

The Chromatic Polynomials and its Algebraic Properties

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Abstract: *This paper studies various results on chromatic polynomials of graphs. We obtain results on the roots of chromatic polynomials of planar graphs. The main results are chromatic polynomial of a graph is polynomial in integer and the leading coefficient of chromatic polynomial of a graph of order n and size m is one, whose coefficient alternate in sign. Mathematics subject classification 2000: 05CXX, 05C15, 05C30, 05C75*

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1. INTRODUCTION

During the period that the Four Color Problem was unsolved, which spanned more than a century, many approaches were introduced with the hopes that they would lead to a solution of this famous problem. In 1912 George David Birkhoff [5] defined a function $P(M, \lambda)$ that gives the number of proper λ -colorings of a map M for a positive integer λ . As we will see, $P(M, \lambda)$ is a polynomial in λ for every map M and is called the chromatic polynomial of M . Consequently, if it could be verified that $P(M, 4) > 0$ for every map M , then this would have established the truth of the Four Color Conjecture.

In 1932 Hassler Whitney [14] expanded the study of chromatic polynomials from maps to graphs. While Whitney obtained a number of results on chromatic polynomials of graphs, this did not contribute to a proof of the Four Color Conjecture. Renewed interest in chromatic polynomials of graphs occurred in 1968 when Ronald C. Read [13] wrote a survey paper on chromatic polynomials.

2. PRELIMINARIES

2.1. Definition: For a graph G and a positive integer λ the number of different proper λ -colorings of G is denoted by $P(G, \lambda)$ and is called the **Chromatic Polynomial** of G . Two λ -colorings c and c' of G from the same set $\{1, 2, \dots, \lambda\}$ of λ colors are considered different if $c(v) \neq c'(v)$ for some vertex v of G . Obviously, if $\lambda < \chi(G)$, then $P(G, \lambda) = 0$. By convention, $P(G, 0) = 0$. Indeed, we have the following.

2.2. Proposition: Let G be a graph. Then $\chi(G) = k$ if and only if k is the smallest positive integer for which $P(G, k) > 0$.

As an example, we determine the number of ways that the vertices of the graph G of Figure.1 can be colored from the set $\{1, 2, 3, 4, 5\}$. The vertex v can be assigned any of these 5 colors, while w can be assigned any color other than the color assigned to v . That is, w can be assigned any of the 4 remaining colors. Both u and t can be assigned any of the 3 colors not used for v and w . Therefore, the number $P(G, 5)$ of 5-colorings of G is $5 \cdot 4 \cdot 3 = 180$. More

generally, $P(G, \lambda) = \lambda(\lambda - 1)(\lambda - 2)^2$ for every integer λ .

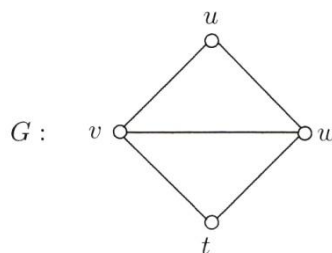


Figure 1. A graph G with $P(G, \lambda) = \lambda(\lambda - 1)(\lambda - 2)^2$

There are some classes of graphs G for which $P(G, \lambda)$ can be easily computed.

3. CHROMATIC POLYNOMIALS

3.1.Theorem: For every positive integer λ

(a) $P(K_n, \lambda) = \lambda(\lambda - 1)(\lambda - 2)\dots(\lambda - n + 1) = \lambda^{(n)}$

(b) $P(K_n, \lambda) = \lambda^n$

Proof: In particular, if $\lambda \geq n$ in Theorem 2(a), then

$$P(K_n, \lambda) = \lambda^{(n)} = \frac{\lambda!}{(\lambda - n)!}$$

We now determine the chromatic polynomial of C_4 in Figure 2. There are λ choices for the color of v_1 . The vertices v_2 and v_4 must be assigned colors different from the assigned to v_1 . The vertices v_2 and v_4 may be assigned the same color or may be assigned different colors. If v_2 and v_4 are assigned the same color, then there are $\lambda - 1$ choices for that color. The vertex v_3 can then be assigned any color except the color assigned to v_2 and v_4 . Hence the number of distinct λ -colorings of C_4 in which v_2 and v_4 are colored the same is $\lambda(\lambda - 1)^2$.

If, on the other hand, v_2 and v_4 are colored differently, then there are $\lambda - 1$ choices for v_2 and $\lambda - 2$ choices for v_4 . Since v_3 can be assigned any color except the two colors assigned to v_2 and v_4 , the number of λ -colorings of C_4 in which v_2

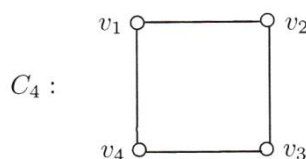


Figure 2. The chromatic polynomial of C_4

and v_4 are colored differently is $\lambda(\lambda - 1)(\lambda - 2)^2$. Hence the number of distinct λ -colorings of C_4 is

$$\begin{aligned} P(C_4, \lambda) &= \lambda(\lambda - 1)^2 + \lambda(\lambda - 1)(\lambda - 2)^2 \\ &= \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3) \\ &= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda \\ &= (\lambda - 1)^4 + (\lambda - 1) \end{aligned}$$

The preceding example illustrates an important observation. Suppose that u and v are nonadjacent vertices in a graph G . The number of λ -colorings of G equals the number of λ -colorings of G in which u and v are colored differently plus the number of λ -colorings of G in which u and v are colored the same. Since the number of λ -colorings of G in which u and v are colored differently is the number of λ -colorings of $G + uv$ while the number of λ -colorings of G in which u and v are colored the same is the number of λ -colorings of the

graph H obtained by identifying u and v (an elementary homomorphism), it follows that

$$P(G, \lambda) = P(G + uv, \lambda) + P(H, \lambda)$$

This observation is summarized below by Erdos, R.J.Wilson [8], chromatic index of graphs.

3.2. Theorem: Let G be a graph containing nonadjacent vertices u and v and let H be the graph obtained from G by identifying u and v . Then

$$P(G, \lambda) = P(G + uv, \lambda) + P(H, \lambda)$$

Proof: Note that if G is a graph of order $n \geq 2$ and size $m \geq 1$, then $G + uv$ has order n and size $m+1$ while H has order $n - 1$ and size at most m . The equation stated in Theorem 2.1 can also be expressed as

$$P(G + uv, \lambda) = P(G, \lambda) - P(H, \lambda)$$

In this context, Theorem 2.1 can be rephrased in terms of an edge deletion and an elementary contraction.

3.3. Corollary: Let G be a graph containing adjacent vertices u and v and let F be the graph obtained from G by identifying u and v . Then

$$P(G, \lambda) = P(G - uv, \lambda) - P(F, \lambda)$$

Proof: By systematically applying Theorem 2.1 to pairs of nonadjacent vertices in a graph G , we eventually arrive at a collection of complete graphs. We now illustrate this. Suppose that we wish to compute the chromatic polynomial of the graph G of Figure 3. For the nonadjacent vertices u and v of G and the graph H obtained by identifying u and v , it follows that by Theorem 2.1 that the chromatic polynomial of G is the sum of the chromatic polynomials of $G + uv$ and H .

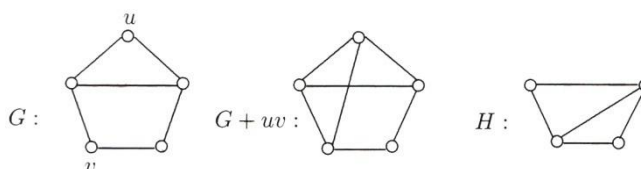
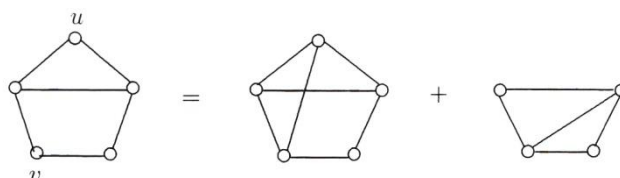


Figure 3

At this point it is useful to adopt a convention introduced by Alexander Zykov [15] and utilized later by Ronald Read [9]. Rather than repeatedly writing the equation that appears in the statement of Theorem 2.1, we represent the chromatic polynomial of a graph by a drawing of the graph and indicate on the drawing which pair u, v of nonadjacent vertices will be separately joined by an edge and identified. So, for the graph G of Figure 3, we have Continuing in this manner, as shown in Figure 4, we obtain



$$P(G, \lambda) = \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda$$

Using this approach, we see that the chromatic polynomial of every graph is the sum of chromatic polynomials of complete graphs. A consequence of this observation is the following.

3.4. Theorem: The chromatic polynomial $P(G, \lambda)$ of a graph G is a polynomial in λ .

Proof. There are some interesting properties possessed by the chromatic polynomial of every graph. In fact, if G is a graph of order n and size m , then the chromatic polynomial

$P(G, \lambda)$ of G can be expressed as

$$P(G, \lambda) = c_0 \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \dots + c_{n-1} \lambda + c_n$$

$$= \lambda^{(5)} + 4\lambda^{(4)} + 3\lambda^{(3)} = \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda$$

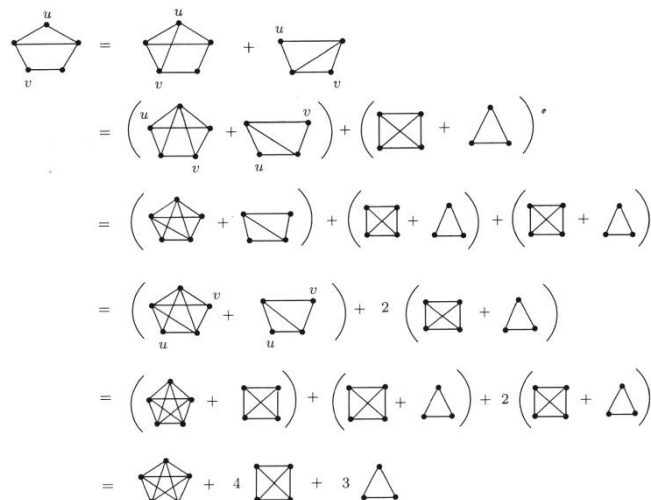


Figure 5. $P(G, \lambda) = P(G + uv, \lambda) + P(H, \lambda)$

Where $c_0 = 1$ ($P(G, \lambda)$ is a polynomial of degree n with leading coefficient 1), $c_1 = -m$, $c_i \geq 0$ if i is even with $0 \leq i \leq n$, and $c_i \leq 0$ if i is odd with $1 \leq i \leq n$. Since $P(G, 0) = 0$, it follows that $c_n = 0$.

3.5. Theorem: Let G be a graph of order n and size m . Then $P(G, \lambda)$ is a polynomial of degree n with leading coefficient 1 such that the coefficient of λ^{n-1} is $-m$, and whose coefficients alternate in sign.

Proof: We proceed by induction on m . If $m = 0$, then $G = \overline{K_n}$ and $P(G, \lambda) = \lambda^n$, as we have seen. Then $P(\overline{K_n}, \lambda) = \lambda^n$ has the desired properties.

Assume that the result holds for all graphs whose size is less than m , where $m \geq 1$. Let G be a graph of order m and let $e = uv$ an edge of G . By Corollary 2.2,

$$P(G, \lambda) = P(G - e, \lambda) - P(F, \lambda),$$

Where F is the graph obtained from G by identifying u and v . Since $G - e$ has order n and size $m - 1$, it follows by the induction hypothesis that

$$P(G - e, \lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_{n-1} \lambda + a_n,$$

Where $a_0 = 1$, $a_1 = -(m-1)$, $a_i \geq 0$ if i is even with $0 \leq i \leq n$, $a_i \leq 0$ if i is odd with $1 \leq i \leq n$. Furthermore, since F has order $n-1$ and size m' , where $m' \leq m-1$, it follows that

$$P(F, \lambda) = b_0 \lambda^{n-1} + b_1 \lambda^{n-2} + b_2 \lambda^{n-3} + \dots + b_{n-2} \lambda + b_{n-1},$$

Where $b_0 = 1$, $b_1 = -m'$, $b_i \geq 0$ if i is even with $0 \leq i \leq n-1$, and $b_i \leq 0$ if i is odd with $1 \leq i \leq n-1$. By Corollary

$$P(G, \lambda) = P(G - e, \lambda) - P(F, \lambda)$$

$$= (a_0 \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_{n-1} \lambda + a_n) -$$

$$(b_0 \lambda^{n-1} + b_1 \lambda^{n-2} + b_2 \lambda^{n-3} + \dots + b_{n-2} \lambda + b_{n-1})$$

$$= a_0 \lambda^n + (a_1 - b_0) \lambda^{n-1} + (a_2 - b_1) \lambda^{n-2} + \dots + (a_{n-1} - b_{n-2}) \lambda + (a_n - b_{n-1})$$

Since $a_0 = 1$, $a_1 - b_0 = -(m-1) - 1 = -m$, $a_i - b_{i-1} \geq 0$ if i is even with $2 \leq i \leq n$, and $a_i - b_{i-1} \leq 0$ if i is odd with $0 \leq i \leq n$, $P(G, \lambda)$ has the desired properties and the theorem follows by mathematical induction.

Suppose that a graph G contains an end-vertex v whose only neighbor is u . Then, of course, $P(G-v, \lambda)$ is the number of λ -colorings of $G-v$. The vertex v can then be assigned any of the λ colors except the color assigned to u . This observation gives the following.

3.6. Theorem: If G is a graph containing an end-vertex v , then

$$P(G, \lambda) = (\lambda - 1) P(G - v, \lambda)$$

One consequence of this result is the following.

3.7. Corollary: If T is a tree of order $n \geq 1$, then

$$P(T, \lambda) = \lambda (\lambda - 1)^{n-1}$$

Proof: We proceed by induction on n . For $n = 1$, $T = K_1$ and certainly $P(K_1, \lambda) = \lambda$. Thus the basis step of the induction is true. Suppose that $P(T', \lambda) = \lambda(\lambda-1)^{n-2}$ for every tree T' of order $n-1 \geq 1$ and let T be a tree of order n . Let v be an end-vertex of T . Thus $T-v$ is a tree of order $n-1$. By Theorem 3.2 and the induction hypothesis

$$P(T, \lambda) = (\lambda - 1)P(T - v, \lambda) = (\lambda - 1)[\lambda(\lambda - 1)^{n-2}] = \lambda(\lambda - 1)^{n-1} \text{ as desired.}$$

Two graphs are chromatically equivalent if they have the same chromatic polynomial.

In this section we'll summarize some well-known facts about the chromatic polynomial's coefficients, roots and substitutions as well as their relations to some graph-theoretic properties of G .

4. COEFFICIENTS OF THE CHROMATIC POLYNOMIALS

Claim 3.1: The lead coefficient of $\text{chr}(G, k)$ is always 1.

Proof: Use the partitioning argument from theorem 2.5. The only partition that contributes to the lead coefficient is the one with n parts, giving an addend of $k(k-1)\dots(k-n+1)$ where the coefficient of k^n is 1.

Claim 3.2: The coefficient of k^{n-1} in $\text{chr}(G, k)$ is the negative of the number of edges.

Proof: Apply the deletion-contraction argument. By contracting an edge we obtain a graph on $n-1$ vertices, whose chromatic polynomial has 1 as the coefficient of k^{n-1} according to our previous claim. Therefore deleting the edge should augment the coefficient of k^{n-1} by 1, finally reaching in zero as all edges are removed. So initially it had to be the negative of the number of edges.

Claim 3.3: The constant term, i.e. the coefficient of 1 in $\text{chr}(G, k)$ is always zero.

Proof: Substituting $k = 0$ into the chromatic polynomial yields 0 since G cannot be colored using 0 colors.

Claim 3.4: The coefficient of k in $\text{chr}(G, k)$ is non-zero if and only if G is connected.

Proof: We know that the chromatic polynomial of a disconnected graph is the product of that of its components. If we have at least two terms, each being divisible by k , then their product is divisible by k^2 , thus its coefficient of k is zero.

For connected graphs we'll prove the slightly stronger result that the coefficient of k is positive if n is odd and negative if n is even. This works by induction based on the deletion-contraction argument. We can always select an edge that is not a bridge unless the graph is a tree. otherwise both deletion and contraction gives a connected graph so we may continue the induction.

Lemma 3.5: Let $c(F)$ denote the number of components in a spanning subgraph F . Then

$$\text{chr}(G, k) = \sum_{F \subseteq E(G)} (-1)^{|F|} k^{c(F)}$$

Proof: The number of ways we may assign colors to $V(G)$ so that vertices connected by F -edges do share the same color is $k^{c(F)}$. This is the number of colorings that *violate* the vertex coloring

condition for all edges in F . Since the chromatic polynomial counts the colorings that violate this condition for no edges in $E(G)$, the result follows from the principle of inclusion-exclusion.

Claim 3.6: The coefficients of the chromatic polynomial alternate in sign. That is, for the coefficient a_m of k^m we have $a_m \geq 0$ if $n \equiv m(2)$ and $a_m \leq 0$ otherwise.

Proof: The claim holds for the empty graph and is preserved during a deletion-contraction step.

Claim 3.7: For a connected graph G the coefficients satisfy.

$$1 = |a_n| < |a_{n-1}| < \cdots < \left| a_{\lfloor \frac{n}{2} \rfloor + 1} \right|$$

Proof: We would like to show $|a_{m+1}| < |a_m|$ for $m > \frac{n}{2}$.

For trees we have $\text{chr}(T_n, k) = k(k-1)^{n-1}$ from section 1.4, so $a_m = (-1)^{n-m} \binom{n-1}{m-1}$ and thus the claim is $\binom{n-1}{m} < \binom{n-1}{m-1}$. Rearranging transforms this to $n-m < m$ which we have assumed.

Otherwise we may select a non-bridge edge as in the proof of claim 3.4 and apply deletion-contraction. Our previous claim tells us that the corresponding coefficients of $\text{chr}(G \setminus e, k)$ and $\text{chr}(G/e, k)$ have opposite signs, and therefore their absolute values add up. For the contracted graph we have

$$0 = |a'_n| < 1 = |a'_{n-1}| < |a'_{n-2}| < \cdots < \left| a'_{\lfloor \frac{n-1}{2} \rfloor + 1} \right|,$$

where the last index is no more than $\lfloor \frac{n}{2} \rfloor + 1$, so both $G \setminus e$ and G/e satisfy the inequalities in the claim and the final addition also preserves them.

This claim is suspected to be possibly strengthened:

Conjecture 3.8: (unimodal conjecture). There exists some k such that

$$|a_n| \leq |a_{n-1}| \leq \cdots \leq |a_{k+1}| \leq |a_k| \geq |a_{k-1}| \geq \cdots \geq |a_2| \geq |a_1|$$

This claim has been verified for a few classes of graphs, but remains generally unknown. For some related results see [12].

5. ROOTS OF CHROMATIC POLYNOMIALS

A nonnegative integer root k of the chromatic polynomial means noncolorability with k colors by definition. It follows that $k \in \mathbb{N}$ is a root if and only if $k < \chi(G)$.

Lemma 4.1: The derivative of the chromatic polynomial satisfies $(-1)^n \text{chr}'(G, 1) > 0$ for any biconnected graph and ≥ 0 for any connected graph G .

Proof: For connected graphs that are not biconnected there exists a cut vertex v and we may write the chromatic polynomial $\text{chr}(G, q)$ as a product $\frac{\text{chr}(G_1, q) \text{chr}(G_2, q)}{q}$ according to known claim.

Neither G_1 nor G_2 is empty, thus both terms have a root at 1, so 1 is at least a double root of $\text{chr}(G, q)$ and therefore its derivative also has a root at 1. It follows that the claim is satisfied with an equality.

At this point it is enough to consider biconnected graphs. For K_2 we have $\text{chr}'(K_2, 1) = 1$. Otherwise both $G \setminus e$ and G/e are connected for any edge e , so we can obtain the weaker claim by using the deletion-contraction argument. To prove the stronger one, we'll show that there exists an edge 2 for which G/e is also biconnected.

The only possible cut vertex of G/e is the contracted one, since any other vertex would also separate G . So we are looking for such an $e = ij$ that removing i and j from $V(G)$ doesn't cut G apart.

There exists a longest path in G : let i be one of its endpoints and j its neighbor. Suppose that the removal of i and j leaves a disconnected graph and pick two between the two, and they have to traverse i and j respectively. The one going through i can be used to extend the selected longest path, implying a contradiction.

Therefore if we pick e as the final segment of a longest path, G/e is also biconnected and thus the claim holds.

Claim 4.2: The chromatic polynomial has no real root greater than $n - 1$.

Proof: the chromatic polynomial is a sum of terms having the form $q(q-1)\dots(q-p+1)$ where $1 \leq p \leq n$, each of them possibly occurring multiple times. It is easy to see that such a term increases strictly monotonically for $q > n - 1 \geq p - 1$, and so does their sum as well.

Since $\text{chr}(G, q)$ is nonnegative for $q = n - 1$ and strictly increasing afterwards, it can have no root $> n - 1$.

Claim 4.3: The chromatic polynomial of a graph has no negative real roots.

Proof: By claim 3.6. we have

$$\text{chr}(G, q) = \sum_{m=1}^n a_m q^m$$

where $a_m \geq 0$ if $n \equiv m(2)$ and $a_m \leq 0$ otherwise. Thus $(-1)^n a_m q^m \geq 0$ for any $q < 0$. We also know that $a_n = 1 > 0$, and therefore $(-1)^n \text{chr}(G, q) > 0$ which implies that q cannot be a root.

Claim 4.4: The chromatic polynomial has no real roots between 0 and 1.

Proof: It suffices to deal with connected graphs. We show that $(-1)^n \text{chr}(G, q) < 0$ for any $0 < q < 1$. This statement can be easily checked for trees where $\text{chr}(G, q) = q(q-1)^{n-1}$ and otherwise it follows from the deletion-contraction property.

An extension of these claims has been proved by Jackson [13]:

Conjecture 4.5: Planar graphs have no real roots in $[4, \infty)$.

They have already solved the weaker version that $[5, \infty)$ contains no real roots. Since then, Appel and Haken have proved the four-color theorem which states that 4 is neither a root [3][4]. For the remaining interval $(4, 5)$, however, the question is still wide open.

Tutte has shown that for planar graphs $\text{chr}(G, \varphi + 2) > 0$ where $\varphi = \frac{\sqrt{5}+1}{2}$ is the golden ratio [12]. He hoped that it takes us closer to the four-color theorem since $\varphi + 2 \approx 3.618$ is close to 4, but unfortunately there are some planar graphs having a root between the two. In fact, Royle has shown that there are roots arbitrarily close to 4 from below [9].

Theorem 4.6: The roots of all chromatics polynomials are dense in \mathbb{C} .

Despite this claim for general graphs, there exist root-free zones if we make some restrictions. These are particularly important from the point of view of statistical mechanics.

Theorem 4.7: There exists a universal constant C such that if G has maximum degree D , then all complex roots of $\text{chr}(G, q)$ satisfy $|q| < CD$.

A similar bound $|q| < CD + 1$ exists for the second largest degree. For the third largest degree, no such claim can be made: there are arbitrarily large chromatic roots even when all except two vertices have degree 2.

6. SUBSTITUTIONS

For $k \in \mathbb{N}$ $\text{chr}(G, k)$ means the number of k -coloring of G by definition. But there are some further locations where the evaluation of the chromatic polynomial is interesting.

Claim 5.1: $|\text{chr}(G, -1)|$ gives the number of acyclic orientations of G .

Proof. Denote the number of acyclic orientations of G by $f(G)=(-1)^n \text{chr}(G, -1)$. For the empty graph we have $f(\overline{K_n})=1$ and thus the proposition holds. Now consider a nonempty graph G with an edge e selected. Suppose we have an orientation \vec{G}_e on all edges except e and we would like to find out how many ways (i.e. 0,1 or 2) we can extend it to an acyclic orientation \vec{G} of G .

Notice that if there exists such an acyclic extension at all, removing the edge e won't break it. On the other hand, if we can't add e in either direction because both would close a directed path in \vec{G}_e into a cycle, then these two paths make up a cycle in \vec{G}_e by themselves. Therefore \vec{G}_e is an acyclic orientation of $G \setminus e$ if and only if e can be added in at least one direction.

\vec{G}_e specifies an orientation on G/e too. If it contains a cycle passing through the contracted point, one of the two possible orientations of e will extend this cycle into a larger one. And a cycle avoiding the contracted point will be kept intact. Thus if \vec{G}_e is cyclic, pointing e in at least one of the two directions will create a cycle. If \vec{G}_e however, both directions of e will result in an acyclic graph, since any cycle in \vec{G} would have been preserved during the contraction.

Thus if \vec{G}_e specifies an acyclic orientation on 0, 1 or 2 of the graphs $G \setminus e$ and G/e , then it can be extended to G in 0, 1 or 2 ways respectively. This argument shows that $f(G)=f(G \setminus e) + f(G/e)$.

Since the same recursion holds for $f(G)$ and $(-1)^n \text{chr}(G, -1)$ and they are equal for empty graphs, they have to be always equal.

Claim 5.2: $\text{chr}'(G,0)$ returns the chromatic invariant $\beta(G)$.

Proof: The derivative of a polynomial at zero equals the linear coefficient. So the claim follows and the properties proved for $\beta(G)$ in section 3.1 apply.

The chromatic polynomial also exhibits interesting behavior at the so-called *Beraha numbers* $B_n = 2 + 2 \cos\left(\frac{2\pi}{n}\right)$, but they are outside the scope of this study.

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