

## What Is $i^i$ ( $i$ to the power of $i$ )

$$e^{i\pi} + 1 = 0$$

$$e^{i\pi} = -1$$

$$\sqrt{e^{i\pi}} = \sqrt{-1}$$

$$(e^{i\pi})^{1/2} = i$$

$$e^{i\pi/2} = i$$

$$(e^{i\pi/2})^i = i^i$$

$$e^{i^2\pi/2} = i^i$$

$$e^{-\pi/2} = i^i$$

$$i^i = e^{-\pi/2} \sim 0.20788 \sim \frac{1}{5}$$

2.38. Consider a discrete-time LTI system with impulse response  $h[n]$  given by

$$h[n] = \alpha^n u[n]$$

- (a) Is this system causal?  
(b) Is this system BIBO stable?

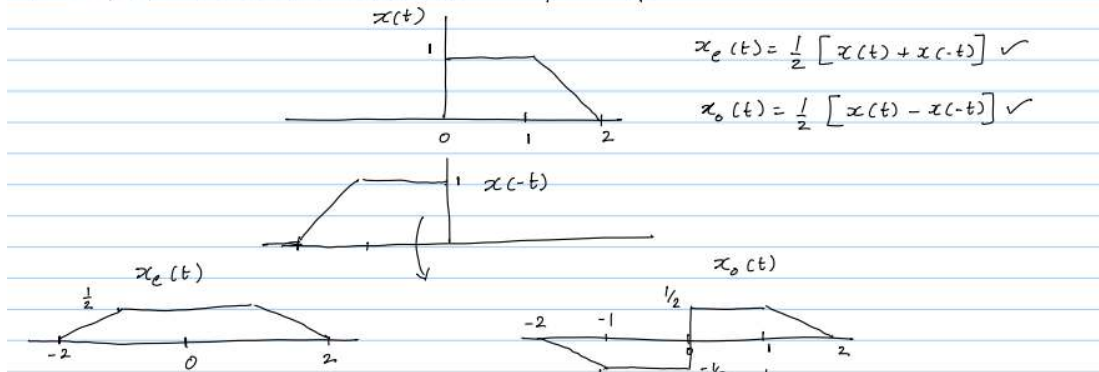
(a) Since  $h[n] = 0$  for  $n < 0$ , the system is causal.

(b) Using Eq. (1.91) (Prob. 1.19), we have

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} |\alpha^k u[k]| = \sum_{k=0}^{\infty} |\alpha|^k = \frac{1}{1-|\alpha|} \quad |\alpha| < 1$$

Therefore, the system is BIBO stable if  $|\alpha| < 1$  and unstable if  $|\alpha| \geq 1$ .

Example: Find the even and odd parts of



1.6. Find the even and odd components of  $x(t) = e^{jt}$ .

Let  $x_e(t)$  and  $x_o(t)$  be the even and odd components of  $e^{jt}$ , respectively.

$$e^{jt} = x_e(t) + x_o(t)$$

From Eqs. (1.5) and (1.6) and using Euler's formula, we obtain

$$x_e(t) = \frac{1}{2}(e^{jt} + e^{-jt}) = \cos t$$

$$x_o(t) = \frac{1}{2}(e^{jt} - e^{-jt}) = j \sin t$$

Example 2. What is the energy from  $t = 0$  to  $t = 10$  in  $x(t) = e^{j\omega t}$ ? Integrating gives

$$E = \int_0^{10} |x(t)|^2 dt = \int_0^{10} |e^{j\omega t}|^2 dt = \int_0^{10} 1 dt = 10.$$

1.3. Given the continuous-time signal specified by

$$x(t) = \begin{cases} 1 - |t| & -1 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

determine the resultant discrete-time sequence obtained by uniform sampling of  $x(t)$  with a sampling interval of (a) 0.25 s, (b) 0.5 s, and (c) 1.0 s.

It is easier to take the graphical approach for this problem. The signal  $x(t)$  is plotted in Fig. 1-21(a). Figs. 1-21(b) to (d) give plots of the resultant sampled sequences obtained for the three specified sampling intervals.

(a)  $T_s = 0.25$  s. From Fig. 1-21(b) we obtain

$$x[n] = \{\dots, 0, 0.25, 0.5, 0.75, 1, 0.75, 0.5, 0.25, 0, \dots\}$$

↑

(b)  $T_s = 0.5$  s. From Fig. 1-21(c) we obtain

$$x[n] = \{\dots, 0, 0.5, 1, 0.5, 0, \dots\}$$

↑

(c)  $T_s = 1$  s. From Fig. 1-21(d) we obtain

$$x[n] = \{\dots, 0, 1, 0, \dots\} = \delta[n]$$

↑

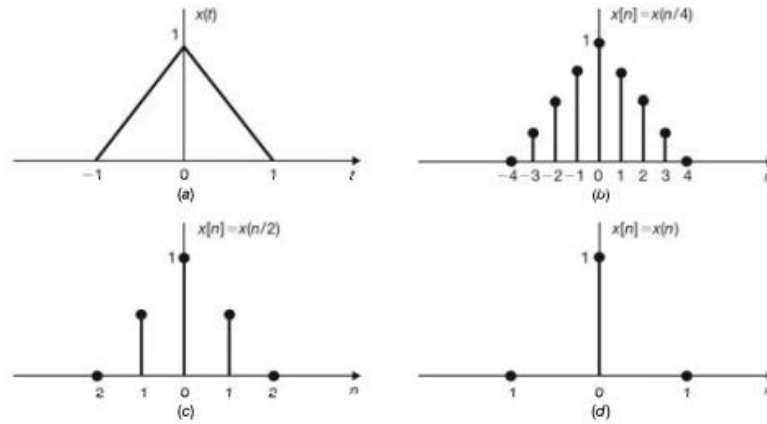
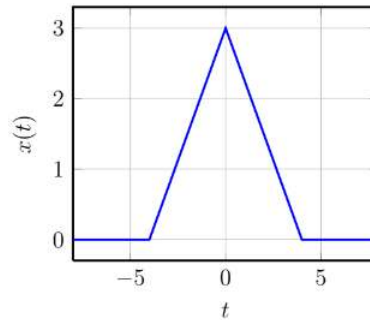


Fig. 1-21

Any signal with finite energy (i.e.,  $E_\infty < \infty$ ) has power  $P_\infty = 0$  and is sometimes called an “energy-type” signal. Any signal with  $0 < P_\infty < \infty$  has  $E_\infty = \infty$  and is sometimes called a “power-type” signal.

$$x(t) = \begin{cases} 3 \left(1 - \frac{t}{4}\right) & \text{if } 0 < t \leq 4 \\ 3 \left(1 + \frac{t}{4}\right) & \text{if } -4 < t \leq 0 \\ 0 & \text{otherwise} \end{cases}$$



**Example 1.3** For the above signal, what is the total energy? Integrating gives

$$\begin{aligned} E_\infty &= \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-4}^4 |x(t)|^2 dt \\ &= 2 \int_0^4 3^2 \left(1 - \frac{t}{4}\right)^2 dt = 18 \int_0^4 \left(1 - \frac{t}{4}\right)^2 dt \\ &= 18 \int_0^4 \left(1 - \frac{t}{2} + \frac{t^2}{16}\right) dt = 18 \left(4 - \frac{16}{4} + \frac{4^3}{48}\right) = 24. \end{aligned}$$

**Example 1**

Find the fundamental frequency of the following continuous signal

$$x(t) = \cos\left(\frac{10\pi}{3}t\right) + \sin\left(\frac{5\pi}{4}t\right)$$

The frequencies and periods of the two terms are, respectively,

$$\omega_1 = \frac{10\pi}{3}, f_1 = \frac{5}{3}, T_1 = \frac{3}{5}$$

$$\text{and } \omega_2 = \frac{5\pi}{4}, f_2 = \frac{5}{8}, T_2 = \frac{8}{5}$$

The fundamental frequency  $f_0$  is the GCD of  $f_1 = 5/3$  and  $f_2 = 5/8$

$$f_0 = \text{GCD}\left(\frac{5}{3}, \frac{5}{8}\right) = \text{GCD}\left(\frac{40}{24}, \frac{15}{24}\right) = \frac{5}{24}$$

Alternatively, the period of the fundamental  $T_0$  is the LCM of  $T_1 = \frac{3}{5}$  and  $T_2 = \frac{8}{5}$

$$T_0 = \text{LCM}\left(\frac{3}{5}, \frac{8}{5}\right) = \frac{24}{5}$$

Now we get  $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0} = \frac{5\pi}{12}$  and the signal can be written as

$$x(t) = \cos\left(8\frac{5\pi}{12}t\right) + \sin\left(3\frac{5\pi}{12}t\right) = \cos(8\omega_0 t) + \sin(3\omega_0 t)$$

i.e., the two terms are the 3<sup>rd</sup> and 8<sup>th</sup> harmonic of the fundamental frequency  $\omega_0$ , respectively.

**1.36.** The discrete-time system shown in Fig. 1-36 is known as the *unit delay* element. Determine whether the system is (a) memoryless, (b) causal, (c) linear, (d) time-invariant, or (e) stable.

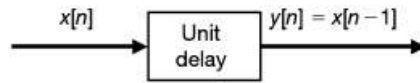


Fig. 1-36 Unit delay element

(a) The system input-output relation is given by

$$y[n] = \mathbf{T}\{x[n]\} = x[n-1] \quad (1.111)$$

Since the output value at  $n$  depends on the input values at  $n-1$ , the system is not memoryless.

(b) Since the output does not depend on the future input values, the system is causal.

(c) Let  $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$ . Then

$$\begin{aligned} y[n] &= \mathbf{T}\{\alpha_1 x_1[n] + \alpha_2 x_2[n]\} = \alpha_1 x_1[n-1] + \alpha_2 x_2[n-1] \\ &= \alpha_1 y_1[n] + \alpha_2 y_2[n] \end{aligned}$$

Thus, the superposition property (1.68) is satisfied and the system is linear.

(d) Let  $y_1[n]$  be the response to  $x_1[n] = x[n-n_0]$ . Then

$$y_1[n] = \mathbf{T}\{x_1[n]\} = x_1[n-1] = x[n-1-n_0]$$

$$\text{and } y[n-n_0] = x[n-n_0-1] = x[n-1-n_0] = y_1[n]$$

Hence, the system is time-invariant.

(e) Since

$$|y[n]| = |x[n-1]| \leq k \quad \text{if } |x[n]| \leq k \text{ for all } n$$

the system is BIBO stable.

**1.16.** Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

(a)  $x(t) = \cos\left(t + \frac{\pi}{4}\right)$

(b)  $x(t) = \sin \frac{2\pi}{3}t$

(c)  $x(t) = \cos \frac{\pi}{3}t + \sin \frac{\pi}{4}t$

(d)  $x(t) = \cos t + \sin \sqrt{2}t$

(e)  $x(t) = \sin^2 t$

(f)  $x(t) = e^{j[(\pi/2)t - 1]}$

(g)  $x[n] = e^{j(\pi/4)n}$

(h)  $x[n] = \cos \frac{1}{4}n$

(i)  $x[n] = \cos \frac{\pi}{3}n + \sin \frac{\pi}{4}n$

(j)  $x[n] = \cos^2 \frac{\pi}{8}n$

(a)  $x(t) = \cos\left(t + \frac{\pi}{4}\right) = \cos\left(\omega_0 t + \frac{\pi}{4}\right) \rightarrow \omega_0 = 1$

$x(t)$  is periodic with fundamental period  $T_0 = 2\pi / \omega_0 = 2\pi$ .

(b)  $x(t) = \sin \frac{2\pi}{3}t \rightarrow \omega_0 = \frac{2\pi}{3}$

$x(t)$  is periodic with fundamental period  $T_0 = 2\pi / \omega_0 = 3$ .

(c)  $x(t) = \cos \frac{\pi}{3}t + \sin \frac{\pi}{4}t = x_1(t) + x_2(t)$

where  $x_1(t) = \cos(\pi/3)t = \cos \omega_1 t$  is periodic with  $T_1 = 2\pi / \omega_1 = 6$  and  $x_2(t) = \sin(\pi/4)t = \sin \omega_2 t$  is periodic with  $T_2 = 2\pi / \omega_2 = 8$ . Since  $T_1 / T_2 = \frac{6}{8} = \frac{3}{4}$  is a rational number,  $x(t)$  is periodic with fundamental period  $T_0 = 4T_1 = 3T_2 = 24$ .

(d)  $x(t) = \cos t + \sin \sqrt{2}t = x_1(t) + x_2(t)$

where  $x_1(t) = \cos t = \cos \omega_1 t$  is periodic with  $T_1 = 2\pi / \omega_1 = 2\pi$  and  $x_2(t) = \sin \sqrt{2}t = \sin \omega_2 t$  is periodic with  $T_2 = 2\pi / \omega_2 = \sqrt{2}\pi$ . Since  $T_1 / T_2 = \sqrt{2}$  is an irrational number,  $x(t)$  is nonperiodic.

(e) Using the trigonometric identity  $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$ , we can write

$$x(t) = \sin^2 t = \frac{1}{2} - \frac{1}{2} \cos 2t = x_1(t) + x_2(t)$$

where  $x_1(t) = \frac{1}{2}$  is a dc signal with an arbitrary period and  $x_2(t) = -\frac{1}{2} \cos 2t = -\frac{1}{2} \cos \omega_2 t$  is periodic with  $T_2 = 2\pi / \omega_2 = \pi$ . Thus,  $x(t)$  is periodic with fundamental period  $T_0 = \pi$ .

(f)  $x(t) = e^{j[(\pi/2)t - 1]} = e^{-j} e^{j(\pi/2)t} = e^{-j} e^{j\omega_0 t} \rightarrow \omega_0 = \frac{\pi}{2}$

$x(t)$  is periodic with fundamental period  $T_0 = 2\pi / \omega_0 = 4$ .

(g)  $x[n] = e^{j(\pi/4)n} = e^{j\Omega_0 n} \rightarrow \Omega_0 = \frac{\pi}{4}$

Since  $\Omega_0 / 2\pi = \frac{1}{8}$  is a rational number,  $x[n]$  is periodic, and by Eq. (1.55) the fundamental period is  $N_0 = 8$ .

(h)  $x[n] = \cos \frac{1}{4}n = \cos \Omega_0 n \rightarrow \Omega_0 = \frac{1}{4}$

Since  $\Omega_0 / 2\pi = 1/8\pi$  is not a rational number,  $x[n]$  is nonperiodic.

(i)  $x[n] = \cos \frac{\pi}{3}n + \sin \frac{\pi}{4}n = x_1[n] + x_2[n]$

where

$$x_1[n] = \cos \frac{\pi}{3}n = \cos \Omega_1 n \rightarrow \Omega_1 = \frac{\pi}{3}$$

$$x_2[n] = \sin \frac{\pi}{4}n = \cos \Omega_2 n \rightarrow \Omega_2 = \frac{\pi}{4}$$

Since  $\Omega_1 / 2\pi = \frac{1}{6}$  (= rational number),  $x_1[n]$  is periodic with fundamental period  $N_1 = 6$ , and since  $\Omega_2 / 2\pi = \frac{1}{8}$  (= rational number),  $x_2[n]$  is periodic with fundamental period  $N_2 = 8$ . Thus, from the result of Prob. 1.15,  $x[n]$  is periodic and its fundamental period is given by the least common multiple of 6 and 8, that is,  $N_0 = 24$ .

(j) Using the trigonometric identity  $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ , we can write

$$x[n] = \cos^2 \frac{\pi}{8}n = \frac{1}{2} + \frac{1}{2} \cos \frac{\pi}{4}n = x_1[n] + x_2[n]$$

where  $x_1[n] = \frac{1}{2} = \frac{1}{2}(1)^n$  is periodic with fundamental period  $N_1 = 1$  and  $x_2[n] = \frac{1}{2} \cos(\pi/4)n = \frac{1}{2} \cos \Omega_2 n \rightarrow \Omega_2 = \pi/4$ . Since  $\Omega_2 / 2\pi = \frac{1}{8}$  (= rational number),  $x_2[n]$  is periodic with fundamental period  $N_2 = 8$ . Thus,  $x[n]$  is periodic with fundamental period  $N_0 = 8$  (the least common multiple of  $N_1$  and  $N_2$ ).