number of disk accesses required to find any key

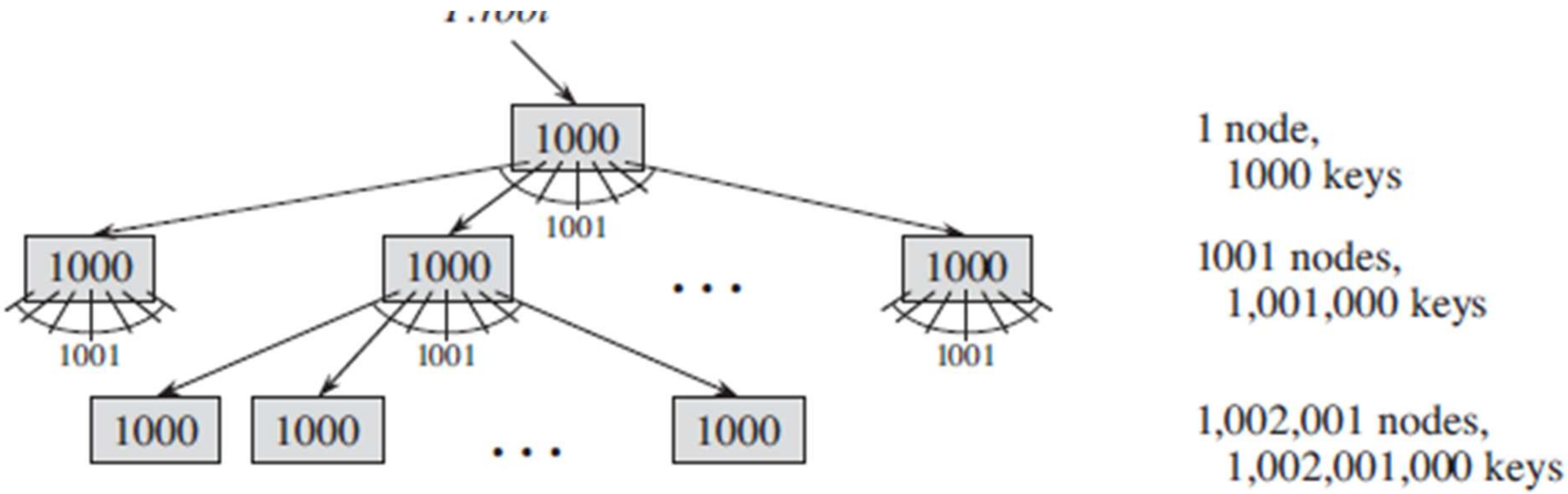


Figure 18.3 A B-tree of height 2 containing over one billion keys. Shown inside each node x is $x.n$, the number of keys in x. Each internal node and leaf contains 1000 keys. This B-tree has 1001 nodes at depth 1 and over one million leaves at depth 2.

A ***B-tree*** T is a rooted tree (whose root is $T.root$) having the following properties:

1. Every node x has the following attributes:

   a. $x.n$, the number of keys currently stored in node x,

   b. the $x.n$ keys themselves, $x.key_1, x.key_2, \ldots, x.key_{x.n}$, stored in nondecreasing order, so that $x.key_1 \le x.key_2 \le \cdots \le x.key_{x.n}$,

   c. $x.leaf$, a boolean value that is TRUE if x is a leaf and FALSE if x is an internal node.

2. Each internal node x also contains $x.n + 1$ pointers $x.c_1, x.c_2, \ldots, x.c_{x.n+1}$ to its children. Leaf nodes have no children, and so their c_i attributes are undefined.
3. The keys $x.key_i$ separate the ranges of keys stored in each subtree: if k_i is any key stored in the subtree with root $x.c_i$, then

$$k_1 \le x.key_1 \le k_2 \le x.key_2 \le \cdots \le x.key_{x.n} \le k_{x.n+1} .$$

4. All leaves have the same depth, which is the tree's height h.
5. Nodes have lower and upper bounds on the number of keys they can contain. We express these bounds in terms of a fixed integer $t \ge 2$ called the ***minimum degree*** of the B-tree:

   a. Every node other than the root must have at least $t - 1$ keys. Every internal node other than the root thus has at least t children. If the tree is nonempty, the root must have at least one key.

   b. Every node may contain at most $2t - 1$ keys. Therefore, an internal node may have at most $2t$ children. We say that a node is ***full*** if it contains exactly $2t - 1$ keys.[2]

- What values of t makes this b-tree legal?

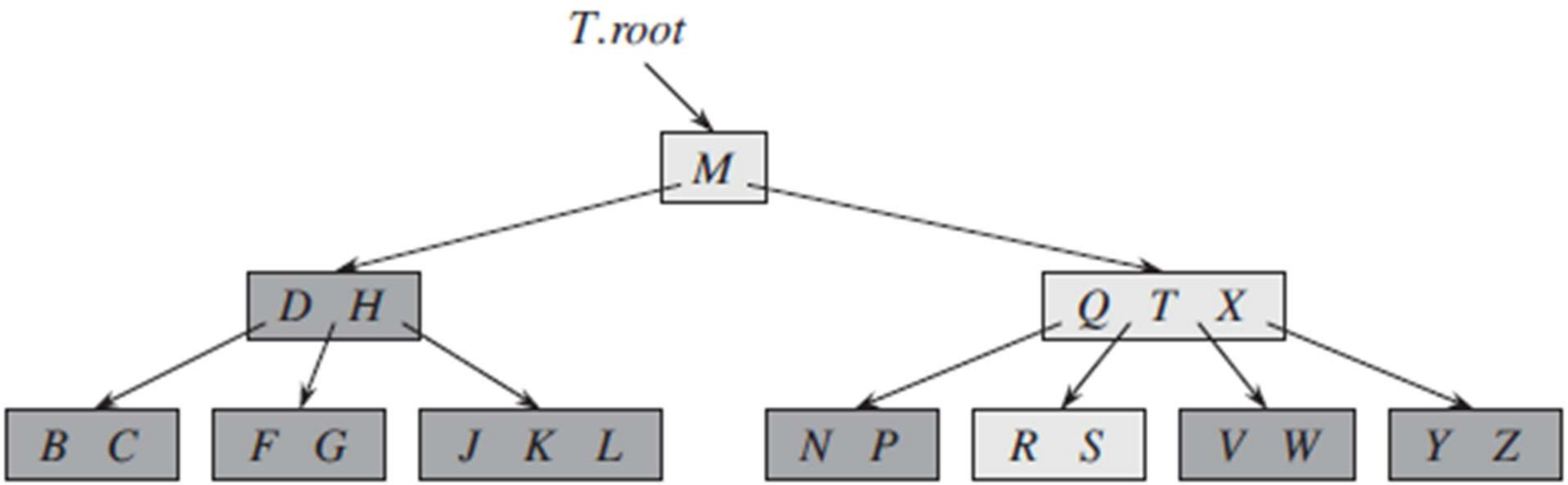


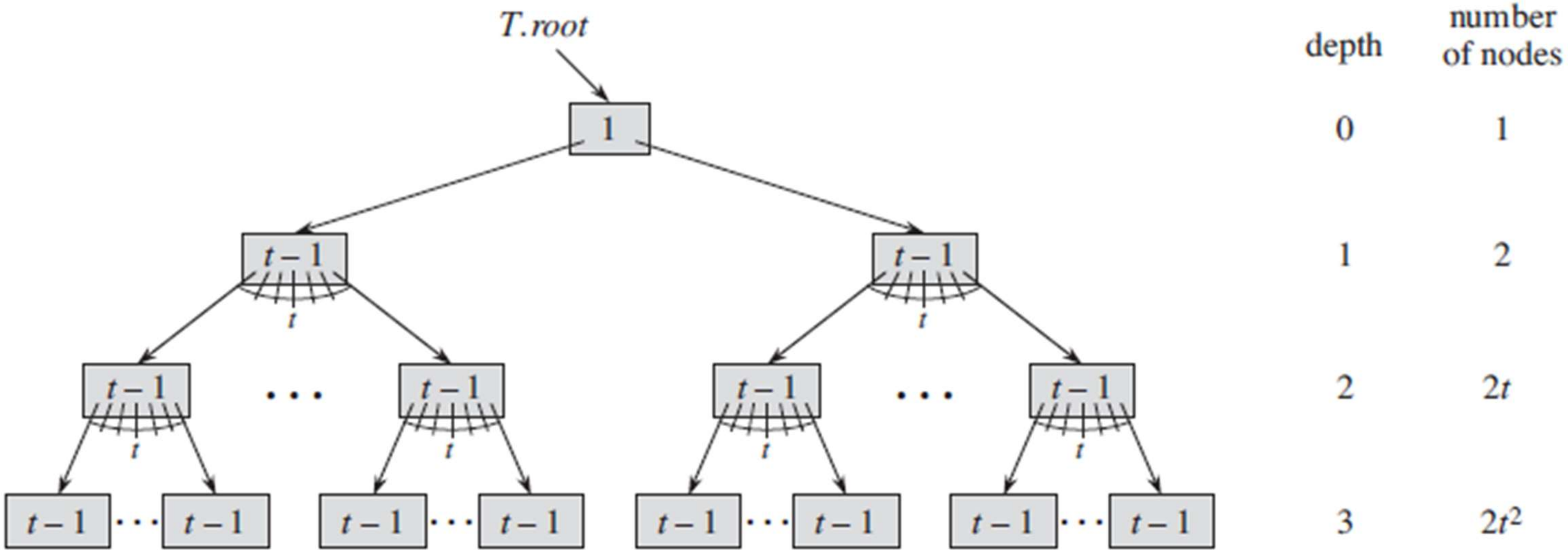
Figure 18.1 A B-tree whose keys are the consonants of English. An internal node x containing $x.n$ keys has $x.n + 1$ children. All leaves are at the same depth in the tree. The lightly shaded nodes are examined in a search for the letter R.

Proof?

Theorem 18.1

If $n \geq 1$, then for any n-key B-tree T of height h and minimum degree $t \geq 2$,

$$h \leq \log_t \frac{n+1}{2} .$$



$$
\begin{aligned}
n &\geq 1 + (t-1)\sum_{i=1}^{h} 2t^{i-1} \\
&= 1 + 2(t-1)\left(\frac{t^h - 1}{t-1}\right) \\
&= 2t^h - 1 .
\end{aligned}
$$

Basic Operations on B-Trees

- In this section, we present the details of the operations B-TREE-SEARCH, BTREE- CREATE, and B-TREE-INSERT. In these procedures, we adopt two conventions:
  - The root of the B-tree is always in main memory, so that we never need to perform a DISK-READ on the root; we do have to perform a DISK-WRITE of the root, however, whenever the root node is changed.
  - Any nodes that are passed as parameters must already have had a DISK-READ operation performed on them.
- The procedures we present are all “one-pass” algorithms that proceed downward from the root of the tree, without having to back up.

B-Tree Search

```
B-TREE-SEARCH(x, k)
i = 1
while i ≤ x.n and k > x.key_i
    i = i + 1
if i ≤ x.n and k == x.key_i
    return (x, i)
elseif x.leaf
    return NIL
else DISK-READ(x.c_i)
    return B-TREE-SEARCH(x.c_i, k)
```

- See a sample
- What about complexity?

Create an empty B-tree

To build a B-tree T, we first use B-TREE-CREATE to create an empty root node and then call B-TREE-INSERT to add new keys. Both of these procedures use an auxiliary procedure ALLOCATE-NODE, which allocates one disk page to be used as a new node in $O(1)$ time. We can assume that a node created by ALLOCATE-NODE requires no DISK-READ, since there is as yet no useful information stored on the disk for that node.

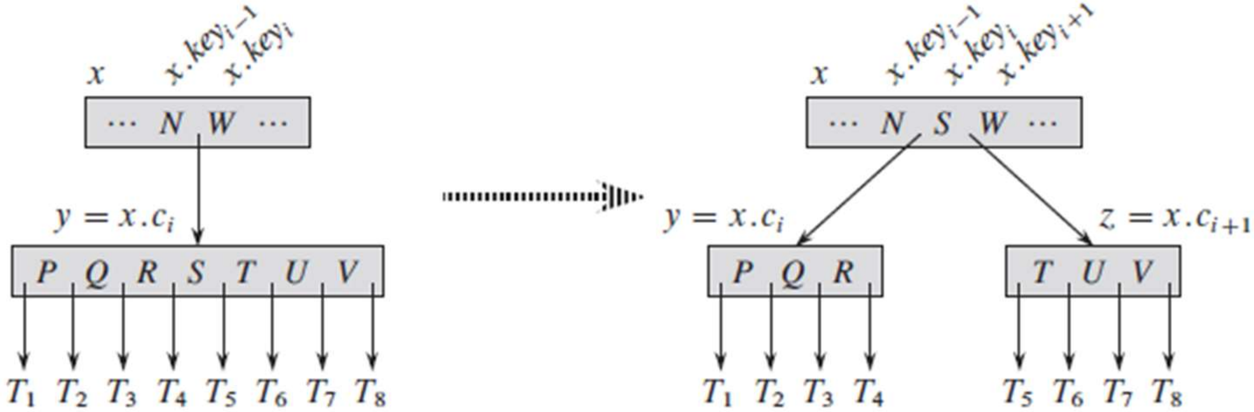
```
B-TREE-CREATE(T)
x = ALLOCATE-NODE()
x.leaf = TRUE
x.n = 0
DISK-WRITE(x)
T.root = x
```

B-TREE-CREATE requires $O(1)$ disk operations and $O(1)$ CPU time.

Inserting a key into a B-Tree

- As with binary search trees, we search for the leaf position at which to insert the new key.
- With a B-tree, however, we cannot simply create a new leaf node and insert it, as the resulting tree would fail to be a valid B-tree.
- Instead, we insert the new key into an existing leaf node. Since we cannot insert a key into a leaf node that is full, we introduce an operation that ***splits*** a full node y (having 2t-1 keys) around its ***median key*** y:*key*t into two nodes having only t-1 keys each.
- The median key moves up into y's parent to identify the dividing point between the two new trees.
- But if y's parent is also full, we must split it before we can insert the new key, and thus we could end up splitting full nodes all the way up the tree.

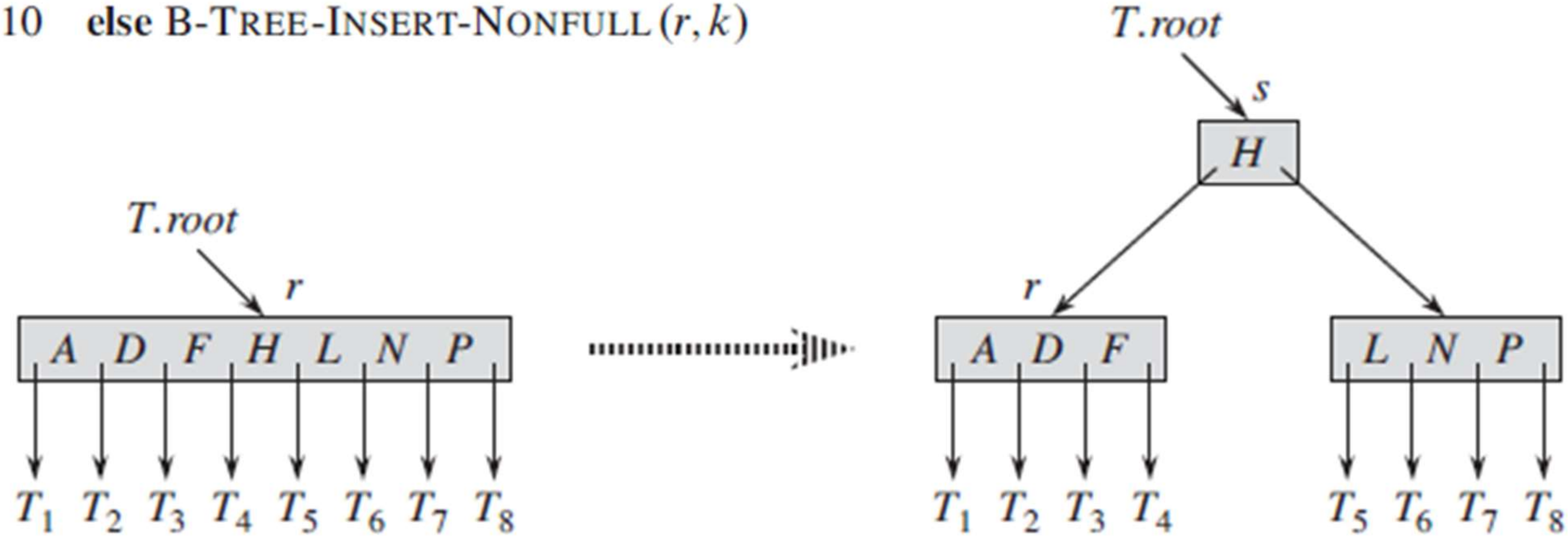
```
B-TREE-SPLIT-CHILD(x, i)
z = ALLOCATE-NODE()
y = x.c_i
z.leaf = y.leaf
z.n = t - 1
for j = 1 to t - 1
    z.key_j = y.key_{j+t}
if not y.leaf
    for j = 1 to t
        z.c_j = y.c_{j+t}
y.n = t - 1
for j = x.n + 1 downto i + 1
    x.c_{j+1} = x.c_j
x.c_{i+1} = z
for j = x.n downto i
    x.key_{j+1} = x.key_j
x.key_i = y.key_t
x.n = x.n + 1
DISK-WRITE(y)
DISK-WRITE(z)
DISK-WRITE(x)
```



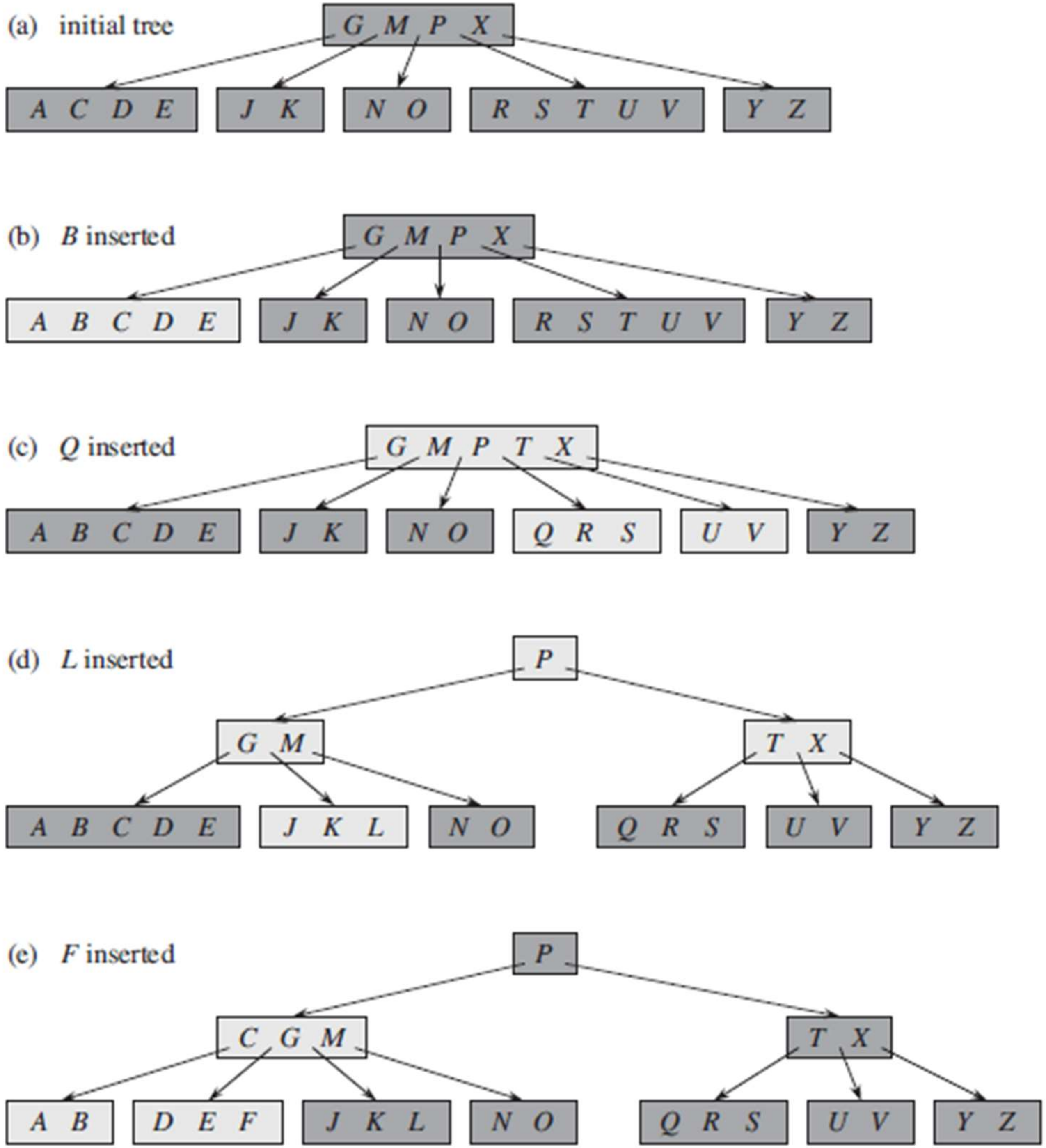
```
B-TREE-INSERT(T, k)
r = T.root
if r.n == 2t − 1
    s = ALLOCATE-NODE()
    T.root = s
    s.leaf = FALSE
    s.n = 0
    s.c_1 = r
    B-TREE-SPLIT-CHILD(s, 1)
    B-TREE-INSERT-NONFULL(s, k)
else B-TREE-INSERT-NONFULL(r, k)
```

The auxiliary recursive procedure B-TREE-INSERT-NONFULL inserts key k into node x, which is assumed to be nonfull when the procedure is called. The operation of B-TREE-INSERT and the recursive operation of B-TREE-INSERT-NONFULL guarantee that this assumption is true.

```
B-TREE-INSERT(T, k)
r = T.root
if r.n == 2t − 1
    s = ALLOCATE-NODE()
    T.root = s
    s.leaf = FALSE
    s.n = 0
    s.c_1 = r
    B-TREE-SPLIT-CHILD(s, 1)
    B-TREE-INSERT-NONFULL(s, k)
else B-TREE-INSERT-NONFULL(r, k)
```



```
B-TREE-INSERT-NONFULL(x, k)
i = x.n
if x.leaf
    while i ≥ 1 and k < x.key_i
        x.key_{i+1} = x.key_i
        i = i − 1
    x.key_{i+1} = k
    x.n = x.n + 1
    DISK-WRITE(x)
else while i ≥ 1 and k < x.key_i
        i = i − 1
    i = i + 1
    DISK-READ(x.c_i)
    if x.c_i.n == 2t − 1
        B-TREE-SPLIT-CHILD(x, i)
        if k > x.key_i
            i = i + 1
    B-TREE-INSERT-NONFULL(x.c_i, k)
```


(a) initial tree
G M P X
A C D E
J K
N O
R S T U V
Y Z
(b) B inserted
G M P X
A B C D E
J K
N O
R S T U V
Y Z
(c) Q inserted
G M P T X
A B C D E
J K
N O
Q R S
U V
Y Z
(d) L inserted
P
G M
T X
A B C D E
J K L
N O
Q R S
U V
Y Z
(e) F inserted
P
C G M
T X
A B
D E F
J K L
N O
Q R S
U V
Y Z

Deleting a key from a B-Tree

- Deletion from a B-tree is analogous to insertion but a little more complicated, because we can delete a key from any node—not just a leaf—and when we delete a key from an internal node, we will have to rearrange the node's children.
- As in insertion, we must guard against deletion producing a tree whose structure violates the B-tree properties.
- Just as we had to ensure that a node didn't get too big due to insertion, we must ensure that a node doesn't get too small during deletion (except that the root is allowed to have fewer than the minimum number $t - 1$ of keys).

Deleting a key from a B-Tree