BLM2502 Theory of Computation

Spring 2015

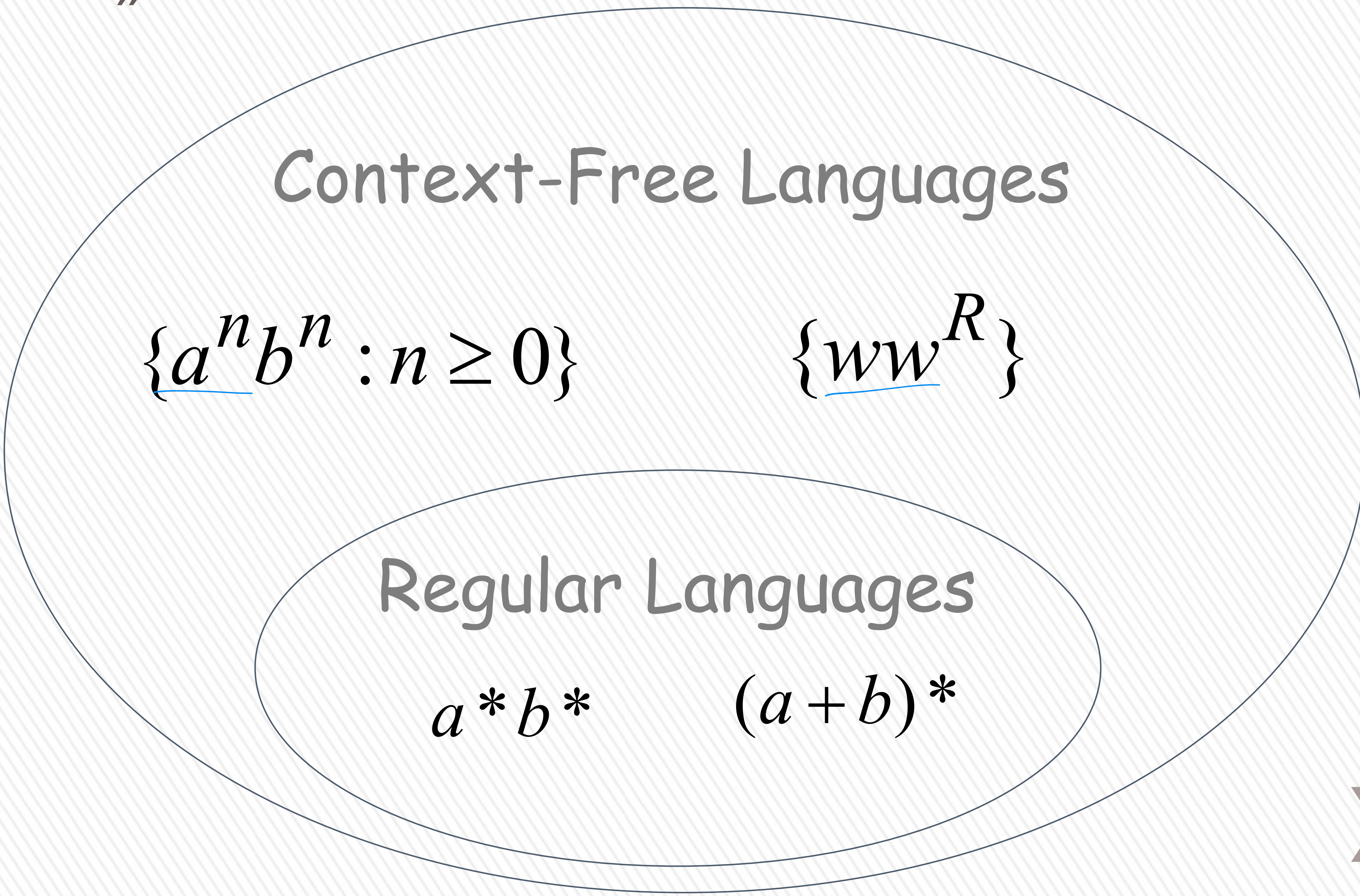
BLM2502 Theory of Computation

- **Course Outline**

Week	Content
1	Introduction to Course
2	Computability Theory, Complexity Theory, Automata Theory, Set Theory, Relations, Proofs, Pigeonhole Principle
3	Regular Expressions
4	Finite Automata
5	Deterministic and Nondeterministic Finite Automata
6	Epsilon Transition, Equivalence of Automata
7	Pumping Theorem
8	April 10 - 14 week is the first midterm week
9	**Context Free Grammars**
10	**Parse Tree, Ambiguity,**
11	Pumping Theorem
12	Turing Machines, Recognition and Computation, Church-Turing Hypothesis
13	Turing Machines, Recognition and Computation, Church-Turing Hypothesis
14	May 22 – 27 week is the second midterm week
15	Review
16	Final Exam date will be announced

Context-Free Languages

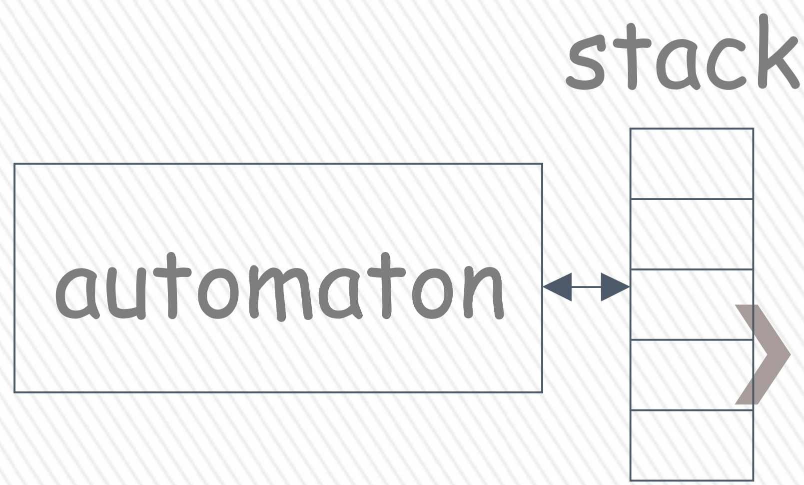
»


Context-Free Languages
$\{a^n b^n : n \geq 0\}$
$\{ww^R\}$
Regular Languages
a^*b^*
$(a+b)^*$

Context-Free Languages

Context-Free Grammars

Pushdown Automata



Context-Free Grammars

Grammars

- Grammars express languages
- Example: the English language grammar

$$\langle sentence \rangle \rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle$$

$$\langle noun\_phrase \rangle \rightarrow \langle article \rangle \ \langle noun \rangle$$

$$\langle predicate \rangle \rightarrow \langle verb \rangle$$

ingilizce grameri

$$\langle article \rangle \rightarrow a$$

$$\langle article \rangle \rightarrow the$$

$$\langle noun \rangle \rightarrow cat$$

$$\langle noun \rangle \rightarrow dog$$

$$\langle verb \rangle \rightarrow runs$$

$$\langle verb \rangle \rightarrow sleeps$$

» Derivation of string “the dog walks” :

$$
\begin{aligned}
\langle sentence \rangle &\Rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle \\
&\Rightarrow \langle noun\_phrase \rangle \ \langle verb \rangle \\
&\Rightarrow \langle article \rangle \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow the \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow the \ \ dog \ \ \langle verb \rangle \\
&\Rightarrow the \ \ dog \ \ sleeps
\end{aligned}
$$

» Derivation of string "a cat runs" :

$$
\begin{aligned}
\langle sentence\rangle &\Rightarrow \langle noun\_phrase\rangle\ \langle predicate\rangle \\
&\Rightarrow \langle noun\_phrase\rangle\ \langle verb\rangle \\
&\Rightarrow \langle article\rangle\ \langle noun\rangle\ \langle verb\rangle \\
&\Rightarrow a\ \ \langle noun\rangle\ \langle verb\rangle \\
&\Rightarrow a\ \ cat\ \ \langle verb\rangle \\
&\Rightarrow a\ \ cat\ \ runs
\end{aligned}
$$

» Language of the grammar:

L = { "a cat runs",
"a cat sleeps",
"the cat runs",
"the cat sleeps",
"a dog runs",
"a dog sleeps",
"the dog runs",
"the dog sleeps" }

Productions

$$\langle noun \rangle \to \overbrace{cat}^{\text{Sequence of Terminals (symbols)}}$$

$$\langle sentence \rangle \to \underbrace{\langle noun\_phrase \rangle \ \langle predicate \rangle}_{\text{Sequence of Variables}}$$

Variables

→ nonterminal

Another Example

Sequence of
terminals and variables

Grammar:

$$S \to \overbrace{aSb}^{} \mid \varepsilon$$

$$S \to \varepsilon$$

$aaSbb$

aSb $\quad aa\cancel{S}bb$

Variable

The right side
may be ε

» Grammar:

$$S \to aSb$$

$$S \to \varepsilon$$

aSb

aaSbb

aabb

» Derivation of string : ab

$$S \Rightarrow aSb \Rightarrow ab$$

$$S \to aSb \qquad\qquad S \to \varepsilon$$

» Grammar:

$$S \to aSb$$

$$S \to \varepsilon$$

» Derivation of string :

$$aabb$$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

$$S \to aSb \qquad\qquad S \to \varepsilon$$

Grammar: $$S \to aSb$$

$$S \to \varepsilon$$

Other derivations:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb$$
$$\Rightarrow aaaaSbbbb \Rightarrow aaaabbbb$$

Grammar:

$$S \rightarrow aSb$$

$$S \rightarrow \varepsilon$$

Language of the grammar:

$$L = \{a^n b^n : n \geq 0\}$$

A Convenient Notation

» We write: $S \overset{*}{\Rightarrow} aaabbb$

for one or more derivation steps

» Instead of:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

In general we write:

$$w_1 \overset{*}{\Rightarrow} w_n$$

$$S \overset{*}{\Rightarrow} aabb$$

If: $w_1 \Rightarrow w_2 \Rightarrow w_3 \Rightarrow \cdots \Rightarrow w_n$

in zero or more derivation steps

Trivially: $w \overset{*}{\Rightarrow} w$

Example Grammar

$$S \to aSb$$

$$S \to \varepsilon$$

Possible Derivations

$$S \overset{*}{\Rightarrow} \varepsilon$$

$$S \overset{*}{\Rightarrow} ab$$

$$S \overset{*}{\Rightarrow} aaabbb$$

$$S \overset{*}{\Rightarrow} aaSbb \overset{*}{\Rightarrow} aaaaaSbbbbb$$

Another convenient notation:

$$S \to aSb \mid \varepsilon$$

$$S \to \varepsilon$$

$$S \to aSb \mid \varepsilon$$

or

$$\langle article \rangle \to a$$

$$\langle article \rangle \to the$$

$$\langle article \rangle \to a \mid the$$

Formal Definition

Gramerde birsürü kural olduğu için hangi kuraldan başlamamız gerektiğini bilmemiz lazım

Grammar: $G = (V, T, S, P)$

- V: Set of variables
- T: Set of terminal symbols
- S: Start variable
- P: Set of productions

Context-Free Grammar: $G = (V, T, S, P)$

All productions in P are of the form

$$A \to s$$

->terminals in right

Variable ->Variables in left

String of variables and terminals

Example for Context-Free Grammar

$$\boxed{S \to aSb \mid \varepsilon}$$

V = S, L, P

T = {a, b}

S = L

P = { S -> aSL, S -> aSaL, S -> {} }

productions

$$P = \{S \to aSb,\ \ S \to \varepsilon\}$$

$$G = (V, T, S, P)$$

$$V = \{S\}$$

variables

büyük harfle yazılır

$$T = \{a, b\}$$

terminals

start variable

Language of a Grammar:

» For a grammar G with start variable S

terminaller

$$L(G) = \{w : \; S \overset{*}{\Rightarrow} w, \quad w \in T^*\}$$

String of terminals or ε

Example:

context-free grammar G : $\boxed{S \to aSb \mid \varepsilon}$

$$L(G) = \{a^n b^n : \quad n \geq 0\}$$

Since, there is derivation

$$S \overset{*}{\Rightarrow} a^n b^n \quad \text{for any } n \geq 0$$

$L = \{0^* 1^*\}$

Context-Free Language:

regular Language

Context Language'inin alt kümesidir

$S \to \varepsilon \mid 0S \mid 1S1$

- A language L is context-free
- if there is a context-free grammar G
- with $L = L(G)$

›

Example:

$$L = \{a^n b^n : \quad n \geq 0\}$$

is a context-free language

since context-free grammar G :

$$\boxed{S \rightarrow aSb \mid \varepsilon}$$

generates $L(G) = L$

Another Example

Context-free grammar G :

→ Palindrom

$$S \rightarrow aSa \mid bSb \mid \varepsilon \mid a \mid b$$

aSa →

Example derivations:

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abba$$

aSa

abSba

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abaSaba \Rightarrow abaaba$$

$$L(G) = \{ww^R : \quad w \in \{a,b\}^*\}$$

Palindromes of even length

Another Example

Context-free grammar G:

$$S \to aSb \mid SS \mid \varepsilon$$

→ ε-eelerle

aSb

aSSb

aSSSSb

Example derivations:

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow ab$$

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow abaSb \Rightarrow abab$$

$$L(G) = \{w \;:\; n_a(w) = n_b(w), \text{ and } n_a(v) \geq n_b(v) \text{ in any prefix } v\}$$

Describes matched parentheses: $()((()))(())$ $\quad a = (, \quad b =)$

Derivation Trees

Derivation Order

Consider the following example grammar with 5 productions:

$$1.\ S \rightarrow AB \qquad 2.\ A \rightarrow aaA \qquad 4.\ B \rightarrow Bb$$

$$3.\ A \rightarrow \varepsilon \qquad 5.\ B \rightarrow \varepsilon$$

aaA

1. $S \rightarrow AB$

2. $A \rightarrow aaA$

3. $A \rightarrow \varepsilon$

4. $B \rightarrow Bb$

5. $B \rightarrow \varepsilon$

soldan başlıyoruz

Leftmost derivation order of string aab:

$$S \overset{1}{\Rightarrow} AB \overset{2}{\Rightarrow} aaAB \overset{3}{\Rightarrow} aaB \overset{4}{\Rightarrow} aaBb \overset{5}{\Rightarrow} aab$$

At each step, we substitute the leftmost variable

1 Derivation trees

-ORDER

Left

Right

1. $S \rightarrow AB$

2. $A \rightarrow aaA$

3. $A \rightarrow \varepsilon$

4. $B \rightarrow Bb$

5. $B \rightarrow \varepsilon$

first

Rightmost derivation order of string aab :

$$S \overset{1}{\Rightarrow} AB \overset{4}{\Rightarrow} ABb \overset{5}{\Rightarrow} Ab \overset{2}{\Rightarrow} aaAb \overset{3}{\Rightarrow} aab$$

At each step, we substitute the rightmost variable

$$1.\ S \to AB \qquad 2.\ A \to aaA \qquad 4.\ B \to Bb$$

$$3.\ A \to \varepsilon \qquad 5.\ B \to \varepsilon$$

Leftmost derivation of **aab**:

$$S \stackrel{1}{\Rightarrow} AB \stackrel{2}{\Rightarrow} aaAB \stackrel{3}{\Rightarrow} aaB \stackrel{4}{\Rightarrow} aaBb \stackrel{5}{\Rightarrow} aab$$

Rightmost derivation of **aab** :

$$S \stackrel{1}{\Rightarrow} AB \stackrel{4}{\Rightarrow} ABb \stackrel{5}{\Rightarrow} Ab \stackrel{2}{\Rightarrow} aaAb \stackrel{3}{\Rightarrow} aab$$

Consider the same example grammar:

$$S \to AB \qquad A \to aaA \mid \varepsilon \quad B \to Bb \mid \varepsilon$$

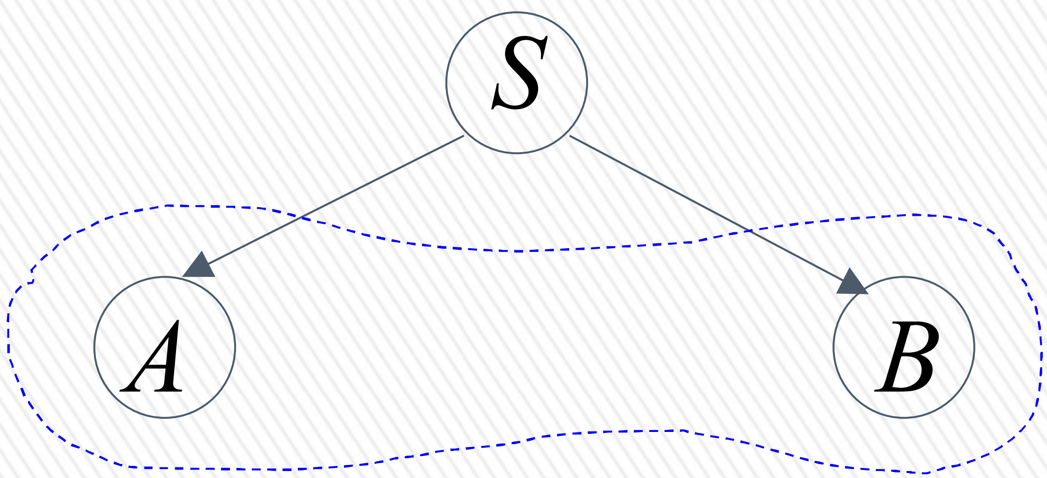
And a derivation of **aab**:

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aab$$

Derivation Tree

$$S \to AB \qquad A \to aaA \mid \varepsilon \quad B \to Bb \mid \varepsilon$$

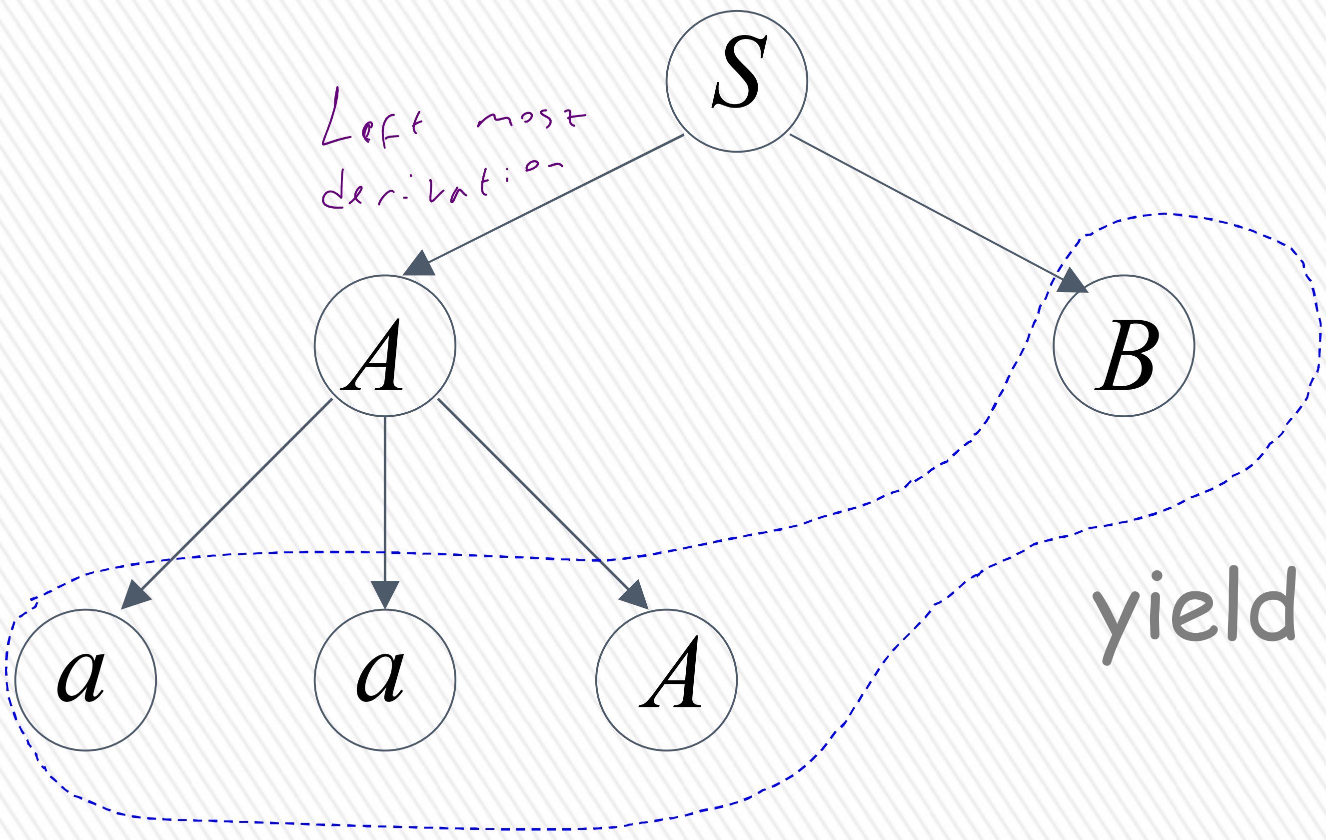
$$S \Rightarrow AB$$



yield ***AB***

$$S \to AB \qquad A \to aaA \mid \varepsilon \quad B \to Bb \mid \varepsilon$$

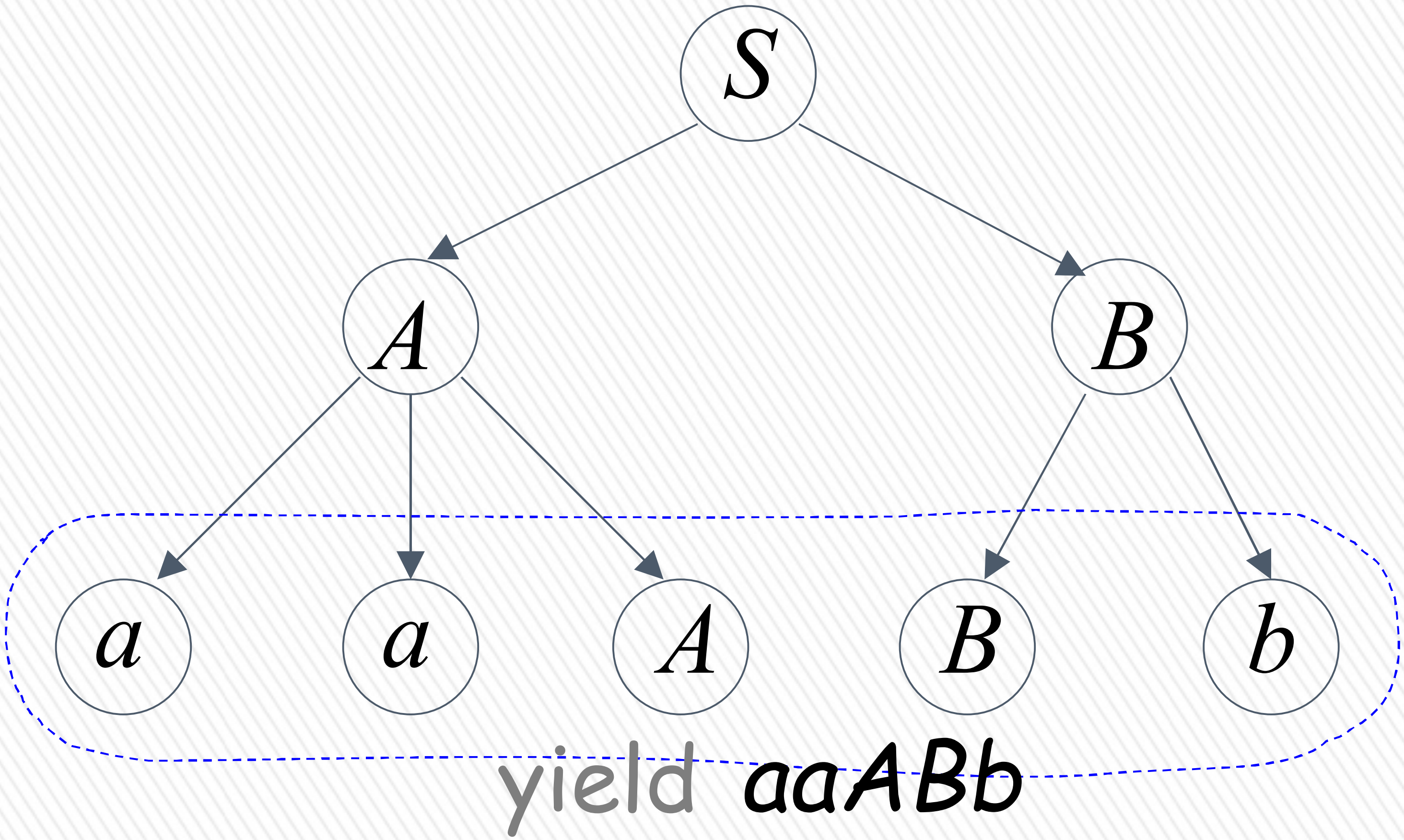
$$S \Rightarrow AB \Rightarrow aaAB$$



yield **aaAB**

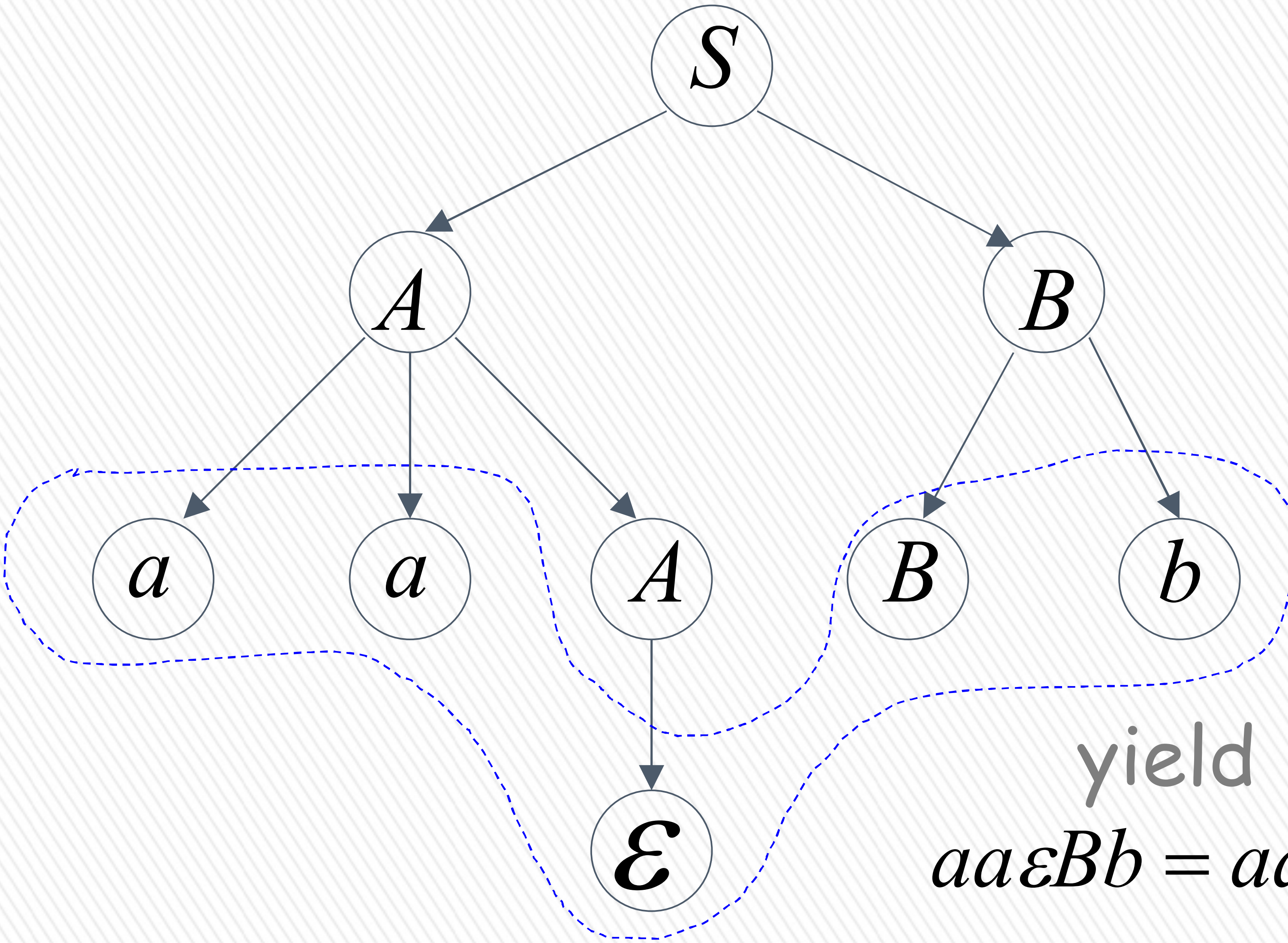
$$S \to AB \qquad A \to aaA \mid \varepsilon \quad B \to Bb \mid \varepsilon$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb$$



$$S \to AB \qquad A \to aaA \mid \varepsilon \quad B \to Bb \mid \varepsilon$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb$$



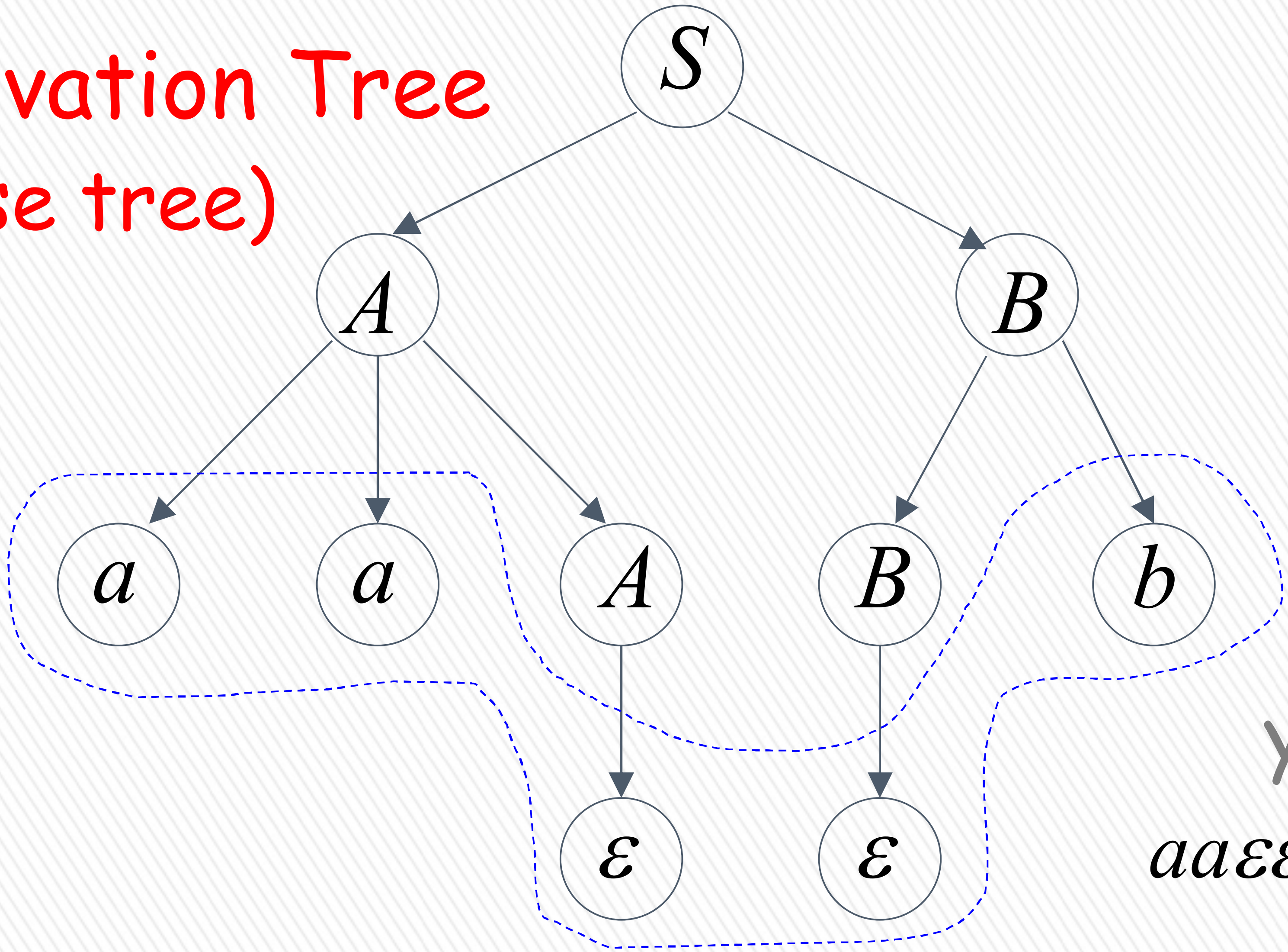
yield

$$aa\varepsilon Bb = aaBb$$

$$S \to AB \qquad A \to aaA \mid \varepsilon \quad B \to Bb \mid \varepsilon$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aab$$

Derivation Tree (parse tree)



yield

$$aa\varepsilon\varepsilon b = aab$$

Sometimes, derivation order doesn't matter

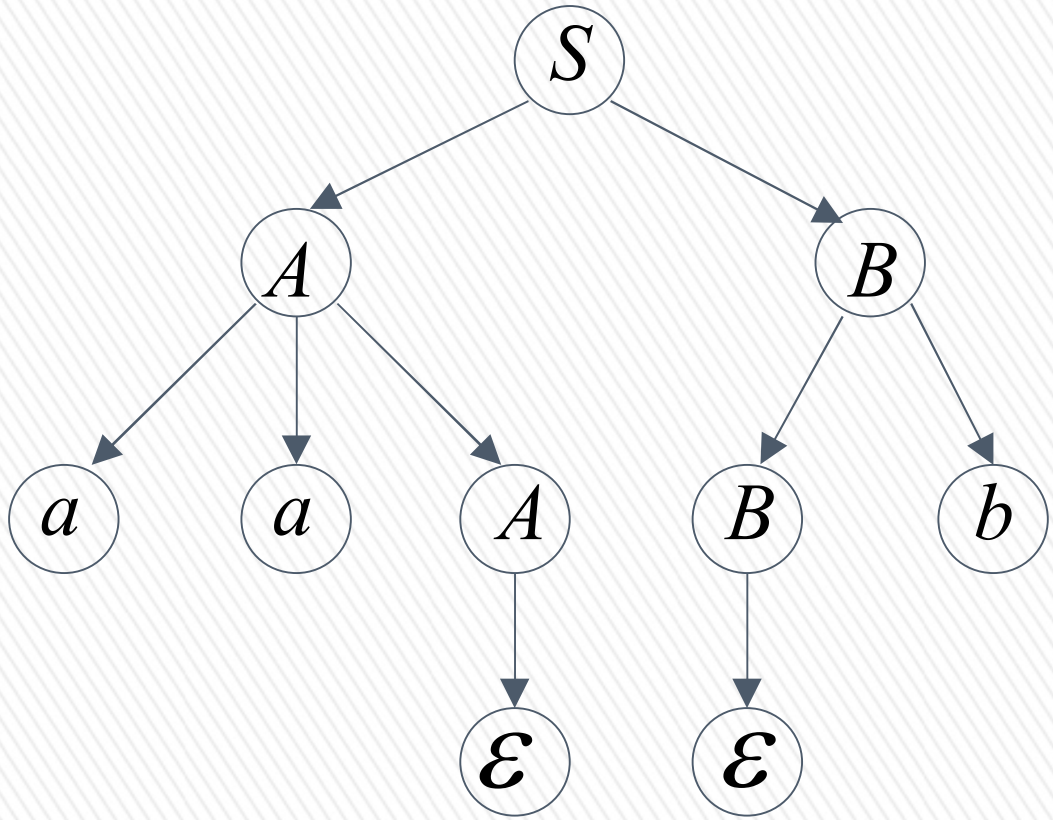
Leftmost derivation:

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaB \Rightarrow aaBb \Rightarrow aab$$

Rightmost derivation:

$$S \Rightarrow AB \Rightarrow ABb \Rightarrow Ab \Rightarrow aaAb \Rightarrow aab$$

Give same derivation tree



→ Ağaçlar benzer değilse

Ambiguity

Grammar for mathematical expressions

$$\boxed{E \rightarrow E + E \;\;|\;\; E * E \;|\; (E) \;|\; a}$$

Example strings:

$$(a + a) * a + (a + a * (a + a))$$

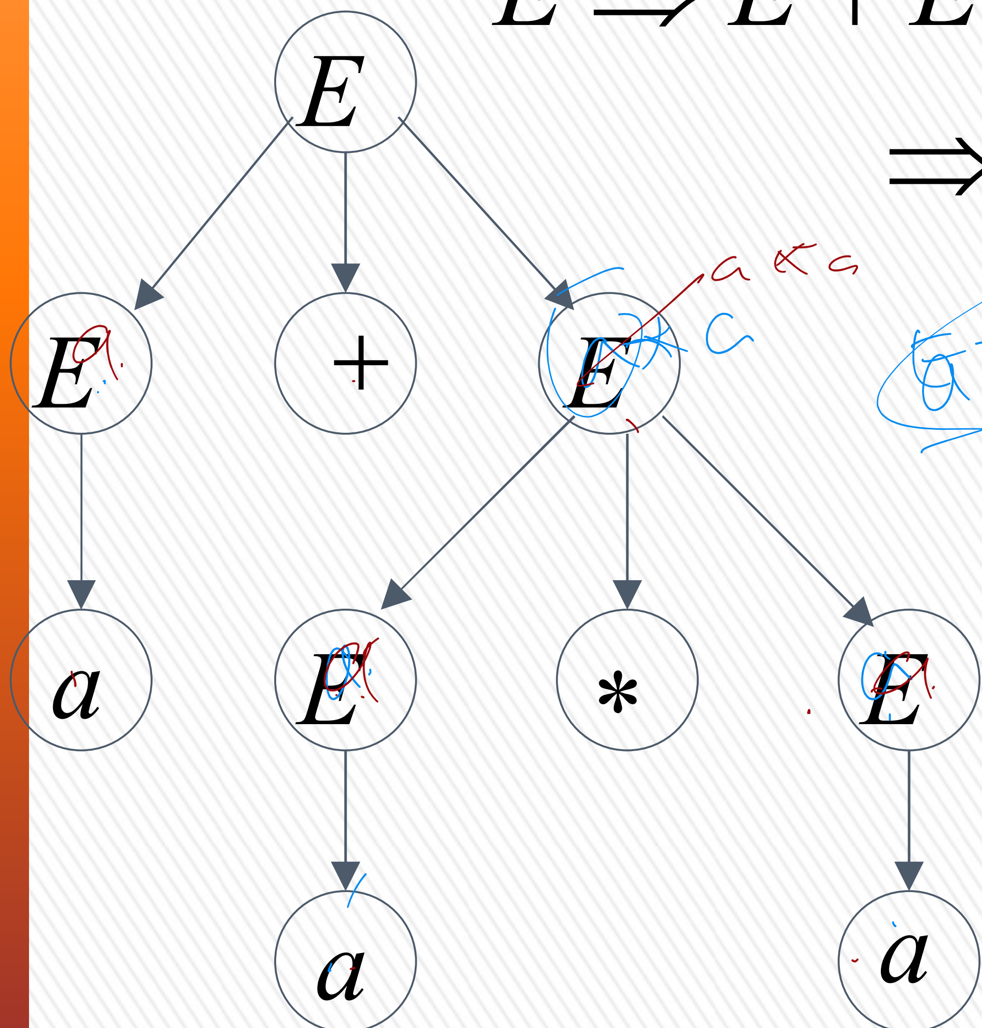
Denotes any number

$$E \rightarrow E+E \mid E*E \mid (E) \mid a$$

a + a* a

$$E \Rightarrow E+E \Rightarrow a+E \Rightarrow a+E*E$$

$$\Rightarrow a+a*E \Rightarrow a+a*a$$



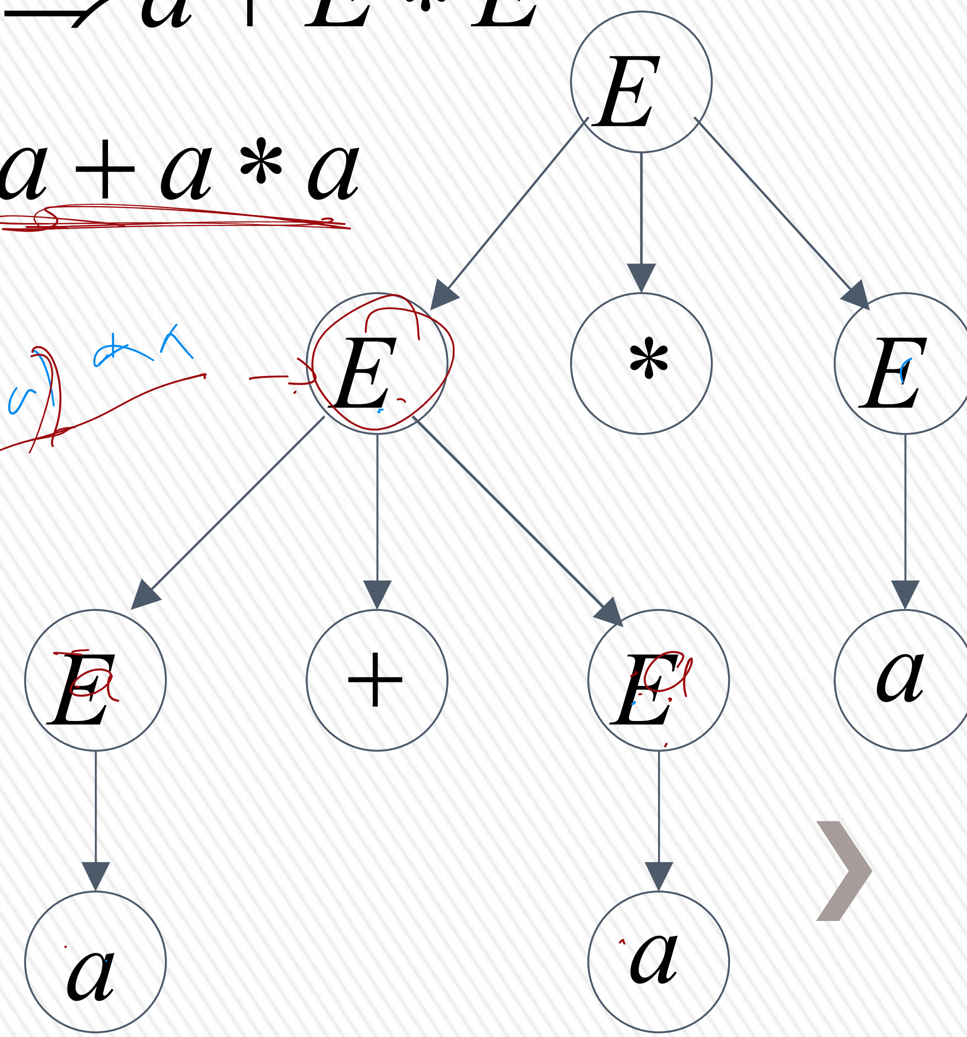
a+(a+a)

A leftmost derivation for $a+a*a$

$$E \rightarrow E+E \;\mid\; E*E \;\mid\; (E) \;\mid\; a$$

$$E \Rightarrow E*E \Rightarrow E+E*E \Rightarrow a+E*E$$
$$\Rightarrow a+a*E \Rightarrow a+a*a$$

$(a+a)$

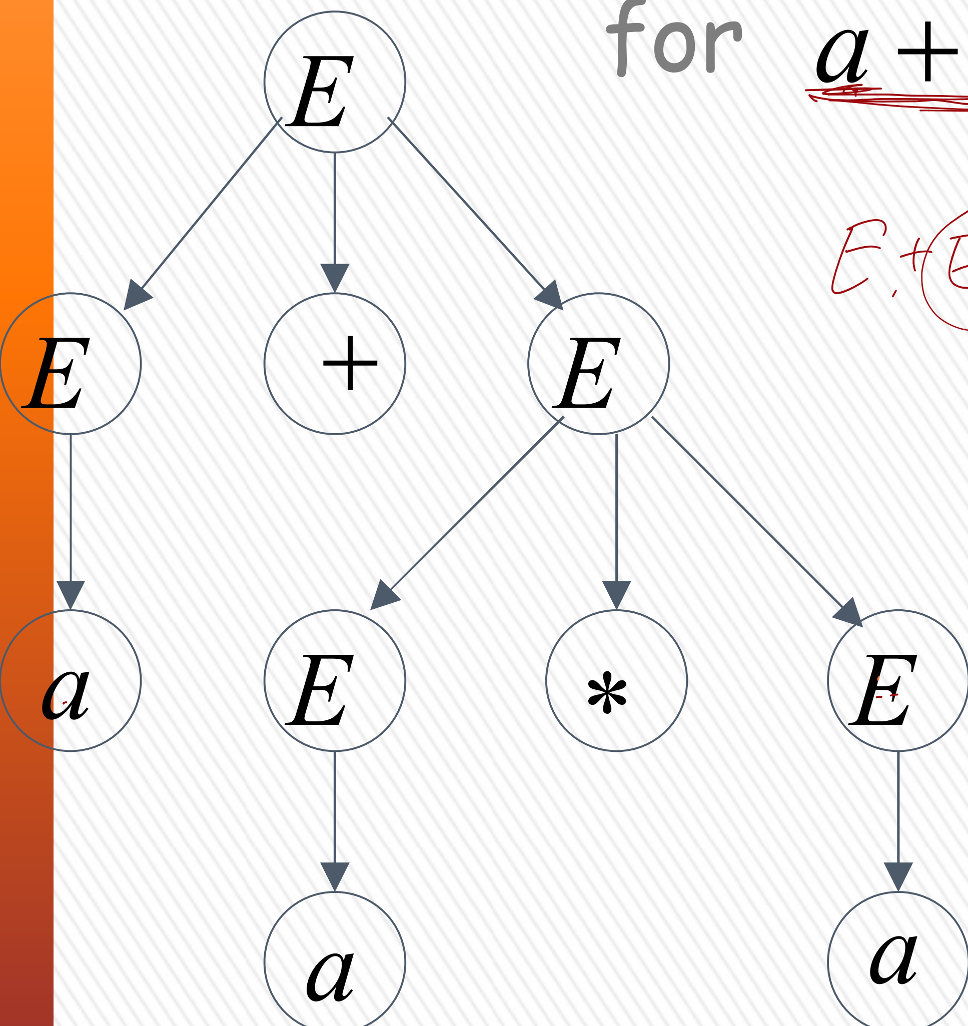


Another leftmost derivation for $a+a*a$

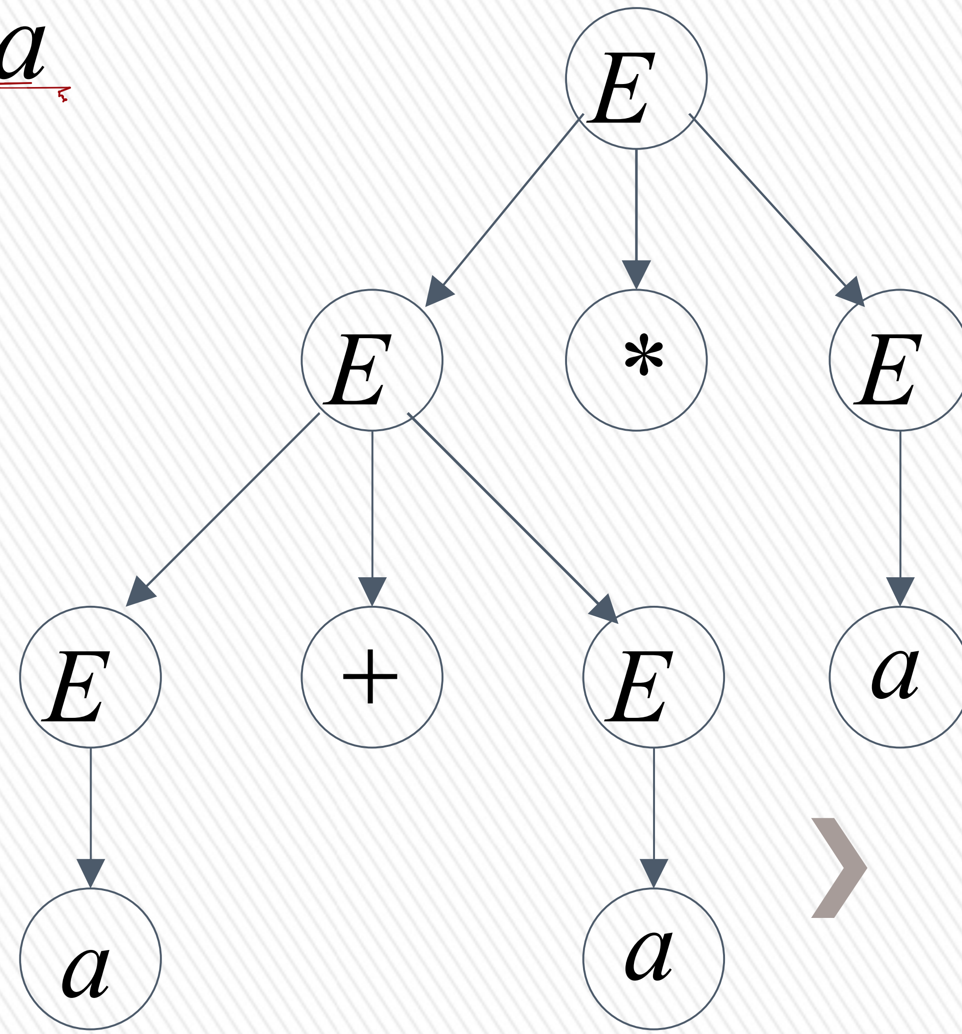
$$E \rightarrow E+E \mid E*E \mid (E) \mid a$$

Two derivation trees

for $a+a*a$

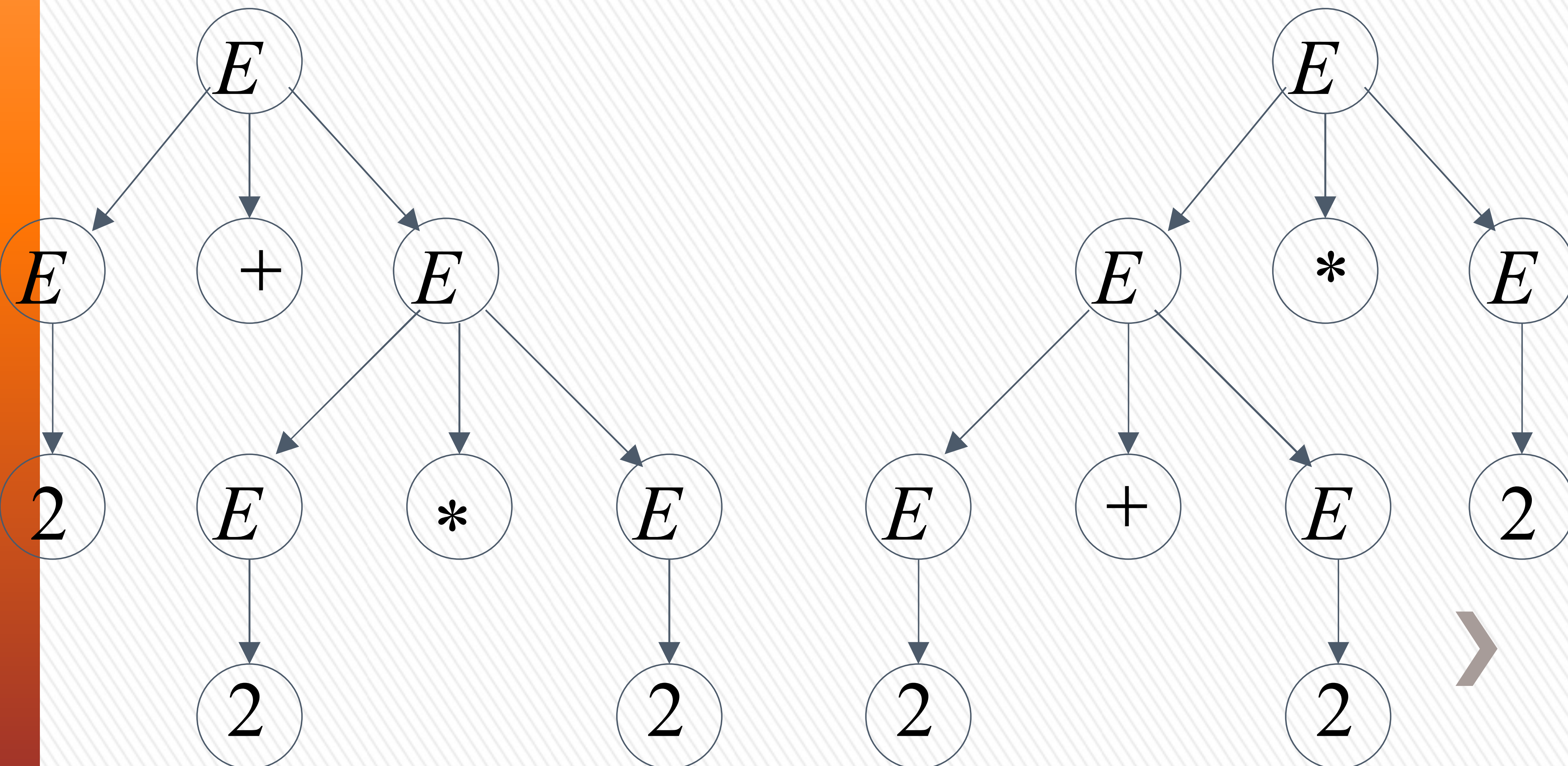


E+E



take $a = 2$

$$a + a * a = 2 + 2 * 2$$



Good Tree

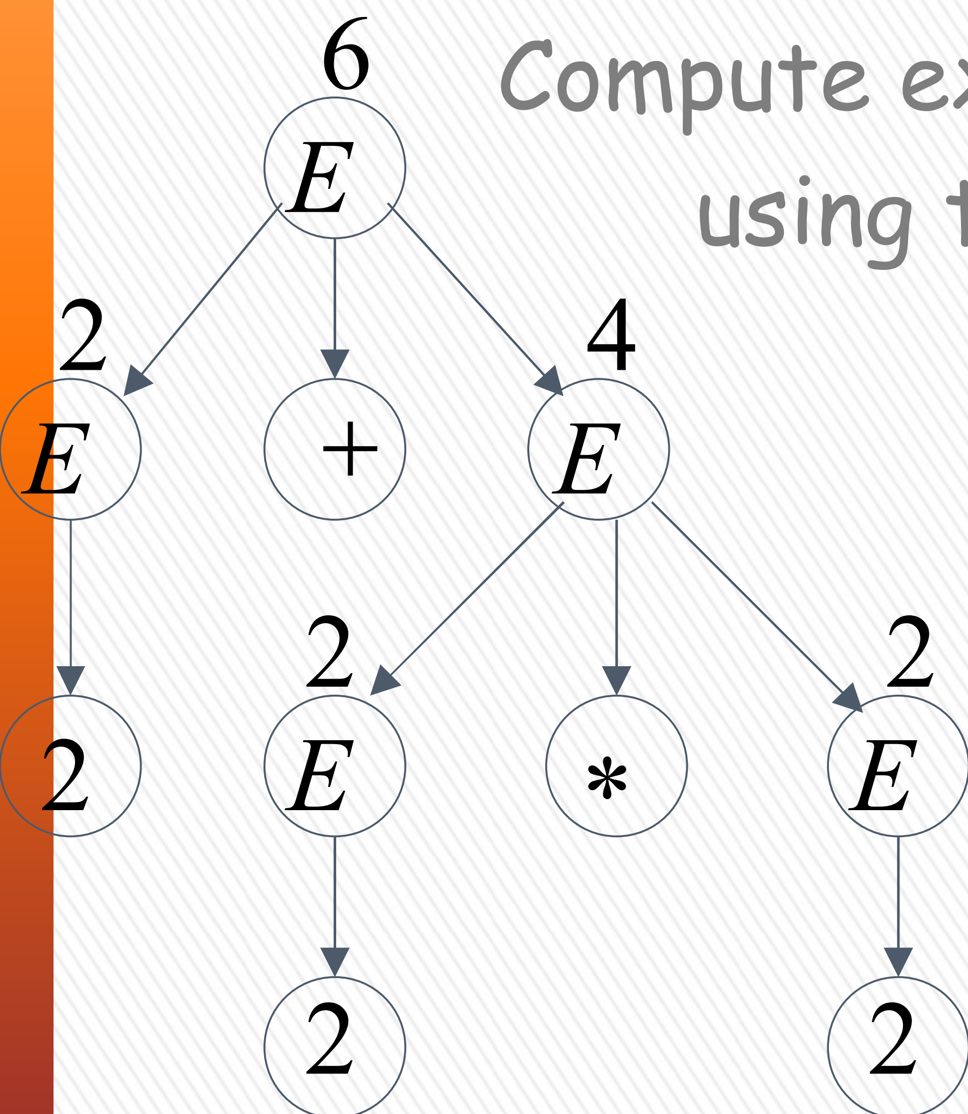
$$2+2*2=6$$

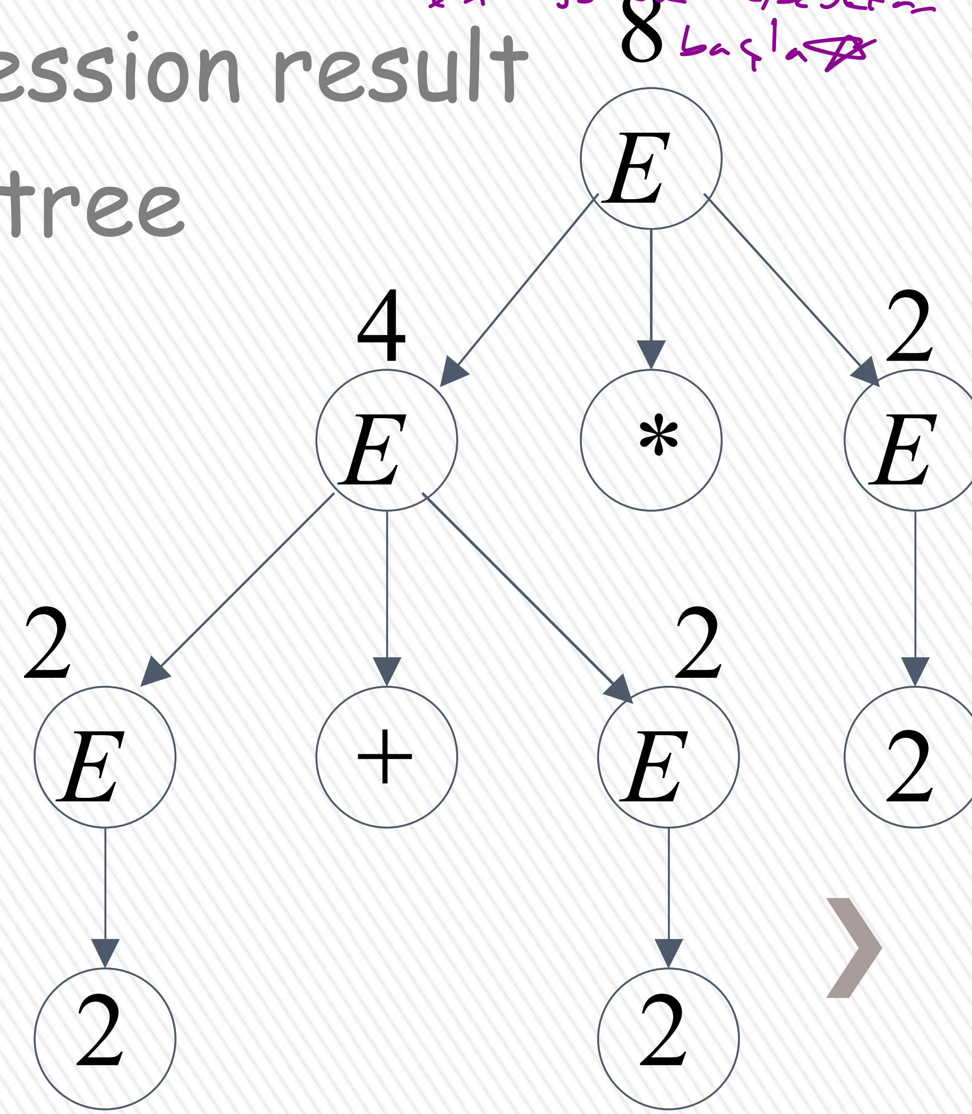
Bad Tree

$$2+\underbrace{2*2}=8$$

önce derinliği
en yüksek çocuklar
başla

Compute expression result using the tree





Two different derivation trees
may cause problems in applications which
use the derivation trees:

- Evaluating expressions

- In general, in compilers
  for programming languages

Ambiguous Grammar: → 2 farklı ağaç çıkıyorsa Ambiguous

A context-free grammar G is ambiguous if there is a string $w \in L(G)$ which has:

two different derivation trees

or

two leftmost derivations

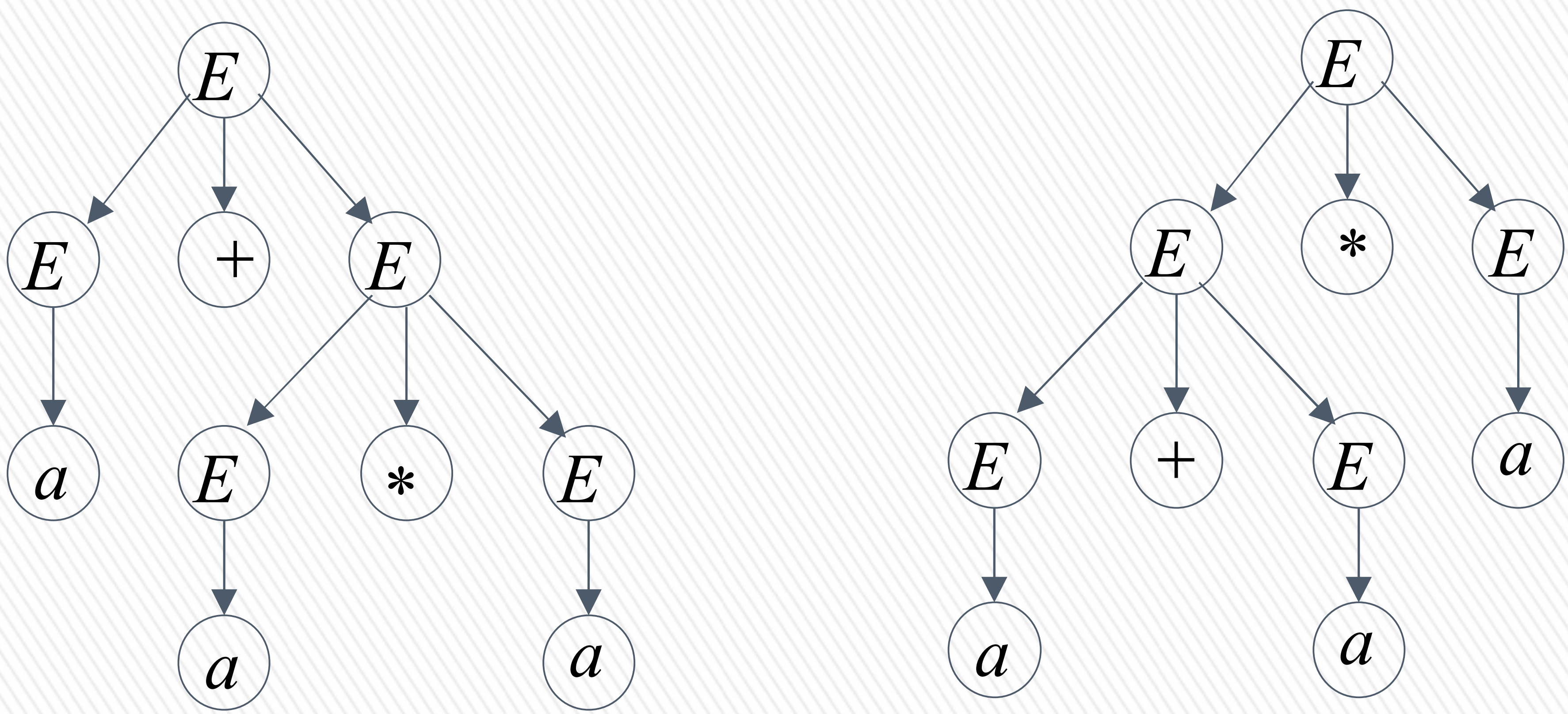
(Two different derivation trees give two different leftmost derivations and vice-versa)

Example:

$$E \to E + E \;\mid\; E * E \;\mid\; (E) \;\mid\; a$$

this grammar is ambiguous since

string $a + a * a$ has two derivation trees



$$E \to E + E \;\; | \; E * E \; | \; (E) \; | \; a$$

this grammar is ambiguous also because string $a + a * a$ has two leftmost derivations

$$E \Rightarrow E + E \Rightarrow a + E \Rightarrow a + E * E$$
$$\Rightarrow a + a * E \Rightarrow a + a * a$$

$$E \Rightarrow E * E \Rightarrow E + E * E \Rightarrow a + E * E$$
$$\Rightarrow a + a * E \Rightarrow a + a * a$$

Another ambiguous grammar:

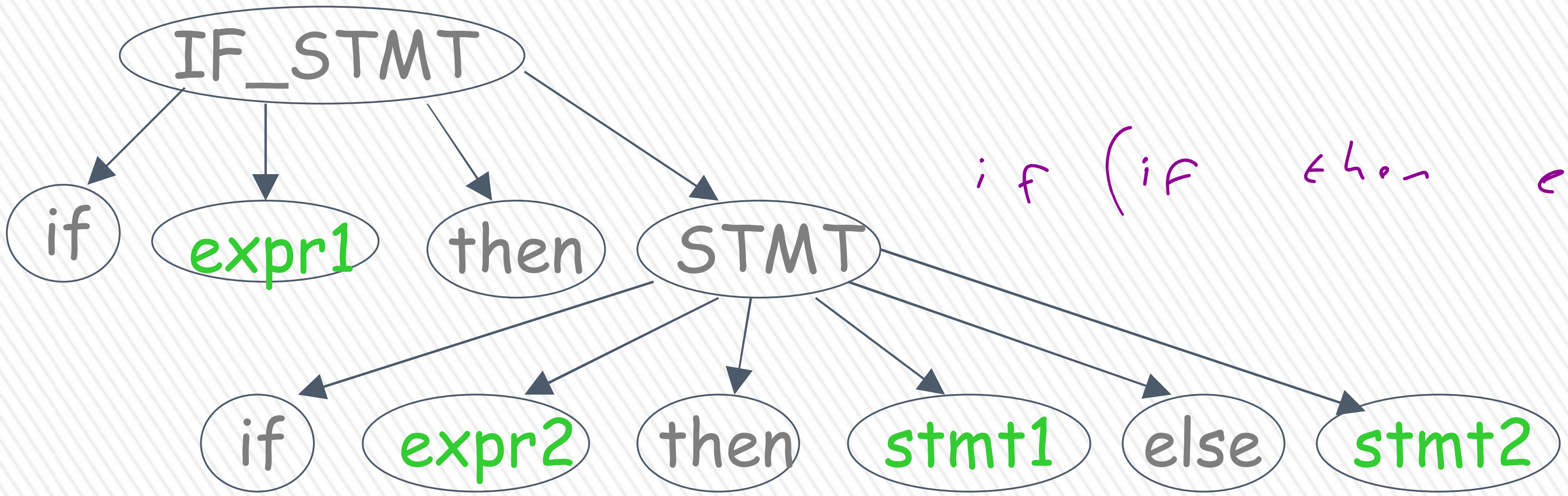
$$\text{IF\_STMT} \rightarrow \text{if EXPR then STMT}$$
$$\mid \text{ if EXPR then STMT else STMT}$$

Variables

Terminals

Very common piece of grammar in programming languages

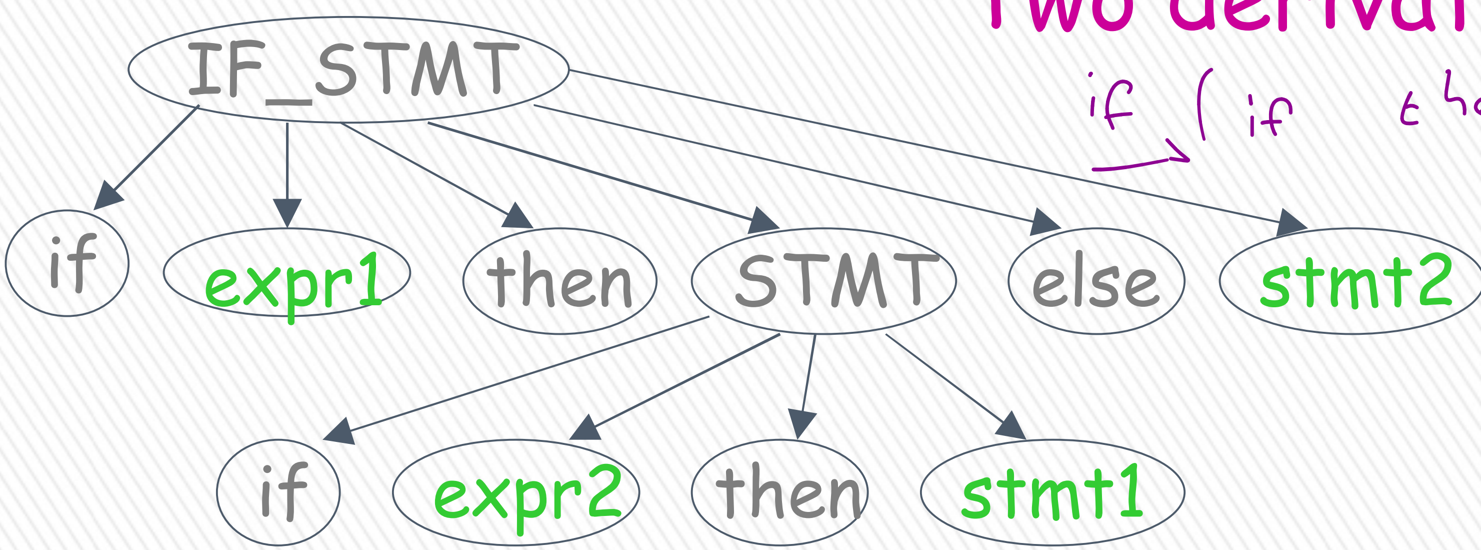
If expr1 then if expr2 then stmt1 else stmt2



if (if then else)

Two derivation trees

if (if then) else



In general, ambiguity is bad
and we want to remove it

Sometimes it is possible to find
a non-ambiguous grammar for a language

But, in general we cannot do so

A successful example:

Ambiguous
Grammar

$$E \to E + E$$
$$E \to E * E$$
$$E \to (E)$$
$$E \to a$$

değişken arttı

Equivalent

Non-Ambiguous
Grammar

$$E \to E + T \mid T$$
$$T \to T * F \mid F$$
$$F \to (E) \mid a$$

generates the same language

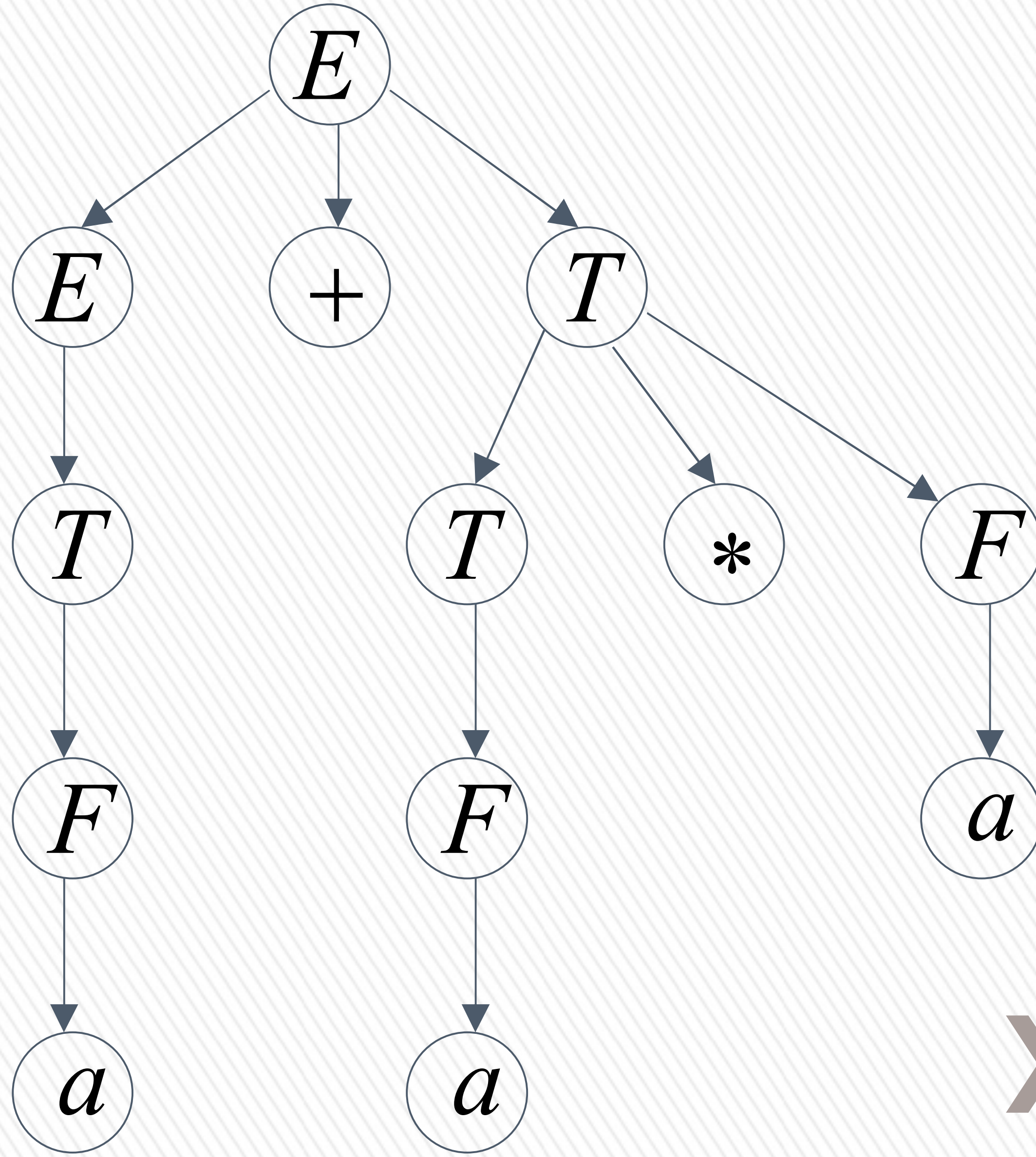
$$E \Rightarrow E+T \Rightarrow T+T \Rightarrow F+T \Rightarrow a+T \Rightarrow a+T*F$$

$$\Rightarrow a+F*F \Rightarrow a+a*F \Rightarrow a+a*a$$

$$E \rightarrow E+T \mid T$$

$$T \rightarrow T*F \mid F$$

$$F \rightarrow (E) \mid a$$



Unique derivation tree for $a+a*a$

An un-successful example:

$$L = \{a^n b^n c^m\} \cup \{a^n b^m c^m\}$$

$$n, m \geq 0$$

işin özü şu diller ambiguoussa birleşimi de ambiguoustur,

L is inherently ambiguous:

every grammar that generates this language is ambiguous

Example (ambiguous) grammar for L:

$$L = \{a^n b^n c^m\} \cup \{a^n b^m c^m\}$$

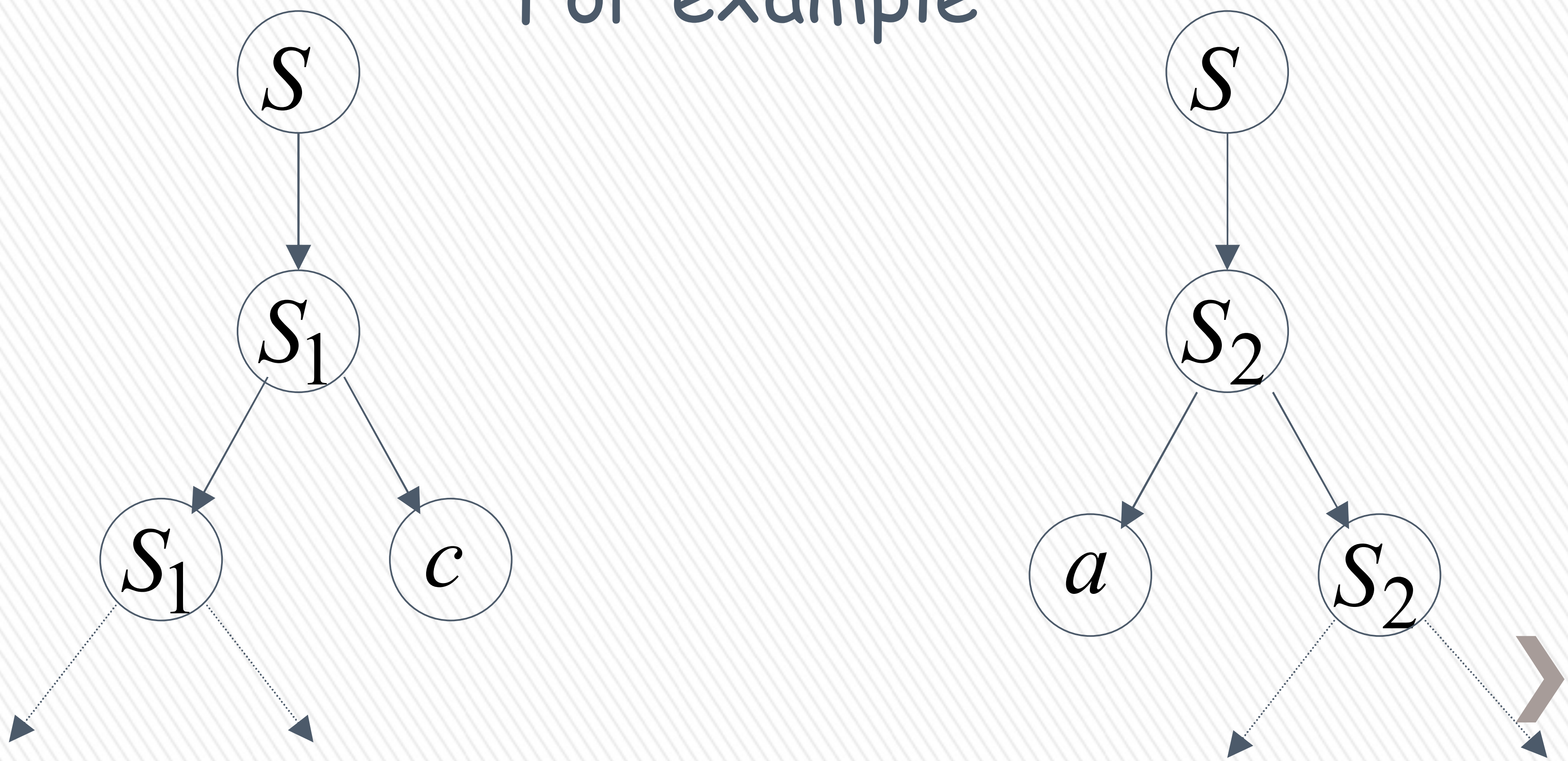
$$S \rightarrow S_1 \mid S_2 \qquad S_1 \rightarrow S_1 c \mid A \qquad S_2 \rightarrow a S_2 \mid B$$

$$A \rightarrow aAb \mid \lambda \qquad B \rightarrow bBc \mid \lambda$$

The string $a^n b^n c^n \in L$

has always two different derivation trees (for any grammar)

For example



BLM2502 Theory of Computation

?