CENG 222
Statistical Methods for Computer Engineering

Week 4

Chapter 4

Continuous Distributions:

Probability density, Uniform and Exponential Distributions

Continuous R.V.s

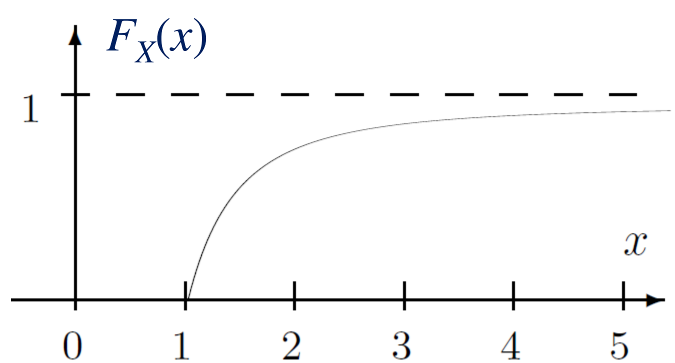
- A continuous random variable may assume any real value in an interval:
  - (a,b), (a,+∞), (-∞,+∞), etc.
- *Examples:*
  - *Time*
  - *Temperature*
  - *Length*
  - *Weight*

Point events have 0 probabilities

- Since there are infinitely many outcomes associated with a continuous random variable, the probability of a specific outcome is 0.
  - $P(X = x) = 0$
- In this case, probabilities of intervals of outcomes are of interest
  - E.g, $P(c < X \leq d)$ or $P(X > d)$
- $P(X < x) = P(X \leq x)$

cdf of continuous r.v.s

- $F_X(x)$ has the same meaning as in the discrete case
  - $F_X(x) = P(X \leq x) = P(X < x)$
- But unlike the discrete cdfs, continuous cdfs do not have jumps, since $P(X = x) = 0$.
- cdfs of continuous r.v.s are continuous functions



Probability density function (pdf)

- Given the cdf $F_X(x)$ as a continuous and non-decreasing functions, the pdf is defined as:
  - $f_X(x) = F_X'(x) = \frac{dF}{dx}$
  - The distribution is called continuous if it has a density
  - $F_X(x)$ is an antiderivative of the density
  - $\int_a^b f_X(x) = F_X(b) - F_X(a) = P(a < X < b)$
  - $\int_{-\infty}^b f_X(x) = F_X(b)$ and $\int_{-\infty}^{+\infty} f_X(x) = 1$

Example 4.1

- Lifetime (in years) of some electronic component is a r.v with the following pdf:
  - $f_X(x) = \begin{cases} 0, & x < 1 \\ \frac{k}{x^3}, & x \geq 1 \end{cases}$
- What is k?
- Find the cdf.
- What is the probability for the lifetime to exceed 5 years?

Joint and Marginal densities

- The joint cdf for two rvs is defined as:
  - $F_{X,Y}(x, y) = P(X \leq x \cap Y \leq y)$
- The joint density function is then given as
  - $f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y)$
- Marginal distributions can be computed from the joint pdf as:
  - $f_X(x) = \int_y f_{X,Y}(x, y) dy$
- Two continuous rvs are independent if the joint pdf is a product of marginal pdfs:
  - $f_{X,Y}(x, y) = f_X(x) f_Y(y)$

Expectation and variance

- Expectation
  - $E(X) = \mu = \int x f_X(x) dx$
- Variance
  - $Var(X) = \int (x-\mu)^2 f_X(x) dx = \int x^2 f_X(x) dx - \mu^2$
- Example 4.2
  - $f_X(x) = 2x^{-3}$ for $x \geq 1$
  - Compute expectation and variance

Some important continuous distributions

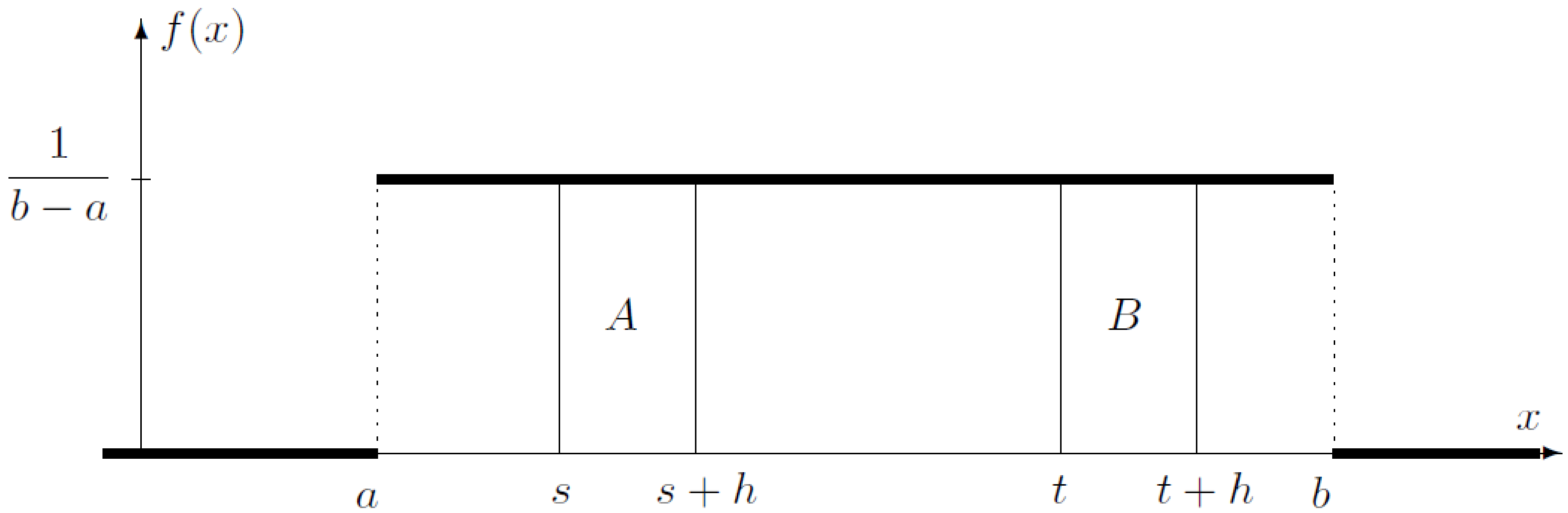
- Uniform
- Exponential
  - related to Poisson, continuous case of Geometric distribution
- Gamma
- Normal

Uniform distribution

- Parameters: interval [a,b]
- Constant density
  - $f_X(x) = \frac{1}{b-a}$
- Expectation
  - $E(X) = \frac{a+b}{2}$
- Variance
  - $Var(X) = \frac{(b-a)^2}{12}$

The Uniform property

- The probability of an interval within [a,b] is only determined by its width, not by its location.



Standard Uniform distribution

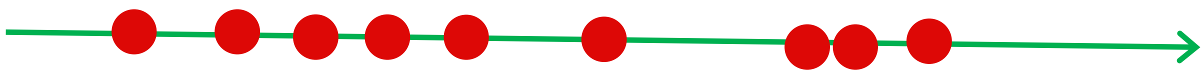
- $[a,b] = [0,1]$ is called Standard Uniform distribution
- If X is a *Uniform*(a,b) rv then $Y=(X-a)/(b-a)$ is the Standard Uniform rv.

Exponential distribution

- Used to model time: waiting time, interarrival time, failure time, etc.
- Can be considered as the continuous version of the geometric distribution which counts the number of trials before success.
- Related to Poisson distribution
  - λ parameter has the same meaning in both distributions
  - λ = avg. # of events in a time unit

Exponential dist. vs Poisson dist.

- Rare events



- N_1 = # of events in 1 min = Poisson (λ)
- N_2 = # of events in 2 mins = Poisson (2λ)
- N_t = # of events in t mins = Poisson ($t\lambda$)

- X = Time between events = Exponential (λ)
- X_1 = Time of the first event = Exponential (λ)

Exponential cdf

- Can be derived from the Poisson pmf
  - $f_X(x) = e^{-\lambda} \frac{\lambda^x}{x!}$
- "The waiting time for the next event is greater than t time units" is the same as saying "0 events occur in t time units". If X is a rv that shows the number of events in t time units (X is a Poisson rv with $t\lambda$)
  - $f_X(0) = e^{-\lambda t} \frac{(\lambda t)^0}{0!} = e^{-\lambda t}$

Exponential cdf

- Exponential cdf $F_T(t)$ shows the total probability that waiting time is less than t.
- If $f_X(0)$ shows the probability of 0 events in t time units, then:
  - $F_T(t) = 1 - f_X(0) = 1 - e^{-\lambda t}$

Exponential pdf

- Is the derivative of the cdf $F_T(t)$
  - $f_T(t) = F_T'(t) = \lambda e^{-\lambda t} \qquad t > 0$

Exponential distribution summary

- Parameter: λ – the number of event per time unit
- Density
  - $f_X(x) = \lambda e^{-\lambda x}, \quad x > 0$
- Expectation
  - $E(X) = \frac{1}{\lambda}$
- Variance
  - $Var(X) = \frac{1}{\lambda^2}$

Memoryless property

- What is the chance that an electronic component **A** survives x hours?
  - X = time to failure = Exponential(λ)
  - $P(X > x) = 1 - F_X(x) = e^{-\lambda x}$
- Another component **B** did not fail for t hours. What is the probability that it will survive another x hours?
  - $P(X > t + x \mid X > t) = ?$

Memoryless property

- $P(X > t + x \mid X > t) = \frac{P(X>t+x \cap X>t)}{P(X>t)}$

$$= \frac{P(X>t+x)}{P(X>t)}$$

$$= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x}$$

- Same as $P(X > x)$!!
- This is called the memoryless property
  - Exponential distribution is the only continuous distribution with this property