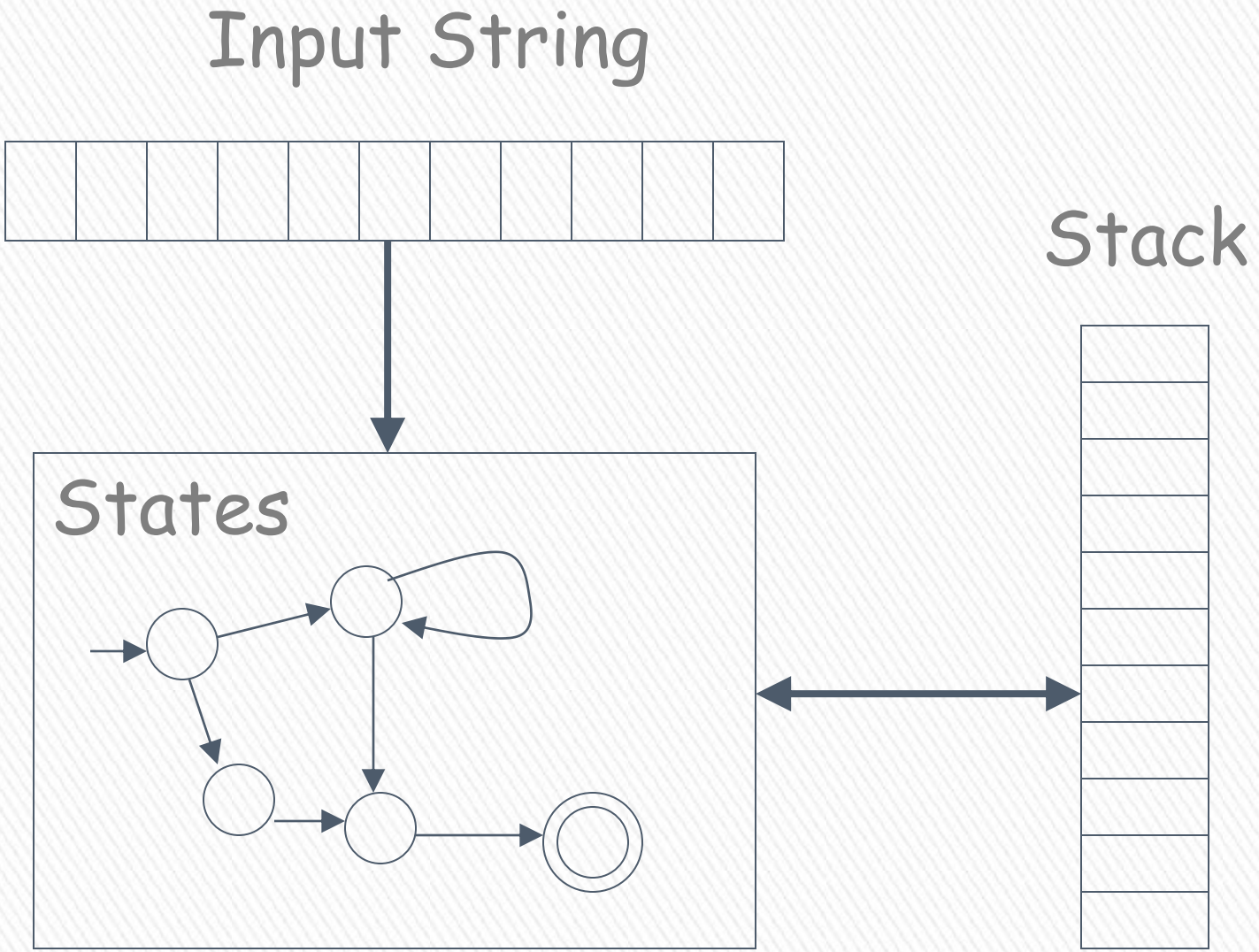
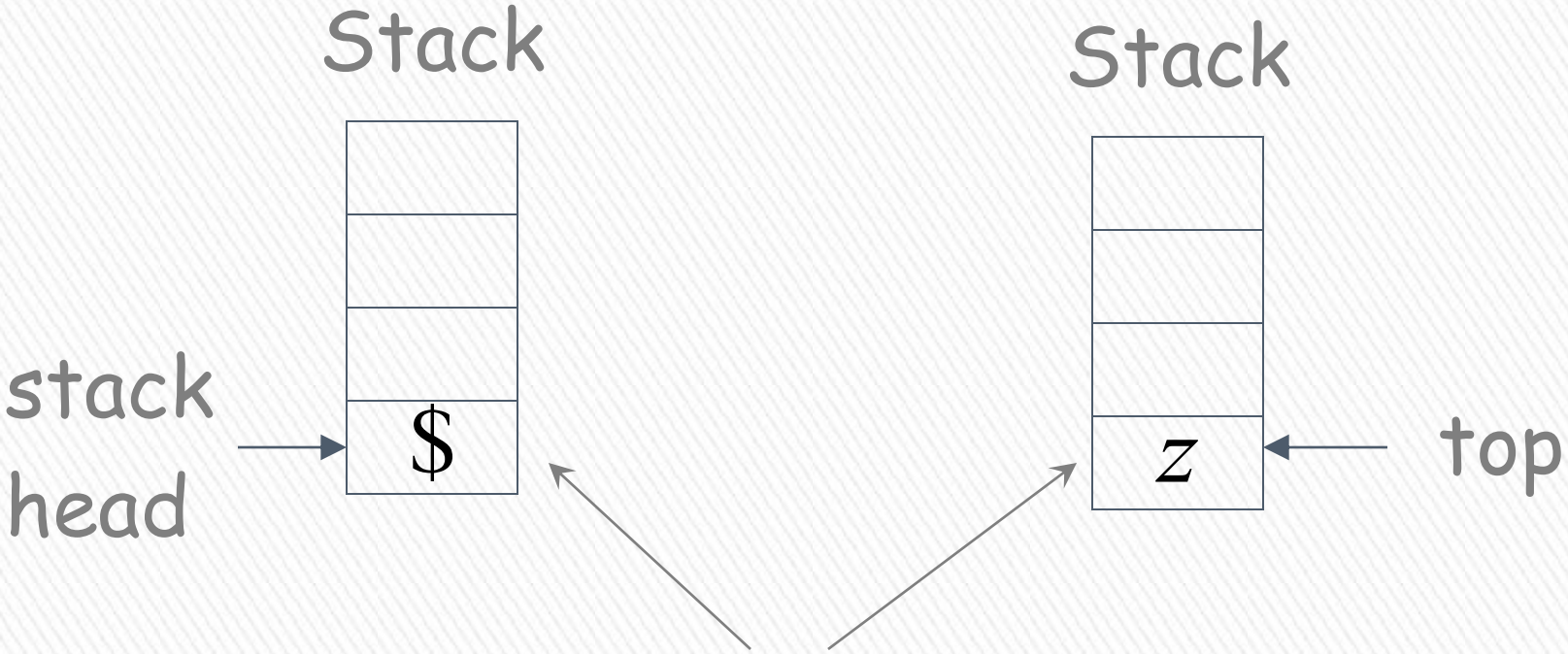
BLM2502 Theory of Computation

Pushdown Automata
PDA

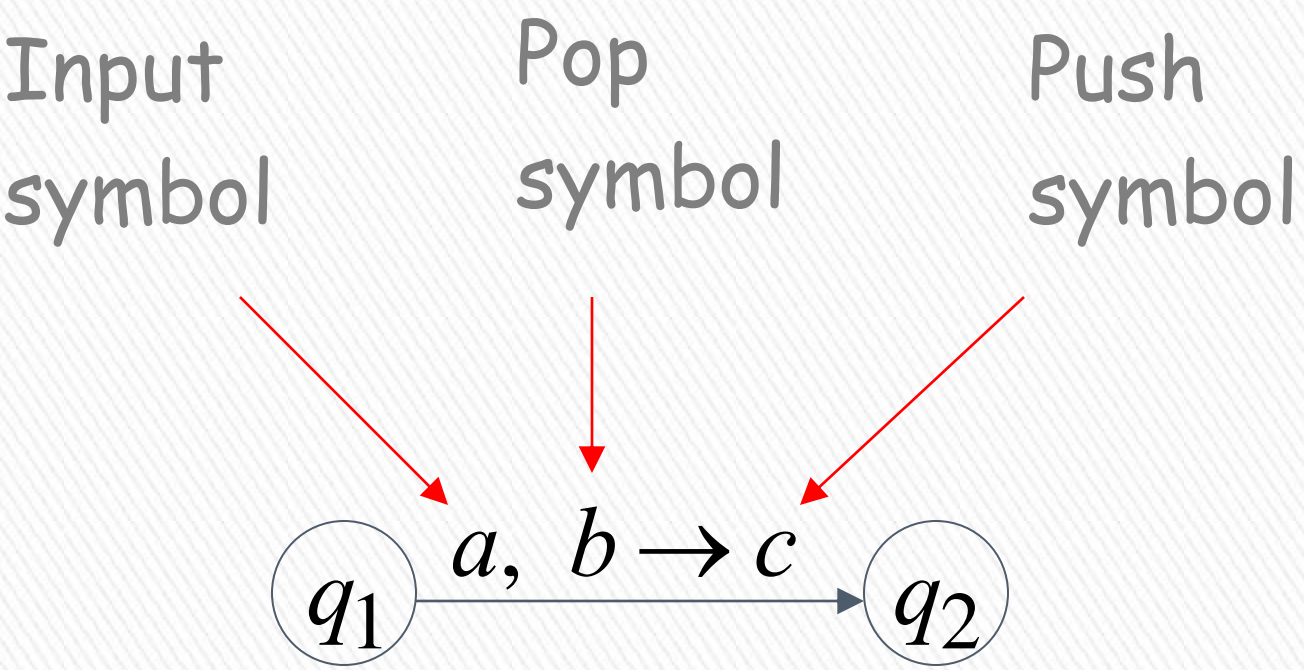
Pushdown Automaton -- PDA



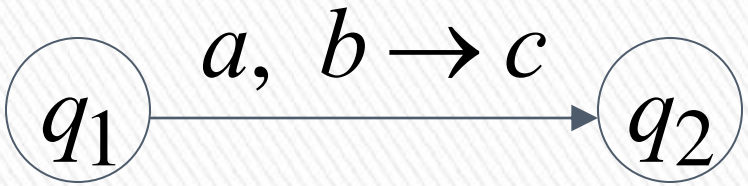
Initial Stack Symbol



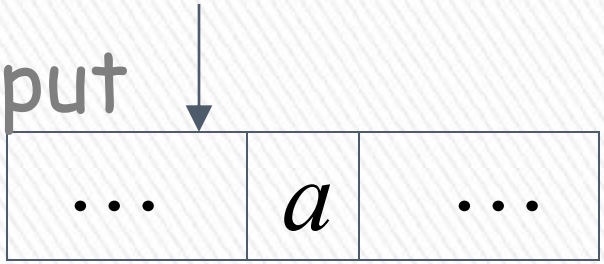
bottom special symbol
Appears at time 0

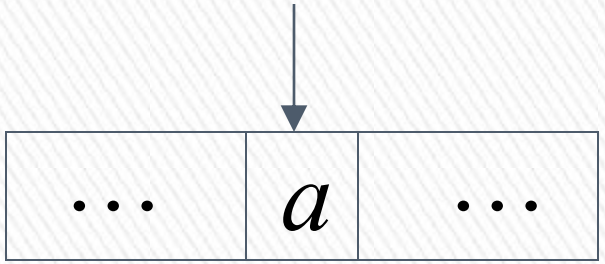


The States

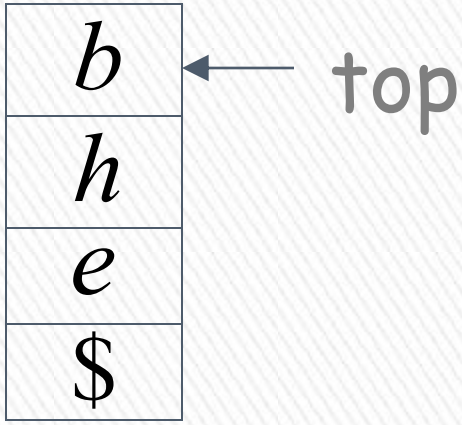


input



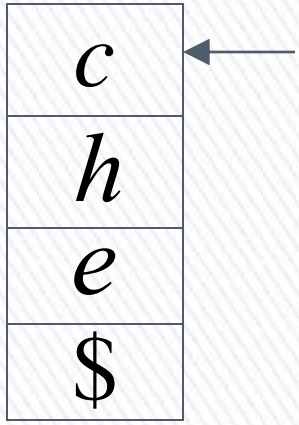


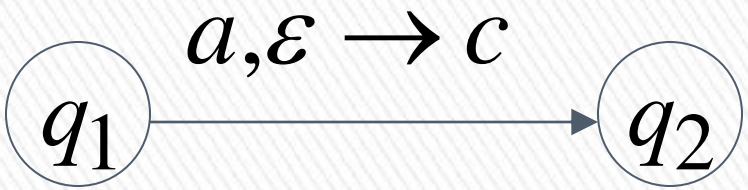
stack



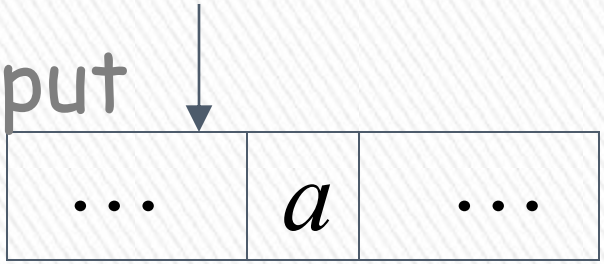
top

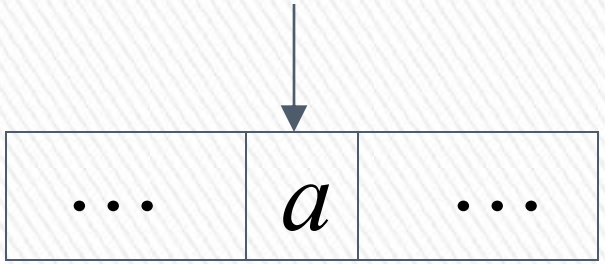
Replace



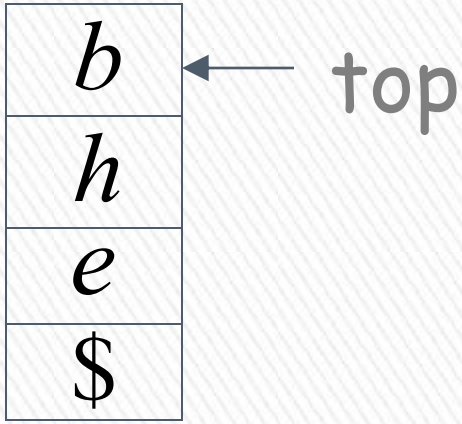


input



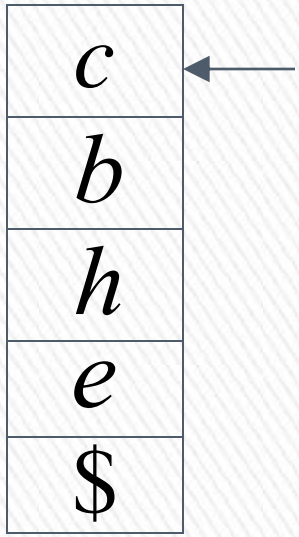


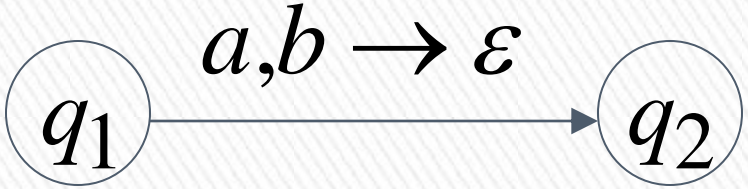
stack



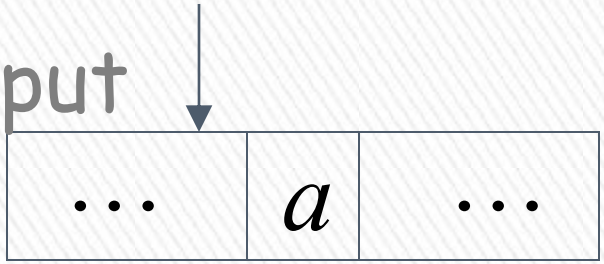
top

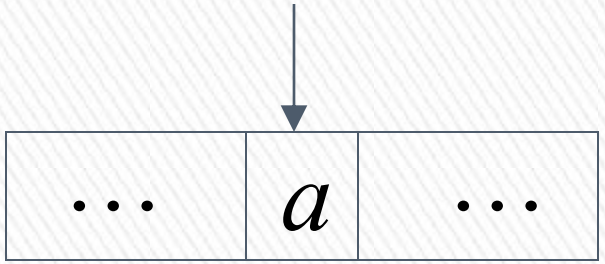
Push



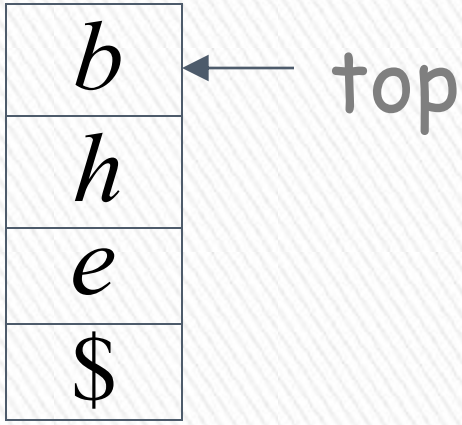


input



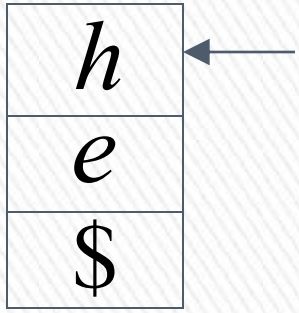


stack

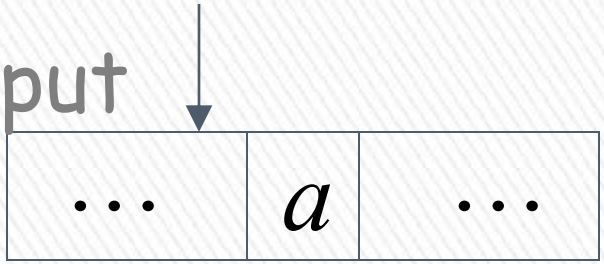


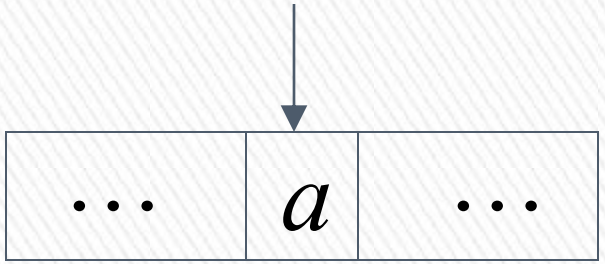
top

Pop

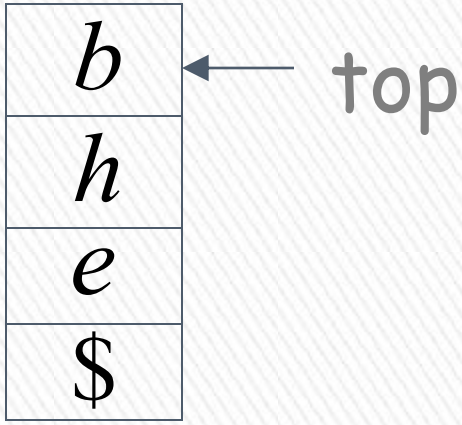


input



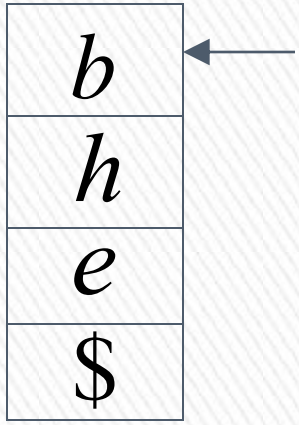


stack

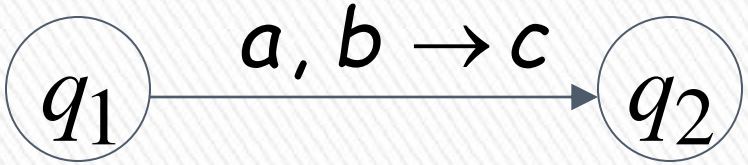


top

No Change



Pop from Empty Stack



Stack underflow

input

stack

top

Pop

Automaton halts!

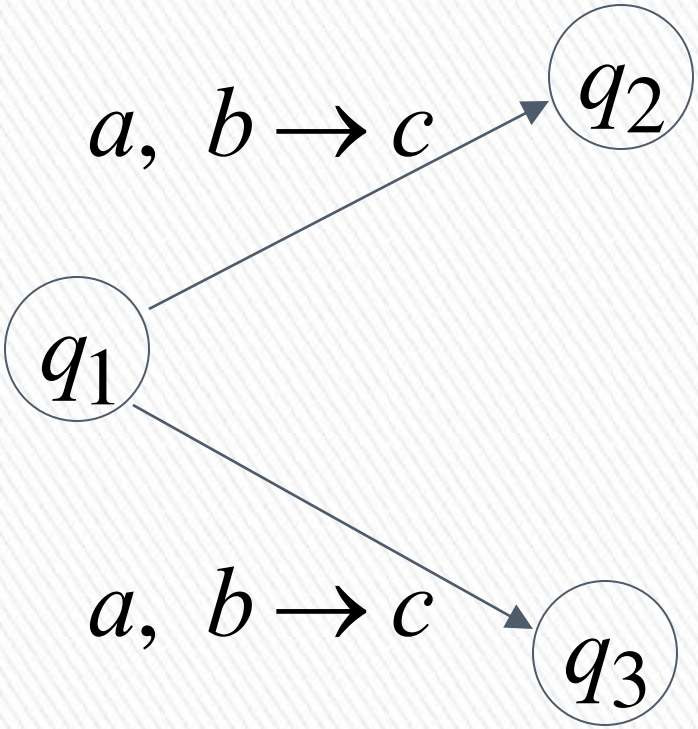
If the automaton attempts to pop from empty stack then it halts and rejects input

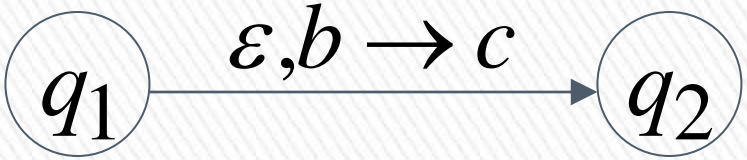
Non-Determinism

PDAs are non-deterministic

Allowed non-deterministic transitions

NFA
i.e.

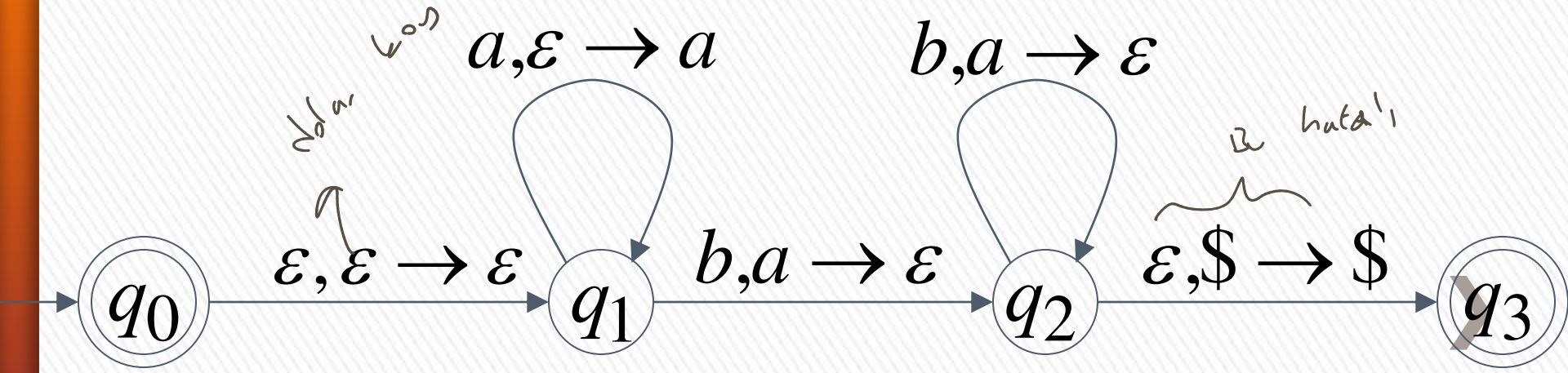




ε – transition

Example PDA

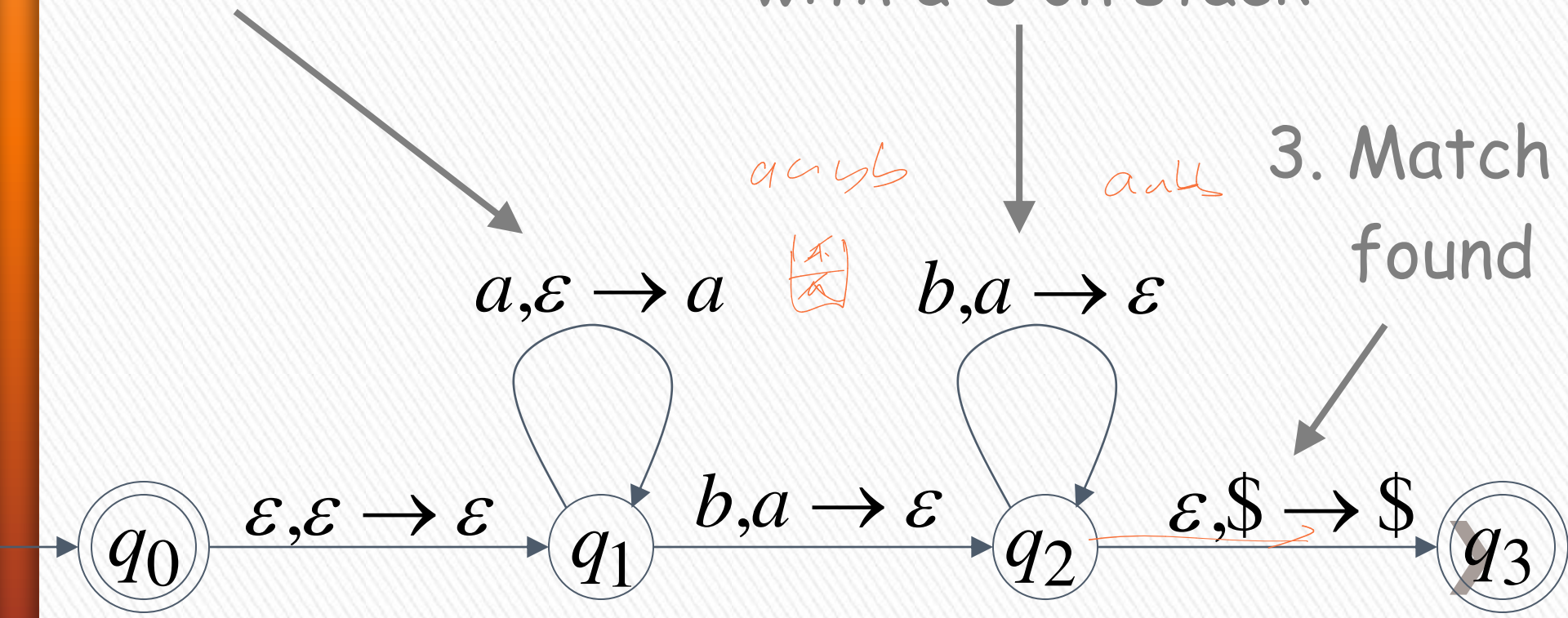
PDA M: $L(M) = \{a^n b^n : n \geq 0\}$



$$L(M) = \{a^n b^n : n \geq 0\}$$

Basic Idea:

1. Push the a's on the stack
2. Match the b's on input with a's on stack
3. Match found



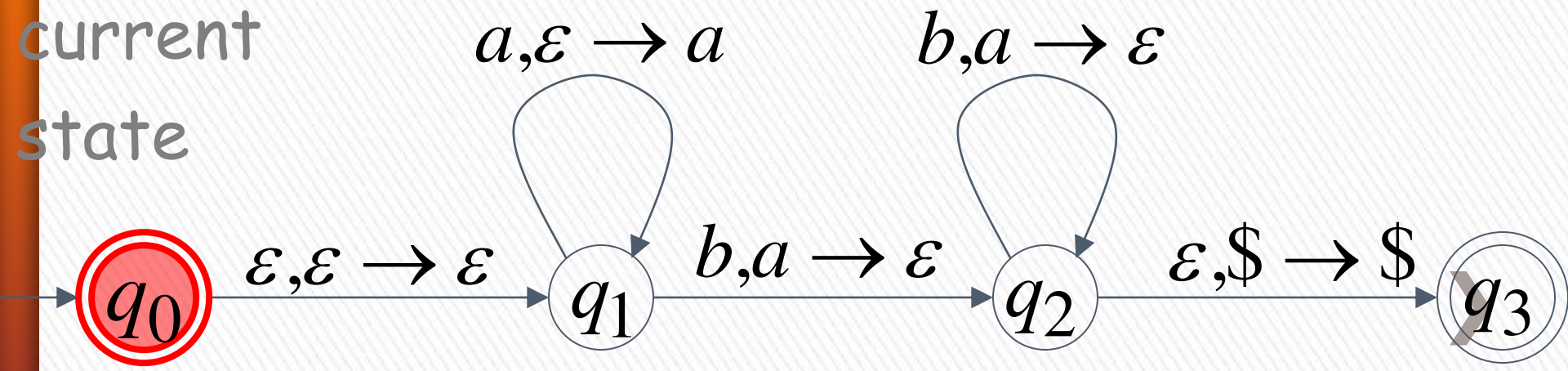
Execution Example: Time 0

Input

a	a	a	b	b	b

Stack

current state

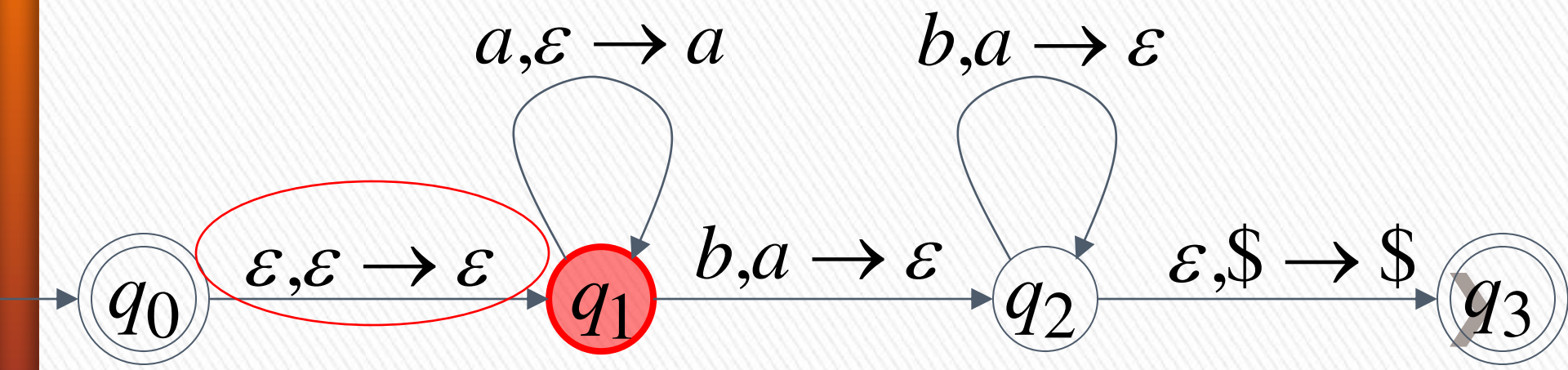


Time 1

Input

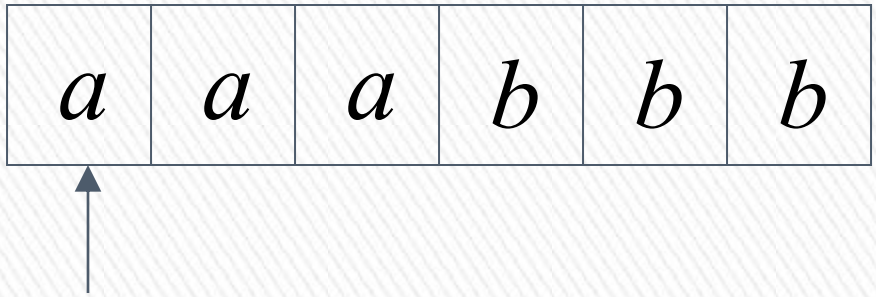
a	a	a	b	b	b

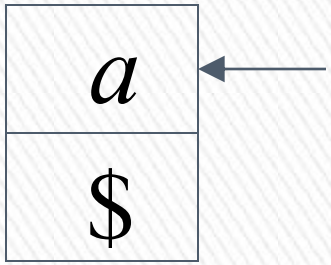
Stack



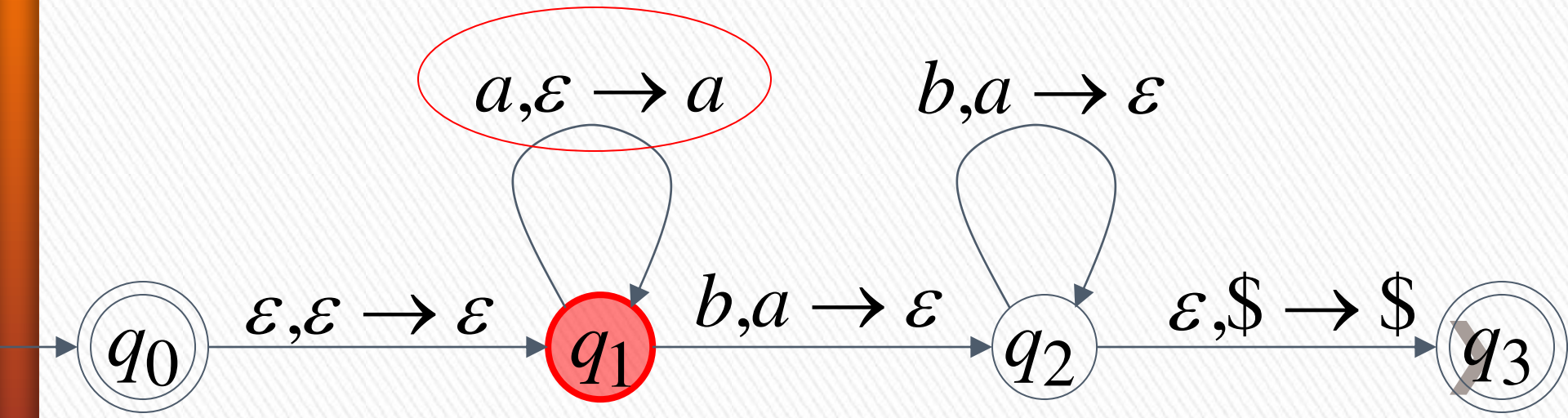
Time 2

Input





Stack



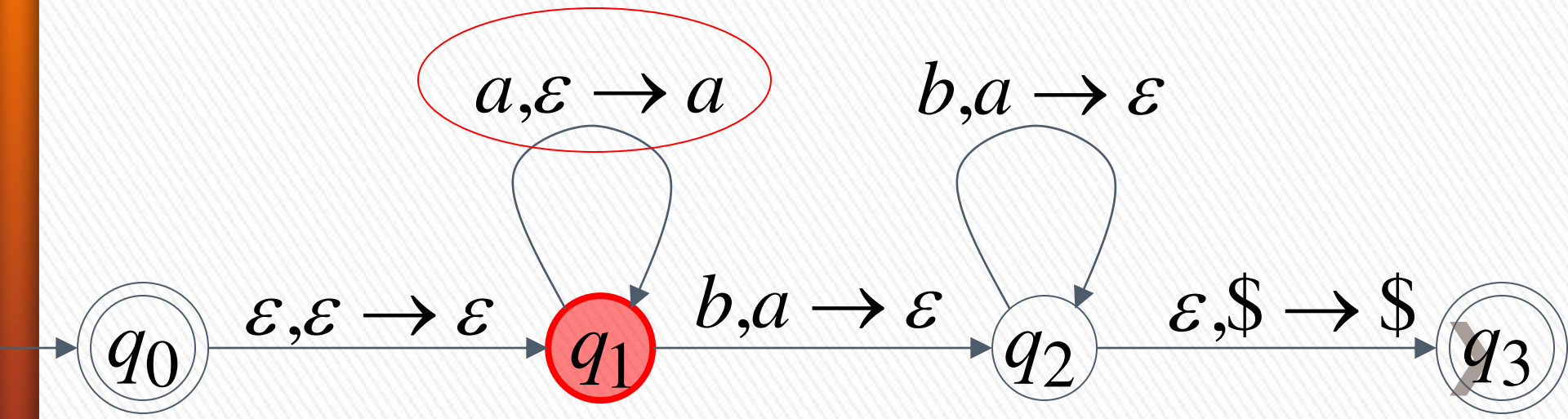
Time 3

Input

a	a	a	b	b	b

a
a
$

Stack



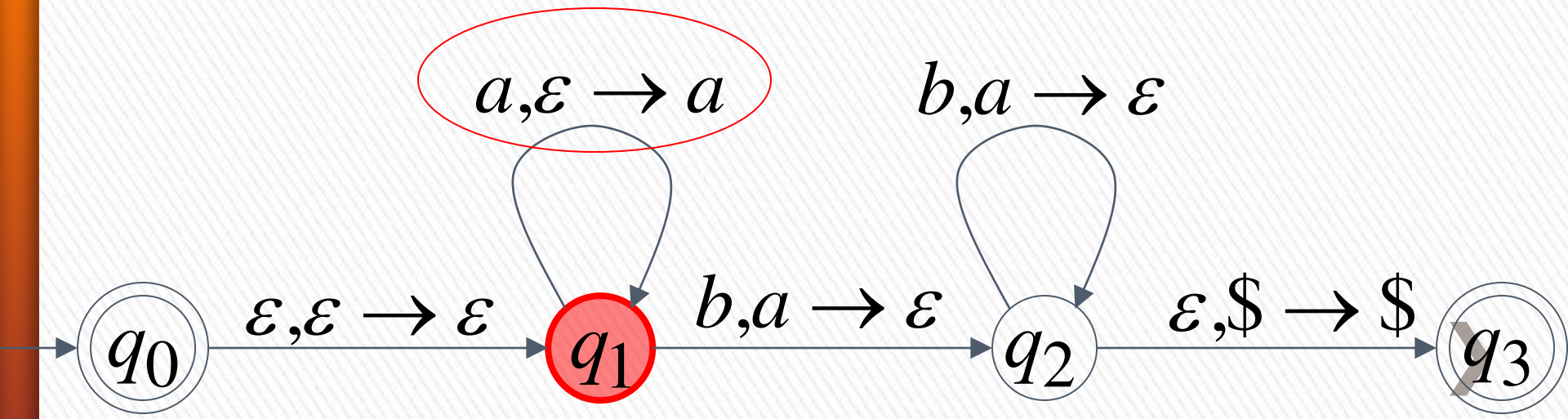
Time 4

Input

a	a	a	b	b	b

a
a
a
$

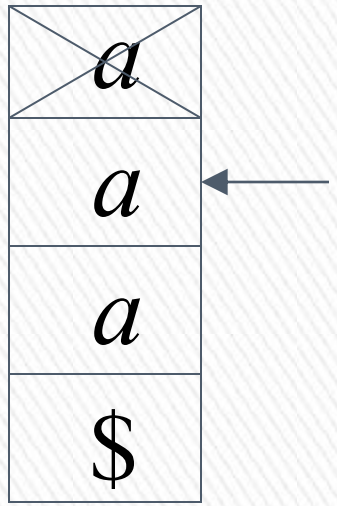
Stack



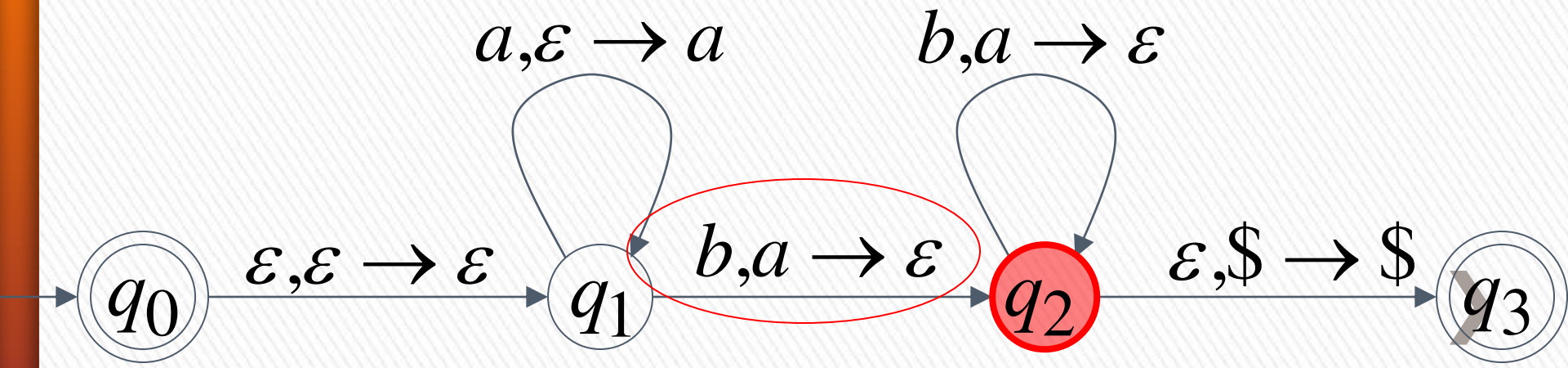
Time 5

Input

a	a	a	b	b	b

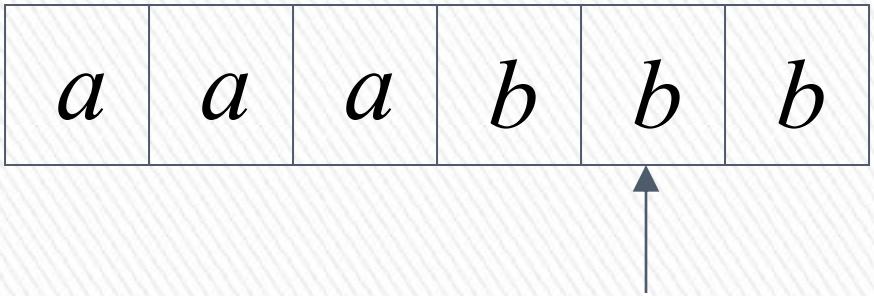


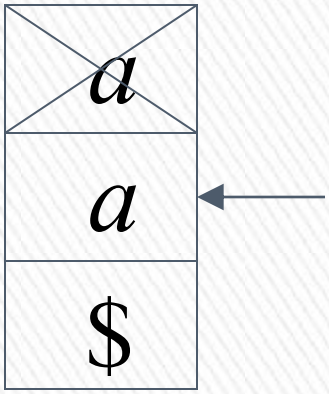
Stack



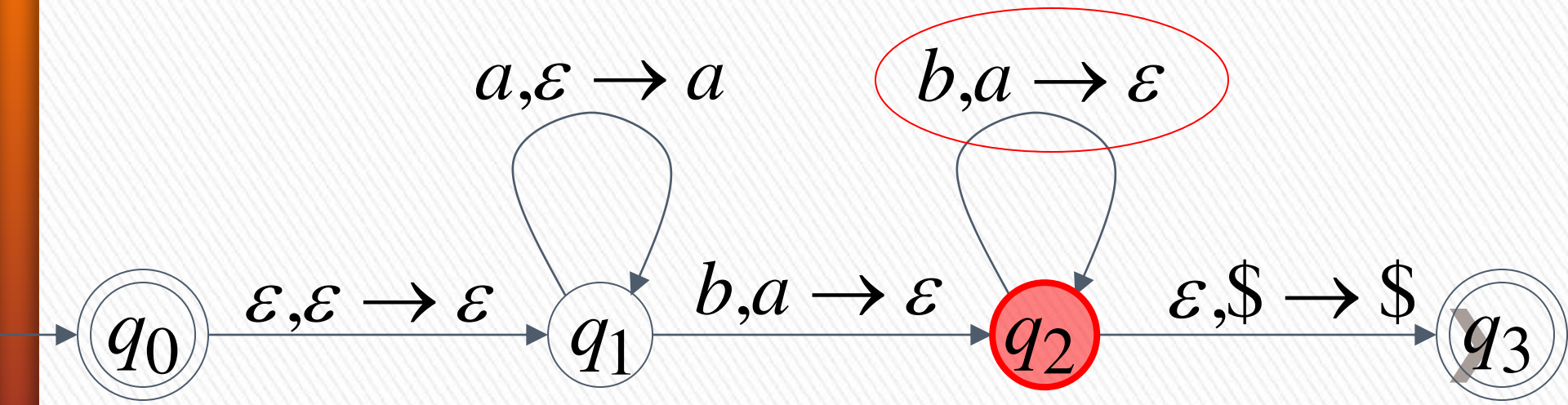
Time 6

Input



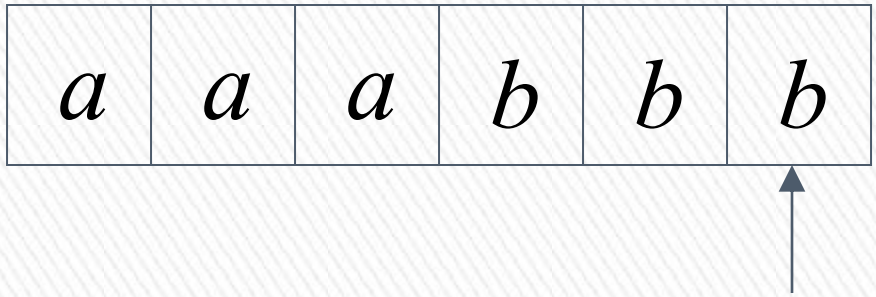


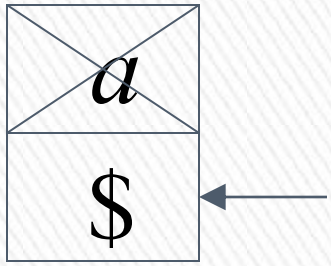
Stack



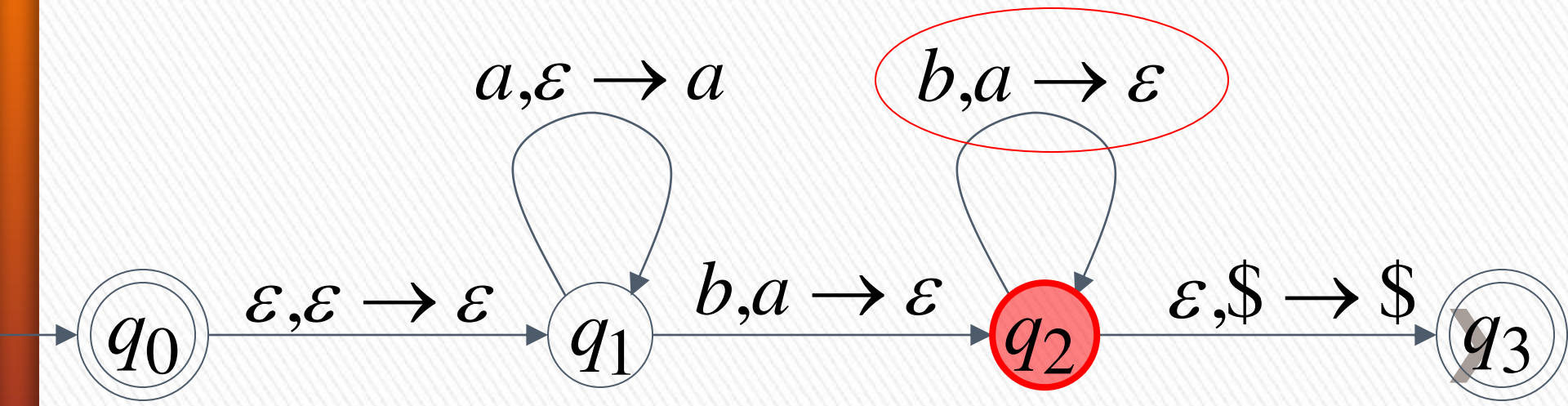
Time 7

Input



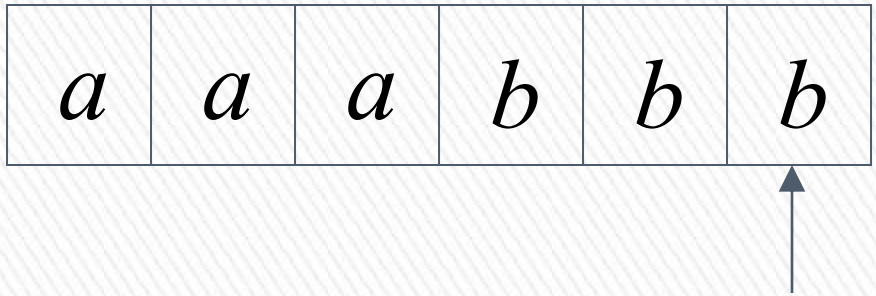


Stack

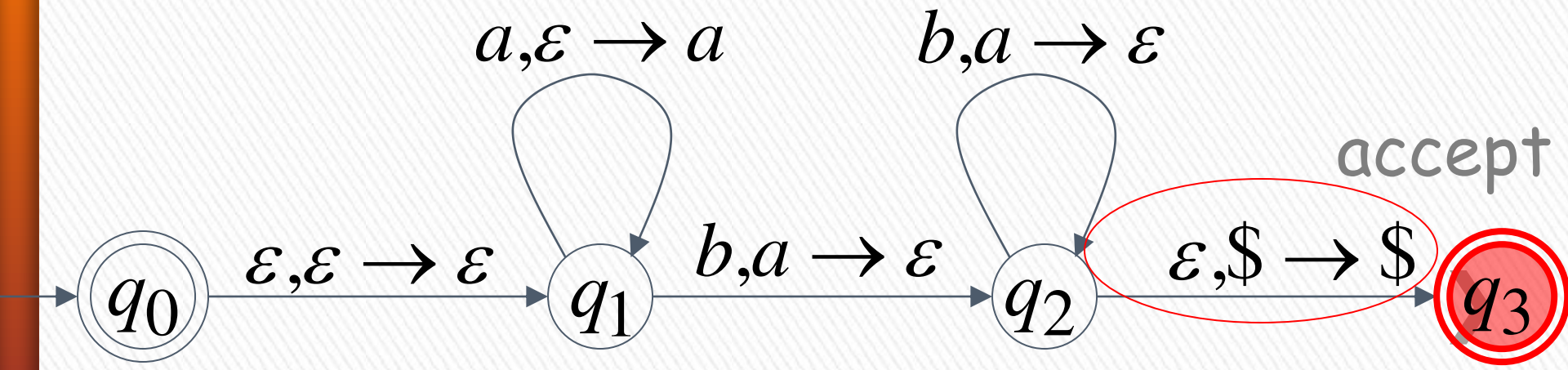


Time 8

Input



Stack



A string is accepted if there is a computation such that:

All the input is consumed

AND

The last state is an accepting state

we do not care about the stack contents at the end of the accepting computation

Rejection Example: Time 0

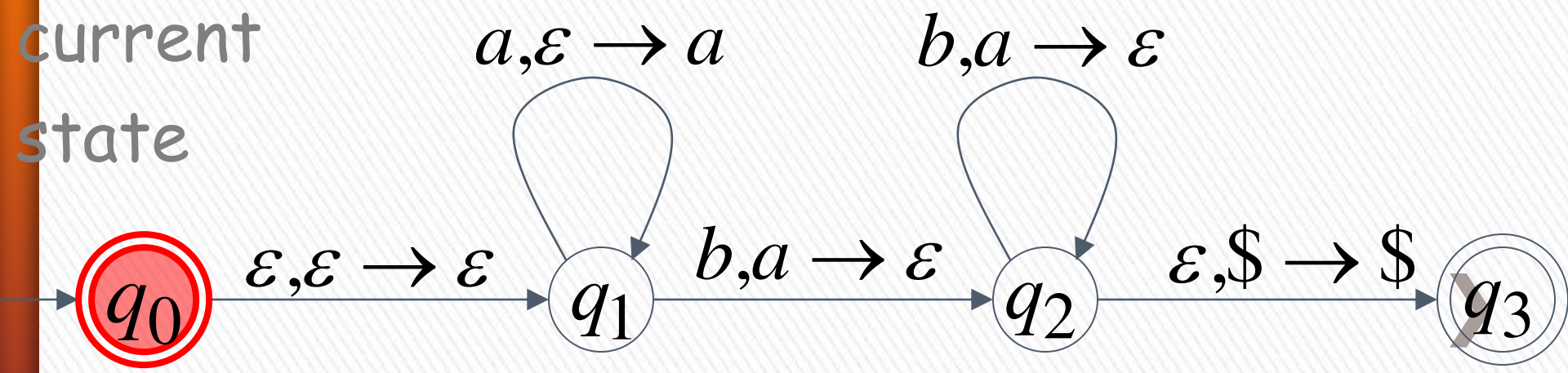
Input

a	a	b

$\$$

Stack

current state



Rejection Example: Time 1

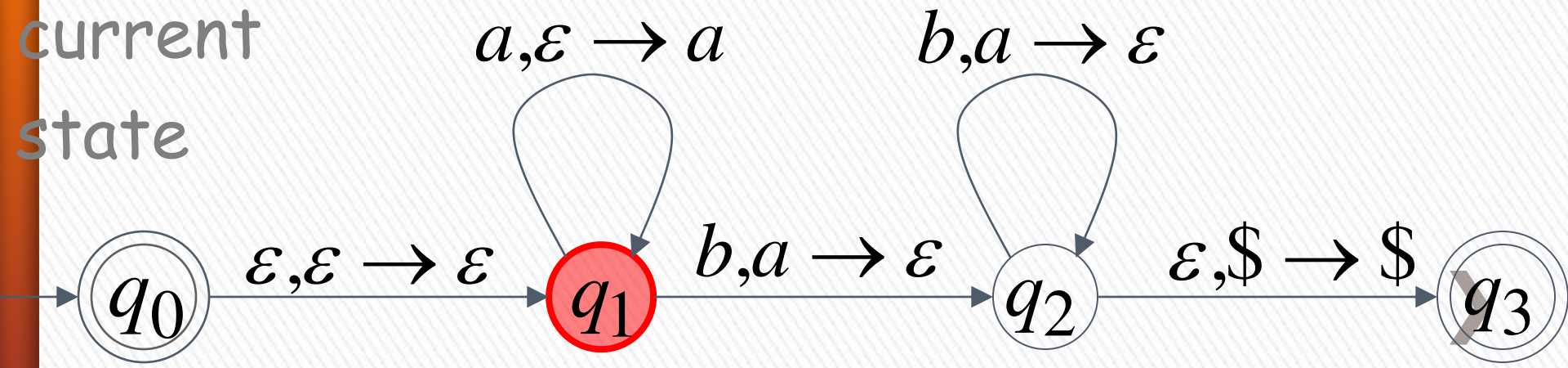
Input

a	a	b

$

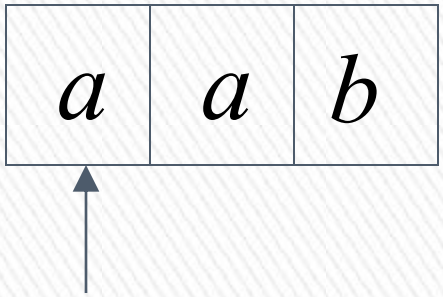
Stack

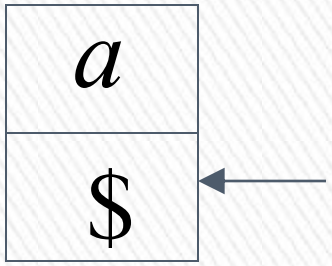
current state



Rejection Example: Time 2

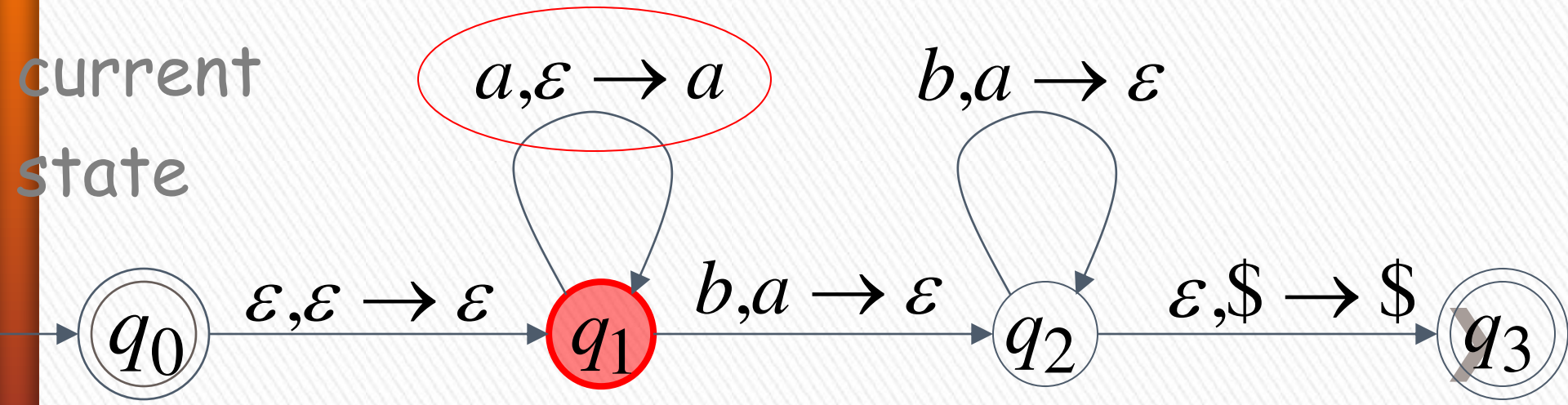
Input





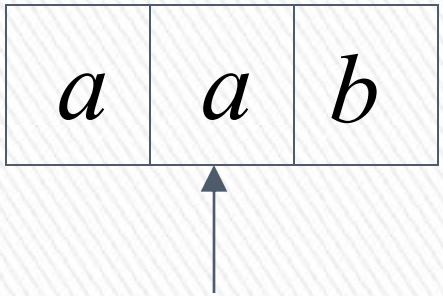
Stack

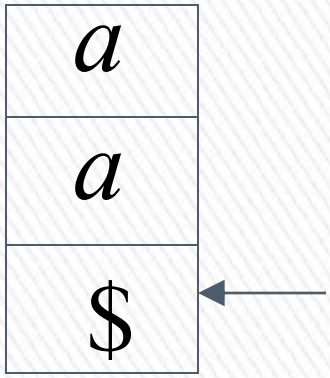
current state



Rejection Example: Time 3

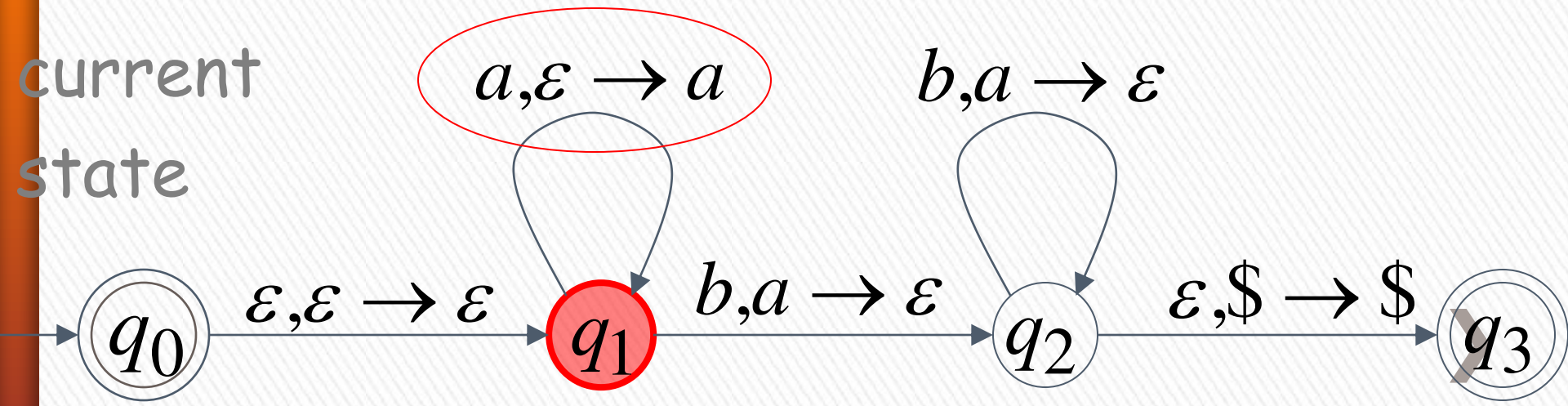
Input





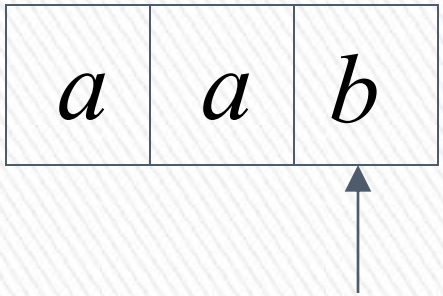
Stack

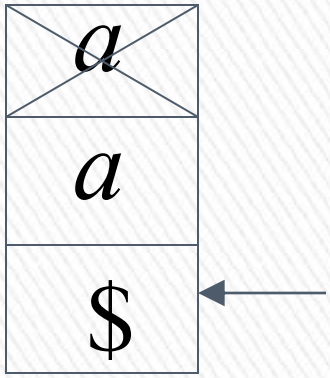
current state



Rejection Example: Time 4

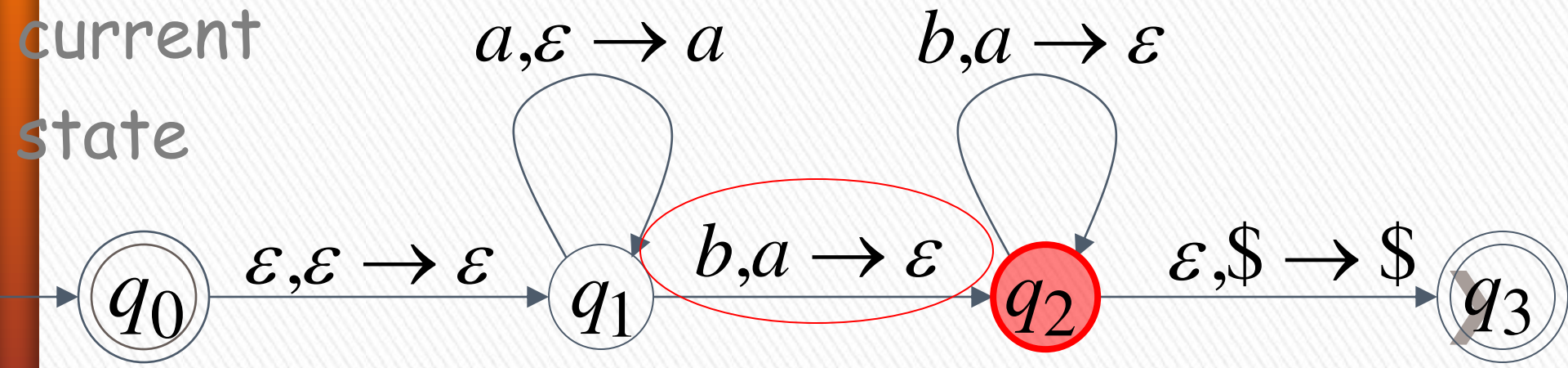
Input



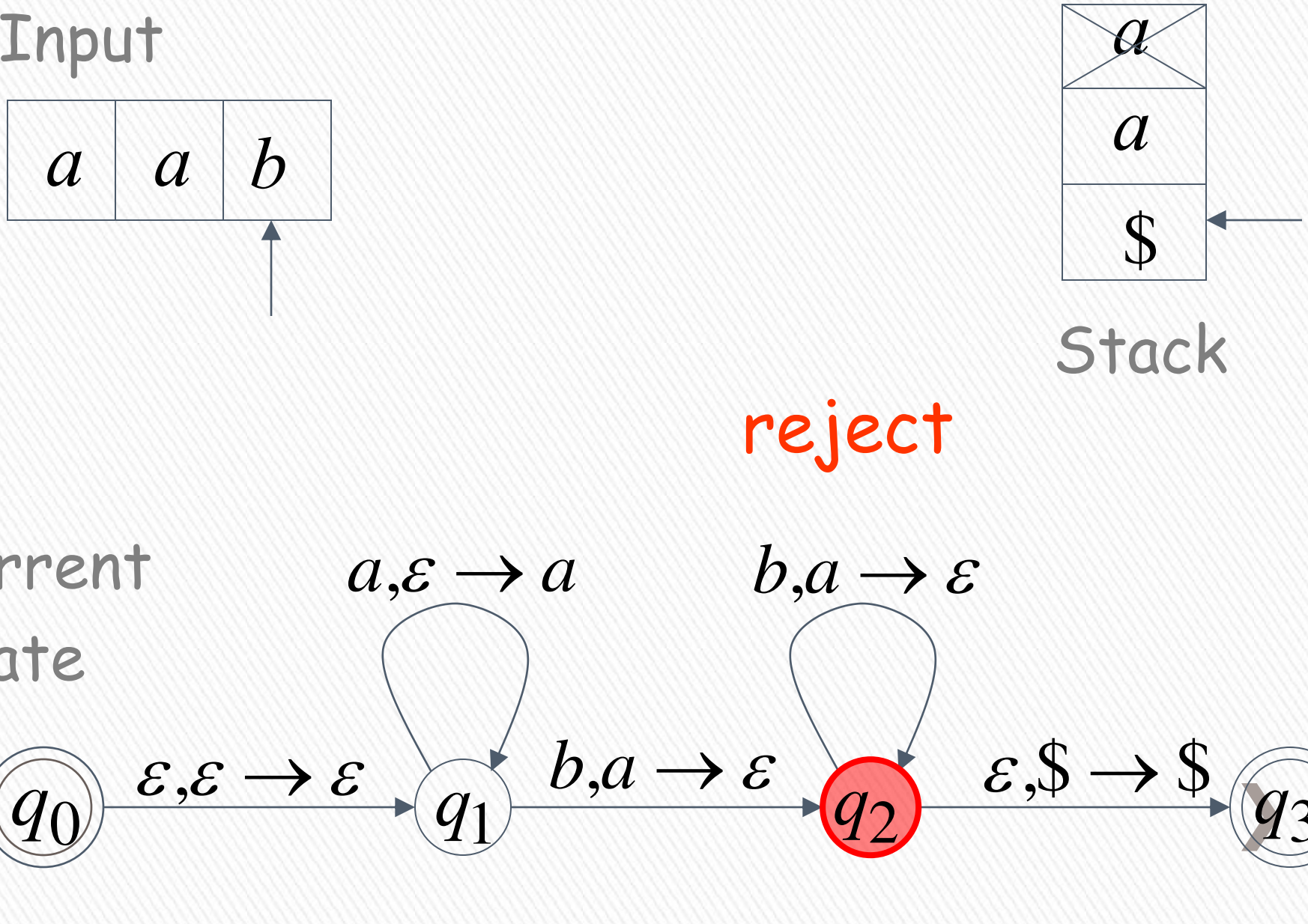


Stack

current state

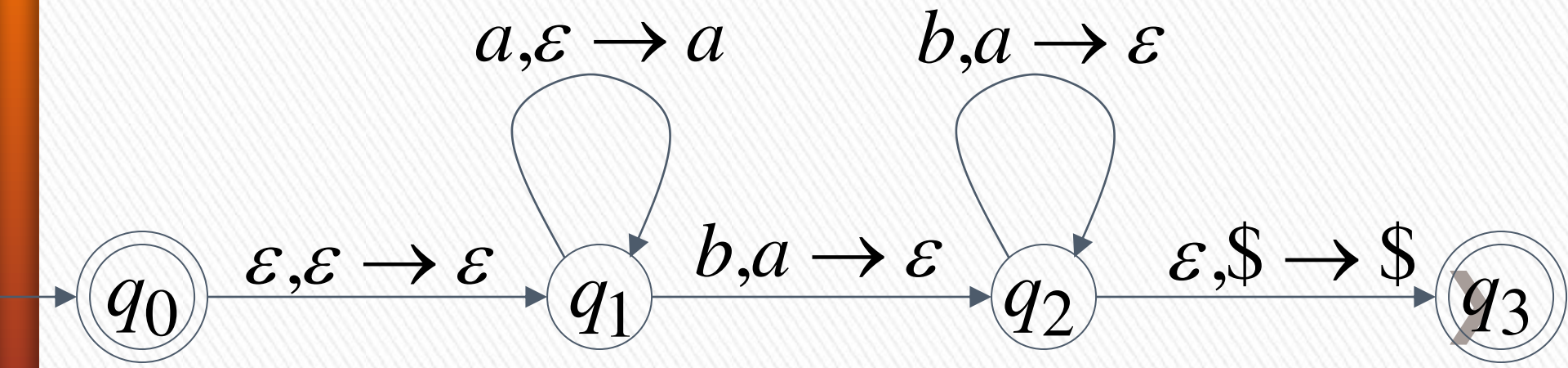


Rejection Example: Time 4



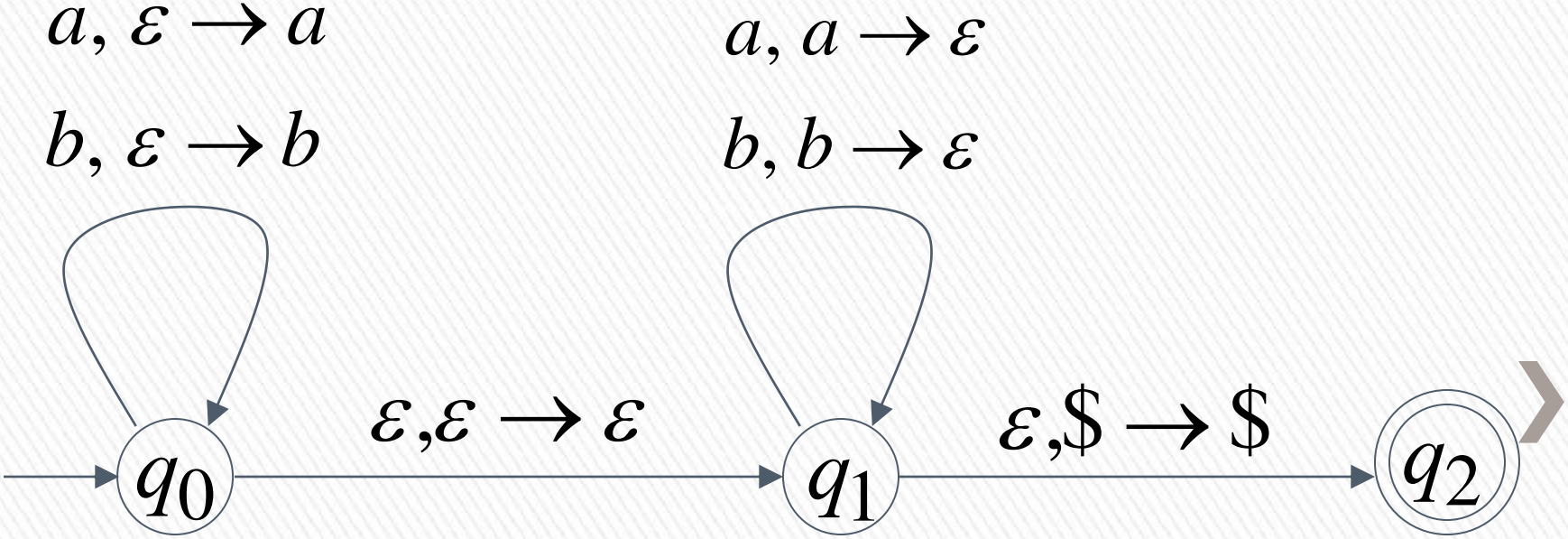
There is no accepting computation for aab

The string aab is rejected by the PDA



Another PDA example

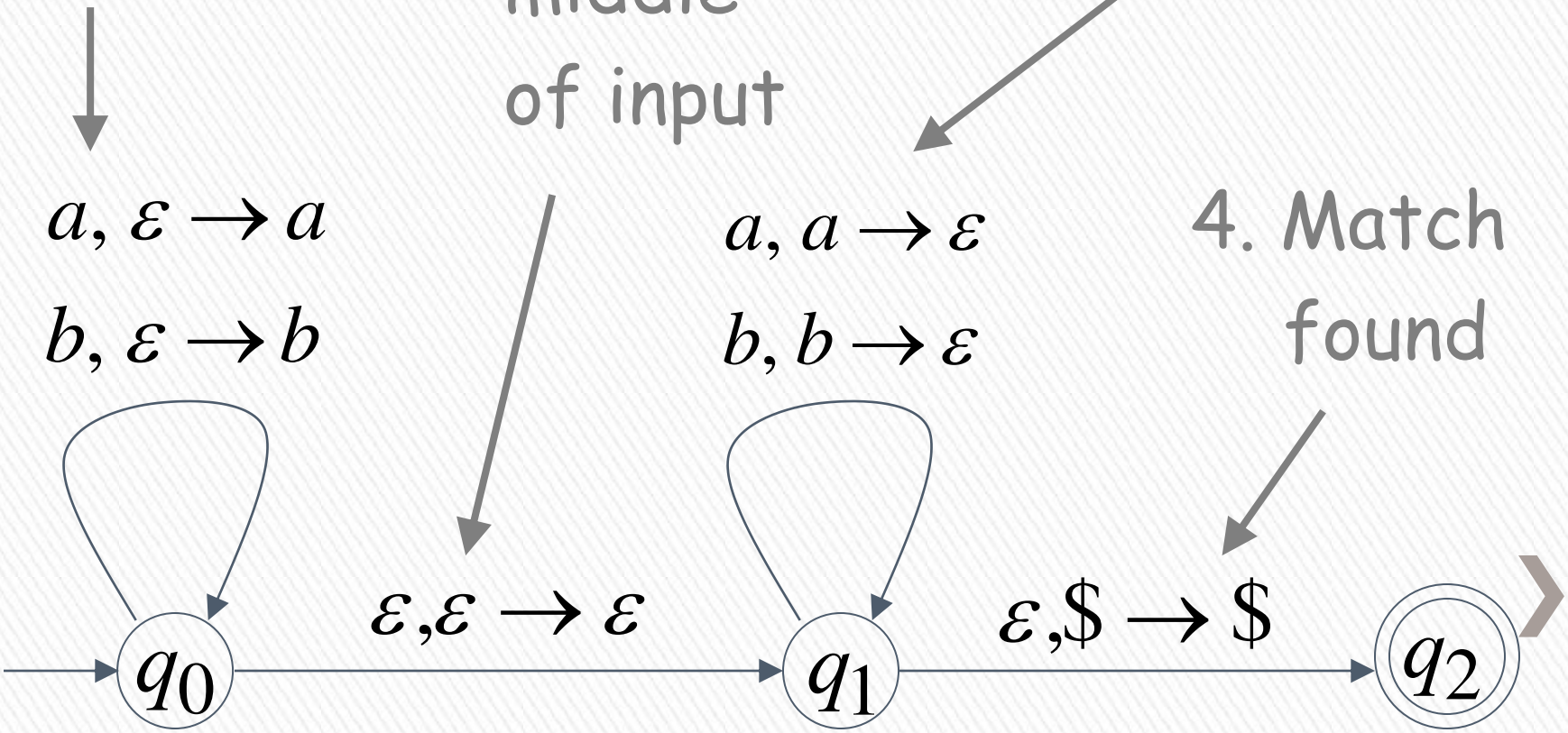
PDA M: $L(M) = \{vv^R : v \in \{a,b\}^*\}$



Basic Idea:

$$L(M) = \{vv^R : v \in \{a,b\}^*\}$$

1. Push v on stack
2. Guess middle of input
3. Match v^R on input with v on stack
4. Match found



Execution Example: Time 0

Input

a	b	b	a

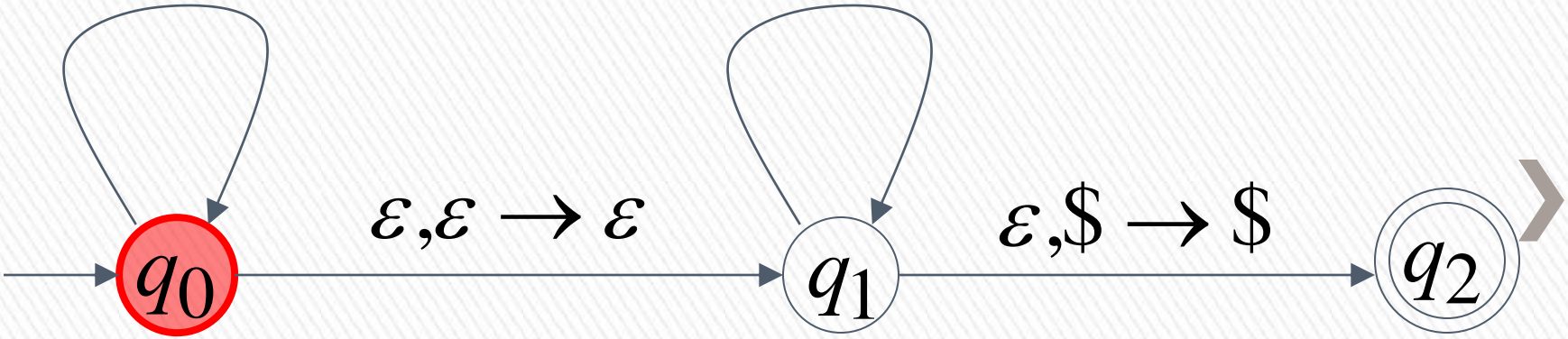
Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

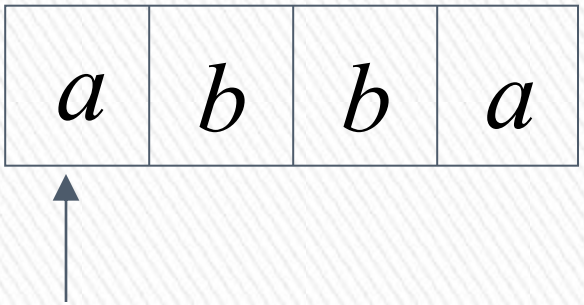
$a, a \to \varepsilon$

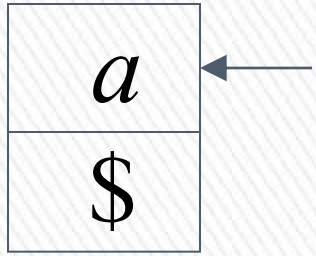
$b, b \to \varepsilon$



Time 1

Input





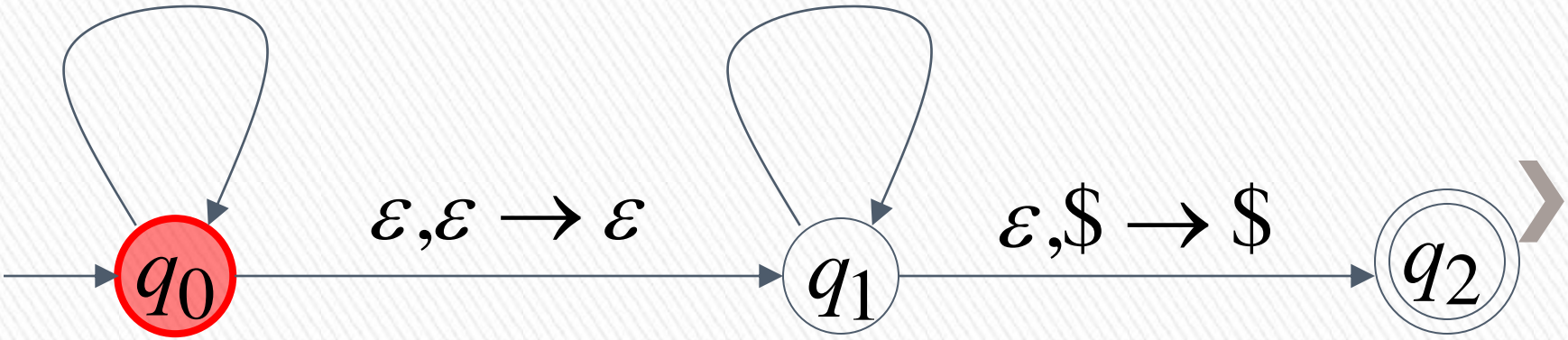
Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

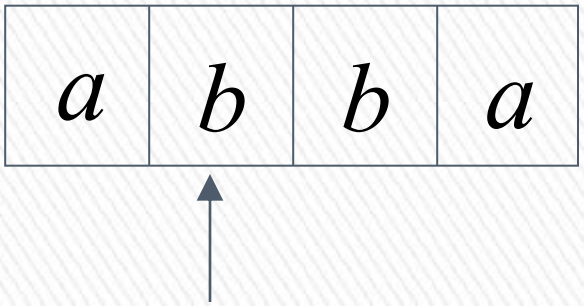
$a, a \to \varepsilon$

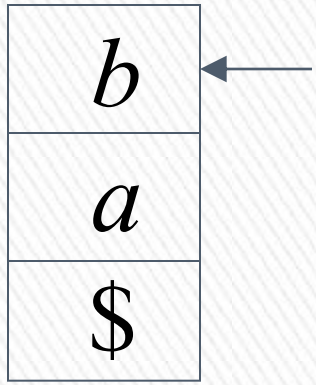
$b, b \to \varepsilon$



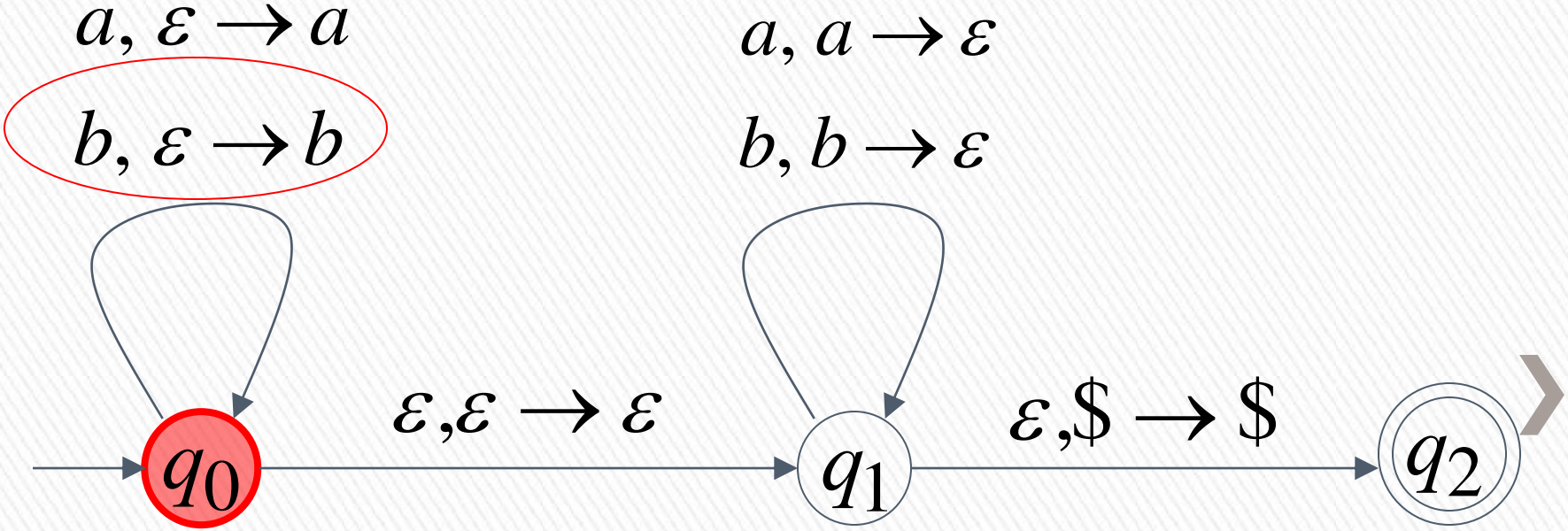
Time 2

Input





Stack



Time 3

Input

a	b	b	a

Guess the middle of string

b
a
\$

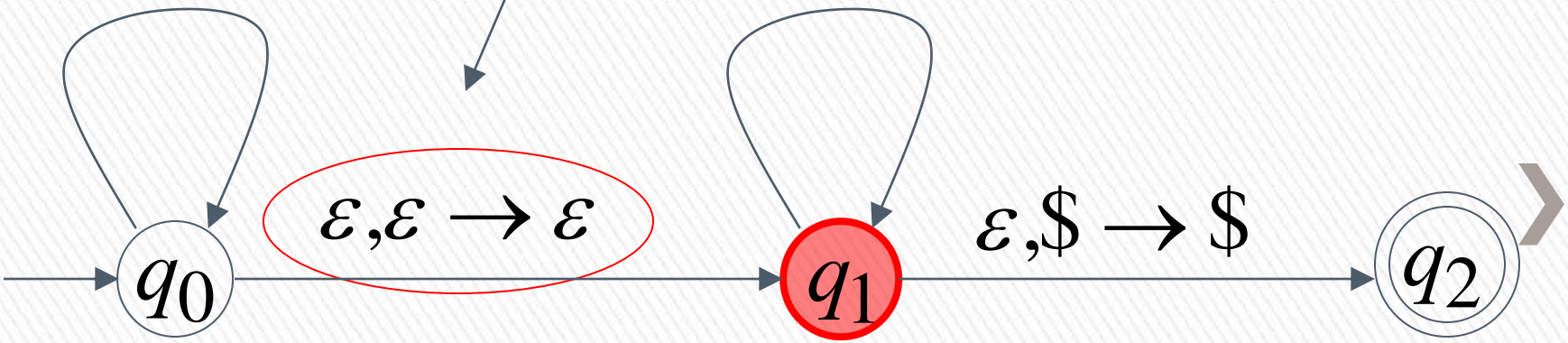
Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

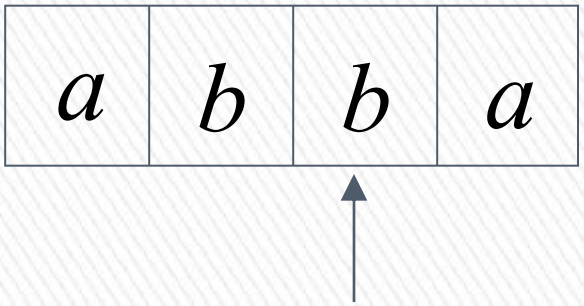
$a, a \to \varepsilon$

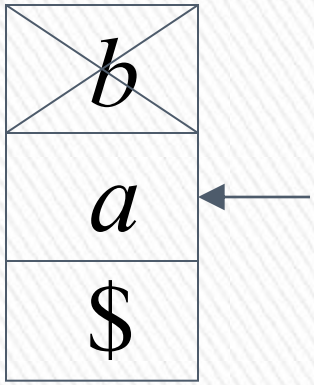
$b, b \to \varepsilon$



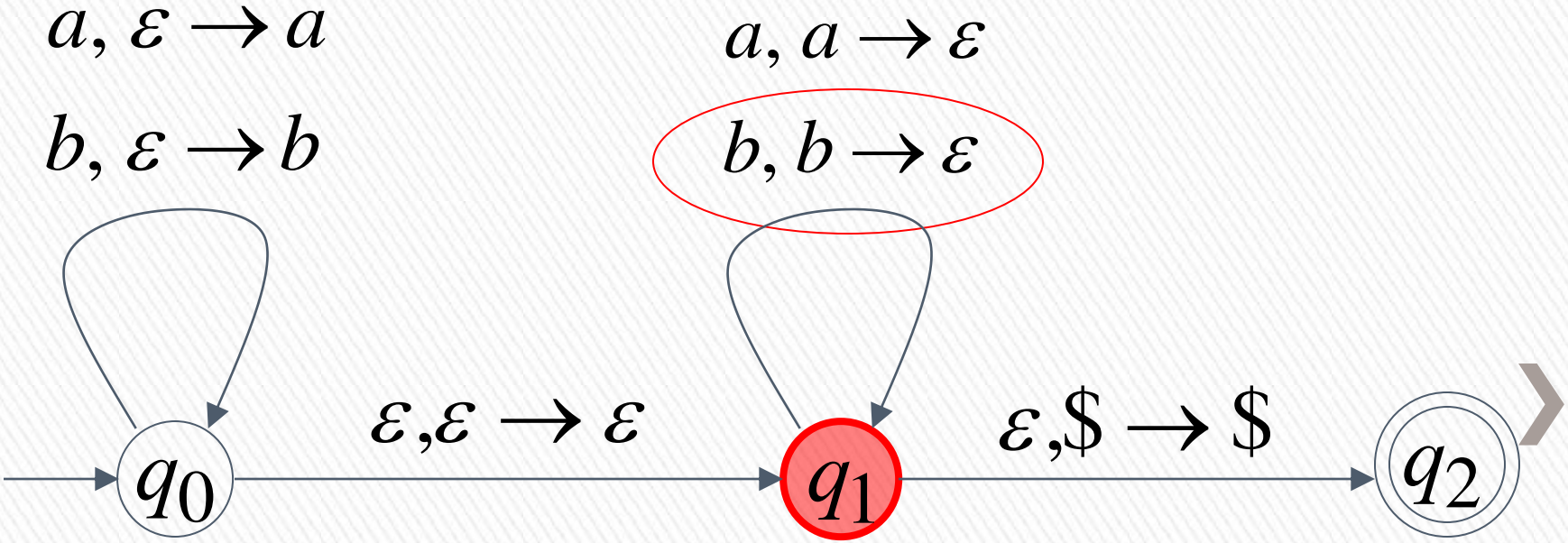
Time 4

Input



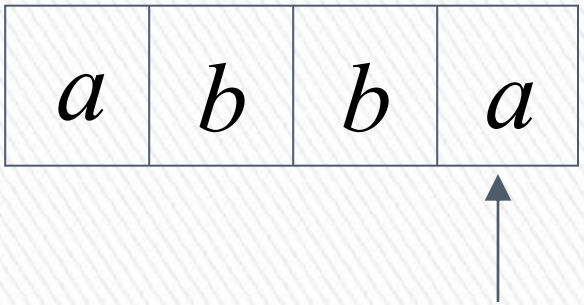


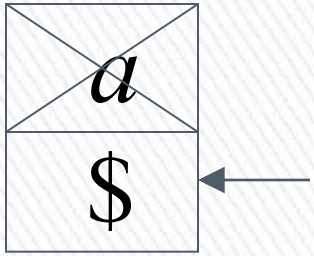
Stack



Time 5

Input





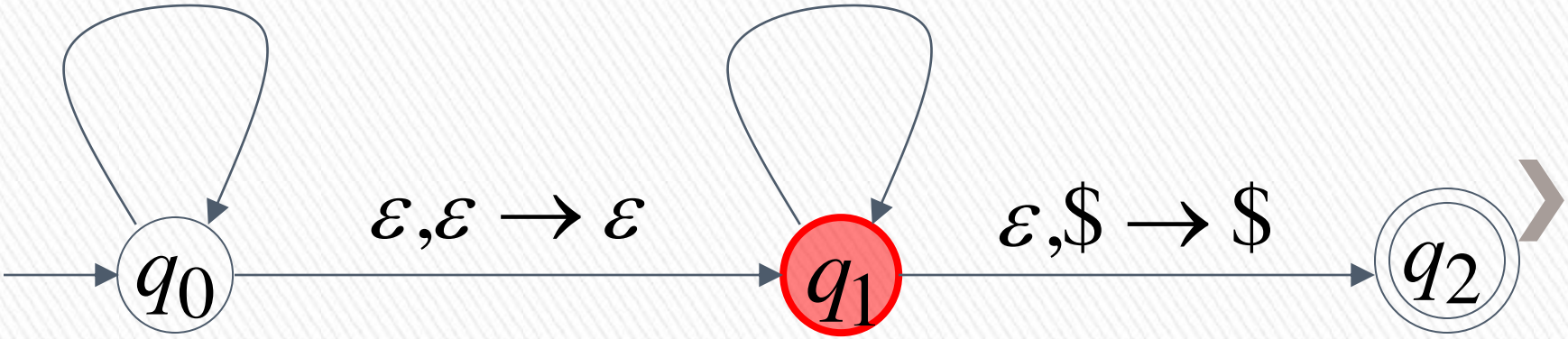
Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

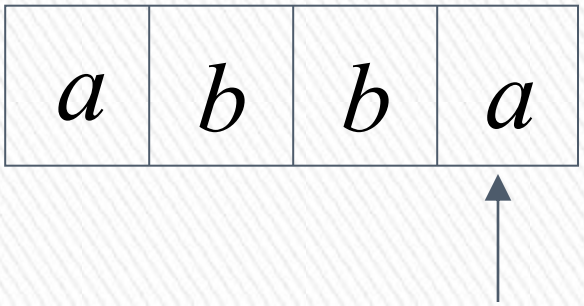
$a, a \to \varepsilon$

$b, b \to \varepsilon$



Time 6

Input



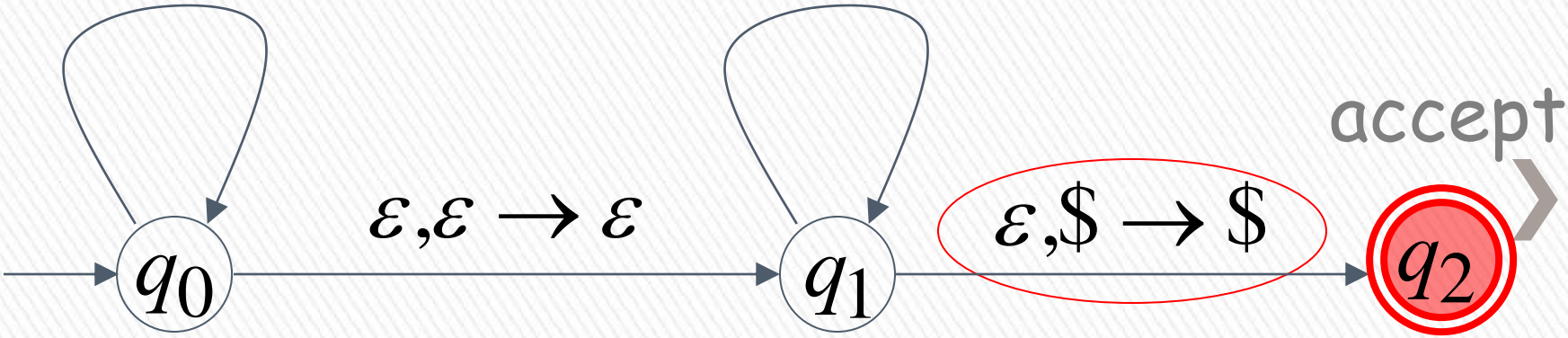
Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

$a, a \to \varepsilon$

$b, b \to \varepsilon$



Rejection Example: Time 0

Input

a	b	b	b

$\$$

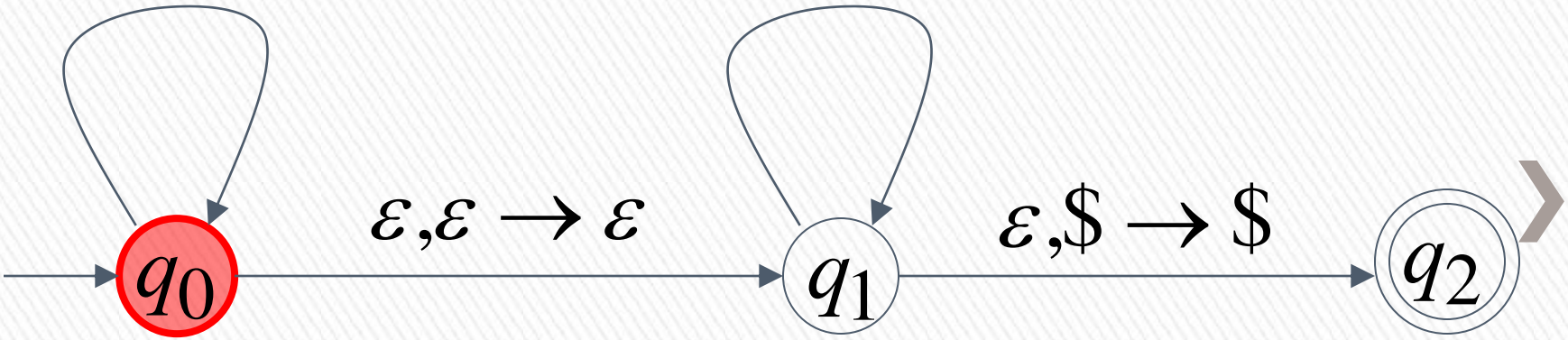
Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

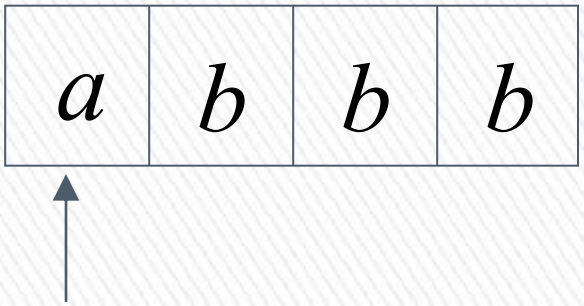
$a, a \to \varepsilon$

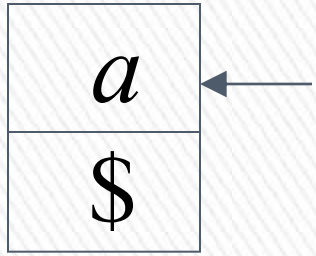
$b, b \to \varepsilon$



Time 1

Input





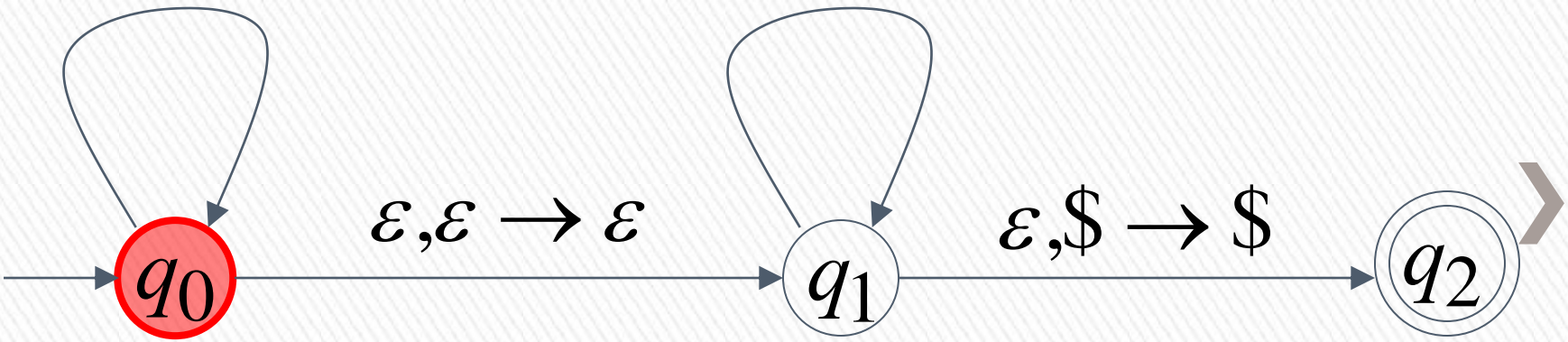
Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

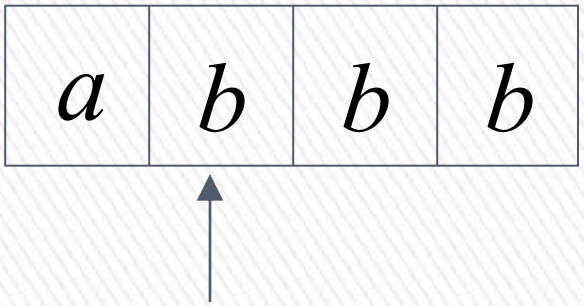
$a, a \to \varepsilon$

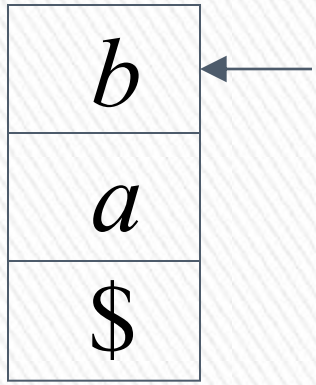
$b, b \to \varepsilon$



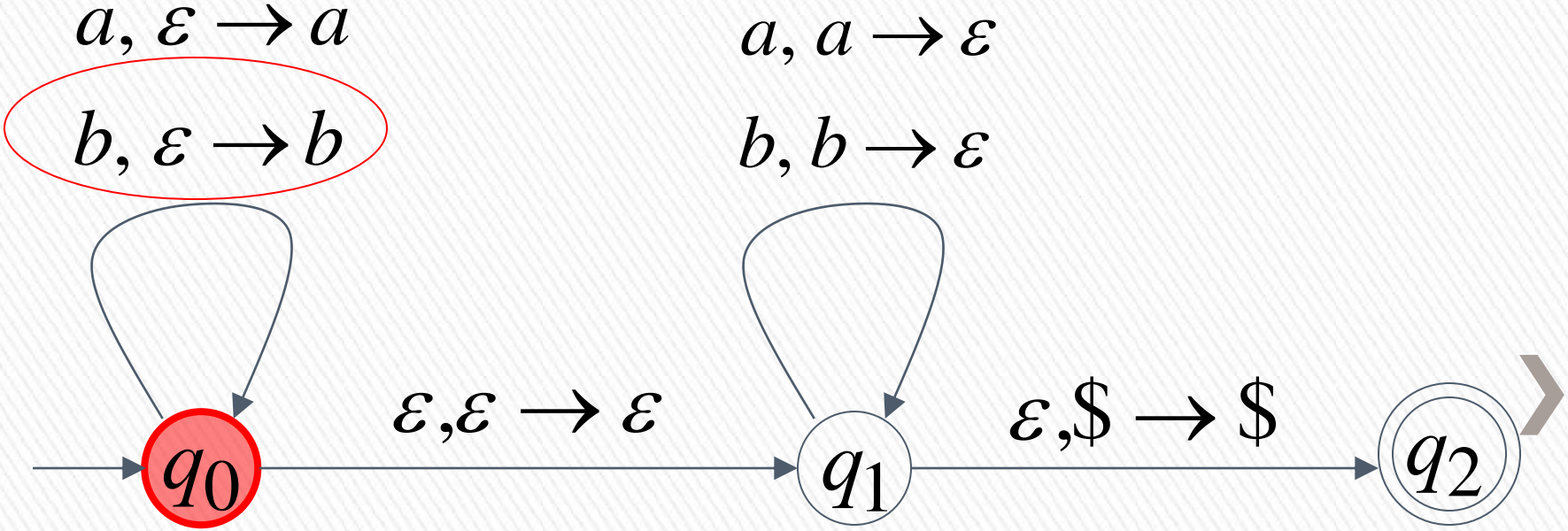
Time 2

Input



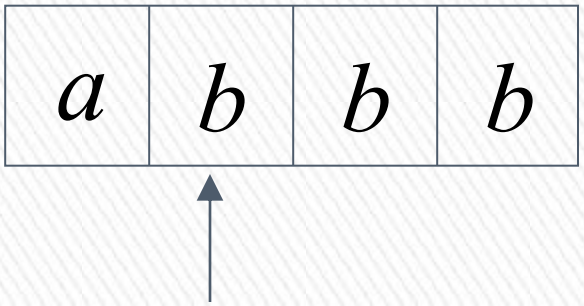


Stack

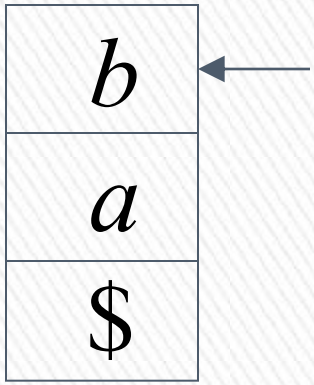


Time 3

Input



Guess the middle of string



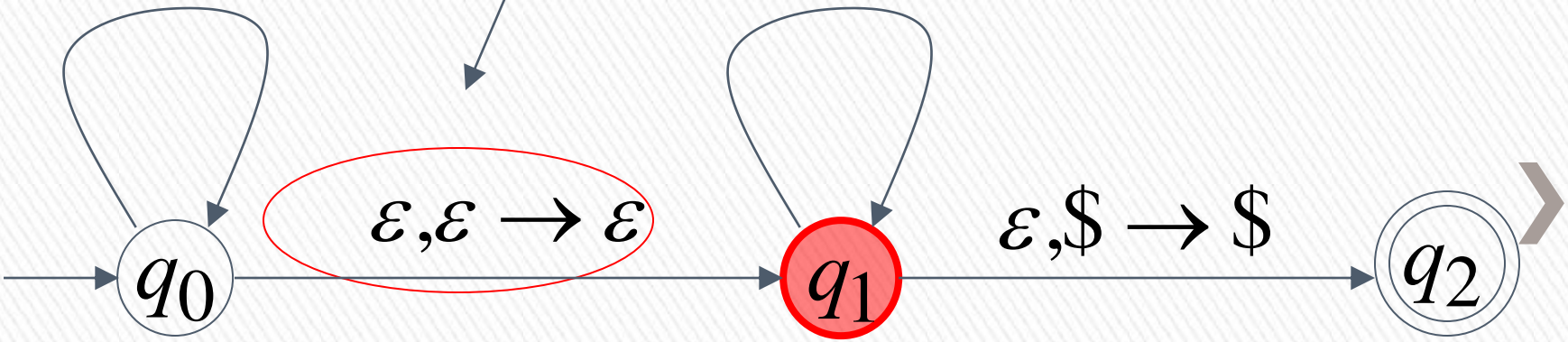
Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

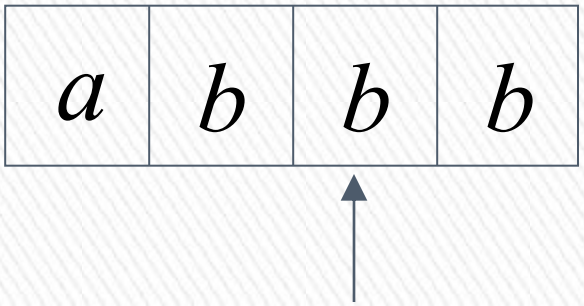
$a, a \to \varepsilon$

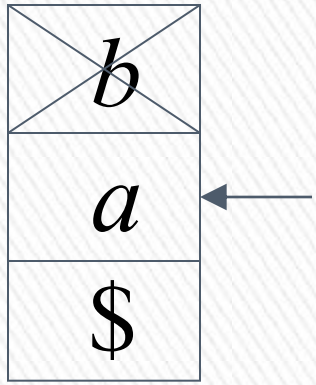
$b, b \to \varepsilon$



Time 4

Input





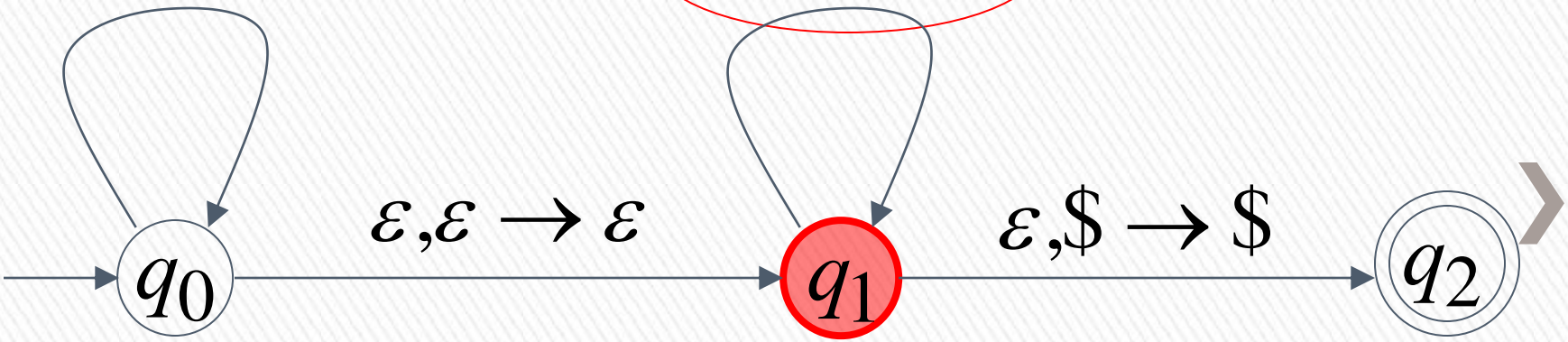
Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

$a, a \to \varepsilon$

$b, b \to \varepsilon$



Time 5

There is no possible transition.

Input

a	b	b	b

Input is not consumed

a
$

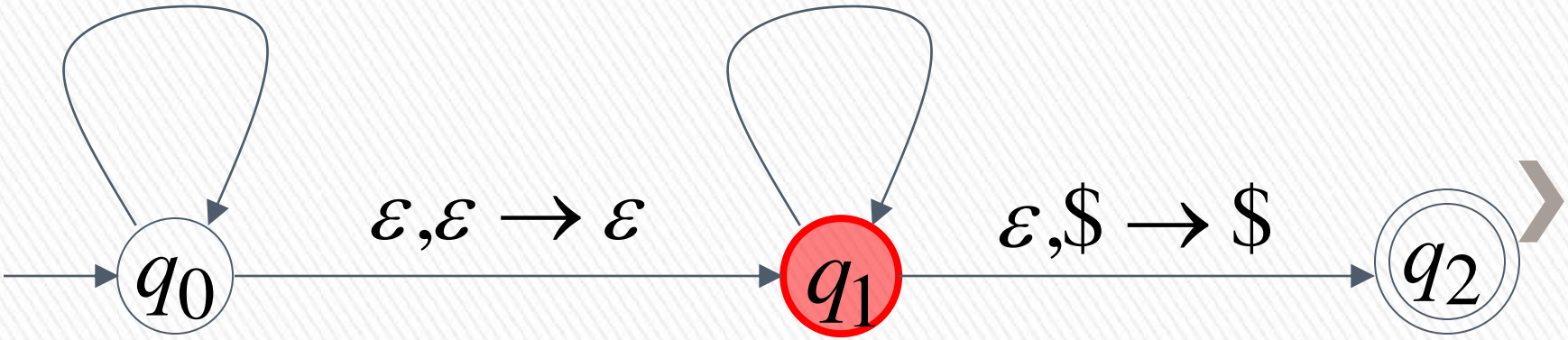
Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

$a, a \to \varepsilon$

$b, b \to \varepsilon$



Another computation on same string:

Input

Time 0

a	b	b	b

$\$$

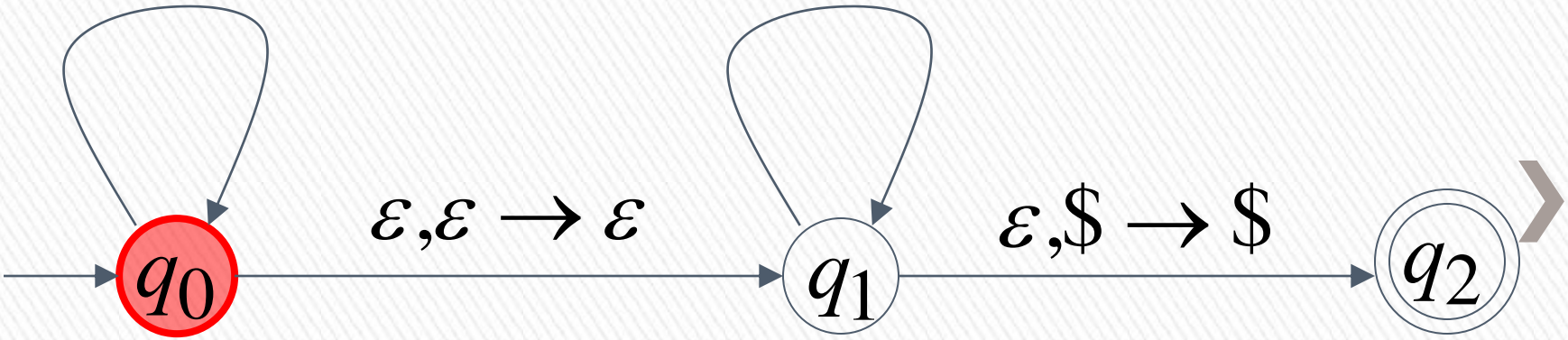
Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

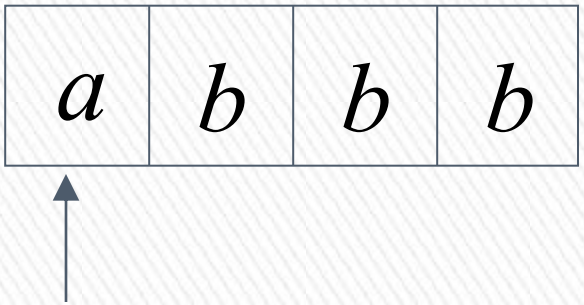
$a, a \to \varepsilon$

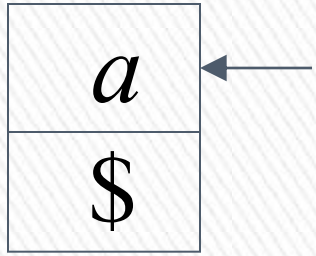
$b, b \to \varepsilon$



Time 1

Input





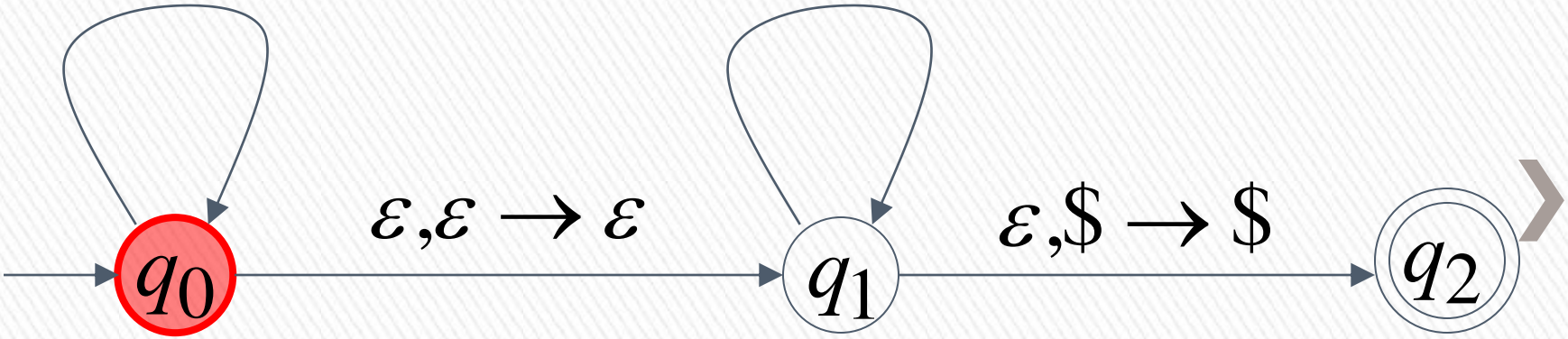
Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

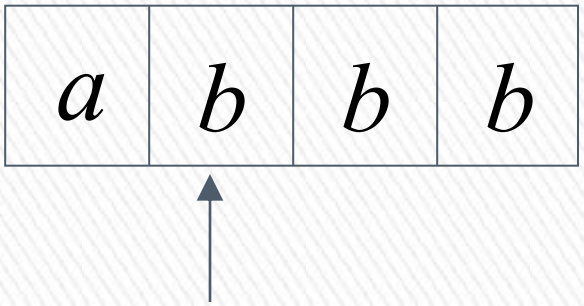
$a, a \to \varepsilon$

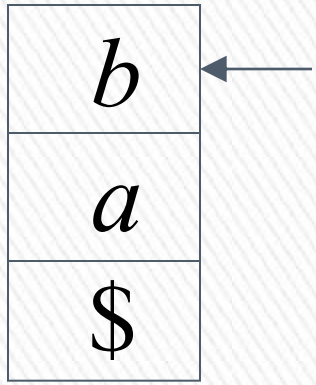
$b, b \to \varepsilon$



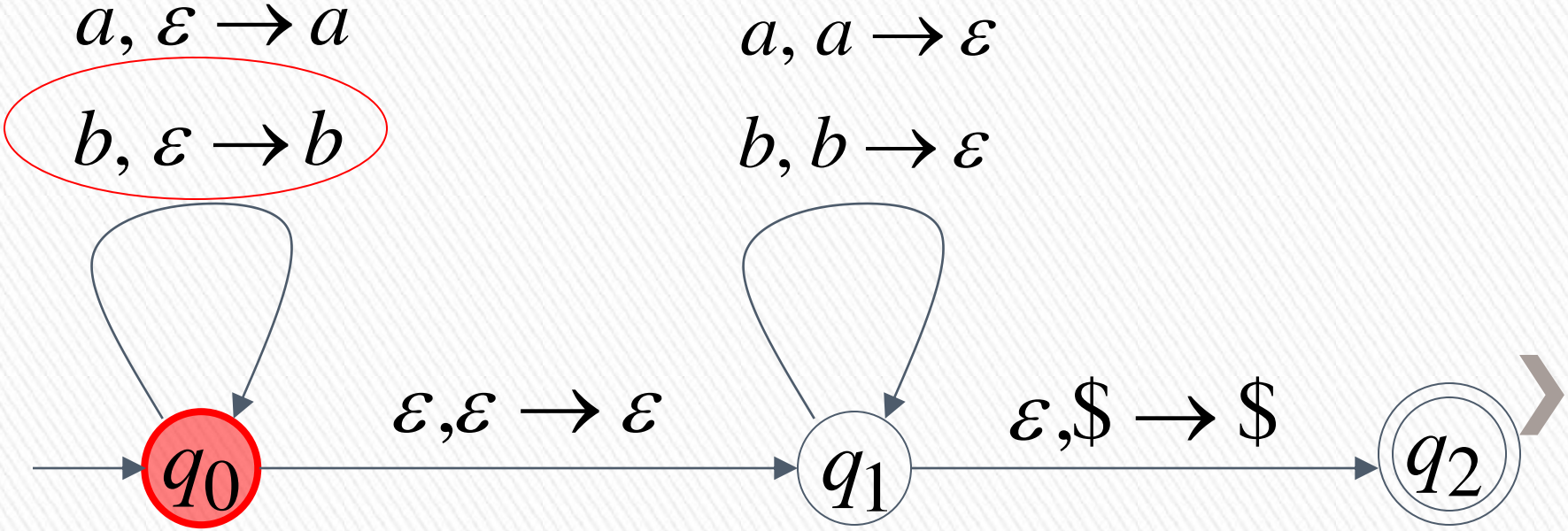
Time 2

Input





Stack



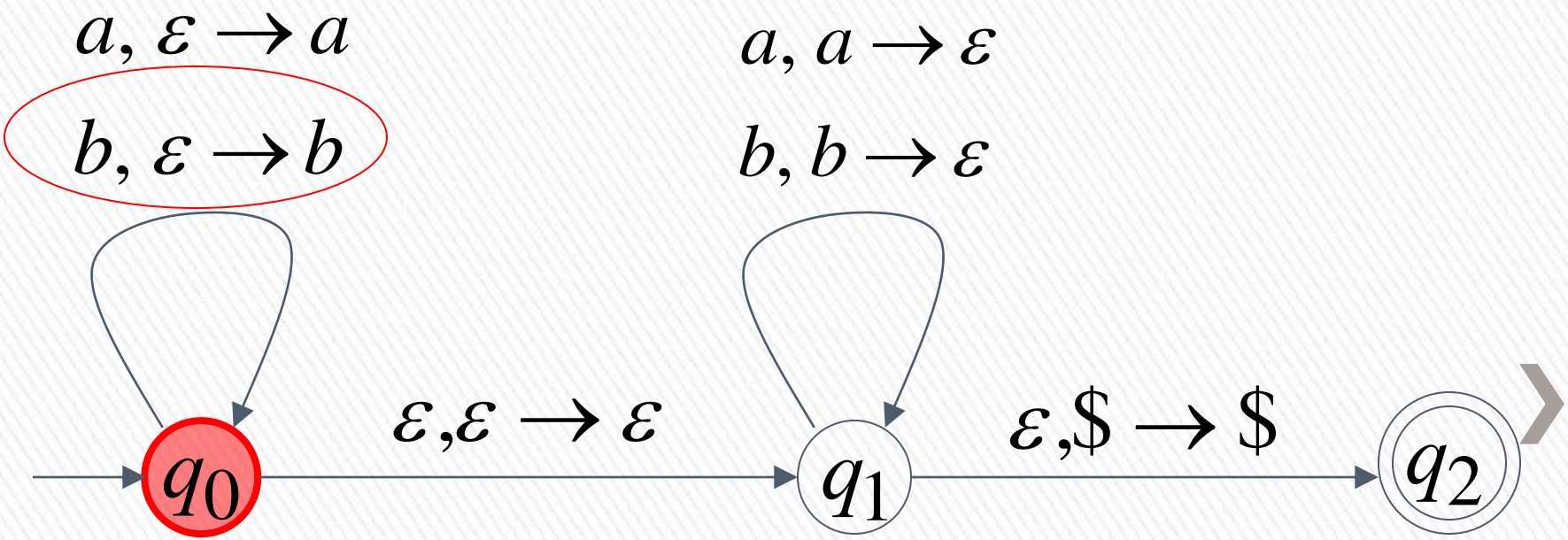
Time 3

Input

a	b	b	b

b
b
a
\$

Stack



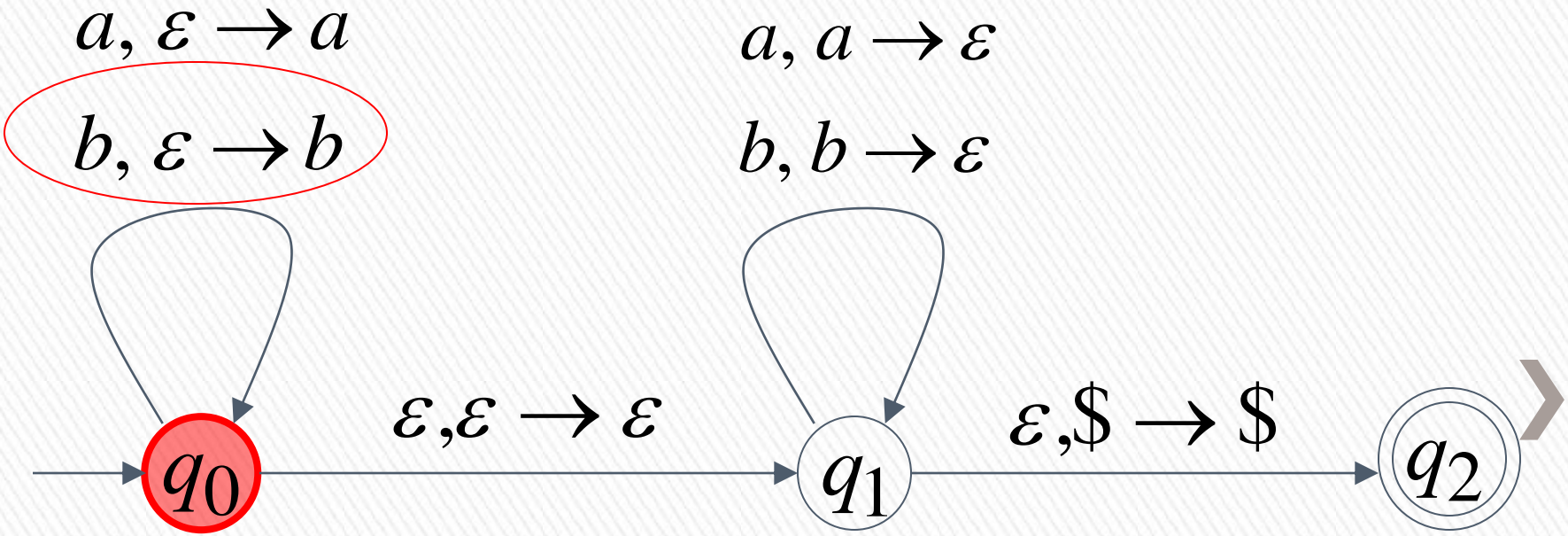
Time 4

Input

a	b	b	b

b
b
b
a
$\$$

Stack



Time 5

Input

a	b	b	b

No accept state is reached

b
b
b
a
\$

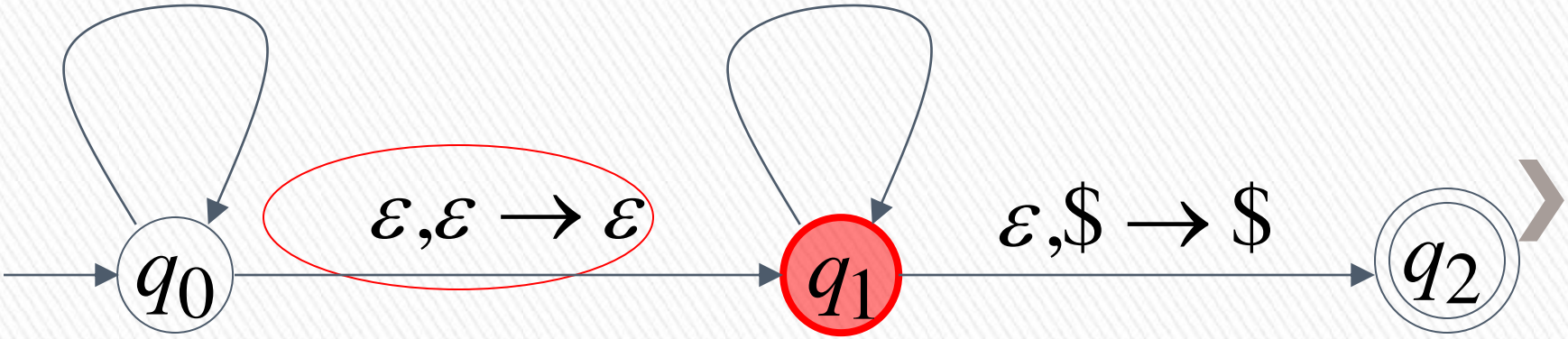
Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

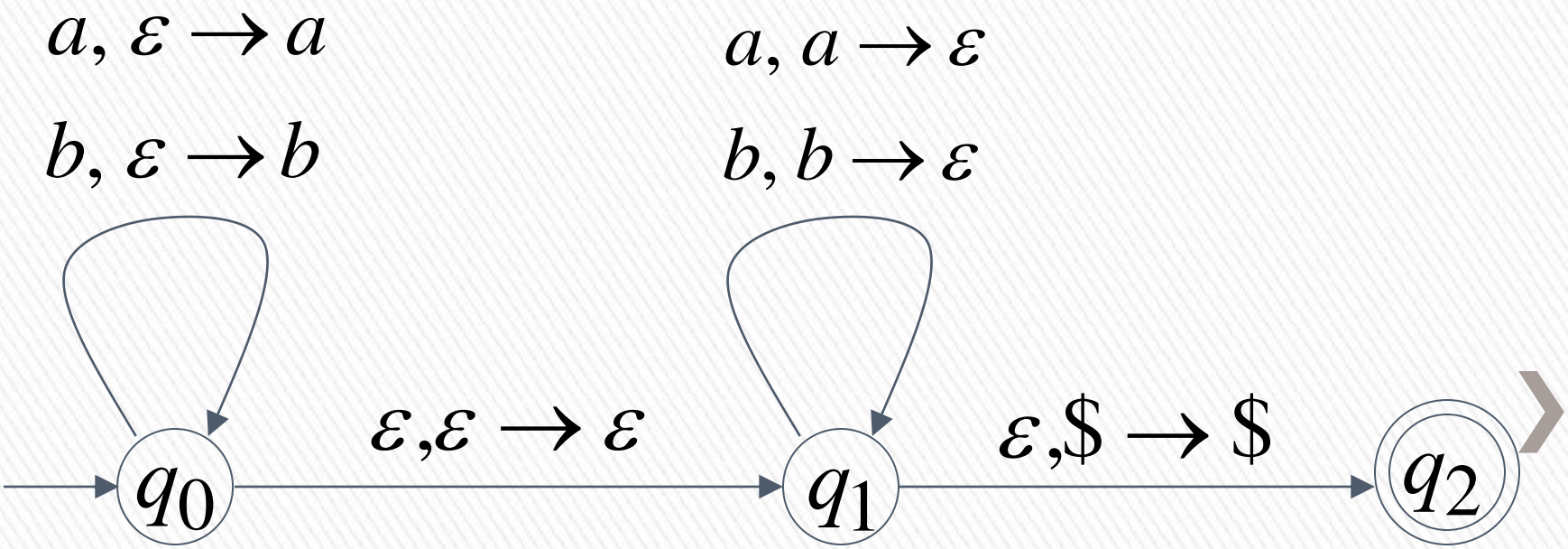
$a, a \to \varepsilon$

$b, b \to \varepsilon$

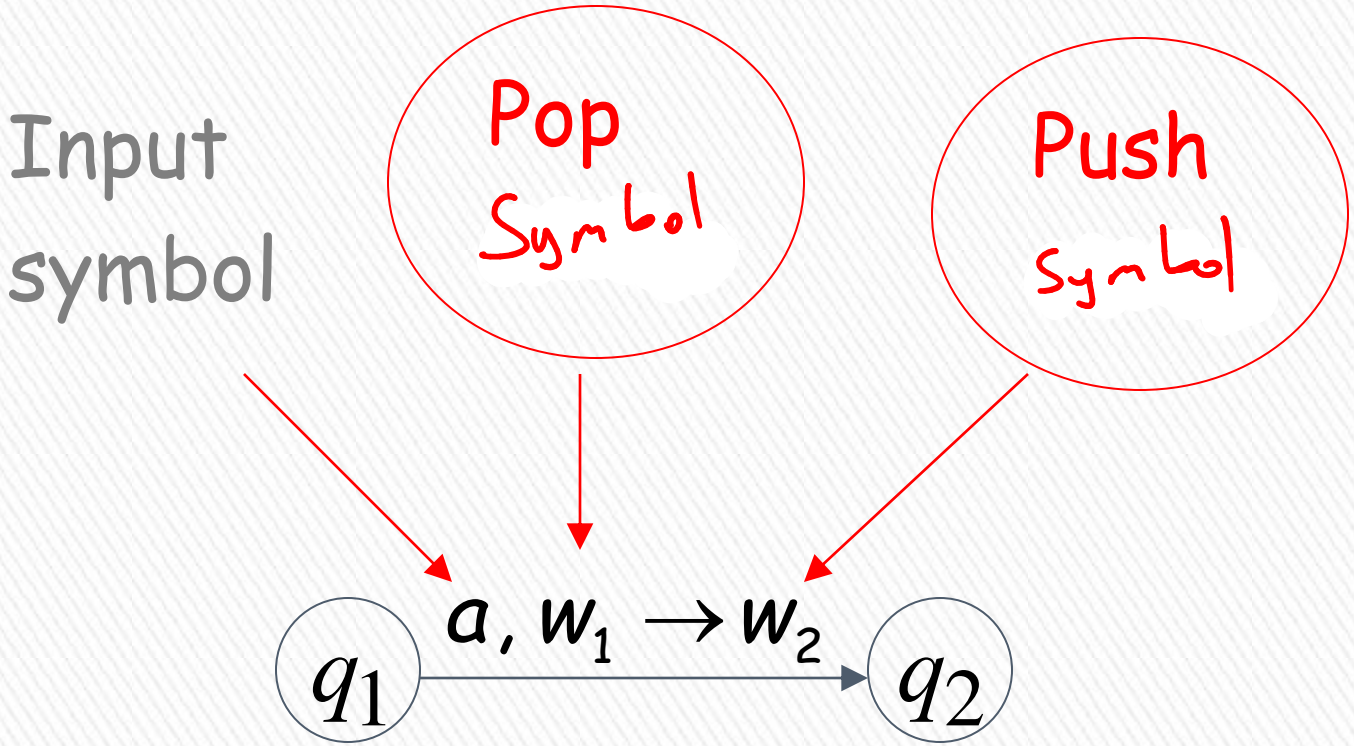


There is no computation that accepts string $abbb$

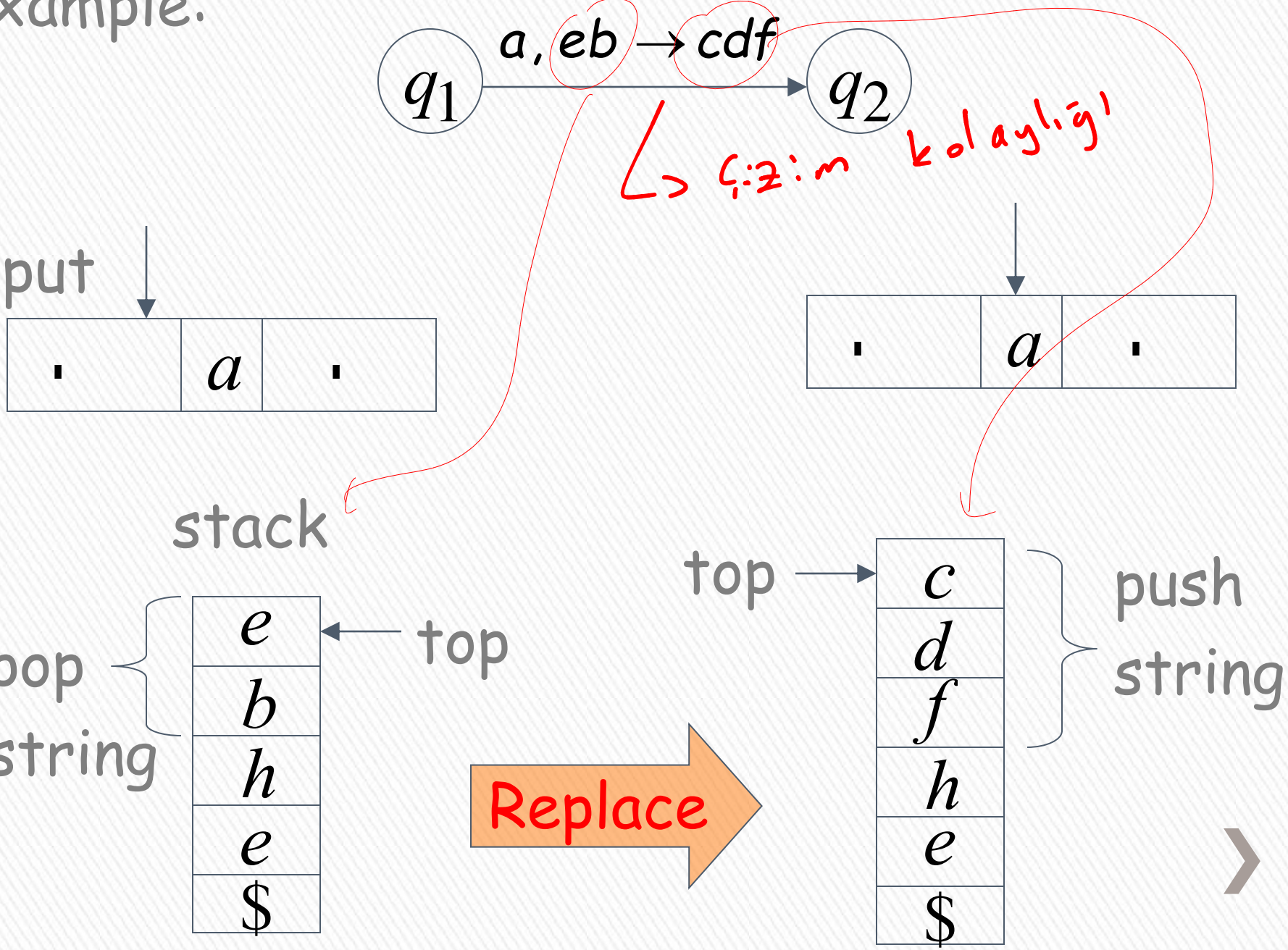
$$abbb \notin L(M)$$



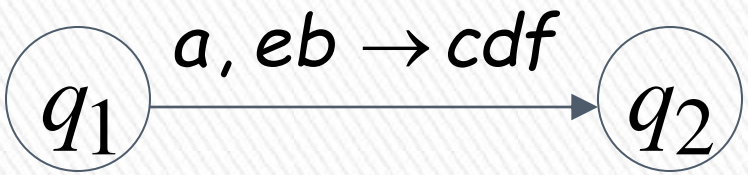
Pushing & Popping Strings

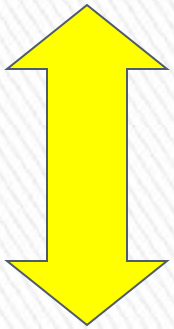


Example:



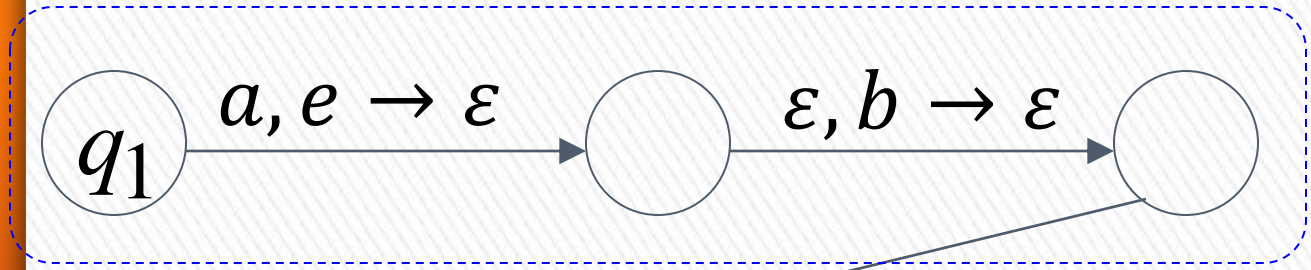
q_1
a, eb → cdf
q_2
↳ çizim kolaylığı
input
a
stack
top
pop string
e
b
h
e
$
Replace
top
c
d
f
h
e
$
push string

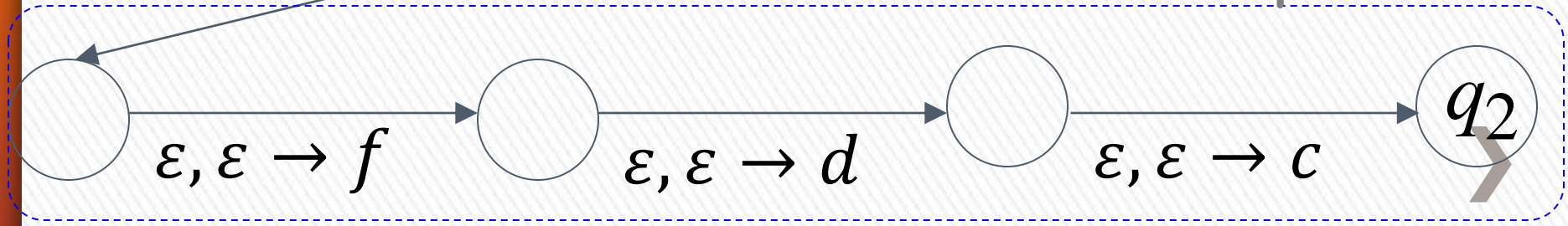




Equivalent transitions

pop





Another PDA example

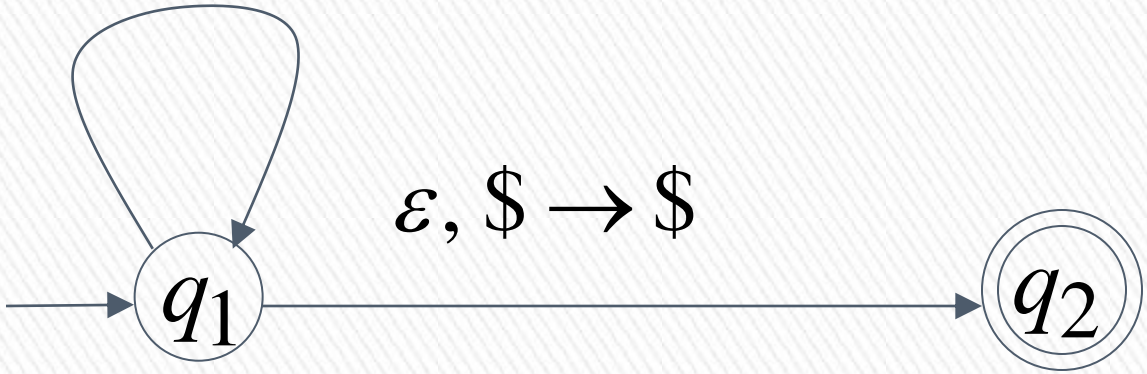
$$L(M) = \{w \in \{a,b\}^* : \ n_a(w) = n_b(w)\}$$

PDA M

$a, \$ \to 0\$$ $\quad b, \$ \to 1\$$

$a, 0 \to 00$ $\quad b, 1 \to 11$

$a, 1 \to \varepsilon$ $\quad b, 0 \to \varepsilon$



Execution Example: Time 0

Input

a	b	b	b	a	a

Stack

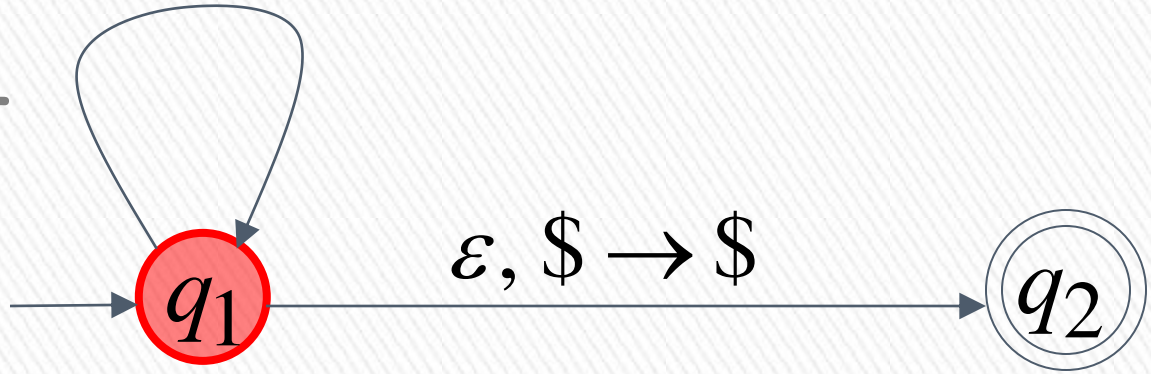
$\$$

$a, \$ \to a\$$ $\quad b, \$ \to b\$$

$a, a \to aa$ $\quad b, b \to bb$

$a, b \to \varepsilon$ $\quad b, a \to \varepsilon$

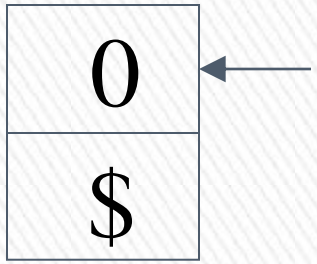
current state



Time 1

Input

a	b	b	b	a	a

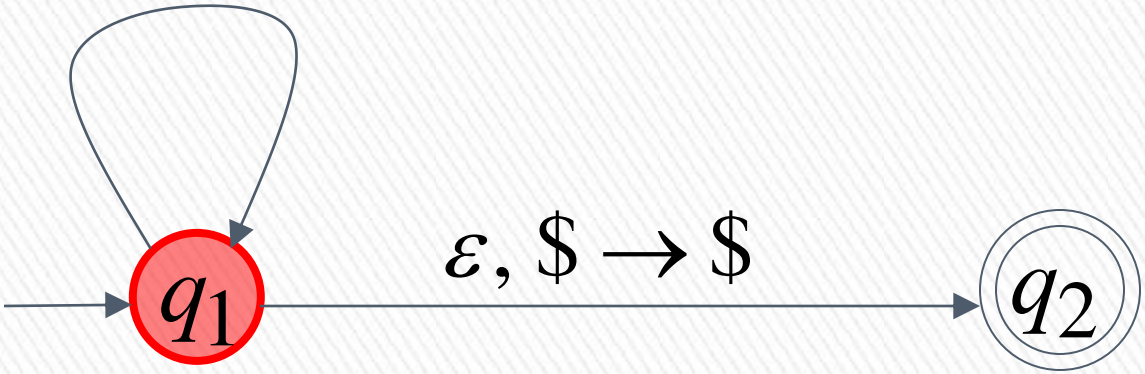


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

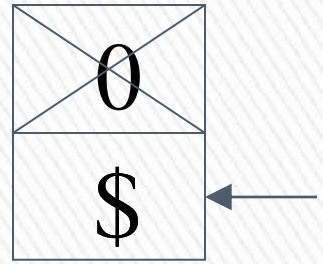
$a, 1 \rightarrow \varepsilon$ $b, 0 \rightarrow \varepsilon$



Time 3

Input

a	b	b	b	a	a

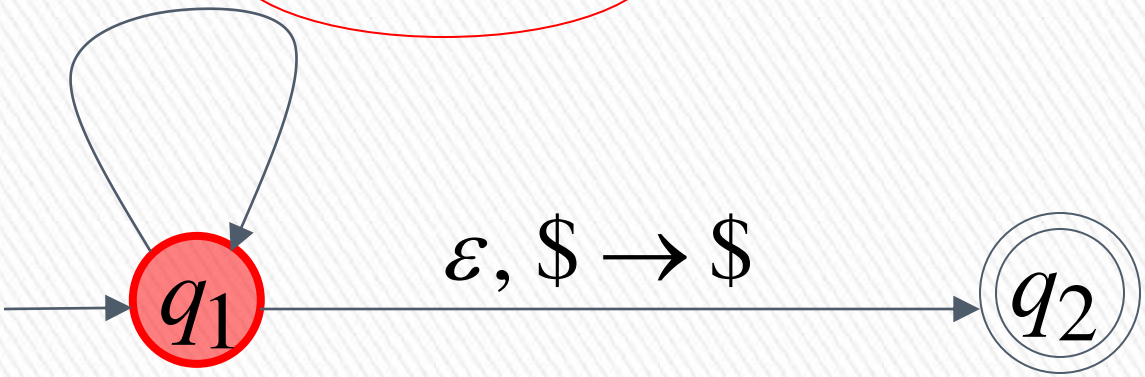


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

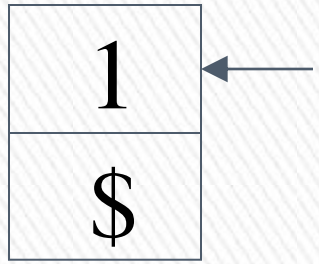
$a, 1 \rightarrow \varepsilon$ $b, 0 \rightarrow \varepsilon$



Time 4

Input

a	b	b	b	a	a

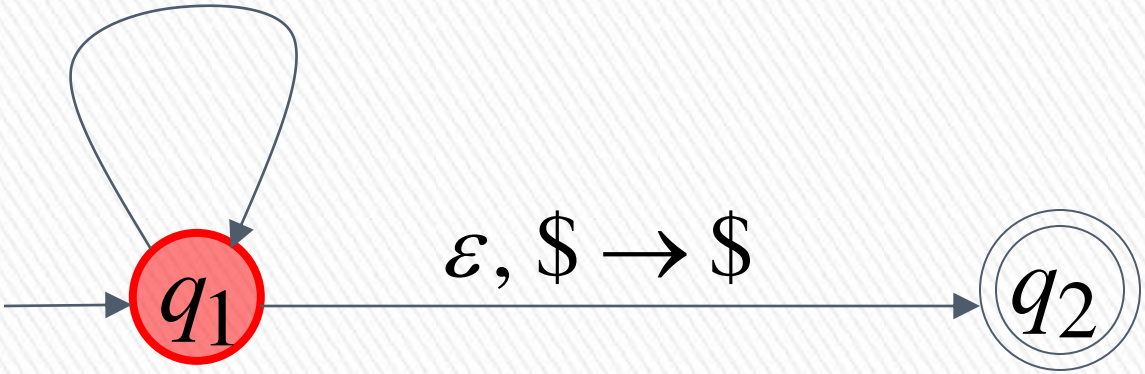


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

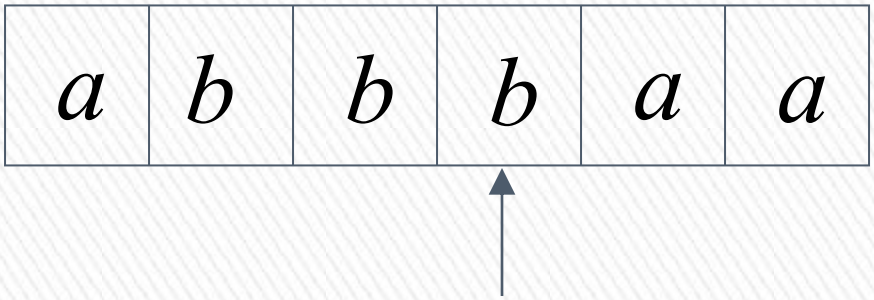
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

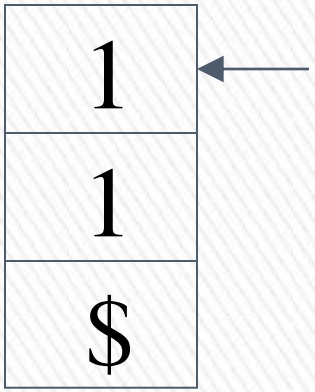
$a, 1 \rightarrow \varepsilon$ $b, 0 \rightarrow \varepsilon$



Time 5

Input



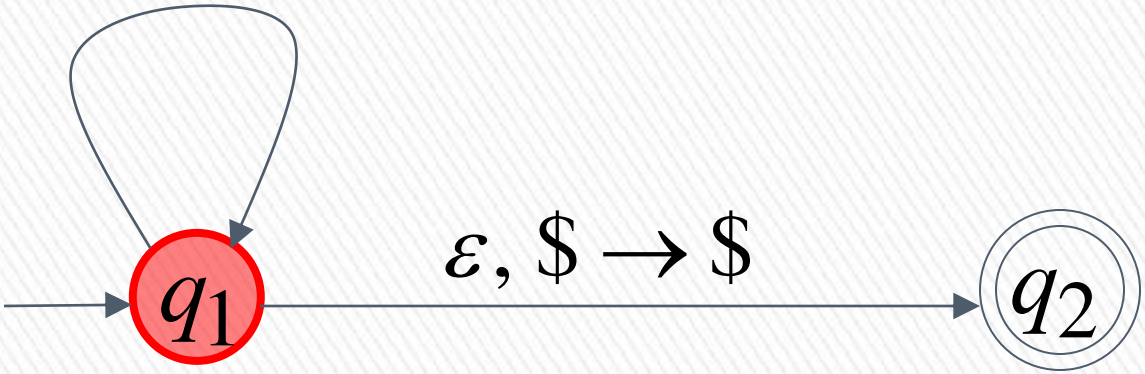


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

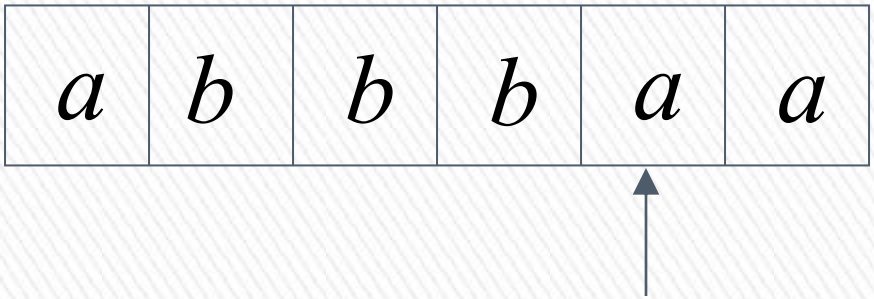
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

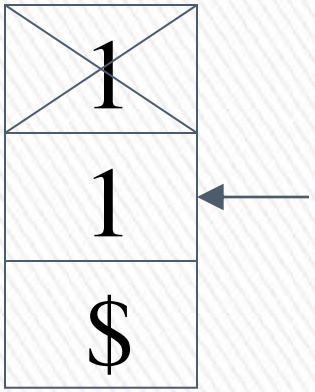
$a, 1 \rightarrow \varepsilon$ $b, 0 \rightarrow \varepsilon$



Time 6

Input



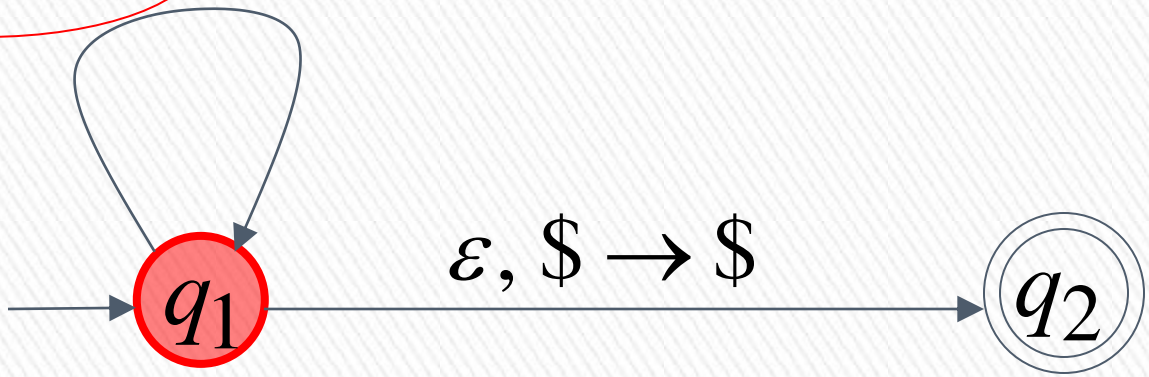


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

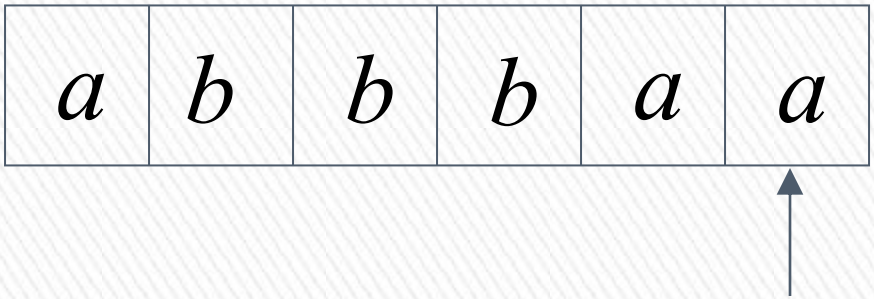
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

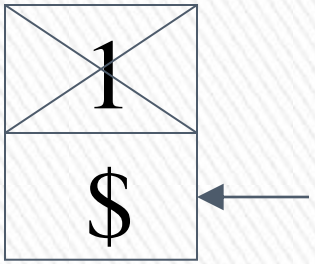
$a, 1 \rightarrow \varepsilon$ $b, 0 \rightarrow \varepsilon$



Time 7

Input



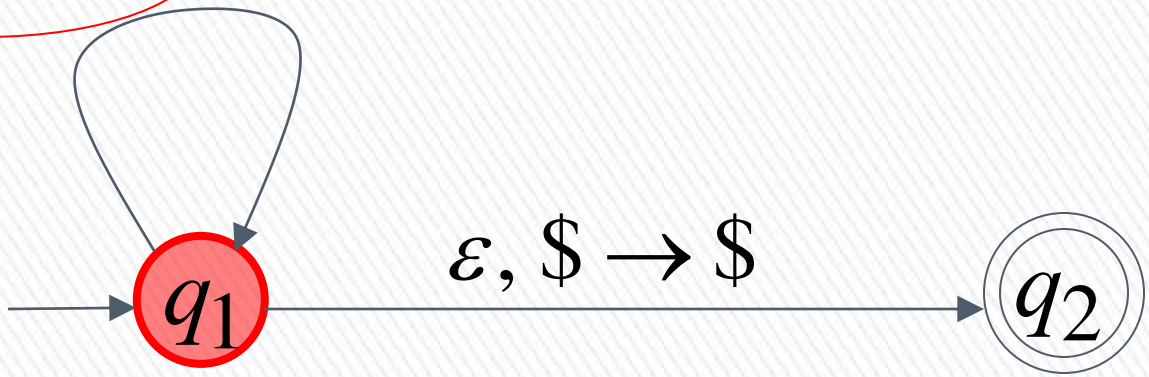


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

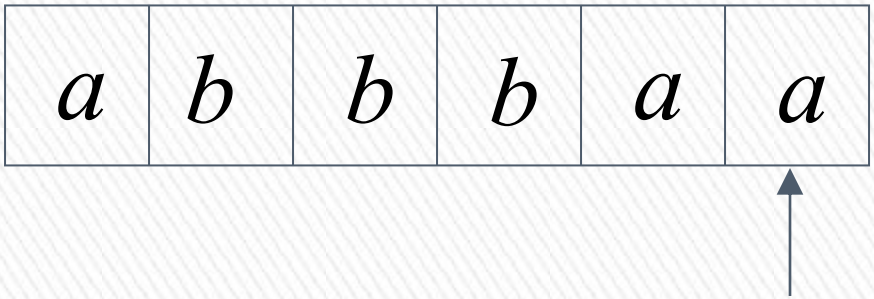
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \varepsilon$ $b, 0 \rightarrow \varepsilon$



Time 8

Input

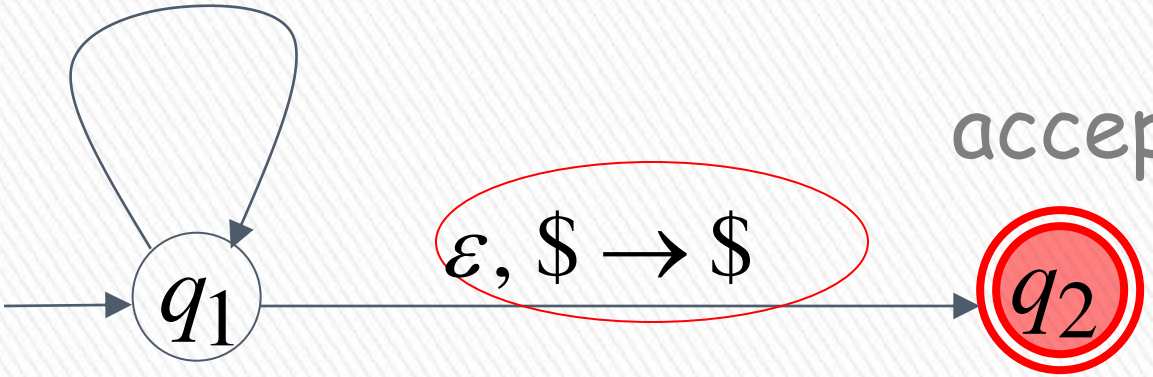


Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

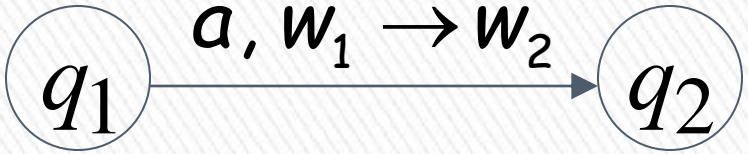
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \varepsilon$ $b, 0 \rightarrow \varepsilon$



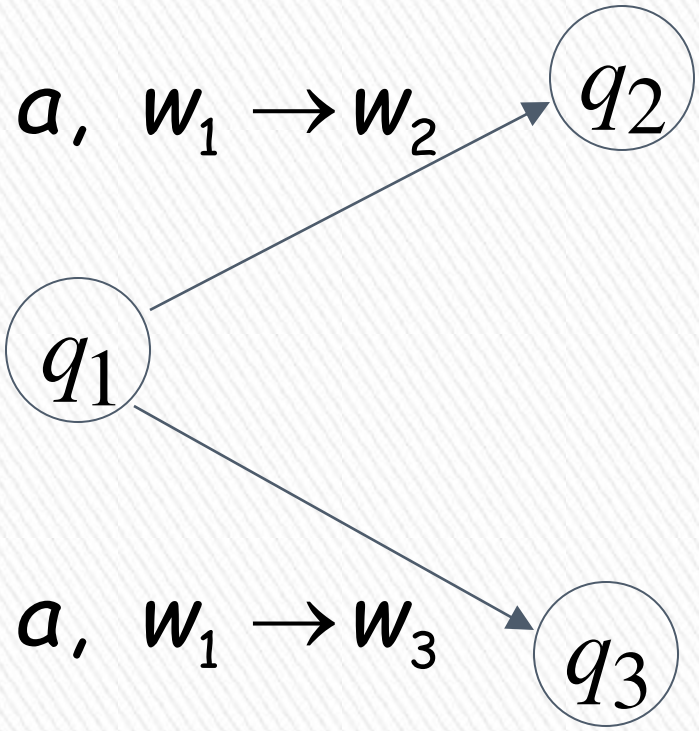
accept

Formalities for PDAs



Transition function:

$$\delta(q_1, a, w_1) = \{(q_2, w_2)\}$$



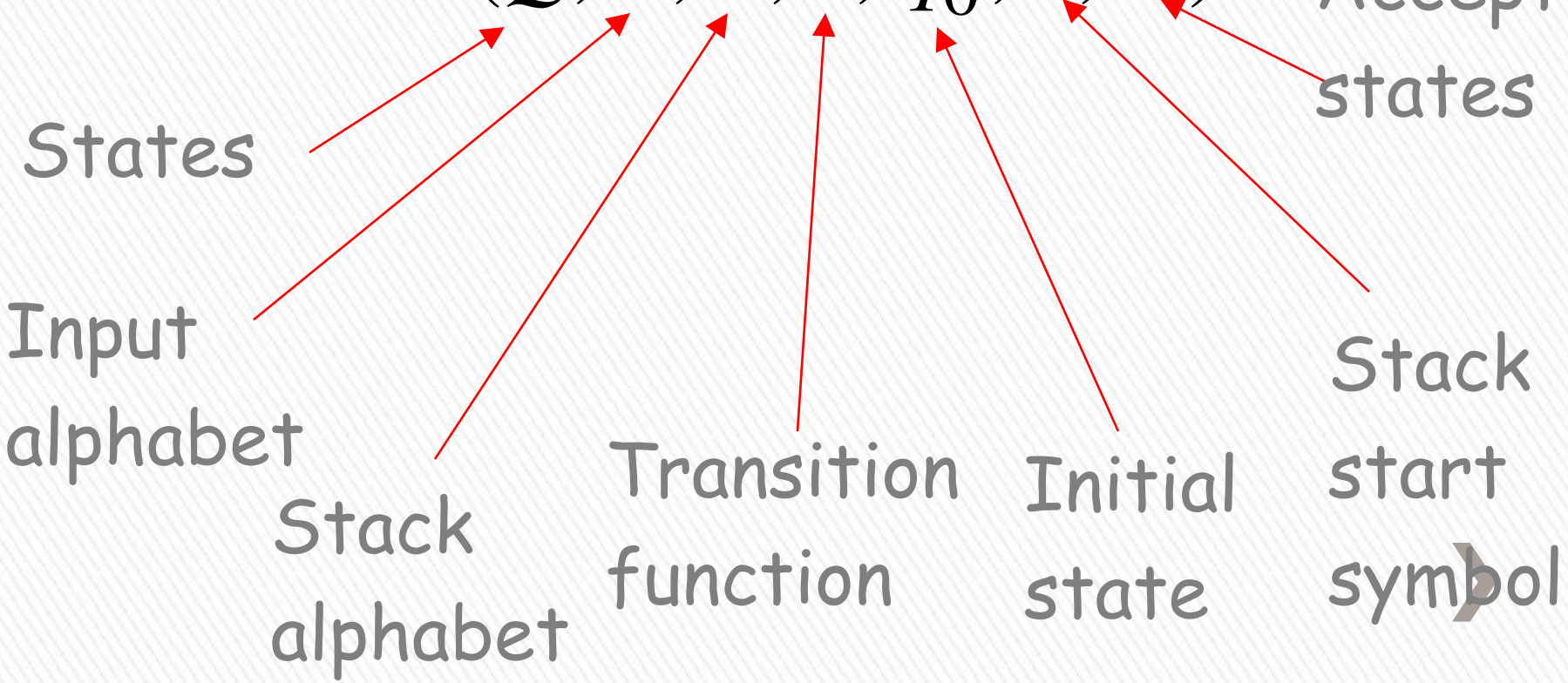
Transition function:

$$\delta(q_1, a, w_1) = \{(q_2, w_2), (q_3, w_3)\}$$

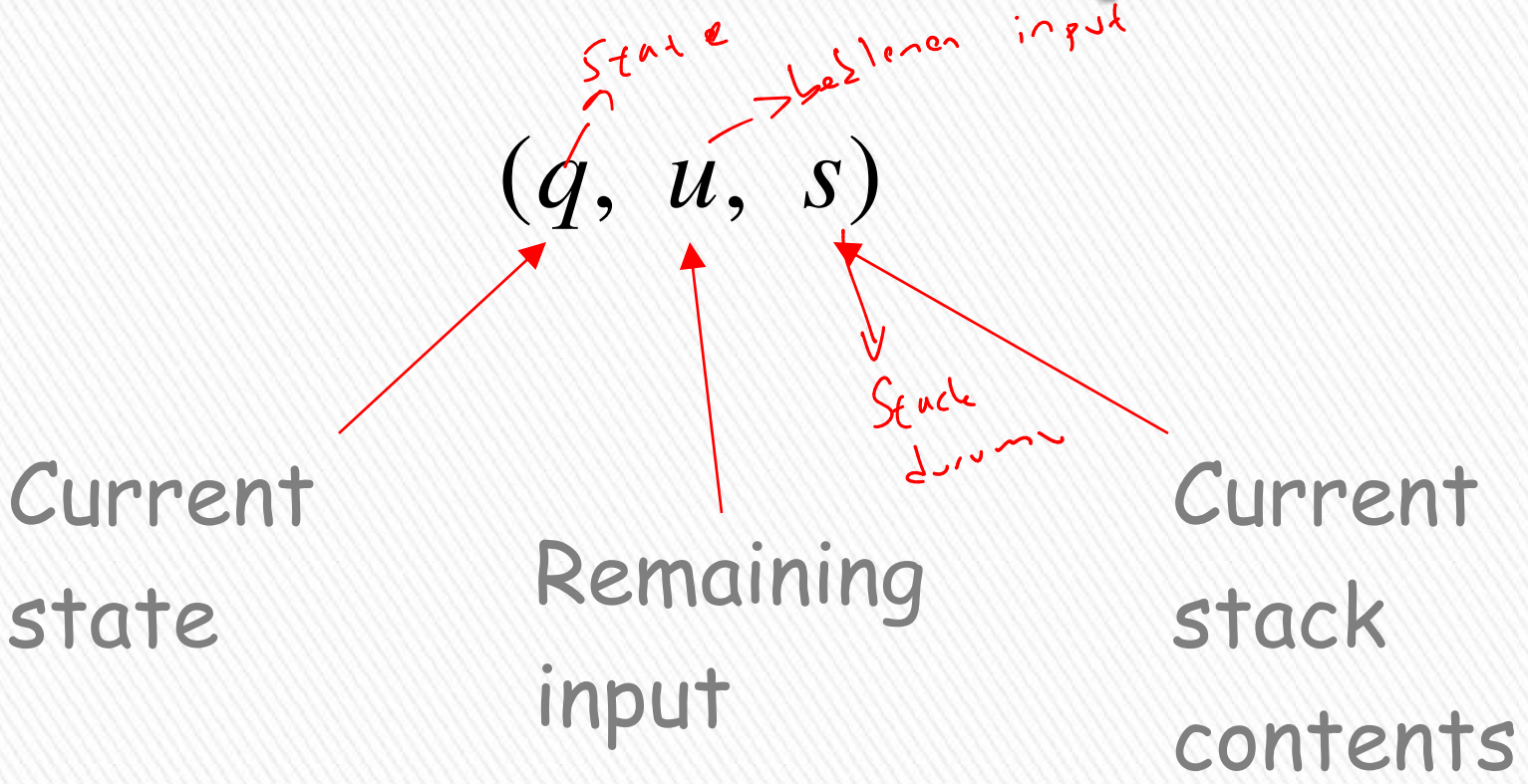
Formal Definition

Pushdown Automaton (PDA)

$$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$$



Instantaneous Description



Example: **Instantaneous Description**

$$(q_1, bbb, aaa\$)$$

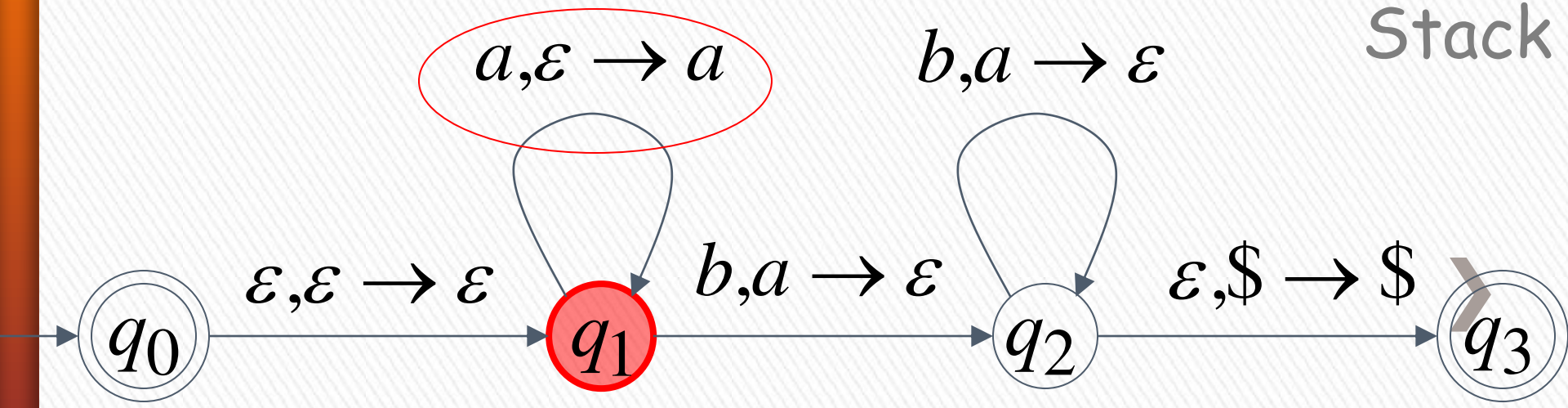
Time 4:

Input

a	a	a	b	b	b

a
a
a
$\$$

Stack

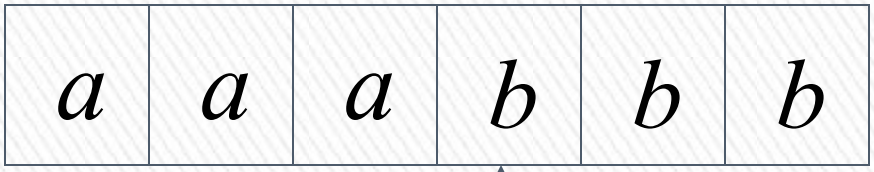


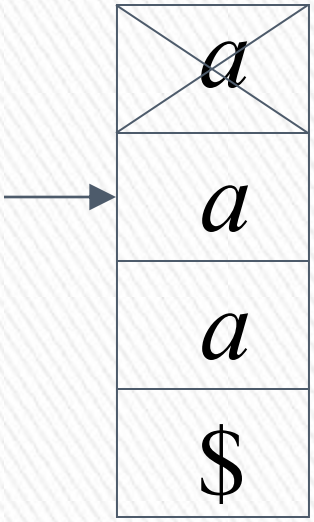
Example: **Instantaneous Description**

$$(q_2, bb, aa\$)$$

Time 5:

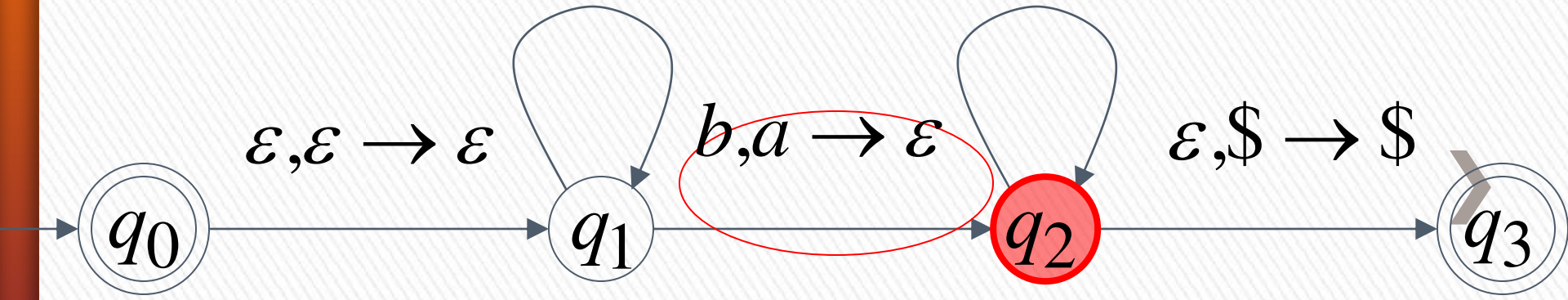
Input





Stack

$a, \varepsilon \rightarrow a$ $\quad$ $b, a \rightarrow \varepsilon$



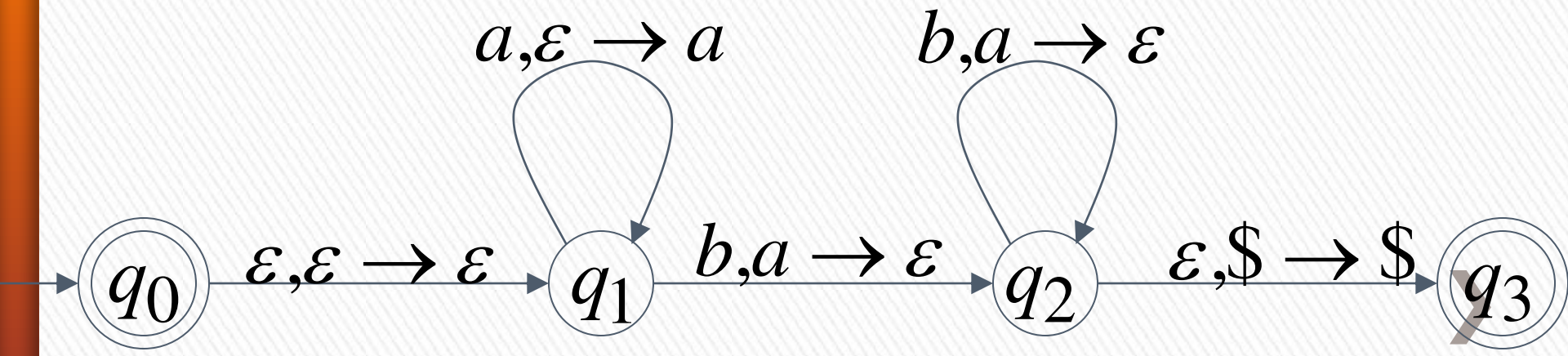
We write:

$$(q_1, bbb, aaa\$) \succ (q_2, bb, aa\$)$$

Time 4 Time 5

A computation:

$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$
$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$
$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \varepsilon, \$) \succ (q_3, \varepsilon, \$)$$



$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$
$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$
$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \varepsilon, \$) \succ (q_3, \varepsilon, \$)$$

For convenience we write:

$$(q_0, aaabbb, \$) \overset{*}{\succ} (q_3, \varepsilon, \$)$$

Language of PDA

Language $L(M)$ accepted by PDA M:

$$L(M) = \{w: \ (q_0, w, z) \overset{*}{\succ} (q_f, \varepsilon, s)\}$$

Initial state

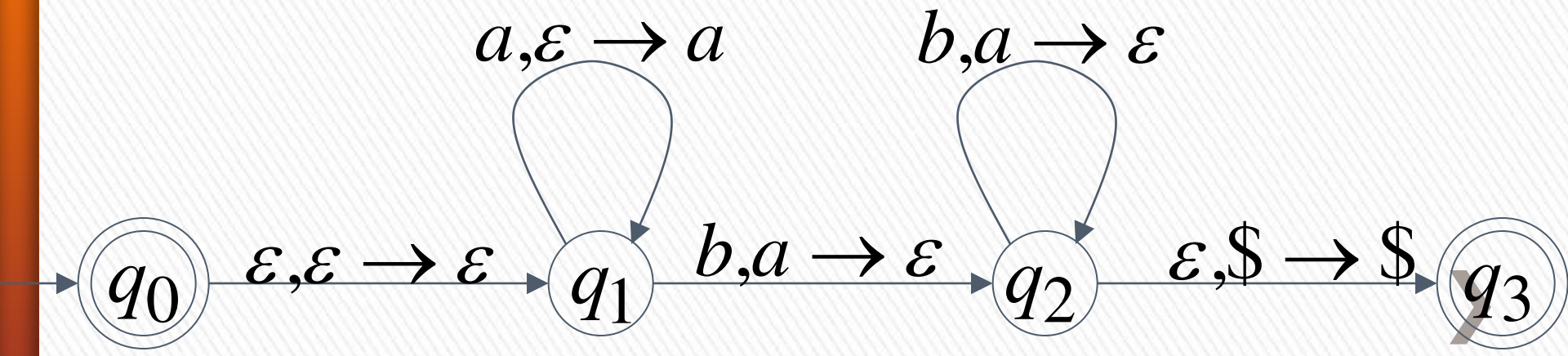
Accept state

Example:

$$(q_0, aaabbb, \$) \overset{*}{\vdash} (q_3, \varepsilon, \$)$$

$$aaabbb \in L(M)$$

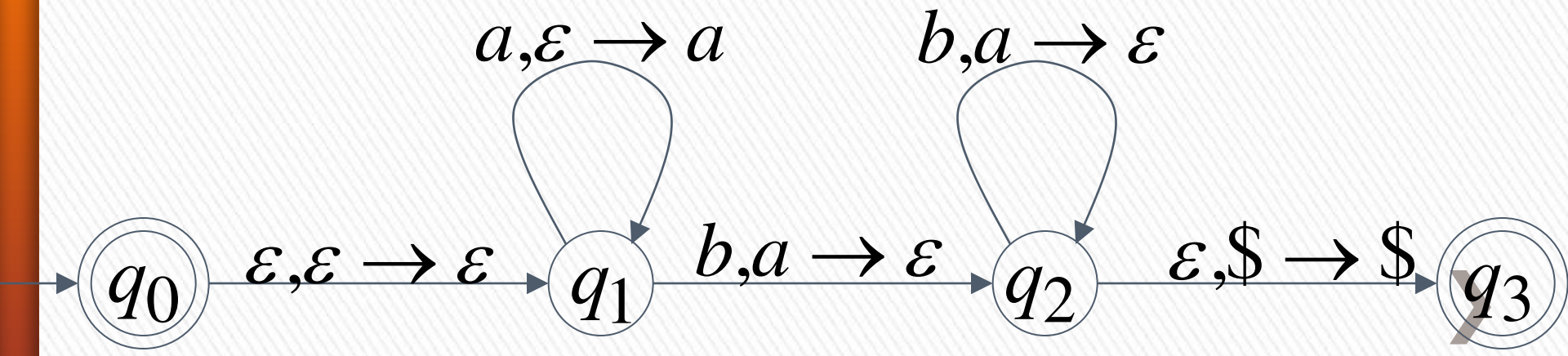
PDA **M** :



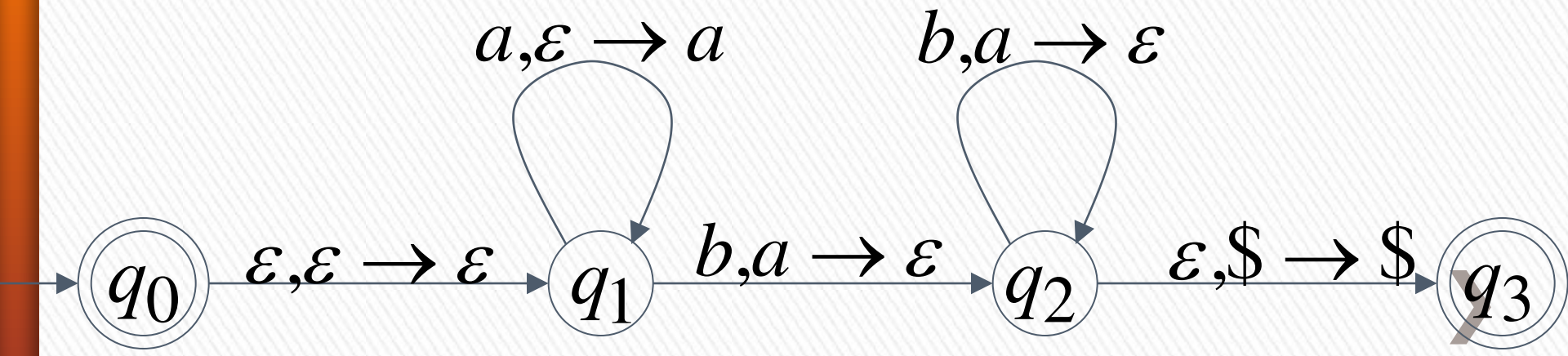
$$(q_0, a^n b^n, \$) \overset{*}{\succ} (q_3, \varepsilon, \$)$$

$$a^n b^n \in L(M)$$

PDA M :



Therefore:

$$L(M) = \{a^n b^n : n \geq 0\}$$

PDA M :