BLM1612 - Circuit Theory

The Instructors:

Dr. Öğretim Üyesi Erkan Uslu
euslu@yildiz.edu.tr
Dr. Öğretim Üyesi Hamza Osman İlhan
hoilhan@yildiz.edu.tr

RC and RL Circuits

First Order Circuits

Objectives of Lecture

- Explain the operation of a RC circuit in dc circuits
  – As the capacitor stores energy when voltage is first applied to the circuit or the voltage applied across the capacitor is increased during the circuit operation.
  – As the capacitor releases energy when voltage is removed from the circuit or the voltage applied across the capacitor is decreased during the circuit operation.
- Explain the operation of a RL circuit in dc circuit
  – As the inductor stores energy when current begins to flow in the circuit or the current flowing through the inductor is increased during the circuit operation.
  – As the inductor releases energy when current stops flowing in the circuit or the current flowing through the inductor is decreased during the circuit operation.

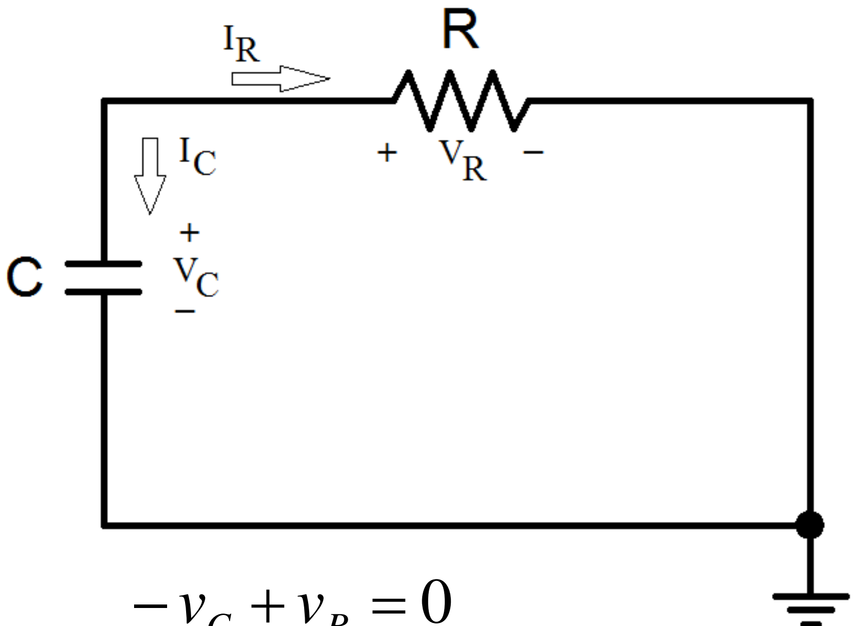
Natural Response

- The behavior of the circuit with no external sources of excitation.
  - There is stored energy in the capacitor or inductor at *time* = 0 s.
  - For $t > 0$ s, the stored energy is released
    - Current flows through the circuit and voltages exist across components in the circuit as the stored energy is released.
    - The stored energy will decays to zero as time approaches infinite, at which point the currents and voltages in the circuit become zero.

RC Circuit

- Suppose there is some charge on a capacitor at time $t = 0$ s.
  - This charge could have been stored because a voltage or current source had been in the circuit at $t < 0$ s, but was switched off at $t = 0$ s.
- We can use the equations relating voltage and current to determine how the charge on the capacitor is removed as a function of time.
  - The charge flows from one plate of the capacitor through the resistor R to the other plate to neutralize the charge on the opposite plate of the capacitor.

Equations for RC Circuit



$$-v_C + v_R = 0$$

$$i_C = -i_R$$

$$i_C = C\frac{dv_C}{dt}$$

$$i_R = \frac{v_R}{R}$$

$$C\frac{dV_C}{dt} + \frac{V_R}{R} = 0$$

$$V_R = V_C$$

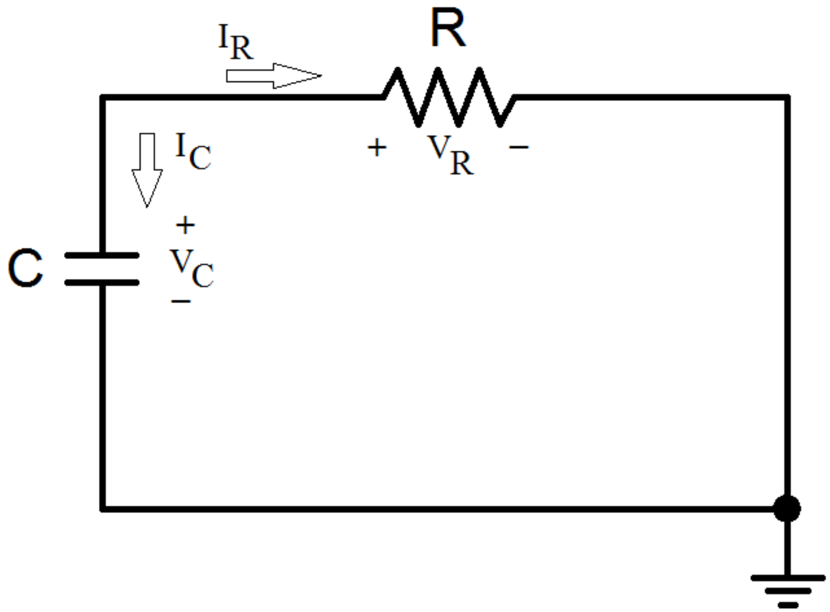
$$\frac{dV_C}{dt} + \frac{V_C}{RC} = 0$$

$$\frac{1}{V_C}\frac{dV_C}{dt} + \frac{1}{RC} = 0$$

$$\frac{dV_C}{V_C} = -\frac{1}{RC}dt$$

$$\ln\left(V_C\right) = -\frac{t}{RC} + \ln\left(V_C\big|_{t=t_o}\right)$$

Equations for RC Circuit



If $V_o = V_C\big|_{t=0s}$ and $\tau = RC$

$$V_C(t) = V_o e^{-\frac{t}{\tau}} \quad \text{when } t \geq 0s$$

$$I_R(t) = -I_C(t) = \frac{V_o}{R} e^{-\frac{t}{\tau}}$$

$$p_R(t) = V_R I_R = \frac{V_o^{\,2}}{R} e^{-\frac{2t}{\tau}}$$

Since the voltages are equal and the currents have the opposite sign, the power that is dissipated by the resistor is the power that is being released by the capacitor.

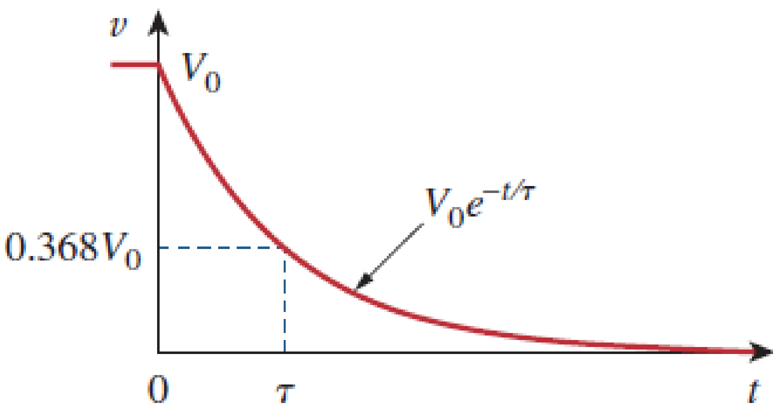
$$w(t) = \int_{0s}^{t} p_R(t)dt = \frac{CV_o^{\,2}}{2}\left[1 - e^{-\frac{2t}{\tau}}\right]$$

The Key to Working with a Source-Free RC Circuit Is Finding:

- The initial voltage $v(0) = V_0$ across the capacitor.
  - Can be obtained by inserting a d.c. source to the circuit for a time much longer than τ (at least $t = -5\tau$) and then removing it at $t = 0$.
    - Capacitor
      - Open Circuit Voltage
- The time constant τ.
  - In finding the time constant $\tau = RC$, R is often the Thevenin equivalent resistance at the terminals of the capacitor;
    - that is, we take out the capacitor C and find $R = R_{Th}$ at its terminals

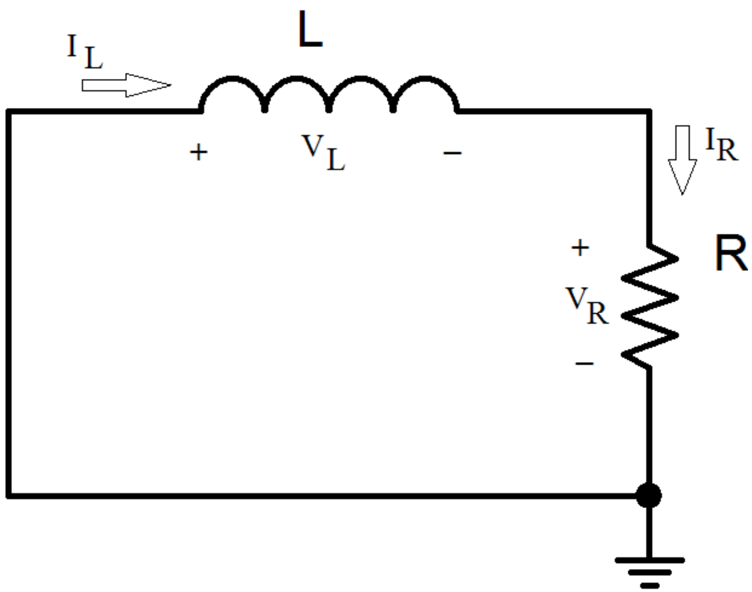
Time constant

- The natural response of a capacitive circuit refers to the behavior (in terms of voltages) of the circuit itself, with no external sources of excitation.
  - The natural response depends on the nature of the circuit alone, with no external sources.
    - In fact, the circuit has a response only because of the energy initially stored in the capacitor.
- The voltage response of the *RC* circuit



  - Time constant, $\tau = RC$
    - The time required for the voltage across the capacitor to decay by a factor of 1/e or 36.8% of its initial value.

Equations for RL Circuits



$$V_L + V_R = 0$$
$$I_L = I_R$$

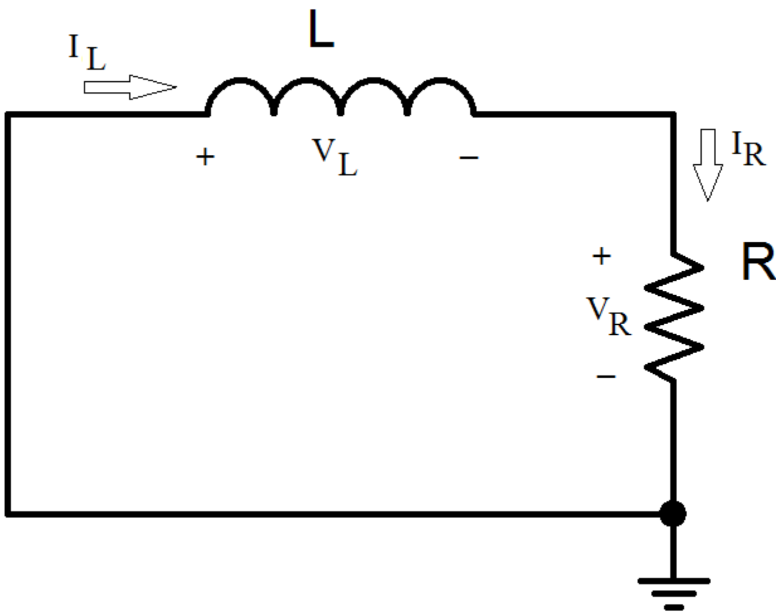
$$V_L = L\frac{dI_L}{dt}$$
$$I_R = V_R / R$$

$$L\frac{dI_L}{dt} + RI_R = 0$$

$$\frac{dI_L}{dt} + \frac{RI_L}{L} = 0$$
$$\frac{1}{I_L}\frac{dI_L}{dt} + \frac{R}{L} = 0$$

$$\frac{dI_L}{I_L} = -\frac{R}{L}dt$$
$$\ln\left(I_L\right) = -\frac{R}{L}t + \ln\left(I_L\big|_{t=0s}\right)$$

Equations for RL Circuit



If $I_o = I_L\big|_{t=0s}$ and $\tau = \dfrac{L}{R}$

$$I_L(t) = I_o e^{-\frac{t}{\tau}} \quad \text{when } t \geq 0s$$

$$V_R(t) = -V_L(t) = RI_o e^{-\frac{t}{\tau}}$$

$$p_R(t) = V_R I_R = RI_o{}^2 e^{-\frac{2t}{\tau}}$$

$$w(t) = \int_{0s}^{t} p_R(t)dt = \frac{LI_o{}^2}{2}\left[1 - e^{-\frac{2t}{\tau}}\right]$$

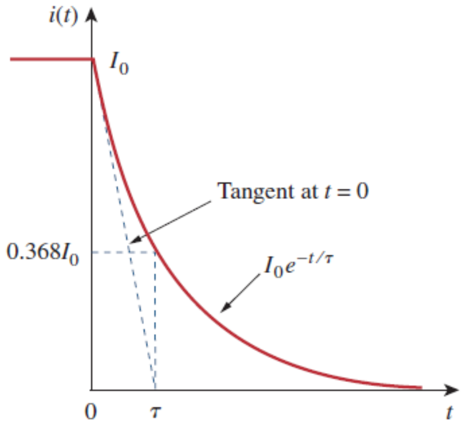
Since the voltages are equal and the currents have the opposite sign, the power that is dissipated by the resistor is the power that is being released by the inductor.

The Key to Working with a Source-Free RL Circuit Is Finding:

- The initial current $i(0) = I_0$ through the inductor.
  - Can be obtained by inserting a d.c. source to the circuit for a time much longer than τ (at least t = -5τ) and then removing it at $t = 0$.
    - Inductor
      - Short Circuit Current
- The time constant τ.
  - In finding the time constant $\tau = L/R$, R is often the Thevenin equivalent resistance at the terminals of the inductor;
    - that is, we take out the inductor L and find $R = R_{Th}$ at its terminals

Time constant

- The natural response of an inductive circuit refers to the behavior (in terms of currents) of the circuit itself, with no external sources of excitation.
  - The natural response depends on the nature of the circuit alone, with no external sources.
    - In fact, the circuit has a response only because of the energy initially stored in the inductor.
- The current response of the *RL* circuit



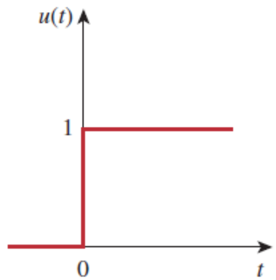
  - Time constant, $\tau = L/R$
    - The time required for the current in the inductor to decay by a factor of 1/e or 36.8% of its initial value.

Singularity Functions

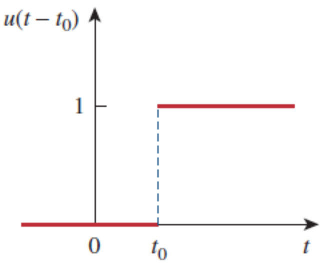
- Singularity functions (also called switching functions) are very useful in circuit analysis.
- They serve as good approximations to the switching signals that arise in circuits with switching operations.
- They are helpful in the neat, compact description of some circuit phenomena,
  - especially the step response of *RC* or *RL* circuits
- Singularity functions are functions that either are discontinuous or have discontinuous derivatives.

Unit Step Function

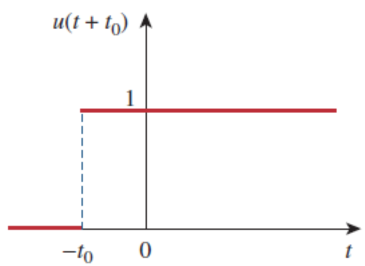
- The unit step function ($u(t)$) is 0 for negative values of t and 1 for positive values of t.



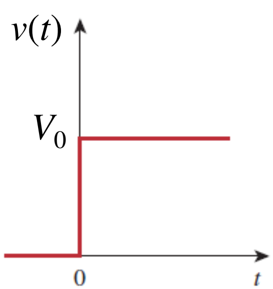
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$



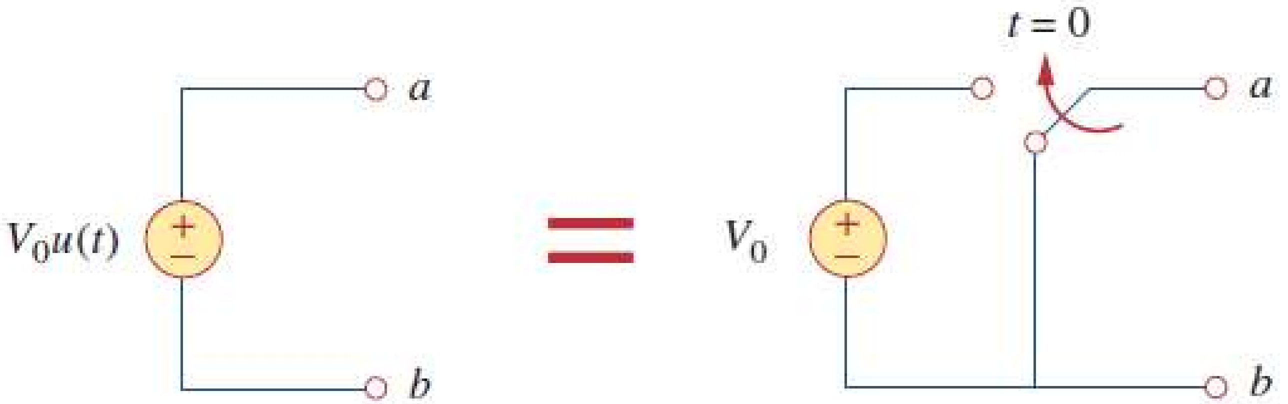
$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases}$$



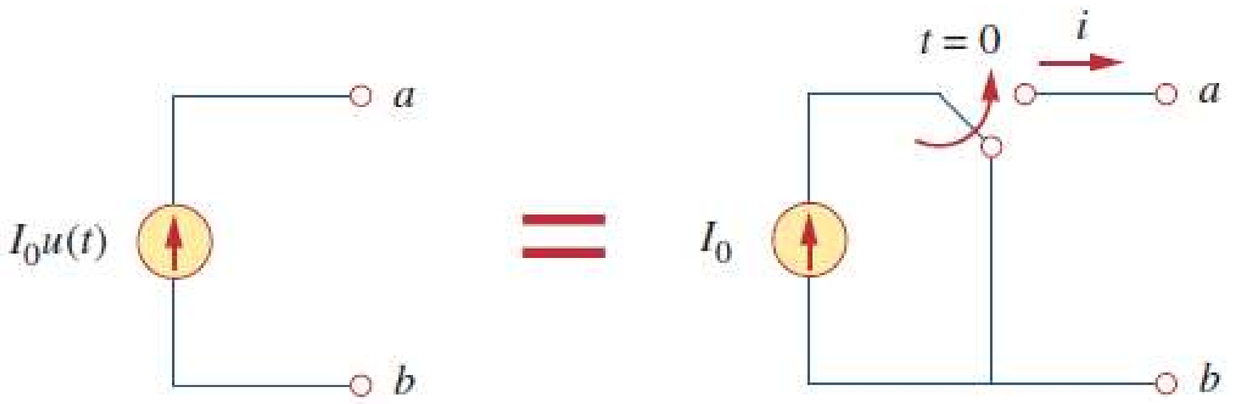
$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases} \quad \Longrightarrow \quad v(t) = V_0 u(t - t_0)$$

Unit Step Function

- Voltage source of $V_0u(t)$ and its equivalent circuit.

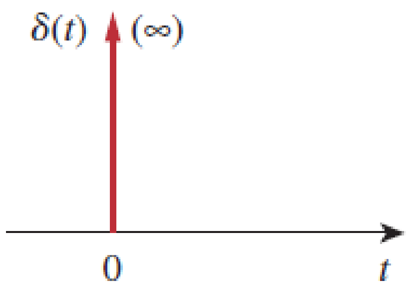


- Current source of $I_0u(t)$ and its equivalent circuit.



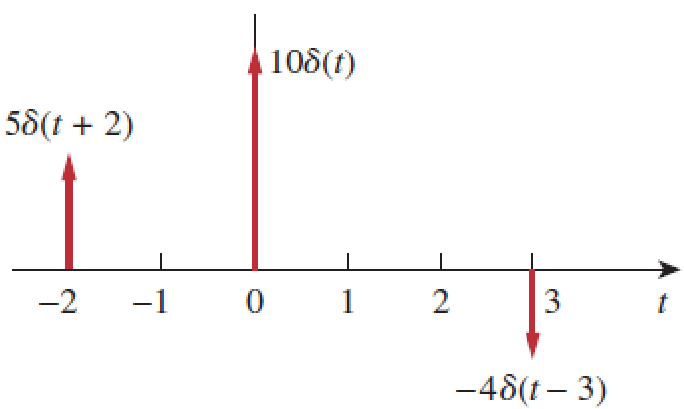
Unit Impulse Function

- The derivative of the unit step function $u(t)$ is the unit impulse function ($\delta(t)$)



$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$

$$\int_{0^-}^{0^+} \delta(t)\, dt = 1$$



$$x(t) = 5\delta(t+2) + 10\delta(t) - 4\delta(t-3)$$

Integration of Unit Functions

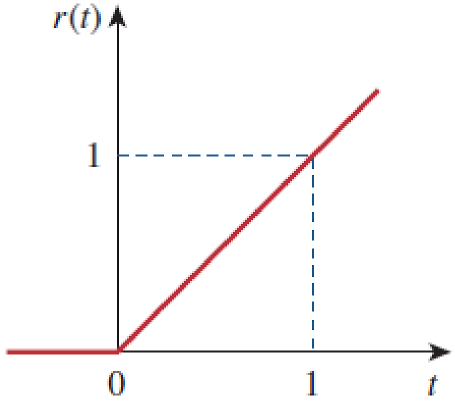
- To illustrate how the impulse function affects other functions, let us evaluate the integral

$$\int_a^b f(t)\delta(t - t_0)\,dt$$

$$\int_a^b f(t)\delta(t - t_0)\,dt = \int_a^b f(t_0)\delta(t - t_0)\,dt$$
$$= f(t_0)\int_a^b \delta(t - t_0)\,dt = f(t_0)$$

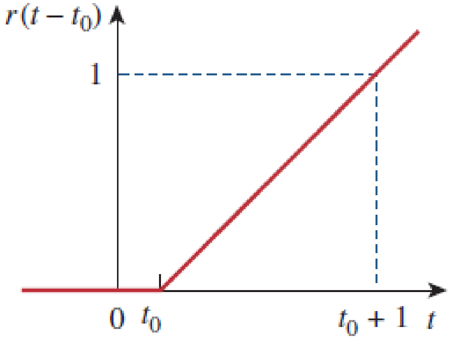
- This is a highly useful property of the impulse function known as the sampling or sifting property.

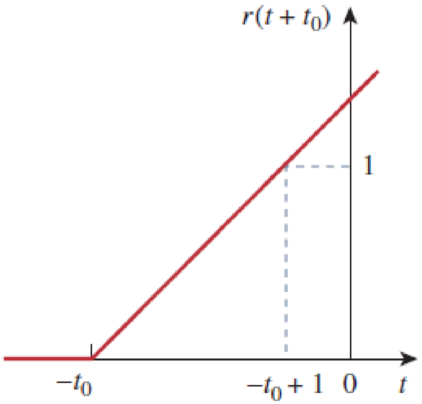
Unit Ramp Function



$$r(t) = \int_{-\infty}^{t} u(\lambda)\,d\lambda = tu(t)$$

$$\boxed{r(t) = \begin{cases} 0, & t \le 0 \\ t, & t \ge 0 \end{cases}}$$





$$r(t - t_0) = \begin{cases} 0, & t \le t_0 \\ t - t_0, & t \ge t_0 \end{cases}$$

$$r(t + t_0) = \begin{cases} 0, & t \le -t_0 \\ t + t_0, & t \ge -t_0 \end{cases}$$

Relationships of singularity functions

- The three singularity functions (impulse, step, and ramp) are related by differentiation as

$$\delta(t) = \frac{du(t)}{dt}, \qquad u(t) = \frac{dr(t)}{dt}$$

- or by integration as

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)\, d\lambda, \qquad r(t) = \int_{-\infty}^{t} u(\lambda)\, d\lambda$$

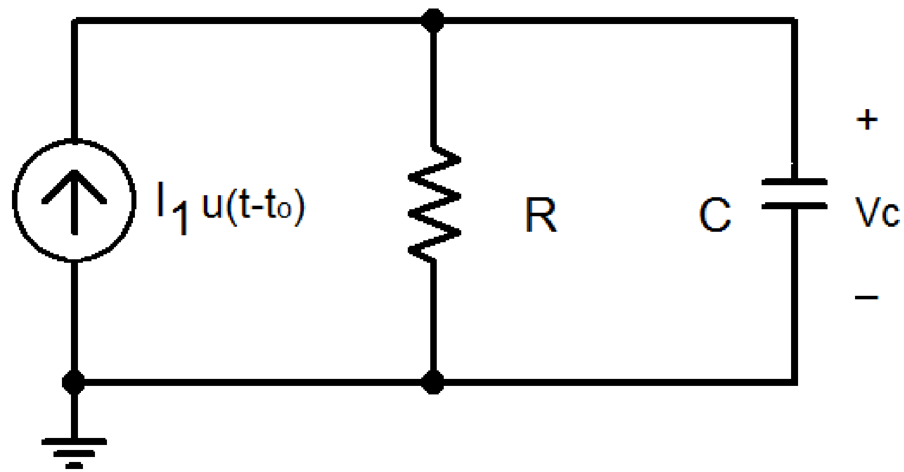
Transient responses of RC and RL circuits

- AKA a forced response to an independent source
- Capacitor and inductor store energy when there is:
  - a transition in a unit step function source, $u(t\text{-}t_o)$
  - a voltage or current source is switched into the circuit.

RC Circuit

$I_C = 0A$ when $t < t_o$

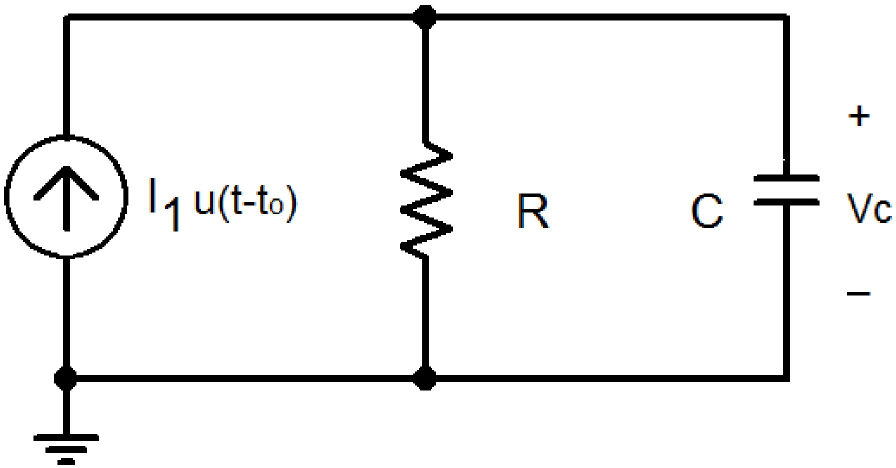
$V_C = 0V$ when $t < t_o$



Because $I_1 = 0A$ (replace it with an open circuit).

RC Circuit

- Find the final condition of the voltage across the capacitor.



  – Replace C with an open circuit and determine the voltage across the terminal.

$I_C = 0A$ when $t \sim \infty$ s
$V_C = V_R = I_1R$ when $t \sim \infty$ s

RC Circuit

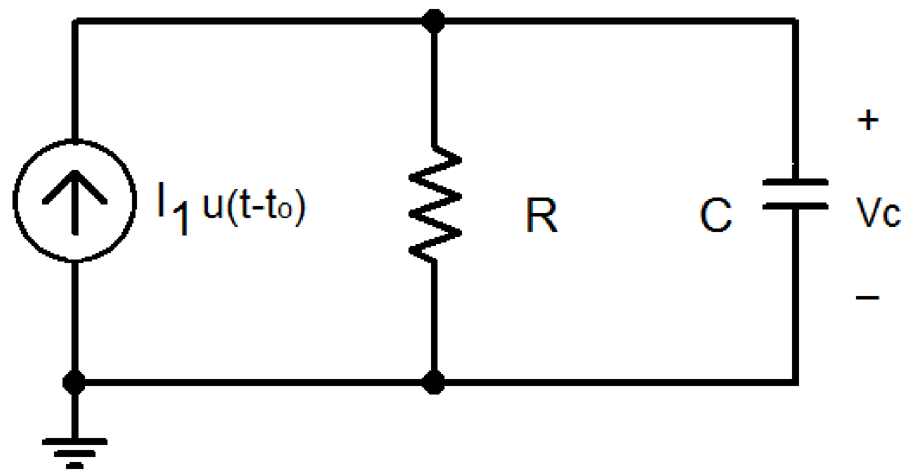
- In the time between t_o and $t = \infty$ s, the capacitor stores energy and currents flow through R and C.

$$V_C = V_R$$

$$I_C = C\frac{dV_C}{dt}$$

$$I_R = \frac{V_R}{R}$$

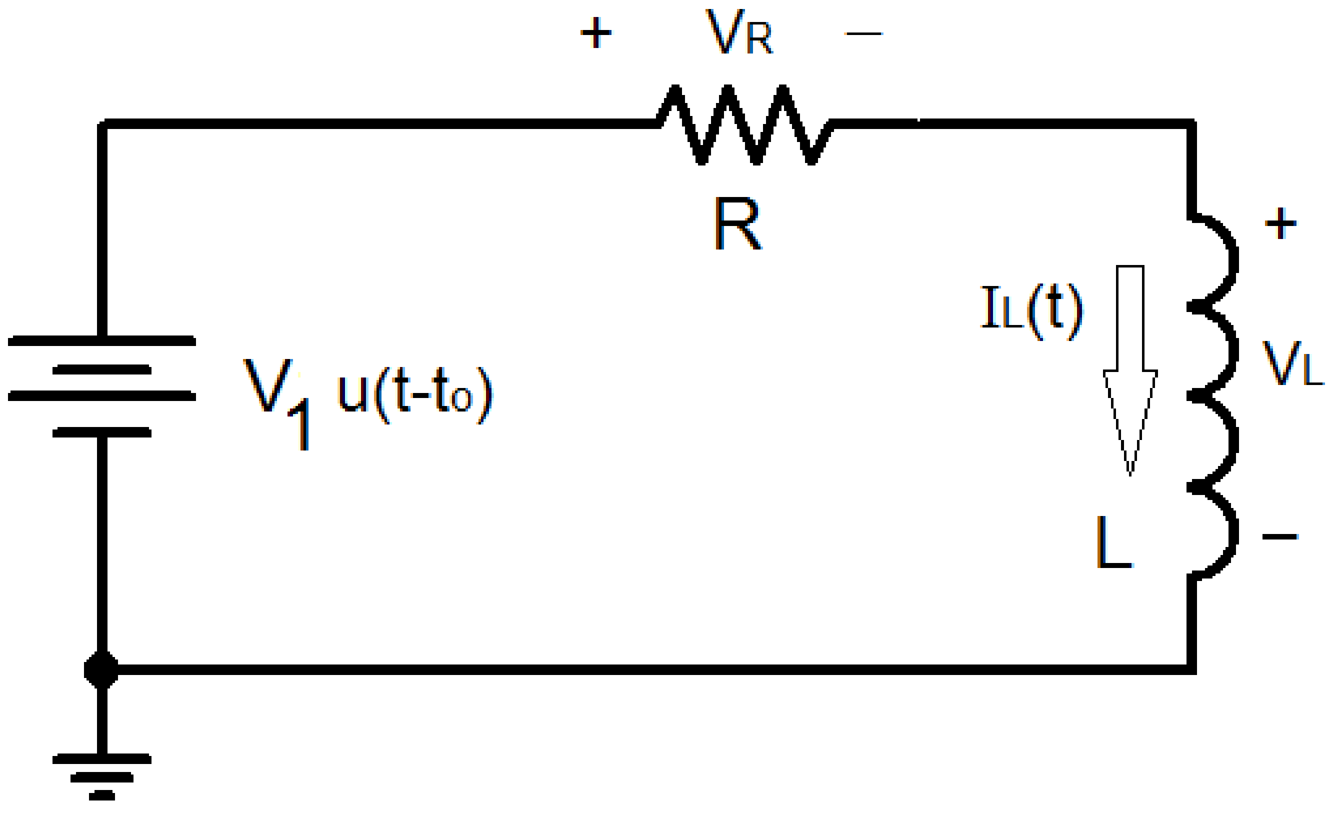
$$I_R + I_C - I_1 = 0$$



$$\frac{V_C}{R} + C\frac{dV_C}{dt} - I_1 = 0$$

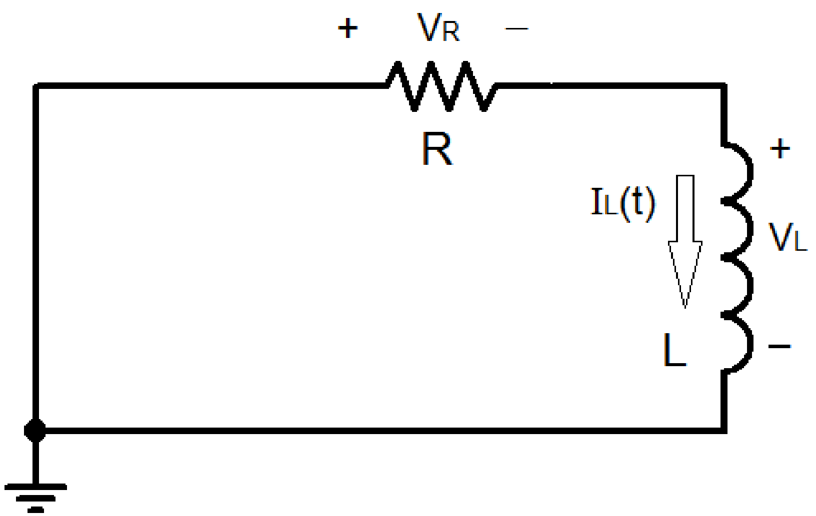
$$V_C(t) = RI_1\left[1 - e^{-\frac{t-t_0}{\tau}}\right] \qquad \tau = RC$$

RL Circuit



RL Circuit

- Initial condition is not important as the magnitude of the voltage source in the circuit is equal to 0V when $t \leq t_o$.

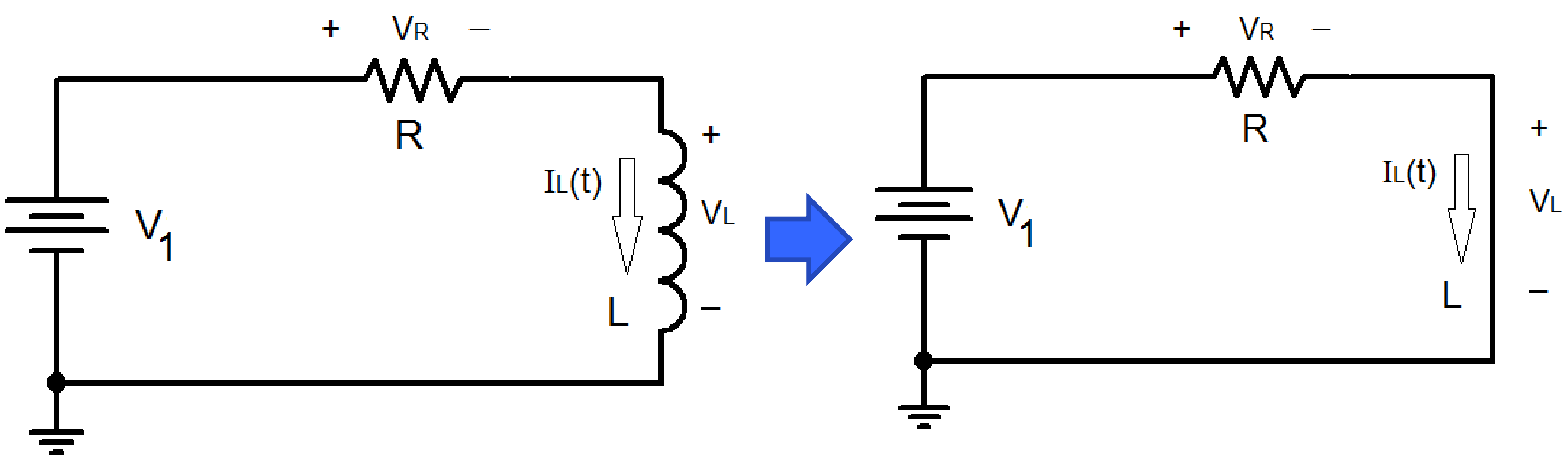


  – Since the voltage source has only been turned on at $t = t_o$, the circuit at $t \leq t_o$ is as shown on the left.

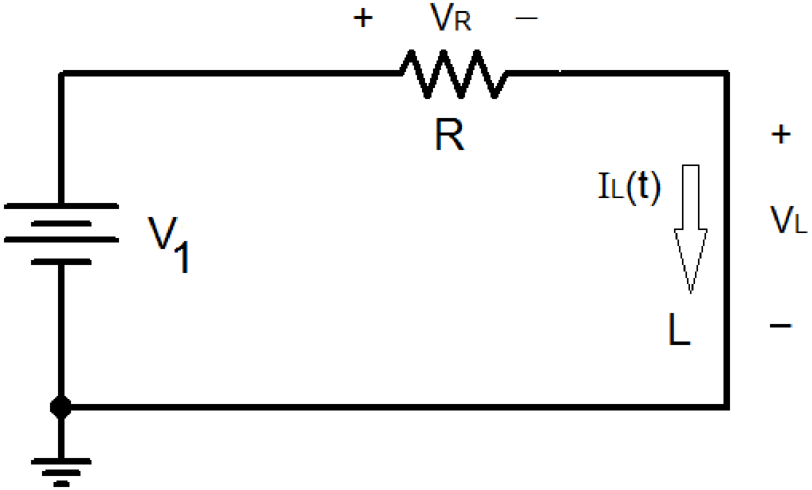
    - As the inductor has not stored any energy because no power source has been connected to the circuit as of yet, all voltages and currents are equal to zero.

RL Circuit

- So, the final condition of the inductor current needs to be calculated after the voltage source has switched on.
  - Replace L with a short circuit and calculate $I_L(\infty)$.



Final Condition

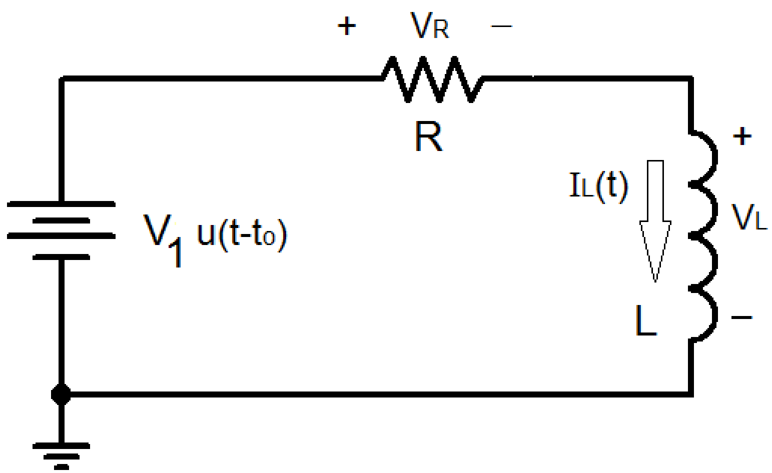


$$V_L(\infty) = 0V$$

$$I_L(\infty) = I_R$$

$$I_R = \frac{V_1}{R}$$

RL Circuit



$$-V_1 + V_L + V_R = 0$$

$$I_L = I_R = V_R / R$$

$$V_L = L\frac{dI_L}{dt}$$

$$\frac{dI_L}{dt} + RI_R - V_1 = 0$$

$$\frac{dI_L}{dt} + \frac{R}{L}I_L - \frac{V_1}{L} = 0$$

$$I_L(t) = \frac{V_1}{R}\left[1 - e^{-(t-t_o)/\tau}\right]$$

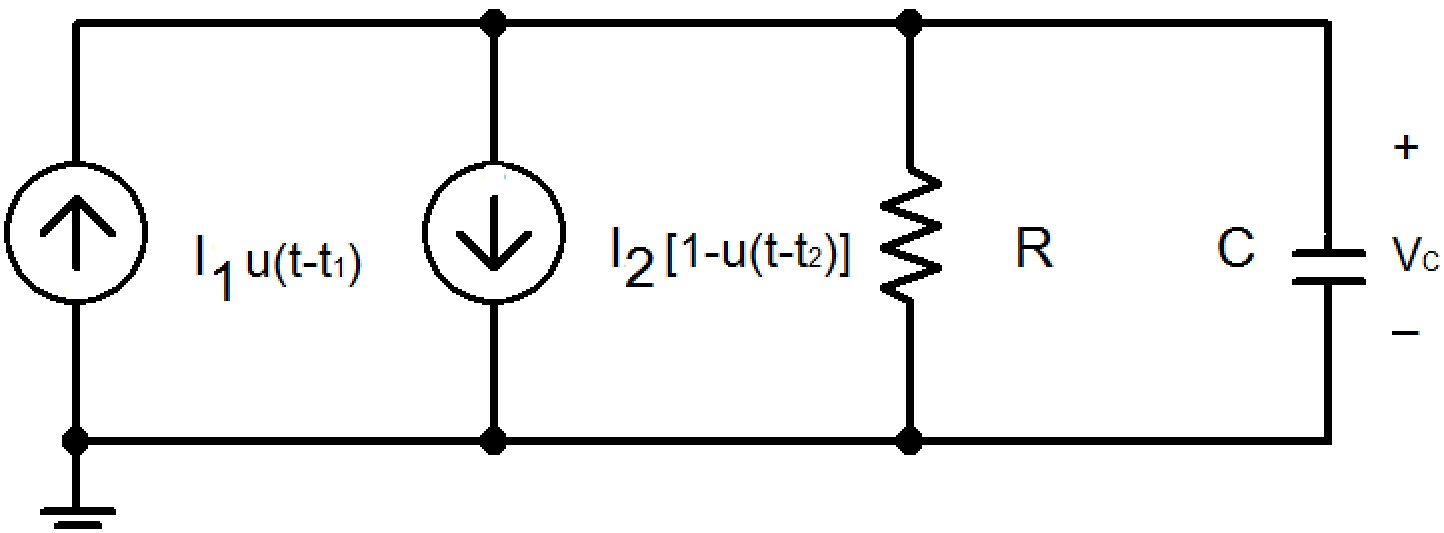
$$\tau = \frac{L}{R}$$

Complete Response

- Is equal to the natural response of the circuit plus the forced response
  - Use superposition to determine the final equations for voltage across components and the currents flowing through them.
- Typically, it is assumed that the currents and voltages in a circuit have reached steady-state once 5τ have passed after a change has been made to the value of a current or voltage source in the circuit.

Example 01…

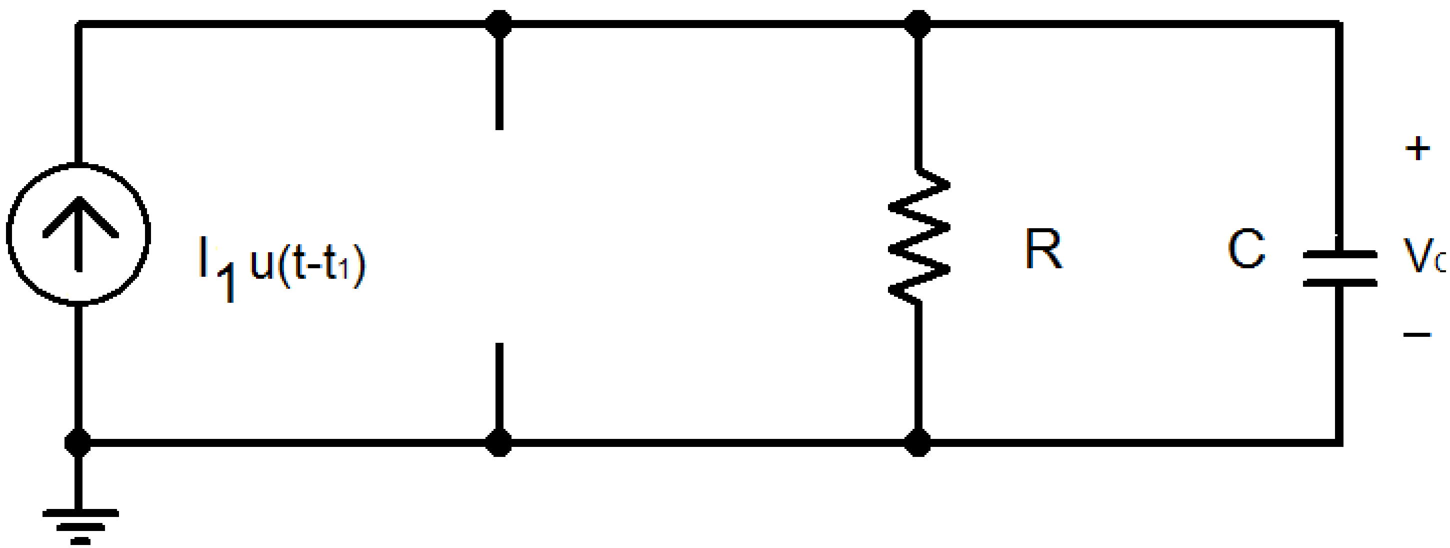
- Suppose there were two unit step function sources in the circuit.



…Example 01…

- The solution for Vc would be the result of superposition where:
  - $I_2 = 0A$, I_1 is left on
    - The solution is a forced response since I_1 turns on at $t = t_1$
  - $I_1 = 0A$, I_2 is left on
    - The solution is a natural response since I_2 turns off at $t = t_2$

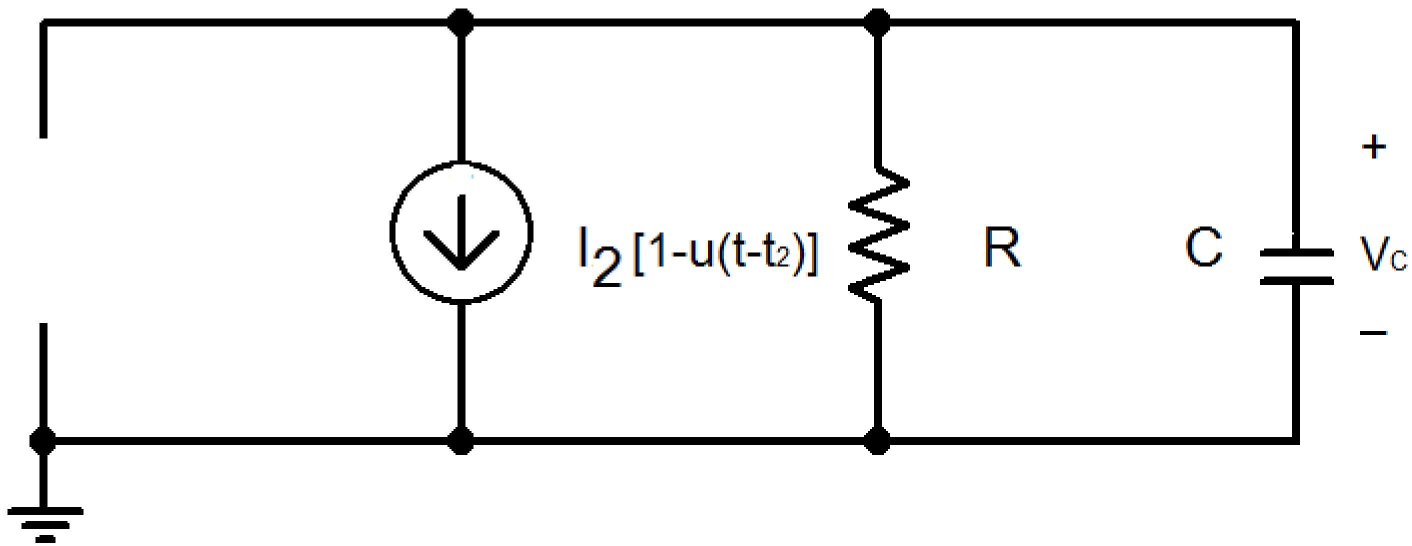
…Example 01…



$$V_C(t) = 0V \quad \text{when } t < t_1$$

$$V_C(t) = RI_1\left[1 - e^{-\frac{(t-t_1)}{RC}}\right] \quad \text{when } t > t_1$$

…Example 01…



$$V_C(t) = -RI_2 \quad \text{when } t < t_2$$

$$V_C(t) = -RI_2 e^{-\frac{(t-t_2)}{RC}} \quad \text{when } t > t_2$$

…Example 01…

- If $t_1 < t_2$

$$V_C(t) = 0V - RI_2 \qquad \text{when } t < t_1$$

$$V_C(t) = RI_1\left[1 - \mathrm{e}^{-\frac{(t-t_1)}{RC}}\right] - RI_2 \qquad \text{when } t_1 < t < t_2$$

$$V_C(t) = RI_1\left[1 - \mathrm{e}^{-\frac{(t-t_1)}{RC}}\right] - RI_2\mathrm{e}^{-\frac{(t-t_2)}{RC}} \qquad \text{when } t > t_2$$

General Equations

- When a voltage or current source changes its magnitude at t = 0s in a simple RC or RL circuit.
  - Equations for a simple RC circuit

$$V_C(t) = V_C(\infty) + \left[V_C(0) - V_C(\infty)\right]e^{-t/\tau}$$

$$I_C(t) = \frac{C}{\tau}\left[V_C(\infty) - V_C(0)\right]e^{-t/\tau}$$

$$\tau = RC$$

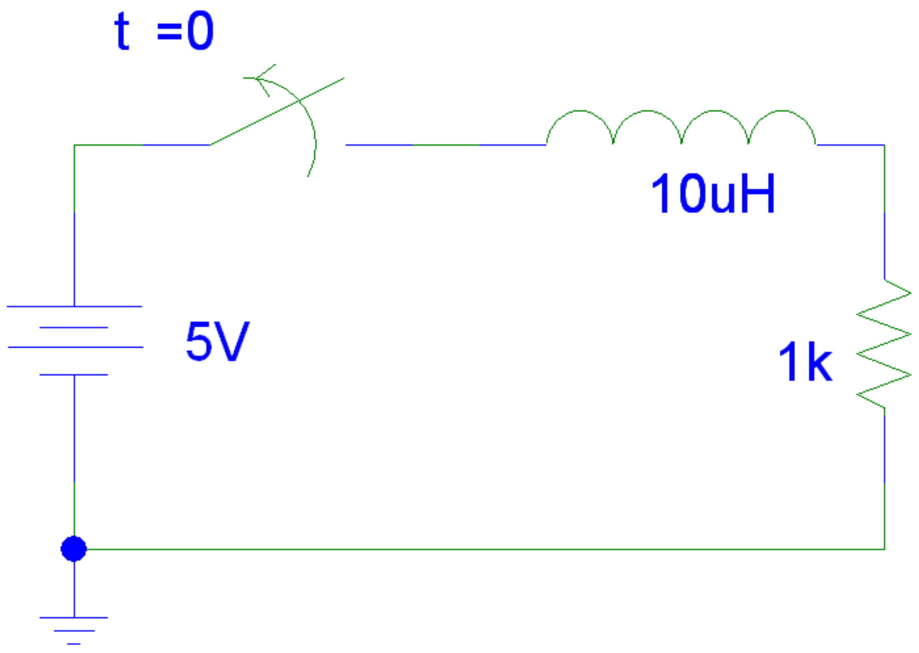
  - Equations for a simple RL circuit

$$I_L(t) = I_L(\infty) + \left[I_L(0) - I_L(\infty)\right]e^{-t/\tau}$$

$$V_L(t) = \frac{L}{\tau}\left[I_L(\infty) - I_L(0)\right]e^{-t/\tau}$$

$$\tau = L / R$$

Example 02…

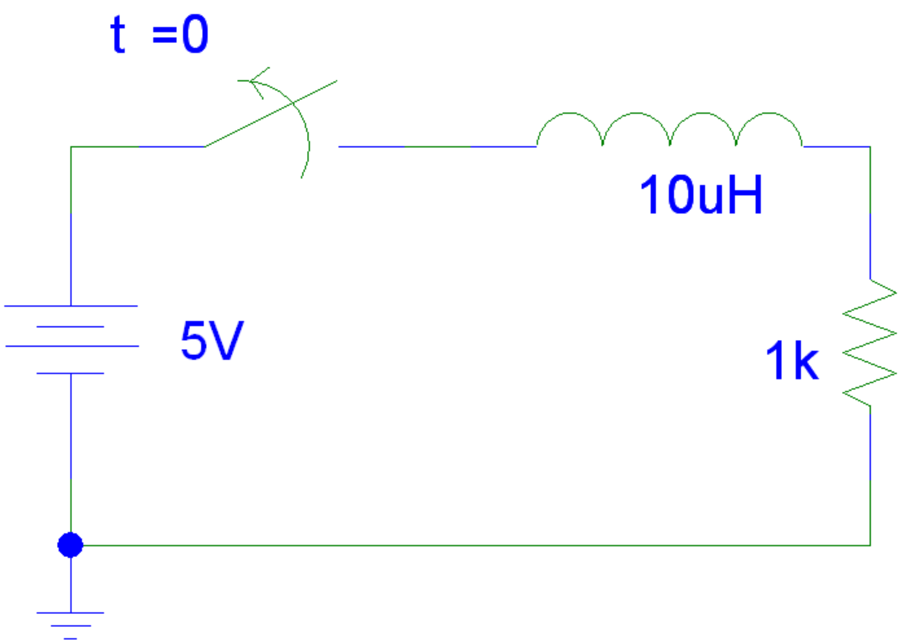


t < 0

$V_L = 0V$
$V_R = 5V$
$I_L = I_R = 5mA$

V(t) = 5V [1 - u(t)]

…Example 02



$V(t) = 5V\,[1 - u(t)]$

t > 0

$\tau = L/R = 10mH/1k\Omega = 10\ ns$

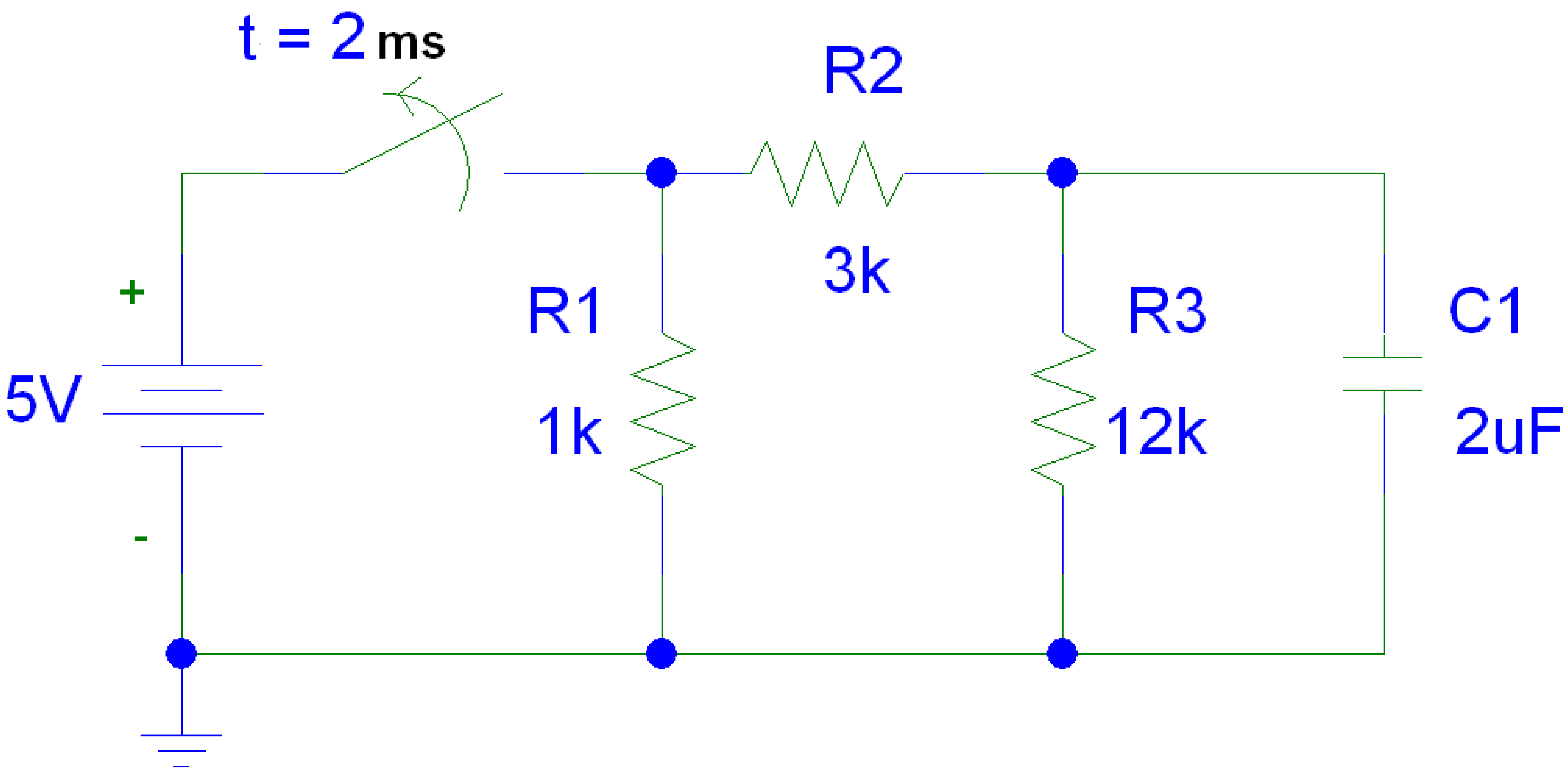
$I_L = I_R = i(0)e^{-t/\tau} = 5mA\ e^{-t/10ns}$

$V_R = 1k\Omega\ I_R\ = 5V\ e^{-t/10ns}$

$V_L = L(dI_L/dt) = -5V\ e^{-t/10ns}$

Note $V_R + V_L = 0\ V$

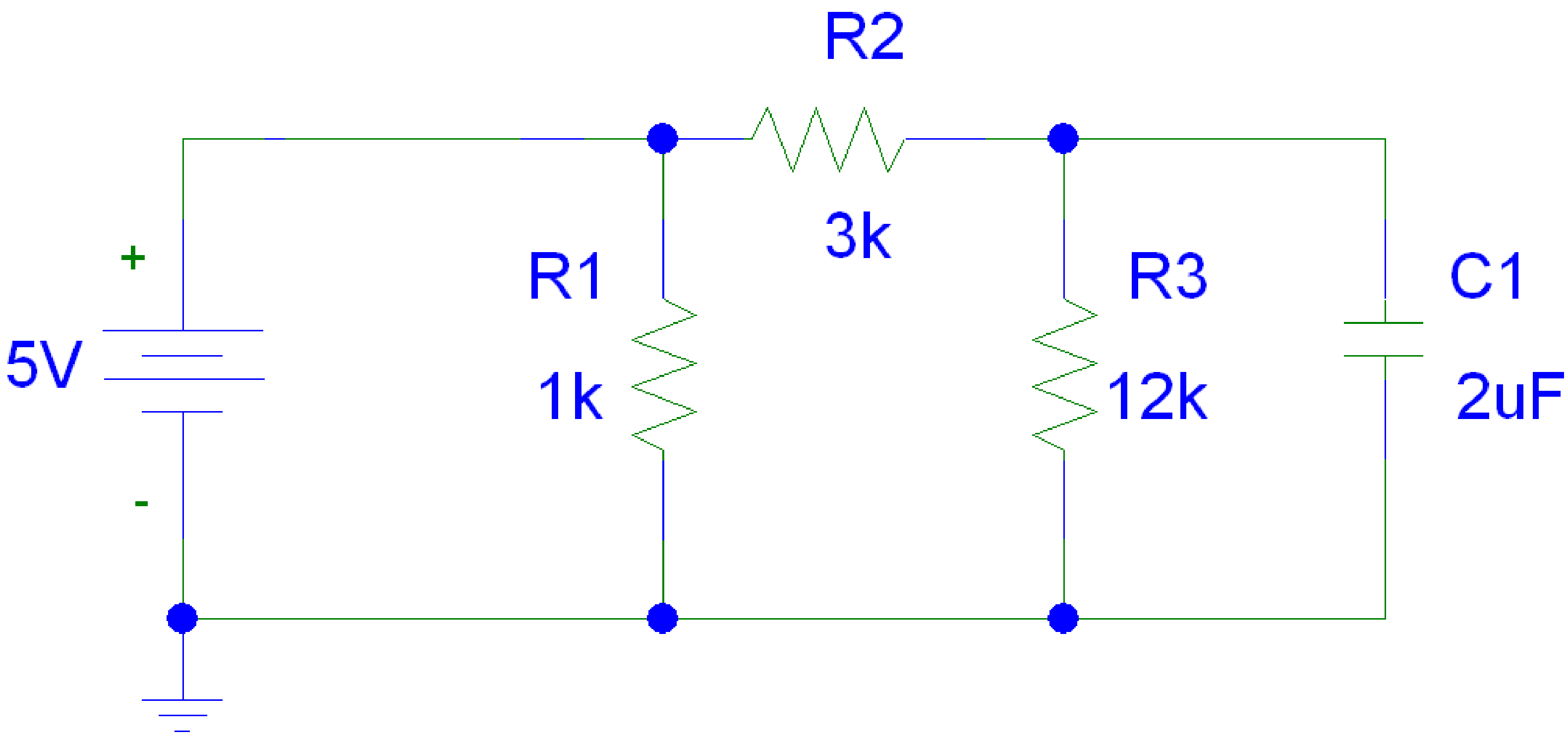
Example 03…



$V(t) = 5V\,[1 - u(t - 2ms)]$

…Example 03…

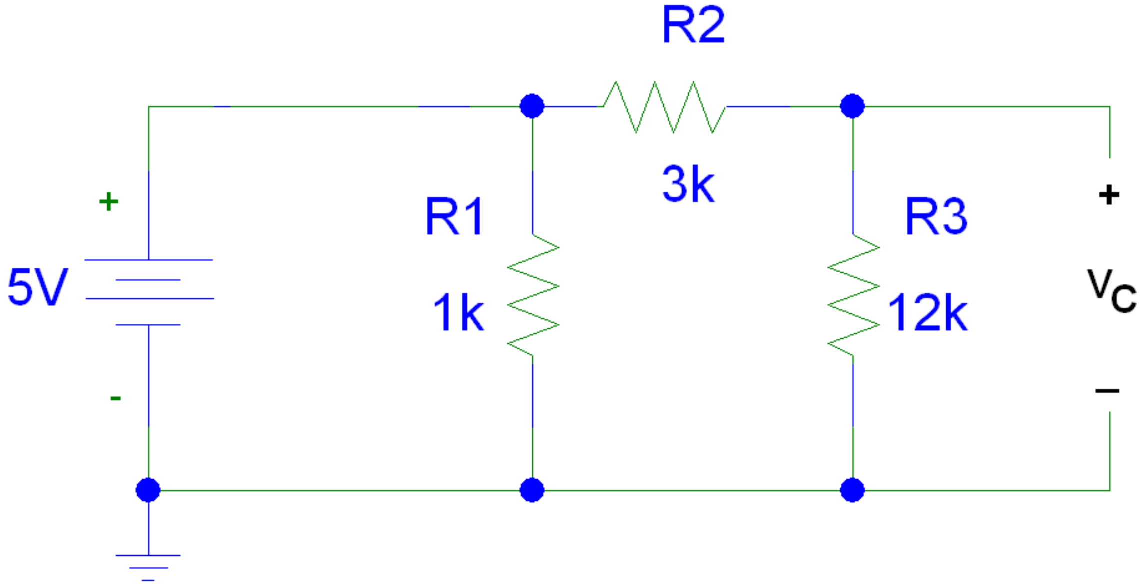
t < 2ms



…Example 03…

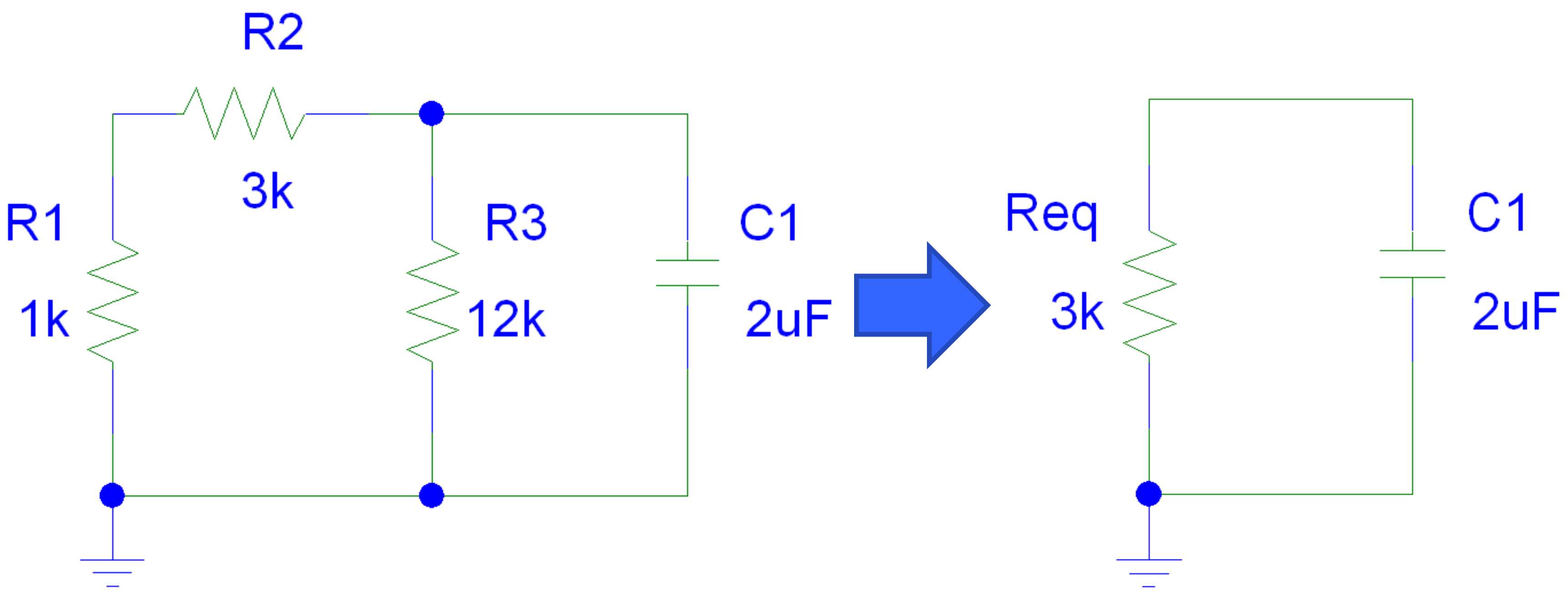
- **t < 2ms**
  - C1 is an open.
    - The voltage across the capacitor is equal to the voltage across the 12kΩ resistor.

$V_C = [12k\Omega / 15k\Omega]\ 5V = 4V$



…Example 03…

t > 2ms



…Example 03

t > 2ms

$\tau = R_{eq}C = 3k\Omega(2\mu F) = 6$ ms

$V_C = V_C(2ms)e^{-(t-2ms)/\tau} = 4V\ e^{-(t-2ms)/6ms}$

$V_R = V_C$

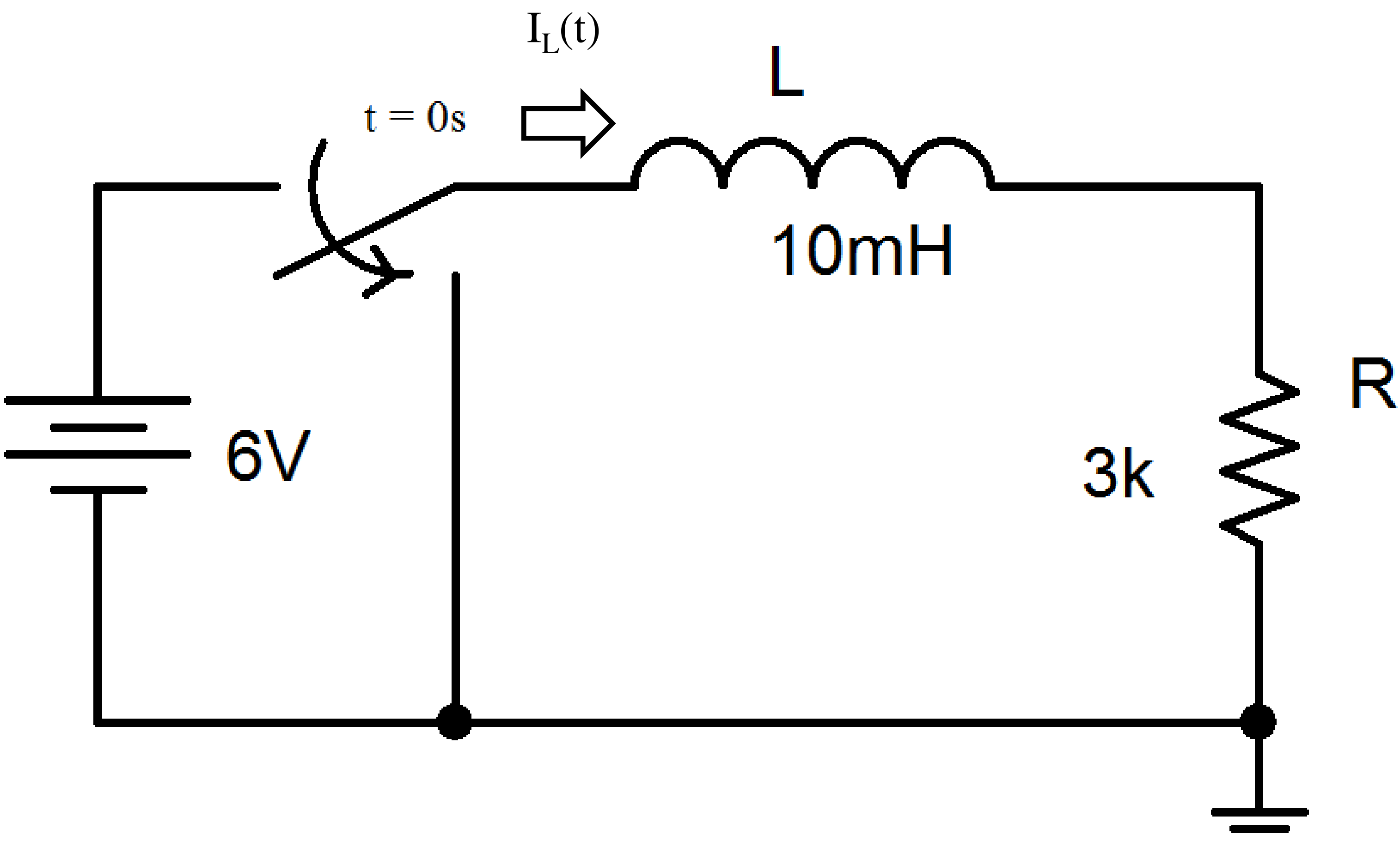
$I_C = C\ dV_c/dt = 2\mu F(-4V/6ms)\ e^{-(t-2ms)/6ms}$

$= -1.33\ e^{-(t-2ms)/6ms}$ mA

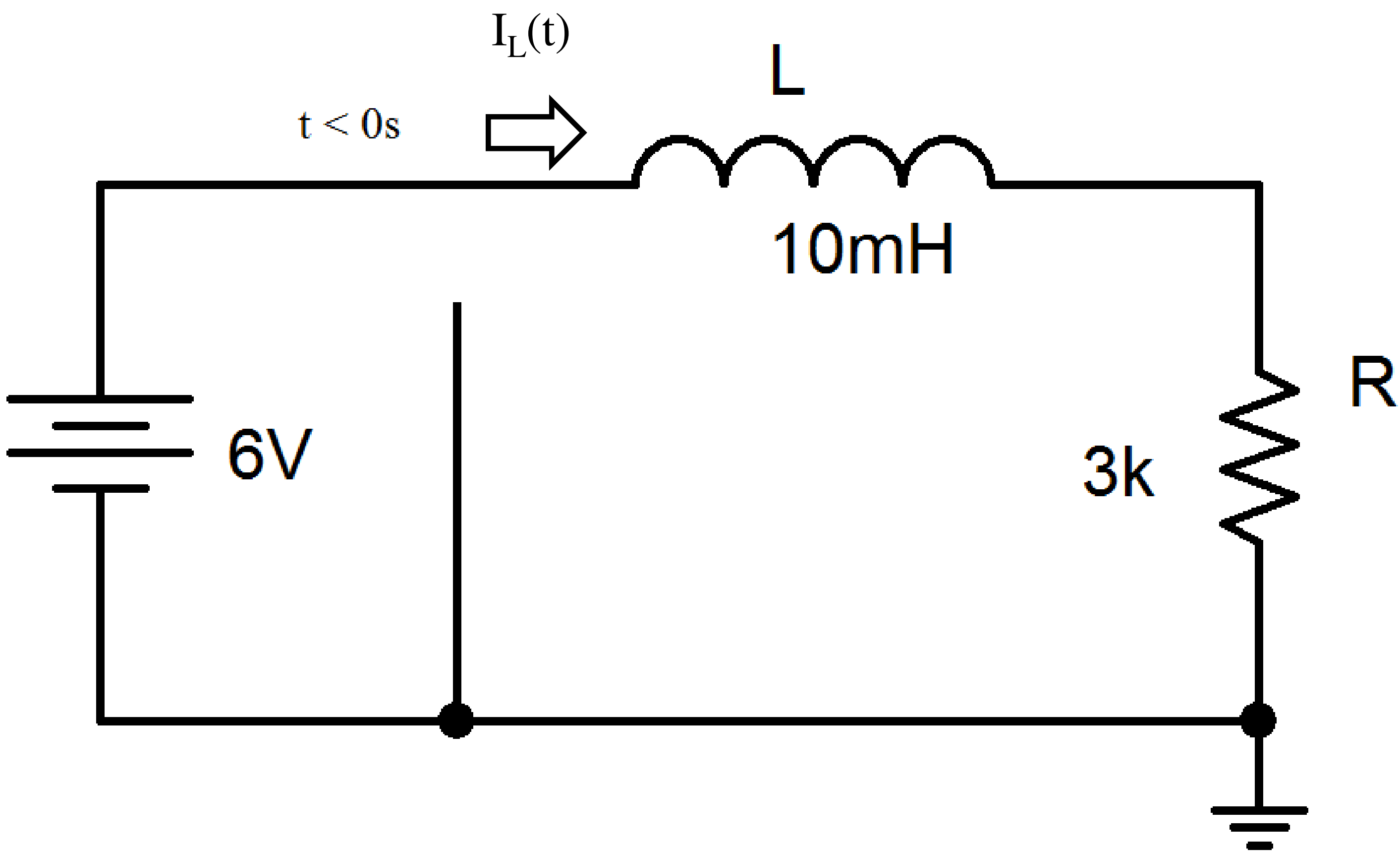
$I_R = -\ I_C = 1.33\ e^{-(t-2ms)/6ms}$ mA

Note $I_R + I_L = 0$ mA

Example 04…

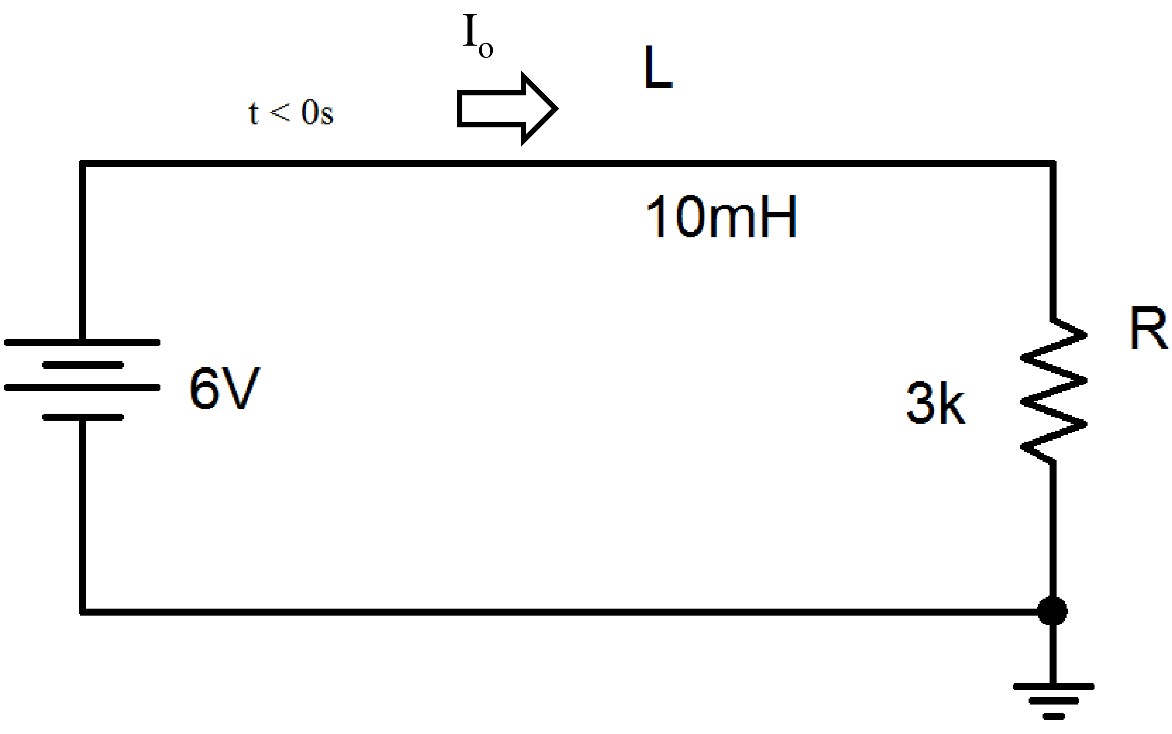


…Example 04…



…Example 04…

Find the initial condition.



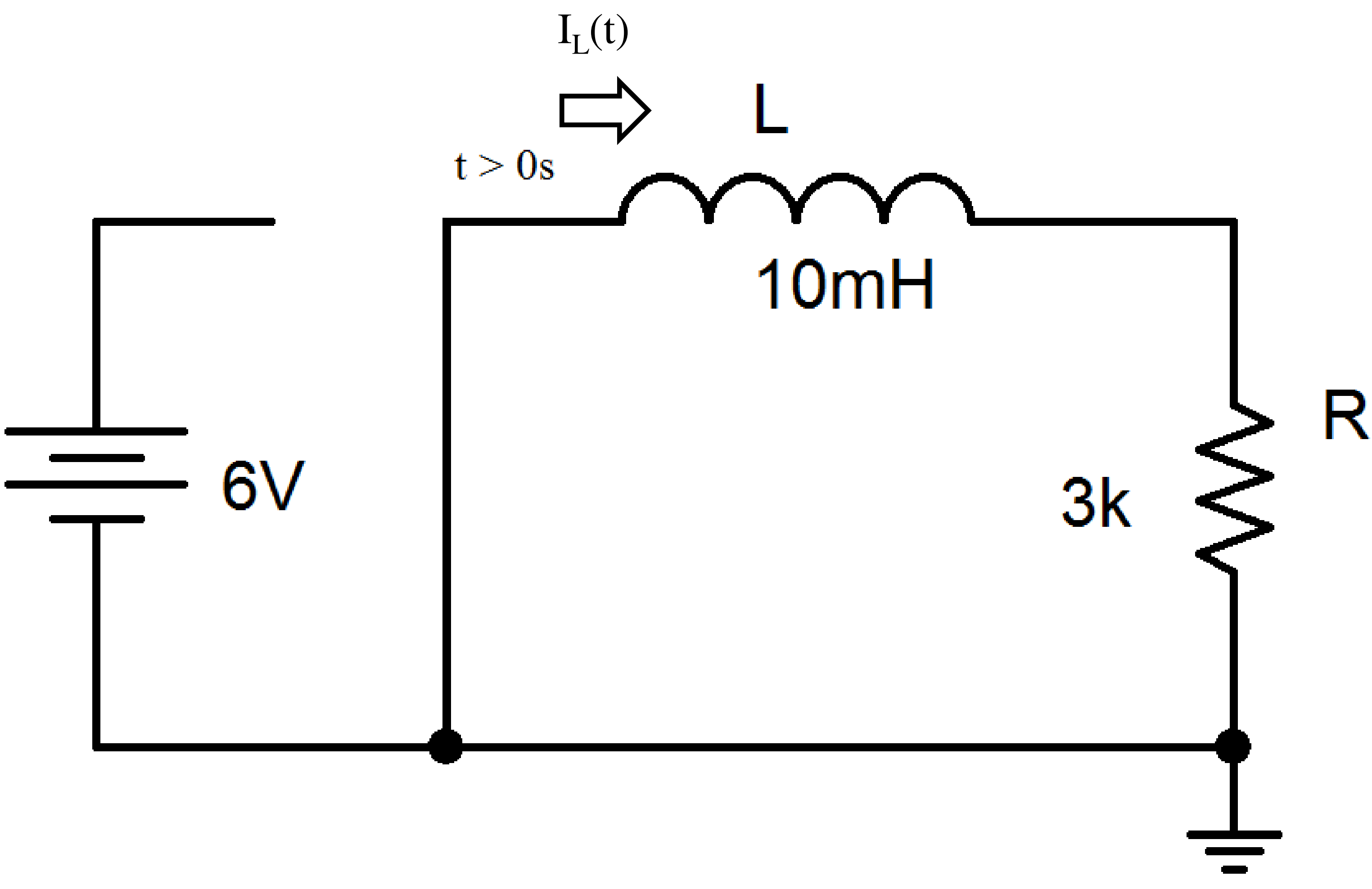
$t < 0s$

$V_L = 0V$
$V_R = 6V$
$I_L = I_R = 2mA$

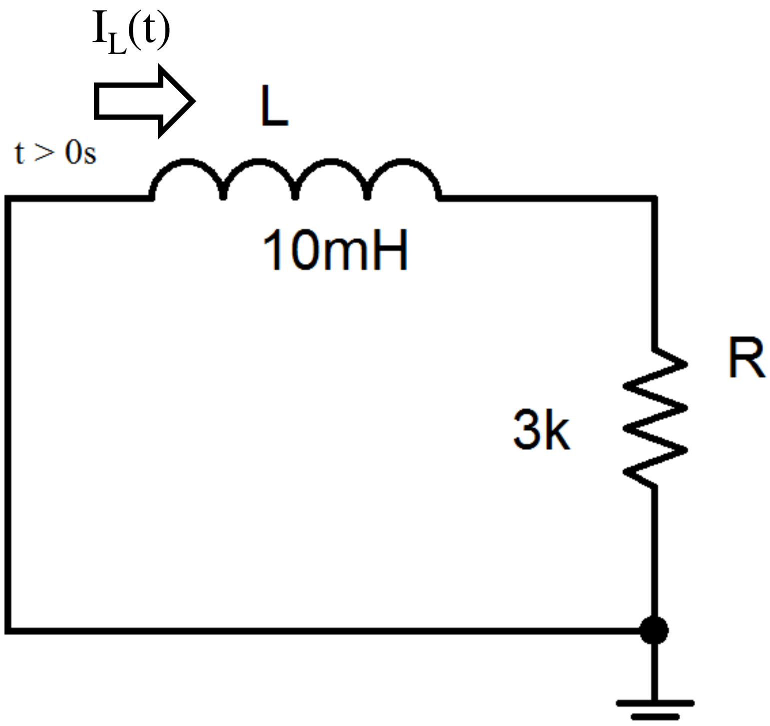
Therefore,

$I_o = 2mA$

…Example 04…



…Example 04



t > 0s

$t = L/R = 10mH/3k\Omega/ =$
$3.33\mu s$

$I_L = I_R = I_o e^{-t/\tau} = 2mA\ e^{-(t/3.33\mu s)}$

$V_R = 3k\Omega\ I_R = 6V\ e^{-(t/3.33\mu s)}$

$V_L = L\ dI_L/dt = -6V\ e^{-(t/3.33\mu s)}$

Note $V_R + V_L = 0$ V