İntegrasyon Teknikleri

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Ör: $\int (x^5+4x^2)^3 \cdot (5x^4+8x)\,dx = ?$

$x^5+4x^2 = u$.
$(5x^4+8x)dx = du$.

$\Rightarrow \int (x^5+4x^2)^3 \cdot (5x^4+8x)\,dx = \int u^3\,du$

$= \frac{u^4}{4} + C$

$= \frac{(x^5+4x^2)^4}{4} + C$

Ör: $\int x^2 \sin x^3\,dx = ?$

$x^3 = u$.
$3x^2dx = du$.

$\int x^2 \sin x^3\,dx = \int \sin u \cdot \frac{du}{3} = \frac{1}{3}\int \sin u\,du$

$= \frac{1}{3}\cdot(-\cos u) + C$

$= -\frac{1}{3}\cos u + C$

$= -\frac{1}{3}\cos x^3 + C$.

Or $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{du}{u} = \ln|u| + c.$

$= \ln|\sin x| + c.$

$\sin x = u$

$\cos x\,dx = du$

Or: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int -\frac{du}{u} = -\ln|u| + c.$

$= -\ln|\cos x| + c$

$\cos x = u$

$-\sin x\,dx = du$

Or: $\int \sec x\,dx = ?$

$\sec x + \tan x = u.$

$(\sec x \tan x + \sec^2 x)\,dx = du.$

$$\int \sec x\,dx = \int \sec x \cdot \frac{(\sec x + \tan x)}{\sec x + \tan x}\,dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

$\int \operatorname{cosec} x \, dx = ?$

$\operatorname{cosec} x + \cot x = u$

$(-\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x)\, dx = du$

$$\int \operatorname{cosec} x \, dx = \int \frac{\operatorname{cosec} x (\operatorname{cosec} x + \cot x)}{(\operatorname{cosec} x + \cot x)} dx$$

$$= \int \frac{-du}{u}$$

$$= -\ln|u| + c$$

$$= -\ln|\operatorname{cosec} x + \cot x| + C$$

$$\star\star \int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a}.$$

Ex: $\int \frac{dx}{9+x^2} = \int \frac{dx}{9.(1+\frac{x^2}{9})} = \frac{1}{9}\int \frac{dx}{1+(\frac{x}{3})^2}$

$$= \frac{1}{9}\int \frac{3\,du}{1+u^2}.$$

$\frac{x}{3} = u$

$\frac{dx}{3} = du$

$$= \frac{1}{3}\arctan u + C$$

$$= \frac{1}{3}\arctan.u + C.$$

$$= \frac{1}{3}\arctan\frac{x}{3} + C$$

Ex: $\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{9.(1-\frac{x^2}{9})}} = \int \frac{dx}{3\sqrt{1-(\frac{x}{3})^2}}.$

$$= \frac{1}{3}\int \frac{dx}{\sqrt{1-(\frac{x}{3})^2}}$$

$\frac{x}{3} = u.$

$\frac{dx}{3} = du$

$$= \frac{1}{\cancel{3}}\int \frac{\cancel{3}\,du}{\sqrt{1-u^2}}$$

$$= \int \frac{du}{\sqrt{1-u^2}}$$

$$= \arcsin u + C.$$

$$= \arcsin\frac{x}{3} + C.$$

$\star\star$ $\int \frac{dx}{\sqrt{a^2-x^2}} = \text{Arcsin}\frac{x}{a} + C$

Ör: $\int \frac{\sin 2x}{\sqrt{1-\sin^4 x}}\,dx = ?$

$\sin^2 x = u$

$\underbrace{2\sin x\cos x\,dx}_{\sin 2x\,dx} = du.$

$$\int \frac{\sin 2x}{\sqrt{1-\sin^4 x}}\,dx = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C.$$

$$= \arcsin(\sin^2 x) + C$$

Ör: $\int \sqrt{x}\,\sin^2\left(x^{\frac{3}{2}}-1\right)dx = ?$

$x^{3/2} - 1 = u.$

$\frac{3}{2}\,x^{1/2}\,dx = du$

$$\int \sqrt{x}\,\sin\left(x^{3/2}-1\right)dx = \int \frac{2}{3}\cdot\sin^2 u\,du.$$

$$= \int \frac{2}{3}\cdot\left(\frac{1-\cos 2u}{2}\right)du.$$

$$= \frac{1}{3}\int (1-\cos 2u)\,du$$

$$= \frac{1}{3}\cdot\left(u - \frac{\sin 2u}{2}\right) + C.$$

$$= \frac{1}{3}\cdot\left(x^{3/2}-1 - \frac{\sin 2\cdot(x^{3/2}-1)}{2}\right) + C.$$

$$= \frac{1}{3}\cdot(x^{3/2}-1) - \frac{\sin 2(x^{3/2}-1)}{6} + C$$

İntegrasyon Teknikleri

Kısmi İntegrasyon

$$\int u\,dv = uv - \int v\,du$$

Belirli İntegralde: $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$.

LAPTÜ
- Üstel fonksiyon
- Trigonometrik fonksiyon
- Polinom
- (Arcsin, Arccos, Arctan, Arcot...) Ters Trigonometrik Fonk.
- Logaritmik fonksiyon.

Ör: $\int x\cdot\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$

$\cos x\,dx = dv$

$\Downarrow$

$\int \cos x\,dx = \int dv$

$\Downarrow$

$\sin x = v$

$x = u$

$\Downarrow$

$dx = du$

Ör: $\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int dx = x\ln x - x + c$

* $\ln x = u$, * $dx = dv$.

$\frac{1}{x}dx = du$, $\int dx = \int dv$

$x = v$.

Ör: $\int \underbrace{x^2}_{u}\underbrace{e^x dx}_{dv} = ?$

* $x^2 = u$, * $e^x dx = dv$

$2x\,dx = du$, $\int e^x dx = \int dv$

$e^x = v$

(*) $\Rightarrow \int x^2 e^x dx = x^2e^x - \int e^x \cdot 2x\,dx = x^2e^x - 2\underbrace{\int xe^x dx}_{\text{Tekrar Kısmi İntegrasyon alalım}}$

$\int \underbrace{x}_{u}\underbrace{e^x dx}_{dv}$ integrali için kısmi integrasyon alalım.

* $x = u$, * $e^x dx = dv$.

$dx = du$, $\int e^x dx = \int dv$

$e^x = v$

$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x$

(*) $\Rightarrow \int x^2e^x dx = x^2e^x - 2[xe^x - e^x] + C = x^2e^x - 2xe^x + 2e^x + C$

Örn: $\int_0^4 x e^{-x} dx = ?$

* $x = u$, $e^{-x} dx = dv$.

$dx = du$

$\int e^{-x} dx = \int dv$

$-e^{-x} = v$.

$$\int_0^4 x e^{-x} dx = -x e^{-x} \Big|_0^4 + \int_0^4 e^{-x} dx$$

$$= (-4e^{-4} - 0) - e^{-x} \Big|_0^4$$

$$= -4e^{-4} - (e^{-4} - e^0)$$

$$= -4e^{-4} - (e^{-4} - 1)$$

$$= -5e^{-4} + 1$$

$$= 1 - 5e^{-4}$$

Örn: $\int \cos^n x\,dx = ?$ (indirgeme formülü bulmuz)

$$\int \cos^n x\,dx = \int \underbrace{\cos^{n-1}x}_{u}\ \underbrace{\cos x\,dx}_{dv}$$

$* \cos^{n-1}x = u$, $* \cos x\,dx = dv$

$-(n-1)\cos^{n-2}x \cdot \sin x\,dx = du$ $\quad \Rightarrow \int \cos x\,dx = \int dv$

$\sin x = v$

$$\int \cos^n x\,dx = \int \cos^{n-1}x \cos x\,dx = \cos^{n-1}x \sin x + (n-1)\int \sin^2 x \cdot \cos^{n-2}x\,dx$$

$$= \cos^{n-1}x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2}x\,dx$$

$$\int \cos^n x\,dx = \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx$$

$$n\int \cos^n x\,dx = \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,dx$$

$$\int \cos^n x\,dx = \frac{\cos^{n-1}x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$ (indirgeme formülü)

Örn: $\int \cos^3 x\,dx = \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\,dx$

$$= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\sin x + C.$$

Trigonometrik İntegraller

$\int \sin^m x \cos^n x \, dx$

* m tek, n çift ise.

$u = \cos x$
$du = -\sin x \, dx$
} dönüşümü ve $\sin^2 x + \cos^2 x = 1$ özdeşliği kullanılarak integral çözülür.

* n tek, m çift ise.

$u = \sin x$
$du = \cos x \, dx$
} dönüşümü ve $\sin^2 x + \cos^2 x = 1$ özdeşliği kullanılarak integral çözülür.

* m ve n ikisi de çift ise.

$\sin^2 x = \frac{1-\cos 2x}{2}$, $\cos^2 x = \frac{1+\cos 2x}{2}$
} $\cos 2x$'in düşük katlarına indirgenir.

* m ve n tek ise herhangi birine u denir.

Ör: $\int \sin^3 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x . \sin x \, dx$

$\cos x = u$
$-\sin x . dx = du$

$= \int (1 - \cos^2 x) . \cos^2 x . \sin x . dx$

$= \int (1 - u^2) . u^2 . (-du)$

$= \int (u^4 - u^2) du$

$= \frac{u^5}{5} - \frac{u^3}{3} + C.$

$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.$

Ör: $\int \cos^5 x\,dx = \int \cos^4 x \cdot \cos x \cdot dx$

$\sin x = u$
$+\cos x \cdot dx = du$

$= \int (\cos^2 x)^2 \cdot \cos x\,dx$

$= \int (1-\sin^2 x)^2 \cdot \cos x\,dx$

$= \int (1-u^2)^2 \cdot du$

$= \int (u^4 - 2u^2 + 1)\,du$

$= \frac{u^5}{5} - \frac{2}{3}u^3 + u + c.$

$= \frac{\sin^5 x}{5} - \frac{2}{3}\sin^3 x + \sin x + c.$

Ör: $\int \sin^2 x \cos^4 x \cdot dx = \int \left(\frac{1-\cos 2x}{2}\right) \cdot \left(\frac{1+\cos 2x}{2}\right)^2 dx$

$= \int \left(\frac{1-\cos 2x}{2}\right) \cdot \left(\frac{\cos^2 2x + 2\cos 2x + 1}{4}\right) dx$

$= \frac{1}{8} \cdot \int (\cos^2 2x + 2\cos 2x + 1 - \cos^3 2x - 2\cos^2 2x - \cos 2x)\,dx$

$= \frac{1}{8} \cdot \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\,dx$

$= \frac{1}{8}\left[x + \frac{\sin 2x}{2} - \int \cos^2 2x\,dx + \int \cos^3 2x\,dx\right]$

$= \frac{1}{8}\left[x + \frac{\sin 2x}{2} - \frac{x}{2} - \frac{\sin 4x}{8} - \frac{\sin 2x}{2} + \frac{\sin^3 2x}{6}\right] + C = \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C$

$\int \cos^2 2x\,dx = \int \left(\frac{1+\cos 4x}{2}\right) dx = \frac{x}{2} + \frac{\sin 4x}{8}.$

$\int \cos^3 2x\,dx = \int \cos^2 2x \cdot \cos 2x\,dx = \int (1-\sin^2 2x) \cdot \cos 2x\,dx = \int (1-u^2) \cdot \frac{du}{2}$

$\left(\begin{array}{l}\sin 2x = u \\ 2\cdot\cos 2x\,dx = du\end{array}\right)$

$= \frac{1}{2} \cdot \left(u - \frac{u^3}{3}\right) + C = \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6}$

Kare Köklerden Kurtulmak

Hatırlatma
* $\cos^2\theta = \frac{1+\cos 2\theta}{2}$
* $2\cos^2\theta = 1+\cos 2\theta$

Örn: $\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx = ?$

$4x = 2\theta$ dersek $\Rightarrow 2x = \theta \Rightarrow 2dx = d\theta$.

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx = \int_0^{\pi/2} \sqrt{1+\cos 2\theta}\, \frac{d\theta}{2}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sqrt{1+\cos 2\theta}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sqrt{2\cos^2\theta}\, d\theta$$

$$= \frac{1}{2}\sqrt{2}\int_0^{\pi/2} \cos\theta\, d\theta$$

$$= \frac{\sqrt{2}}{2}\left(\sin\theta\Big|_0^{\pi/2}\right)$$

$$= \frac{\sqrt{2}}{2}\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= \frac{\sqrt{2}}{2}.$$

Tanx ve Secx Kuvvetlerinin İntegralleri

* $1+\tan^2x=\sec^2x$

Ör: $\int \tan^2x\,dx = ?$

I.yol $\int \tan^2x\,dx = \int(\tan^2x+1-1)\,dx = \int(\tan^2x+1)\,dx - \int dx$

$= \tan x - x + C$

II.yol $\int \tan^2x\,dx = \int(\sec^2x-1)\,dx = \int \sec^2x\,dx - \int dx$

$= \tan x - x + C$

Ör: $\int \tan^4x\,dx = \int \tan^2x.\tan^2x\,dx = \int \tan^2x.(\sec^2x-1)\,dx$

$= \underbrace{\int \tan^2x\sec^2x\,dx}_{I_1} - \underbrace{\int \tan^2x.dx}_{I_2}$

$= \frac{\tan^3x}{3} - \tan x + x + C$

$I_1 = \int \tan^2x\sec^2x.dx = \int u^2du = \frac{u^3}{3}+C = \frac{\tan^3x}{3}+C$

$\tan x = u$.

$\sec^2x\,dx = du$

$I_2 = \int \tan^2x\,dx = \tan x - x + C$

Or: $\int \sec^3 x\,dx = \int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x\,dx}_{dv}$

* $\sec x = u$, $\sec^2 x\,dx = dv$

$\sec x \tan x\,dx = du$ $\tan x = v$

$$\int \sec^3 x\,dx = \int \sec x \sec^2 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\,dx$$

$$= \sec x \tan x - \int (\sec^3 x - \sec x)\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + c.$$

$$\int \sec^3 x\,dx = \frac{\sec x \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x| + c.$$

* $\int \sin mx \sin nx\, dx$, $\int \sin mx \cos nx\, dx$, $\int \cos mx \cos nx\, dx$

$$\sin mx . \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$

$$\cos mx \cos nx = \frac{1}{2}\cdot\left[\cos(m-n)x + \cos(m+n)x\right]$$

$$\sin mx . \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$$

Ör: $\int \sin 3x \cos 5x\, dx = ?$

$$\int \sin 3x \cos 5x\, dx = \int \frac{1}{2}\cdot\left[\sin(-2x) + \sin 8x\right] dx$$

$$= \frac{1}{2}\int (-\sin 2x + \sin 8x)\, dx$$

$$= \frac{1}{2}\left(\frac{\cos 2x}{2} - \frac{\cos 8x}{8}\right) + C.$$

$$= \frac{\cos 2x}{4} - \frac{\cos 8x}{16} + C.$$

Trigonometrik Değişken Dönüşümleri

$x = a\tan\theta$ dönüşümü,

$\Downarrow$

$\sqrt{a^2+x^2}$

$\Downarrow$

$\sqrt{a^2+a^2\tan^2\theta}$

$\Downarrow$

$\sqrt{a^2.(1+\tan^2\theta)}$

$a.|\sec\theta|$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

aralığında

$\tan\theta$ nın tersi vardır

$x = a\tan\theta$

$\theta = \tan^{-1}\left(\frac{x}{a}\right)$

ya da

$\theta = \arctan\left(\frac{x}{a}\right)$

dır.

$x = a\sin\theta$ dönüşümü,

$\Downarrow$

$\sqrt{a^2-x^2}$

$\Downarrow$

$\sqrt{a^2-a^2\sin^2\theta}$

$\Downarrow$

$\sqrt{a^2.(1-\sin^2\theta)}$

$a.|\cos\theta|$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

aralığında

$\sin\theta$ nın tersi vardır.

$x = a\sin\theta$

$\theta = \sin^{-1}\left(\frac{x}{a}\right)$

ya da

$\theta = \arcsin\left(\frac{x}{a}\right)$

dır

$x = a\sec\theta$ dönüşümü.

$\Downarrow$

$\sqrt{x^2-a^2}$

$\Downarrow$

$\sqrt{a^2\sec^2\theta-a^2}$

$\Downarrow$

$\sqrt{a^2.(\sec^2\theta-1)}$

$a.|\tan\theta|$

$0 \leq \theta < \frac{\pi}{2}, \frac{x}{a} \geq 1$

$\frac{\pi}{2} < \theta \leq \pi, \frac{x}{a} \leq -1$

aralıklarında

$\sec\theta$ nın tersi vardır.

$x = a\sec\theta$

$\theta = \sec^{-1}\left(\frac{x}{a}\right)$

ya da

$\theta = \operatorname{arcsec}\left(\frac{x}{a}\right)$

dır.

Ön: $\int \frac{dx}{\sqrt{4+x^2}} = ?$

$x = 2\tan\theta \quad , \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

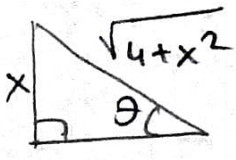
$dx = 2\sec^2\theta \, d\theta.$

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta \, d\theta}{\sqrt{4+4\tan^2\theta}}$$

$$= \int \frac{2\sec^2\theta \cdot d\theta}{\sqrt{4\cdot(1+\tan^2\theta)}} \quad (\sec^2\theta)$$

$$= \int \frac{2\sec^2\theta}{2|\sec\theta|} d\theta$$

$$= \int \frac{\sec\theta}{\sec\theta} d\theta$$



$$= \int \sec\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C.$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + c.$$

Ör: $\int \frac{x^2 dx}{\sqrt{9-x^2}} = ?$

$x = 3\sin\theta$, $-\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$

$dx = 3\cos\theta\, d\theta$

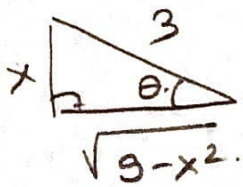
$$\int \frac{x^2 dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{\sqrt{9-9\sin^2\theta}}$$

$$= \int \frac{27\sin^2\theta\cos\theta}{\sqrt{9.(1-\sin^2\theta)}}\, d\theta \qquad (1-\sin^2\theta = \cos^2\theta)$$

$$= \int \frac{27\sin^2\theta\cos\theta}{3|\cos\theta|}\, d\theta$$

$$= \int \frac{9\sin^2\theta\cos\theta}{\cos\theta}\, d\theta$$

$$= \int 9\sin^2\theta\, d\theta$$



$$= \int 9.\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{9}{2}\theta - \frac{9}{4}\underbrace{\sin 2\theta}_{2.\sin\theta.\cos\theta} + C$$

$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{9}{4_2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C$$

$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{\sqrt{9-x^2}}{2} + C.$$

Ör: $\int \frac{dx}{\sqrt{25x^2-4}} = ?$

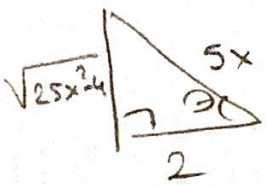
$x = \frac{2}{5}\sec\theta$

$dx = \frac{2}{5}\sec\theta.\tan\theta\, d\theta.$

$$\int \frac{dx}{\sqrt{25x^2-4}} = \int \frac{\frac{2}{5}\sec\theta\tan\theta.d\theta}{\sqrt{4\sec^2\theta-4}}$$

$$= \int \frac{\frac{2}{5}\sec\theta.\tan\theta.d\theta}{\sqrt{4.(\sec^2\theta-1)}} \quad (\sec^2\theta-1 = \tan^2\theta)$$

$$= \int \frac{\frac{2}{5}\sec\theta.\tan\theta.d\theta}{2\tan\theta}$$



$$= \frac{1}{5}\int \sec\theta.d\theta$$

$$= \frac{1}{5}\ln|\sec\theta+\tan\theta| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C$$