



BLM2502

Theory of

Computation

Spring 2015

BLM2502 Theory of Computation

» Course Outline

| » Week | Content |
|--------|---|
| » 1 | Introduction to Course |
| » 2 | Computability Theory, Complexity Theory, Automata Theory, Set Theory, Relations, Proofs, Pigeonhole Principle |
| » 3 | Regular Expressions |
| » 4 | Finite Automata |
| » 5 | Deterministic and Nondeterministic Finite Automata |
| » 6 | Epsilon Transition, Equivalence of Automata |
| » 7 | Pumping Theorem |
| » 8 | April 10 - 14 week is the first midterm week |
| » 9 | Context Free Grammars |
| » 10 | Parse Tree, Ambiguity, |
| » 11 | Pumping Theorem |
| » 12 | Turing Machines, Recognition and Computation, Church-Turing Hypothesis |
| » 13 | Turing Machines, Recognition and Computation, Church-Turing Hypothesis |
| » 14 | May 22 – 27 week is the second midterm week |
| » 15 | Review |
| » 16 | Final Exam date will be announced |



Context-Free Languages

»

Context-Free Languages

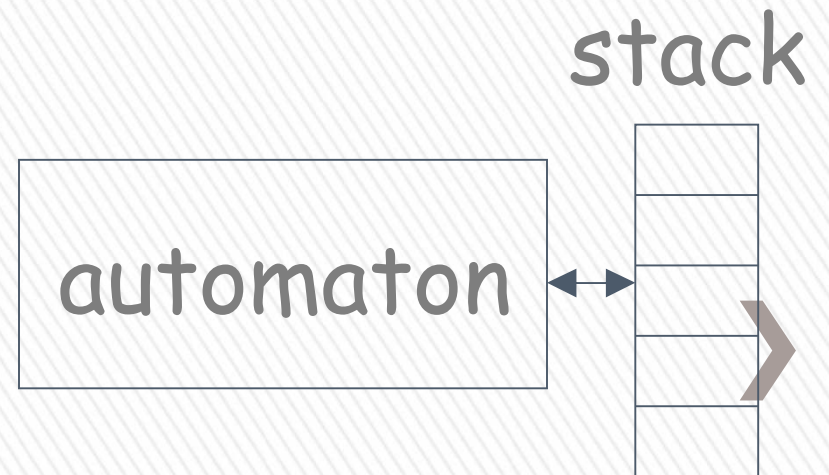
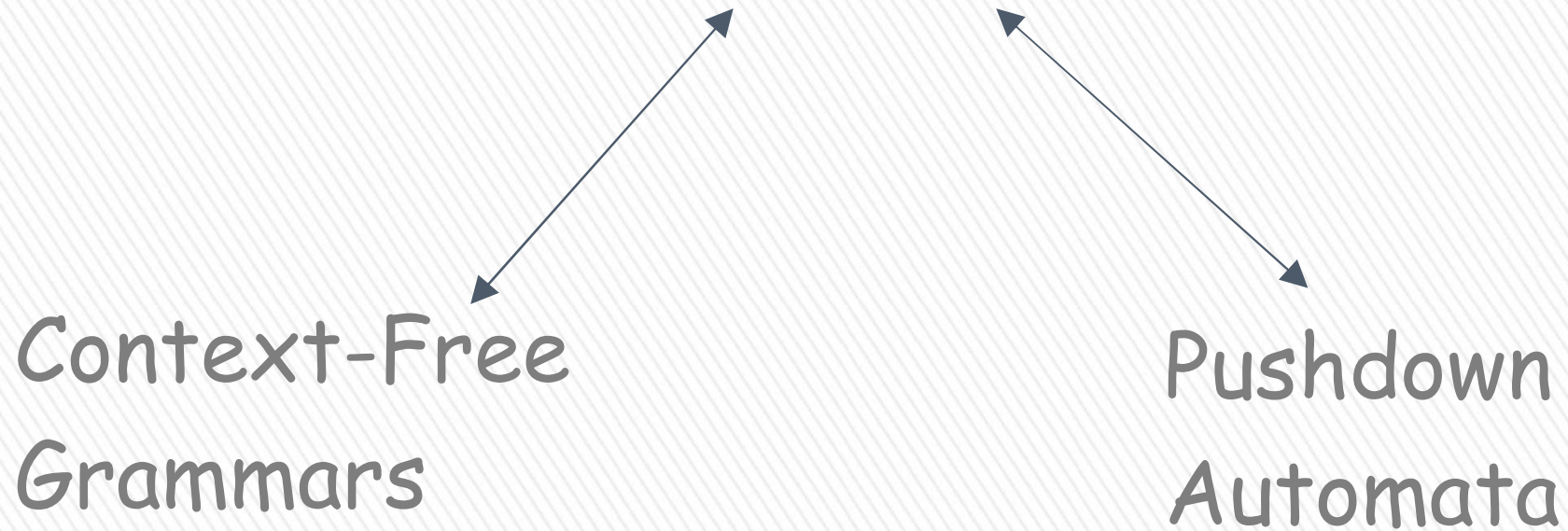
$$\{a^n b^n : n \geq 0\} \quad \{ww^R\}$$

Regular Languages

$$a^* b^* \quad (a + b)^*$$



Context-Free Languages





Context-Free Grammars

Grammars

- » Grammars express languages
- » Example: the English language grammar

$\langle sentence \rangle \rightarrow \langle noun_phrase \rangle \langle predicate \rangle$

$\langle noun_phrase \rangle \rightarrow \langle article \rangle \langle noun \rangle$

$\langle predicate \rangle \rightarrow \langle verb \rangle$



$\langle \textit{article} \rangle \rightarrow a$

$\langle \textit{article} \rangle \rightarrow \textit{the}$

$\langle \textit{noun} \rangle \rightarrow \textit{cat}$

$\langle \textit{noun} \rangle \rightarrow \textit{dog}$

$\langle \textit{verb} \rangle \rightarrow \textit{runs}$

$\langle \textit{verb} \rangle \rightarrow \textit{sleeps}$



» Derivation of string “the dog walks” :

$\langle sentence \rangle \Rightarrow \langle noun_phrase \rangle \langle predicate \rangle$
 $\Rightarrow \langle noun_phrase \rangle \langle verb \rangle$
 $\Rightarrow \langle article \rangle \langle noun \rangle \langle verb \rangle$
 $\Rightarrow the \langle noun \rangle \langle verb \rangle$
 $\Rightarrow the \ dog \langle verb \rangle$
 $\Rightarrow the \ dog \ sleeps$



» Derivation of string “a cat runs” :

$\langle sentence \rangle \Rightarrow \langle noun_phrase \rangle \langle predicate \rangle$
 $\Rightarrow \langle noun_phrase \rangle \langle verb \rangle$
 $\Rightarrow \langle article \rangle \langle noun \rangle \langle verb \rangle$
 $\Rightarrow a \langle noun \rangle \langle verb \rangle$
 $\Rightarrow a \ cat \langle verb \rangle$
 $\Rightarrow a \ cat \ runs$



» Language of the grammar:

$$L = \{ \text{"a cat runs"}, \\ \text{"a cat sleeps"}, \\ \text{"the cat runs"}, \\ \text{"the cat sleeps"}, \\ \text{"a dog runs"}, \\ \text{"a dog sleeps"}, \\ \text{"the dog runs"}, \\ \text{"the dog sleeps"} \}$$


Productions

Sequence of
Terminals (symbols)

$\langle \textit{noun} \rangle \rightarrow \textit{cat}$

$\langle \textit{sentence} \rangle \rightarrow \langle \textit{noun_phrase} \rangle \langle \textit{predicate} \rangle$

Variables

Sequence of Variables



Another Example

Sequence of
terminals and variables

Grammar:

$$S \rightarrow aSb$$

$$S \rightarrow \varepsilon$$

Variable

The right side
may be ε



» Grammar:

$$S \rightarrow aSb$$

$$S \rightarrow \varepsilon$$

» Derivation of string : ab

$$S \Rightarrow aSb \Rightarrow ab$$
$$S \rightarrow aSb \qquad S \rightarrow \varepsilon$$



» Grammar:

$$S \rightarrow aSb$$

$$S \rightarrow \varepsilon$$

» Derivation of string :

aabb

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$



$$S \rightarrow aSb$$

$$S \rightarrow \varepsilon$$



Grammar: $S \rightarrow aSb$

$S \rightarrow \varepsilon$

Other derivations:

$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbbb$

$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb$
 $\Rightarrow aaaaSbbbb \Rightarrow aaabbbbb$



Grammar: $S \rightarrow aSb$
 $S \rightarrow \varepsilon$

Language of the grammar:

$$L = \{a^n b^n : n \geq 0\}$$



A Convenient Notation

» We write: $S \xRightarrow{*} aaabbb$

for one or more derivation steps

» Instead of:

$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaasbbb \Rightarrow aaabbbb$



In general we write:

$$w_1 \xRightarrow{*} w_n$$

If: $w_1 \Rightarrow w_2 \Rightarrow w_3 \Rightarrow \cdots \Rightarrow w_n$

in zero or more derivation steps

Trivially: $w \xRightarrow{*} w$



Example Grammar

$$S \rightarrow aSb$$

$$S \rightarrow \varepsilon$$

Possible Derivations

$$S \overset{*}{\Rightarrow} \varepsilon$$

$$S \overset{*}{\Rightarrow} ab$$

$$S \overset{*}{\Rightarrow} aaabbb$$

$$S \overset{*}{\Rightarrow} aaSbb \overset{*}{\Rightarrow} aaaaaSbbbb$$



Another convenient notation:

$$S \rightarrow aSb$$

$$S \rightarrow \varepsilon$$



$$S \rightarrow aSb \mid \varepsilon$$

$$\langle \textit{article} \rangle \rightarrow a$$

$$\langle \textit{article} \rangle \rightarrow \textit{the}$$



$$\langle \textit{article} \rangle \rightarrow a \mid \textit{the}$$



Formal Definition

Grammar: $G = (V, T, S, P)$

Set of
variables

Set of
terminal
symbols

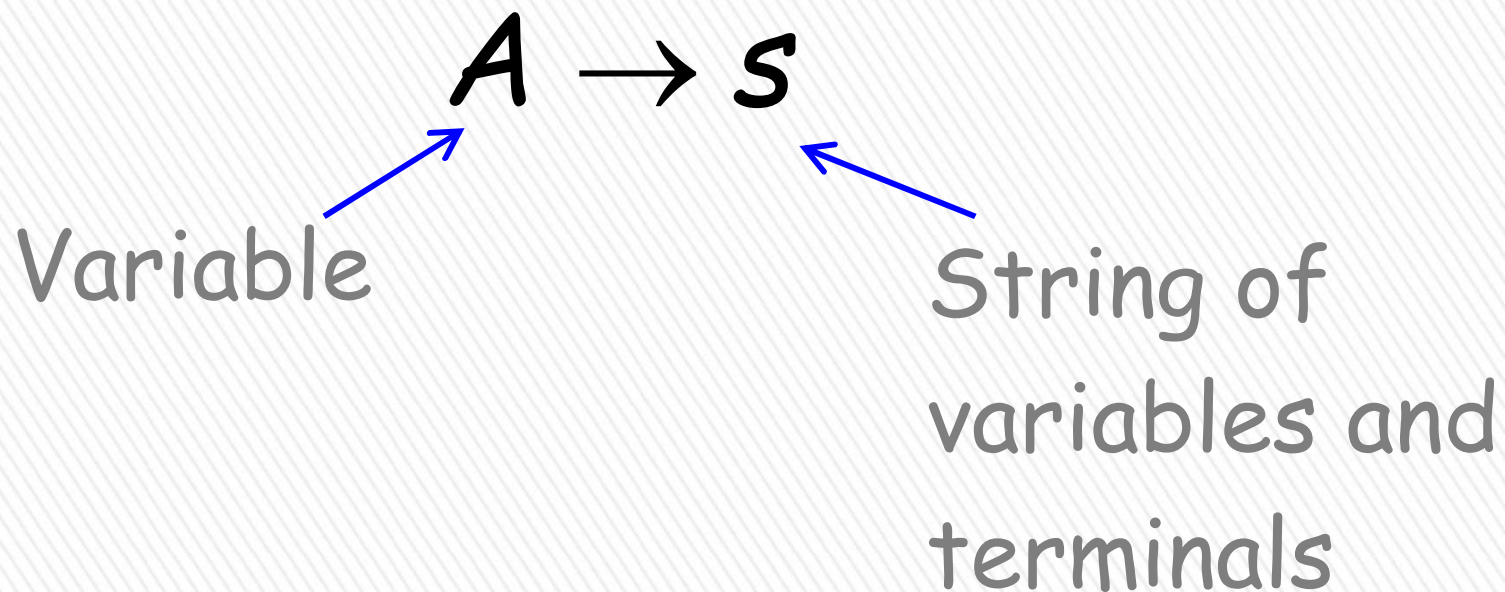
Start
variable

Set of
productions



Context-Free Grammar: $G = (V, T, S, P)$

All productions in P are of the form



Example for Context-Free Grammar

$$S \rightarrow aSb \mid \varepsilon$$

productions

$$P = \{S \rightarrow aSb, S \rightarrow \varepsilon\}$$

$$G = (V, T, S, P)$$

$$V = \{S\}$$

variables

$$T = \{a, b\}$$

terminals

start variable



Language of a Grammar:

» For a grammar G with start variable S

$$L(G) = \{w: S \Rightarrow^* w, \quad w \in T^*\}$$

String of terminals or ε



Example:

context-free grammar $G : \boxed{S \rightarrow aSb \mid \varepsilon}$

$$L(G) = \{a^n b^n : n \geq 0\}$$

Since, there is derivation

$$S \xRightarrow{*} a^n b^n \quad \text{for any } n \geq 0 \rangle$$

Context-Free Language:

- » A language L is context-free
- » if there is a context-free grammar G
- » with $L = L(G)$



Example:

$$L = \{a^n b^n : n \geq 0\}$$

is a context-free language

since context-free grammar G :

$$S \rightarrow aSb \mid \varepsilon$$

generates $L(G) = L$



Another Example

Context-free grammar G :

$$S \rightarrow aSa \mid bSb \mid \varepsilon$$

Example derivations:

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abba$$

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abaSaba \Rightarrow abaaba$$

$$L(G) = \{ww^R : w \in \{a,b\}^*\}$$

Palindromes of even length



Another Example

Context-free grammar G :

$$S \rightarrow aSb \mid SS \mid \varepsilon$$

Example derivations:

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow ab$$

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow abaSb \Rightarrow abab$$

$$L(G) = \{w : n_a(w) = n_b(w),$$

$$\text{and } n_a(v) \geq n_b(v)$$

in any prefix v \}

Describes
matched

parentheses:

$() ((())) (())$ $a = ($, $b =)$



Derivation Trees

Derivation Order

Consider the following example grammar with 5 productions:

- | | | |
|-----------------------|--------------------------------|--------------------------------|
| 1. $S \rightarrow AB$ | 2. $A \rightarrow aaA$ | 4. $B \rightarrow Bb$ |
| | 3. $A \rightarrow \varepsilon$ | 5. $B \rightarrow \varepsilon$ |



| | | |
|-----------------------|--------------------------------|--------------------------------|
| 1. $S \rightarrow AB$ | 2. $A \rightarrow aaA$ | 4. $B \rightarrow Bb$ |
| | 3. $A \rightarrow \varepsilon$ | 5. $B \rightarrow \varepsilon$ |

Leftmost derivation order of string *aab*:

$$\begin{array}{ccccccccc} & 1 & & 2 & & 3 & & 4 & & 5 \\ S & \Rightarrow & AB & \Rightarrow & aaAB & \Rightarrow & aaB & \Rightarrow & aaBb & \Rightarrow & aab \end{array}$$

At each step, we substitute the
leftmost variable



| | | |
|-----------------------|--------------------------------|--------------------------------|
| 1. $S \rightarrow AB$ | 2. $A \rightarrow aaA$ | 4. $B \rightarrow Bb$ |
| | 3. $A \rightarrow \varepsilon$ | 5. $B \rightarrow \varepsilon$ |

Rightmost derivation order of string *aab* :

$$\begin{array}{ccccccccc}
 & 1 & & 4 & & 5 & & 2 & & 3 \\
 S & \Rightarrow & AB & \Rightarrow & ABb & \Rightarrow & Ab & \Rightarrow & aaAb & \Rightarrow & aab
 \end{array}$$

At each step, we substitute the
rightmost variable



- | | | |
|-----------------------|--------------------------------|--------------------------------|
| 1. $S \rightarrow AB$ | 2. $A \rightarrow aaA$ | 4. $B \rightarrow Bb$ |
| | 3. $A \rightarrow \varepsilon$ | 5. $B \rightarrow \varepsilon$ |

Leftmost derivation of aab :

$$\begin{array}{ccccccccc} & 1 & & 2 & & 3 & & 4 & & 5 \\ S & \Rightarrow & AB & \Rightarrow & aaAB & \Rightarrow & aaB & \Rightarrow & aaBb & \Rightarrow & aab \end{array}$$

Rightmost derivation of aab :

$$\begin{array}{ccccccccc} & 1 & & 4 & & 5 & & 2 & & 3 \\ S & \Rightarrow & AB & \Rightarrow & ABb & \Rightarrow & Ab & \Rightarrow & aaAb & \Rightarrow & aab \end{array} \quad \triangleright$$

Consider the same example grammar:

$$S \rightarrow AB \qquad A \rightarrow aaA \mid \varepsilon \qquad B \rightarrow Bb \mid \varepsilon$$

And a derivation of *aab*:

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aab$$

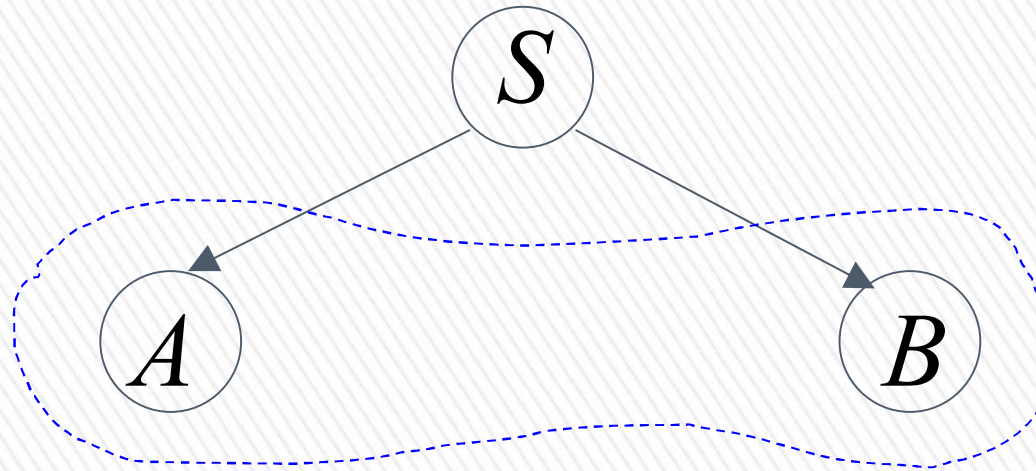


Derivation Tree

$$S \rightarrow AB$$

$$A \rightarrow aaA \mid \varepsilon \quad B \rightarrow Bb \mid \varepsilon$$

$$S \Rightarrow AB$$



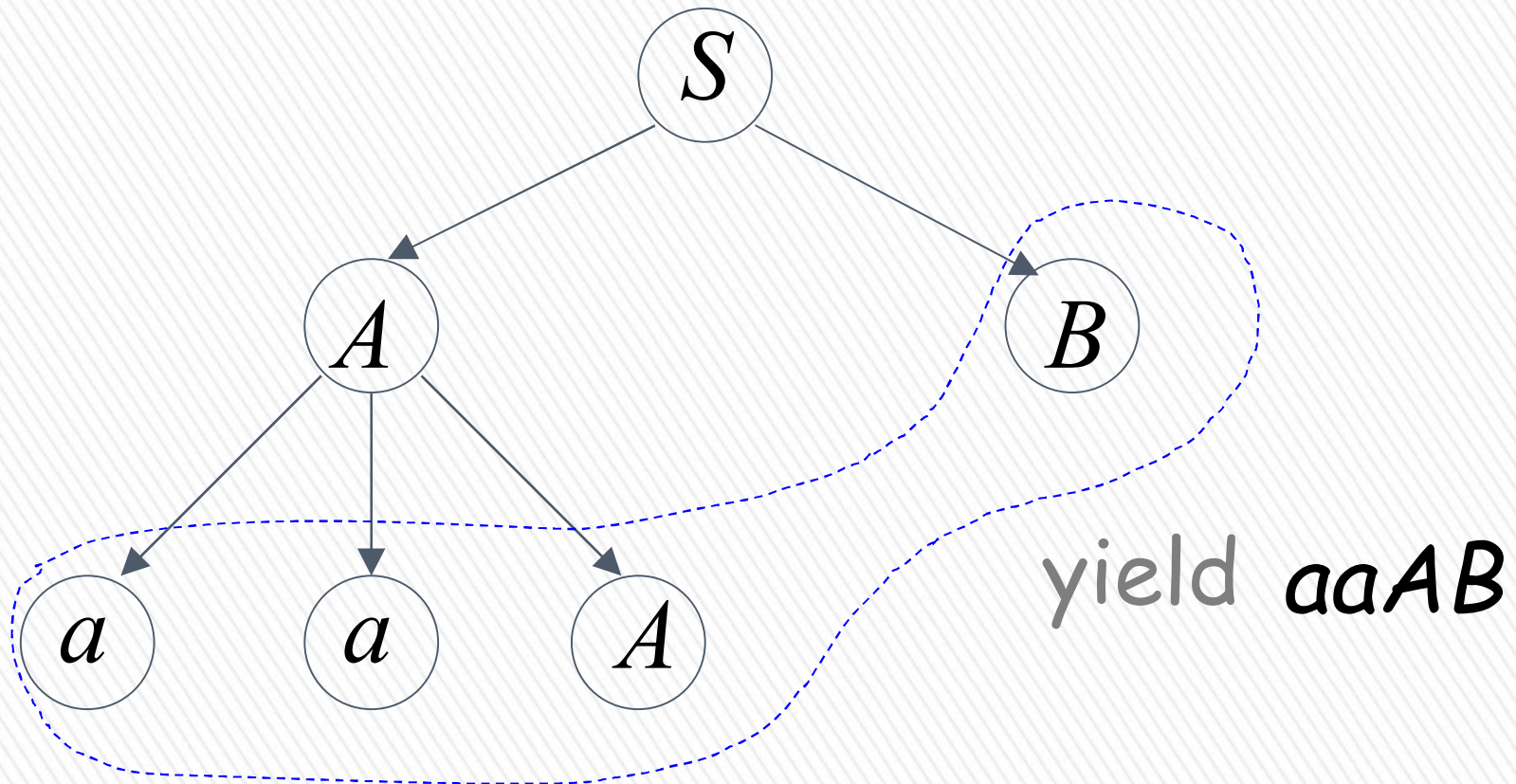
yield AB



$$S \rightarrow AB$$

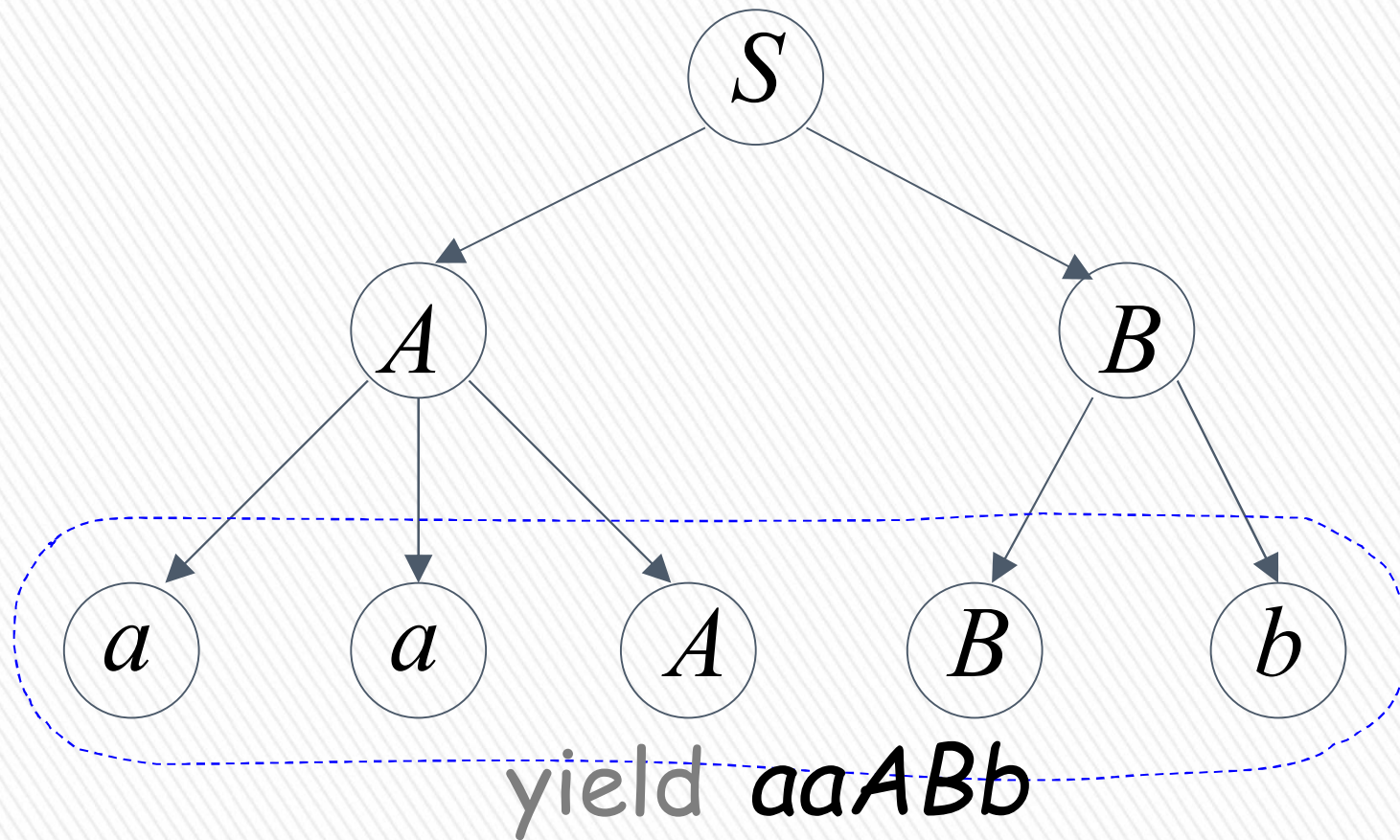
$$A \rightarrow aaA \mid \varepsilon \quad B \rightarrow Bb \mid \varepsilon$$

$$S \Rightarrow AB \Rightarrow aaAB$$



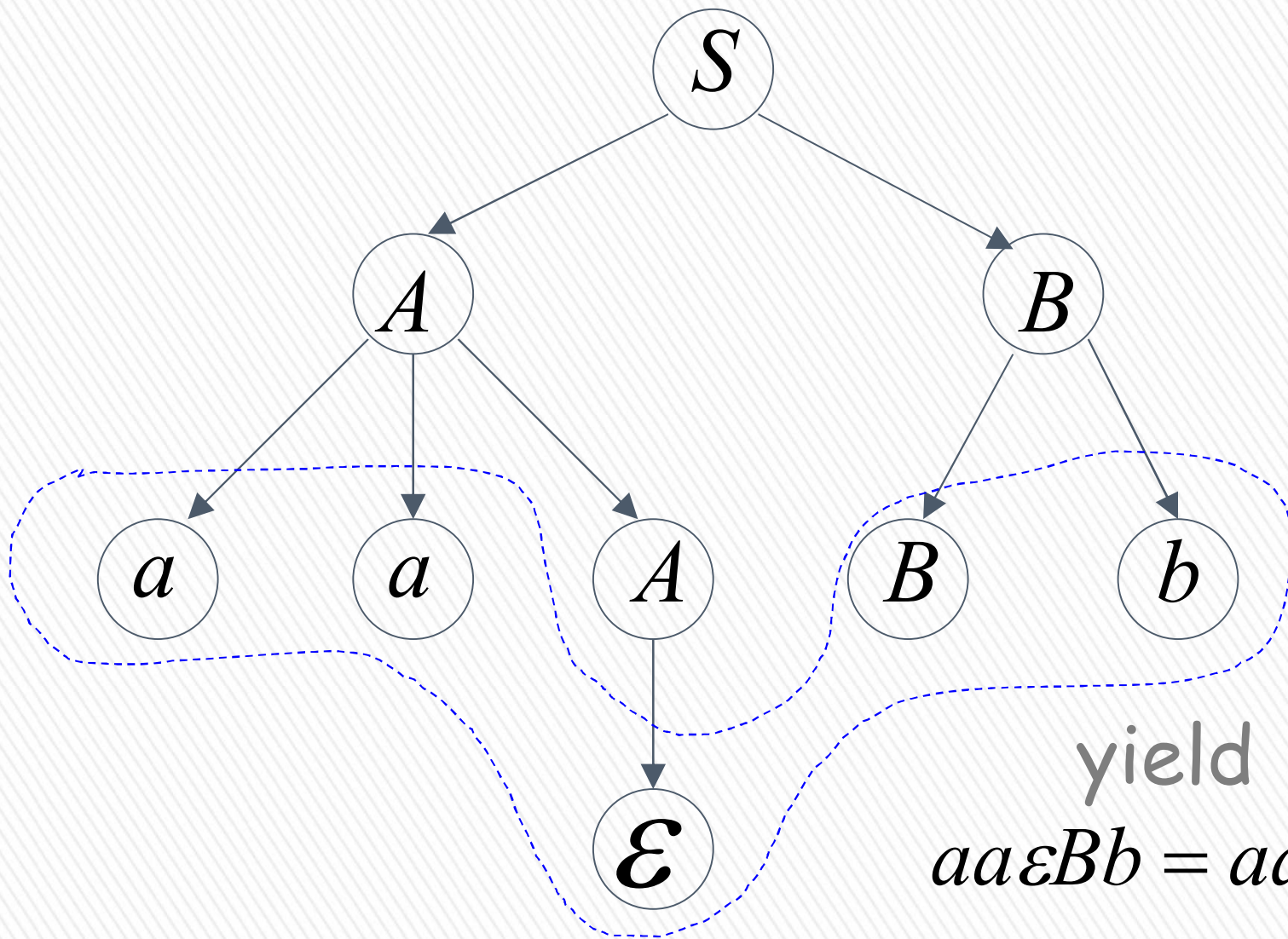
$$S \rightarrow AB \qquad A \rightarrow aaA \mid \varepsilon \qquad B \rightarrow Bb \mid \varepsilon$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb$$



$$S \rightarrow AB \qquad A \rightarrow aaA \mid \varepsilon \qquad B \rightarrow Bb \mid \varepsilon$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb$$

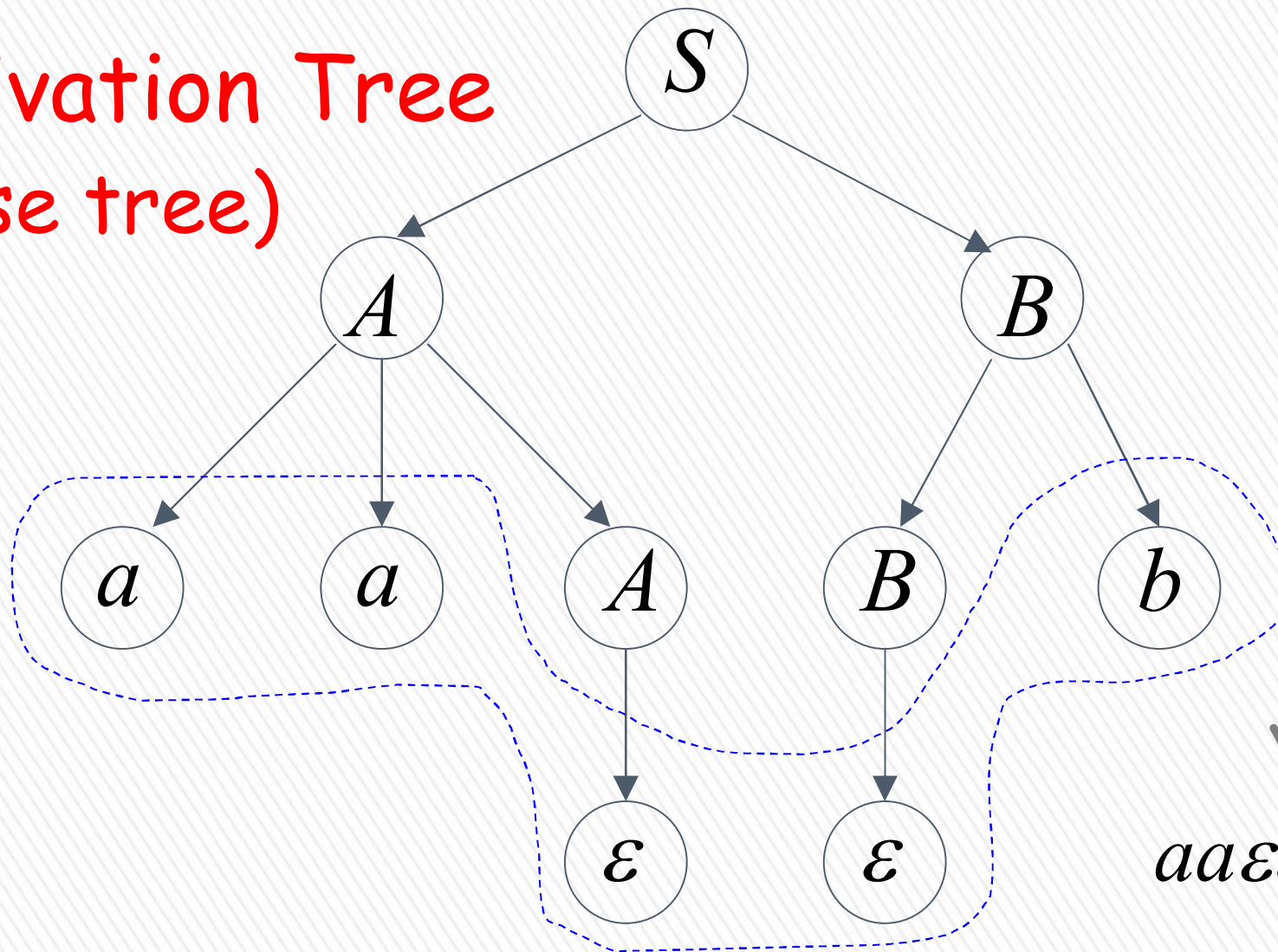


$$S \rightarrow AB$$

$$A \rightarrow aaA \mid \varepsilon \quad B \rightarrow Bb \mid \varepsilon$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aab$$

Derivation Tree
(parse tree)



yield \triangleright
 $aa\varepsilon\varepsilon b = aab$

Sometimes, derivation order doesn't matter

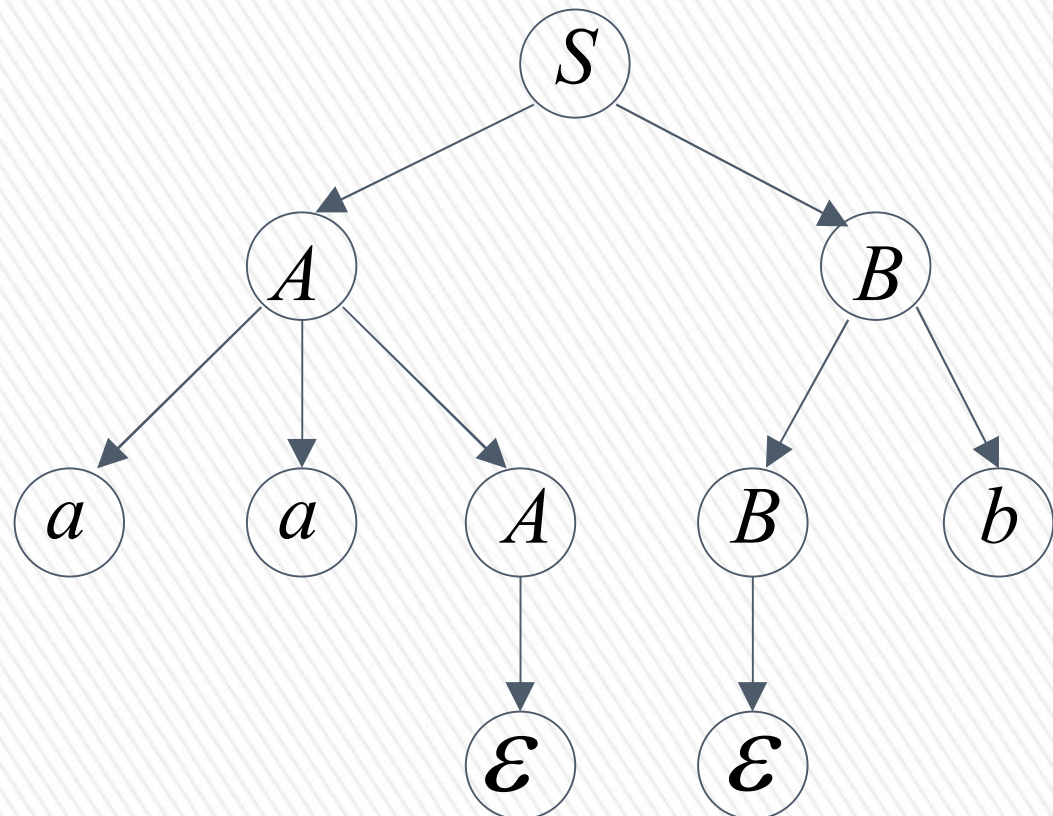
Leftmost derivation:

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaB \Rightarrow aaBb \Rightarrow aab$$

Rightmost derivation:

$$S \Rightarrow AB \Rightarrow ABb \Rightarrow Ab \Rightarrow aaAb \Rightarrow aab$$

Give same
derivation tree





Ambiguity

Grammar for mathematical expressions

$$E \rightarrow E + E \mid E * E \mid (E) \mid a$$

Example strings:

$$(a + a) * a + (a + a * (a + a))$$

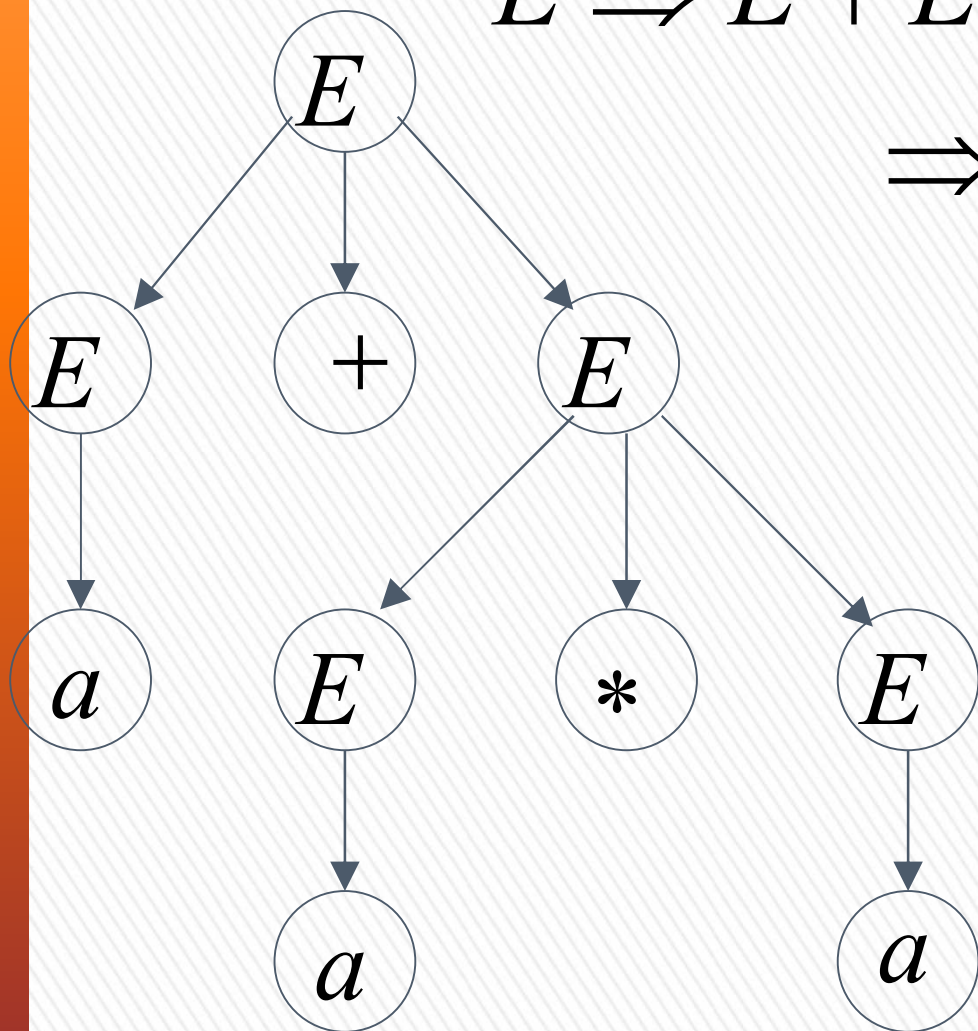


Denotes any number



$$E \rightarrow E + E \mid E * E \mid (E) \mid a$$

$$\begin{aligned} E &\Rightarrow E + E \Rightarrow a + E \Rightarrow a + E * E \\ &\Rightarrow a + a * E \Rightarrow a + a * a \end{aligned}$$



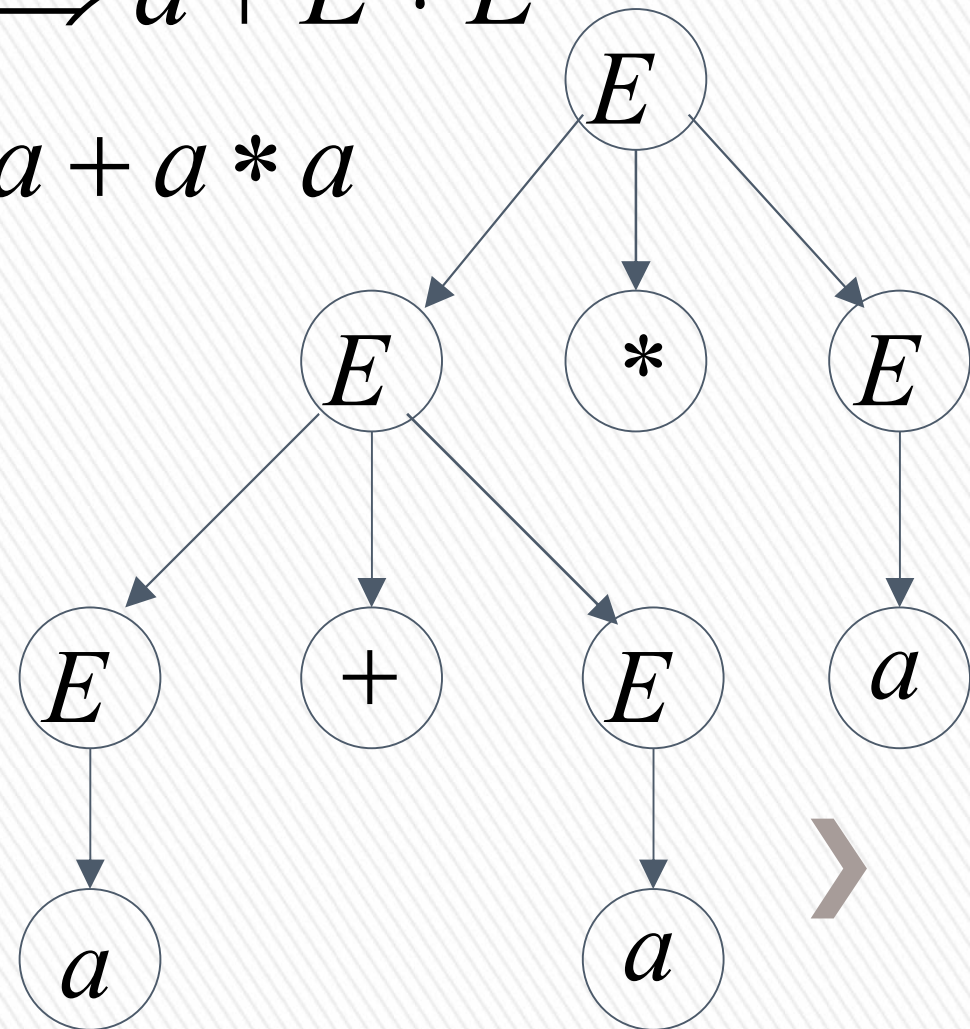
A leftmost derivation
for $a + a * a$



$$E \rightarrow E + E \mid E * E \mid (E) \mid a$$

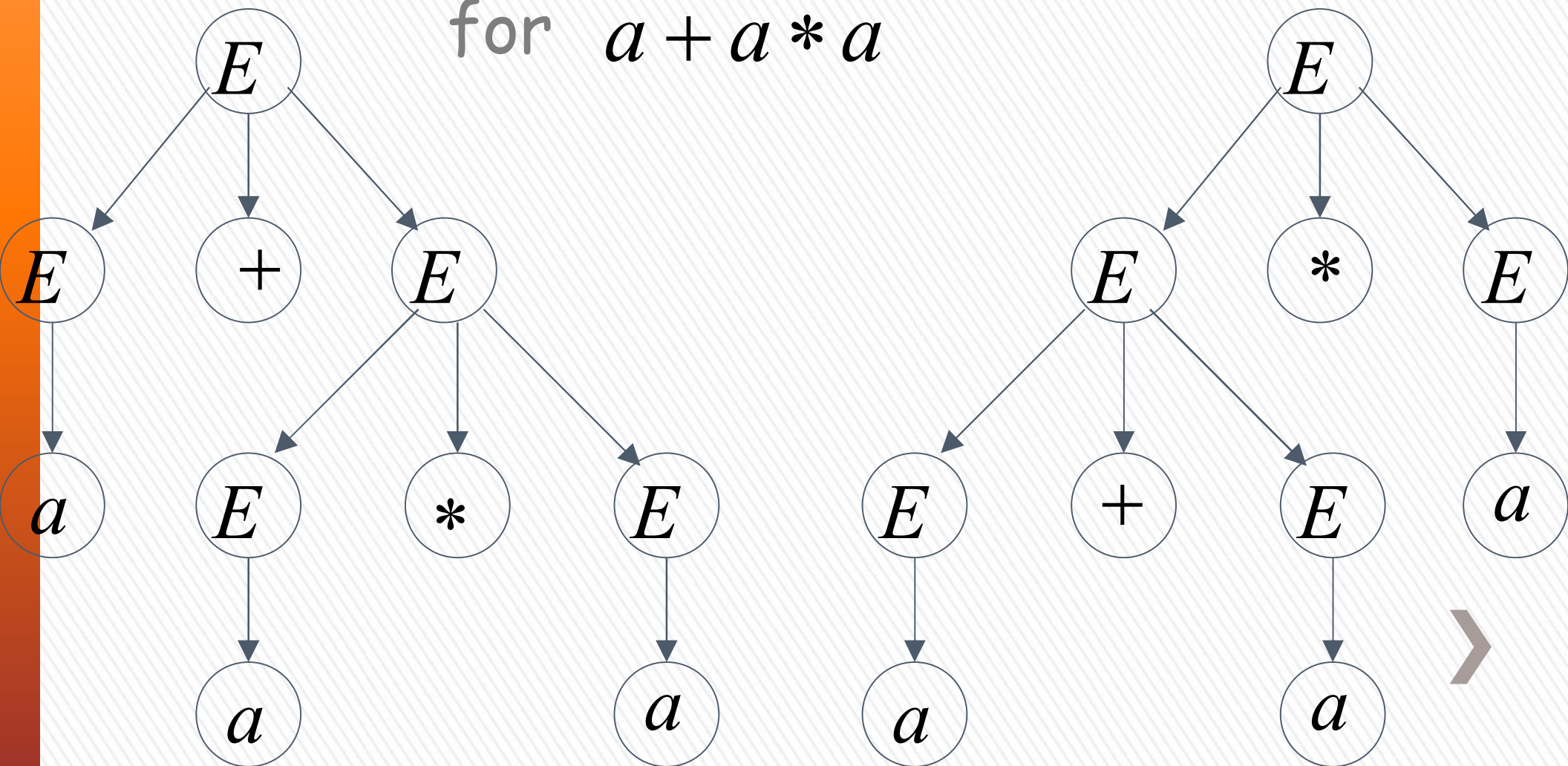
$$E \Rightarrow E * E \Rightarrow E + E * E \Rightarrow a + E * E \\ \Rightarrow a + a * E \Rightarrow a + a * a$$

Another
leftmost derivation
for $a + a * a$



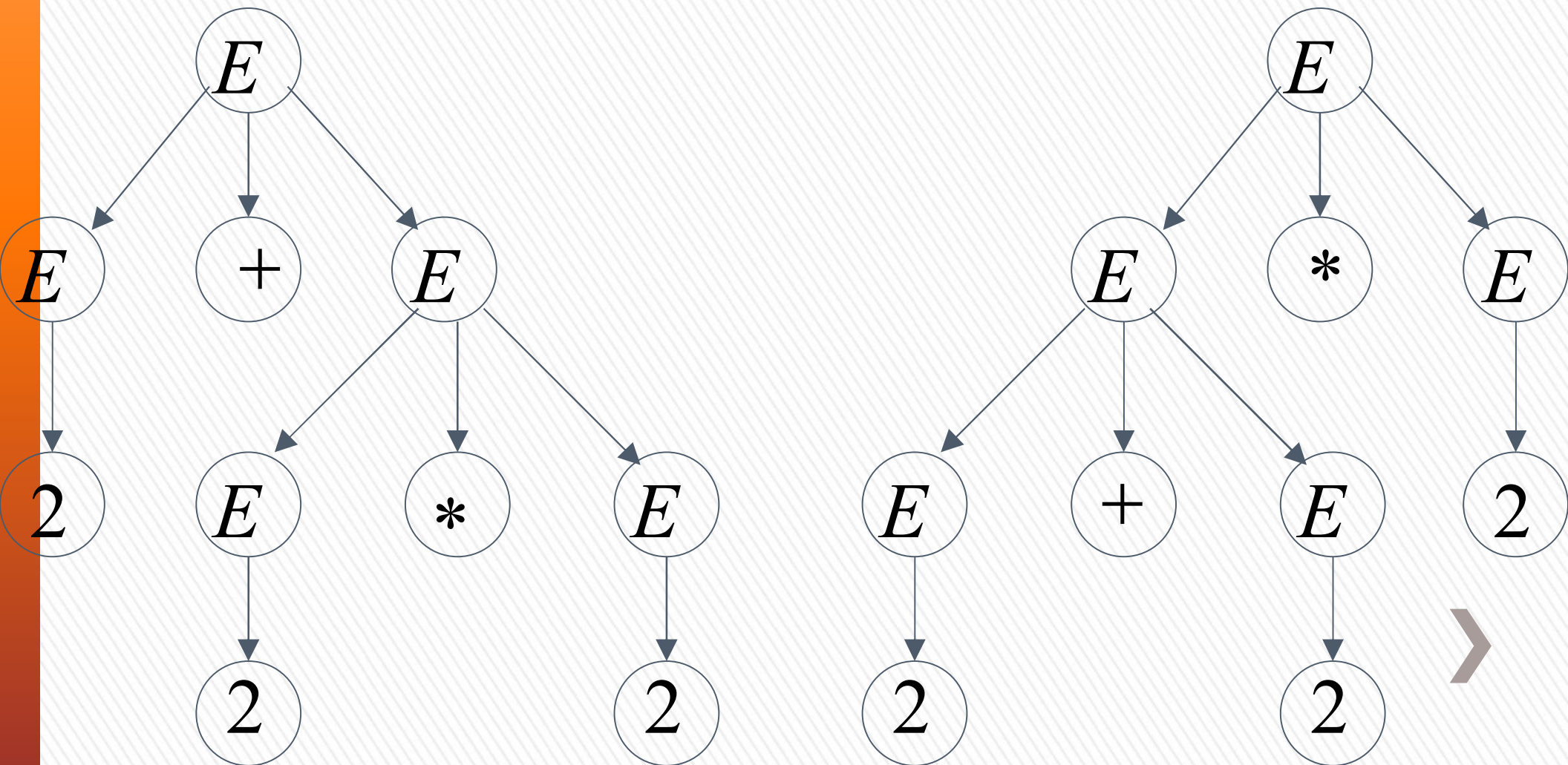
$$E \rightarrow E + E \mid E * E \mid (E) \mid a$$

Two derivation trees
for $a + a * a$



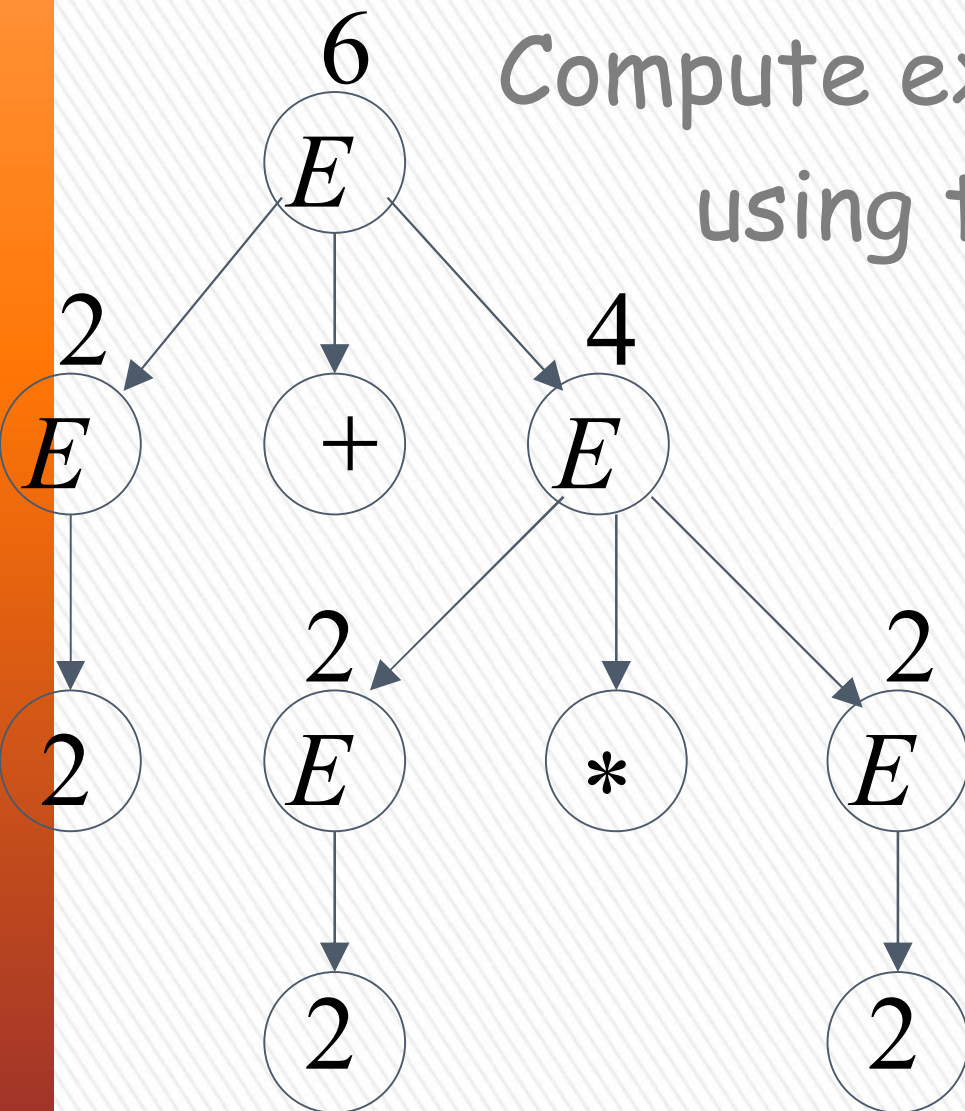
take $a = 2$

$$a + a * a = 2 + 2 * 2$$



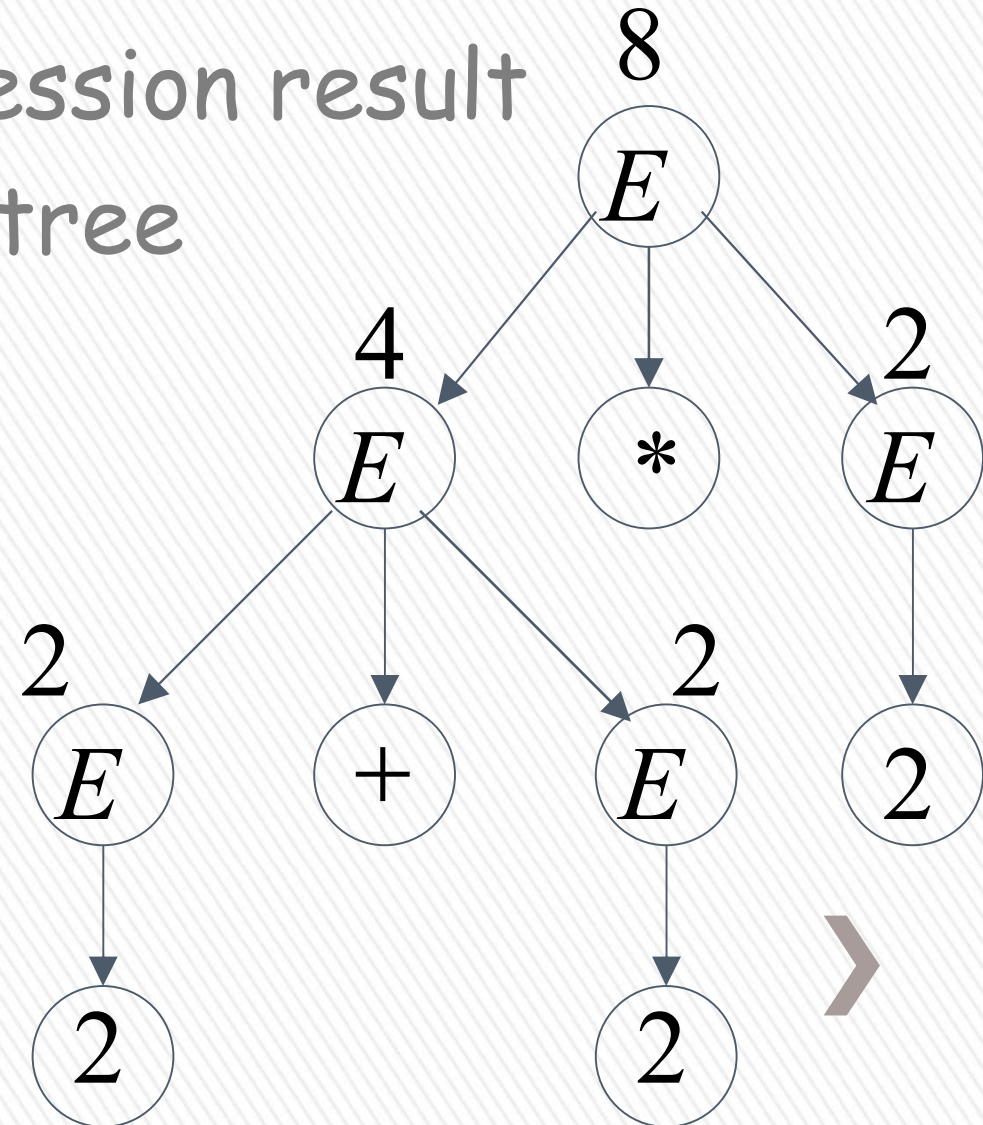
Good Tree

$$2 + 2 * 2 = 6$$



Bad Tree

$$2 + 2 * 2 = 8$$



Compute expression result
using the tree

Two different derivation trees
may cause problems in applications which
use the derivation trees:

- Evaluating expressions
- In general, in compilers
for programming languages



Ambiguous Grammar:

A context-free grammar G is ambiguous if there is a string $w \in L(G)$ which has:

two different derivation trees

or

two leftmost derivations

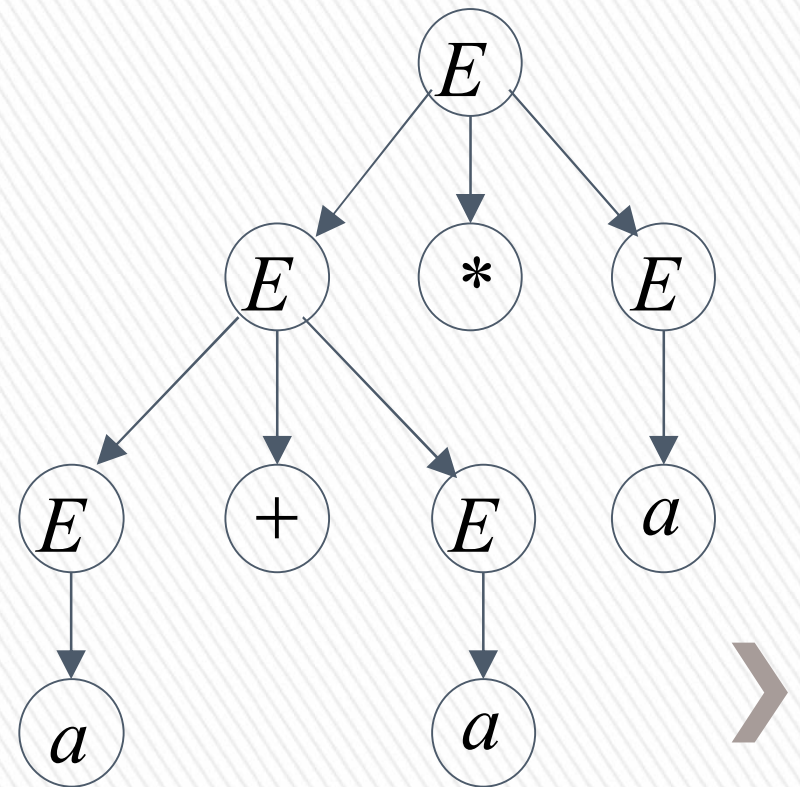
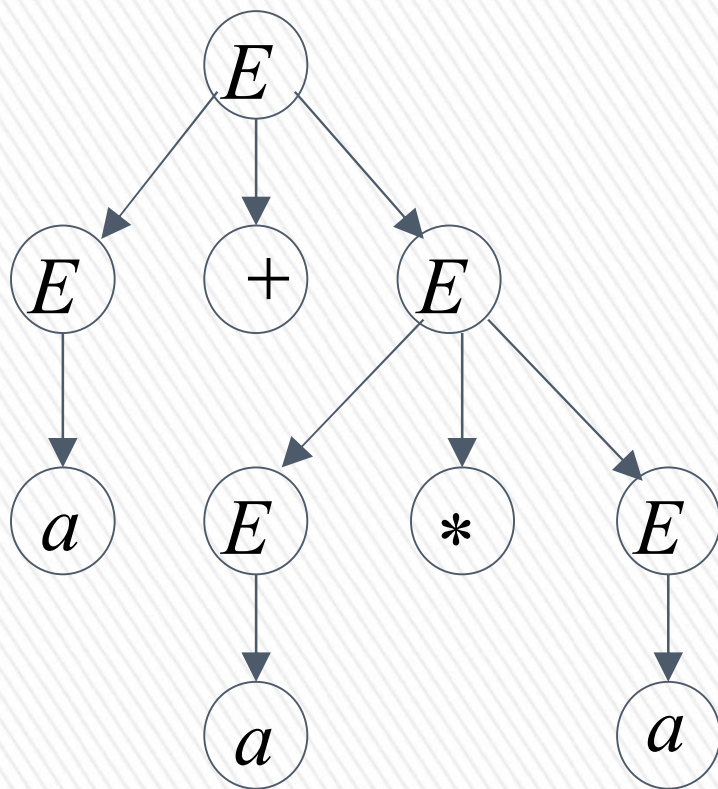
(Two different derivation trees give two different leftmost derivations and vice-versa)



Example:

$$E \rightarrow E + E \mid E * E \mid (E) \mid a$$

this grammar is ambiguous since
string $a + a * a$ has two derivation trees



$$E \rightarrow E + E \mid E * E \mid (E) \mid a$$

this grammar is ambiguous also because
string $a + a * a$ has two leftmost derivations

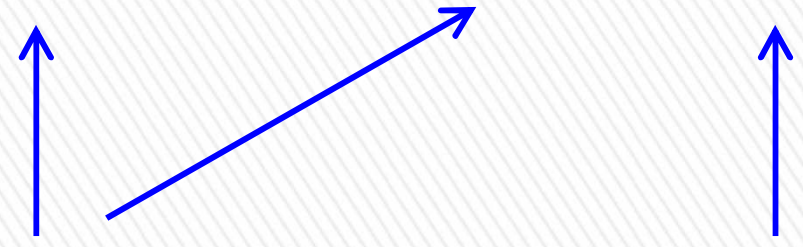
$$\begin{aligned} E &\Rightarrow E + E \Rightarrow a + E \Rightarrow a + E * E \\ &\Rightarrow a + a * E \Rightarrow a + a * a \end{aligned}$$

$$\begin{aligned} E &\Rightarrow E * E \Rightarrow E + E * E \Rightarrow a + E * E \\ &\Rightarrow a + a * E \Rightarrow a + a * a \end{aligned}$$



Another ambiguous grammar:

IF_STMT \rightarrow if EXPR then STMT
 | if EXPR then STMT else STMT

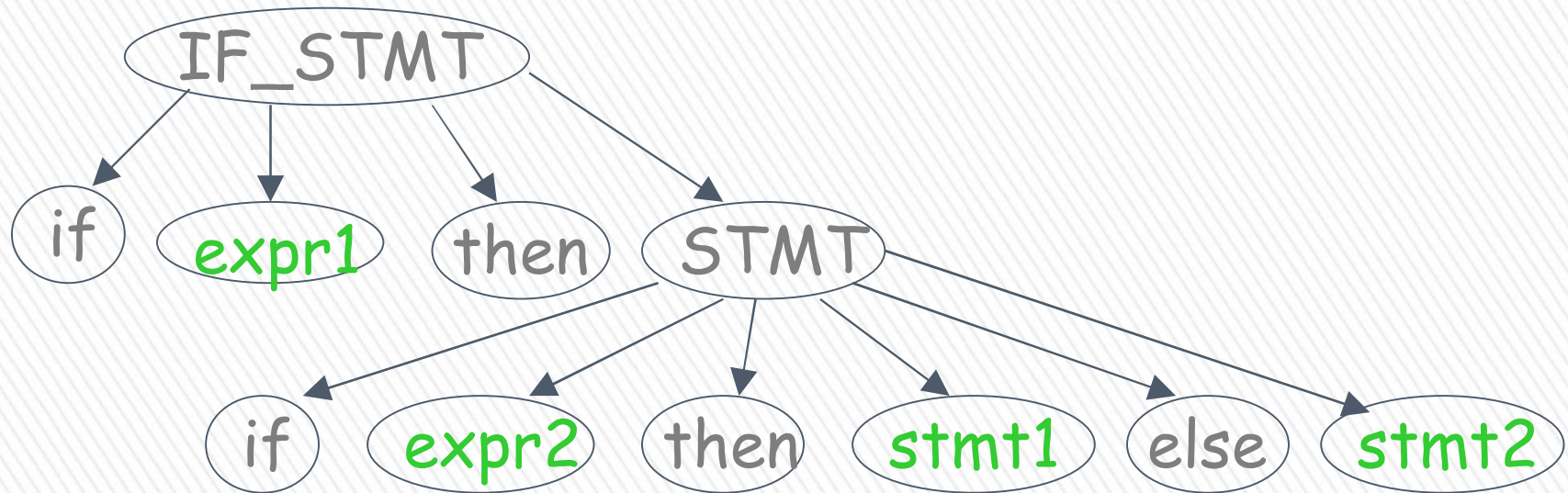


Variables Terminals

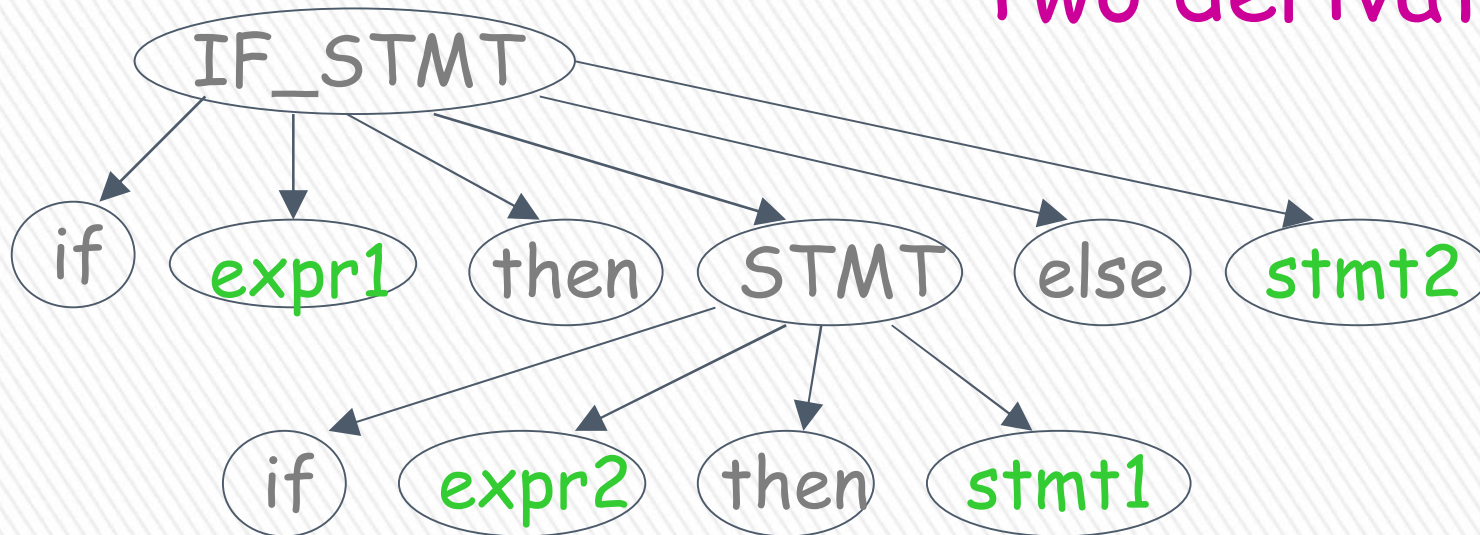
Very common piece of grammar
in programming languages



If *expr1* then if *expr2* then *stmt1* else *stmt2*



Two derivation trees



In general, ambiguity is bad
and we want to remove it

Sometimes it is possible to find
a non-ambiguous grammar for a language

But, in general we cannot do so



A successful example:

Ambiguous
Grammar

$$E \rightarrow E + E$$

$$E \rightarrow E * E$$

$$E \rightarrow (E)$$

$$E \rightarrow a$$

Equivalent

Non-Ambiguous
Grammar

$$E \rightarrow E + T \mid T$$

$$T \rightarrow T * F \mid F$$

$$F \rightarrow (E) \mid a$$

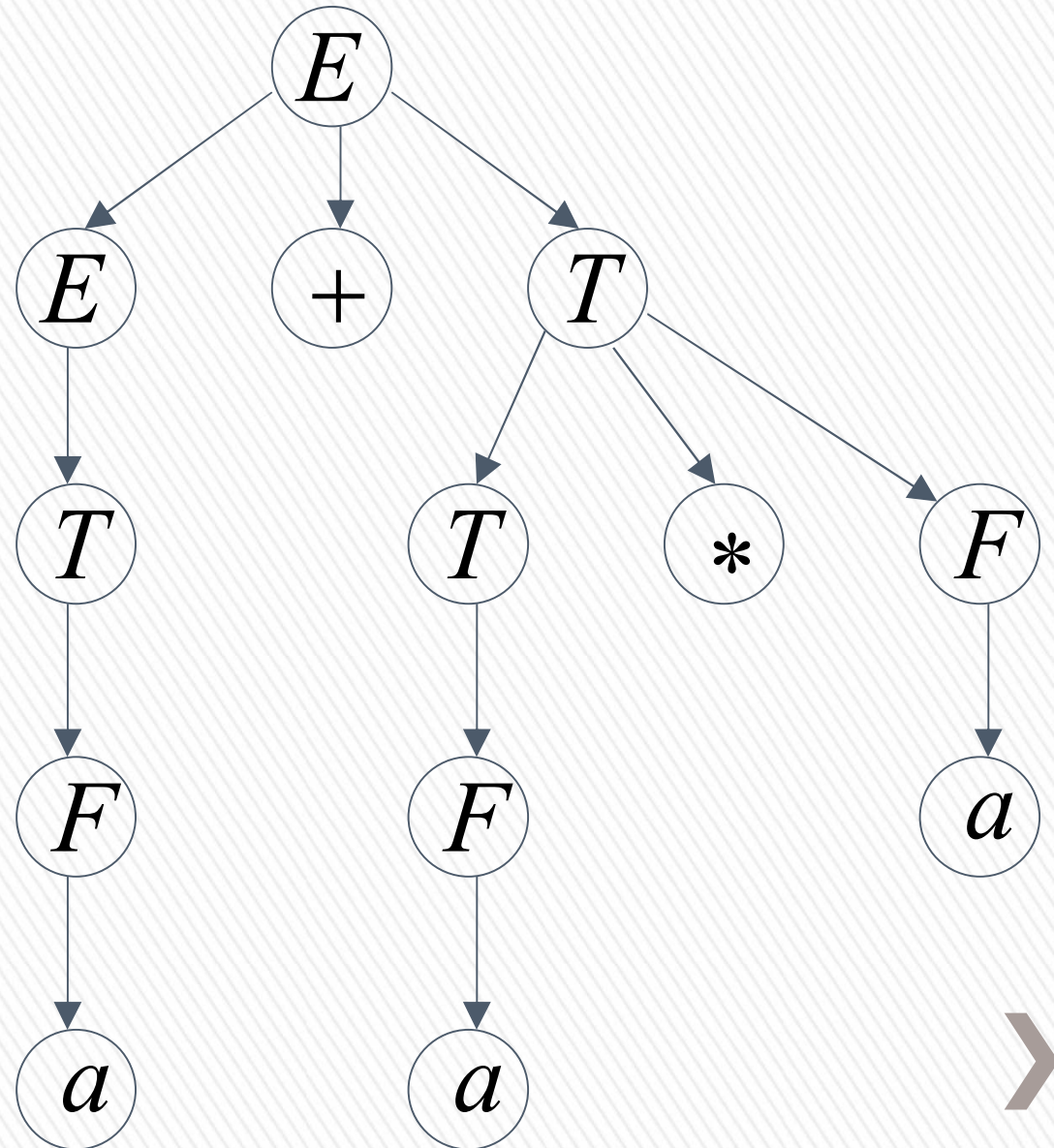
generates the same
language



$$\begin{aligned}
 E &\Rightarrow E + T \Rightarrow T + T \Rightarrow F + T \Rightarrow a + T \Rightarrow a + T * F \\
 &\Rightarrow a + F * F \Rightarrow a + a * F \Rightarrow a + a * a
 \end{aligned}$$

$$\begin{aligned}
 E &\rightarrow E + T \mid T \\
 T &\rightarrow T * F \mid F \\
 F &\rightarrow (E) \mid a
 \end{aligned}$$

Unique
derivation tree
for $a + a * a$



An un-successful example:

$$L = \{a^n b^n c^m\} \cup \{a^n b^m c^m\}$$

$$n, m \geq 0$$

L is inherently ambiguous:

every grammar that generates this language is ambiguous



Example (ambiguous) grammar for L :

$$L = \{a^n b^n c^m\} \cup \{a^n b^m c^m\}$$


$$S \rightarrow S_1 \mid S_2$$


$$S_1 \rightarrow S_1 c \mid A$$

$$A \rightarrow aAb \mid \lambda$$


$$S_2 \rightarrow aS_2 \mid B$$

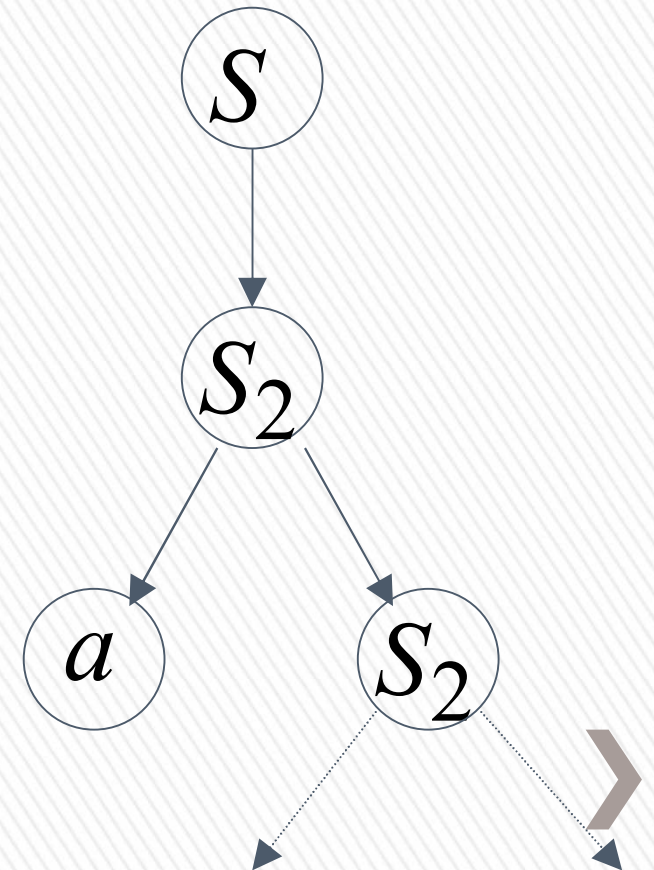
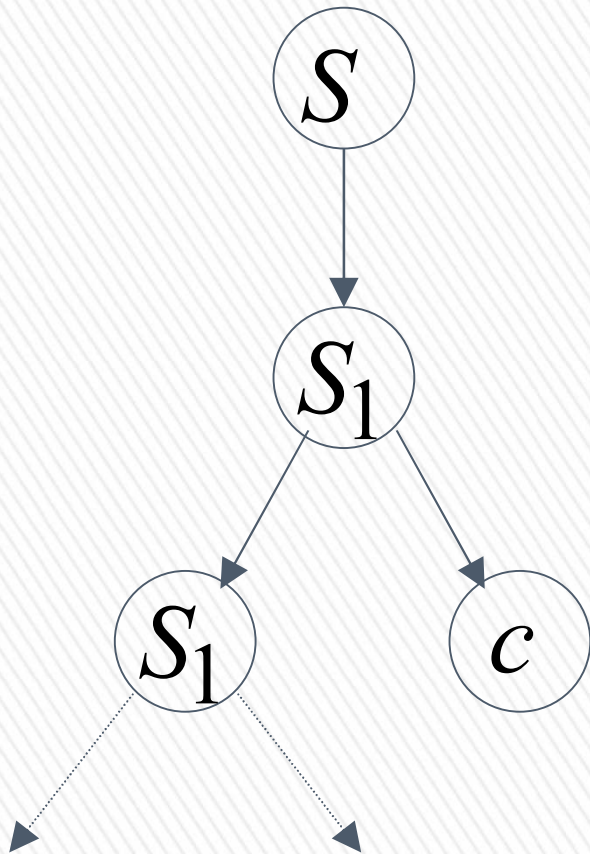
$$B \rightarrow bBc \mid \lambda$$



The string $a^n b^n c^n \in L$

has always two different derivation trees
(for any grammar)

For example



BLM2502 Theory of Computation

