MAT 1072 / Matematik 2

Parametrik Denklemler / Kutupsal Koordinatlar

① $x = 4\sin t$, $y = 2\cos t$ eğrisinin $t = \frac{\pi}{4}$ deki teğetinin denklemini bulunuz.

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\sin t}{4\cos t}\bigg|_{t=\frac{\pi}{4}} = -\frac{1}{2}$$

$$t = \frac{\pi}{4} \implies \begin{array}{l} x_0 = 2\sqrt{2} \\ y_0 = \sqrt{2} \end{array}$$

Denklem: $y - y_0 = m(x - x_0) \implies y - \sqrt{2} = -\frac{1}{2}(x - 2\sqrt{2}) \implies y = -\frac{1}{2}x + 2\sqrt{2}$

② $x = x(t)$, $y = y(t)$ olarak tanımlandıklarını kabul ederek,

$$\left.\begin{array}{l} t^2\sin x + x^3 = e^t \\ y = t\sin t - 2t \end{array}\right\}$$ parametrik denklemleri ile verilen eğrinin $t=0$ daki teğet doğrusunun eğimini bulunuz.

$t = 0 \implies (x, y) = (1, 0)$

$t^2\sin x + x^3 = e^t \implies 2t\sin x + t^2\cos x \frac{dx}{dt} + 3x^2\frac{dx}{dt} = e^t$

$$\left.\begin{array}{l} t=0 \\ x=1 \end{array}\right\} 0 + 0 + \frac{3dx}{dt} = 1 \implies \frac{dx}{dt}\bigg|_{t=0} = \frac{1}{3}$$

$y = t\sin t - 2t \implies \frac{dy}{dt} = \sin t + t\cos t - 2 \implies \frac{dy}{dt}\bigg|_{t=0} = -2$

$$\frac{dy}{dx}\bigg|_{t=0} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2}{\frac{1}{3}} = -6$$

③ $\left.\begin{array}{l} x = 2t^2 + 3 \\ y = t^4 \end{array}\right\}$ parametrik denklemleri ile verilen eğrinin $t = -1$ noktasındaki teğet doğrusunun denklemini yazınız.

$t = -1 \implies \left.\begin{array}{l} x = 5 \\ y = 1 \end{array}\right\}$ $y - 1 = m \cdot (x - 5)$

$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t^3}{4t} = t^2 \implies m\big|_{t=-1} = 1 \implies y - 1 = x - 5$

$\implies y = x - 4$

④ $x = 8\cos t + 8t\sin t$
$y = 8\sin t - 8t\cos t$ } parametrizasyonu ile verilen eğrinin uzunluğunu bulunuz

$0 \leq t \leq \frac{\pi}{2}$

$$L = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$\frac{dx}{dt} = -8\sin t + 8\sin t + 8t\cos t \Rightarrow \left(\frac{dx}{dt}\right)^2 = 64t^2\cos^2 t$

$\frac{dy}{dt} = 8\cos t - 8\cos t + 8t\sin t \Rightarrow \left(\frac{dy}{dt}\right)^2 = 64t^2\sin^2 t$

$$L = \int_0^{\pi/2} \sqrt{64t^2} = \int_0^{\pi/2} 8t \, dt = 4t^2 \Big|_0^{\pi/2} = \pi^2$$

⑤ $-1 \leq t \leq 0$ olmak üzere $x(t) = t^2$, $y(t) = 1 - t^2$ ile çizilmiş yolun uzunluğunu bulunuz.

$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = -2t$

$$L = \int_{-1}^0 \sqrt{(2t)^2 + (-2t)^2} \, dt = 2\sqrt{2} \int_{-1}^0 |t| \, dt = -2\sqrt{2} \int_{-1}^0 t \, dt = \sqrt{2}$$

⑥ $x = 2t^2 + 3$
$y = t^4$ } parametrik denklemi ile verilen eğrinin $t = -1$ deki normal doğrusunun denklemini bulunuz.

$t = -1$ } $x = 5$
$y = 1$ $\Rightarrow$ (5,1) noktasından geçen normal doğrunun denklemi:

$$y - 1 = m_N (x - 5)$$

$m_T = \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{4t^3}{4t} = t^2 \Rightarrow m_T\Big|_{t=-1} = (-1)^2 = 1$
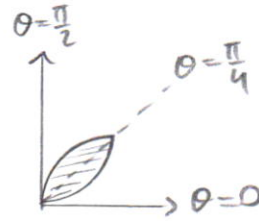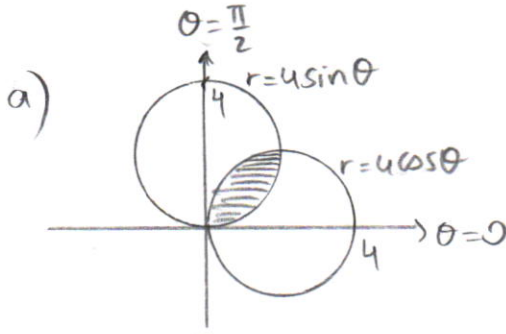
$m_T \cdot m_N = -1 \Rightarrow m_N = -1$

$y - 1 = -1(x - 5) \Rightarrow y = 6 - x$

7) **a)** $r = 4\cos\theta$ ⎱ eğrilerinin sınırladığı ortak alanı
$\quad\quad r = 4\sin\theta$ ⎰

b) $r = 4\cos\theta$ içinde ⎱ kalan alanı
$\quad r = 4\sin\theta$ dışında ⎰

c) $r = 4\cos\theta$ dışında ⎱ kalan alanı bulunuz. (integralleri hesaplamayın)
$\quad r = 4\sin\theta$ içinde ⎰

a)

$\theta = \dfrac{\pi}{2}$, $r = 4\sin\theta$, $r = 4\cos\theta$, $\theta = 0$

$\theta = \dfrac{\pi}{2}$, $\theta = \dfrac{\pi}{4}$, $\theta = 0$

$4\cos\theta = 4\sin\theta$

$\theta = \dfrac{\pi}{4}$

$$A = \frac{1}{2}\int_0^{\pi/4}(4\sin\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2}(4\cos\theta)^2\,d\theta$$

b)

$\theta = \dfrac{\pi}{2}$, $r = 4\sin\theta$, $\theta = \dfrac{\pi}{4}$, $\theta = 0$, $r = 4\cos\theta$
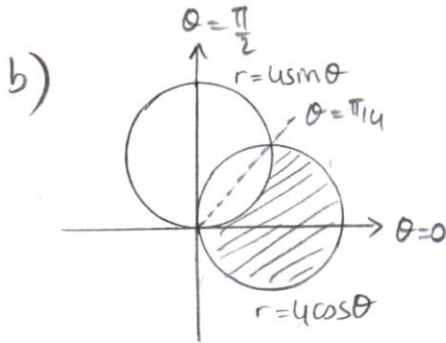
1.yol: $\quad A = \dfrac{1}{2}\displaystyle\int_{-\pi/2}^{\pi/2}(4\cos\theta)^2\,d\theta - \dfrac{1}{2}\left[\displaystyle\int_0^{\pi/4}(4\sin\theta)^2\,d\theta + \displaystyle\int_{\pi/4}^{\pi/2}(4\cos\theta)^2\,d\theta\right]$

2.yol: $\quad A = \dfrac{1}{2}\displaystyle\int_{-\pi/2}^{\pi/4}(4\cos\theta)^2\,d\theta - \dfrac{1}{2}\displaystyle\int_0^{\pi/4}(4\sin\theta)^2\,d\theta$

c)

$\theta = \dfrac{\pi}{2}$, $\theta = \dfrac{\pi}{4}$, $r = 4\sin\theta$, $\theta = 0$, $r = 4\cos\theta$
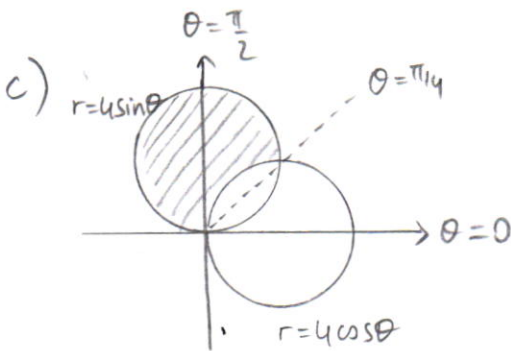
1.yol:

$$A = \frac{1}{2}\int_0^{\pi}(4\sin\theta)^2\,d\theta - \frac{1}{2}\left[\int_0^{\pi/4}(4\sin\theta)^2\,d\theta + \int_{\pi/4}^{\pi/2}(4\cos\theta)^2\,d\theta\right]$$

2.yol:

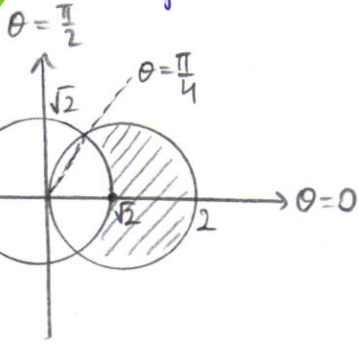$$A = \frac{1}{2}\int_{\pi/4}^{\pi}(4\sin\theta)^2\,d\theta - \frac{1}{2}\int_{\pi/4}^{\pi/2}(4\cos\theta)^2\,d\theta$$

⑧ $r = 2\cos\theta$ eğrisinin içinde, $r = \sqrt{2}$ eğrisinin dışında kalan alanı bulunuz.
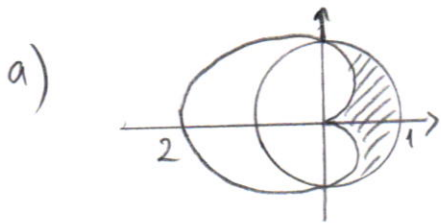

$2\cos\theta = \sqrt{2} \Rightarrow \cos\theta = \dfrac{\sqrt{2}}{2} \Rightarrow \theta = \dfrac{\pi}{4}$

$$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/4}(2\cos\theta)^2\,d\theta - \frac{1}{2}\int_0^{\pi/4}(\sqrt{2})^2\,d\theta$$

$$= \frac{1}{2}\left(\int_0^{\pi/4}(4\cos^2\theta - 2)\,d\theta\right) = \frac{1}{2}\int_0^{\pi/4} 2\cos 2\theta\,d\theta$$

$$= \frac{\sin 2\theta}{2}\Big|_0^{\pi/4} = \frac{1}{2} \Rightarrow A = 1$$

⑨ a) $r = 1$ eğrisinin içinde $r = 1-\cos\theta$ eğrisinin dışında kalan

b) $r = 1$ " dışında $r = 1-\cos\theta$ " içinde "

c) $r = 1$ ve $r = 1-\cos\theta$ eğrilerinin sınırladığı ortak alanı bulunuz.

(İnt. hesaplamayın)

a)

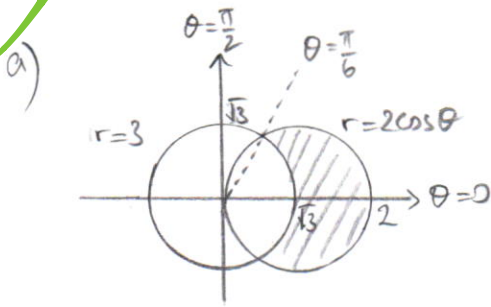$$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/2} 1^2\,d\theta - \frac{1}{2}\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta$$

b)

$$\frac{A}{2} = \frac{1}{2}\int_{\pi/2}^{\pi}\left[(1-\cos\theta)^2 - 1^2\right]d\theta$$
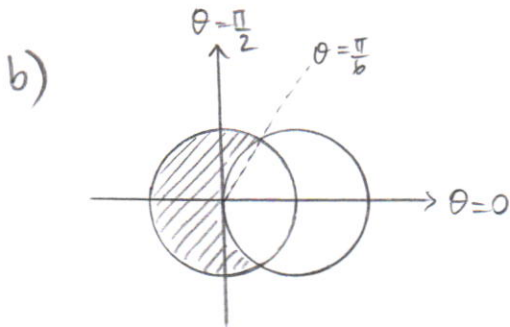
c)
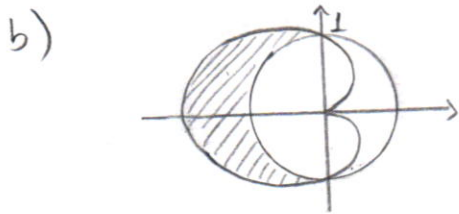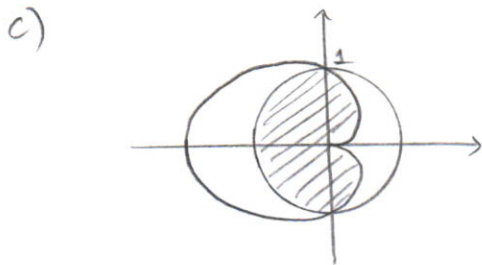
$$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/2}^{\pi} 1^2\,d\theta$$

10) a) $r = 2\cos\theta$ içinde, $r = \sqrt{3}$ dışında kalan,

b) $r = 2\cos\theta$ dışında, $r = \sqrt{3}$ içinde kalan,

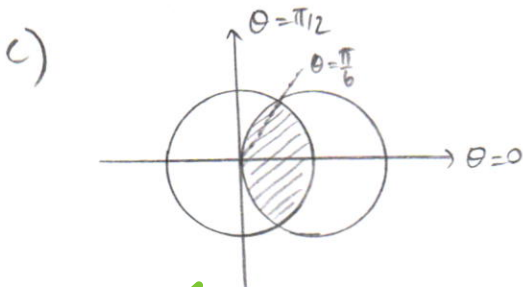c) $r = 2\cos\theta$ ve $r = \sqrt{3}$ sınırladığı ortak alanı bulunuz.

a)

$2\cos\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{6}$

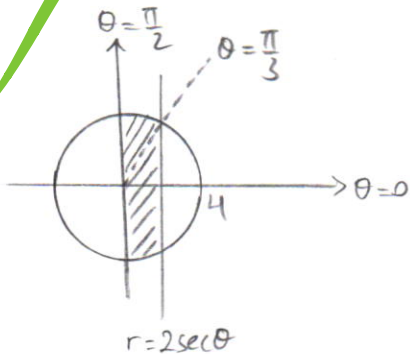$$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/6}\left((2\cos\theta)^2 - 3\right)d\theta$$

b)

$$\frac{A}{2} = \frac{1}{2}\int_{\pi/6}^{\pi}3\,d\theta - \int_{\pi/6}^{\pi/2}(2\cos\theta)^2\,d\theta$$

c)

$$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/6}3\,d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2}(2\cos\theta)^2\,d\theta$$

11) $r = 4$, $\theta = \frac{\pi}{2}$, $r = 2\sec\theta$ arasında kalan alanı hesaplayınız.
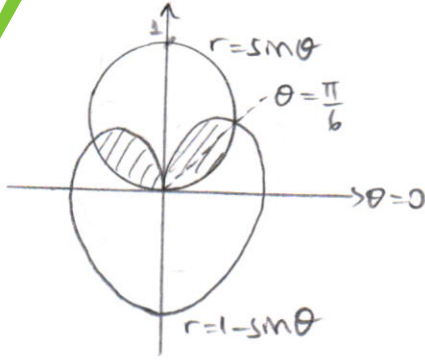

$r = 2\sec\theta$

$r = 2\sec\theta = \dfrac{2}{\cos\theta} \Rightarrow r\cos\theta = 2 \Rightarrow x = 2$ doğrusu

$\left.\begin{array}{l} r = 4 \\ r = \dfrac{2}{\cos\theta} \end{array}\right\} \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/3}(2\sec\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2}4^2\,d\theta \Rightarrow A = 4\tan\theta\Big|_0^{\pi/3} + 16\theta\Big|_{\pi/3}^{\pi/2} = 4\sqrt{3} + \frac{8\pi}{3}$$

(12) $r = 1 - \sin\theta$ ve $r = \sin\theta$ eğrilerinin sınırladığı bölgenin alanını hesaplayın.
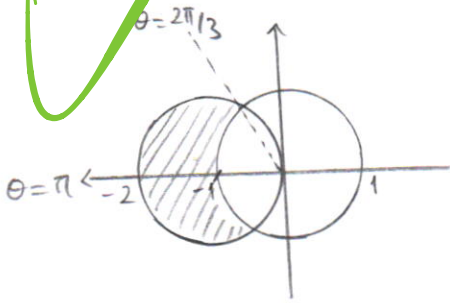

$1 - \sin\theta = \sin\theta \Rightarrow \theta = \frac{\pi}{6}$ (ve $\frac{5\pi}{6}$)

$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/6}(\sin\theta)^2 d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2}(1-\sin\theta)^2 d\theta$

$A = \int_0^{\pi/6}\frac{1-\cos2\theta}{2}d\theta + \int_{\pi/6}^{\pi/2}\frac{1}{2}(3 - 4\sin\theta - \cos2\theta)d\theta$

$= \left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right) + \left(\frac{\pi}{2} - \frac{7\sqrt{3}}{8}\right) = \frac{7\pi}{12} - \sqrt{3}$
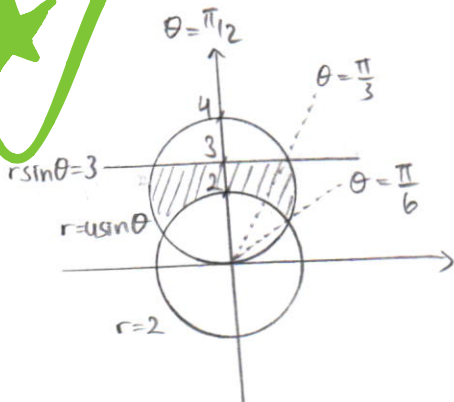
(13) $r = -2\cos\theta$ çemberinin içinde, $r = 1$ çemberinin dışında kalan alanı bulunuz.

$-2\cos\theta = 1 \Rightarrow \theta = \frac{2\pi}{3}$

$\frac{A}{2} = \frac{1}{2}\int_{2\pi/3}^{\pi}(-2\cos\theta)^2 d\theta - \frac{1}{2}\int_{2\pi/3}^{\pi}1^2 d\theta$

(14) $r = 2$, $r = 4\sin\theta$ ve $r\sin\theta = 3$ ile sınırlı bölgenin alanını veren belirli integrali yazınız.

$r\sin\theta = 3 \Rightarrow y = 3$ doğrusu

$4\sin\theta = \frac{3}{\sin\theta} \Rightarrow \sin^2\theta = \frac{3}{4} \Rightarrow \sin\theta = \mp\frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$
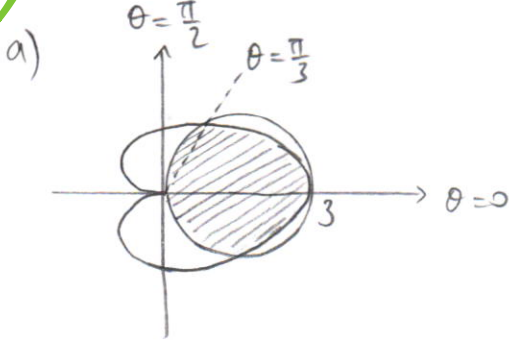
$4\sin\theta = 2 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

$\frac{A}{2} = \frac{1}{2}\int_{\pi/6}^{\pi/3}\left[(4\sin\theta)^2 - 2^2\right]d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2}\left[\left(\frac{3}{\sin\theta}\right)^2 - 2^2\right]d\theta$

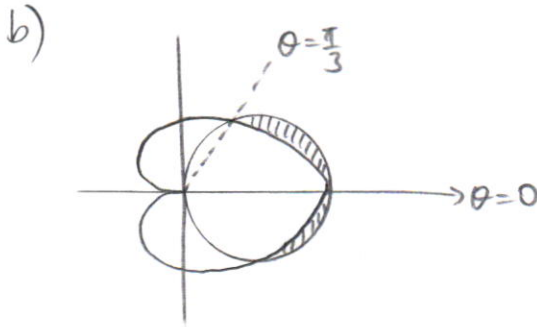(15) a) $r=3\cos\theta$ ve $r=1+\cos\theta$ eğrilerinin sınırladığı ortak alanı

b) $r=3\cos\theta$ nın içinde $r=1+\cos\theta$ nın dışında kalan alanı

c) $r=3\cos\theta$ nın dışında $r=1+\cos\theta$ nın içinde kalan alanı
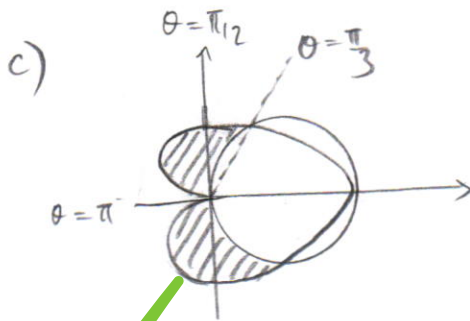
veren integralleri yazın.

a)

$$1+\cos\theta = 3\cos\theta \Rightarrow \theta = \frac{\pi}{3}$$

$$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/3}(1+\cos\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2}(3\cos\theta)^2\,d\theta$$
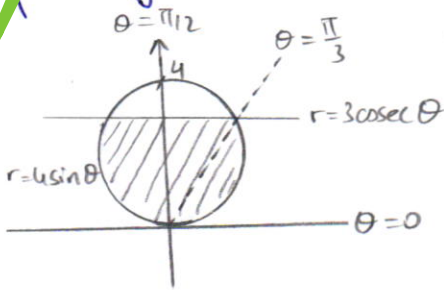
b)

$$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/3}[(3\cos\theta)^2 - (1+\cos\theta)^2]\,d\theta$$

c)

$$\frac{A}{2} = \frac{1}{2}\int_{\pi/3}^{\pi}(1+\cos\theta)^2\,d\theta - \frac{1}{2}\int_{\pi/3}^{\pi/2}(3\cos\theta)^2\,d\theta$$

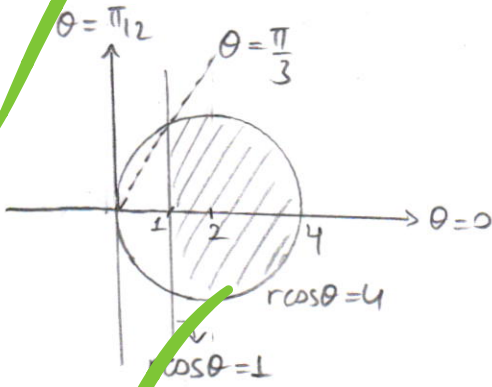(16) $r=4\sin\theta$, $r=3\csc\theta$ ve $\theta=0$ ile sınırlı bölgenin alanını veren integrali yazınız.

$$r = 3\csc\theta = \frac{3}{\sin\theta} \Rightarrow y=3 \text{ doğrusu}$$

$$4\sin\theta = \frac{3}{\sin\theta} \Rightarrow \theta = \frac{\pi}{3}$$

$$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/3}(4\sin\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2}(3\csc\theta)^2\,d\theta$$

(17) $r=4\cos\theta$ ile $r\cos\theta \geq 1$ sınırladığı ortak alanı bulunuz.
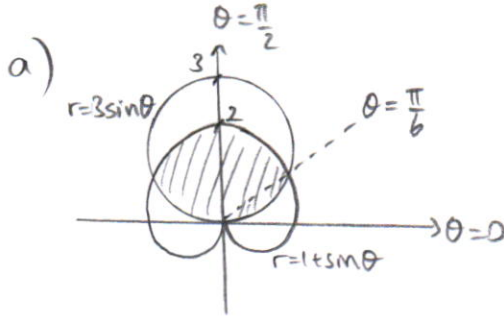

$r\cos\theta \geq 1 \implies x \geq 1$

$\left.\begin{array}{l} r=4\cos\theta \\ r=\dfrac{1}{\cos\theta} \end{array}\right\} \cos\theta=\dfrac{1}{2} \implies \theta=\dfrac{\pi}{3}$

$\dfrac{A}{2}=\dfrac{1}{2}\displaystyle\int_0^{\pi/3}(4\cos\theta)^2 d\theta-\dfrac{1}{2}\int_0^{\pi/3}\sec^2\theta\, d\theta$

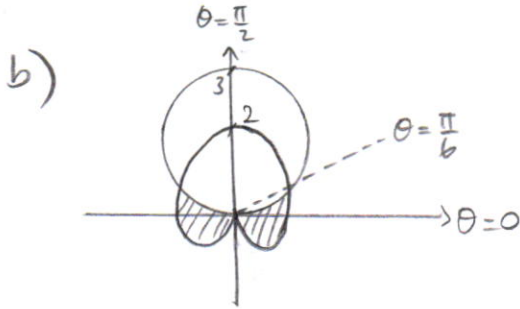(18) a) $r=3\sin\theta$ ve $r=1+\sin\theta$ eğrilerinin sınırladığı ortak alanı

b) $r=3\sin\theta$ dışında, $r=1+\sin\theta$ içinde kalan alanı

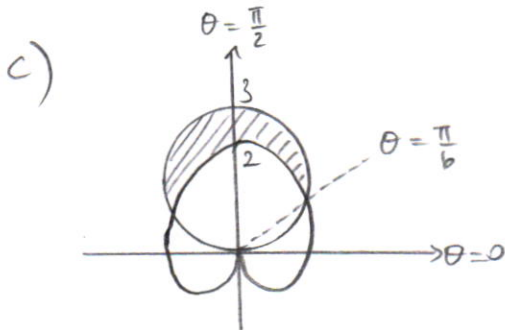c) $r=3\sin\theta$ içinde, $r=1+\sin\theta$ dışında kalan alanı bulunuz.

a)

$1+\sin\theta=3\sin\theta \implies \theta=\dfrac{\pi}{6}$

$\dfrac{A}{2}=\dfrac{1}{2}\displaystyle\int_0^{\pi/6}(3\sin\theta)^2+\dfrac{1}{2}\int_{\pi/6}^{\pi/2}(1+\sin\theta)^2 d\theta$

b)

$\dfrac{A}{2}=\dfrac{1}{2}\displaystyle\int_{-\frac{\pi}{2}}^{\pi/6}(1+\sin\theta)^2 d\theta-\dfrac{1}{2}\int_0^{\pi/6}(3\sin\theta)^2 d\theta$

c)

$\dfrac{A}{2}=\dfrac{1}{2}\displaystyle\int_{\pi/6}^{\pi/2}[(3\sin\theta)^2-(1+\sin\theta)^2]\,d\theta$