Spring 2017

BLM2502 Theory of Computation

» Book

- Michael Sipser, Introduction to the Theory of Computation (3E), Thomson
- No handouts, its your responsibily to take notes.

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- In this Theory of Computation course we will try to answer the following questions:
- What are the mathematical properties of computer hardware and software?
- What is a computation and what is an algorithm? Can we give rigorous mathematical definitions of these notions?
- What are the limitations of computers? Can "everything" be computed? (As we will see, the answer to this question is "no".)
- **Purpose of the Theory of Computation:**

Develop formal mathematical models of computation that reflect real-world computers.

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- Complexity Theory
- The main question asked in this area is "What makes some problems computationally hard and other problems easy?"
- Informally, a problem is called "easy", if it is efficiently solvable. Examples of "easy" problems are
  - Sorting a sequence of, e.g, 1,000,000 numbers,
  - Searching for a name in a telephone directory
  - Computing the fastest way to drive from Esenler to Beşiktaş.

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- Complexity Theory
- On the other hand, a problem is called "hard", if it cannot be solved efficiently, or if we don't know whether it can be solved efficiently. Examples of "hard" problems are
  - Time table scheduling for all courses
  - Factoring a 300-digit integer into its prime factors
  - Computing a layout for chips in VLSI.
- **Central Question in Complexity Theory:**

Classify problems according to their degree of "difficulty". Give a rigorous proof that problems that seem to be "hard" are really "hard".

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- Computability Theory
- In the 1930’s, scientists discovered that some of the fundamental mathematical problems cannot be solved by a “computer”. (computers were invented in 1940s). For example: "Is an arbitrary mathematical statement true or false?”
- To attack such a problem, we need formal definitions of the notions of
  - computer
  - algorithm
  - computation

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- Computability Theory
- The theoretical models that were proposed in order to understand solvable and unsolvable problems led to the development of real computers.
- Central Question in Computability Theory:

  Classify problems as being solvable or unsolvable.

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- Automata Theory
- Automata Theory deals with definitions and properties of different types of “computation models”. Examples :
  - Finite Automata. These are used in text processing, compilers, and hardware design.
  - Context-Free Grammars. These are used to define programming languages and in Artificial Intelligence.
  - Turing Machines. These form a simple abstract model of a “real” computer, such as your PC at home.
- Central Question in Automata Theory:

  Do these models have the same power, or can one model solve more problems than the other?

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- Set Theory
- A set is a collection of well-defined objects.
  - The set of **natural numbers** is **N** = {1, 2, 3, . . .}.
  - The set of **integers** is **Z** = {. . . ,−3,−2,−1, 0, 1, 2, 3, . . .}.
  - The set of **rational numbers** is **Q** = {m/n : m $\in$ **Z**, n $\in$ **Z**, n $\neq$ 0}.
    - What is irrational numbers?
  - The set of **real numbers** is denoted by **R**.
  - If A and B are sets, then A is a **subset** of B, written as A $\subseteq$ B, if every element of A is also an element of B.
  - If B is a set, then the **power set** *P*(B) of B is defined to be the set of all subsets of B:
    - *P*(B) = {A : A $\subseteq$ B}.
    - Observe $\phi \in$ P(B) and B $\in$ P(B).

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- Set Theory
  - If A and B are two sets, then
    - their union is defined as
      - $A \cup B = \{x : x \in A \text{ or } x \in B\}$,
    - their intersection is defined as
      - $A \cap B = \{x : x \in A \text{ and } x \in B\}$,
    - their difference is defined as
      - $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$,
    - the Cartesian product of A and B is defined as
      - $A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$,
    - the complement of A is defined as
      - $\bar{A} = \{x : x \notin A\}$.

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» Set Theory

- A **binary relation** on two sets A and B is a subset of $A \times B$.
- A **function** f from A to B, denoted by $f : A \rightarrow B$, is a binary relation *R*, having the property that for each element $a \in A$, there is exactly one ordered pair in *R*, whose first component is a. We will also say that $f(a) = b$, or f maps a to b, or the image of a under f is b. The set A is called the **domain** of f, and the set
  $\{b \in B : \text{there is an } a \in A \text{ with } f(a) = b\}$ is called the **range** of f.
- A binary relation $R \subseteq A \times A$ is an **equivalence relation**, if it satisfies the following three conditions:
  - R is **reflexive**: For every element in $a \in A$, we have $(a, a) \in R$.
  - R is **symmetric**: For all a and b in A, if $(a, b) \in R$, then $(b, a) \in R$.
  - R is **transitive**: For all a, b, and c in A, if $(a, b) \in R$ and $(b, c) \in R$, then also $(a, c) \in R$.

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- Boolean Logic
- The Boolean values are 1 and 0, that represent true and false, respectively. The basic Boolean operations:
- **negation** (or NOT), represented by $\neg$,
- **conjunction** (or AND), represented by Λ,
- **disjunction** (or OR), represented by V,
- **exclusive-or** (or XOR), represented by $\oplus$,
- **equivalence**, represented by $\leftrightarrow$ or $\Leftrightarrow$,
- **implication**, represented by $\rightarrow$ or $\Rightarrow$. ,

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» Truth Table (0=false, 1=true)

NOT	AND	OR	XOR	equivalence	implication
$\neg\ 0 = 1$	$0 \wedge 0 = 0$	$0 \vee 0 = 0$	$0 \oplus 0 = 0$	$0 \Leftrightarrow 0 = 1$	$0 \Rightarrow 0 = 1$
$\neg\ 1 = 0$	$0 \wedge 1 = 0$	$0 \vee 1 = 1$	$0 \oplus 1 = 1$	$0 \Leftrightarrow 1 = 0$	$0 \Rightarrow 1 = 1$
	$1 \wedge 0 = 0$	$1 \vee 0 = 1$	$1 \oplus 0 = 1$	$1 \Leftrightarrow 0 = 0$	$1 \Rightarrow 0 = 0$
	$1 \wedge 1 = 1$	$1 \vee 1 = 1$	$1 \oplus 1 = 0$	$1 \Leftrightarrow 1 = 1$	$1 \Rightarrow 1 = 1$

» Implication (if … then …)

- antecedent (condition)->consequence (promise)

» E.g.

- p: “you take out the trash”.
- q:”you get a dollar”
- p=>q is false only if you take out the trash but don’t get a dolar.

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- Proof Techniques
- In mathematics, a **theorem** is a statement that is true. A **proof** is a sequence of mathematical statements that form an argument to show that a theorem is true.
  - **Axioms**: assumptions about the underlying mathematical structures
  - **Hypotheses**: a supposition or proposed explanation made based on limited evidence as a starting point for further investigation.
  - **Theorem**: described above
  - **Lemmas**: previously proved theorems
  - **Corollaries**: Special cases of theorem
- There is no specified way of producing a proof, but there are some generic strategies that could be of help.

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- Proof Techniques
- Tips:
  - **Read** and completely understand the statement of the theorem to be proved. Most often this is the hardest part. **Rewrite** the statement in your own words.
  - Sometimes, theorems contain theorems inside them. For example, “Property A if and only if property B”, requires showing **two statements**:
    - (a) If property A is true, then property B is true ($A \rightarrow B$).
    - (b) If property B is true, then property A is true ($B \rightarrow A$).
  - Another example is the theorem “Set A equals set B.” To prove this, we need to prove that $A \subseteq B$ and $B \subseteq A$. That is, we need to show that each element of set A is in set B, and that each element of set B is in set A.

- Proof Techniques
- Tips:
  - Try to **work out a few simple cases** of the theorem just to get a grip on it (i.e., crack a few simple cases first).
  - Try to **write down the proof** once you have it. This is to ensure the correctness of your proof. Often, mistakes are found at the time of writing.
  - Finding proofs takes time, we do not come prewired to produce proofs. **Be patient**, think, express and write clearly and try to be precise as much as possible.

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- Proof Techniques
  - Direct Proofs or Constructive Proofs or Proof by Construction
  - Nonconstructive Proofs
  - Proofs by Contradiction
  - Proofs by Induction
  - Pigeon Principle

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- Proof Techniques - **Direct Proofs**
- As the name suggests, in a direct proof of a theorem, we just approach the theorem directly.
- **Theorem:** If n is an odd positive integer, then n^2 is odd as well.
- **Proof:** An odd positive integer n can be written as: $n = 2k + 1$, for some integer $k \geq 0$. Then:
  - $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.
  - Since $2(2k^2 + 2k)$ is even, and "even plus one is odd", we can conclude that
  - n^2 is odd.

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» Proof Techniques - Constructive Proofs

(Proof By Construction)

» Many theorems state that a particular type of object exists

» One way to prove is to find a way to construct one such object

» This technique is called proof by construction

» **Theorem:** There exists a rational number p which can be expressed as a^b, with a and b both irrational.

A *constructive* proof of the above theorem on irrational powers of irrationals would give an actual example, such as:

$$a = \sqrt{2}\,, \quad b = \log_2 9\,, \quad a^b = 3\,.$$

The square root of 2 is irrational, and 3 is rational. $\log_2 9$ is also irrational: if it were equal to $\frac{m}{n}$, then, by the properties of logarithms, 9^n would be equal to 2^m, but the former is odd, and the latter is even.

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- Proof Techniques - **Proof by Contradiction**
- The proof by contradiction is grounded in the fact that **any proposition must be either true or false, but not both true and false at the same time**.
- One common way to prove a theorem is to assume that the theorem is false, and then show that this assumption leads to an obviously false consequence (also called a contradiction)
- This type of reasoning is used frequently in everyday life.

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- Proof Techniques - **Proof by Contradiction**
- Let us define a number is rational if it can be expressed as p/q where p and q are integers; if it cannot, then the number is called irrational.
- **Theorem:** $\sqrt{2}$ (the square root of 2) is irrational.
- **Proof:** Assume that $\sqrt{2}$ is rational. Then, it can be written as p/q for some positive integers p and q such that **p and q does not have a common factor**.
- Then, we have $p^2/q^2 = 2$, or $2q^2 = p^2$
- (continued in next page)

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» Proof Techniques - **Proof by Contradiction**

- » Since $2q^2$ is an even number, p^2 is also an even number
- » This implies that **p is an even number** (why?)
- » So, $p = 2r$ for some integer r, and so, $2q^2 = p^2 = (2r)^2 = 4r^2$
- » This implies $2r^2 = q^2$
- » So, **q is an even number**
- » Something wrong happens!.. We now have:
- » "p and q does not have common factor"
- » AND
- » "**p and q have common factor**"
- » **This is a contradiction**
- » Thus, the assumption is wrong, so that $\sqrt{2}$ is irrational

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- Proof Techniques - **Proof by Induction**
- For each positive integer n, let P(n) be a mathematical statement that depends on n. Assume we wish to prove that P(n) is true for all positive integers n. A proof by induction of such a statement is carried out as follows:
- **Basis**: Prove that P(1) is true.
- **Induction step**: Prove that for all $n \geq 1$, the following holds: If P(n) is true, then P(n + 1) is also true.
- In the induction step, we choose an arbitrary integer n 1 and assume that P(n) is true; this is called the induction hypothesis. Then we prove that P(n + 1) is also true.

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- Proof Techniques - **Proof by Induction**
- **Theorem:** For all positive integers n, we have

  $1 + 2 + 3 + . . . + n = n(n + 1)/2$
- **Proof:** Start with the basis of the induction. If $n = 1$, then the left-hand side is equal to 1, and so is the right-hand side. So the theorem is true for $n = 1$.
- For the induction step, let $n \geq 1$ and assume that the theorem is true for n, i.e., assume that

  $1 + 2 + 3 + . . . + n = n(n + 1)/2$
- We have to prove that the theorem is true for $n + 1$

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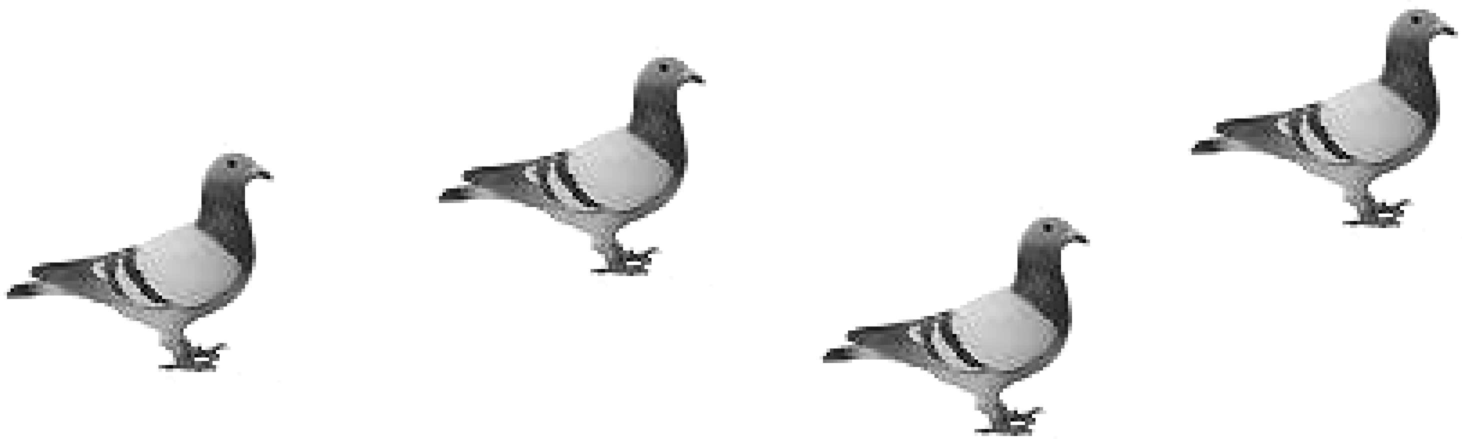
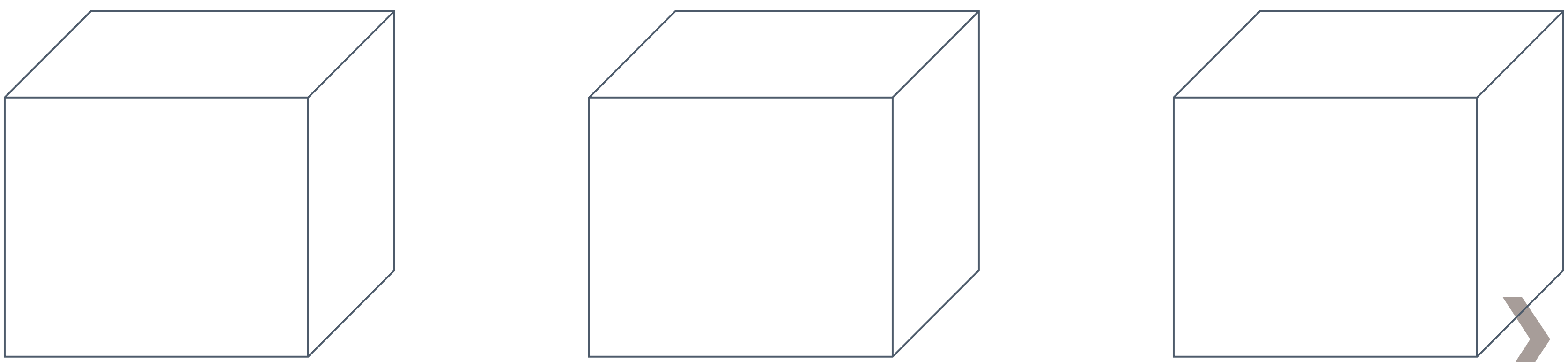
- Proof Techniques – **Pigeonhole Principle**
- If n + 1 or more objects are placed into n boxes, then there is at least one box containing two or more objects.
- In other words, if A and B are two sets such that $|A| > |B|$, then there is no one-to-one function from A to B.

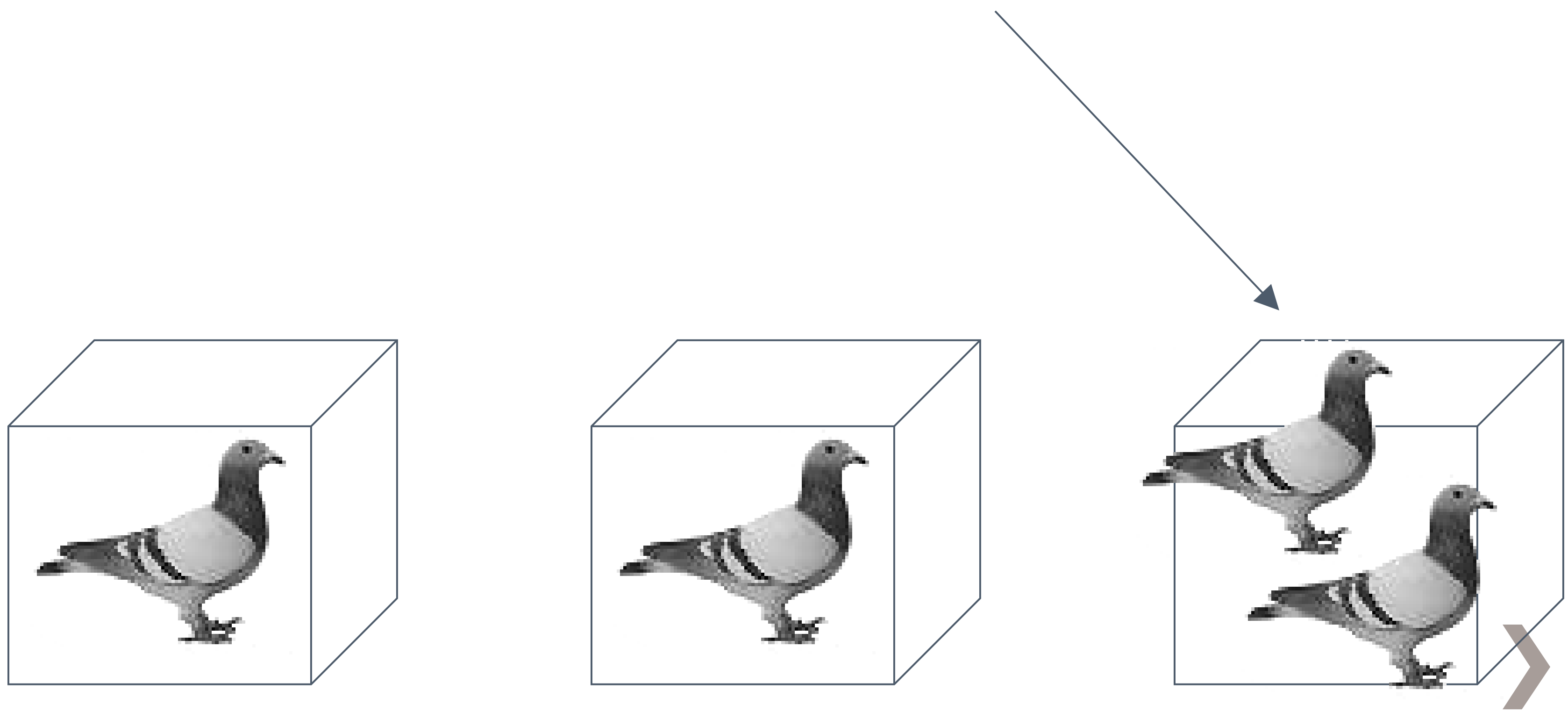
pigeons



pigeonholes



A pigeonhole must
contain at least two pigeons



- Proof techniques: Pigeonhole Principle.
- Example: Prove that if seven distinct numbers are selected from{1,2,...,11},then some two of these numbers sum to 12
- All numbers from 1 to 11 can be put into following **6 pigeonholes**: {1,11},{2,10},{3,9},{4,8},{5,7},{6}.
- We select **7** distinct numbers (**pigeons**). First 6 pigeon can be put to different pigeonholes. But after that we have to put to an existing pigeonhole. The pigeonhole of 6 can hold only one pigeon ☺.

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- Proof Techniques – Pigeonhole Principle
- **Theorem:** Let n be a positive integer. Every sequence of $n^2 + 1$ distinct natural numbers contains a subsequence of length $n + 1$ that is either increasing or decreasing.
- **Proof:** For example consider the sequence

  (8,**11**,**9**,1,4,**6**,12,10,**5**,7)

of $10 = 3^2 + 1$ numbers. This sequence contains a decreasing subsequence of length $4 = 3 + 1$, shown. There are other subsequences of length 4, too.

Proof: Let $a_1, a_2, \ldots, a_{n^2+1}$ be a sequence of $n^2 + 1$ distinct real numbers. Associate an ordered pair with each term of the sequence, namely, associate (i_k, d_k) to the term a_k, where i_k is the length of the longest increasing subsequence starting at a_k, and d_k is the length of the longest decreasing subsequence starting at a_k.

Suppose that there are no increasing or decreasing subsequences of length $n + 1$. Then i_k and d_k are both positive integers less than or equal to n, for $k = 1, 2, \ldots, n^2 + 1$. Hence, by the product rule there are n^2 possible ordered pairs for (i_k, d_k). By the pigeonhole principle, two of these $n^2 + 1$ ordered pairs are equal. In other words, there exist terms a_s and a_t, with $s < t$ such that $i_s = i_t$ and $d_s = d_t$. We will show that this is impossible. Because the terms of the sequence are distinct, either $a_s < a_t$ or $a_s > a_t$. If $a_s < a_t$, then, because $i_s = i_t$, an increasing subsequence of length $i_t + 1$ can be built starting at a_s, by taking a_s followed by an increasing subsequence of length i_t beginning at a_t. This is a contradiction. Similarly, if $a_s > a_t$, the same reasoning shows that d_s must be greater than d_t, which is a contradiction. ◁

Sequence=(8,11,9,1,4,6,12,10,5,7)

If there are no sequences of length , there are only pairs. Then since we have distinct numbers. At least 2 pairs are equal by pigeonhole principle. But this is not possible as numbers are **distinct**.