Homework 2 Solution: First-Order Differential Equations in Practical Scenarios

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Questions

1 Pipe Flow Regulation

Differential Equation:

$$\frac{dy}{dt} + 2y = e^t$$

1.1 Integration Factor Formula:

The integrating factor µ(x) is given by

$$\mu(x) = e^{\int P(x)\,dx},$$

If we use this formula (we do not add constant), $P(x) = 2$:

$$\mu(t) = e^{\int 2\,dt} = e^{2t}$$

We are multiplying with integrating factor both sides.

$$e^{2t}y' + 2e^{2t}y = e^{3t}$$

$$\frac{d}{dx}\big(\mu(x)y\big) = \mu(x)Q(x), \quad \frac{d}{dt}\big(e^{2t}v\big) = e^{2t}Q(t)$$

The solution for a first-order linear differential equation using the integrating factor method is:

$$\mu(x)y = \int \mu(x)Q(x)\,dx + C, \quad e^{2t}v = \int e^{2t}Q(t)\,dt + C$$

where $\mu(x)$ is the integrating factor, and C is the constant of integration.

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)\,dx + C\right), \quad y = \frac{1}{e^{2t}}\left(\int e^{2t}Q(x)\,dx + C\right)$$

The result of the integral is

$$y = \frac{e^t}{3} + \frac{C}{e^{2t}}$$

2 Sediment Accumulation in a River

Differential Equation:

$$\frac{dy}{dx} = xy^2$$

2.1 Separable Differential Equation:

This differential equation is a **Separable differential equation**.

$$\frac{dy}{y^2} = xdx, \quad \int y^{-2}dy = \int xdx, \quad -y^{-1} = \frac{x^2}{2} + C$$

From here we reach the following conclusion:

$$y = \frac{-2}{x^2 + 2C} = \frac{-1}{\frac{x^2}{2} + C}$$

3 Stability of Retaining Walls

Differential Equation:

$$(x^2 + y)\, dx + (x + y^2)\, dy = 0,$$

3.1 Exact Differential Equation:

This differential equation is **exact differential equation** is a first-order differential equation that can be expressed in the form:

$$M(x, y)\, dx + N(x, y)\, dy = 0,$$

where $M(x, y)$ and $N(x, y)$ are functions of x and y.
In our differential equation $M(x, y)$ and $N(x, y)$ stand for $M(x, y) = x^2 + y$ and $N(x, y) = y^2 + x$ respectively.

3.1.1 Condition for Exactness

For the equation to be exact, the mixed partial derivatives of the potential function $F(x, y)$ must be equal. That is:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$$

Condition is satisfied, there exists a function $F(x, y)$ such that:

$$\frac{\partial F}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x, y).$$

The solution to the exact differential equation is given by:

$$F(x, y) = \int (x^2 + y)\, dx = \frac{x^3}{3} + xy + h(y),$$

h(y) is function of y and constant in this equation.

$$\frac{\partial F}{\partial y} = N(x, y)$$

$$x + h'(y) = x + y^2 \quad \text{such that} \quad h'(y) = y^2$$

Let's take the integral $h(y) = \int y^2 = \frac{y^3}{3} + C$ If we substitute

$$y(t) = F(x, y) = \frac{x^3}{3} + xy + \frac{y^3}{3} = C$$

4 Beam Loading Analysis

Differential Equation:

$$\frac{dy}{dx} + y = xy^2$$

4.1 Bernoulli Differential Equation:

This differential equation is a **Bernoulli differential equation**. The Bernoulli differential equation is a first-order and a special type of nonlinear differential equation. Such equations can be transformed into a linear form using an appropriate substitution.

$$\frac{dx}{dy} + P(x)y = Q(x)y^n$$

$$v = y^{1-n}$$

In our differential equation, the coefficient of y, that is n, is 2;

$$v = y^{1-2} = y^{-1}$$

Let's differentiate v;

$$v' = -y^2 x'$$

We will make changes to the differential equation.

$$-y^2 x' - y^{-1} = -x, \quad P(x) = 1, \quad Q(x) = x$$

$$v' - v = -x, \quad P(x) = -1, \quad Q(x) = -x$$

4.2 Integration Factor Formula:

The integrating factor µ(x) is given by

$$\mu(x) = e^{\int P(x)\, dx}$$

If we use this formula (we do not add constant):

$$\mu(x) = e^{\int -1\, dx} = e^{-x}$$

We are multiplying with integrating factor both sides.

$$e^{-x} v' - e^{-x} v = -xe^{-x}$$

$$\frac{d}{dx}\big(\mu(x)y\big) = \mu(x)Q(x), \quad \frac{d}{dx}\big(e^{-x}v\big) = e^{-x}Q(x)$$

The solution for a first-order linear differential equation using the integrating factor method is:

$$\mu(x)y = \int \mu(x)Q(x)\,dx + C, \quad e^{-x}v = \int e^{-x}Q(x)\,dx + C$$

where $\mu(x)$ is the integrating factor, and C is the constant of integration.

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)\,dx + C\right), \quad v = \frac{1}{e^{-x}}\left(\int e^{-x}Q(x)\,dx + C\right)$$

The result of the integral is $v = x + 1 + ce^{x}$ therefore

$$y = \frac{1}{v} = \frac{1}{x + 1 + ce^{x}}$$

5 Concrete Pouring Rates

Differential Equation:

$$\frac{dy}{dt} = 2t$$

5.1 Separable Differential Equation:

This differential equation is a **Separable differential equation**.

$$dy = 2tdt, \quad \int dy = \int 2tdt, \quad y = t^2 + C$$

From here we reach the following conclusion, t = 0, y = 3 from find C constant.

$$C = 3$$

Height change with time $y = t^2 + 3$

6 Heat Dissipation in a Processor

Differential Equation:

$$\frac{dT}{dt} = -k(T - T_{\mathrm{a}})$$

The temperature T(t) of the processor satisfies Newton's Law of Cooling and k ¿ 0.

6.1 Newton's Law of Cooling

Newton's Law of Cooling describes the rate at which the temperature of an object changes in relation to the temperature of its surrounding environment. It states that the rate of change of the temperature of an object is proportional to the difference between the object's temperature and the ambient temperature.

6.2 Separable Differential Equation:

This differential equation is a **Separable differential equation**.

$$\frac{dT}{T - T_{\mathrm{a}}} = -kdt$$

Lets take integration,

$$\int \frac{dT}{T - T_{\mathrm{a}}} = \int -k\,dt \quad \text{such that} \quad \ln(T - T_{\mathrm{a}}) = -kt + C$$

To solve for T, exponentiation is applied to both sides of the equation:

$$|T - T_{\mathrm{a}}| = e^{-kt+C}$$

If the initial temperature is T(0) = T_0,
For $T \geq T_{\mathrm{a}}$

$$T(t) = Ce^{-kt} + T_{\mathrm{a}}, \quad T(0) = T_0 = C + T_{\mathrm{a}}, \quad C = T_0 - T_{\mathrm{a}}$$

$$T(t) = T_0 e^{-kt} - T_{\mathrm{a}} e^{-kt} + T_{\mathrm{a}}$$

For $T < T_{\mathrm{a}}$

$$T(t) = T_{\mathrm{a}} - Ce^{-kt}, \quad T(0) = T_0 = T_{\mathrm{a}} - C, \quad C = T_{\mathrm{a}} - T_0$$

$$T(t) = T_{\mathrm{a}} e^{-kt} - T_0 e^{-kt} + T_{\mathrm{a}}$$

7 Network Traffic Modeling

Differential Equation:

$$\frac{dP}{dt} + P = e^{-2t}$$

7.1 Integration Factor Formula:

The integrating factor µ(x) is given by

$$\mu(x) = e^{\int P(x)\,dx}$$

If we use this formula (we do not add constant), $P(x) = 1$:

$$\mu(t) = e^{\int 1\,dt} = e^t$$

We are multiplying with integrating factor both sides.

$$e^t P' + 2e^t P = e^{-t}$$

$$\frac{d}{dx}\big(\mu(x)y\big) = \mu(x)Q(x), \quad \frac{d}{dt}\big(e^t P\big) = e^t Q(t)$$

The solution for a first-order linear differential equation using the integrating factor method is:

$$\mu(x)y = \int \mu(x)Q(x)\,dx + C, \quad e^t P = \int e^t Q(t)\,dt + C$$

where $\mu(x)$ is the integrating factor, and C is the constant of integration.

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)\,dx + C\right), \quad y = \frac{1}{e^{2t}}\left(\int e^{2t}Q(x)\,dx + C\right)$$

The result of the integral is

$$P = -e^{-2t} + Ce^{-t}$$

If the initial packet arrival rate is P(0) = P_0

$$P(0) = -1 + C = P_0, \quad C = P_0 + 1 \quad \text{such that} \quad P = -e^{-2t} + P_0 e^{-t} + e^{-t}$$

8 Gradient Descent for Optimization

Given the quadratic loss function:

$$f(w) = w^2 + 2w + 5,$$

the gradient descent update rule is:

$$w_{n+1} = w_n - \alpha\frac{df}{dw}, \quad w_{n+1} - w_n = -\alpha\frac{df}{dw}, \quad \frac{dw}{dt} = -\alpha\frac{df}{dw}$$

where $\frac{df}{dw} = 2w + 2$ The initial value is $w_0 = 3$ and the learning rate is $\alpha = 0.1$(constant).

Differential Equation:

$$\frac{dw}{dt} = -\alpha(2w + 2)$$

8.1 Separable Differential Equation:

This differential equation is a **Separable differential equation**.

$$\frac{dw}{2w+2} = -\alpha dt, \quad \int \frac{dw}{2w+2} = \int -\alpha, \quad \ln|2w+2| = -2\alpha t + C$$

To solve for w, exponentiation is applied to both sides of the equation:

$$|2w+2| = e^{-2\alpha\, t + C_1}$$

$$C_2 = e^{C_1}, \quad w(t) = -1 + \frac{C_2 e^{-2\alpha\, t}}{2}$$

From here we reach the following conclusion, $w_0 = 3$, $\alpha = 0.1$ from find C constant.

$$3 = -1 + \frac{C_2}{2}, \quad C_2 = 8$$

The continuous solution obtained from the differential equation is:

$$w(t) = -1 + 4e^{-0.2\, t}$$

To discretize it for iterations, we use $t = n\Delta t$ where n represents the iteration number. Typically, $\Delta t = 1$ is assumed since each iteration corresponds to one time step. Thus, the equation becomes:

$$w_n = -1 + 4e^{-0.2\, n}$$

8.2 Calculating Iterations for w_n

w_1 Calculation ($n = 1$)

$$w_1 = -1 + 4e^{-2(0.1)(1)}$$
$$w_1 = -1 + 4e^{-0.2}$$

Since $e^{-0.2} \approx 0.81873$:

$$w_1 = -1 + 4(0.81873) = -1 + 3.27492 = 2.27492$$

Approximately:

$$w_1 \approx 2.27$$

w_2 Calculation ($n = 2$)

$$w_2 = -1 + 4e^{-2(0.1)(2)}$$
$$w_2 = -1 + 4e^{-0.4}$$

Since $e^{-0.4} \approx 0.67032$:

$$w_2 = -1 + 4(0.67032) = -1 + 2.68128 = 1.68128$$

Approximately:

$$w_2 \approx 1.68$$

w_3 Calculation ($n = 3$)

$$w_3 = -1 + 4e^{-2(0.1)(3)}$$

$$w_3 = -1 + 4e^{-0.6}$$

Since $e^{-0.6} \approx 0.54881$:

$$w_3 = -1 + 4(0.54881) = -1 + 2.19524 = 1.19524$$

Approximately:

$$w_3 \approx 1.20$$