BLM3620 Digital Signal Processing

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Lecture #11 – Fast Fourier Transform and Windowing

- Fast Fourier Transform
- Examples
- Windowing
- MATLAB Applications

Course Materials

Important Materials:

- James H. McClellan, R. W. Schafer, M. A. Yoder, *DSP First Second Edition*, Pearson, 2015.

- Lizhe Tan, Jean Jiang, *Digital Signal Processing: Fundamentals and Applications*, Third Edition, Academic Press, 2019.

Auxilary Materials:

- Prof. Sarp Ertürk, *Sayısal İşaret İşleme*, Birsen Yayınevi.

- Prof. Nizamettin Aydin, DSP Lecture Notes.

- J. G. Proakis, D. K. Manolakis, *Digital Signal Processing Fourth Edition*, Peason, 2014.

- J. K. Perin, *Digital Signal Processing*, *Lecture Notes,* Standford University, 2018.

Syllabus

Week	Lectures
1	Introduction to DSP and MATLAB
2	Sinuzoids and Complex Exponentials
3	Spectrum Representation
4	Sampling and Aliasing
5	Discrete Time Signal Properties and Convolution
6	Convolution and FIR Filters
7	Frequency Response of FIR Filters
8	Midterm Exam
9	Discrete Time Fourier Transform and Properties
10	Discrete Fourier Transform and Properties
11	Fast Fourier Transform and Windowing
12	z- Transforms
13	FIR Filter Design and Applications
14	IIR Filter Design and Applications
15	Final Exam

For more details -> Bologna page: http://www.bologna.yildiz.edu.tr/index.php?r=course/view&id=5730&aid=3

Review & Recall

Discrete Fourier Transform (DFT)

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

Inverse DFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn}$$

Discrete-time Fourier Transform (DTFT)

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\hat{\omega} n}$$

Inverse DTFT

$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\hat{\omega}}) e^{j\hat{\omega} n} d\hat{\omega}$$

Recap: MATLAB Code for DFT

```
clc; clear all;
%%
x = [1 2 2 1 1 2 3 4];
N = length(x);
X = zeros(1,N);

for k = 0:N-1
    for n = 0:N-1
        X(k+1) = X(k+1) + x(n+1)*exp(-j*(2*pi/N)*k*n);
    end
end

X
fft(x)
```

$$X[k]=\sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn}$$

What is the algorithm complexity here?

Fast Fourier Transform (FFT)

FFT is an algorithm that allows to obtain the same DFT results with a faster way.

Most of applications in real world use FFT instead of DFT.

Algorithm complexity of DFT is O(N^2). It requires *N* complex-multiplication for each *k* sample.

Therefore, if N=2^{10} -> $N^2 = 2^{20}$. Too much computational complexity!!!

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

Fast Fourier Transform (FFT)

Algorithm complexity of DFT is O(N^2).

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

Algorithm complexity of FFT is $\frac{N}{2}$O $(log_2\, N)$

<u>Main Idea:</u>

Divide N-length signal into two N/2-length sub-parts (even and odd indices) until reaching 2-length signals.

Can be calculated using two ways:

1) Decimation in time
2) Decimation in frequency

Fast Fourier Transform (FFT)

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

Let’s define a Phase Factor:

$$\boxed{W_N^k = e^{-j\frac{2\pi}{N}k}}$$

Using the symmetry and periodicity properties of DFT, we can write:

$$\boxed{W_N^{k+N/2} = e^{-j\frac{2\pi}{N}(k+N/2)} = -W_N^k} \qquad \boxed{W_N^{k+N} = e^{-j\frac{2\pi}{N}(k+N)} = W_N^k}$$

Benefitting from these properties, we can calculate FFT in a faster way.

Decimation in time algorithm is also known as Radix-2 algorithm.

Derivation of FFT Equations

N-point DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$

$$W_N^k = e^{-j\frac{2\pi}{N}k}$$

Decompose the signal into even part (x[2n]) and odd part (x[2n+1]):

$$X[k] = \sum_{n=0}^{N/2-1} x[2n]\, W_N^{k2n} + \sum_{n=0}^{N/2-1} x[2n+1]\, W_N^{k(2n+1)}$$

$$W_N^{k(2n)} = e^{-j\frac{2\pi 2kn}{N}} = e^{-j\frac{2\pi kn}{N/2}} = W_{N/2}^{kn}$$

By using this property, we can rewrite the equation as:

$$X[k] = \sum_{n=0}^{N/2-1} x[2n]\, W_{N/2}^{kn} + W_N^k \sum_{n=0}^{N/2-1} x[2n+1]\, W_{N/2}^{kn}$$

Derivation of FFT Equations

$$X[k] = \sum_{n=0}^{N/2-1} x[2n]\, W_{N/2}^{kn} + W_N^k \sum_{n=0}^{N/2-1} x[2n+1]\, W_{N/2}^{kn}$$

$$W_N^k = e^{-j\frac{2\pi}{N}k}$$

Here, we can rename the signals as $x_1[n] = x[2n]$ ve $x_2[n] = x[2n+1]$

$$X[k] = \sum_{n=0}^{N/2-1} x_1[n]\, W_{N/2}^{kn} + W_N^k \sum_{n=0}^{N/2-1} x_2[n]\, W_{N/2}^{kn}$$

This bring us to the final equation:

$$X[k] = X_1[k] + W_N^k X_2[k]$$

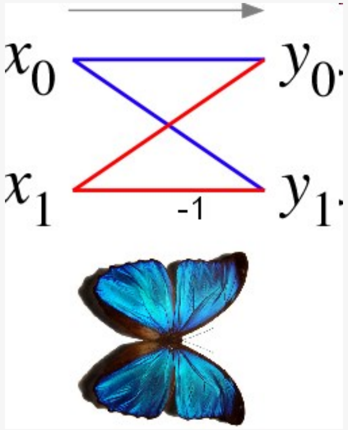
$$X[k+N/2] = X_1[k] - W_N^k X_2[k]$$

Conjugate symmetry property

Derivation of FFT Equations

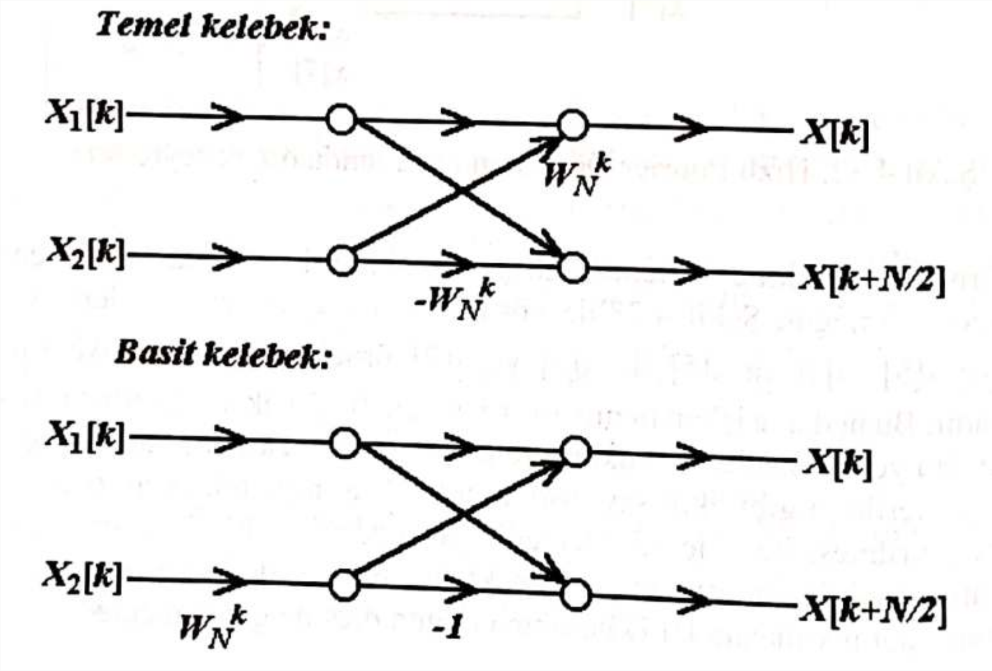
$$W_N^k = e^{-j\frac{2\pi}{N}k}$$

Butterfly FFT



$$X[k] = X_1[k] + W_N^k X_2[k]$$

$$X[k+N/2] = X_1[k] - W_N^k X_2[k]$$



Algorithm of FFT

<u>8-pt FFT algorithm:</u>

- Önce tek ve çift örnekleri belirleyerek iki parçaya ayır -> $(x_t, x_ç)$
- Sonra tek ve çift örnekleri de ikiye ayır -> $(x_{tt}, x_{tç}, x_{çt}, x_{çç})$
- Her ikili çift için AFD'sini hesapla. -> $(X_{tt}, X_{tç}, X_{çt}, X_{çç})$
- Aşağıdaki denklemleri kullanarak tek ve çift örneklerin AFD'sini bul. -> $(X_t, X_ç)$

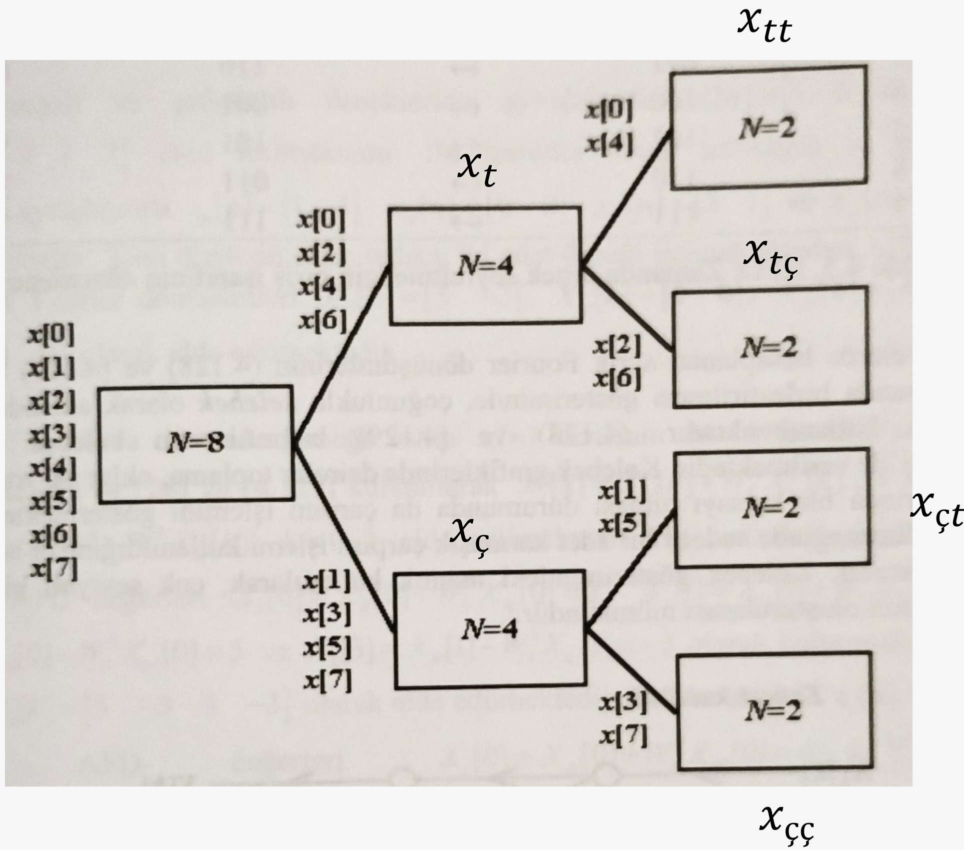
$$X_t[k] = X_{tt}[k] + W_4^k X_{tç}[k]$$

$$X_t[k + N/2] = X_{tt}[k] - W_4^k X_{tç}[k]$$

- Tek ve çift örneklerin AFD'sini kullanarak denklemlerle sonucu bul. -> (X)

$$X[k] = X_t[k] + W_8^k X_ç[k]$$

$$X[k + N/2] = X_t[k] - W_8^k X_ç[k]$$



Example:

Hızlı Fourier dönüşümü kullanarak $\boldsymbol{x[n]} = [\mathbf{1\ \ 3\ \ 0\ \ 2\ \ 4\ \ 1\ \ 0\ \ 2}]$ işaretinin ayrık Fourier dönüşümünü hesaplayınız.

$x_t[n] = [1 \quad 0 \quad 4 \quad 0]$

$x_ç[n] = [3 \quad 2 \quad 1 \quad 2]$

$x_{tt}[n] = [1 \quad 4]$

$x_{tç}[n] = [0 \quad 0]$

$x_{çt}[n] = [3 \quad 1]$

$x_{çç}[n] = [2 \quad 2]$

$$X_{tt}[0] = \sum_{n=0}^{1} x_{tt}[n]\, e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{1} x_{tt}[n]\, e^{-j\pi 0 n} = 5$$

$$X_{tt}[1] = \sum_{n=0}^{1} x_{tt}[n]\, e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{1} x_{tt}[n]\, e^{-j\pi n} = 1 - 4 = -3$$

$\Rightarrow$

$X_{tt}[k] = [5 \;\; -3]$

$X_{çt}[k] = [4 \quad 2]$

$X_{tç}[k] = [0 \quad 0]$

$X_{çç}[k] = [4 \quad 0]$

Example:

$X_{tt}[k] = [5 \quad -3]$ $\quad X_{tç}[k] = [0 \quad 0]$ $\quad X_{çt}[k] = [4 \quad 2]$ $\quad X_{tç}[k] = [4 \quad 0]$

$X_t[k] = ?$ $\quad X_ç[k] = ?$

$$X_t[k] = X_{tt}[k] + W_4^k X_{tç}[k], \qquad k = 0,1$$

$$X_t[k+2] = X_{tt}[k] - W_4^k X_{tç}[k], \qquad k = 0,1$$

$$W_4^k = e^{-j\frac{2\pi}{N}k} = e^{-j\frac{\pi}{2}k}$$

$$X_t[0] = X_{tt}[0] + W_4^0 X_{tç}[0] = 5$$
$$X_t[1] = X_{tt}[1] + W_4^1 X_{tç}[1] = -3$$
$$X_t[0+2] = X_{tt}[0] - W_4^0 X_{tç}[0] = 5$$
$$X_t[1+2] = X_{tt}[1] - W_4^1 X_{tç}[1] = -3$$

$$X_t[k] = [5 \quad -3 \quad 5 \quad -3]$$
$$X_ç[k] = [8 \quad 2 \quad 0 \quad 2]$$

Example:

$$X_t[k] = [5 \quad -3 \quad 5 \quad -3] \qquad\qquad X_ç[k] = [8 \quad 2 \quad 0 \quad 2]$$

$$X[k] = ?$$

$$X[k] = X_t[k] + W_8^k X_ç[k], \qquad k = 0,1,2,3$$

$$X[k+4] = X_t[k] - W_8^k X_ç[k], \qquad k = 0,1,2,3$$

$$W_8^k = e^{-j\frac{2\pi}{N}k} = e^{-j\frac{\pi}{4}k}$$

$$X[0] = X_t[0] + W_8^0 X_ç[0] = 5 + 8e^{-0.25\pi 0} = 13$$

$$X[1] = X_t[1] + W_8^1 X_ç[1] = -3 + 2e^{-0.25\pi 1} = 1.5858 - j1.412$$

$$X[2] = X_t[2] + W_8^2 X_ç[2] = 5 + 0e^{-0.25\pi 2} = 5$$

$$X[3] = X_t[3] + W_8^3 X_ç[3] = -3 + 2e^{-0.25\pi 3} = -4.4142 - j1.412$$

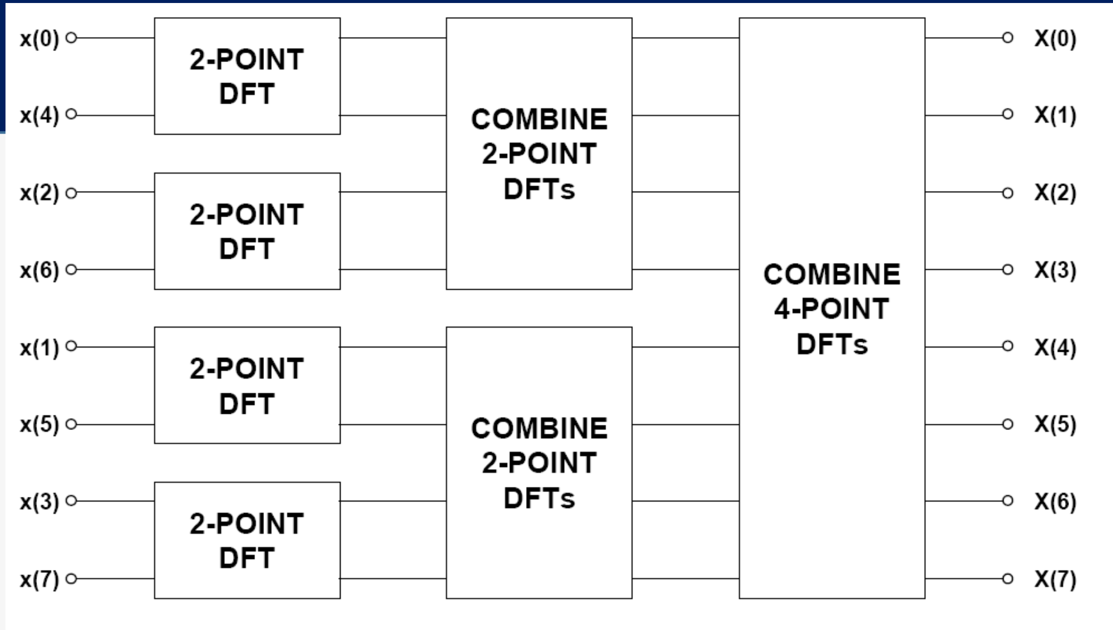
$$X[4] = X_t[0] - W_8^0 X_ç[0] = 5 - 8e^{-(j0.25\pi)0} = -3,$$

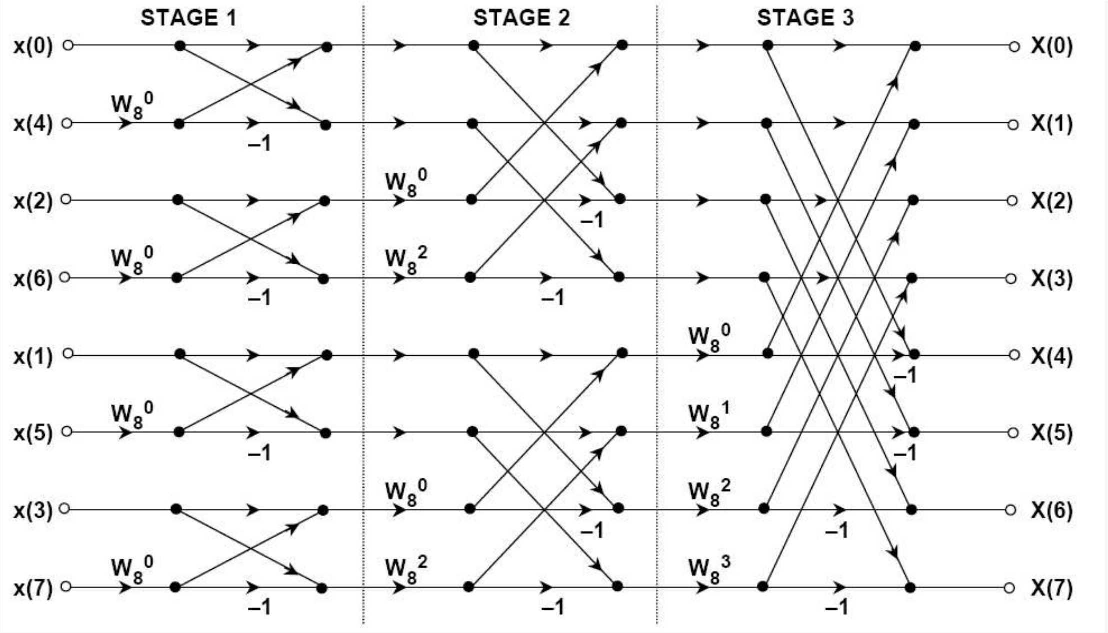
$$X[5] = X_t[1] - W_8^1 X_ç[1] = -3 - 2e^{-(j0.25\pi)1} = -4.4142 + j1.4142,$$

$$X[6] = X_t[2] - W_8^2 X_ç[2] = 5 - 0e^{-(j0.25\pi)2} = 5,$$

$$X[7] = X_t[3] - W_8^3 X_ç[3] = -3 - 2e^{-(j0.25\pi)3} = -1.5858 + j1.4142$$

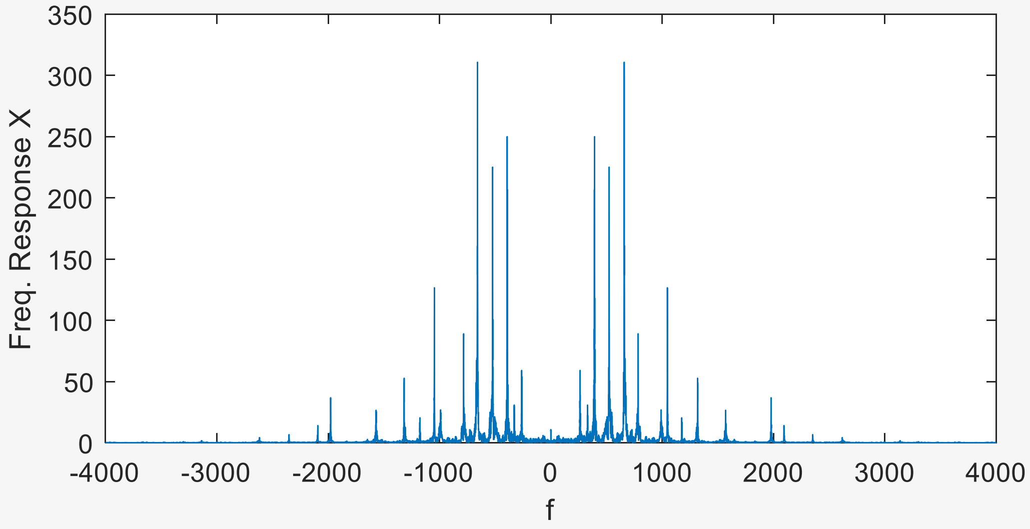
- Input signal must be properly re-ordered using a *bit reversal* algorithm
- In-place computation
- Number of stages: $log_2 N$
- Stage 1: all the twiddle factors are 1
- Last Stage: the twiddle factors are in sequential order

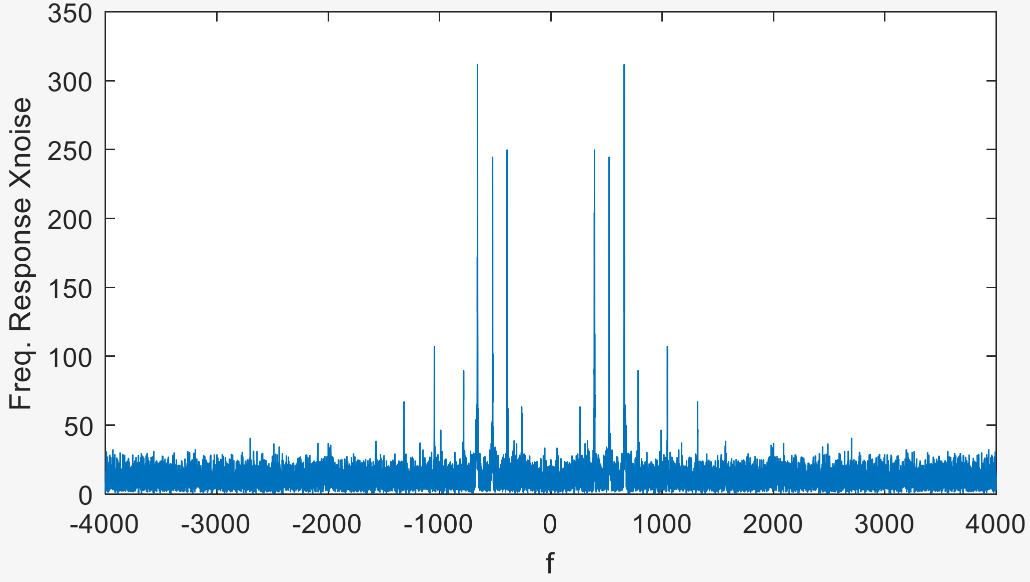




Application – 1: FFT of a noisy sound

```
clc; clear all;
%% Load Sound
load ('piano2.mat');
N = 2^14; x = x(1:N);
%% FFT of the signal
X = fftshift(fft(x,N));
wHat = (-N/2:N/2-1) * Fs/N;
plot(wHat,abs(X)); xlabel('f');
ylabel('Freq. Response X');
%% Add Noise
Xnoise = awgn(x,20);
Xnoisef = fftshift(fft(Xnoise,N));
figure(2);
plot(wHat,abs(Xnoisef));
xlabel('f'); ylabel('Freq. Response
Xnoise');
```





Application – 2: Filtering in Fourier Domain

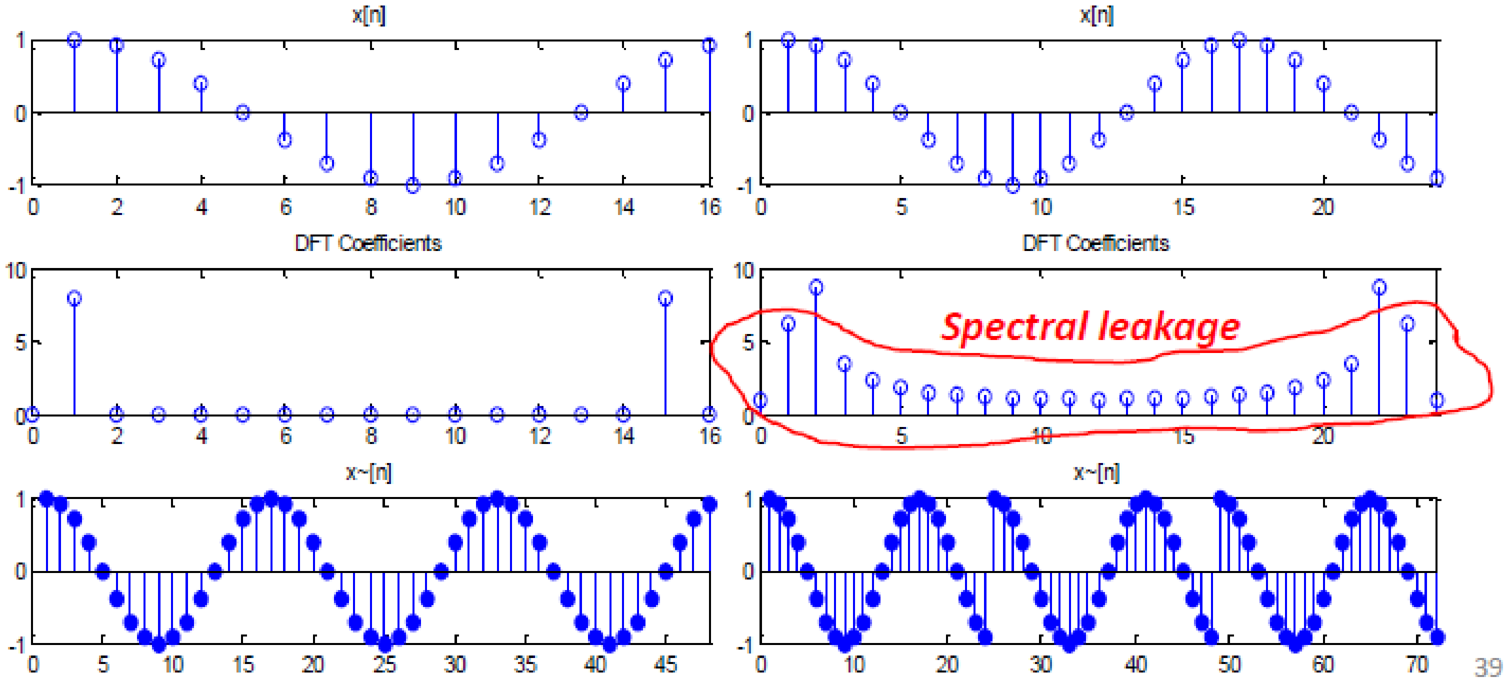
```
N = 2^14; x = x(1:N);
f = (-N/2:N/2-1) * Fs/N;

%% FFT of the signal
X = fftshift(fft(x,N));
Xnoise = awgn(x,35);
Xnoisef = fftshift(fft(Xnoise,N));
figure(1);
plot(f,abs(Xnoisef));
xlabel('f'); ylabel('Freq. Response Xnoise');
sound(real(Xnoise));

%% Generate any signal in FFT domain
H = zeros(size(X));
H (4097:12279)=1;
figure(2); plot(f,abs(Xnoisef.*H)); xlabel('f'); ylabel('Freq. Response Result');
y = ifft(ifftshift(Xnoisef.*H));
sound(real(y));
```

Convolution	$\sum_{m=0}^{N-1} h[m]x[((n-m))_N]$	$H[k]X[k]$

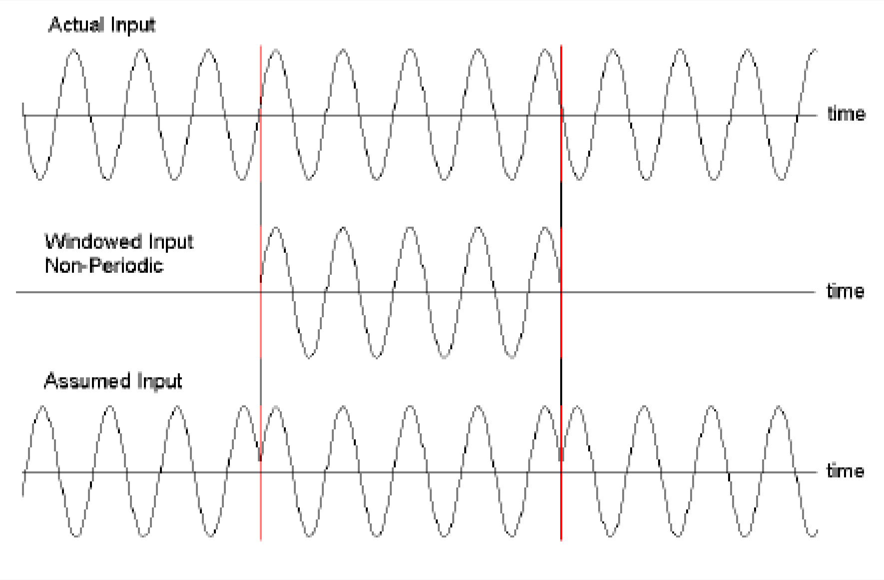
Convolution in time = Multiplication in freq.

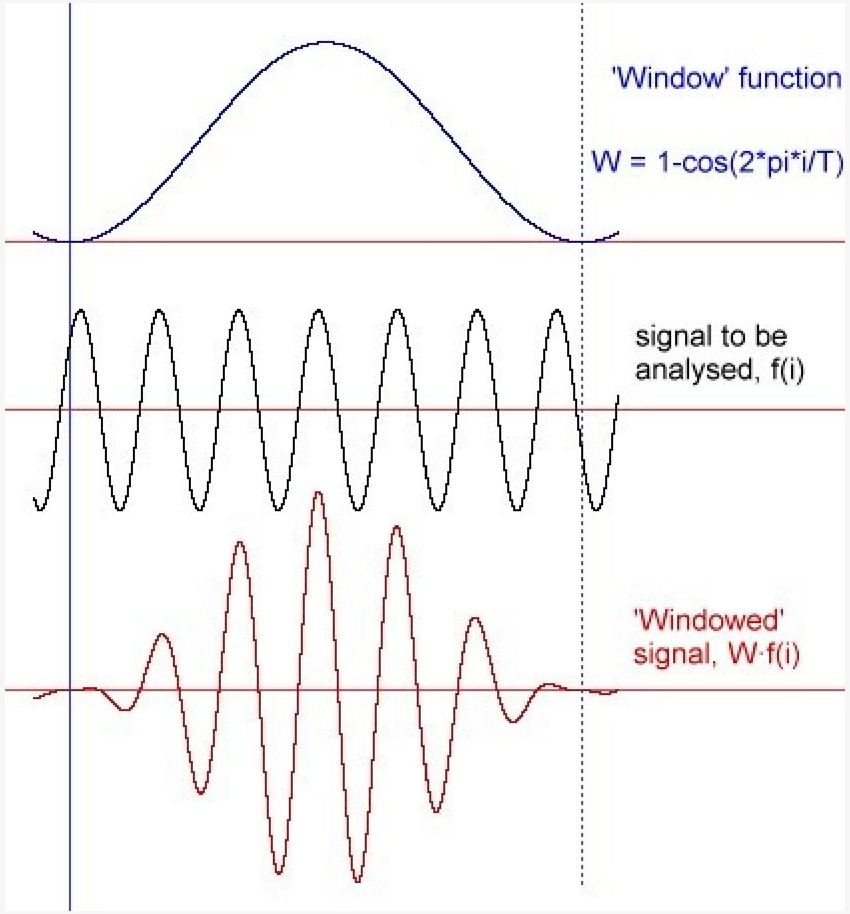
Windowing

- *Periodicity* causes unwanted spectral effects in frequency domain
- This issue is called as ***spectral leakage.***



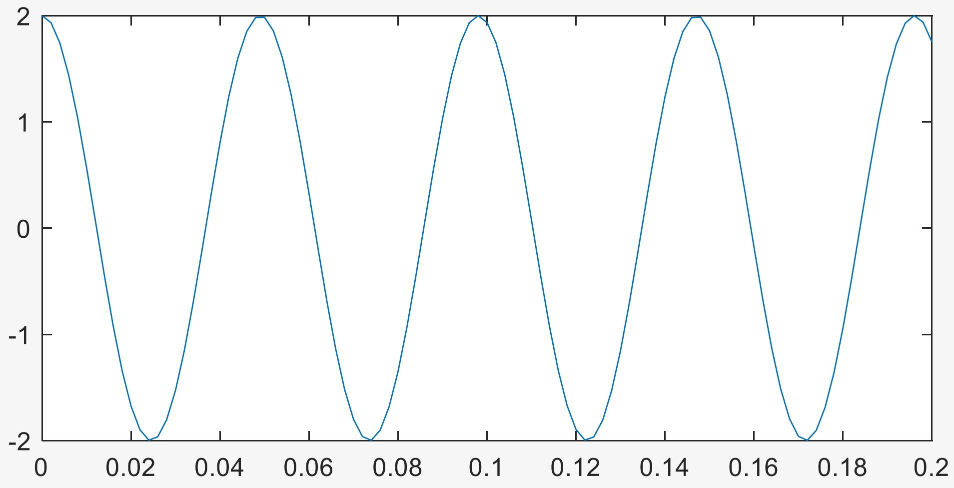
Windowing

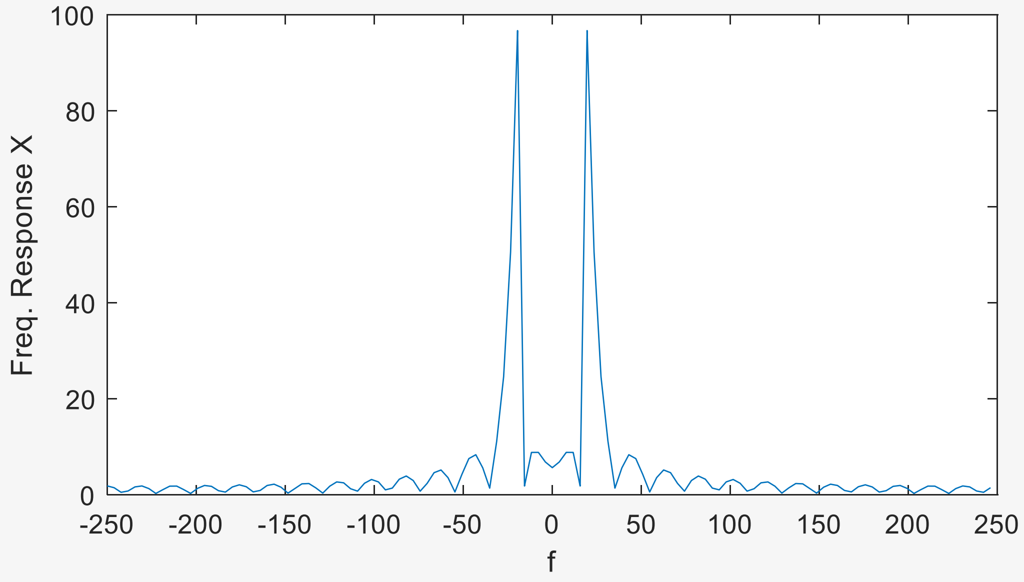

Actual Input
time
Windowed Input
Non-Periodic
time
Assumed Input
time


'Window' function
W = 1-cos(2*pi*i/T)
signal to be
analysed, f(i)
'Windowed'
signal, W·f(i)

Example

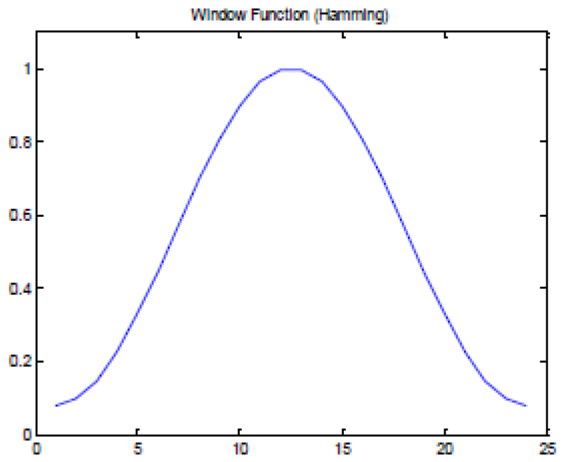
```
clc; clear all;

%% Load Sound
Fs = 500;
n = 0:1/Fs:0.2;
x = 2*cos(40.8*pi*n);
plot(n,x);
N = 128;

%% FFT of the signal
X = fftshift(fft(x,N));
f = (-N/2:N/2-1) * Fs/N;
plot(f,abs(X)); xlabel('f'); ylabel('Freq.
Response X');
```





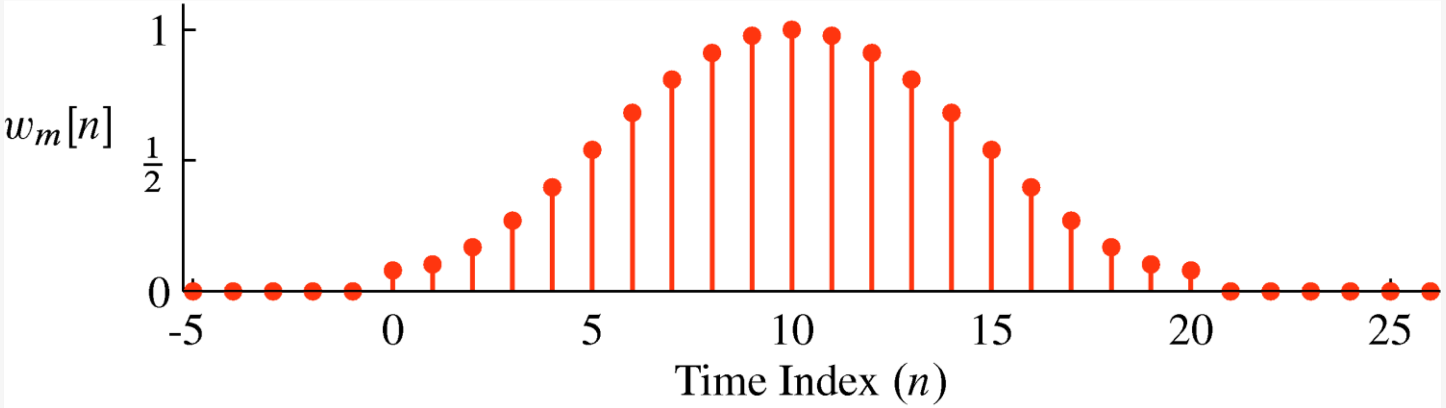
Windowing

- Windowing is utilized to overcome this effect.
- Well known window functions:
  - Triangular
  - Trapezoid
  - Hamming
  - Hanning
  - Blackman
  - Parzen
  - Welch
  - Nuttall
  - Kaiser



Window Filter Design

- Plot of Length-21 Hamming window

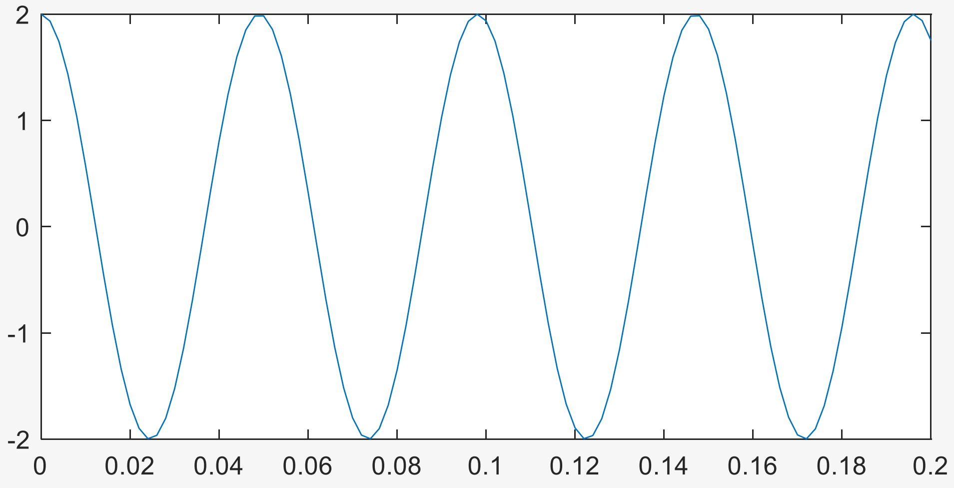


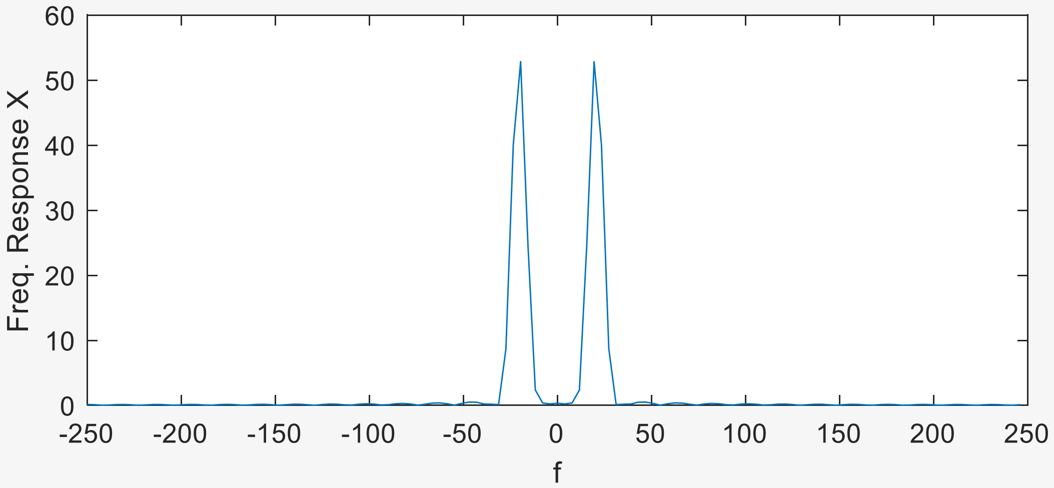
Hamming Window

$$w_m[n] = \begin{cases} 0 & n < 0 \\ 0.54 - 0.46\cos(2\pi(n)/(L-1)) & 0 \le n < L \\ 0 & n \ge L \end{cases}$$

Example

```
clc; clear all;

%% Load Sound
Fs = 500;
n = 0:1/Fs:0.2;
x = 2*cos(40.8*pi*n);
plot(n,x);
N = 128;

%% FFT of the signal
w = hamming(101)';
x = x.*w;
X = fftshift(fft(x,N));
f = (-N/2:N/2-1) * Fs/N;
plot(f,abs(X)); xlabel('f'); ylabel('Freq. Response X');
```





Homework: Spectrogram (aka Short-Time FT)

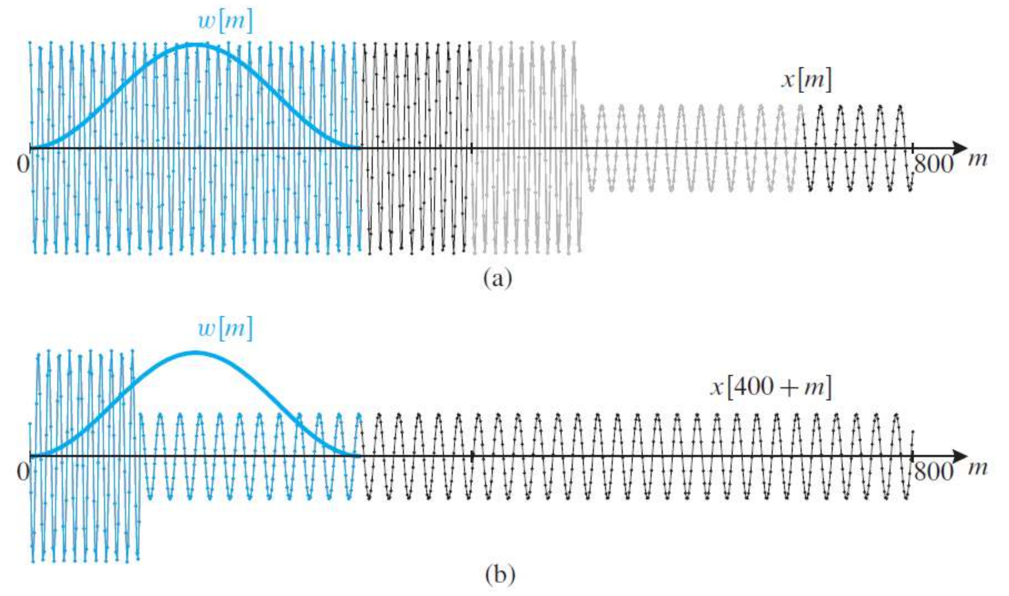
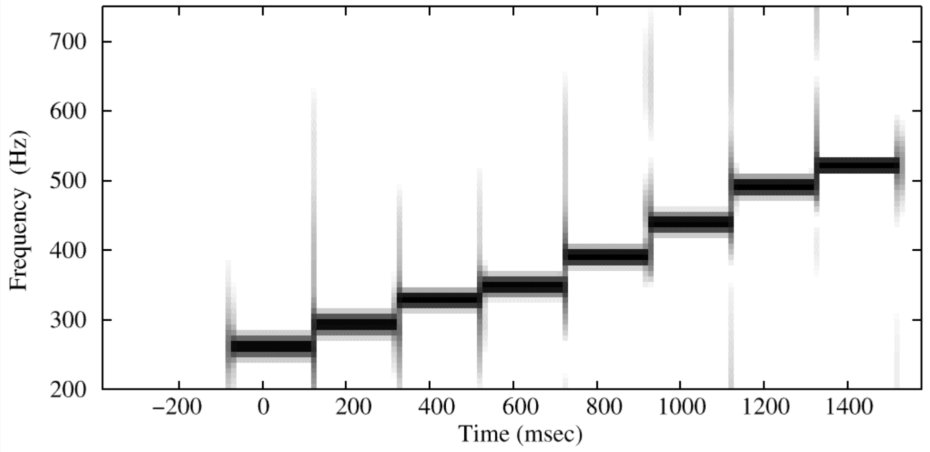


Figure 8-19 Time-dependent Hann windowing of a signal using a length-301 window. (a) The signal $x[0+m] = x[m]$ and fixed window $w[m]$. (b) Fixed window $w[m]$ with signal shifted by 400 samples (i.e., $x[400+m]$).



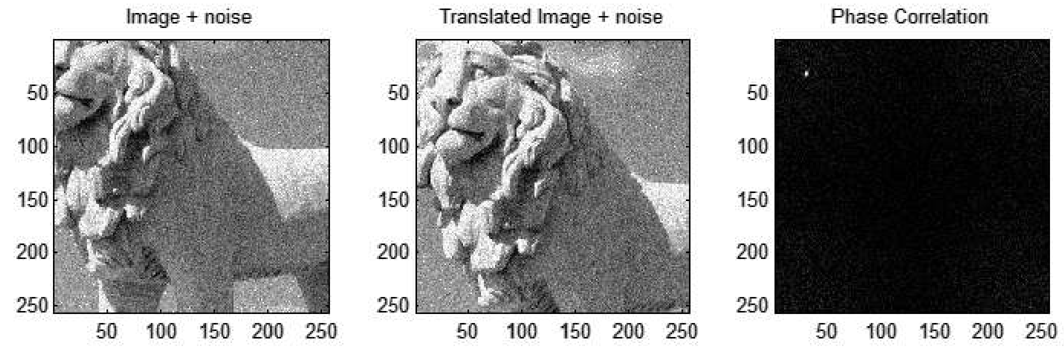
Write a code that generates spectrogram of a short song:

1- User can select the window length (*L*),

2- You must indicate the Frequency and Time on figure,

3- You must analyze the effect of window length.

Application-3: Video Stabilization



Given two input images g_a and g_b:

Apply a window function (e.g., a Hamming window) on both images to reduce edge effects (this may be optional depending on the image characteristics). Then, calculate the discrete 2D Fourier transform of both images.

$$\mathbf{G}_a = \mathcal{F}\{g_a\}, \ \mathbf{G}_b = \mathcal{F}\{g_b\}$$

Calculate the cross-power spectrum by taking the complex conjugate of the second result, multiplying the Fourier transforms together elementwise, and normalizing this product elementwise.

$$R = \frac{\mathbf{G}_a \circ \mathbf{G}_b^*}{|\mathbf{G}_a \circ \mathbf{G}_b^*|}$$

Where $\circ$ is the Hadamard product (entry-wise product) and the absolute values are taken entry-wise as well. Written out entry-wise for element index (j,k):

$$R_{jk} = \frac{G_{a,jk} \cdot G_{b,jk}^*}{|G_{a,jk} \cdot G_{b,jk}^*|}$$

Obtain the normalized cross-correlation by applying the inverse Fourier transform.

$$r = \mathcal{F}^{-1}\{R\}$$

Determine the location of the peak in r.

$$(\Delta x, \Delta y) = \arg\max_{(x,y)}\{r\}$$

Image Registeration using Phase Correlation

1- Select two image

2- Apply windowing to each image

3- Compute FFTs of Im1 and Im2

4- Obtain Phase Correlation Matrix

5- Find the Location of Max. Point

6- Stabilize the selected image Im2

```
%% PC
Ga = fft2(win.*I1);
Gb = fft2(win.*I2);
R = ifft2(Ga.*conj(Gb)./abs(Ga.*conj(Gb)));
```

$$R_{jk} = \frac{G_{a,jk} \cdot G^*_{b,jk}}{|G_{a,jk} \cdot G^*_{b,jk}|}$$

Obtain the normalized cross-correlation by applying the inverse Fourier transform.

$$r = \mathcal{F}^{-1}\{R\}$$

First Try in a Shifted Image

```
clc; clear all;
I1 = double(imread ('cameraman.tif'));
I2 = I1(5:end,6:end);
I1 = I1(1:end-4,1:end-5);

figure(1); imshow(I1,[]);
figure(2); imshow(I2,[]);
%% FFT
win =
hamming(size(I1,1))*hamming(size(I1,2))';
Ga = fft2(win.*I1);
Gb = fft2(win.*I2);
R = ifft2(Ga.*conj(Gb)./abs(Ga.*conj(Gb)));
[dX, dY] = find(R==max(max(R)));

%% Crop Im.
I2 = cropIm(I2,dX,dY);
figure(3); imshow(I2,[]);
```

I1

I2

I2Corr

Video Stabilization

1- Select two sequential image

2- Apply windowing to each image

3- Compute FFTs of Im1 and Im2

4- Obtain Phase Correlation Matrix

5- Find the Location of Max. Point

6- Stabilize the selected image Im2

```
%% Read video frames
k=1;
v = VideoReader('18AF.avi');
vidHeight = v.Height;
vidWidth = v.Width;
s = struct('cdata',zeros(vidHeight,vidWidth,'uint8'),
'colormap',[]);

while hasFrame(v)
    s(k).cdata = readFrame(v);
    k = k+1;
end
```

```
I2rgb = s(k).cdata;
```