BLM2502 Theory of Computation

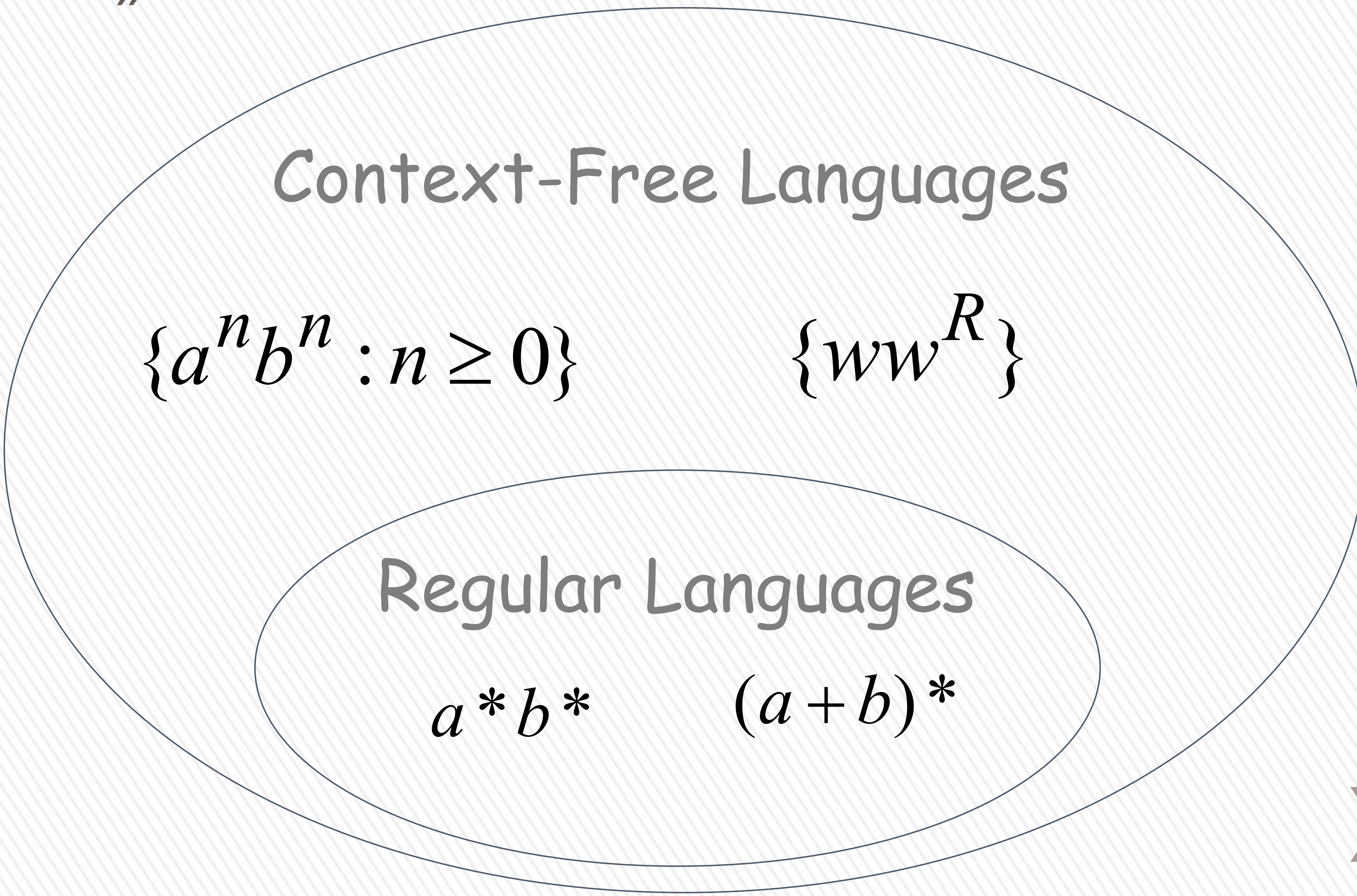
Spring 2015

BLM2502 Theory of Computation

- **Course Outline**

Week	Content
1	Introduction to Course
2	Computability Theory, Complexity Theory, Automata Theory, Set Theory, Relations, Proofs, Pigeonhole Principle
3	Regular Expressions
4	Finite Automata
5	Deterministic and Nondeterministic Finite Automata
6	Epsilon Transition, Equivalence of Automata
7	Pumping Theorem
8	April 10 - 14 week is the first midterm week
9	**Context Free Grammars**
10	**Parse Tree, Ambiguity,**
11	Pumping Theorem
12	Turing Machines, Recognition and Computation, Church-Turing Hypothesis
13	Turing Machines, Recognition and Computation, Church-Turing Hypothesis
14	May 22 – 27 week is the second midterm week
15	Review
16	Final Exam date will be announced

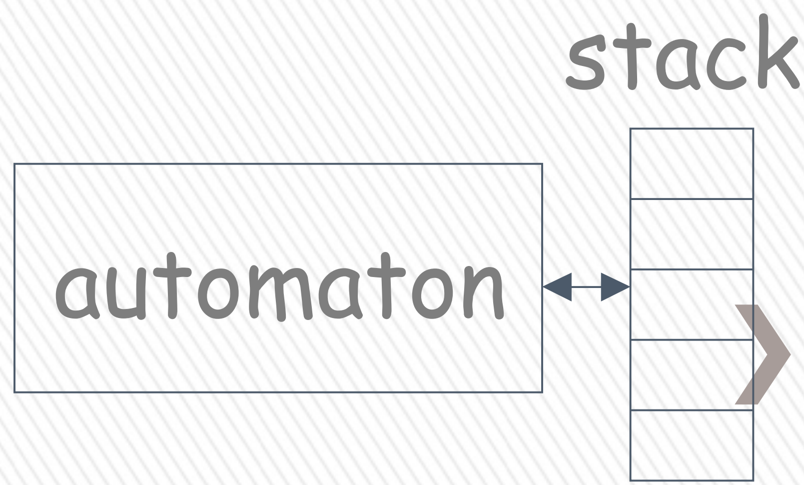
Context-Free Languages


Context-Free Languages
$\{a^n b^n : n \geq 0\}$
$\{ww^R\}$
Regular Languages
a^*b^*
$(a+b)^*$

Context-Free Languages

Context-Free Grammars

Pushdown Automata



Context-Free Grammars

Grammars

» Grammars express languages

» Example: the English language grammar

$$\langle sentence \rangle \to \langle noun\_phrase \rangle \ \langle predicate \rangle$$

$$\langle noun\_phrase \rangle \to \langle article \rangle \ \langle noun \rangle$$

$$\langle predicate \rangle \to \langle verb \rangle$$

$$\langle article \rangle \rightarrow a$$

$$\langle article \rangle \rightarrow the$$

$$\langle noun \rangle \rightarrow cat$$

$$\langle noun \rangle \rightarrow dog$$

$$\langle verb \rangle \rightarrow runs$$

$$\langle verb \rangle \rightarrow sleeps$$

» Derivation of string "the dog walks":

$$
\begin{aligned}
\langle sentence \rangle &\Rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle \\
&\Rightarrow \langle noun\_phrase \rangle \ \langle verb \rangle \\
&\Rightarrow \langle article \rangle \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow the \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow the \ \ dog \ \langle verb \rangle \\
&\Rightarrow the \ \ dog \ \ sleeps
\end{aligned}
$$

» Derivation of string "a cat runs" :

$$
\begin{aligned}
\langle sentence \rangle &\Rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle \\
&\Rightarrow \langle noun\_phrase \rangle \ \langle verb \rangle \\
&\Rightarrow \langle article \rangle \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow a \ \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow a \ \ cat \ \ \langle verb \rangle \\
&\Rightarrow a \ \ cat \ \ runs
\end{aligned}
$$

» Language of the grammar:

L = { "a cat runs",
"a cat sleeps",
"the cat runs",
"the cat sleeps",
"a dog runs",
"a dog sleeps",
"the dog runs",
"the dog sleeps" }

Productions

Sequence of Terminals (symbols)

$$\langle noun \rangle \to \overbrace{cat}$$

$$\langle sentence \rangle \to \underbrace{\langle noun\_phrase \rangle \ \langle predicate \rangle}$$

Variables

Sequence of Variables

Another Example

Grammar:

$$S \rightarrow \overbrace{aSb}^{\text{Sequence of terminals and variables}}$$

$$S \rightarrow \varepsilon$$

Variable

The right side may be ε

» Grammar:

$$S \to aSb$$

$$S \to \varepsilon$$

» Derivation of string $: ab$

$$S \Rightarrow aSb \Rightarrow ab$$

$$S \to aSb \qquad\qquad S \to \varepsilon$$

» Grammar:

$$S \to aSb$$

$$S \to \varepsilon$$

» Derivation of string :

$$aabb$$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

$$S \to aSb \qquad\qquad S \to \varepsilon$$

Grammar: $S \to aSb$

$S \to \varepsilon$

Other derivations:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb$$
$$\Rightarrow aaaaSbbbb \Rightarrow aaaabbbb$$

Grammar:

$$S \to aSb$$

$$S \to \varepsilon$$

Language of the grammar:

$$L = \{a^n b^n : n \geq 0\}$$

A Convenient Notation

- We write: $S \overset{*}{\Rightarrow} aaabbb$

for one or more derivation steps

- Instead of:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

In general we write:

$$w_1 \overset{*}{\Rightarrow} w_n$$

If: $$w_1 \Rightarrow w_2 \Rightarrow w_3 \Rightarrow \cdots \Rightarrow w_n$$

in zero or more derivation steps

Trivially: $$w \overset{*}{\Rightarrow} w$$

Example Grammar

$$S \rightarrow aSb$$

$$S \rightarrow \varepsilon$$

Possible Derivations

$$S \overset{*}{\Rightarrow} \varepsilon$$

$$S \overset{*}{\Rightarrow} ab$$

$$S \overset{*}{\Rightarrow} aaabbb$$

$$S \overset{*}{\Rightarrow} aaSbb \overset{*}{\Rightarrow} aaaaaSbbbbb$$

Another convenient notation:

$$S \to aSb$$

$$S \to \varepsilon$$

$$\Longrightarrow \quad S \to aSb \mid \varepsilon$$

$$\langle article \rangle \to a$$

$$\langle article \rangle \to the$$

$$\Longrightarrow \quad \langle article \rangle \to a \mid the$$

Formal Definition

Grammar: $G = (V, T, S, P)$

Set of variables

Set of terminal symbols

Start variable

Set of productions

Context-Free Grammar: $G = (V, T, S, P)$

All productions in P are of the form

$$A \rightarrow s$$

Variable

String of variables and terminals

Example for Context-Free Grammar

$$\boxed{S \to aSb \mid \varepsilon}$$

productions

$$P = \{S \to aSb,\ S \to \varepsilon\}$$

$$G = (V, T, S, P)$$

$$V = \{S\}$$

variables

$$T = \{a, b\}$$

terminals

start variable

Language of a Grammar:

» For a grammar G with start variable S

$$L(G) = \{w: \quad S \overset{*}{\Rightarrow} w, \quad w \in T^*\}$$

String of terminals or ε

Example:

context-free grammar G : $\boxed{S \to aSb \mid \varepsilon}$

$$L(G) = \{a^n b^n : \quad n \geq 0\}$$

Since, there is derivation

$$S \stackrel{*}{\Rightarrow} a^n b^n \quad \text{for any } n \geq 0$$

Context-Free Language:

» A language L is context-free

» if there is a context-free grammar G

» with $L = L(G)$

Example:

$$L = \{a^n b^n : \quad n \geq 0\}$$

is a context-free language

since context-free grammar G :

$$\boxed{S \to aSb \mid \varepsilon}$$

generates $L(G) = L$

Another Example

Context-free grammar G :

$$S \rightarrow aSa \mid bSb \mid \varepsilon$$

Example derivations:

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abba$$

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abaSaba \Rightarrow abaaba$$

$$L(G) = \{ww^R : \quad w \in \{a,b\}^*\}$$

Palindromes of even length

Another Example

Context-free grammar G:

$$S \rightarrow aSb \mid SS \mid \varepsilon$$

Example derivations:

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow ab$$

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow abaSb \Rightarrow abab$$

$$L(G) = \{w \quad : n_a(w) = n_b(w),$$

$$\text{and } \ n_a(v) \geq n_b(v)$$

$$\text{in any } \ \text{prefix } \ v\}$$

Describes matched parentheses: ( ) ((( ))) (( )) $a = ($, $\mathrm{b} =)$

Derivation Trees

Derivation Order

Consider the following example grammar with 5 productions:

1. $S \to AB$	2. $A \to aaA$	4. $B \to Bb$
	3. $A \to \varepsilon$	5. $B \to \varepsilon$

$$1.\ S \to AB \qquad 2.\ A \to aaA \qquad 4.\ B \to Bb$$

$$3.\ A \to \varepsilon \qquad 5.\ B \to \varepsilon$$

Leftmost derivation order of string *aab*:

$$S \overset{1}{\Rightarrow} AB \overset{2}{\Rightarrow} aaAB \overset{3}{\Rightarrow} aaB \overset{4}{\Rightarrow} aaBb \overset{5}{\Rightarrow} aab$$

At each step, we substitute the leftmost variable

1. $S \to AB$	2. $A \to aaA$	4. $B \to Bb$
	3. $A \to \varepsilon$	5. $B \to \varepsilon$

Rightmost derivation order of string aab :

$$S \overset{1}{\Rightarrow} AB \overset{4}{\Rightarrow} ABb \overset{5}{\Rightarrow} Ab \overset{2}{\Rightarrow} aaAb \overset{3}{\Rightarrow} aab$$

At each step, we substitute the rightmost variable

1. $S \to AB$	2. $A \to aaA$	4. $B \to Bb$
	3. $A \to \varepsilon$	5. $B \to \varepsilon$

Leftmost derivation of **aab**:

$$S \overset{1}{\Rightarrow} AB \overset{2}{\Rightarrow} aaAB \overset{3}{\Rightarrow} aaB \overset{4}{\Rightarrow} aaBb \overset{5}{\Rightarrow} aab$$

Rightmost derivation of **aab** :

$$S \overset{1}{\Rightarrow} AB \overset{4}{\Rightarrow} ABb \overset{5}{\Rightarrow} Ab \overset{2}{\Rightarrow} aaAb \overset{3}{\Rightarrow} aab$$

Consider the same example grammar:

$$S \to AB \qquad A \to aaA \mid \varepsilon \quad B \to Bb \mid \varepsilon$$

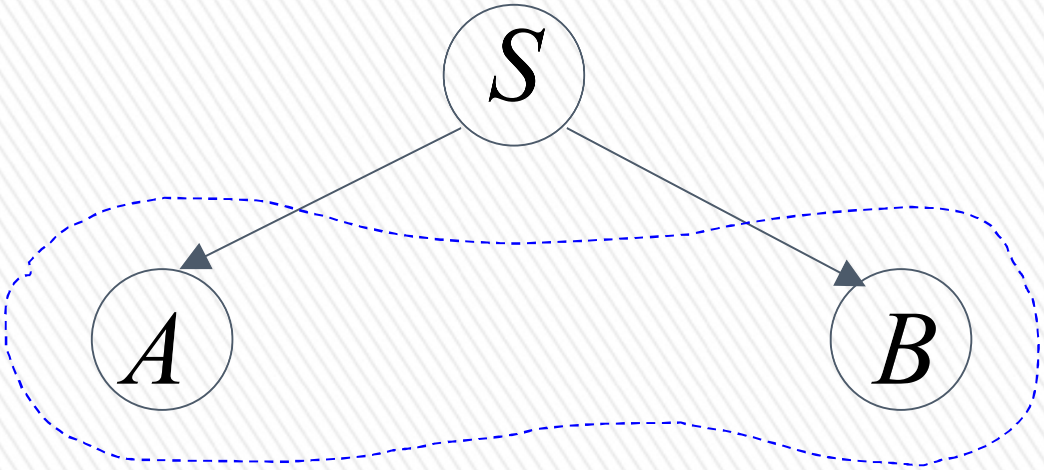
And a derivation of **aab**:

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aab$$

Derivation Tree

$$S \to AB \qquad A \to aaA \mid \varepsilon \quad B \to Bb \mid \varepsilon$$

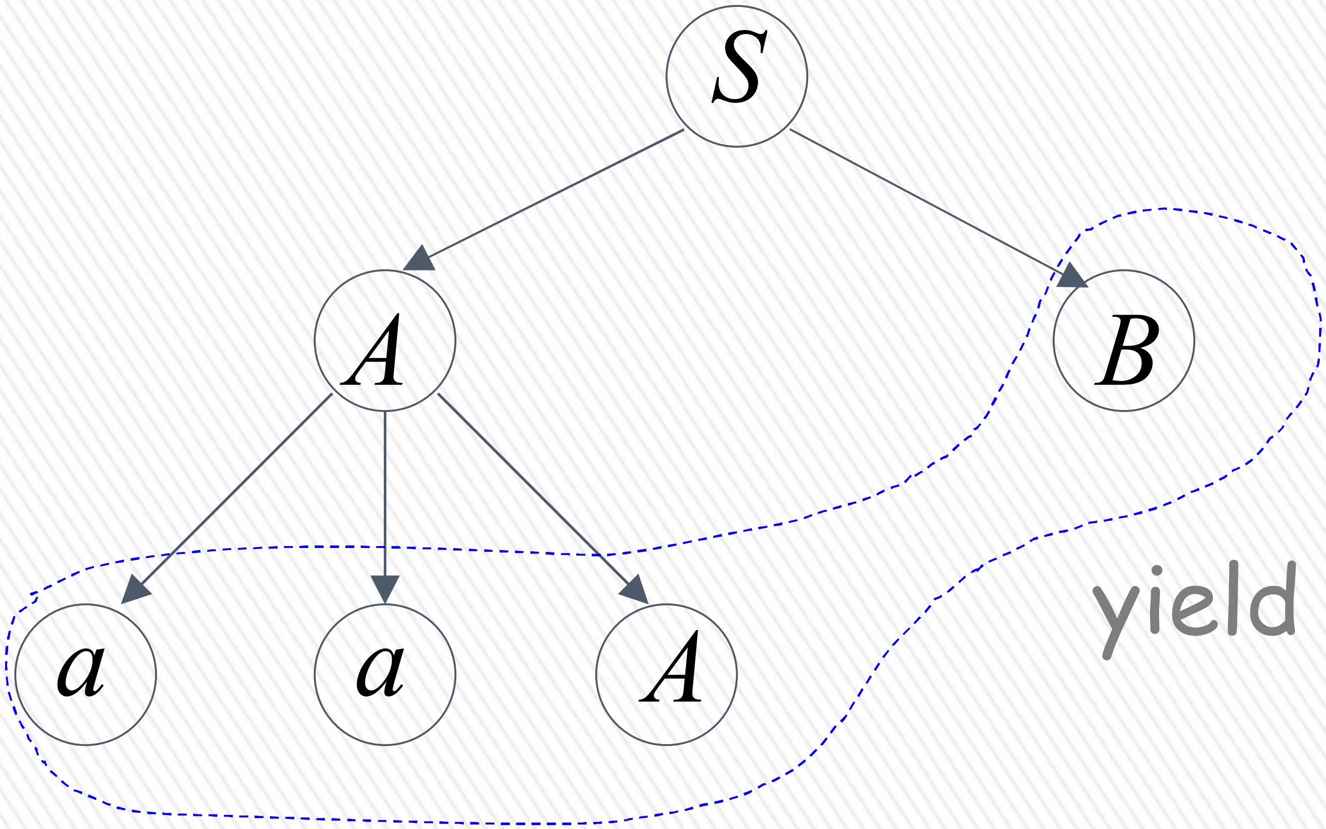
$$S \Rightarrow AB$$



yield ***AB***

$$S \to AB \qquad A \to aaA \mid \varepsilon \quad B \to Bb \mid \varepsilon$$

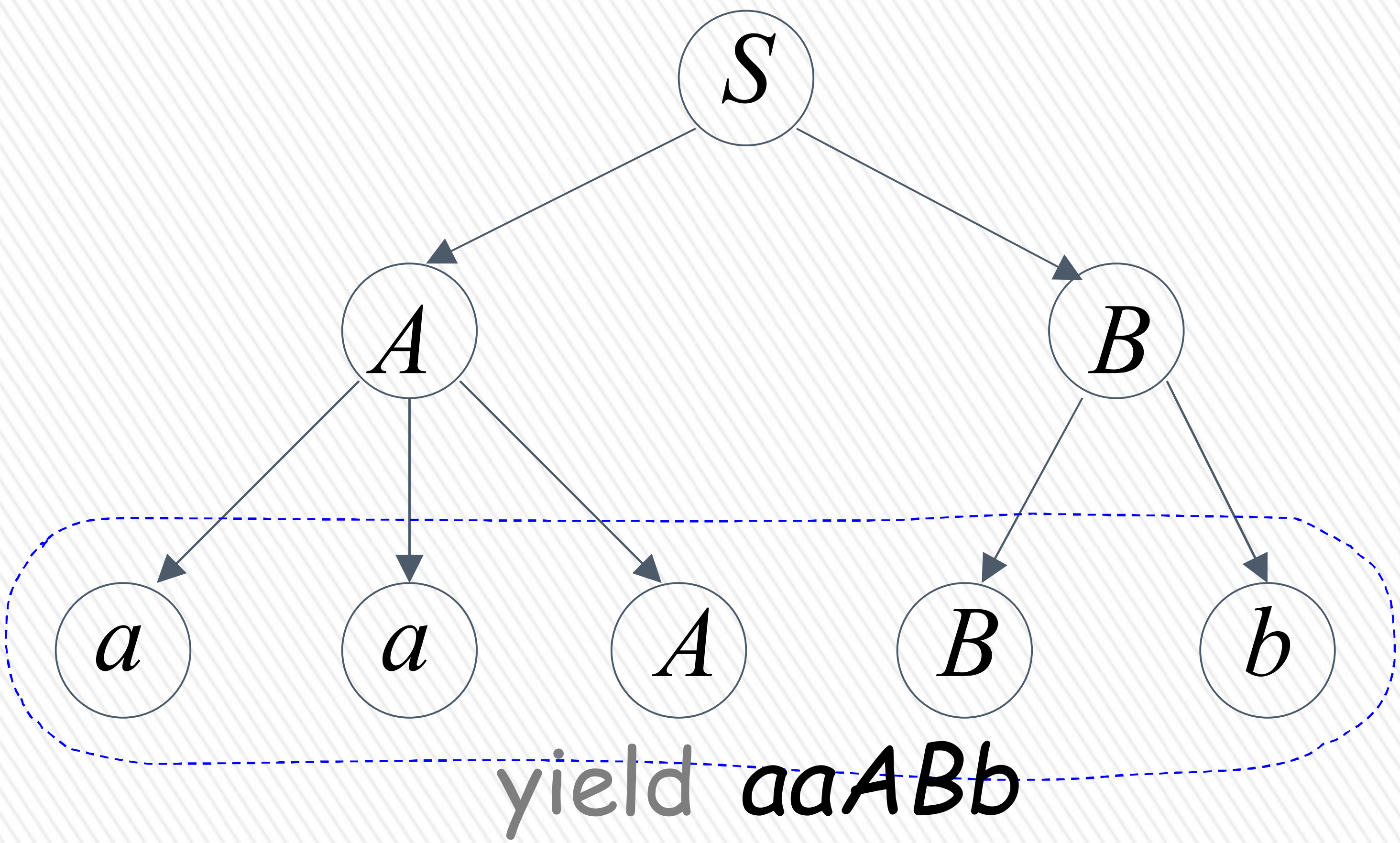
$$S \Rightarrow AB \Rightarrow aaAB$$



yield **aaAB**

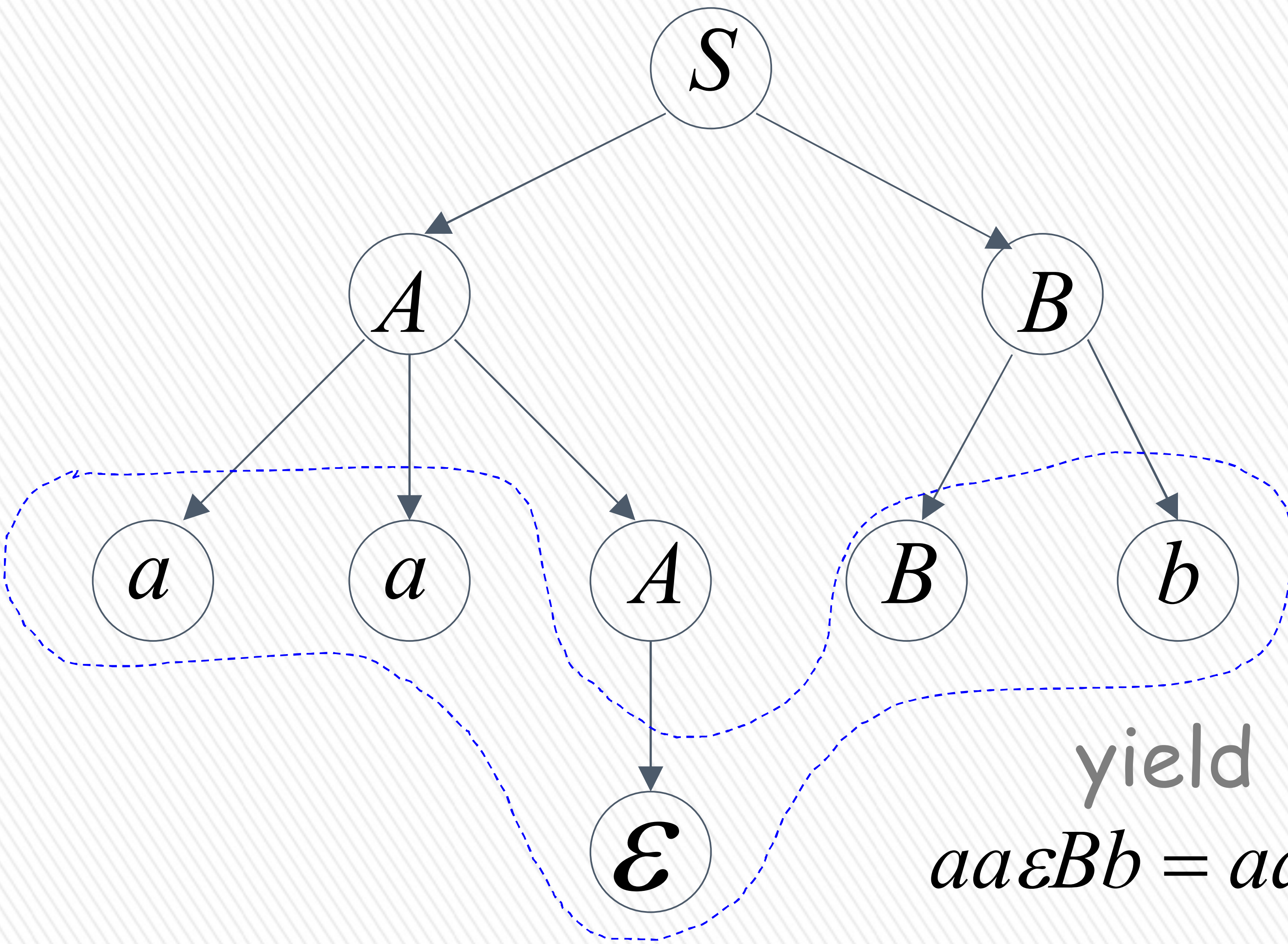
$$S \to AB \qquad A \to aaA \mid \varepsilon \quad B \to Bb \mid \varepsilon$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb$$



$$S \to AB \qquad A \to aaA \mid \varepsilon \quad B \to Bb \mid \varepsilon$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb$$



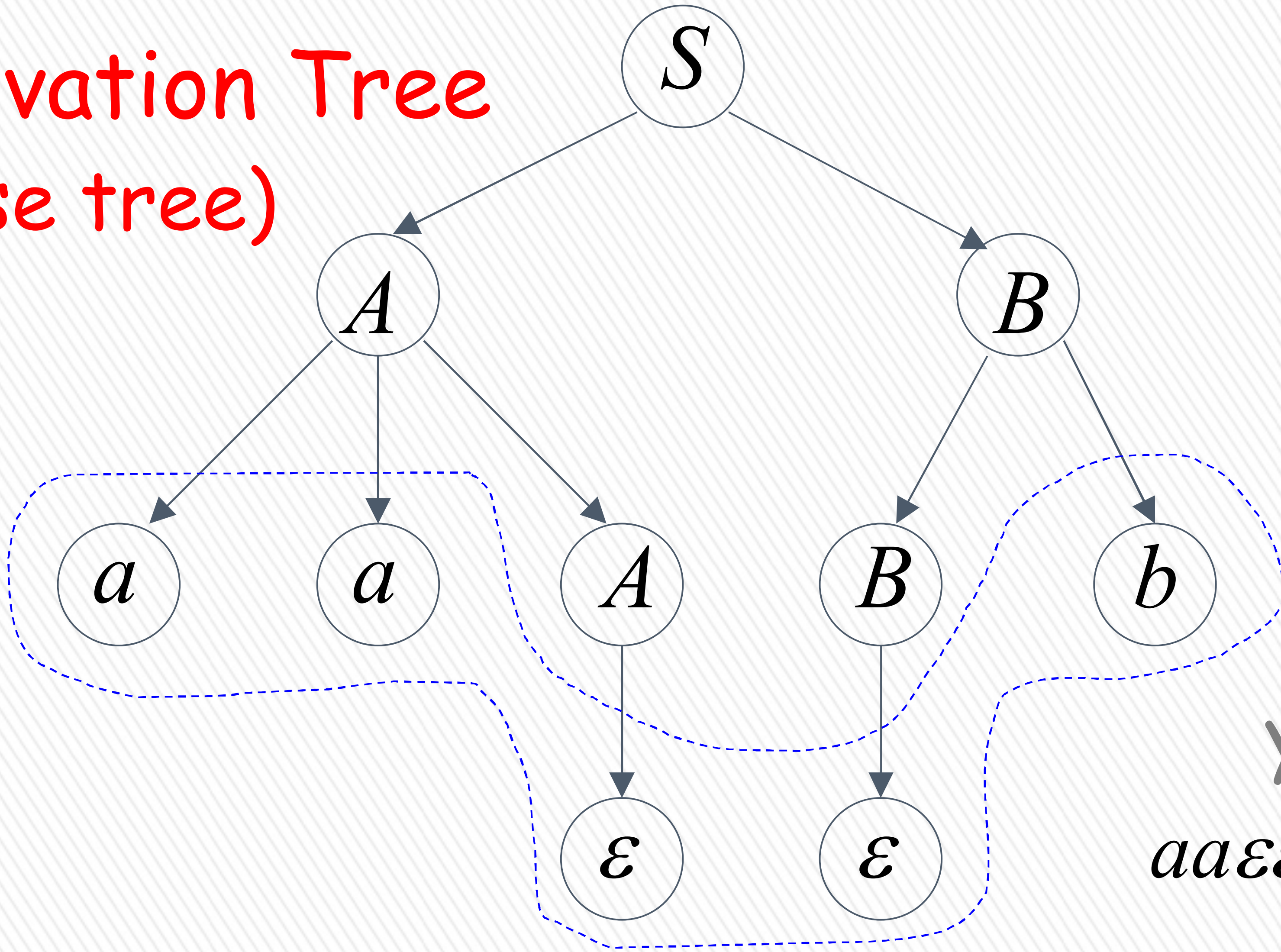
yield

$$aa\varepsilon Bb = aaBb$$

$$S \to AB \qquad A \to aaA \mid \varepsilon \quad B \to Bb \mid \varepsilon$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aab$$

Derivation Tree
(parse tree)



yield

$$aa\varepsilon\varepsilon b = aab$$

Sometimes, derivation order doesn't matter

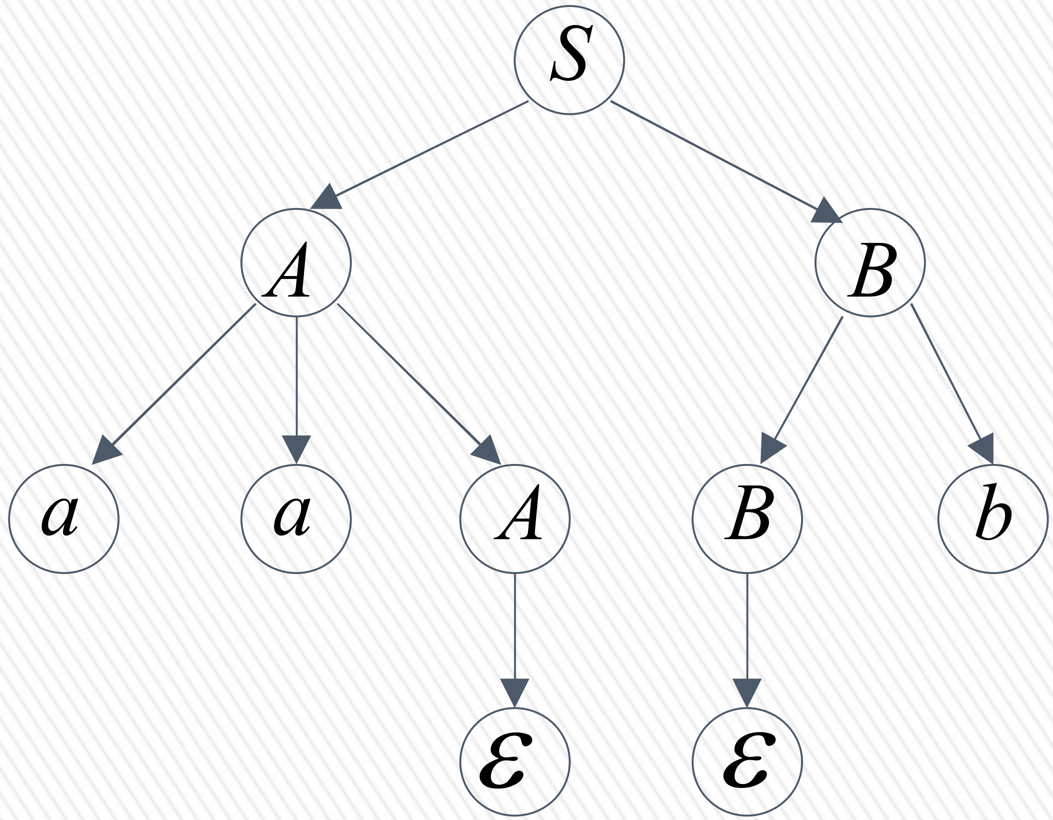
Leftmost derivation:

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaB \Rightarrow aaBb \Rightarrow aab$$

Rightmost derivation:

$$S \Rightarrow AB \Rightarrow ABb \Rightarrow Ab \Rightarrow aaAb \Rightarrow aab$$

Give same derivation tree



Ambiguity

Grammar for mathematical expressions

$$E \rightarrow E + E \quad | \; E * E \; | \; (E) \; | \; a$$

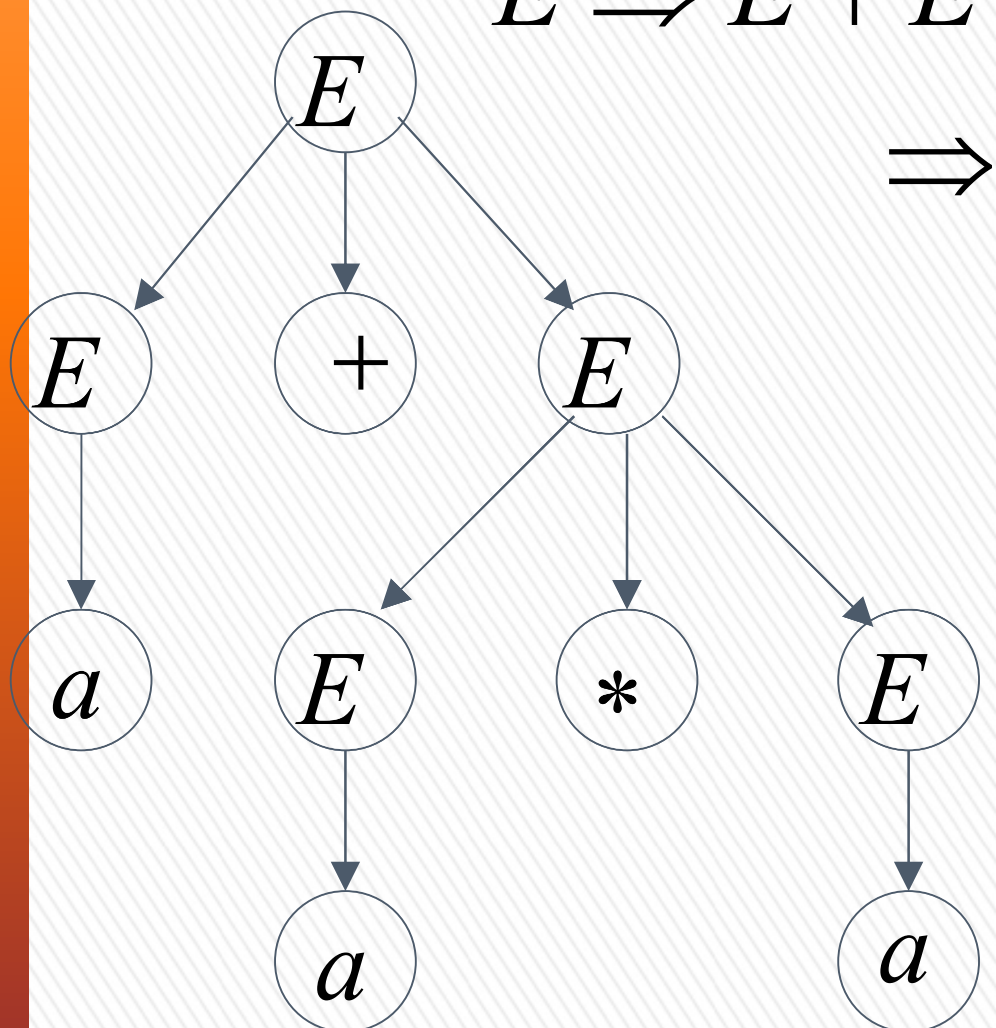
Example strings:

$$(a + a) * a + (a + a * (a + a))$$

Denotes any number

$$E \to E + E \mid E * E \mid (E) \mid a$$

$$E \Rightarrow E + E \Rightarrow a + E \Rightarrow a + E * E$$
$$\Rightarrow a + a * E \Rightarrow a + a * a$$



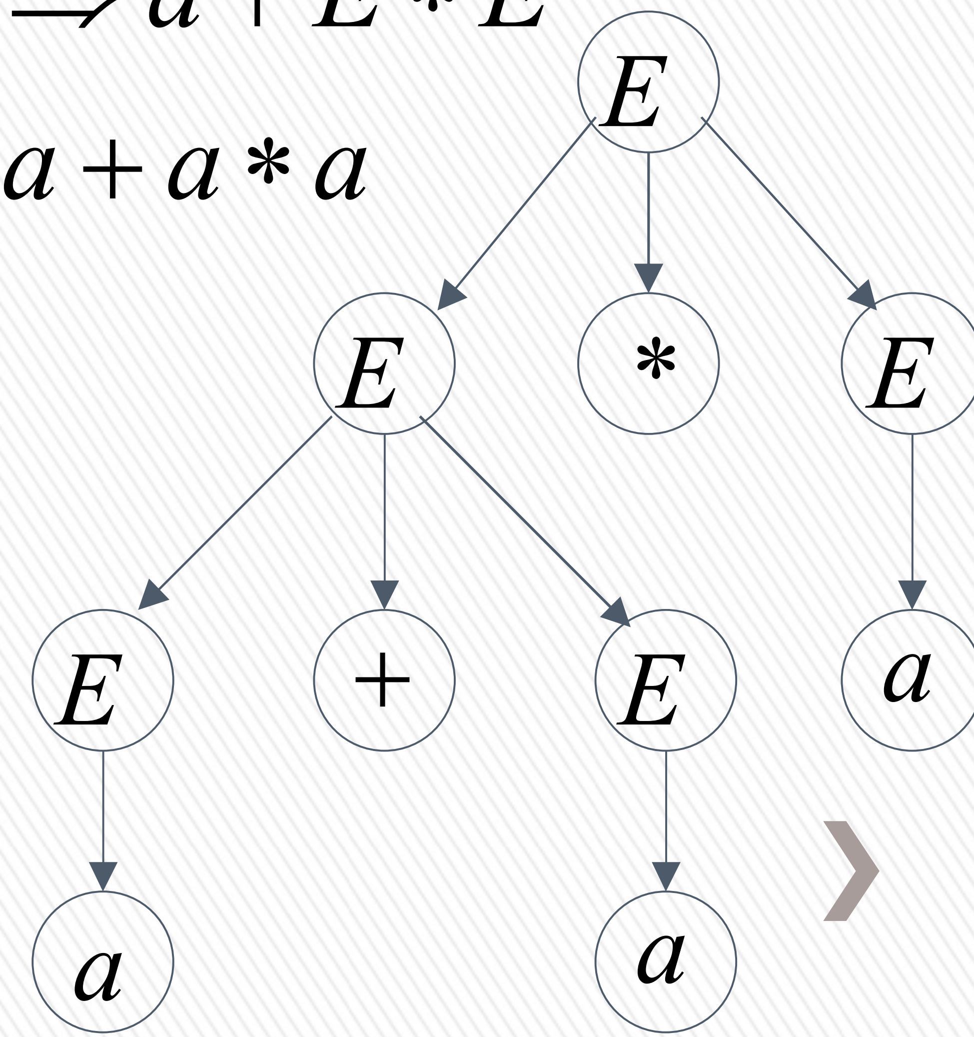
A leftmost derivation for $a + a * a$

$$E \to E + E \mid E * E \mid (E) \mid a$$

$$E \Rightarrow E * E \Rightarrow E + E * E \Rightarrow a + E * E$$

$$\Rightarrow a + a * E \Rightarrow a + a * a$$

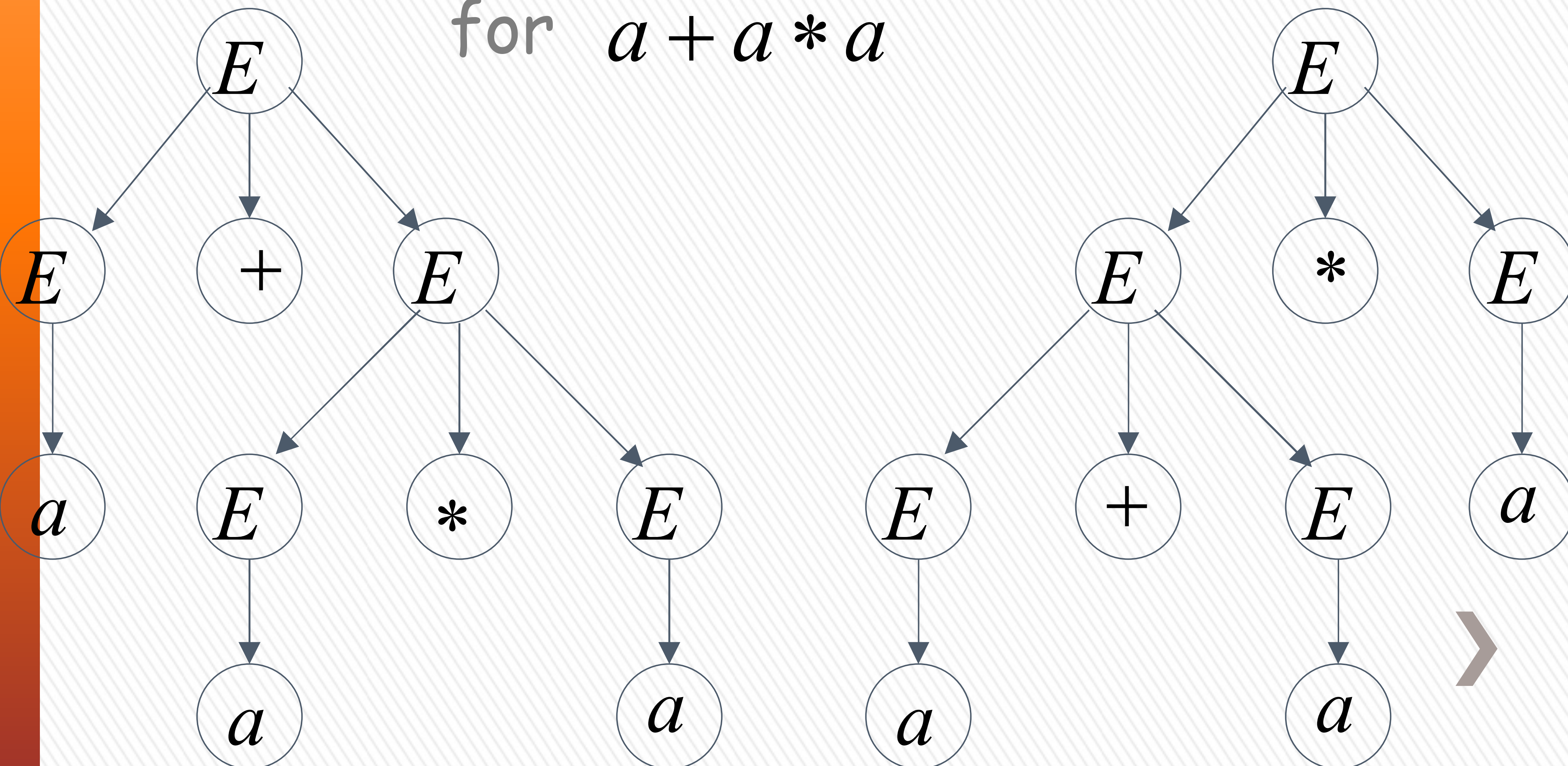
Another leftmost derivation for $a + a * a$



$$E \to E + E \mid E * E \mid (E) \mid a$$

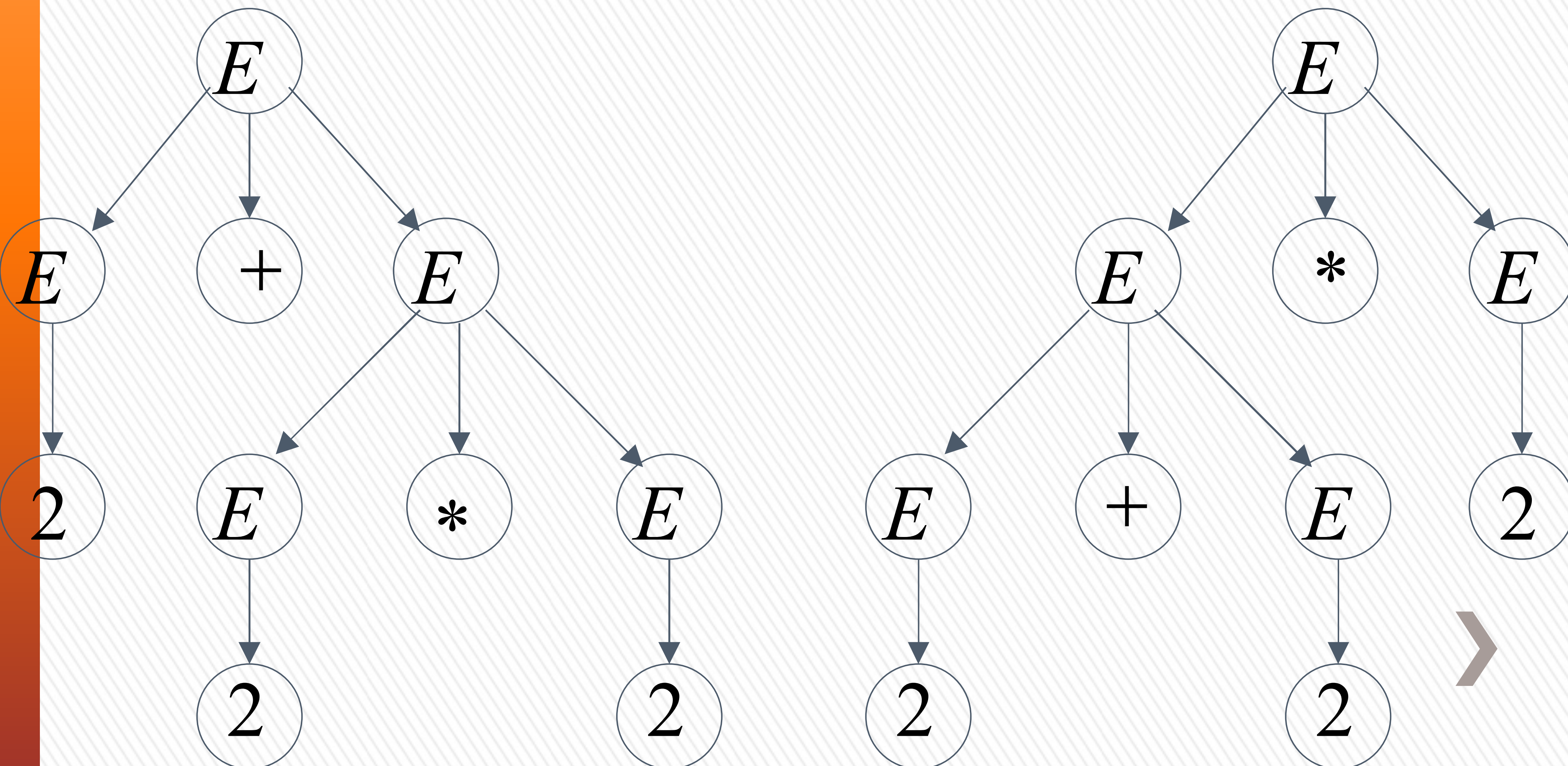
Two derivation trees

for $a + a * a$



take $a = 2$

$$a + a * a = 2 + 2 * 2$$



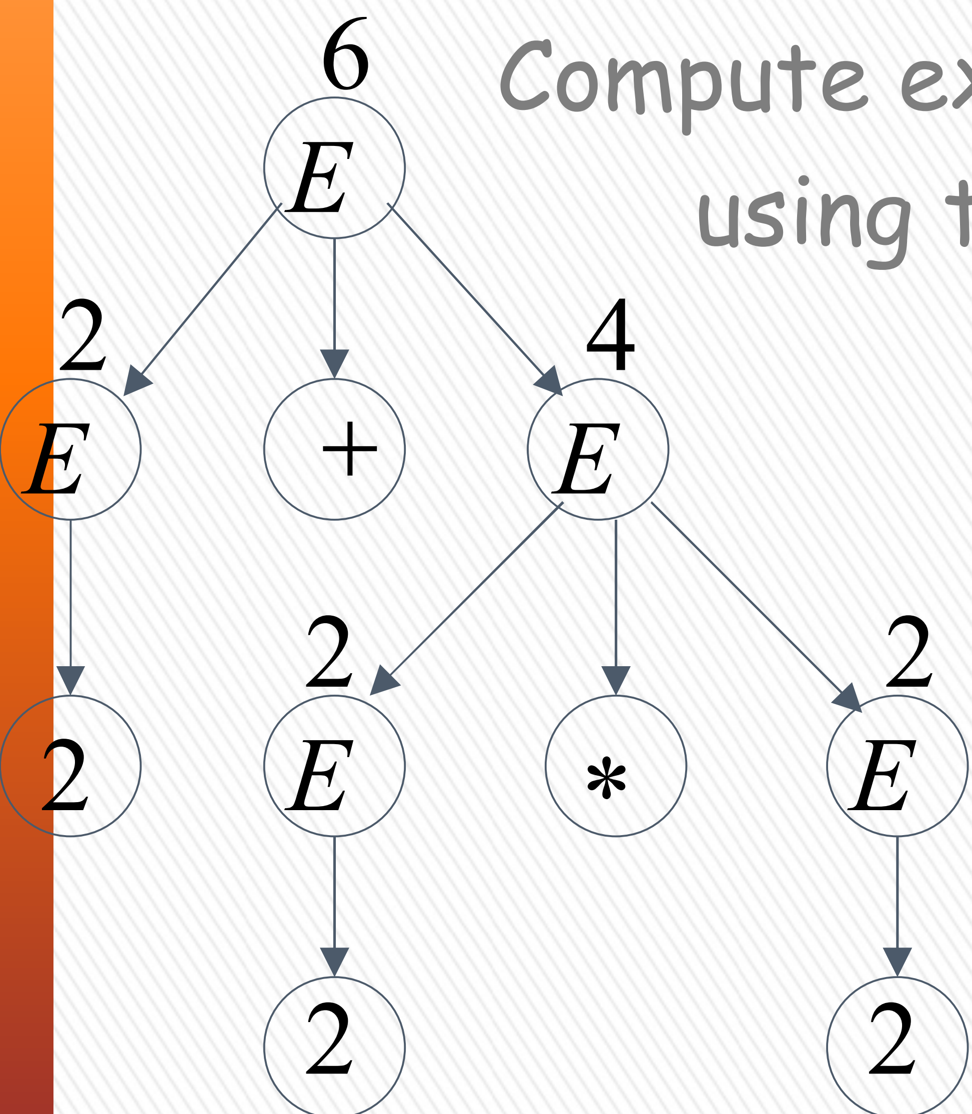
Good Tree

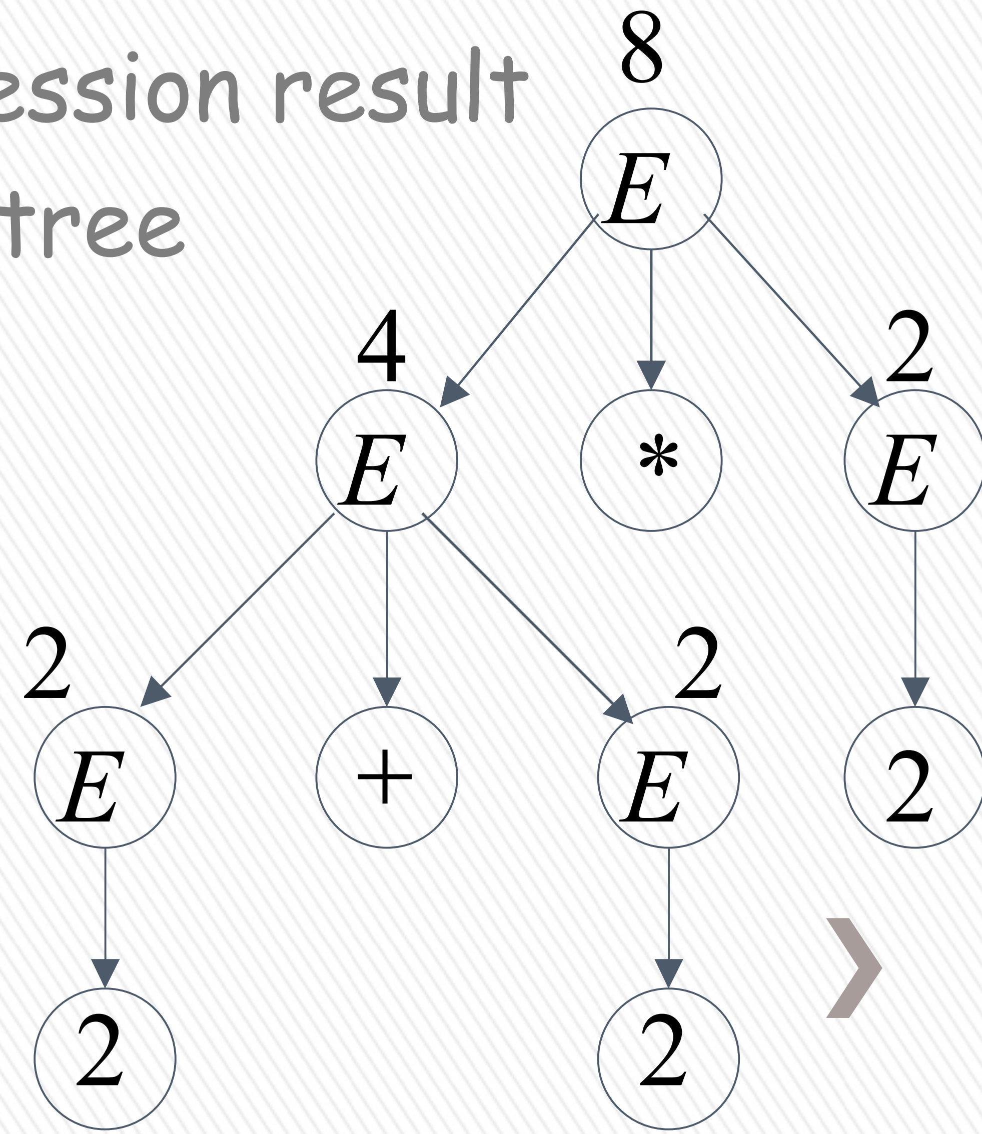
$$2+2*2=6$$

Bad Tree

$$2+2*2=8$$

Compute expression result using the tree





Two different derivation trees
may cause problems in applications which
use the derivation trees:

- Evaluating expressions

- In general, in compilers
  for programming languages

Ambiguous Grammar:

A context-free grammar G is ambiguous if there is a string $w \in L(G)$ which has:

two different derivation trees

or

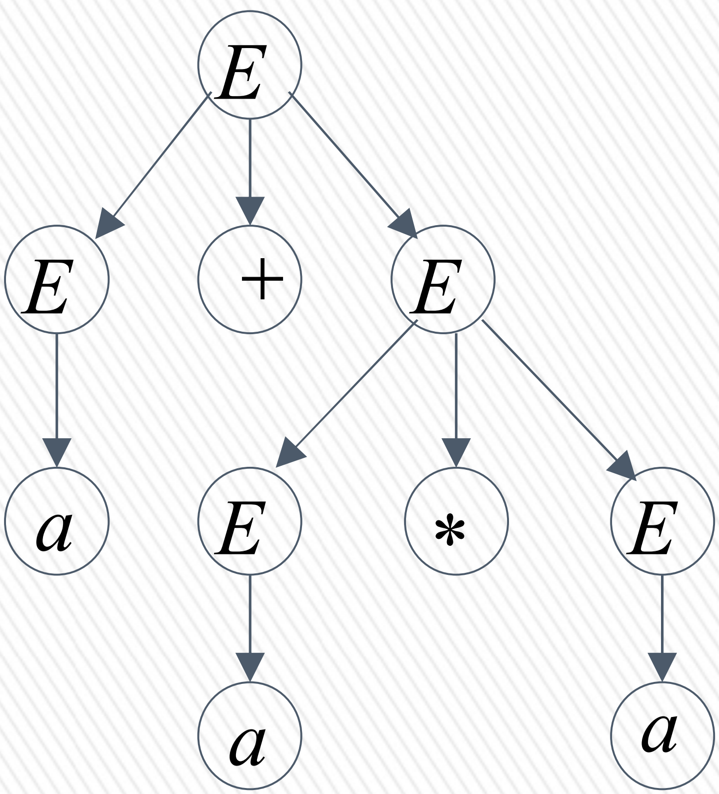
two leftmost derivations

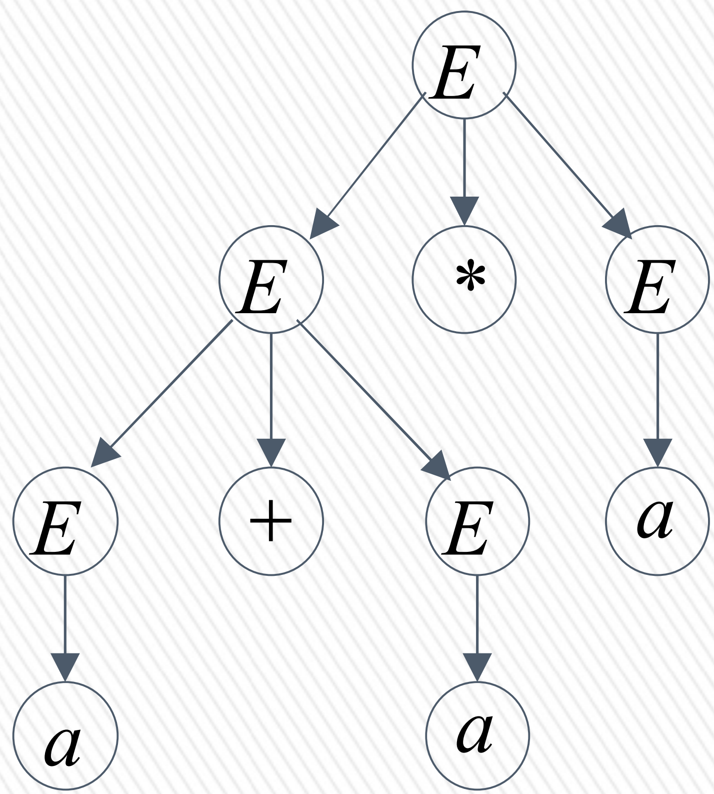
(Two different derivation trees give two different leftmost derivations and vice-versa)

Example: $E \to E + E \mid E * E \mid (E) \mid a$

this grammar is ambiguous since

string $a + a * a$ has two derivation trees





$$E \to E + E \mid E * E \mid (E) \mid a$$

this grammar is ambiguous also because

string $a + a * a$ has two leftmost derivations

$$E \Rightarrow E + E \Rightarrow a + E \Rightarrow a + E * E$$
$$\Rightarrow a + a * E \Rightarrow a + a * a$$

$$E \Rightarrow E * E \Rightarrow E + E * E \Rightarrow a + E * E$$
$$\Rightarrow a + a * E \Rightarrow a + a * a$$

Another ambiguous grammar:

IF_STMT → if EXPR then STMT

| if EXPR then STMT else STMT

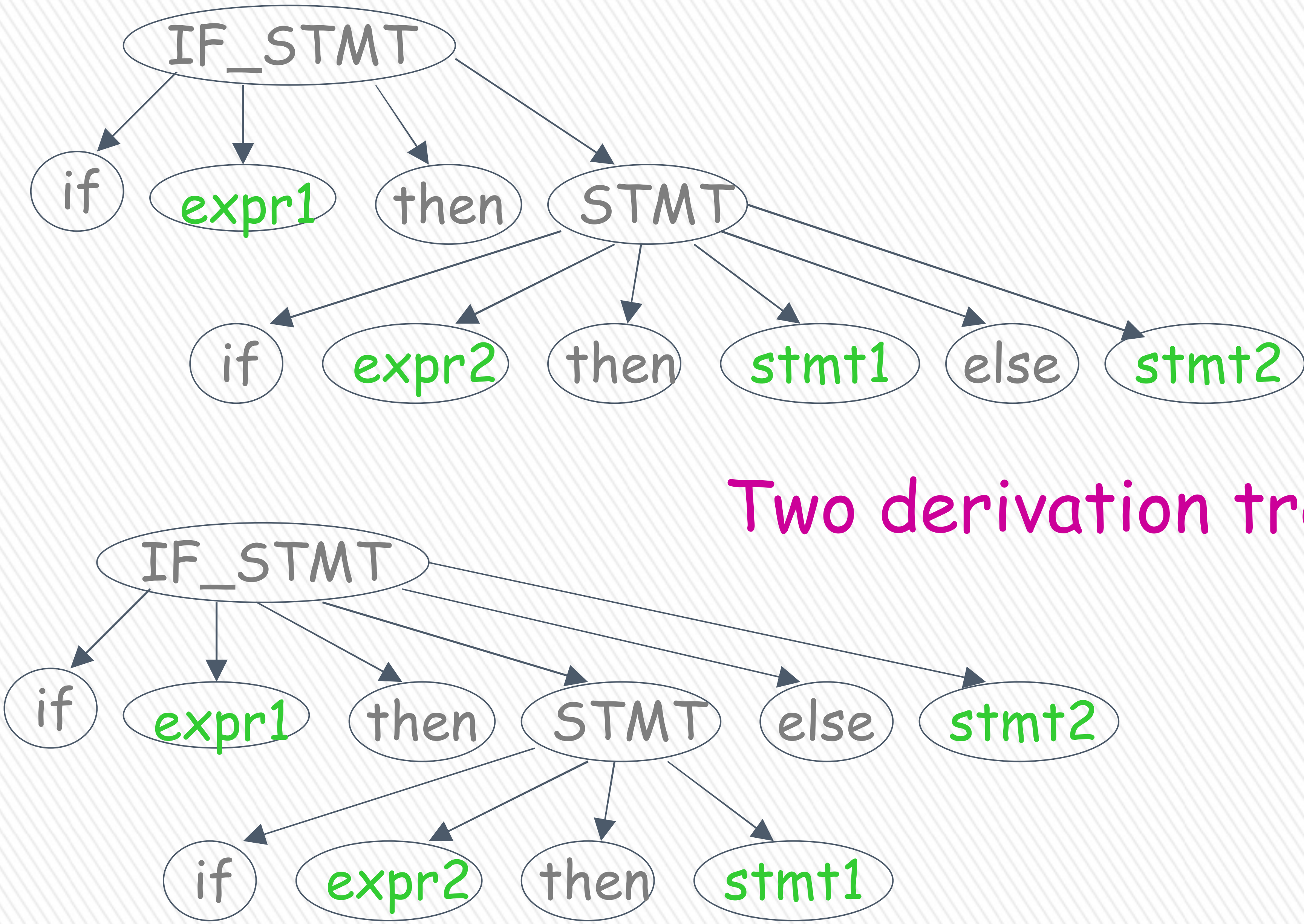
Variables

Terminals

Very common piece of grammar
in programming languages

❯

If expr1 then if expr2 then stmt1 else stmt2



Two derivation trees

In general, ambiguity is bad
and we want to remove it

Sometimes it is possible to find
a non-ambiguous grammar for a language

But, in general we cannot do so

A successful example:

Ambiguous Grammar

$$E \to E + E$$
$$E \to E * E$$
$$E \to (E)$$
$$E \to a$$

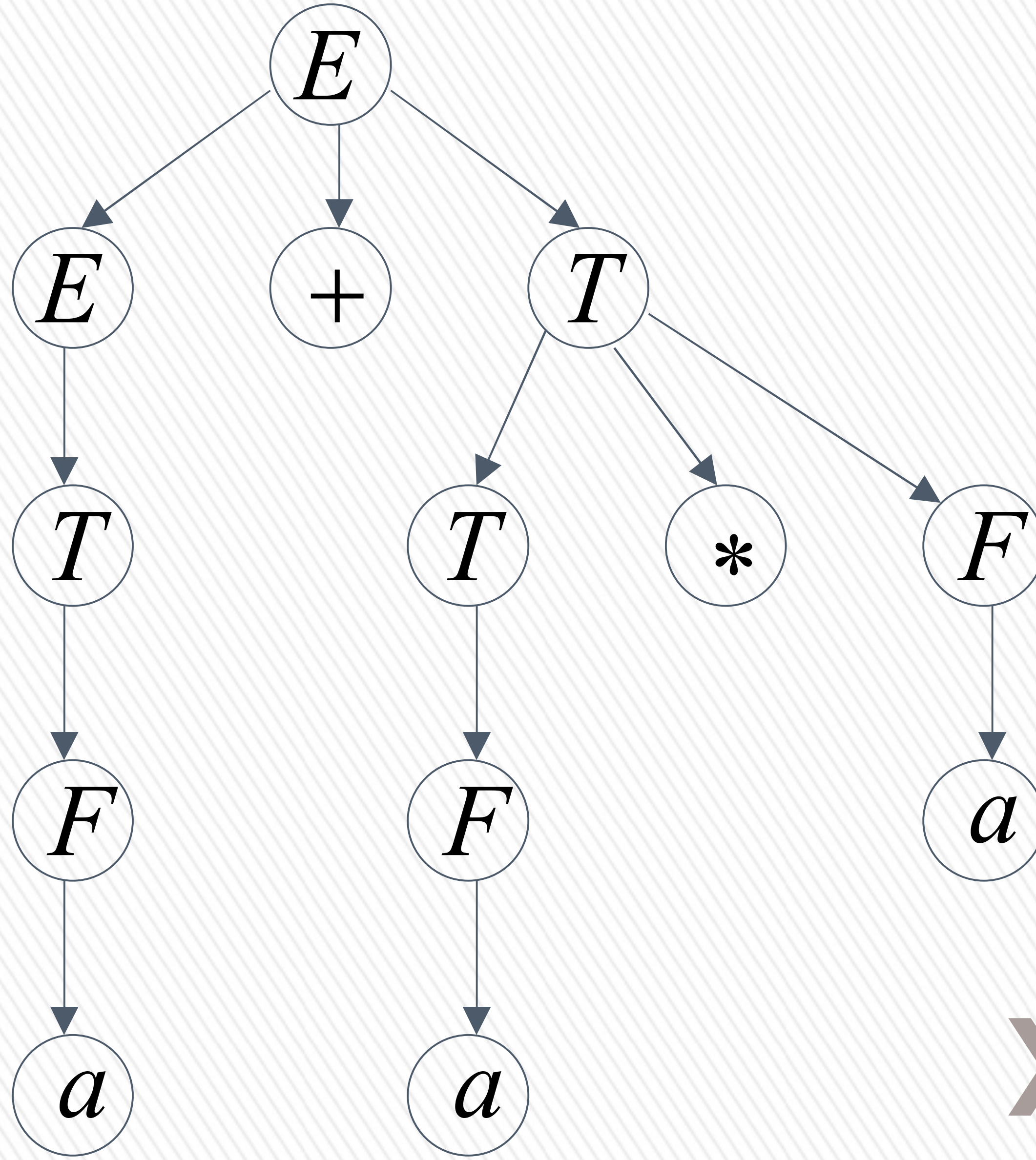
Equivalent Non-Ambiguous Grammar

$$E \to E + T \mid T$$
$$T \to T * F \mid F$$
$$F \to (E) \mid a$$

generates the same language

$$E \Rightarrow E+T \Rightarrow T+T \Rightarrow F+T \Rightarrow a+T \Rightarrow a+T*F$$
$$\Rightarrow a+F*F \Rightarrow a+a*F \Rightarrow a+a*a$$

$$E \rightarrow E+T \mid T$$
$$T \rightarrow T*F \mid F$$
$$F \rightarrow (E) \mid a$$



Unique derivation tree for $a+a*a$

An un-successful example:

$$L = \{a^n b^n c^m\} \cup \{a^n b^m c^m\}$$

$$n, m \geq 0$$

L is inherently ambiguous:

every grammar that generates this language is ambiguous

Example (ambiguous) grammar for L:

$$L = \{a^n b^n c^m\} \cup \{a^n b^m c^m\}$$

$$S \to S_1 \mid S_2$$

$$S_1 \to S_1 c \mid A$$

$$A \to aAb \mid \lambda$$

$$S_2 \to aS_2 \mid B$$

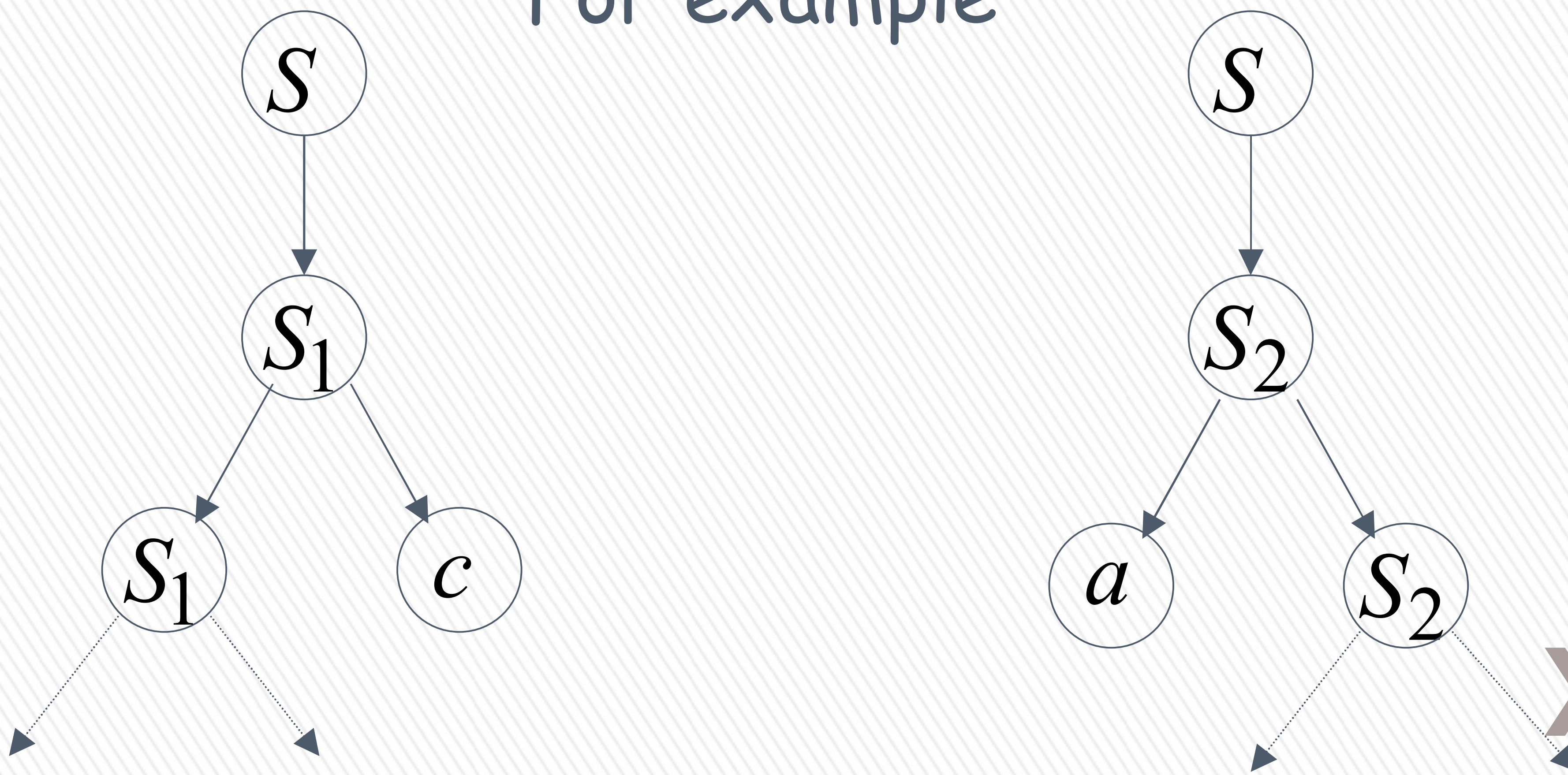
$$B \to bBc \mid \lambda$$

The string $a^n b^n c^n \in L$

has always two different derivation trees

(for any grammar)

For example



BLM2502 Theory of Computation

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