MAT1071 MATEMATİK 1

ALIŞTIRMALAR 2 - LİMİT

1. $$\lim_{n\to\infty} \frac{1-2+3-4+5-6+\dots-2n}{\sqrt{n^2+1}+\sqrt{4n^2-1}} = ?$$

$$\lim_{n\to\infty} \frac{1+3+5+\dots+2n-1-(2+4+6+\dots+2n)}{\sqrt{n^2+1}+\sqrt{4n^2-1}}$$

$$=\lim_{n\to\infty} \frac{n^2-n(n+1)}{\sqrt{n^2+1}+\sqrt{4n^2-1}} = \lim_{n\to\infty} \frac{n^2-n^2-n}{\sqrt{n^2\left(1+\frac{1}{n^2}\right)}+\sqrt{n^2\left(4-\frac{1}{n^2}\right)}}$$

$$=\lim_{n\to\infty} \frac{-n}{\underbrace{|n|}_{n}\left(\sqrt{1+\underbrace{\tfrac{1}{n^2}}_{0}}+\sqrt{4-\underbrace{\tfrac{1}{n^2}}_{0}}\right)} = \lim_{n\to\infty} \frac{-\cancel{n}}{\cancel{n}\left(\sqrt{1+\underbrace{\tfrac{1}{n^2}}_{0}}+\sqrt{4-\underbrace{\tfrac{1}{n^2}}_{0}}\right)} = \frac{-1}{1+2} = -\frac{1}{3}$$

2. $$\lim_{x\to 0} \frac{(1+x)^{1/3}-(1-x)^{1/3}}{x} = ?$$

$$a^3-b^3 = (a-b)(a^2+ab+b^2) \Rightarrow (a-b) = \frac{a^3-b^3}{(a^2+ab+b^2)}$$

$$(1+x)^{1/3}-(1-x)^{1/3} = \frac{\left[(1+x)^{1/3}\right]^3-\left[(1-x)^{1/3}\right]^3}{(1+x)^{2/3}+(1+x)^{1/3}(1-x)^{1/3}+(1-x)^{2/3}}$$

$$\lim_{x\to 0} \frac{(1+x)^{1/3}-(1-x)^{1/3}}{x} = \lim_{x\to 0} \frac{(1+x)-(1-x)}{x\left[(1+x)^{2/3}+(1+x)^{1/3}(1-x)^{1/3}+(1-x)^{2/3}\right]}$$

$$=\lim_{x\to 0} \frac{2\cancel{x}}{\cancel{x}\left[(1+x)^{2/3}+(1+x)^{1/3}(1-x)^{1/3}+(1-x)^{2/3}\right]} = \frac{2}{1+1+1} = \frac{2}{3}$$

3. $\lim_{x\to 0} \dfrac{\sqrt{1-\cos 2x}}{\sqrt{2}\,x} = ?$

$\cos 2x = \cos^2 x - \sin^2 x$ $\qquad \sin^2 x + \cos^2 x = 1$

$$\lim_{x\to 0} \frac{\sqrt{1-\cos^2 x+\sin^2 x}}{\sqrt{2}\,x} = \lim_{x\to 0} \frac{\sqrt{\sin^2 x+\sin^2 x}}{\sqrt{2}\,x}$$

$$= \lim_{x\to 0} \frac{\cancel{\sqrt{2}}\,|\sin x|}{\cancel{\sqrt{2}}\,x} \nearrow \lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

$$\searrow \lim_{x\to 0^-} \frac{|\sin x|}{x} = \lim_{h\to 0} \frac{|\sin(0-h)|}{(0-h)} = \lim_{h\to 0} \frac{|-\sin h|}{-h} = -1$$

$$\left.\begin{array}{l} \lim\limits_{x\to 0^+} \dfrac{|\sin x|}{x} = 1 \\ \lim\limits_{x\to 0^-} \dfrac{|\sin x|}{x} = -1 \end{array}\right\} \neq \text{ olduğundan limit mevcut değildir.}$$

4. $\lim_{x\to -\infty} \left[\dfrac{x^4 \sin(1/x) + x^2}{1+|x|^3}\right] = ?$

$$\lim_{x\to -\infty} \frac{\cancel{x^3}\left[\dfrac{\sin\frac{1}{x}}{\frac{1}{x}} + \dfrac{1}{x}\right]}{\cancel{x^3}\left[\dfrac{1}{x^3} - 1\right]} = \lim_{x\to -\infty} \frac{\overbrace{\dfrac{\sin\frac{1}{x}}{\frac{1}{x}}}^{1} + \cancel{\dfrac{1}{x}}^{0}}{\underbrace{\cancel{\dfrac{1}{x^3}}}_{0} - 1} = \frac{1}{-1} = -1$$

5. $\lim_{x\to \infty} \left(\dfrac{3x^2+2x+1}{x^2+x+2}\right)^{\frac{6x+1}{3x+2}} = ?$

$$\lim_{x\to \infty} \left[\frac{\cancel{x^2}\left(3+\cancel{\frac{2}{x}}^{0}+\cancel{\frac{1}{x^2}}^{0}\right)}{\cancel{x^2}\left(1+\underbrace{\cancel{\frac{1}{x}}}_{0}+\underbrace{\cancel{\frac{2}{x^2}}}_{0}\right)}\right]^{\frac{\cancel{x}\left(6+\cancel{\frac{1}{x}}^{0}\right)}{\cancel{x}\left(3+\underbrace{\cancel{\frac{2}{x}}}_{0}\right)}}$$

$$= \left(\frac{3}{1}\right)^{\frac{6}{3}} = 3^2 = 9$$

6. $\lim_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right]=?$

$$\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)\cdot\frac{\left(\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}\right)}{\left(\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}\right)}$$

$$=\lim_{x\to\infty}\frac{\cancel{x}+\sqrt{x+\sqrt{x}}-\cancel{x}}{\left(\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}\right)}=\lim_{x\to\infty}\frac{\cancel{\sqrt{x}}\sqrt{1+\underbrace{\cancel{\frac{1}{\sqrt{x}}}}_{0}}}{\cancel{\sqrt{x}}\left(\sqrt{1+\underbrace{\cancel{\frac{1}{\sqrt{x}}}}_{0}\sqrt{1+\underbrace{\cancel{\frac{1}{\sqrt{x}}}}_{0}}}+1\right)}=\frac{1}{1+1}=\frac{1}{2}$$

7. $f(x)=4x-5$ fonksiyonu için

eğer $0<|x-3|<\delta$ ise $|f(x)-7|<\varepsilon$ koşulunu sağlayan

$\delta>0$ sayısını $\varepsilon>0$ sayısına bağlı olarak bulunuz.

$$|4x-5-7|=|4x-12|=|4(x-3)|$$
$$=4|x-3|<\varepsilon$$

$|x-3|<\delta$

$$\Rightarrow |x-3|<\frac{\varepsilon}{4}=\delta$$

$\Rightarrow \boxed{\delta=\frac{\varepsilon}{4}}$ seçilebilir.

8. $\lim_{x\to\infty}\frac{x+\cos x}{x+\sin x}=?$

$$\lim_{x\to\infty}\frac{\cancel{x}\left(1+\frac{\cos x}{x}\right)}{\cancel{x}\left(1+\frac{\sin x}{x}\right)}=\lim_{x\to\infty}\frac{1+\overbrace{\frac{\cos x}{x}}^{0}}{1+\underbrace{\frac{\sin x}{x}}_{0}}$$

$\begin{pmatrix}-1\leq\cos x\leq 1\\ -1\leq\sin x\leq 1\end{pmatrix}$

$$=\frac{1+0}{1+0}=1$$

9. $\lim\limits_{x\to 2} \dfrac{2^x+2^{3-x}-6}{2^{-\frac{x}{2}}-2^{1-x}} = ?$

$$\lim_{x\to 2} \frac{2^x+2^3.2^{-x}-6}{\frac{1}{2^{\frac{x}{2}}}-\frac{2}{2^x}} = \lim_{x\to 2} \frac{2^x+2^3.2^{-x}-6}{\frac{1}{2^x}\cdot\left(2^{\frac{x}{2}}-2\right)}$$

$$= \lim_{x\to 2} \frac{2^{2x}-6.2^x+8}{2^{\frac{x}{2}}-2} \qquad \begin{pmatrix} 2^x = t \\ x\to 2 \Rightarrow t\to 4 \end{pmatrix}$$

$$= \lim_{t\to 4} \frac{t^2-6t+8}{\sqrt{t}-2} = \lim_{t\to 4} \frac{(t-4)(t-2)}{\sqrt{t}-2} = \lim_{t\to 4} \frac{\cancel{(\sqrt{t}-2)}(\sqrt{t}+2)(t-2)}{\cancel{(\sqrt{t}-2)}}$$

$$= 4.2 = 8$$

10. $0<k<1$ olmak üzere $\lim\limits_{n\to\infty} \dfrac{n^k \sin^2(n!)}{n+2} = ?$

$$\lim_{n\to\infty} \frac{n^k \sin^2(n!)}{n\left(1+\frac{2}{n}\right)} = \lim_{n\to\infty} \frac{\overbrace{\sin^2(n!)}^{\text{sonlu bir sayı } (0\le A\le 1)}}{\underbrace{n^{1-k}}_{\infty}\left(1+\cancel{\frac{2}{n}}^{\,0}\right)} = 0$$

$$\left[\begin{array}{l} 0\le \sin^2(n!) \le 1 \\ 1-k>0 \\ n\to\infty \Rightarrow n^{1-k}\to\infty \end{array}\right]$$

11. $\lim\limits_{x\to\infty} \dfrac{2\sqrt{x}+3\sqrt[3]{x}+5\sqrt[5]{x}}{\sqrt{3x-2}+\sqrt[3]{2x-3}} = ?$

$$\lim_{x\to\infty} \frac{\cancel{\sqrt{x}}\left(2+3.x^{-1/6}+5.x^{-3/10}\right)}{\cancel{\sqrt{x}}\left(\sqrt{3-\frac{2}{x}}+x^{-1/6}.\sqrt[3]{2-\frac{3}{x}}\right)}$$

$$= \lim_{x\to\infty} \frac{2+\cancel{\frac{3}{x^{1/6}}}^{\,0}+\cancel{\frac{5}{x^{3/10}}}^{\,0}}{\sqrt{3-\cancel{\frac{2}{x}}^{\,0}}+\cancel{\frac{1}{x^{1/6}}}^{\,0}.\sqrt[3]{2-\cancel{\frac{3}{x}}^{\,0}}} = \frac{2}{\sqrt{3}}$$

12. $\lim\limits_{n\to\infty} \dfrac{n!}{(n+1)!-n!} = ?$

$$\lim_{n\to\infty} \frac{n!}{n!(n+1)-n!} = \lim_{n\to\infty} \frac{\cancel{n!}}{\cancel{n!}(n+\cancel{1}-\cancel{1})} = \lim_{n\to\infty} \frac{1}{n} = 0$$

13. $\lim\limits_{n\to\infty} \dfrac{\sqrt[4]{n^5+2}-\sqrt[3]{n^2+1}}{\sqrt[5]{n^4+2}-\sqrt{n^3+1}} = ?$

$$\lim_{n\to\infty} \frac{n^{\frac{5}{4}}\left(\sqrt[4]{1+\frac{2}{n^5}}\right)-n^{2/3}\left(\sqrt[3]{1+\frac{1}{n^2}}\right)}{n^{4/5}\left(\sqrt[5]{1+\frac{2}{n^4}}\right)-n^{3/2}\sqrt{1+\frac{1}{n^3}}}$$

(en yüksek dereceli terim $n^{3/2}$)

$$= \lim_{n\to\infty} \frac{\cancel{n^{3/2}}\left[\underset{0}{n^{-1/6}}\sqrt[4]{1+\underset{0}{\cancel{\tfrac{2}{n^5}}}}-\underset{0}{n^{-5/6}}\sqrt[3]{1+\underset{0}{\cancel{\tfrac{1}{n^2}}}}\right]}{\cancel{n^{3/2}}\left[\underset{0}{n^{-7/10}}\sqrt[5]{1+\underset{0}{\cancel{\tfrac{2}{n^4}}}}-\sqrt{1+\underset{0}{\cancel{\tfrac{1}{n^3}}}}\right]} = \frac{0-0}{0-1} = 0$$

14. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+1}-\sqrt[3]{x^2+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4-1}} = ?$

$$\lim_{x\to\infty} \frac{x\sqrt{1+\underset{0}{\cancel{\tfrac{1}{x^2}}}}-x^{2/3}\sqrt[3]{1+\underset{0}{\cancel{\tfrac{1}{x^2}}}}}{x\sqrt[4]{1+\underset{0}{\cancel{\tfrac{1}{x^4}}}}-x^{4/5}\sqrt[5]{1-\underset{0}{\cancel{\tfrac{1}{x^4}}}}} = \frac{1-0}{1-0} = 1$$

15. $\lim\limits_{x\to 1} \dfrac{\sqrt{1-\cos 2(x-1)}}{x-1} = ?$

$x-1=t$

$x\to 1 \Rightarrow t\to 0$

$$\lim_{t\to 0} \frac{\sqrt{1-\cos 2t}}{t} = \lim_{t\to 0} \frac{\sqrt{1-\cos^2 t+\sin^2 t}}{t} = \lim_{t\to 0} \frac{\sqrt{2}\,|\sin t|}{t}$$

$\lim\limits_{t\to 0} \dfrac{\sqrt{2}\,|\sin t|}{t}$

$\lim\limits_{t\to 0^+} \sqrt{2}\dfrac{\sin t}{t} = \sqrt{2}$

$\lim\limits_{t\to 0^-} -\sqrt{2}\dfrac{\sin t}{t} = -\sqrt{2}$

$\neq$ olduğundan limit mevcut değildir.

16. $\lim\limits_{t\to 0} \dfrac{\sin^2 t}{(1+\cos t)\,t} = ?$

$$\lim_{t\to 0} \frac{\sin^2 t}{t^2} \cdot \frac{t}{1+\cos t} = \lim_{t\to 0} \underbrace{\left(\frac{\sin t}{t}\right)^2}_{1} \cdot \underbrace{\frac{t}{1+\cos t}}_{0} = 1.0 = 0$$

17. $\lim\limits_{x\to +\infty} \dfrac{\sin x}{x} = ?$

Her $x \in \mathbb{R}$ $\quad -1 \leq \sin x \leq 1$

Her $x > 0$ için $\quad -\dfrac{1}{x} \leq \dfrac{\sin x}{x} \leq \dfrac{1}{x}$

$\lim\limits_{x\to \infty} -\dfrac{1}{x} = 0$, $\lim\limits_{x\to \infty} \dfrac{1}{x} = 0$ ve Sıkıştırma teoreminden,

$\lim\limits_{x\to \infty} \dfrac{\sin x}{x} = 0$ dır.