Homework 1 (To be delivered during the 1st midterm exam)

BLM 2502: Theory of Computations — Spring 2020

Print family (or last) name: ——————————————————————————

Print given (or first) name: ——————————————————————————

Print given student number: ——————————————————————————

I see that this homework has 17 questions in total 16 pages.

I agree that I have to submit my homework solution before the deadline (i.e., the time of the 1st midterm exam) otherwise my homework solution will not be graded. I accept that ***I will add the signed version of this instruction page as a first page into my homework solution***; otherwise my homework solution will not be graded. I know that ***I have to give my solutions in the empty-white spaces just below the questions***; otherwise my homework solution will not be graded. I will take care of the readability of my solutions, from which I may lose 10 points. For any proofs, I am sure to provide a step-by-step argument, with justifications for every step. I understand that, during solving this homework, it is prohibited to exchange information about solutions with any other person in any way, including by talking or exchanging solutions / papers.

I know that the course book is "Introduction to the theory of computation, 2nd Ed., Massachusetts Institute of Technology, by Micheal Sipser."

I have read, understand and accept all of the instructions above. On my honor, I pledge that I have not violated the provisions of the Academic Integrity Code of Yıldız Technical University.

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Signature and Date

1	2	3	4	5	6	7	8	9	10
10 pts	**20 pts**	**15 pts**	**15 pts**	**15 pts**	**15 pts**	**10 pts**	**20 pts**	**20 pts**	**20 pts**

11	12	13	14	15	16	17
20 pts	**20 pts**	**20 pts**	**20 pts**	**20 pts**	**20 pts**	**20 pts**

Total
300 pts

1) **[10 Points]** Why do we need computation? Why do we need programming language for computation? Why do we need automats / machines to recognizes/accept programming language?

Computation is any numerical or logical process / calculation that outputs something meaningful. So we need it to think, understand and build complex models such as algorithms. In the past people think only humans can do computation. But now when we say "computation" mostly computers will come to mind. Because of that; like we people need languages to communicate, we need such as languages to communicate with computers aka machines. We call this kind of languages "programming languages". If a machine can understand more complex language, this means it has more computational power. By "understanding" we mean recognizing / accepting the given input. So in the end we need machines which understands our "programming language" to do computations we want to get results.

2) **[20 Points]** For any $n \in \mathbb{N}$, prove that the following equality is valid.

$$1^6 + 2^6 + 3^6 + \cdots + n^6 = \frac{n}{42}(n+1)(2n+1)(3n^4 + 6n^3 - 3n + 1)$$

we're gonna prove this equation by induction. For this we need to prove it first for small numbers. lets prove it for 1 and 2.

$n=1 \quad 1^6 = \frac{1}{42}(1+1)(2+1)(3+6-3+1) \longrightarrow 1 = \frac{1}{42} \cdot 2 \cdot 3 \cdot 7 = \frac{42}{42} = 1$, true for 1.

$n=2 \quad 1^6 + 2^6 = \frac{2}{42}(2+1)(4+1)(3.16 + 6.8 - 3.2 + 1) \longrightarrow 65 = \frac{2}{42} \cdot 3 \cdot 5 \cdot 91 = \frac{15 \cdot 91}{21} = 65$ true for 2.

now we're going to check that if this is true for $(n+1)$ when we assume that this is true for n.

we're going to use open form of formula $\longrightarrow \frac{n}{42}(n+1)(2n+1)(3n^4+6n^3-3n+1) = \frac{1}{42}(6n^7+21n^6+21n^5-7n^3+n)$

if the following is holds then this means this formula is true for $(n+1)$ when its true for n.

$$\frac{1}{42}(6n^7+21n^6+21n^5-7n^3+n) + (n+1)^6 = \frac{1}{42}\left(6(n+1)^7 + 21(n+1)^6 + 21(n+1)^5 - 7(n+1)^3 + (n+1)\right)$$

$$\frac{1}{42}\left(6n^7+21n^6+21n^5-7n^3+n+42(n^6+6n^5+15n^4+20n^3+15n^2+6n+1)\right) = \frac{1}{42}\begin{pmatrix} 6(n^7+7n^6+21n^5+35n^4+35n^3+21n^2+7n+1) + \\ 21(n^6+6n^5+15n^4+20n^3+15n^2+6n+1) + \\ 21(n^5+5n^4+10n^3+10n^2+5n+1) - 7(n^3+3n^2+3n+1) + n+1 \end{pmatrix}$$

$$\frac{1}{42}(6n^7+63n^6+273n^5+630n^4+833n^3+630n^2+253n+42) = 2\,\frac{1}{42}(6n^7+63n^6+273n^5+630n^4+833n^3+630n^2+253n+42)$$

we know that formula true for 2 and any n+1 if it is for n. and since we can reach any number from 2, this is also for any $n \in \mathbb{N}$. proved by induction. $\square$

3) **[15 Points]** Let α and β be two positive integer numbers. If $\alpha^2 - \beta^2$ is not odd, then $\alpha + \beta \geq 2$

proof by contradiction.

Lets say this assumption is false and when a^2-b^2 is even $a+b<2$. if we try to prove this and fail, then we can say the original assumption is correct.

if $(a-b)(a+b)$ is even, this means;

a) $(a-b)(a+b)=0$

which means $a=b$, and since both positive min a and b can be 1,

$a+b \geq 1+1 \quad 1+1=2<2$ false

b) $(a-b)(a+b)=2k$ or $-2k \rightarrow$ $(a-b)$ or $(a+b)$ or both need to be even

i) $a-b$ even $\rightarrow a=b+2k, k>0 \quad a+b=2b+2k<2$ false

$b=a+2k, k>0 \quad a+b=2a+2k<2$ false

ii) $a+b$ even $\rightarrow$ a and b odd

$a=2k+1 \quad b=2n+1 \quad a+b=2k+1+2n+1<2$ false

a and b even

$a=2k \quad b=2n \quad a+b=2k+2n<2$ false

4) **[15 Points]** Let a, b, c, d be integers. If $a > c$ and $b > c$, then prove that $\max(a, b) - c$ is always positive.

proof by contradiction. let say this statement is not true. in this case $\max(a,b)-c$ would be equal to 0 (zero) or negative number.

if $\max(a,b)-c=0$

$\max(a,b)=c$

since $a>c$ and $b>c$

a or b cannot be equal to c

cannot be true

if $\max(a,b)-c<0$

$\max(a,b)<c$

since $a>c$ and $b>c$

a or b cannot be smaller than c

cannot be true

since these statements cannot be true, we left with only one statement which means $\max(a,b)-c$ is always >0 aka positive.

5) **[15 points]** Given two sets X and Y. The Cartesian product of X and Y, written as $X \times Y$, is defined as the set of pairs (x, y) where $x \in X$ and $y \in Y$. Then, find a mathematical closed-form expression to write $|X \times Y|$ in terms of $|X|$ and $|Y|$.

$X \times Y = \{(x,y) \mid x \in X \text{ and } y \in Y\}$

$|X \times Y| = |X| \cdot |Y|$

6) **[15 points]** Let us given two disjoint sets X and Y, and then their joint set S is $S = X \cup Y$. Sum of the elements in a set S is denoted by $\sum(S)$ while their product is by $\prod(S)$. Accordingly, what are $\sum(S)$ and $\prod(S)$ in terms of $\sum(X)$, $\sum(Y)$, $\prod(X)$ and $\prod(Y)$. Conclude from this what $\sum(\emptyset)$ and $\prod(\emptyset)$ should be ($\emptyset$ is the empty set).

$$\Sigma(S) = \Sigma(X \cup Y) = \Sigma(X) + \Sigma(Y) - \Sigma(X \cap Y) \quad \text{since } X \cap Y = \emptyset \rightarrow = \Sigma(X) + \Sigma(Y)$$

$$\Pi(S) = \Pi(X \cup Y) = \frac{\Pi(X) \cdot \Pi(Y)}{\Pi(X \cap Y)} \quad \text{since } X \cap Y = \emptyset \rightarrow = \Pi(X) \cdot \Pi(Y)$$

This give us $\rightarrow$ $\underline{\Sigma(\emptyset) = 0}$ and $\underline{\Pi(\emptyset) = 1}$

$$\cancel{\Sigma(X)} + \cancel{\Sigma(Y)} - \Sigma(\emptyset) = \cancel{\Sigma(X)} + \cancel{\Sigma(Y)}$$

$$-\Sigma(\emptyset) = 0 \rightarrow \Sigma(\emptyset) = 0 \checkmark$$

$$\frac{\cancel{\Pi(X)} \cdot \cancel{\Pi(Y)}}{\Pi(\emptyset)} = \cancel{\Pi(X)} \cdot \cancel{\Pi(Y)}$$

$$\frac{1}{\Pi(\emptyset)} = 1 \rightarrow \Pi(\emptyset) = 1$$

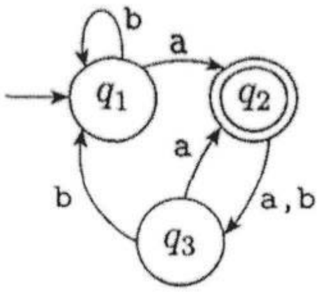
7) **[10 points]** What is the relation between programming language and the power of a machine that recognizes / accepts that programming language? Give an example in your explanation.

As programming language gets more complex; the machine that recognizes/accepts that programming language becomes more powerfull which means it has more computational capability/power.

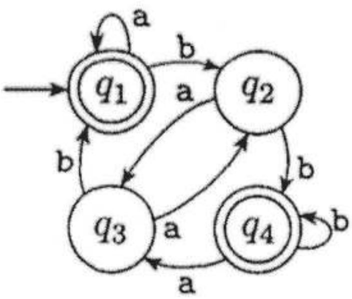
ex: Let say we have $L_1 = \{\}$ and $L_2 = \{w \in \{0,1\}^* \mid w = (1^*(00)^*1^*)^+\}$

we can obviously see that L_2 is more complex than L_1. Lets say we also have two machines; M_1 that recognizes L_1 and M_2 that recognizes L_2. As we can see M_1 would have little computational power as it dont need to do much because it have only one state and no transitions at all. In the other hand M_2 has to deal multiple states with multiple transitions which has also loops in it. By that we can say that M_2 has more computational power than M_1; because M_2 has to recognize more complex language than M_1.

8) **[20 Points]** The following are the state diagrams of two DFAs, M_1 and M_2 . Answer the following questions about each of these machines.



M_1



M_2

a) What is the start state?

for M_1

$= q_1$

for M_2

$= q_1$

b) What is the set of accept states?

for M_1

$F = \{ q_2 \}$

for M_2

$F = \{ q_1, q_4 \}$

c) What sequence of states does the machine go through on input *aabb* ?

for M_1

$\xrightarrow{\varepsilon} q_1 \xrightarrow{a} q_2 \xrightarrow{a} q_3 \xrightarrow{b} q_1 \xrightarrow{b} q_1$

for M_2

$\xrightarrow{\varepsilon} q_1 \xrightarrow{a} q_1 \xrightarrow{a} q_1 \xrightarrow{b} q_2 \xrightarrow{b} q_4$

d) Does the machine accept the string *aabb* ?

for M_1

NO

for M_2

YES

e) Does the machine accept the string ε?

for M_1

NO

for M_2

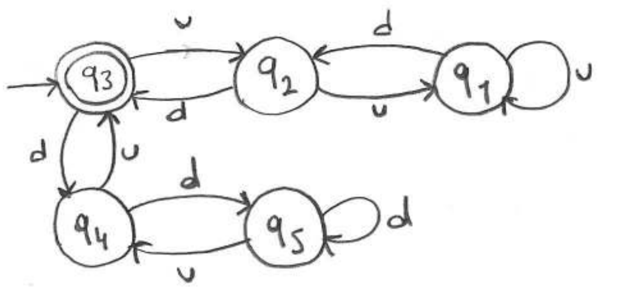
YES

9) **[20 Points]** The formal 5-tupple description of a DFA M is

$$(\{q_1, q_2, q_3, q_4, q_5\}, \{u, d\}, \delta, q_3, \{q_3\}),$$

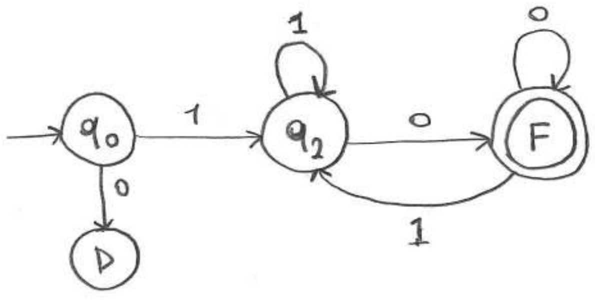
where δ is given by the following table. Give the state diagram of this machine.

	u	d
q_1	q_1	q_2
q_2	q_1	q_3
q_3	q_2	q_4
q_4	q_3	q_5
q_5	q_4	q_5

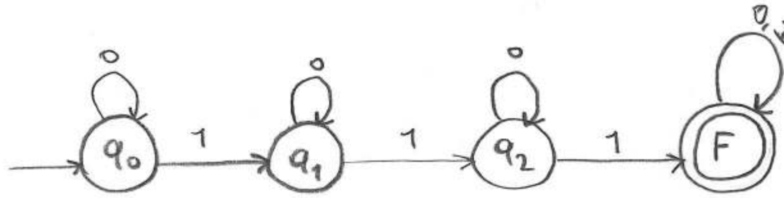


10) **[20 Points]** Give state diagrams of DFAs recognizing the following languages. In all parts the alphabet is {0,1}.

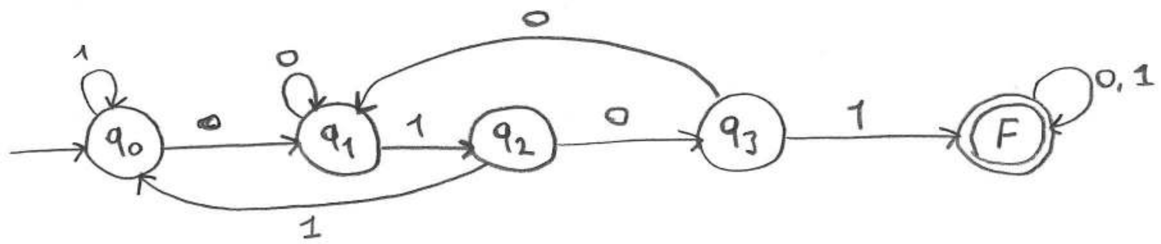
a) {w | w begins with a 1 and ends with a 0}



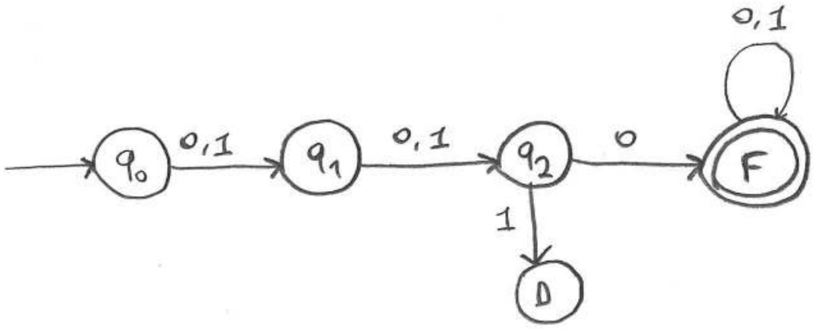
b) {w | w contains at least three 1s}



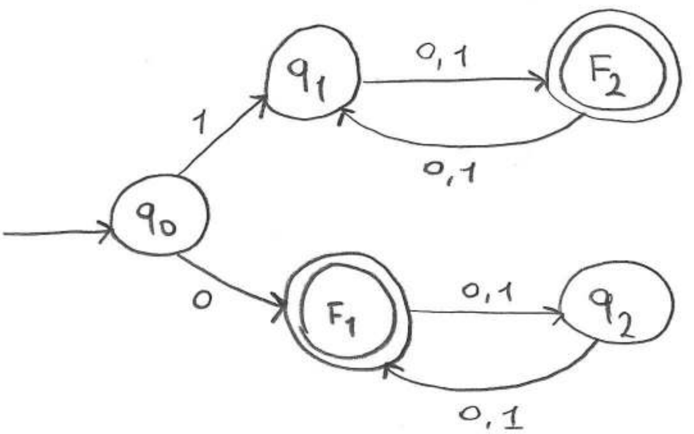
c) {w | w contains the substring 0101, i.e., w = x0101y for some x and y}



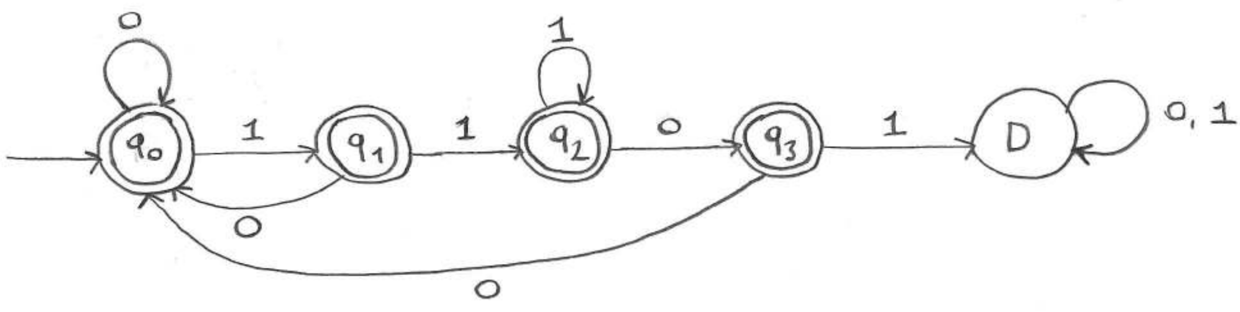
d) {w |w has length at least 3 and its third symbol is a 0}



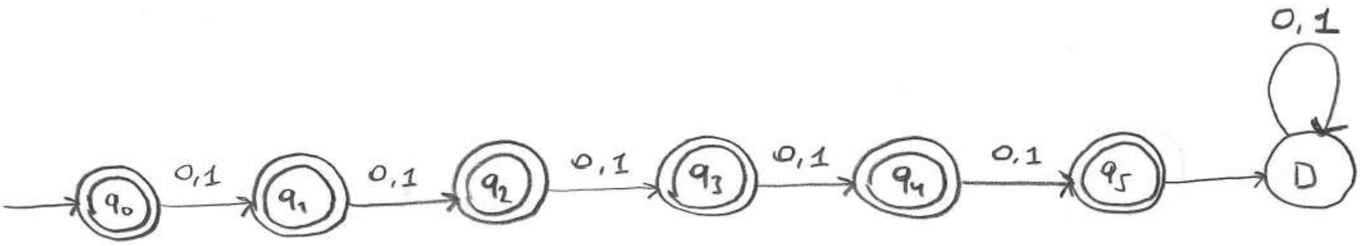
e) {w | w starts with 0 and has odd length, or starts with 1 and has even length}



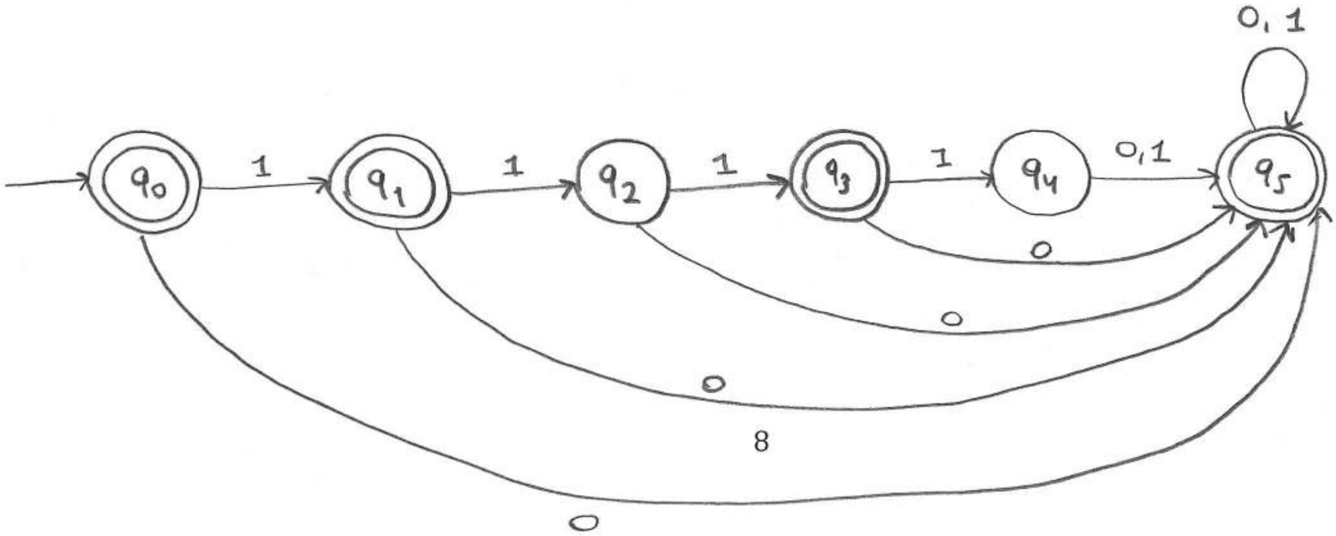
f) {w | w doesn't contain the substring 1101}



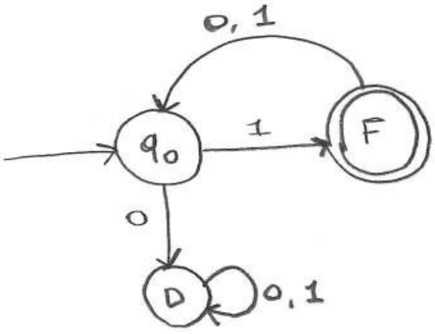
g) {w | w the length of w is at most 5}



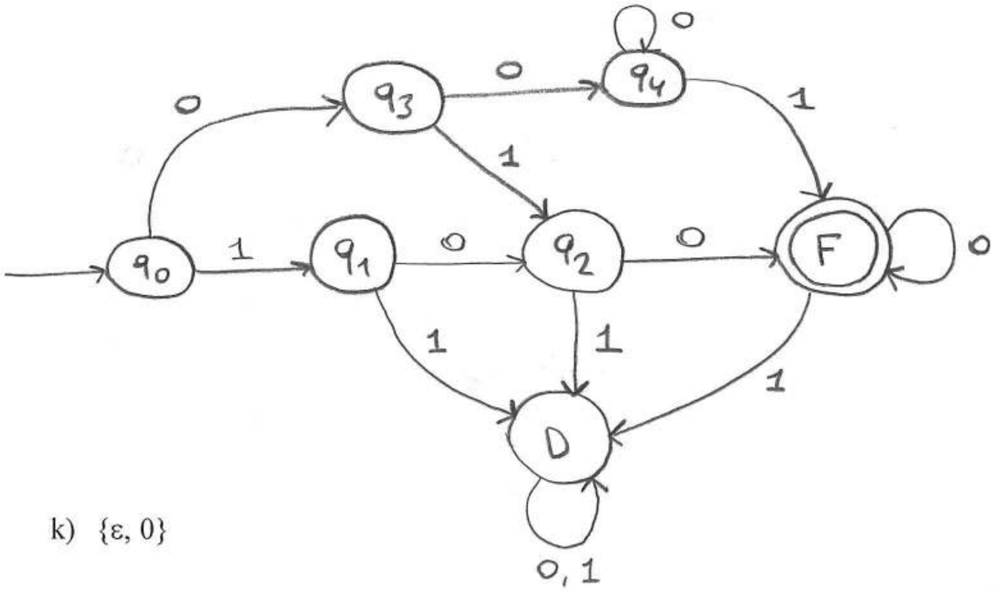
h) {w | w is any string except 11 and 1111}



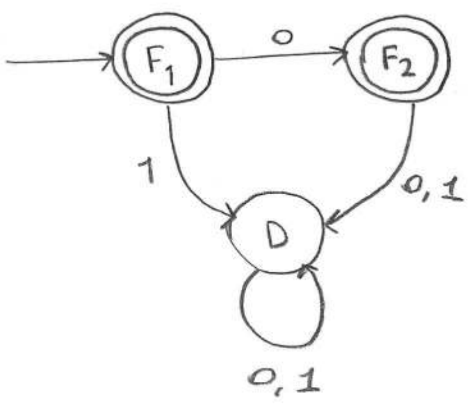
i) {w | every odd position of w is a 1}



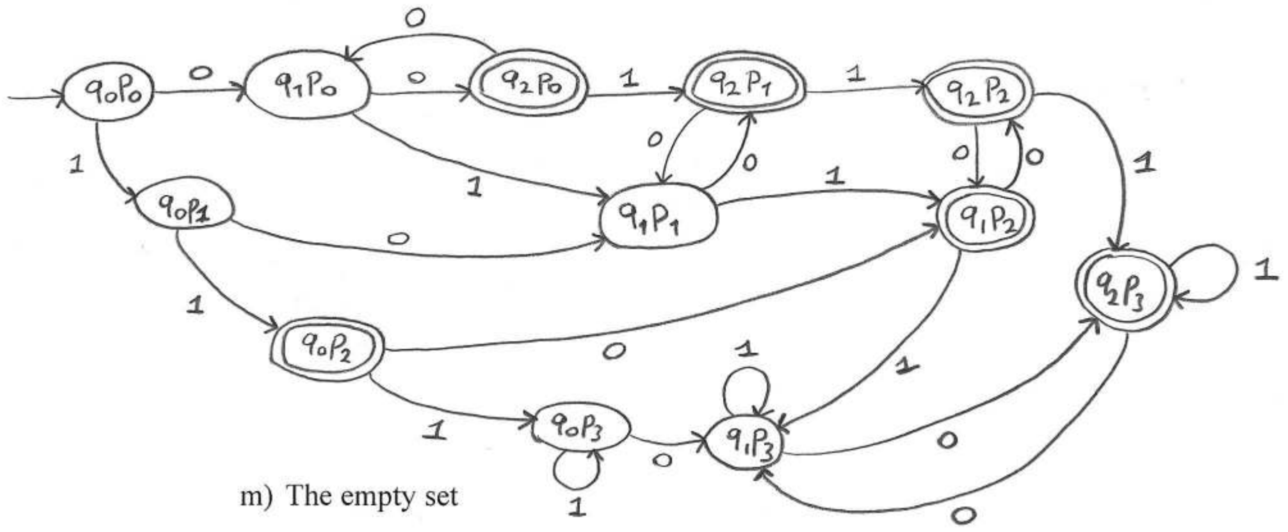
j) {w | w contains at least two 0s and at most one 1}



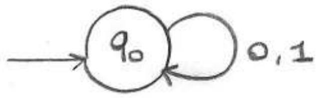
k) {ε, 0}



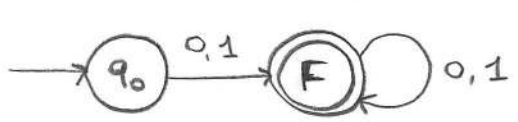
l) {w | w contains an even number of 0s, or contains exactly two ls}



m) The empty set



n) All strings except the empty string



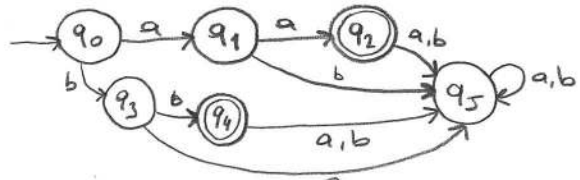
11) **[20 Points]** Show by giving an example that if M is a DFA that recognizes language C, swapping the final and non-final states in M yields a new DFA that recognizes $\bar{C}$.

Lets say M is a DFA and $L(M) = \{aa, bb\}$ over $\Sigma = \{a, b\}$ then $\overline{L(M)}$ would be;

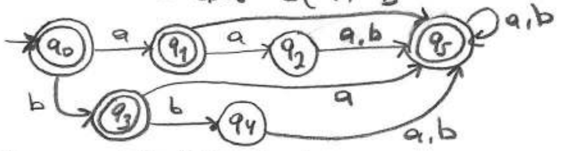
$$\overline{L(M)} = \Sigma^* \setminus L(M) = \{a,b\}^* \setminus \{aa, bb\}$$

Lets draw state diagrams

M that accepts $L(M)$



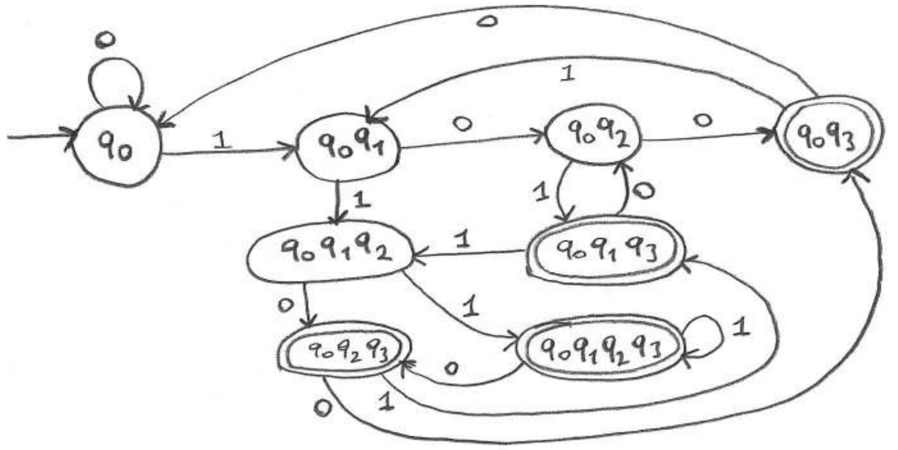
M' that accepts $\overline{L(M)}$



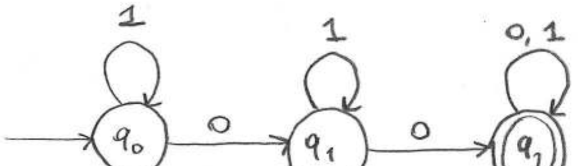
As you can see, by swapping final and non-final states we made a new DFA that accepts everything except $L(M)$. This means our M' accepts $\overline{L(M)}$.

12) **[20 Points]** Design automata (DFA) to accept the following languages:

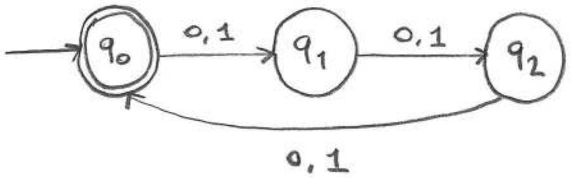
a) $A = \{w \in \{0, 1\}^* : w$ has a 1 in the third position from the right$\}$.



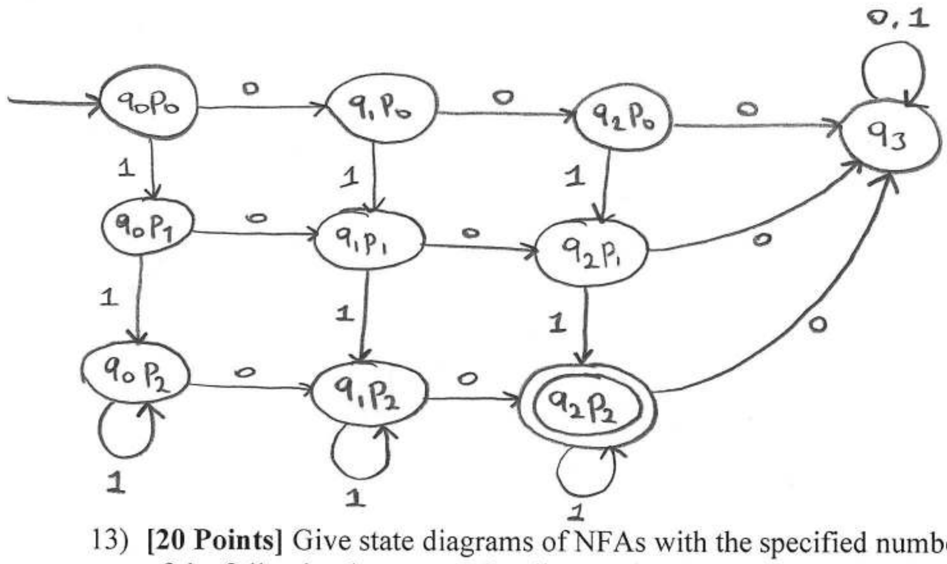
b) $B = \{w \in \{0, 1\}^* : w$ contains at least two 0s$\}$



c) $C = \{w \in \{0, 1\}^* : \text{the length of } w \text{ is divisible by three}\}$

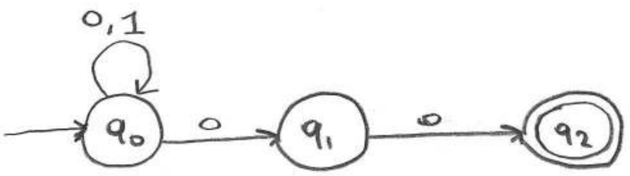


d) $D = \{w \in \{0, 1\}^* : w \text{ contains exactly two 0s and at least two 1s}\}$.

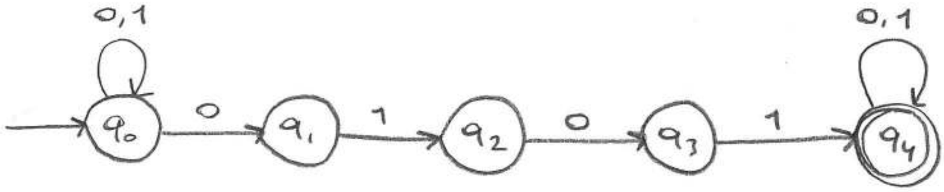


13) **[20 Points]** Give state diagrams of NFAs with the specified number of states recognizing each of the following languages. In all parts the alphabet is $\{0,1\}$.

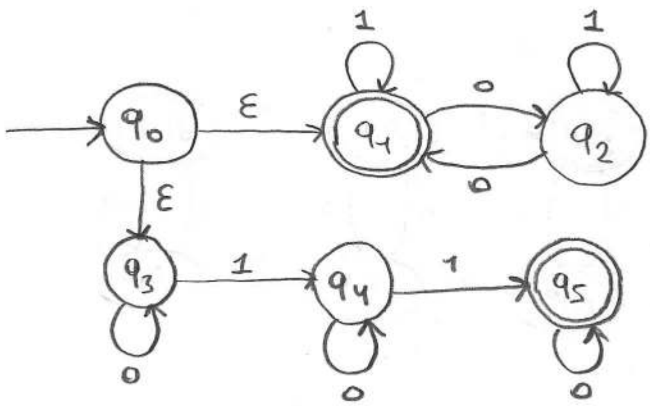
a) The language $\{w \mid w \text{ ends with } 00\}$ with three states



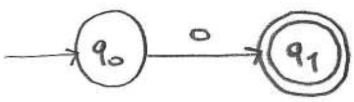
b) The language {w | w contains the substring 0101, i.e., w = x0101y for some x and y} with five states



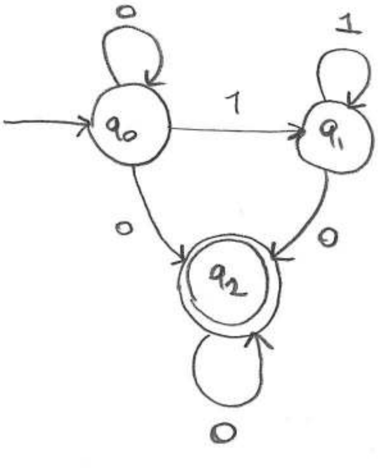
c) The language {w | w contains an even number of 0s, or contains exactly two ls} with six states



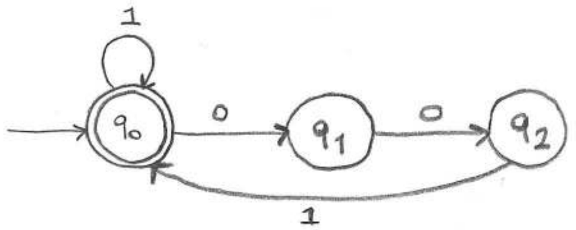
d) The language {0} with two states



e) The language $0^*1^*0^+$ with three states

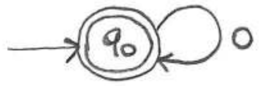


f) The language $1^*(001^+)^*$ with three states



g) The language $\{\varepsilon\}$ with one state

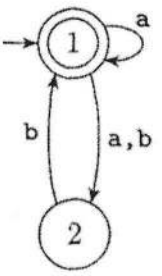
h) The language 0^* with one state



14) **[20 Points]** Use the construction given in Theorem 1.39 in the book to convert the following two non-deterministic finite automata to equivalent deterministic finite automata.

$Q=\{1,2\}$ over $\Sigma=\{a,b\}$

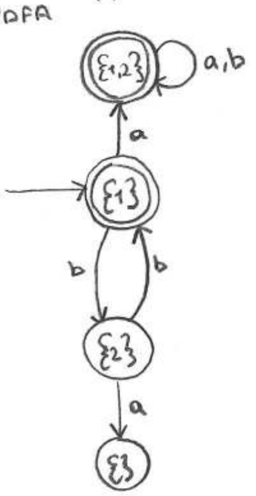
δ	a	b
1	$\{1,2\}$	$\{2\}$
2	$\{\}$	$\{1\}$



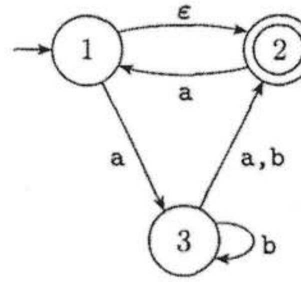
(a)

$P(Q)=\{\{\},\{1\},\{2\},\{1,2\}\} = Q_{DFA}$

δ_{DFA}	a	b
$\{1\}$	$\{1,2\}$	$\{2\}$
$\{2\}$	$\{\}$	$\{1\}$
$\{1,2\}$	$\{1,2\}$	$\{1,2\}$



DFA $q_0=\{1\}$



$Q=\{1,2,3\}$ over $\Sigma=\{a,b\}$

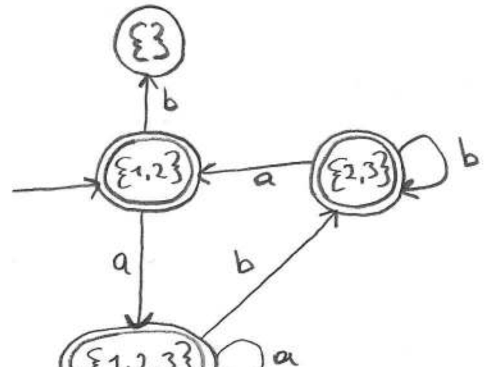
δ	a	b
1	$\{1,2,3\}$	$\{\}$
2	$\{1,2\}$	$\{\}$
3	$\{2\}$	$\{2,3\}$

(b)

$P(Q)=Q_{DFA}=\{\{\},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$

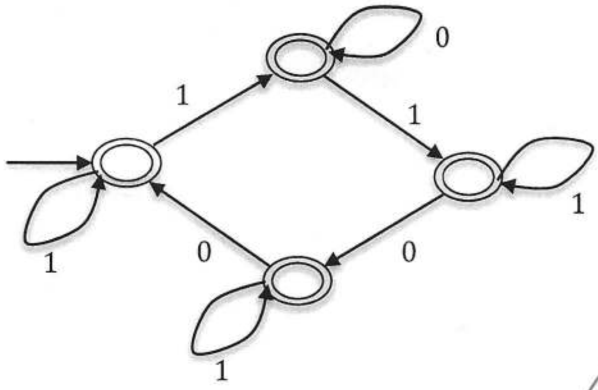
δ_{DFA}	a	b
$\{1,2\}$	$\{1,2,3\}$	$\{\}$
$\{2,3\}$	$\{1,2\}$	$\{2,3\}$
$\{1,2,3\}$	$\{1,2,3\}$	$\{2,3\}$

DFA $q_0=\{1,2\}$



$r = (1^* + 1^*10^* + 1^*10^*11^* + 1^*10^*11^*01^* + 1^*10^*11^*01^*01^*)^*$

15) **[20 points]** For the alphabet $\Sigma_1 = \{0,1\}$, answer the following questions for the automata machine shown below



$L = \{w \in \Sigma^* \mid w \in L(reg)\}$

reg is obtained in option (b)

a) Is the machine DFA or NFA? Why?

Machine is an NFA but not a DFA because for some states; there is more than one transition outs from the state for a symbol in alphabet.
For example initial state has two transition for symbol 1

b) Give its regular expression.

$r = (1^+0^*1^+01^*0)^*1^* \cup 1^+(0^*1^+01^*01^+)^*0^* \cup 1^+0^*1^+(01^*01^+0^*1^+)^* \cup 1^+0^*1^+01^*(01^+0^*1^+01^*)^*1^*$

c) Write the language which is a set consisting of strings that are recognized by this automaton.

$L = \{w \in \{0,1\}^* \mid w = (1^+0^*1^+01^*0)^*1^*\} \cup \{w \in \{0,1\}^* \mid w = 1^+(0^*1^+01^*01^+)^*0^*\}$
$\cup \{w \in \{0,1\}^* \mid w = 1^+0^*1^+(01^*01^+0^*1^+)^*\}$
$\cup \{w \in \{0,1\}^* \mid w = 1^+0^*1^+01^*(01^+0^*1^+01^*)^*1^*\}$

16) **[20 Points]** Give regular expressions describing the following languages:

a) A = { w ∈ {0, 1}* : w contains at least three 1s}.

$r = \cancel{0^*10^*10^*10^*1^*}$ $(0+1)^*1(0+1)^*1(0+1)^*1(0+1)^*$

b) B = { w ∈ {0, 1}* : w contains at least two 1s and at most one 0},

$r = 011^+ \cup 101^+ \cup 11^+01^*$

c) C = { w ∈ {0, 1}* : w contains an even number of 0s and exactly two 1s}.

$r = (00)^* 0101 (00)^* \cup 0(00)^* 0101 (00)^* 0 \cup (00)^* 11 (00)^* \cup 0(00)^* 11 (00)^* 0$
$\cup (00)^* 1010 (00)^* \cup 0(00)^* 1010 (00)^* 0$

d) D = { w ∈ {0, 1}* : w contains an even number of 0s and each 0 is followed by at least one}

$r = \cancel{(1^*(00)^*1^*)^*}$ $\qquad 1^*(01^*01^*)^*1^*$ ✓

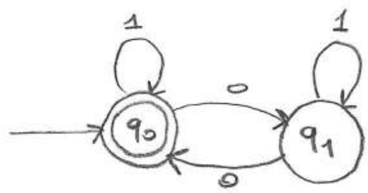
17) **[20 Points]** Design a DFA or NFA for the following languages. $n_0(w)$ denotes the number of zeros in the string w.

a) $L_1 = \{ w \in \{0, 1\}^* : n_0(w) \bmod 2 = 0 \}$,

$Q_1 = \{q_0, q_1\}$ $\Sigma = \{0,1\}$ start state $= q_0$ Final states $= \{q_0\}$

δ_1	0	1
q_0	q_1	q_0
q_1	q_0	q_1

state diagram =

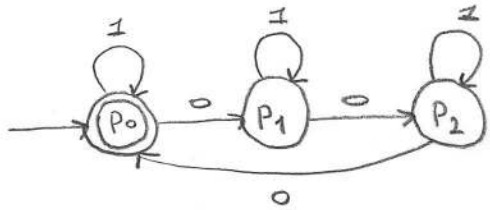


b) $L_2 = \{ w \in \{0, 1\}^* : n_0(w) \bmod 3 = 0 \}$,

$Q_2 = \{P_0, P_1, P_2\}$ $\Sigma = \{0,1\}$ start state $= P_0$ Final States $= \{P_0\}$

δ_2	0	1
P_0	P_1	P_0
P_1	P_2	P_1
P_2	P_0	P_2

state diagram =



c) Based on using the NFA and DFA you designed in the options a and b, design an NFA that recognized the language $L_3 = \{ w \in \{0, 1\}^* : n_0(w) \bmod 6 = 0\}$.

Hint: De Morgan's Laws $L_1 \cap L_2 = \overline{(\overline{L_1} \cup \overline{L_2})}$ can be used for designing an NFA that recognizes the intersection of languages.

$Q_3 = Q_1 \times Q_2 = \{(q_0,P_0), (q_0,P_1), (q_0,P_2), (q_1,P_0), (q_1,P_1), (q_1,P_2)\}$ $\Sigma = \{0,1\}$

start state $= (q_0,P_0)$ Final States $= \{(q_0,P_0)\}$ (only intersection)

δ_3	0	1
q_0P_0	q_1P_1	q_0P_0
q_0P_1	q_1P_2	q_0P_1
q_0P_2	q_1P_0	q_0P_2
q_1P_0	q_0P_1	q_1P_0
q_1P_1	q_0P_2	q_1P_1
q_1P_2	q_0P_0	q_1P_2

state diagram: