



BLM3620 Digital Signal Processing

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Yıldız Technical University – Computer Engineering

Lecture #14 – Filter Design Basics

- Relation of z-Transform and DTFT
- Examples
- Filter Design Using z-Plane
- MATLAB Example

Important Materials:

- James H. McClellan, R. W. Schafer, M. A. Yoder, *DSP First Second Edition*, Pearson, 2015.
- Lizhe Tan, Jean Jiang, *Digital Signal Processing: Fundamentals and Applications*, Third Edition, Academic Press, 2019.

Auxiliary Materials:

- Prof. Sarp Ertürk, *Sayısal İşaret İşleme*, Birsen Yayınevi.
- Prof. Nizamettin Aydın, DSP Lecture Notes.
- J. G. Proakis, D. K. Manolakis, *Digital Signal Processing Fourth Edition*, Pearson, 2014.
- J. K. Perin, *Digital Signal Processing, Lecture Notes*, Stanford University, 2018.

Syllabus



Week	Lectures
1	Introduction to DSP and MATLAB
2	Sinuzoids and Complex Exponentials
3	Spectrum Representation
4	Sampling and Aliasing
5	Discrete Time Signal Properties and Convolution
6	Convolution and FIR Filters
7	Frequency Response of FIR Filters
8	Midterm Exam
9	Discrete Time Fourier Transform and Properties
10	Discrete Fourier Transform and Properties
11	Fast Fourier Transform and Windowing
12	-
13	z- Transforms
14	FIR Filter Design and Applications

Viz.
ve örneği
1 soru

TF örneği

4 soru
kullan

2 dönüşümü

Fourier dönüşümü

DFT - FFT

For more details -> Bologna page: <http://www.bologna.yildiz.edu.tr/index.php?r=course/view&id=5730&aid=3>

Z-Transform EXAMPLE



- ANY SIGNAL has a z-Transform:

$$X(z) = \sum_n x[n]z^{-n}$$

→ Transfer Function

$$H(z) = \sum_n h[n]z^{-n}$$

Example 7.1

n	$n < -1$	-1	0	1	2	3	4	5	$n > 5$
$x[n]$	0	0	2	4	6	4	2	0	0

$$X(z) = ?$$

$$X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4}$$

FREQUENCY RESPONSE ?



- Same Form:

$\hat{\omega}$ – Domain

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^M b_k e^{-j\hat{\omega}k}$$
$$H(e^{j\hat{\omega}}) = \sum_{k=0}^M b_k (e^{j\hat{\omega}})^{-k}$$

$$z = e^{j\hat{\omega}}$$

z – Domain

$$H(z) = \sum_{k=0}^M b_k z^{-k}$$

SAME COEFFICIENTS

If we have single complex exponential signal....



SINUSOIDAL RESPONSE

- $x[n] = \text{SINUSOID} \Rightarrow y[n]$ is SINUSOID
- Get MAGNITUDE & PHASE from $H(z)$

if $x[n] = e^{j\hat{\omega}n}$

then $y[n] = H(e^{j\hat{\omega}})e^{j\hat{\omega}n}$

where $H(e^{j\hat{\omega}}) = H(z)|_{z=e^{j\hat{\omega}}}$

POP QUIZ

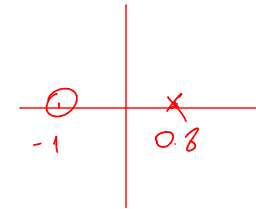
- Given:

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

Handwritten notes for $H(z)$:

$2z + 2 \rightarrow z = -1$ (zeros)
 $1 - 0.8z^{-1} \rightarrow z = 0.8$ (poles)

- Find the Impulse Response, $h[n]$
- Find the output, $y[n]$



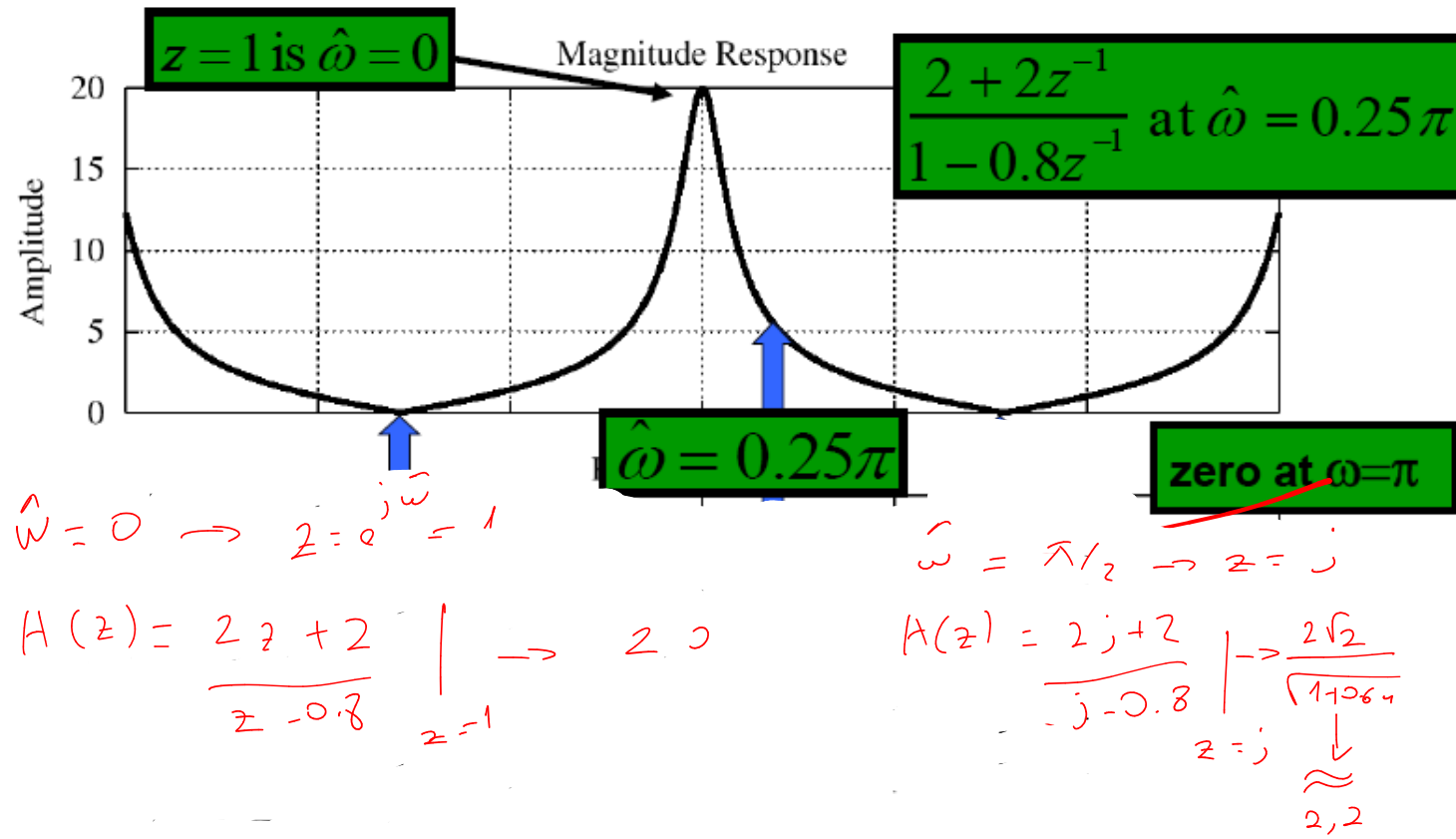
– When

$$x[n] = \cos(0.25\pi n)$$

Exercise-1



Evaluate FREQ. RESPONSE



POP QUIZ: Eval Freq. Resp.

- Given:

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

Handwritten notes:
 $\cos(\pi n)$ olmalı
 $z = e^{j\omega} = -1$

- Find output, $y[n]$, when

– Evaluate at

$$x[n] = \cos(0.25\pi n)$$

$$z = e^{j0.25\pi}$$

Handwritten note: $\rightarrow \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$

Handwritten note: $\downarrow \hat{\omega} = 0.25\pi$

$$H(z) = \frac{2 + 2(\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2})}{1 - 0.8e^{-j0.25\pi}} = 5.182e^{-j1.309}$$

$$y[n] = 5.182 \cos(0.25\pi n - 0.417\pi)$$

$$H(z) = 1 - z^{-1} \rightarrow \frac{z-1}{z} \rightarrow \begin{matrix} \text{zeros} \rightarrow 1 \\ z \rightarrow \text{pole} = 0 \end{matrix}$$

$$* \cos(\pi n) \rightarrow \hat{\omega} = \pi \quad z = -1$$

$$\frac{-2}{-1} = 2$$

$$\text{order}(0) = 0$$

$$2e^{j0} \rightarrow \cos(\pi n) \cdot 2$$

$$A \cos(\pi n) + \cos(0.5\pi n) \rightarrow \text{ikisinin ayrı ayrı hesaplanıp toplanması gerektiği}$$

$$* \cos(0.5\pi n) \rightarrow \hat{\omega} = 0.5\pi \quad z = j$$

$$\frac{j-1}{j} \rightarrow -1-j \rightarrow \sqrt{2} \cdot e^{+j\pi/4} \rightarrow \sqrt{2} \cdot \cos(0.5\pi n + \pi/4) = y[n]$$

$$\begin{matrix} j \\ \searrow \\ j-1 \end{matrix} \rightarrow j-1 \text{ in } \text{acıs'}$$

$$\text{order} \left(\frac{1}{-1} \right) \rightarrow \frac{3\pi}{4}$$

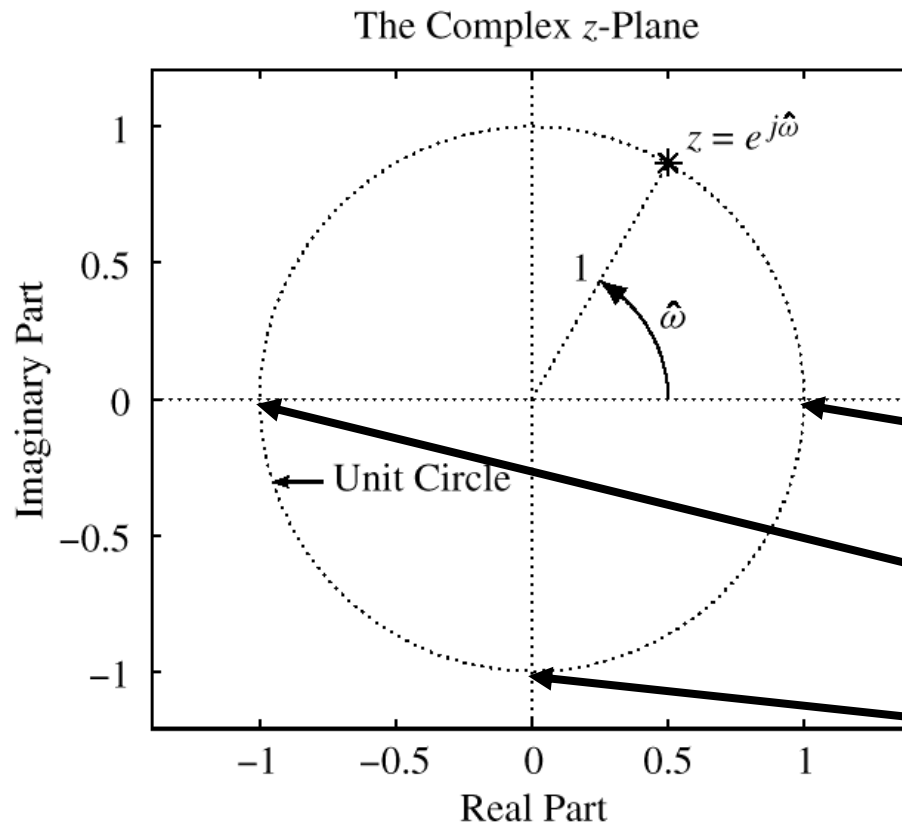
$$\text{pay} - \text{payda} \rightarrow \frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}$$

UNIT CIRCLE: RECAP



- MAPPING BETWEEN

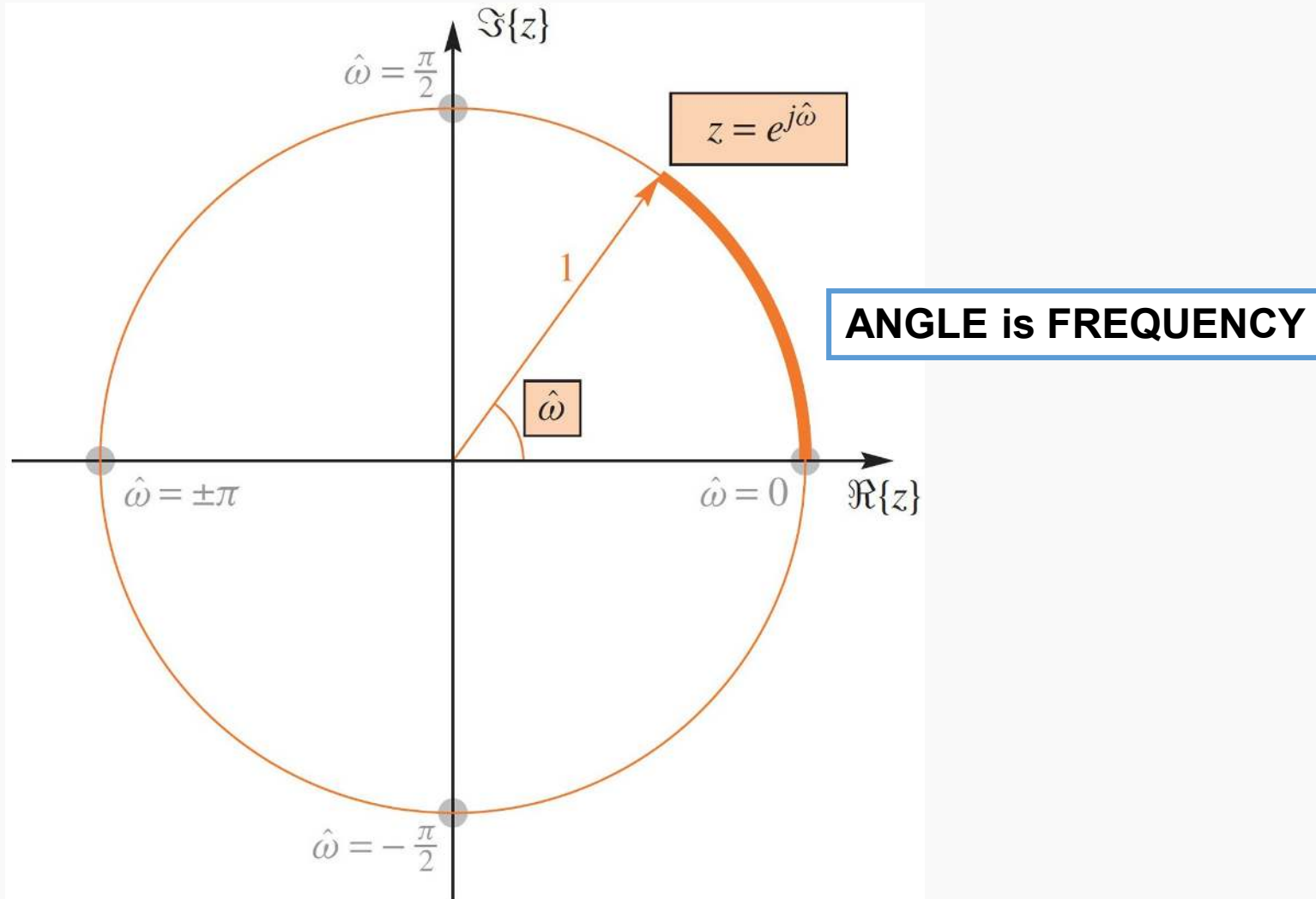
z and $\hat{\omega}$



$$z = e^{j\hat{\omega}}$$

$$\begin{aligned} z = 1 & \leftrightarrow \hat{\omega} = 0 \\ z = -1 & \leftrightarrow \hat{\omega} = \pm \pi \\ z = \pm j & \leftrightarrow \hat{\omega} = \pm \frac{1}{2} \pi \end{aligned}$$

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$



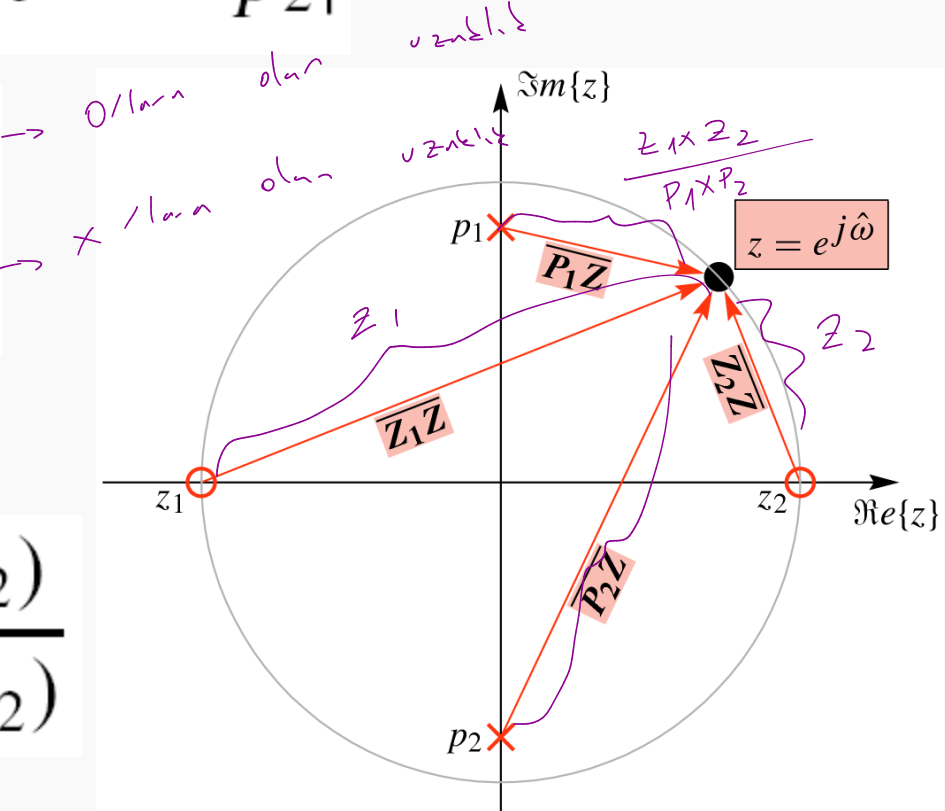
Frequency Response from poles and zeros



$$|H(e^{j\hat{\omega}})| = G \frac{|e^{j\hat{\omega}} - z_1| |e^{j\hat{\omega}} - z_2|}{|e^{j\hat{\omega}} - p_1| |e^{j\hat{\omega}} - p_2|}$$

not: Orjine konulan yollar sonucu çözülemez

$$|H(e^{j\hat{\omega}})| = G \frac{\overline{Z_1 Z} \cdot \overline{Z_2 Z}}{\overline{P_1 Z} \cdot \overline{P_2 Z}}$$



$$H(z) = G \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)}$$

ie

$$H(z) = \frac{z}{z - 0.8}$$

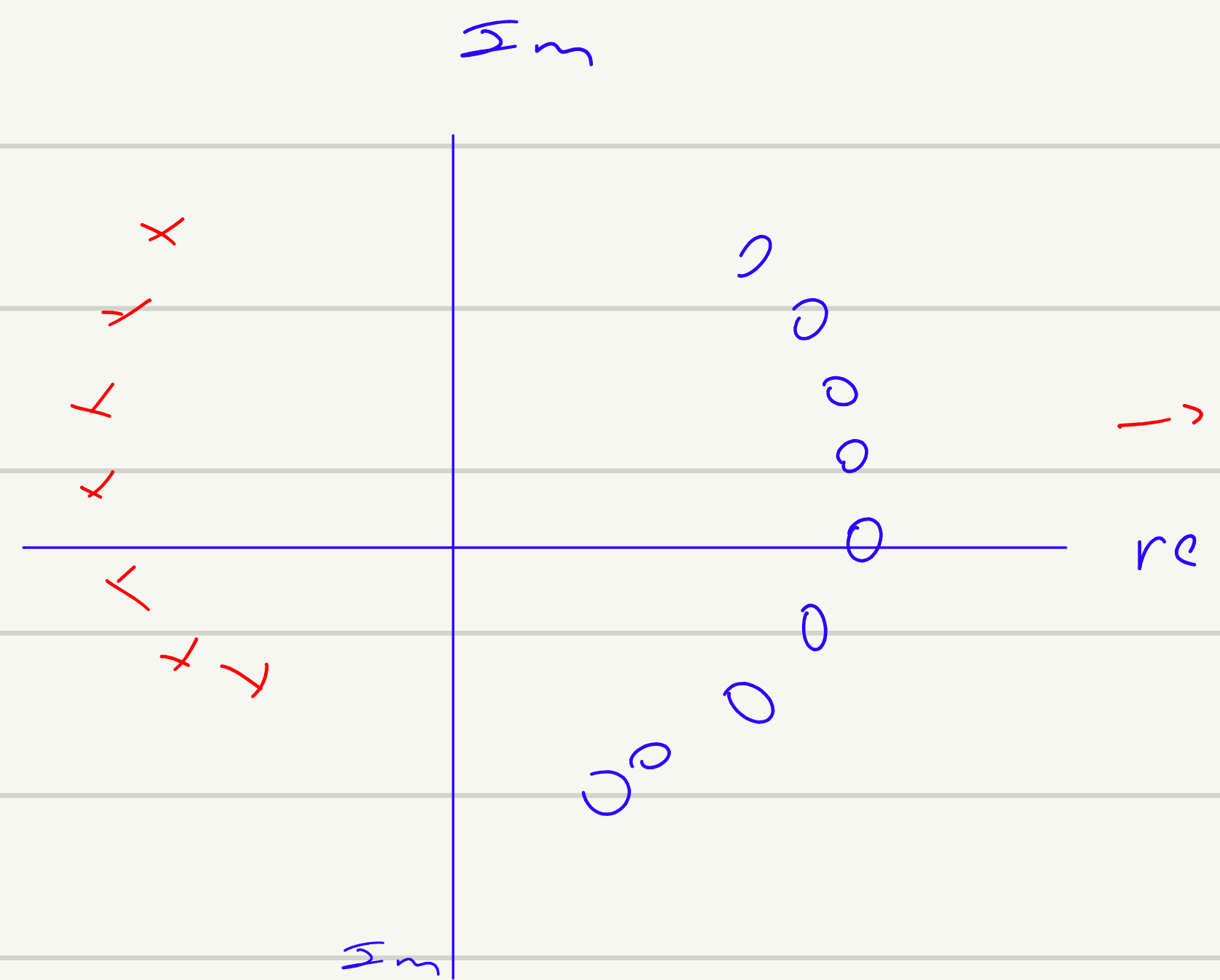
$$\hat{\omega} = \pi/2 \quad z = j$$

$$j - 0.8 \rightarrow \sqrt{1.64} \rightarrow \frac{1}{\sqrt{1.64}} = 0.78$$

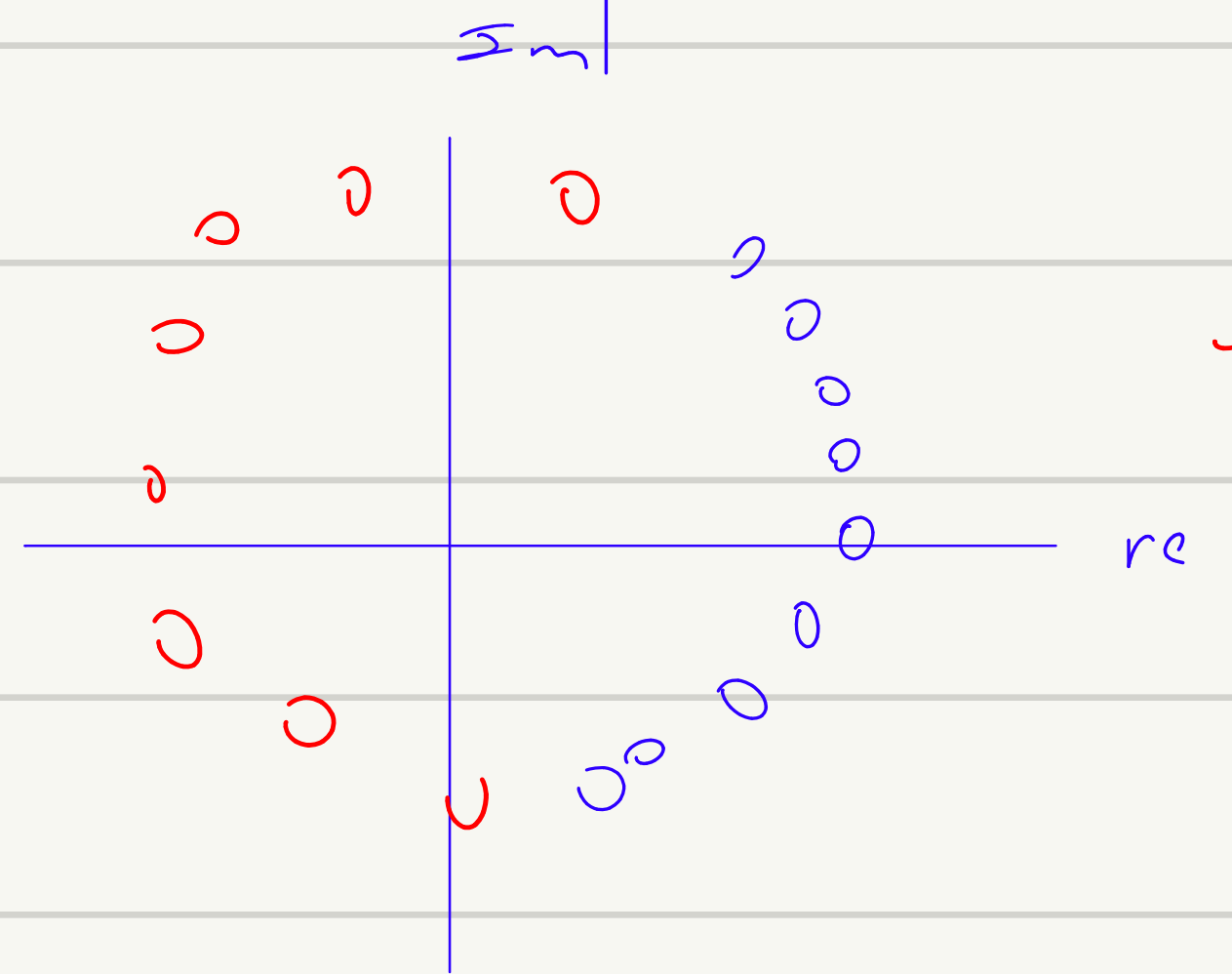
$$\hat{\omega} = \pi \quad z = -1$$

$$\frac{\sqrt{(1-1)^2}}{\sqrt{(-1-0.8)^2}} \rightarrow \frac{1}{1.8}$$

ie



→ gültigkeit: dann bestim. geçiriyor



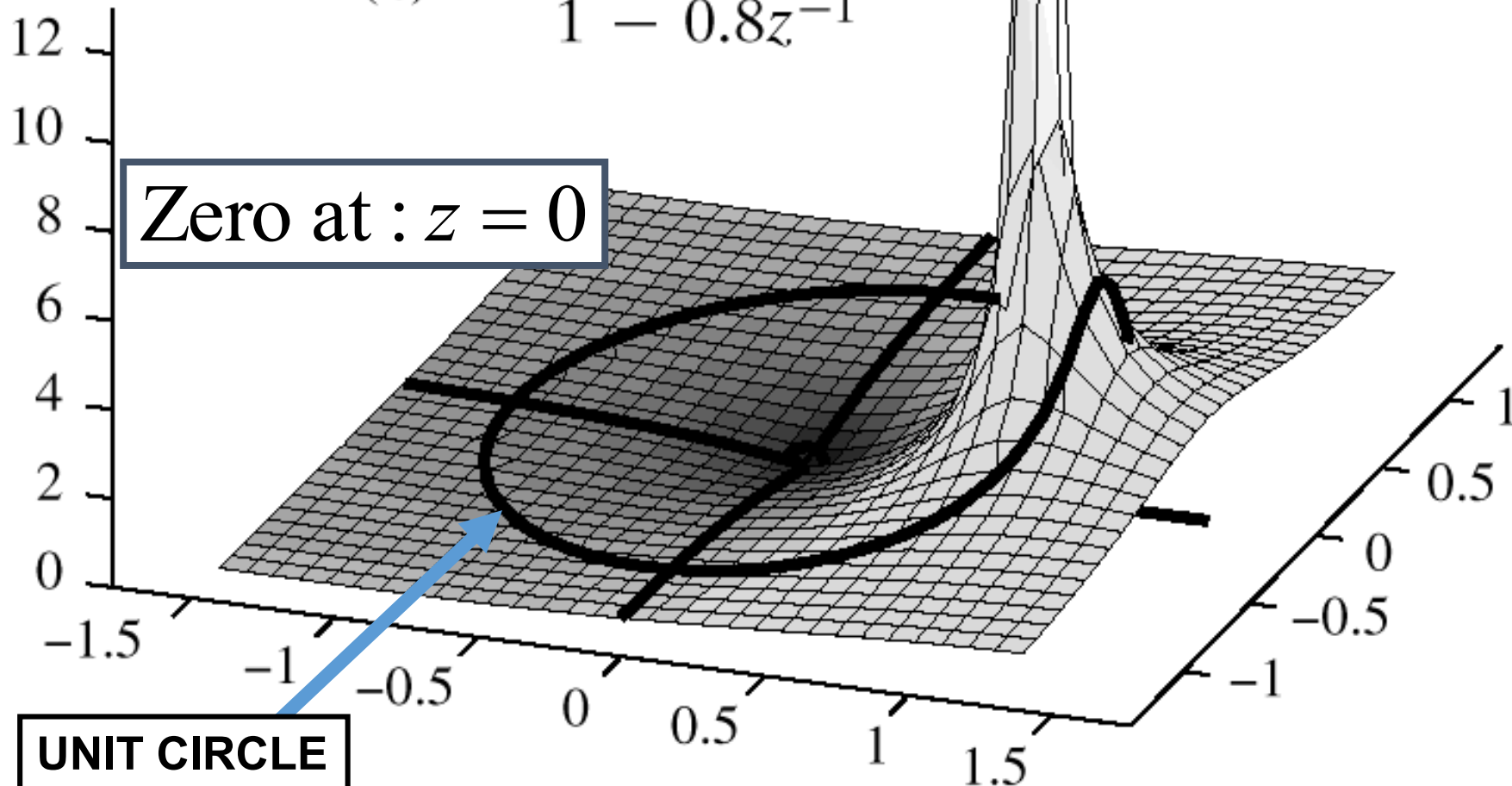
→ Bant geçiriyor

3-D VIEWPOINT: EVALUATE $H(z)$ EVERYWHERE

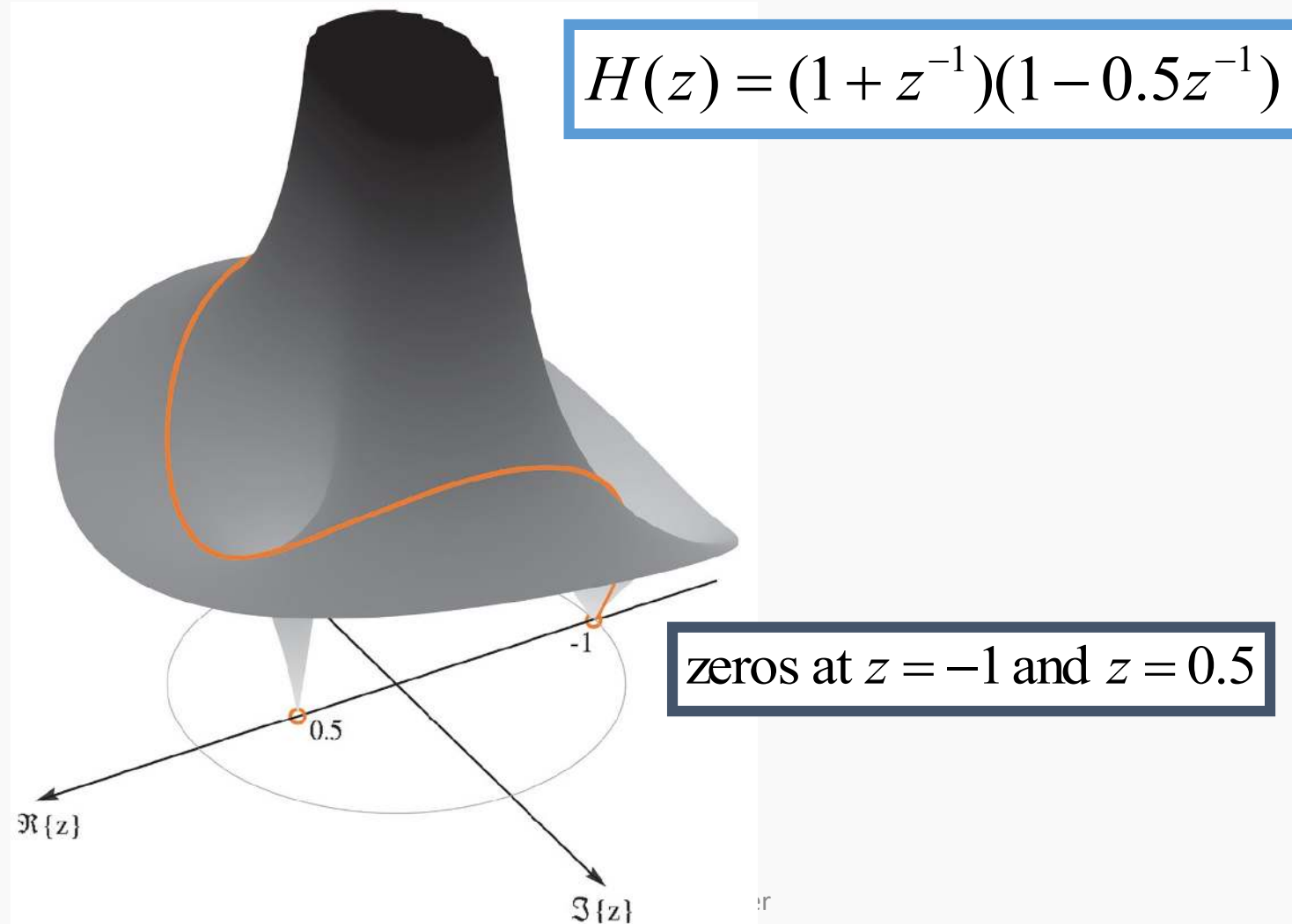
Pole at : $z = 0.8$

$$H(z) = \frac{1}{1 - 0.8z^{-1}}$$

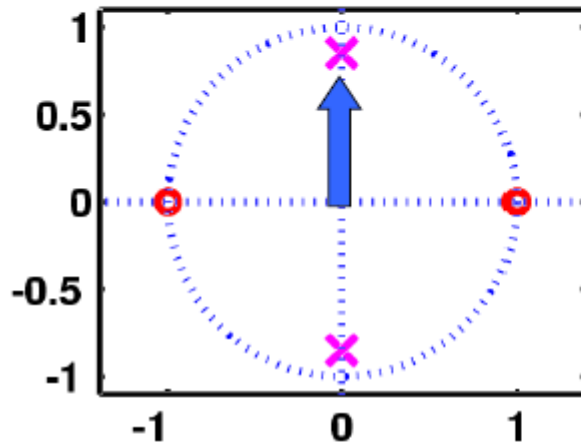
Zero at : $z = 0$



Evaluate $H(z)$ on Unit Circle

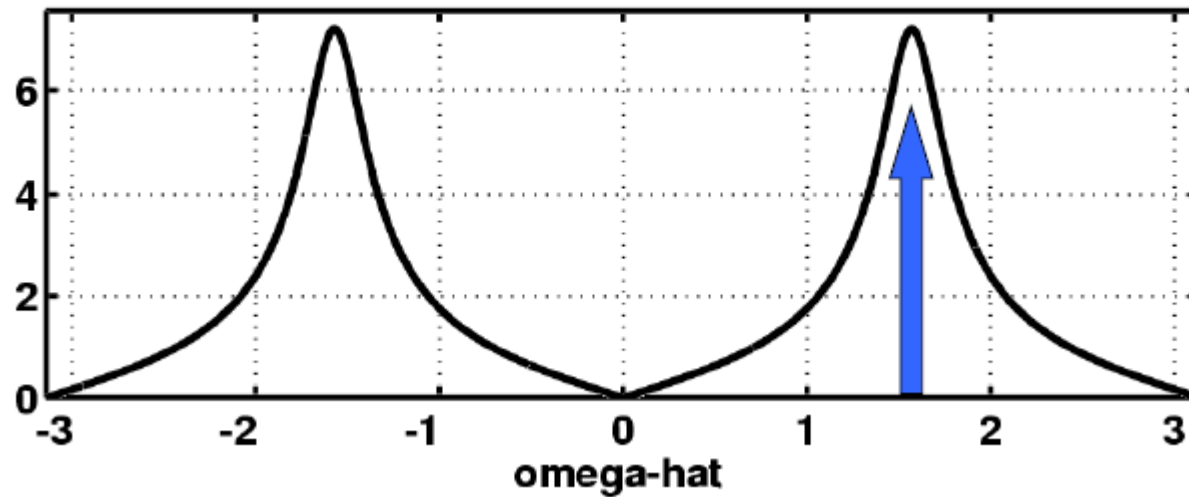


FREQUENCY RESPONSE from POLE-ZERO PLOT



$$H(e^{j\hat{\omega}}) = \frac{1 - e^{-j2\hat{\omega}}}{1 + 0.7225e^{-j2\hat{\omega}}}$$

Magnitude Response

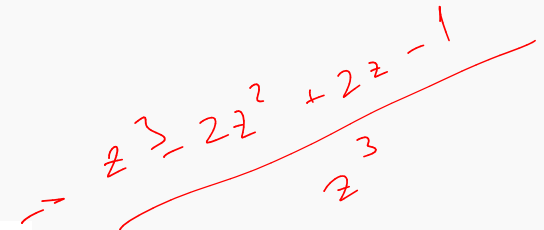


ZEROS of $H(z)$ – example 2



- Find z , where $H(z)=0$
 - Interesting when z is ON the unit circle.

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

Handwritten red text showing the derivation of the polynomial: $z^3 - 2z^2 + 2z - 1$ over z^3 .
$$\frac{z^3 - 2z^2 + 2z - 1}{z^3}$$

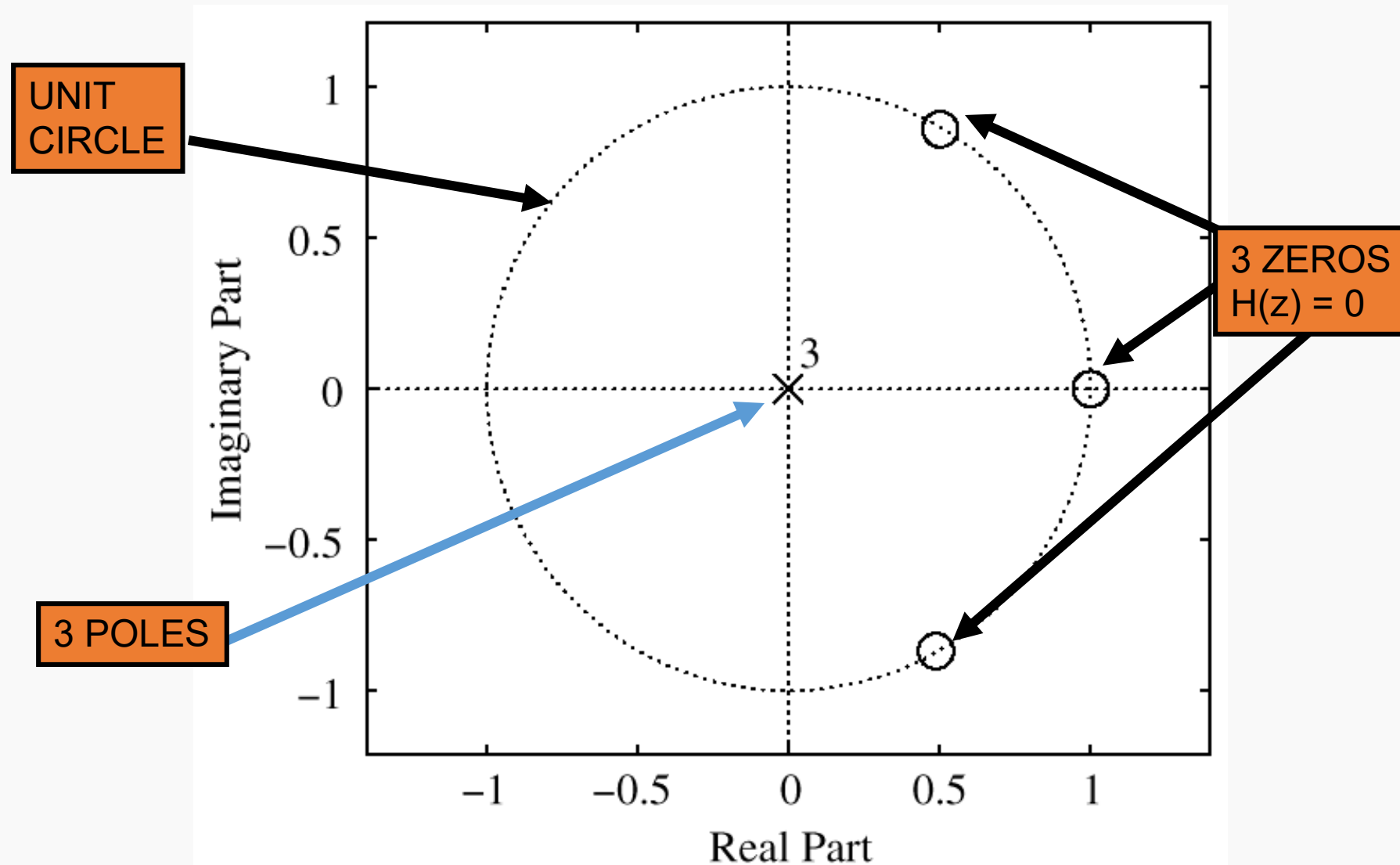
$$H(z) = (1 - z^{-1})(1 - z^{-1} + z^{-2})$$

$$\text{Roots : } z = 1, \frac{1}{2} \pm j \frac{\sqrt{3}}{2}$$

$$e^{\pm j\pi/3}$$

Recall: Roots occur in Conjugate pairs when coefficients are real

PLOT ZEROS in z-DOMAIN



POLES of $H(z)$



- Find z , where
 - FIR only has poles at $z=0$

$$H(z) \rightarrow \infty$$

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(z) = \frac{z^3 - 2z^2 + 2z - 1}{z^3}$$

Three Poles at : $z = 0$

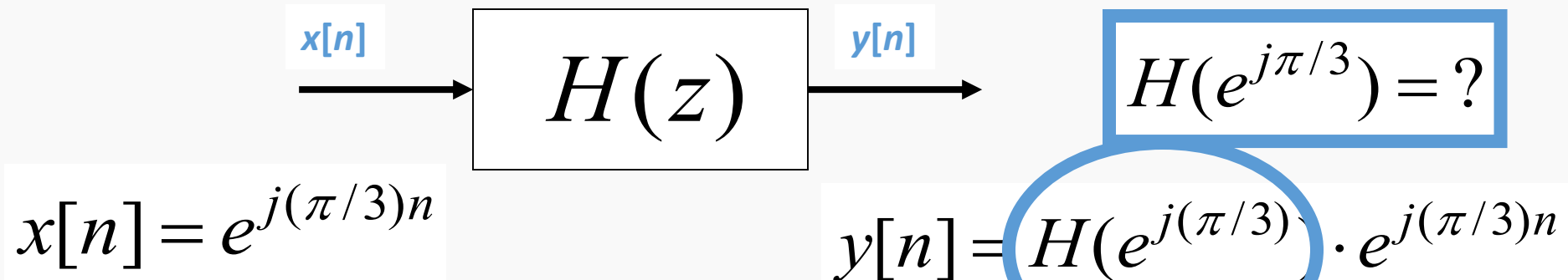
NULLING PROPERTY of $H(z)$



- When $H(z)=0$ on the unit circle.
 - Find inputs $x[n]$ that give zero output

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}}$$



NULLING PROPERTY of $H(z)$



- Evaluate $H(z)$ at the input “frequency”

$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}}$$

$$y[n] = H(e^{j\pi/3}) \cdot e^{j(\pi/3)n}$$

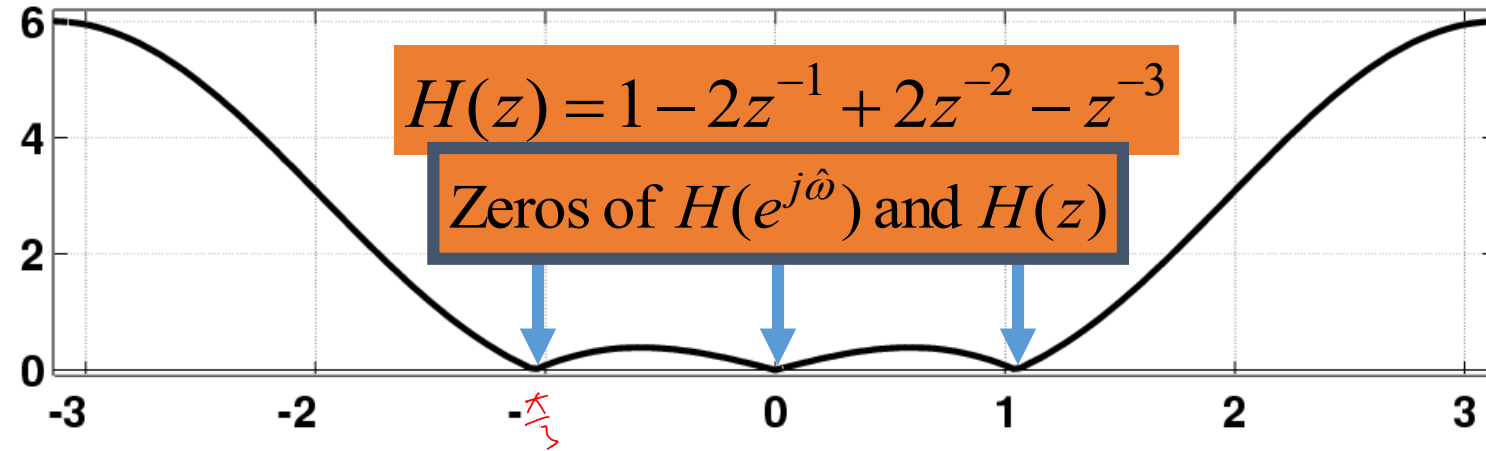
$$y[n] = (1 - 2e^{-j\pi/3} + 2e^{-j2\pi/3} - e^{-j3\pi/3}) \cdot e^{j(\pi/3)n}$$

$$(1 - 2(\frac{1}{2} - j\frac{\sqrt{3}}{2}) + 2(-\frac{1}{2} - j\frac{\sqrt{3}}{2}) - (-1))$$

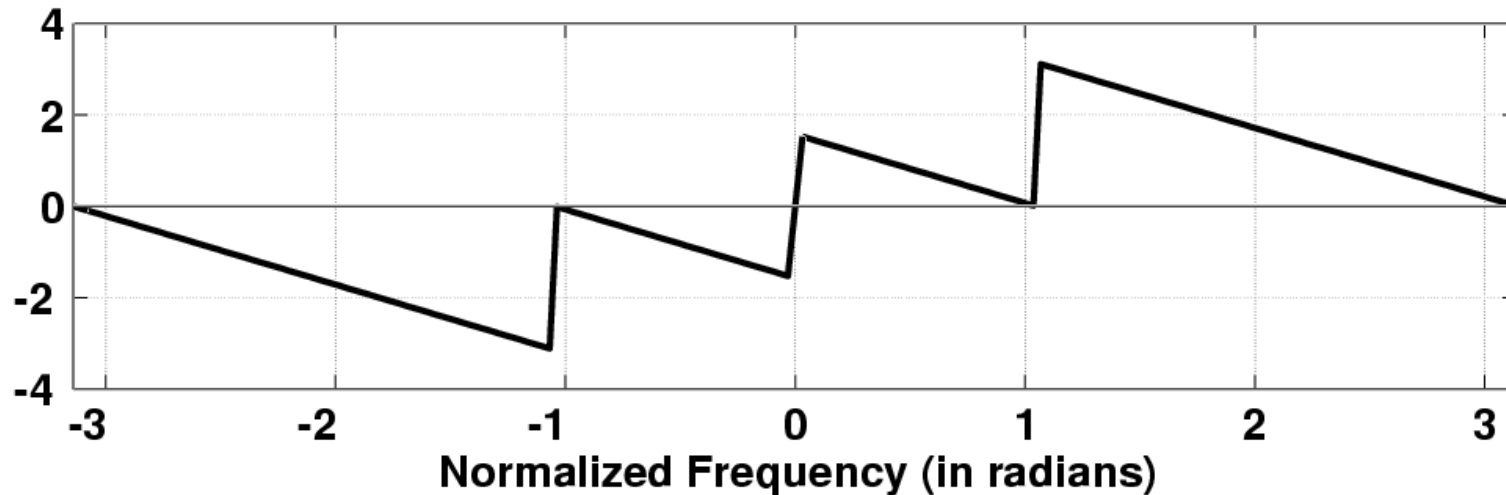
$$y[n] = (1 - 1 + j\sqrt{3} - 1 - j\sqrt{3} + 1) \cdot e^{j(\pi/3)n} = 0$$

FIR Frequency Response

Magnitude of Frequency Response for $h[n] = 1, -2, 2, -1$

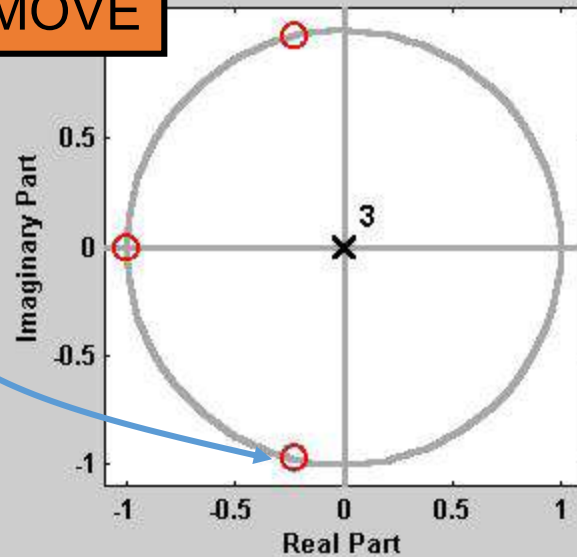


Phase Angle of Frequency Response for $h[n] = 1, -2, 2, -1$



3 DOMAINS MOVIE: FIR

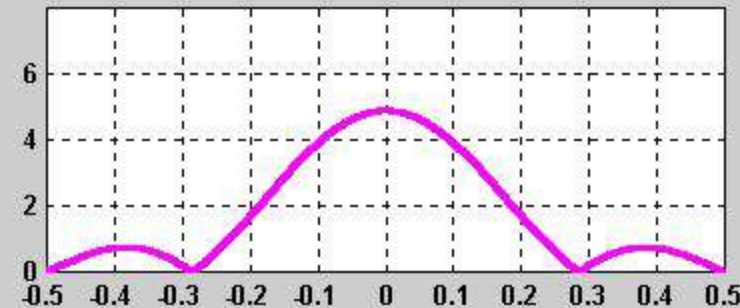
ZEROS MOVE



$$1 + 1.45z^{-1} + 1.45z^{-2} + z^{-3}$$

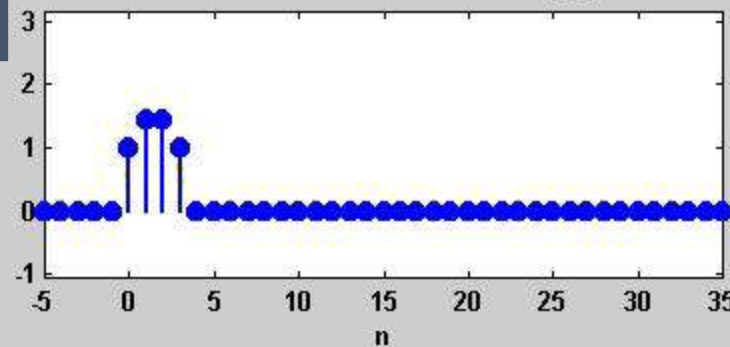
$H(z)$

DTFT: MAGNITUDE RESPONSE

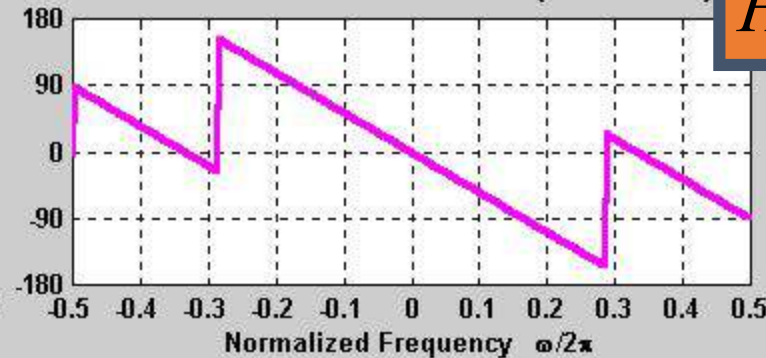


$h[n]$

IMPULSE RESPONSE: $h[n]$



DTFT: PHASE RESPONSE (DEGREES)



$H(e^{j\hat{\omega}})$

4 MOVIES @ WEBSITE

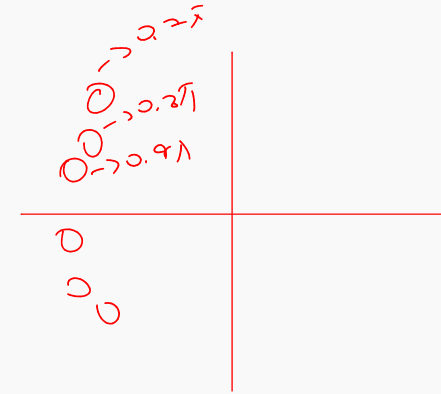


- http://dspfirst.gatech.edu/chapters/07ztrans/demos/3_domain/index.html
- 3 DOMAINS MOVIES: FIR Filters
 - Two zeros moving around UC and inside
 - Three zeros; one held fixed at $z=-1$
 - Ten zeros; 9 equally spaced around UC; one moving
 - Ten zeros; 8 equally spaced around UC; two moving

DESIGN PROBLEM



- Example:
 - Design a Lowpass FIR filter (Find b_k)
 - Reject completely 0.7π , 0.8π , and 0.9π
 - Estimate the filter length needed to accomplish this task. How many b_k ?
- Z POLYNOMIALS provide the TOOLS



NULLING FILTER DESIGN



- PLACE ZEROS to make $y[n] = 0$

Need 6 ZEROS
where $H(z) = 0$

$$H(z_k) = 0, \quad \text{for } z_k = e^{\pm j0.7\pi}, e^{\pm j0.8\pi}, e^{\pm j0.9\pi}$$

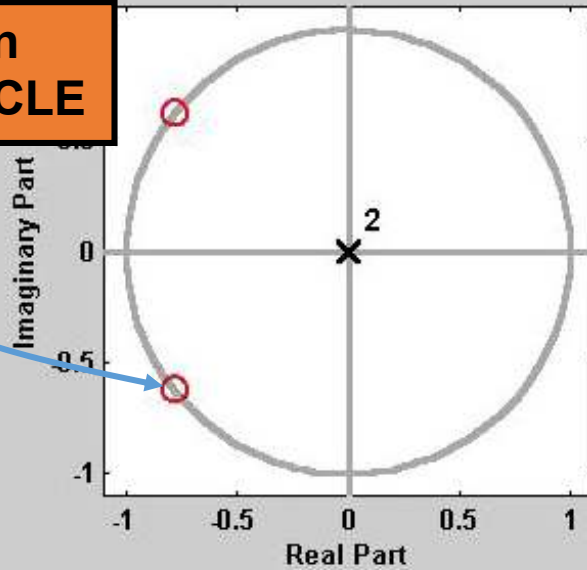
- 6th order FIR has 7 filter coefficients

$$x[n] = e^{j0.8\pi n} \Rightarrow y[n] = H(e^{j0.8\pi})e^{j0.8\pi n}$$

$$H(z) = b_0 + b_1z^{-1} + b_2z^{-2} + b_3z^{-3} + b_4z^{-4} + b_5z^{-5} + b_6z^{-6}$$

3 DOMAINS MOVIE: FIR

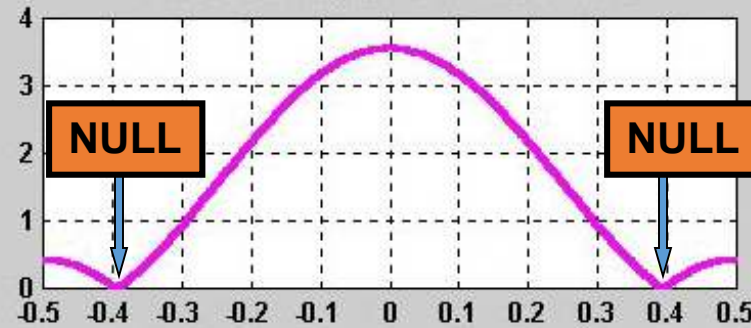
ZEROS on
UNIT-CIRCLE



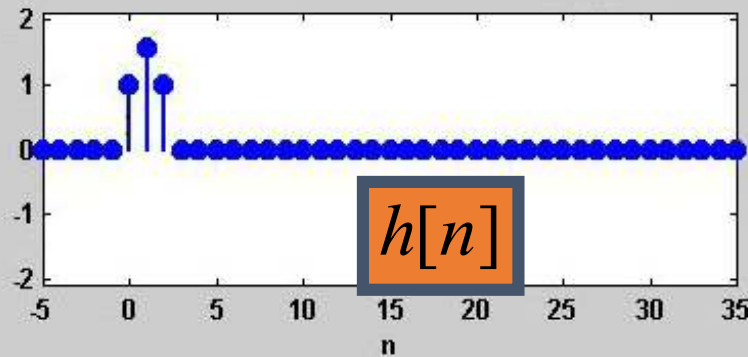
$$1 + 1.56z^{-1} + z^{-2}$$

$$H(z)$$

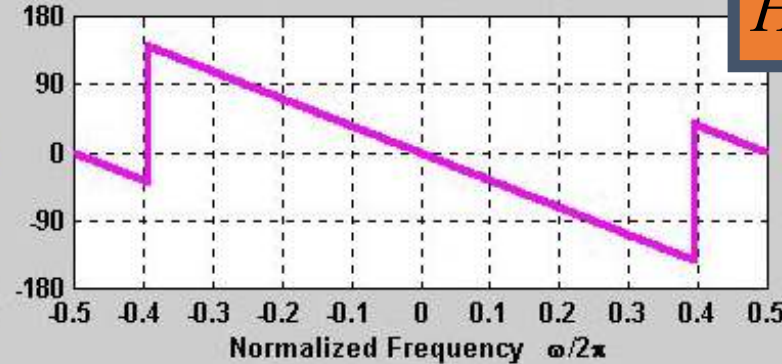
DTFT: MAGNITUDE RESPONSE



IMPULSE RESPONSE: $h[n]$

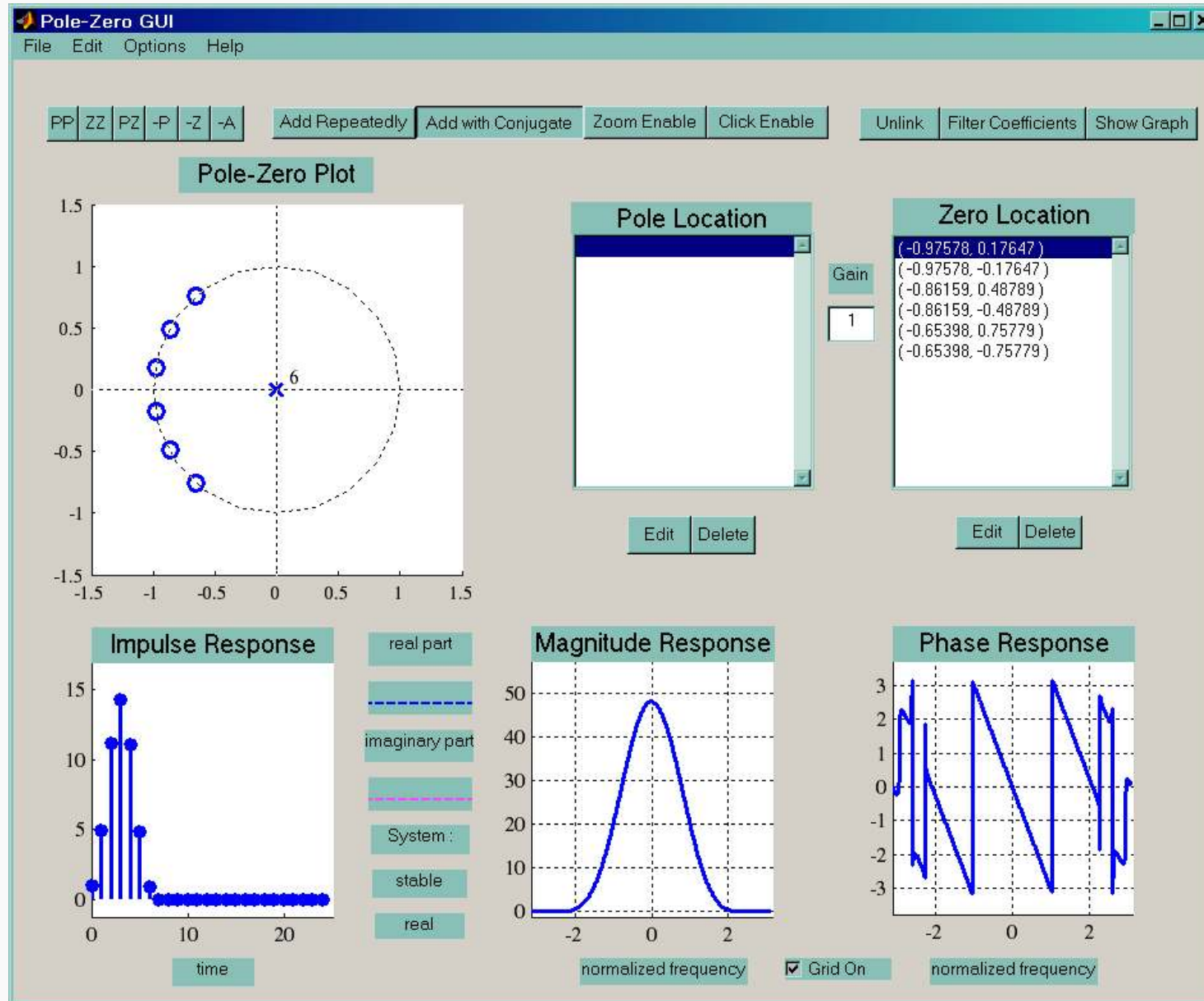


DTFT: PHASE RESPONSE (DEGREES)



$$H(e^{j\hat{\omega}})$$

PeZ Demo: Zero Placing



One zero, two zeros, ...

We usually want filters with real coefficients

$$H(z) = 1 - az^{-1} \Rightarrow H(z) = 0 \text{ @ } z = a$$

If we want to block sinusoid with $\hat{\omega} = \pm 0.8\pi$

$$\begin{aligned} H(z_k) &= 0 \quad \text{for } z_k = e^{\pm j0.8\pi} \\ \Rightarrow H(z) &= z^{-2}(z - e^{j0.8\pi})(z - e^{-j0.8\pi}) \\ &= z^{-2}(z^2 - z(e^{j0.8\pi} + e^{-j0.8\pi}) + 1) \\ &= 1 - 2(\cos 0.8\pi)z^{-1} + z^{-2} = 1 + 1.618z^{-1} + z^{-2} \end{aligned}$$

z^{-2} needed for causality

$$h[0] = 1, \quad h[1] = 1.618, \quad h[2] = 1$$

Block Multiple Frequencies



Want to totally block: $\hat{\omega}_1, \hat{\omega}_2, \dots, \hat{\omega}_m$

$H(z)$ must have zeros at: $z = e^{\pm j\hat{\omega}_1}, e^{\pm j\hat{\omega}_2}, \dots, e^{\pm j\hat{\omega}_m}$

To block $\hat{\omega} = 0$ or π must have zero at $z = 1$ or -1

So, the general form becomes:

$$H(z) = (1 - z^{-1})(1 + z^{-1}) \prod_{n=1}^m (1 - e^{j\hat{\omega}_n} z^{-1})(1 - e^{-j\hat{\omega}_n} z^{-1})$$

to block DC  to block $f_s/2$ 

On the other hand: Not much control over other frequencies

L-pt RUNNING SUM $H(z)$

$$H(z) = \sum_{k=0}^{L-1} z^{-k} = \frac{1 - z^{-L}}{1 - z^{-1}} = \frac{z^L - 1}{z^{L-1}(z - 1)}$$

$$z^L - 1 = 0 \Rightarrow z^L = 1 = e^{j2\pi k}$$

$$z = e^{j(2\pi/L)k} \quad \text{for } k = 1, 2, \dots, L-1$$

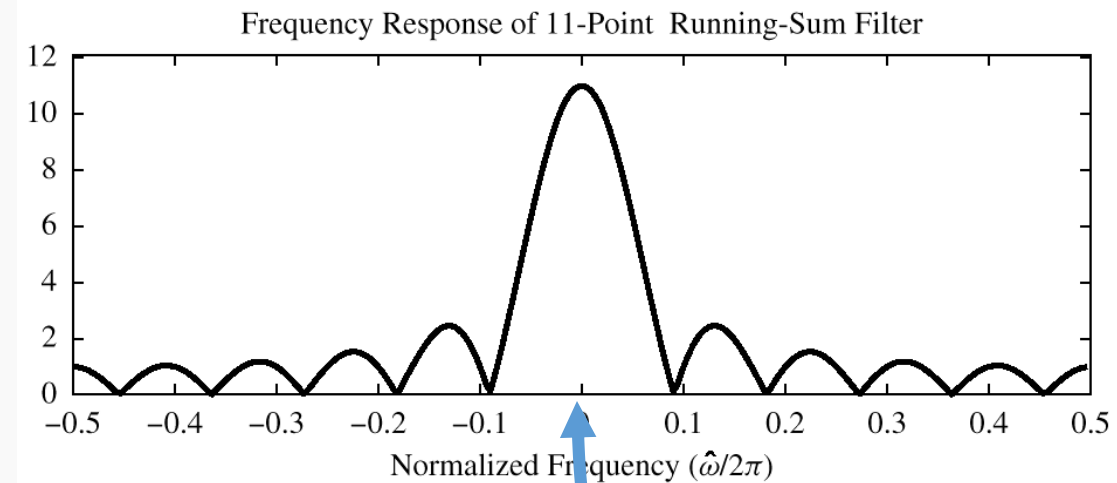
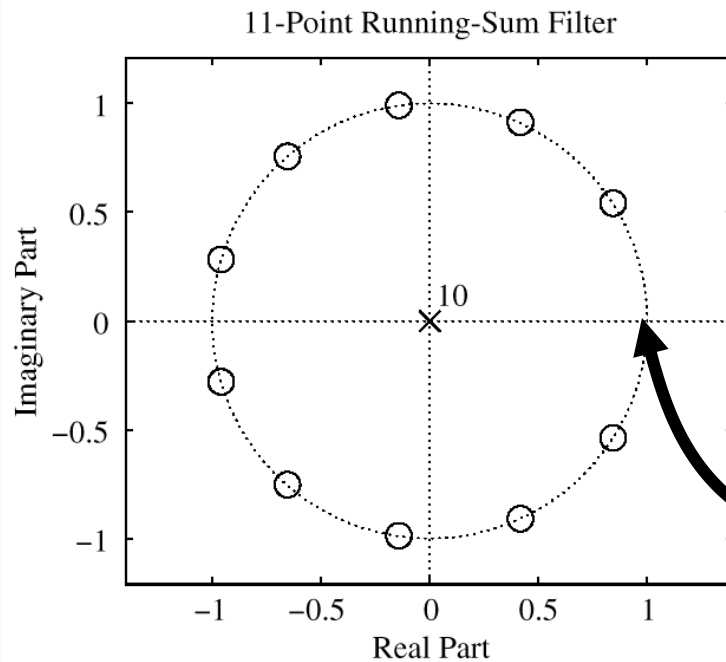
ZEROS on
UNIT CIRCLE

(z-1) in
denominator
cancels k=0 term

11-pt RUNNING SUM $H(z)$

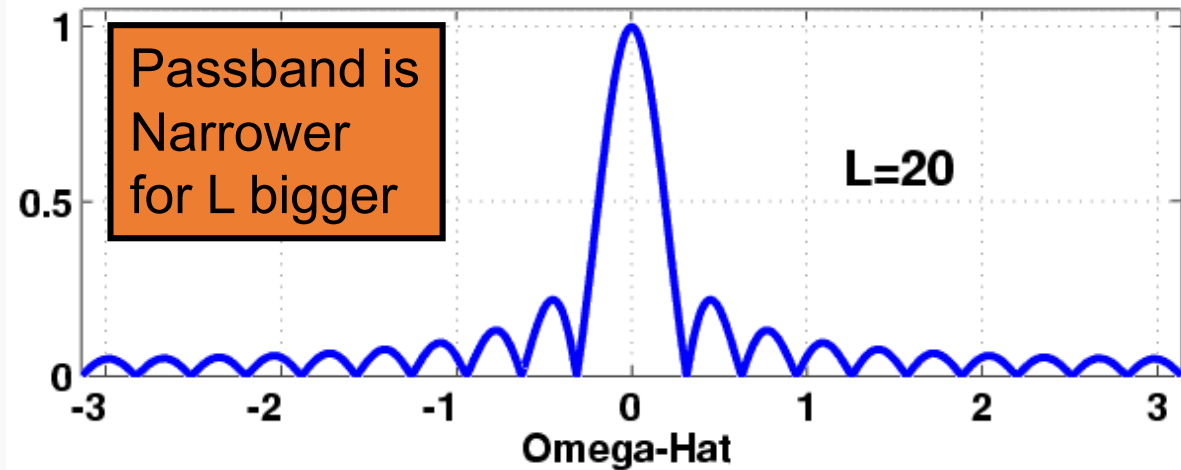
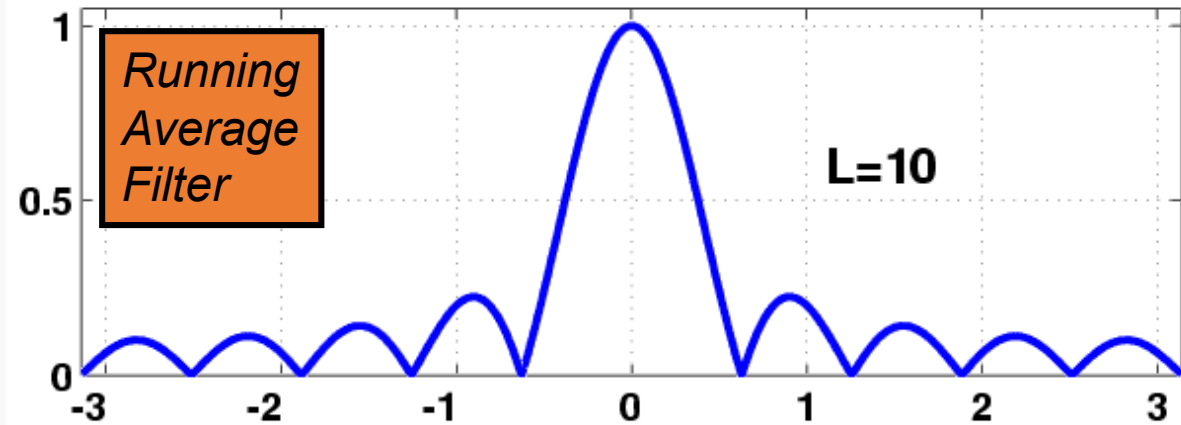
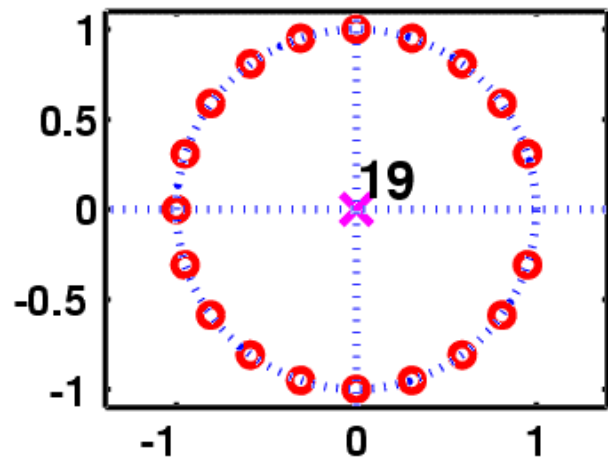
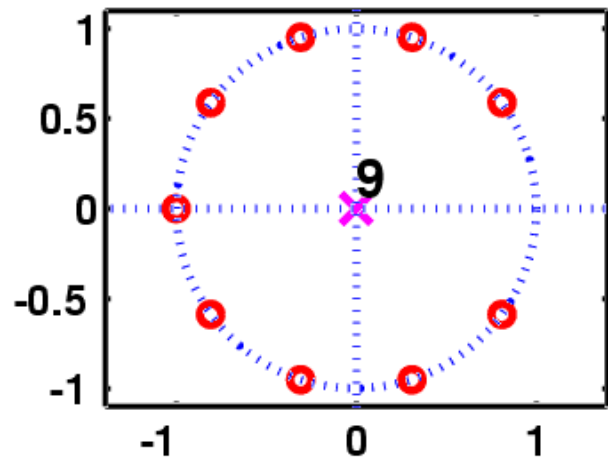
$$H(z) = \sum_{k=0}^{10} z^{-k}$$

$$H(z) = (1 - e^{j2\pi/11}z^{-1})(1 - e^{j4\pi/11}z^{-1}) \cdots (1 - e^{j20\pi/11}z^{-1})$$



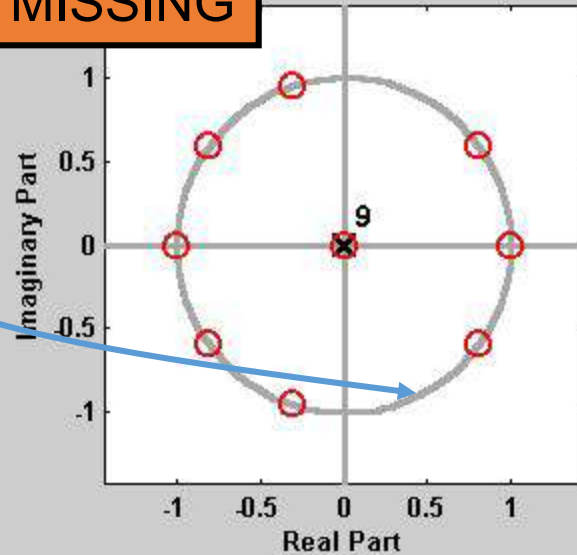
NO zero at $z=1$

FILTER DESIGN: CHANGE L



3 DOMAINS MOVIE: FIR BPF

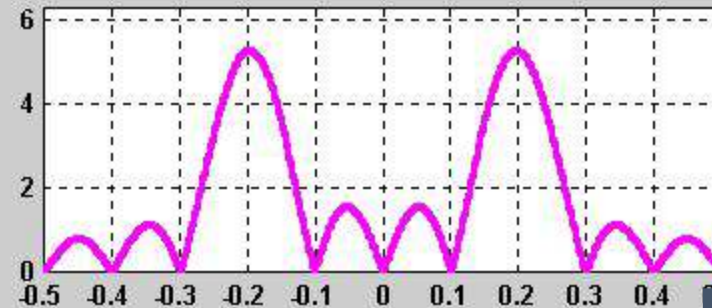
ZEROS MISSING



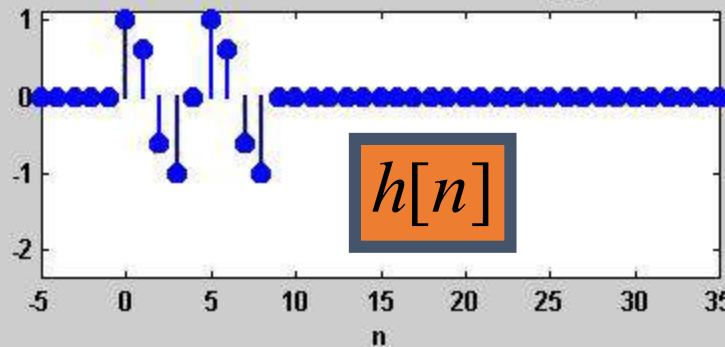
$$1 + 0.618z^{-1} - 0.618z^{-2} - z^{-3} + z^{-5} + 0.618z^{-6} - 0.618z^{-7} - z^{-8}$$

$H(z)$

DTFT: MAGNITUDE RESPONSE

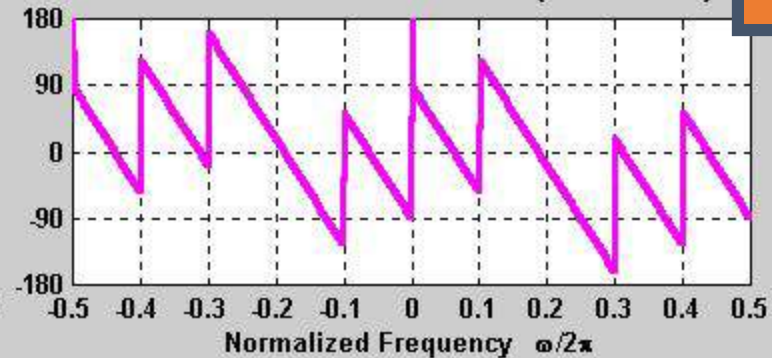


IMPULSE RESPONSE: $h[n]$



$h[n]$

DTFT: PHASE RESPONSE (DEGREES)



$H(e^{j\hat{\omega}})$

Check Website & MATLAB Code



https://dspfirst.gatech.edu/chapters/07ztrans/demos/3_domain/index.html

```
%% load piano sounds
load labtest.mat;
sound(xx,fs);
figure(1); spectrogram(xx);

%% If I want to pass  $w = 0.2\pi$ 
load filter2.mat;
yy = filter( z_values, p_values, xx);
[h,w] = freqz( z_values, p_values, 'whole',120);
plot((w)/pi,abs(h));

sound(yy,fs);
figure(2); spectrogram(yy);
```