

CENG 222

Statistical Methods for Computer Engineering

Week 13

Chapter 6 Stochastic Processes
Markov Processes and Markov Chains

Definitions and Classification

- $X(t, \omega)$ denotes a stochastic process where $t \in T$ is time and $\omega \in S$ is an outcome
- At any fixed time $X_t(\omega)$ is a random variable.
- If we fix an outcome, $X_\omega(t)$ is a function of time and is called a *realization*, a *sample path*, or a *trajectory* of a process $X(t, \omega)$.
- If the set of times is discrete, the process is called a *discrete-time* process. Otherwise, it is called a *continuous-time* process.
- Similarly, if the outcomes are discrete, the process is called *discrete-state* process (and *continuous-state* otherwise)

Example stochastic processes

- Temperature
- Stock value
- Number of jobs in a queue
- Number of internet connections
- Football score
- *Poisson process*
- *Binomial process*
- *Brownian motion*

Markov Process

- A stochastic process $X(t)$ is a Markov process if for any $t_1 < \dots < t_n < t$

$$\begin{aligned} P(X(t) \in A \mid X(t_1) = x_1, \dots, X(t_n) = x_n) \\ = P(X(t) \in A \mid X(t_n) = x_n) \end{aligned}$$

which means

$$P(\text{future} \mid \text{past, present}) = P(\text{future} \mid \text{present})$$

Markov Chain

- A Markov chain is a discrete-time, discrete-state Markov process
- $T = \{0, 1, 2, \dots\}$
- A Markov chain is a random sequence
- $\{X(0), X(1), X(2), \dots\}$
- Markov property implies that the value of $X(t + 1)$ can be predicted by only looking at $X(t)$

Transition probability

- $p_{ij}(t) = P(X(t + 1) = j \mid X(t) = i)$

is the probability of the Markov chain X to make a transition from state i to state j at time t .

- $p_{ij}^{(h)}(t) = P(X(t + h) = j \mid X(t) = i)$

is the h -step transition probability

Homogeneity

- A Markov chain is *homogeneous* if all its transition probabilities are independent of t , i.e., the transition from state i to state j is the same at any time.
- Hence, all the one-step transition probabilities can be represented as an $n \times n$ matrix, if we have n states.

State distribution

- At each time step, we have a probability mass function that shows the likelihood of outcomes/states at that time point.
- P_t is the probability mass function for $X(t)$
- P_0 is the initial distribution
- The distribution of a Markov chain is completely determined by P_0 and the transition probabilities p_{ij}

Things we can compute from P_0 and p_{ij}

- h -step transition probabilities $p_{ij}^{(h)}$
- P_h , i.e. the state distribution at time h .
- The limit of P_h as $h \rightarrow \infty$, i.e., the long-term forecast.

One-step transition probabilities

$$\begin{array}{c}
 P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}
 \end{array}
 \begin{array}{c}
 \text{From} \\
 \text{state:} \\
 1 \\
 2 \\
 \vdots \\
 n
 \end{array}$$

$$\begin{array}{c}
 \text{To state:} \\
 1 \quad 2 \quad \cdots \quad n
 \end{array}$$

h-step transition probabilities

- $P^{(h)} = P^h$
- The h^{th} power of the one-step transition probability matrix gives the h -step transition probability matrix.

The state distribution at time h

- $P_h = P_0 P^h$
- Caution: The state distributions P_h and P_0 are row vectors, i.e., $1 \times n$ matrices; whereas the transition probability matrices P , $P^{(h)}$, and P^h are $n \times n$ matrices.
- The transition probability matrices are always row normalized, i.e., sum of probabilities in a row is 1.
- State distributions are *pmfs*, i.e., the probabilities also add up to 1 in a state distribution.

Steady-state distribution

- The state distribution at the limit is called the steady-state distribution
- $\pi_x = \lim_{h \rightarrow \infty} P_h(x)$
- In the limit, the state distribution does not change from time t to time $t+1$.
- Hence, it can be found by solving

$$\pi = \pi P$$

This equation has infinitely many solutions (scaled by a constant factor c), but a unique state distribution as the solution.

Limit of P^h

- $\Pi = \lim_{h \rightarrow \infty} P^{(h)} = \begin{pmatrix} \pi_1 \pi_2 & \cdots & \pi_n \\ \vdots & \ddots & \vdots \\ \pi_1 \pi_2 & \cdots & \pi_n \end{pmatrix}$
- Each row of the matrix is the steady-state distribution.

Existence of a Steady State

- Periodic Markov chains do not have steady state distributions
- Example:
- $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $P^{(h)} = P^h = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for all odd } h \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{for all even } h \end{cases}$

Regular Markov Chains

- A Markov chain is regular if for some step h , all the h -step transition probabilities between states are strictly greater than 0.
- Any regular Markov chain has a steady-state distribution.
- Example 6.15.
 - If the one-step transition matrix contains 0s, can the Markov chain be regular?
 - Yes, if its h -step step transition matrix contains values all > 0 .