Doğal Logaritma ve Doğal Üstel Fonksiyon

★ $\ln x = \int_1^x \frac{dt}{t}$ $(x>0)$ denklemi ile verilen $f(x)=\ln x$ fonksiyonuna "Doğal Logaritma" denir.

★ $\ln^{-1} x = e^x$ ile tanımlı ($\ln x$ fonksiyonunun ters fonksiyonu) $f(x)=e^x$ fonksiyonuna "Doğal Üstel fonksiyon" denir.

★ $f(x)=a^x$ $(a>0, a\neq 1)$ ile tanımlı fonksiyona "Genel üstel fonksiyon" denir.

★ $f(x)=\log_a x$ fonksiyonuna (a^x fonksiyonunun tersidir) "Genel logaritmik fonksiyon" denir. ($\ln x = \log_e x$ dir)

Özellikleri

① $\ln e = 1$

② $\ln xy = \ln x + \ln y$

③ $\ln \frac{x}{y} = \ln x - \ln y$

④ $\ln x^r = r \ln x$

$\log_a x^r = r \log_a x$

⑤ $\log_a a^x = x$

$\ln e^x = x$

⑥ $e^{\ln x} = x$

$a^{\log_a x} = x$

⑦ $a^x = e^{x \ln a}$

$x^n = e^{n \ln x}$

⑧ $\log_a x = \frac{\ln x}{\ln a}$

⑨ $\ln 1 = 0$

$\log_a 1 = 0$

⑩ $\log_a xy = \log_a x + \log_a y$

⑪ $\log_a \frac{x}{y} = \log_a x - \log_a y$

⑫ $\ln \frac{1}{x} = -\ln x$

$\log_a \frac{1}{x} = -\log_a x$

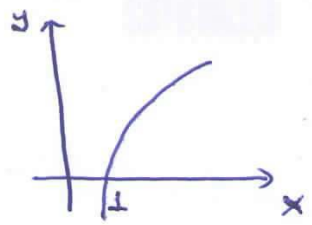
⑬ $\log_a a = 1$

⑭ $\log_a x = y \iff x = a^y$

$\ln x = y \iff x = e^y$

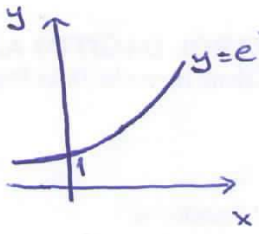
Grafikleri

★ $y = \ln x$



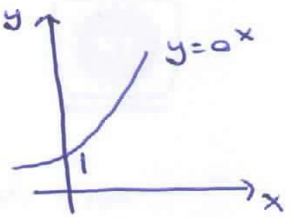
Fonk. Artandır

★ $y = e^x$



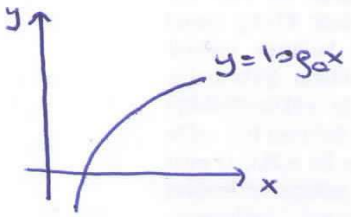
Fonk. Artandır

★ $y = a^x \quad (a > 1)$



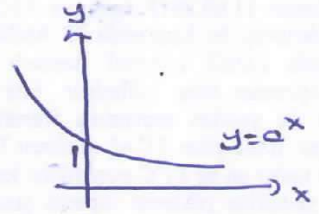
Fonk. Artandır

★ $y = \log_a x \quad (a > 1)$



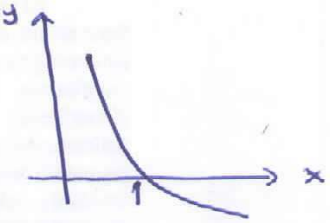
Fonk. Artandır

★ $y = a^x \quad (0 < a < 1)$



Fonk. Azalandır

★ $y = \log_a x \quad (0 < a < 1)$



Fonk. Azalandır

Limitleri

① $\lim_{x \to \infty} \ln x = \infty$ ② $\lim_{x \to 0^+} \ln x = -\infty$ ③ $\lim_{x \to \infty} e^x = \infty$

④ $\lim_{x \to -\infty} e^x = 0$ ⑤ $\lim_{x \to \infty} a^x = 0 \quad (0 < a < 1)$ ⑥ $\lim_{x \to \infty} a^x = \infty \quad (a > 1)$

⑦ $\lim_{x \to -\infty} a^x = \infty \quad (0 < a < 1)$ ⑧ $\lim_{x \to -\infty} a^x = 0 \quad (a > 1)$

⑨ $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$ ⑩ $\lim_{x \to 0^+} (1 + ax)^{1/x} = e^a$ ⑪ $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Türevleri

① $(\ln x)' = \frac{1}{x}$ ② $(\ln f(x))' = \frac{f'(x)}{f(x)}$ ③ $(e^x)' = e^x$

④ $(e^{f(x)})' = f'(x) \cdot e^{f(x)}$ ⑤ $(a^{f(x)})' = f'(x) \cdot a^{f(x)} \ln a$ ⑥ $(\log_a f(x))' = \frac{f'(x)}{f(x) \cdot \ln a}$

Ters Trigonometrik Fonksiyonlar

Altı temel trigonometrik fonksiyon bire-bir değiller. dir; fakat tanım kümelerini bire-bir oldukları aralıklara kısıtlayabiliriz. Bu kısıtlanmış fonksiyonlar artık bire-bir oldukları için tersleri vardır ve aşağıdaki şekilde gösterilirler:

Ters Trig. Fonksiyon	Tanım Kümesi	Görüntü K.
① $f(x) = \text{ArcSin}x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
② $f(x) = \text{ArcCos}x$	$-1 \le x \le 1$	$0 \le y \le \pi$
③ $f(x) = \text{ArcTan}x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
④ $f(x) = \text{ArcCot}x$	$-\infty < x < \infty$	$0 < y < \pi$
⑤ $f(x) = \text{ArcSec}x$	$x \le -1$ veya $x \ge 1$	$0 \le y \le \pi$, $y \neq \frac{\pi}{2}$
⑥ $f(x) = \text{ArcCosec}x$	$x \le -1$ veya $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \neq 0$

✱ $y = \text{ArcSin}x \Rightarrow x = \text{Sin}y$

✱ $y = \text{ArcCos}x \Rightarrow x = \text{Cos}y$

✱ $y = \text{ArcTan}x \Rightarrow x = \text{Tan}y$

✱ $y = \text{ArcCot}x \Rightarrow x = \text{Cot}y$

✱ $y = \text{ArcSec}x \Rightarrow x = \text{Sec}y$

✱ $y = \text{ArcCosec}x \Rightarrow x = \text{Cosec}y$

Örnek:

x	$y = \text{ArcSin}x$	$y = \text{ArcCos}x$
$\frac{1}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$
0	0	$\pi/2$
$\frac{\sqrt{3}}{2}$	$\pi/3$	$\pi/6$

x	$y = \text{ArcSin}x$	$y = \text{ArcCos}x$
$-1/2$	$-\frac{\pi}{6}$	$\frac{2\pi}{3}$
$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$	$\frac{3\pi}{4}$
$-\frac{\sqrt{3}}{2}$	$-\frac{\pi}{3}$	$\frac{5\pi}{6}$

(*) Örnek:

x	y = Arc Tan x
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
0	0
$\frac{\sqrt{3}}{3}$	$\pi/6$

x	y = Arc Tan x
$-\sqrt{3}$	$-\pi/3$
-1	$-\pi/4$
$-\frac{\sqrt{3}}{3}$	$-\frac{\pi}{6}$

Özdeşlikler

① $\text{Arc Cos}x + \text{Arc Cos}(-x) = \pi$

② $\text{Arc Sin}x + \text{Arc Cos}x = \frac{\pi}{2}$

③ $\text{Arc Tan}x + \text{Arc Cot}x = \frac{\pi}{2}$

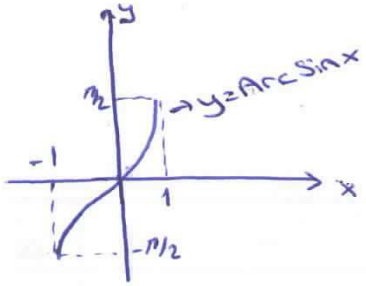
④ $\text{Arc Cosec}x + \text{Arc Sec}x = \frac{\pi}{2}$

Türevleri

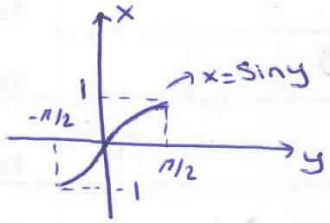
① $y = \text{Arc Sin}x \Rightarrow y' = \frac{1}{\sqrt{1-x^2}}$	$y = \text{Arc Sin}f(x) \Rightarrow y' = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$
② $y = \text{Arc Cos}x \Rightarrow y' = \frac{-1}{\sqrt{1-x^2}}$	$y = \text{Arc Cos}f(x) \Rightarrow y' = -\frac{f'(x)}{\sqrt{1-(f(x))^2}}$
③ $y = \text{Arc Tan}x \Rightarrow y' = \frac{1}{1+x^2}$	$y = \text{Arc Tan}f(x) \Rightarrow y' = \frac{f'(x)}{1+(f(x))^2}$
④ $y = \text{Arc Cot}x \Rightarrow y' = \frac{-1}{1+x^2}$	$y = \text{Arc Cot}f(x) \Rightarrow y' = \frac{-f'(x)}{1+(f(x))^2}$
⑤ $y = \text{Arc Sec}x \Rightarrow y' = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$y = \text{Arc Sec}f(x) \Rightarrow y' = \frac{f'(x)}{\lvert f(x)\rvert\sqrt{(f(x))^2-1}}$
⑥ $y = \text{Arc Cosec}x \Rightarrow y' = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$y = \text{Arc Cosec}(f(x)) \Rightarrow y' = \frac{-f'(x)}{\lvert f(x)\rvert\sqrt{f^2-1}}$

Grafikleri

① $y = \text{Arc}\sin x$ T.K: $[-1,1]$ G.K: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



II. Yol $y = \text{Arc}\sin x \Rightarrow x = \sin y$ olusunu kullanırsak:



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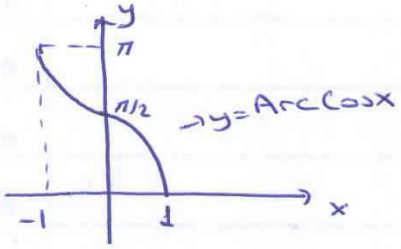
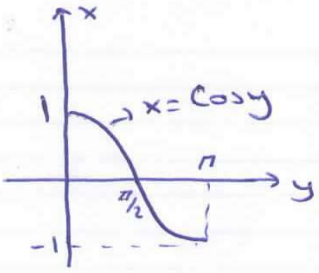
② $y = \text{Arc}\cos x$ T.K: $[-1,1]$ G.K: $[0,\pi]$



veya: $y = \text{Arc}\cos x \Rightarrow x = \cos y$ olusunu kullanırsak:



Hiperbolik fonksiyonlar

* İki üstel fonksiyon e^x ve e^{-x} in birleşimi ile oluşan fonksiyonlardır.

Sinüs hiperbolik fonksiyonu: $\sinh x = \frac{e^x - e^{-x}}{2}$

Cosinüs " " : $\cosh x = \frac{e^x + e^{-x}}{2}$

* Bu temel çiftten hareketle Tanjant, Cotanjant, Secant ve Cosecant Hiperbolik fonk. tanımlanır.
* Hiperbolik fonksiyonlar isimlerini oldukları Trigonometrik fonksiyonlar ile birçok benzerlik gösterirler.

Özdeşlikler

① $\tanh x = \frac{\sinh x}{\cosh x}$	② $\coth x = \frac{\cosh x}{\sinh x}$	③ $\operatorname{sech} x = \frac{1}{\cosh x}$
④ $\operatorname{cosech} x = \frac{1}{\sinh x}$	⑤ $\sinh 2x = 2\sinh x \cosh x$	⑥ $\cosh^2 x - \sinh^2 x = 1$

⑦ $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$

⑧ $\tanh^2 x = 1 - \operatorname{sech}^2 x$	⑨ $\coth^2 x = 1 + \operatorname{cosech}^2 x$

Türevleri

① $y = \sinh x \longrightarrow y' = \cosh x$	$y = \sinh f(x) \rightarrow y' = f'(x).\cosh f(x)$
② $y = \cosh x \rightarrow y' = \sinh x$	$y = \cosh f(x) \rightarrow y' = f'(x)\sinh f(x)$
③ $y = \tanh x \rightarrow y' = \operatorname{sech}^2 x$	$y = \tanh f(x) \rightarrow y' = f'(x).\operatorname{sech}^2 f(x)$
④ $y = \coth x \rightarrow y' = -\operatorname{cosech}^2 x$	$y = \coth f(x) \rightarrow y' = -f'(x)\operatorname{cosech}^2 f(x)$
⑤ $y = \operatorname{sech} x \rightarrow y' = -\operatorname{sech} x \tanh x$	
⑥ $y = \operatorname{cosech} x \rightarrow y' = -\coth x.\operatorname{cosech} x$	

Ters Hiperbolik Fonksiyonlar

- $y = \text{Sinh}^{-1} x \Rightarrow x = \text{Sinh} y$
- $y = \text{Cosh}^{-1} x \Rightarrow x = \text{Cosh} y$
- $y = \text{Tanh}^{-1} x \Rightarrow x = \text{Tanh} y$
- $y = \text{Coth}^{-1} x \Rightarrow x = \text{Coth} y$
- $y = \text{Cosech}^{-1} x \Rightarrow x = \text{Cosech} y$
- $y = \text{Sech}^{-1} x \Rightarrow x = \text{Sech} y$

Özdeşlikler

① $\text{Sech}^{-1} x = \text{Cosh}^{-1} \frac{1}{x}$ ② $\text{Cosech}^{-1} x = \text{Sinh}^{-1} \frac{1}{x}$ ③ $\text{Coth}^{-1} x = \text{Tanh}^{-1} \frac{1}{x}$

Türevleri

① $y = \text{Sinh}^{-1} x \Rightarrow y' = \frac{1}{\sqrt{1+x^2}}$

② $y = \text{Cosh}^{-1} x \Rightarrow y' = \frac{1}{\sqrt{x^2-1}}$

③ $y = \text{Tanh}^{-1} x \Rightarrow y' = \frac{1}{1-x^2}$

④ $y = \text{Coth}^{-1} x \Rightarrow y' = \frac{1}{1-x^2}$

⑤ $y = \text{Sech}^{-1} x \Rightarrow y' = \frac{-1}{x \cdot \sqrt{1-x^2}}$

⑥ $y = \text{Cosech}^{-1} x \Rightarrow y' = \frac{-1}{|x| \cdot \sqrt{1+x^2}}$

Çözümlü Sorular

① $\cosh 2x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$ olduğunu gösteriniz.

$\cosh x = \frac{e^x + e^{-x}}{2} \Rightarrow \cosh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4} \Rightarrow$

$$2\cosh^2 x - 1 = \frac{e^{2x} + 2 + e^{-2x}}{2} - 1 = \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x \checkmark$$

$\sinh x = \frac{e^x - e^{-x}}{2} \Rightarrow \sinh^2 x = \frac{e^{2x} - 2 + e^{-2x}}{4}$

$$\Downarrow$$

$$2\sinh^2 x + 1 = \frac{e^{2x} - 2 + e^{-2x}}{2} + 1 = \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x \checkmark$$

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② $2\cosh(\ln x) = ?$ $\quad \cosh x = \frac{e^x + e^{-x}}{2}$ olduğundan

$$2.\cosh(\ln x) = 2.\frac{e^{\ln x} + e^{-\ln x}}{2} \quad \leftarrow \left(\begin{array}{l} e^{\ln x} = x \\ e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x} \end{array} \right)$$

$$= 2.\frac{x + \frac{1}{x}}{2} = x + \frac{1}{x}$$

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③ $y = \ln(\sinh x) \Rightarrow y' = ?$

$$y' = \frac{(\sinh x)'}{\sinh x} = \frac{\cosh x}{\sinh x} = \coth x$$

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④ $y = \sinh(e^{\cosh x}) \Rightarrow y' = ?$

$$y' = \sinh x \cdot e^{\cosh x} \cdot \cosh(e^{\cosh x})$$

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⑤ $y = \operatorname{sech}(\ln(\cos x)) \Rightarrow y' = ?$

$$y' = \frac{-\sin x}{\cos x} \cdot (-\operatorname{sech}(\ln\cos x) \cdot \tanh(\ln\cos x))$$

⑥ $\operatorname{Sin}\left(\operatorname{Arc}\operatorname{Cos}\frac{\sqrt{2}}{2}\right)=?$ $\Rightarrow \operatorname{Sin}\left(\operatorname{Arc}\operatorname{Cos}\frac{\sqrt{2}}{2}\right)=\operatorname{Sin}\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

⑦ $\operatorname{Sec}\left(\operatorname{Arc}\operatorname{Cos}\frac{1}{2}\right)=?$ $\Rightarrow \operatorname{Sec}\left(\operatorname{Arc}\operatorname{Cos}\frac{1}{2}\right)=\operatorname{Sec}\frac{\pi}{3}=2$

⑧ $\operatorname{Tan}\left(\operatorname{Arc}\operatorname{Sin}\left(-\frac{1}{2}\right)\right)=?$ $\Rightarrow \operatorname{Tan}\left(\operatorname{Arc}\operatorname{Sin}\frac{-1}{2}\right)=\operatorname{Tan}\left(-\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{3}$

⑨ $\lim_{x\to 1^-}\operatorname{Arc}\operatorname{Sin}x=\frac{\pi}{2}$ $\lim_{x\to\infty}\operatorname{Arc}\operatorname{Tan}x=\frac{\pi}{2}$ $\lim_{x\to\infty}\operatorname{Arc}\operatorname{Cosec}x=0$

⑩ $y=\operatorname{Arc}\operatorname{Cos}\frac{1}{x}$ $\Rightarrow$ $y'=-\dfrac{-\frac{1}{x^2}}{\sqrt{1-\frac{1}{x^2}}}$

⑪ $y=\operatorname{Arc}\operatorname{Tan}(\ln x)$ $\Rightarrow y'=\dfrac{\frac{1}{x}}{1+\ln^2 x}$

⑫ $y=\operatorname{Arc}\operatorname{Sin}e^x$ $\Rightarrow$ $y'=\dfrac{e^x}{\sqrt{1-e^{2x}}}$

⑬ $y=\operatorname{Arc}\operatorname{Cos}(\ln\operatorname{Sin}x)$ $\Rightarrow y'=-\dfrac{\frac{\operatorname{Cos}x}{\operatorname{Sin}x}}{\sqrt{1-(\ln\operatorname{Sin}x)^2}}=-\dfrac{\operatorname{Cot}x}{\sqrt{1-\ln^2\operatorname{Sin}x}}$

⑭ $y=\ln(\operatorname{Arctan}x)$ $\Rightarrow y'=\dfrac{\frac{1}{1+x^2}}{\operatorname{Arctan}x}=\dfrac{1}{(1+x^2)\operatorname{Arctan}x}$

(15) $\lim_{x\to 0} \frac{\text{ArcSin}x}{x} = ?$ (F10)

$\text{ArcSin}x = y$, $x \to 0$ $y \to 0$, $x = \text{Sin}y$ $\Rightarrow \lim_{x\to 0} \frac{\text{ArcSin}x}{x} = \lim_{y\to 0} \frac{y}{\text{Sin}y} = 1$

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(16) $\lim_{x\to 0^+} \frac{(\text{Arctan}\sqrt{x})^2}{x\sqrt{x+1}} = ?$

$\text{Arctan}\sqrt{x} = y \Rightarrow \text{Tan}y = \sqrt{x}$, $x \to 0^+ \Rightarrow y \to 0^+$, $1 + \text{Tan}^2 y = \text{Sec}^2 y$

$\lim_{x\to 0^+} \frac{(\text{Arctan}\sqrt{x})^2}{x\sqrt{x+1}} = \lim_{y\to 0^+} \frac{y^2}{(\text{Tan}y)^2 \cdot \sqrt{1+\text{Tan}^2 y}}$

$= \lim_{y\to 0^+} \underbrace{\left(\frac{y}{\text{Tan}y}\right)^2}_{1} \cdot \underbrace{\frac{1}{\text{Sec}y}}_{1} = 1$

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(17) $\lim_{x\to\infty} \left(\frac{x+7}{x+3}\right)^{2x+3} = ?$

$\lim_{x\to\infty} \left(\frac{x+7}{x+3}\right)^{2x+3} = \lim_{x\to\infty} \underbrace{\left[\left(1+\frac{4}{x+3}\right)^{x+3}\right]}_{e^4}^{\frac{2x+3}{x+3}\cdot 2} = (e^4)^2 = e^8$

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(18) $\lim_{x\to\infty} \left(\frac{x}{x+2}\right)^{3x} = ?$

$\lim_{x\to\infty} \left(\frac{x}{x+2}\right)^{3x} = \lim_{x\to\infty} \frac{1}{\underbrace{\left(\left(1+\frac{2}{x}\right)^x\right)^3}_{e^2}} = \frac{1}{e^6}$

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(19) $\text{ArcSin}x$'in türevinin $\frac{1}{\sqrt{1-x^2}}$ olduğunu gösteriniz.

$f(x) = \text{Sin}x$ olsun. $f^{-1}(x) = \text{ArcSin}x$ olur.

$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\text{Cos}(\underbrace{\text{ArcSin}x}_{a})} = \frac{1}{\text{Cos}a} = \frac{1}{\sqrt{1-x^2}}$

$\text{ArcSin}x = a$ olsun. $\Rightarrow x = \text{Sin}a$, $\text{Cos}a = \sqrt{1-x^2}$