BLM2041 Signals and Systems

The Instructors:

Prof. Dr. Nizamettin Aydın

naydin@yildiz.edu.tr

Asist. Prof. Dr. Ferkan Yilmaz

ferkan@yildiz.edu.tr

BLM2041 Signals and Systems

Fourier Transform

LECTURE OBJECTIVES

- Review
  - Frequency Response
  - Fourier Series
- Definition of Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

  - Relation to Fourier Series
- Examples of Fourier transform pairs
- Basic properties of Fourier transforms
  - **Convolution** property
  - **Multiplication** property

WHY use the Fourier transform?

- Manipulate the **Frequency Spectrum**
- Analog Communication Systems
  - AM: Amplitude Modulation; FM
  - What are the **Building Blocks** ?
    - **Abstract Layer**, not implementation
- Ideal Filters
  - mostly BPFs
- Frequency Shifters
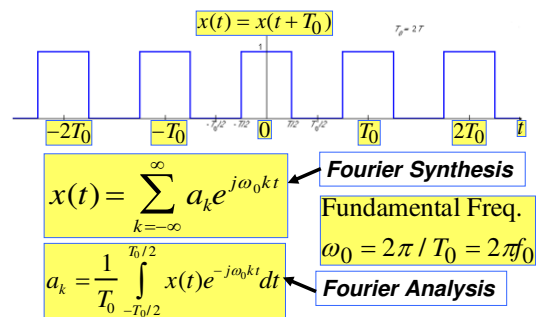 - aka Modulators, Mixers or Multipliers: $x(t) \times p(t)$

Everything = Sum of Sinusoids

- One Square Pulse = Sum of Sinusoids

  - ???????????

- Finite Length
- Not Periodic

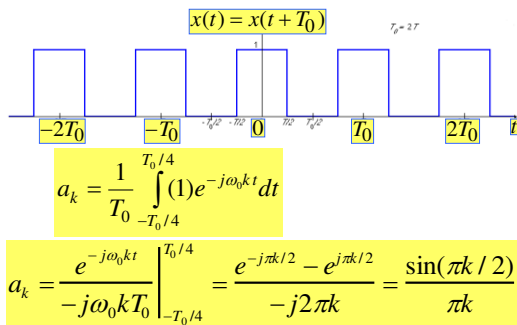- Limit of Square Wave as Period → infinity
  - Intuitive Argument

Fourier Series: Periodic $x(t)$
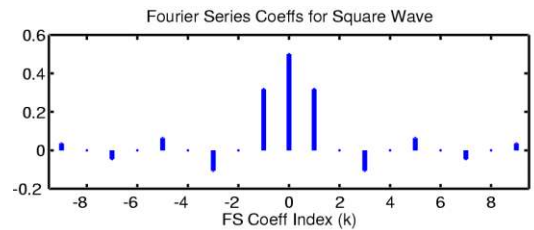
$$x(t) = x(t + T_0)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$

Fourier Synthesis

Fundamental Freq.

$$\omega_0 = 2\pi / T_0 = 2\pi f_0$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)e^{-j\omega_0 kt}dt$$

Fourier Analysis

1

Square Wave Signal

$$x(t) = x(t + T_0)$$

$T_0 = 2T$

$$a_k = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} (1) e^{-j\omega_0 kt} dt$$

$$a_k = \frac{e^{-j\omega_0 kt}}{-j\omega_0 k T_0} \bigg|_{-T_0/4}^{T_0/4} = \frac{e^{-j\pi k/2} - e^{j\pi k/2}}{-j2\pi k} = \frac{\sin(\pi k/2)}{\pi k}$$

7

Spectrum from Fourier Series

$$a_k = \frac{\sin(\pi k/2)}{\pi k} = \begin{cases} \neq 0 & k = 0, \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \end{cases}$$

Fourier Series Coeffs for Square Wave

8

What if $x(t)$ is not periodic?

• Sum of Sinusoids?
  – Non-harmonically related sinusoids
  – Would not be periodic,
    • but would probably be non-zero for all *t*.
• Fourier transform
  – gives a "sum" (actually an integral) that involves **ALL** frequencies
  – can represent signals that are identically zero for negative *t*. !!!!!!!!!

9

Limiting Behavior of FS

10
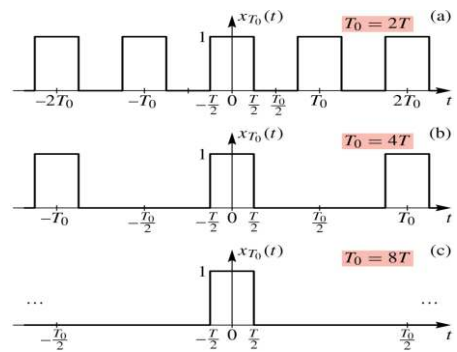
Limiting Behavior of Spectrum

11
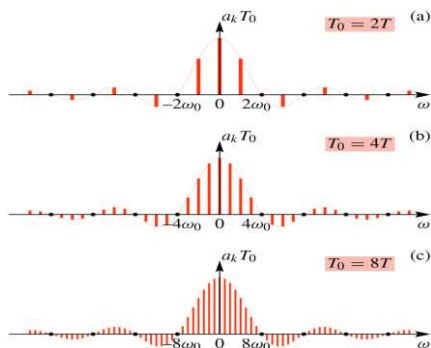
FS in the LIMIT (long period)

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} (T_0 a_k) e^{j\omega_0 kt} \left(\frac{2\pi}{T_0}\right) \mapsto x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

$$\lim_{T_0 \to \infty} \frac{2\pi}{T_0} = d\omega \qquad \lim_{T_0 \to \infty} \frac{2\pi}{T_0} k = \omega \qquad \lim_{T_0 \to \infty} T_0 a_k = X(j\omega)$$

$$T_0 a_k = \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_0 kt} dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

12

2

Fourier Transform Defined

• For non-periodic signals

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega$$

Fourier Synthesis (Inverse Transform)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

Fourier Analysis (Forward Transform)

$$\text{Time - Domain} \Leftrightarrow \text{Frequency - Domain}$$
$$x(t) \Leftrightarrow X(j\omega)$$

13

Example 1

$$x(t) = e^{-at}u(t)$$

$$X(j\omega) = \int_{0}^{\infty} e^{-at}e^{-j\omega t} dt = \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$a > 0$$

$$X(j\omega) = -\frac{e^{-at}e^{-j\omega t}}{a+j\omega}\bigg|_{0}^{\infty} = \frac{1}{a+j\omega}$$
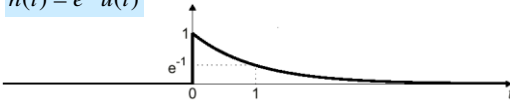
$$X(j\omega) = \frac{1}{a+j\omega}$$

14

Example 1 - Frequency Response

• Fourier Transform of $h(t)$ is
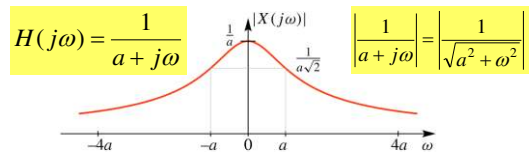 – the Frequency Response

$$h(t) = e^{-t}u(t)$$

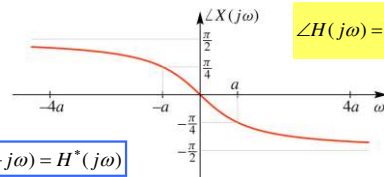$$h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1+j\omega}$$

15

Example 1 - Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{a+j\omega}$$

$$\left|\frac{1}{a+j\omega}\right| = \left|\frac{1}{\sqrt{a^2+\omega^2}}\right|$$

$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

$$H(-j\omega) = H^*(j\omega)$$

16

Example 2

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}$$

$$X(j\omega) = \int_{-T/2}^{T/2} (1)e^{-j\omega t} dt = \int_{-T/2}^{T/2} e^{-j\omega t} dt$$
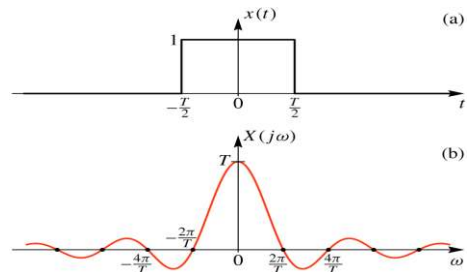
$$X(j\omega) = \frac{e^{-j\omega t}}{-j\omega}\bigg|_{-T/2}^{T/2} = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

17

Example 2

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

18

3

Example 3

$$X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi} \int_{-\omega_b}^{\omega_b} 1e^{j\omega t}d\omega$$

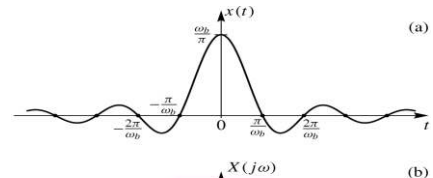$$x(t) = \frac{1}{2\pi} \frac{e^{j\omega t}}{jt}\bigg|_{-\omega_b}^{\omega_b} = \frac{1}{2\pi} \frac{e^{j\omega_b t} - e^{-j\omega_b t}}{jt}$$

$$x(t) = \frac{\sin(\omega_b t)}{\pi t}$$

19

Example 3

$$x(t) = \frac{\sin(\omega_b t)}{\pi t} \iff X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$

20

Example 4

$$x(t) = \delta(t - t_0)$$

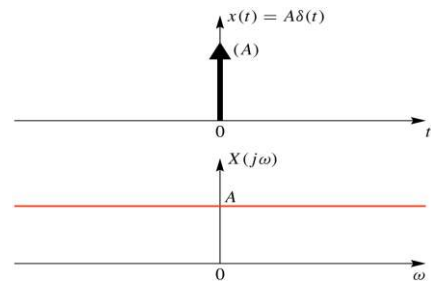$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = 1$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0)e^{-j\omega t}dt = e^{-j\omega t_0}$$

Shifting Property of the Impulse

21

Impulse function – Time and Frequency domains

$$x(t) = \delta(t) \iff X(j\omega) = 1$$

22

Example 5

$$X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

$$x(t) = e^{j\omega_0 t} \iff X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

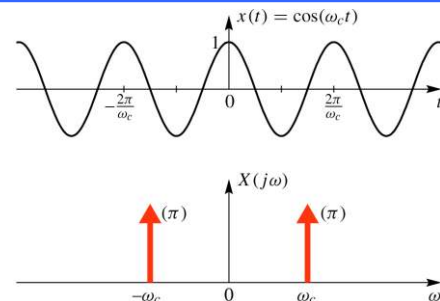$$x(t) = 1 \iff X(j\omega) = 2\pi\delta(\omega)$$

$$x(t) = \cos(\omega_0 t) \iff$$
$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

23

Example 5

$$x(t) = \cos(\omega_c t) \iff X(j\omega) = \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)$$

24

Table of Fourier Transforms

$$x(t) = e^{-at}u(t) \Leftrightarrow X(j\omega) = \frac{1}{a+j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

25

Fourier Transform of a General Periodic Signal
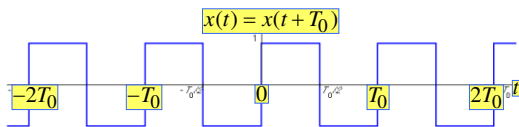
• If $x(t)$ is periodic with period T_0,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt$$

Therefore, since $e^{jk\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - k\omega_0)$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$
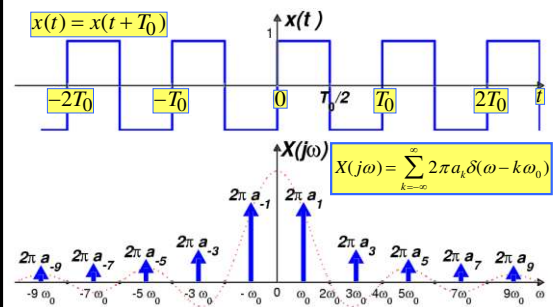
26

Square Wave Signal

$$x(t) = x(t + T_0)$$

$$a_k = \frac{1}{T_0}\int_0^{T_0/2}(1)e^{-j\omega_0 kt}dt + \frac{1}{T_0}\int_{T_0/2}^{T_0}(-1)e^{-j\omega_0 kt}dt$$

$$a_k = \frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\Big|_0^{T_0/2} - \frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\Big|_{T_0/2}^{T_0} = \frac{1 - e^{-j\pi k}}{j\pi k}$$

27

Square Wave Fourier Transform

$$x(t) = x(t + T_0)$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

28

Table of Easy FT Properties

Linearity Property

$$ax_1(t) + bx_2(t) \Leftrightarrow aX_1(j\omega) + bX_2(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d}X(j\omega)$$

Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|}X(j(\frac{\omega}{a}))$$
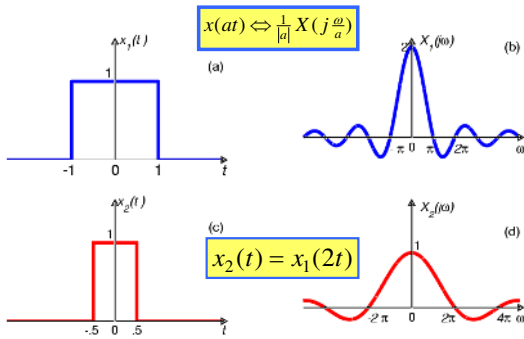
29

Scaling Property

$$x(at) \Leftrightarrow \frac{1}{|a|}X(j\frac{\omega}{a})$$

$$\int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\lambda)e^{-j\omega(\lambda/a)}\frac{d\lambda}{|a|}$$

$$= \frac{1}{|a|}X(j\frac{\omega}{a})$$

$x(2t)$ shrinks; $\frac{1}{2}X(j\frac{\omega}{2})$ expands

30

5

Scaling Property

$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$

$$x_2(t) = x_1(2t)$$

Uncertainty Principle

- Try to make $x(t)$ shorter
  - Then $X(j\omega)$ will get wider
  - Narrow pulses have wide bandwidth

- Try to make $X(j\omega)$ narrower
  - Then $x(t)$ will have longer duration

- **Cannot simultaneously reduce time duration and bandwidth**

Significant FT Properties

$$x(t)*h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

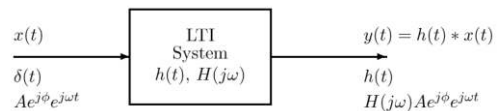$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega)*P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega-\omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

Convolution Property

- Convolution in the time-domain

$$y(t) = h(t)*x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

corresponds to **MULTIPLICATION** in the frequency-domain

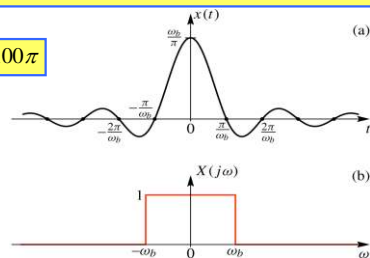$$Y(j\omega) = H(j\omega)X(j\omega)$$

Convolution Example

- Bandlimited **Input** Signal
  - "sinc" function

- Ideal LPF (Lowpass Filter)
  - **h(t)** is a "sinc"

- **Output** is Bandlimited
  - Convolve "sincs"

Ideally Bandlimited Signal

$$x(t) = \frac{\sin(100\pi t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < 100\pi \\ 0 & |\omega| > 100\pi \end{cases}$$
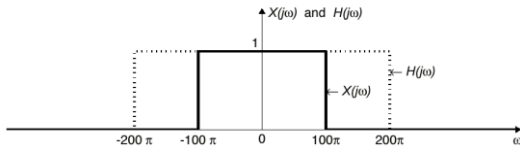
$$\omega_b = 100\pi$$

6

Convolution Example 1

$$x(t)*h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$\frac{\sin(100\pi t)}{\pi t} * \frac{\sin(200\pi t)}{\pi t} = \frac{\sin(100\pi t)}{\pi t}$$

37

Cosine Input to LTI System
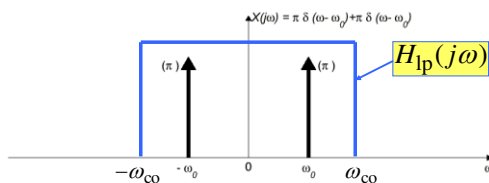
$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$= H(j\omega)[\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$

$$= H(j\omega_0)\pi\delta(\omega - \omega_0) + H(-j\omega_0)\pi\delta(\omega + \omega_0)$$

$$
\begin{aligned}
y(t) &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H(-j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\
&= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H^*(j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\
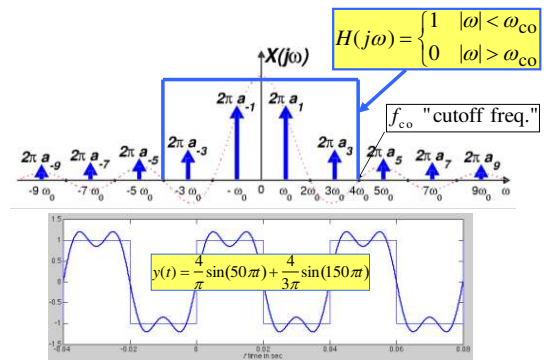&= |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0))
\end{aligned}
$$

38

Ideal Lowpass Filter

$$H_{lp}(j\omega)$$

$$y(t) = x(t) \qquad \text{if } \omega_0 < \omega_{co}$$
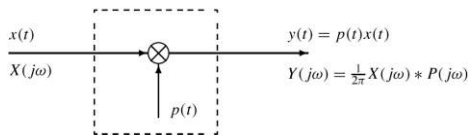
$$y(t) = 0 \qquad \text{if } \omega_0 > \omega_{co}$$

39

Ideal Lowpass Filter

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$

$$f_{co} \text{ "cutoff freq."}$$

$$y(t) = \frac{4}{\pi}\sin(50\pi) + \frac{4}{3\pi}\sin(150\pi)$$

40

Signal Multiplier (Modulator)

$$x(t)$$
$$X(j\omega)$$
$$y(t) = p(t)x(t)$$
$$Y(j\omega) = \frac{1}{2\pi}X(j\omega)*P(j\omega)$$
$$p(t)$$

• Multiplication in the time-domain corresponds to convolution in the frequency-domain.

$$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega - \theta))d\theta$$

41
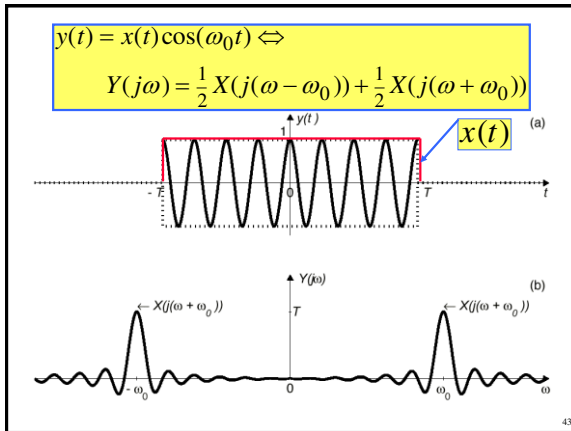
Frequency Shifting Property

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

$$\int_{-\infty}^{\infty} e^{j\omega_0 t}x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega - \omega_0)t}dt$$

$$= X(j(\omega - \omega_0))$$

$$y(t) = \frac{\sin 7t}{\pi t}e^{j\omega_0 t} \Leftrightarrow Y(j\omega) = \begin{cases} 1 & \omega_0 - 7 < \omega < \omega_0 + 7 \\ 0 & elsewhere \end{cases}$$

42

$$y(t) = x(t)\cos(\omega_0 t) \Leftrightarrow$$

$$Y(j\omega) = \frac{1}{2} X(j(\omega - \omega_0)) + \frac{1}{2} X(j(\omega + \omega_0))$$

$x(t)$ (a)

(b)

Differentiation Property

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega)X(j\omega)e^{j\omega t}d\omega$$

Multiply by $j\omega$

$$\frac{d}{dt}\left(e^{-at}u(t)\right) = -ae^{-at}u(t) + e^{-at}\delta(t)$$

$$= \delta(t) - ae^{-at}u(t)$$

$$\Leftrightarrow \frac{j\omega}{a + j\omega}$$

8