⊛ $\{a_n\} = \left\{ \left(\frac{n}{n+1}\right)^{n^2+1} \right\}$ dizisinin yakınsaklığını inceleyin.

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n^2+1} = \lim_{n\to\infty} \left[\underbrace{\left(1-\frac{1}{n+1}\right)^{n+1}}_{e^{-1}} \right]^{\overbrace{\frac{n^2+1}{n+1}}^{\infty}} = e^{-\infty} = 0 \Rightarrow \text{dizi yakınsak}$$

Soru 1. Genel terimi $a_n = n - \frac{1}{2} \ln\left(1+e^{2n}\right)$, $(n=1,2,\ldots)$, olan dizinin limitini bulunuz.

Cevap 1.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left[n - \frac{1}{2} \ln(1+e^{2n}) \right]$$

(05) $$= \lim_{n\to\infty} \left[\ln(e^n) - \ln(1+e^{2n})^{1/2} \right]$$

(05) $$= \lim_{n\to\infty} \ln\left(\frac{e^n}{\sqrt{1+e^{2n}}}\right)$$

(05) $$= \lim_{n\to\infty} \ln\left(\frac{1}{\sqrt{\frac{1}{e^{2n}}+1}}\right)$$

(05) $$= \ln\left(\lim_{n\to\infty} \frac{1}{\sqrt{\frac{1}{e^{2n}}+1}}\right) = \ln\left(\frac{1}{\lim_{n\to\infty}\sqrt{\frac{1}{e^{2n}}+1}}\right)$$

$$= \ln\left(\frac{1}{\sqrt{0+1}}\right) = \ln(1)$$

(05)

$$= 0$$

Soru 3. a) $\displaystyle\sum_{n=0}^{\infty}\frac{\pi^{-n}}{\cos(n\pi)}$ serisinin toplamını bulunuz.*(10puan)*

$$\sum_{n=0}^{\infty}\frac{\pi^{-n}}{\cos(n\pi)} = 1-\frac{1}{\pi}+\frac{1}{\pi^2}-\frac{1}{\pi^3}+\cdots+(-1)^n\pi^{-n}+\cdots=\sum_{n=0}^{\infty}\left(-\frac{1}{\pi}\right)^n$$

geometrik seridir. Burada $a=1$ $r=-\frac{1}{\pi}$ dir.

$|r|<1$ olduğundan serinin toplamı

$$S=\frac{a}{1-r}=\frac{1}{1-\left(-\frac{1}{\pi}\right)}=\frac{\pi}{\pi+1}$$

b) Genel terimi $a_n=\left(\dfrac{3n-1}{3n+2}\right)^n$ olan $\{a_n\}$ dizisinin limitini bulunuz.

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\left(1-\frac{3}{3n+2}\right)^n=\lim_{n\to\infty}\left[\left(1-\frac{3}{3n+2}\right)^{3n+2-2}\right]^{1/3}$$

$$=\lim_{n\to\infty}\left[\left(1-\frac{3}{3n+2}\right)^{3n+2}\cdot\left(1-\frac{3}{3n+2}\right)^{-2}\right]^{1/3}$$

25) 5,232323... sayısının serileri kullanarak iki tamsayının oranı olarak ifade ediniz.

$$5,232323\ldots=5+\frac{23}{100}+\frac{23}{(100)^2}+\frac{23}{(100)^3}+\cdots$$

$$=5+\sum_{n=1}^{\infty}\frac{23}{100}\left(\frac{1}{100}\right)^{n-1}\qquad a=\frac{23}{100}\quad r=\frac{1}{100}$$

$$|r|=\frac{1}{100}<1\Rightarrow\sum_{n=1}^{\infty}\frac{23}{100}\left(\frac{1}{100}\right)^{n-1}=\frac{a}{1-r}=\frac{\frac{23}{100}}{1-\frac{1}{100}}=\frac{23}{99}$$

(Seri yakınsak)

$$5,232323\ldots=5+\frac{23}{99}=\frac{518}{99}$$

(*) $\sum_{n=1}^{\infty} \frac{4}{n^2+4n+3} = ?$

$\left.\frac{4}{n^2+4n+3} = \frac{A}{n+3} + \frac{B}{n+1}\right\} \Rightarrow \boxed{A=-2 \mid B=2}$

$\sum_{n=1}^{\infty} \frac{4}{n^2+4n+3} = 2\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3}$

$S_n = 2\left[\left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right) + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+3}\right)\right]$

$S_n = 2\left[\frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}\right] \Rightarrow \lim_{n\to\infty} S_n = \frac{5}{3} \Rightarrow \sum_{n=1}^{\infty} \frac{4}{n^2+4n+3} = \frac{5}{3}$

(*) $\sum_{k=2}^{\infty} \ln\left(\frac{k-1}{k}\right)$ serisinin toplamını bulup sonucu yorumlayın.

$S_n = \ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \cdots + \ln\frac{n-1}{n}$

$= \ln\left(\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{4}} \cdots \frac{\cancel{n-1}}{n}\right) = \ln\left(\frac{1}{n}\right) \Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln\left(\frac{1}{n}\right) = -\infty$

$\Downarrow$

Seri $-\infty$'a ıraksar.

(*) $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = ?$

$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} = 2 - \frac{6}{5} = \frac{4}{5}$

$a=1 \quad r=\frac{1}{2}$

$|r| = \frac{1}{2} < 1$

Toplam $= \frac{a}{1-r}$

$= \frac{1}{1-\frac{1}{2}} = 2$

$a=1 \quad r=\frac{1}{6}$

$|r| = \frac{1}{6} < 1$

Toplam $= \frac{a}{1-r} = \frac{1}{1-\frac{1}{6}}$

$= \frac{6}{5}$

(*) $\sum_{n=1}^{\infty} \ln\sqrt{n+1} - \ln\sqrt{n}$ serisinin n. kısmi toplamı için bir formül bulunuz ve bu formül yardımıyla serinin yakınsaklığını inceleyiniz.

$$\sum_{n=1}^{\infty} -\ln\sqrt{n} + \ln\sqrt{n+1} \Rightarrow S_n = -\overset{0}{\overline{\ln 1}} + \cancel{\ln\sqrt{2}} - \cancel{\ln\sqrt{2}} + \cancel{\ln\sqrt{3}} \cdots - \cancel{\ln\sqrt{n}} + \ln\sqrt{n+1}$$

$$= \ln\sqrt{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln\sqrt{n+1} = +\infty \Rightarrow \text{Seri } +\infty\text{'a ıraksar}$$

(*) $\sum_{n=1}^{\infty} \mathrm{ArcCos}\frac{1}{n+1} - \mathrm{ArcCos}\frac{1}{n+2}$ serisinin n. kısmi toplamı için bir formül bulup yakınsaklığını inceleyiniz. Yakınsak ise değerini bulunuz.

$$S_n = \mathrm{ArcCos}\frac{1}{2} - \cancel{\mathrm{ArcCos}\frac{1}{3}} + \cancel{\mathrm{ArcCos}\frac{1}{3}} - \cancel{\mathrm{ArcCos}\frac{1}{4}} + \cdots + \cancel{\mathrm{ArcCos}\frac{1}{n+1}} - \mathrm{ArcCos}\frac{1}{n+2}$$

$$S_n = \underbrace{\mathrm{ArcCos}\frac{1}{2}}_{\pi/3} - \mathrm{ArcCos}\frac{1}{n+2} = \frac{\pi}{3} - \mathrm{ArcCos}\frac{1}{n+2}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{\pi}{3} - \underbrace{\mathrm{ArcCos}\underbrace{\frac{1}{n+2}}_{0}}_{\pi/2} = \frac{\pi}{3} - \frac{\pi}{2} = -\frac{\pi}{6} \rightarrow \text{Seri yakınsaktır.}$$

Toplamı $-\frac{\pi}{6}$ dır.

(*) $\{a_n\} = \left\{\left(1+\frac{1}{n^2}\right)^n\right\}$ dizisinin yakınsaklığını inceleyin.

I. Yol

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^n = \lim_{n\to\infty}\left[\underbrace{\left(1+\frac{1}{n^2}\right)^{n^2}}_{e}\right]^{1/n \to 0} = e^0 = 1 \Rightarrow \text{Dizi yakınsaktır.}$$

II. Yol Logaritmik limit ile de çözülebilir.

⑥ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = ?$

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} \qquad A = \frac{1}{2} \quad B = -\frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2}$$

$$S_n = \frac{1}{2}\left[\left(1 - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \cdots + \left(\cancel{\frac{1}{n-2}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right] \Rightarrow \lim_{n\to\infty} S_n = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{4}$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{4}$$

(*) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+3n+1}{n^3(n+1)^3}$ serisinin toplamını bulunuz.

$$\frac{3n^2+3n+1}{n^3.(n+1)^3} = \frac{(1+n)^3 - n^3}{n^3(1+n)^3} = \frac{1}{n^3} - \frac{1}{(n+1)^3} \text{ olduğundan}$$

$$\sum_{n=1}^{\infty} \frac{3n^2+3n+1}{n^3(n+1)^3} = \sum_{n=1}^{\infty} \frac{1}{n^3} - \frac{1}{(n+1)^3} \text{ dır.}$$

$$S_n = \left(1 - \cancel{\frac{1}{2^3}}\right) + \left(\cancel{\frac{1}{2^3}} - \cancel{\frac{1}{3^3}}\right) + \left(\cancel{\frac{1}{3^3}} - \cancel{\frac{1}{4^3}}\right) + \cdots + \left(\cancel{\frac{1}{(n-1)^3}} - \cancel{\frac{1}{n^3}}\right) + \left(\cancel{\frac{1}{n^3}} - \frac{1}{(n+1)^3}\right)$$

$$= 1 - \frac{1}{(n+1)^3}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{(n+1)^3} = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{3n^2+3n+1}{n^3(1+n)^3} = \underline{1}$$

⊛ $\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\cdots=?$

$$\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\cdots=\sum_{n=1}^{\infty}\frac{n+1}{(n+2)!}$$

$$\frac{n+1}{(n+2)!}=\frac{n+2-1}{(n+2)!}=\frac{n+2}{(n+2)!}-\frac{1}{(n+2)!}=\frac{1}{(n+1)!}-\frac{1}{(n+2)!}$$

olduğundan

$$\sum_{n=1}^{\infty}\frac{n+1}{(n+2)!}=\sum_{n=1}^{\infty}\frac{1}{(n+1)!}-\frac{1}{(n+2)!}$$ dir.

$$S_n=\left(\frac{1}{2!}-\cancel{\frac{1}{3!}}\right)+\left(\cancel{\frac{1}{3!}}-\cancel{\frac{1}{4!}}\right)+\cdots+\left(\cancel{\frac{1}{(n+1)!}}-\frac{1}{(n+2)!}\right)=\frac{1}{2!}-\frac{1}{(n+2)!}$$

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{2}-\frac{1}{(n+2)!}=\frac{1}{2}\Rightarrow\frac{2}{3!}+\frac{3}{4!}+\cdots=\underline{\underline{\frac{1}{2}}}$$

⊛ $x=2,131313\ldots=2,\overline{13}$ sayısını serileri kullanarak iki tamsayının oranı olarak ifade ediniz.

$$x=2,\overline{13}=2+\underbrace{\frac{13}{100}+\frac{13}{(100)^2}+\frac{13}{(100)^3}+\cdots}_{\text{Geometrik Seri}}=2+\sum_{n=1}^{\infty}\frac{13}{100}\cdot\left(\frac{1}{100}\right)^{n-1}$$

$a=\frac{13}{100}$ $\quad r=\frac{1}{100}$

$$|r|=\frac{1}{100}<1\Rightarrow\sum_{n=1}^{\infty}\frac{13}{100}\cdot\left(\frac{1}{100}\right)^{n-1}=\frac{a}{1-r}=\frac{\frac{13}{100}}{1-\frac{1}{100}}=\frac{13}{99}$$

seri yakınsaktır

$$x=2+\frac{13}{99}=\frac{211}{99}$$

⊛ $4-1+\frac{1}{4}-\frac{1}{16}+\frac{1}{64}\cdots=?$ $\Rightarrow a=4 \quad r=-\frac{1}{4}$ Geometrik Seridir

$-1/4$ $\quad -1/4 \quad -1/4 \quad -\frac{1}{4}$

$$\sum_{n=1}^{\infty}4\cdot\left(-\frac{1}{4}\right)^{n-1}\Rightarrow|r|=\frac{1}{4}<1$$ Seri $\frac{a}{1-r}$'ye yakınsar.

$$\frac{a}{1-r}=\frac{4}{1-(-\frac{1}{4})}=\frac{16}{5}\Rightarrow 4-1+\frac{1}{4}-\frac{1}{16}\cdots=\underline{\underline{\frac{16}{5}}}$$