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The Chromatic Polynomials and its Algebraic Properties

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Abstract: *This paper studies various results on chromatic polynomials of graphs. We obtain results on the roots of chromatic polynomials of planar graphs. The main results are chromatic polynomial of a graph is polynomial in integer and the leading coefficient of chromatic polynomial of a graph of order n and size m is one, whose coefficient alternate in sign. Mathematics subject classification 2000: 05CXX, 05C15, 05C30, 05C75*

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1. INTRODUCTION

During the period that the Four Color Problem was unsolved, which spanned more than a century, many approaches were introduced with the hopes that they would lead to a solution of this famous problem. In 1912 George David Birkhoff **[5]** defined a function $P(M, \lambda)$ that gives the number of proper λ-colorings of a map M for a positive integer λ. As we will see, $P(M, \lambda)$ is a polynomial in λ for every map M and is called the chromatic polynomial of M. Consequently, if it could be verified that $P(M, 4)>0$ for every map M, then this would have established the truth of the Four Color Conjecture.

In 1932 Hassler Whitney **[14]** expanded the study of chromatic polynomials from maps to graphs. While Whitney obtained a number of results on chromatic polynomials of graphs, this did not contribute to a proof of the Four Color Conjecture. Renewed interest in chromatic polynomials of graphs occurred in 1968 when Ronald C. Read **[13]** wrote a survey paper on chromatic polynomials.

2. PRELIMINARIES

2.1. Definition: For a graph G and a positive integer λ the number of different proper λ-colorings of G is denoted by $P(G, \lambda)$ and is called the **Chromatic Polynomial** of G. Two λ-colorings c and c' of G from the same set $\{1, 2, \ldots, \lambda\}$ of λ colors are considered different if $c(v) \neq c'(v)$ for some vertex v of G. Obviously, if $\lambda < \chi(G)$, then $P(G, \lambda)=0$. By convention, $P(G, 0)=0$. Indeed, we have the following.

2.2. Proposition: Let G be a graph. Then $\chi(G)=k$ if and only if k is the smallest positive integer for which $P(G, k) > 0$.

As an example, we determine the number of ways that the vertices of the graph G of Figure.1 can be colored from the set $\{1, 2, 3, 4, 5\}$. The vertex v can be assigned any of these 5 colors, while w can be assigned any color other than the color assigned to v. That is, w can be assigned any of the 4 remaining colors. Both u and t can be assigned any of the 3 colors not used for v and w. Therefore, the number $P(G, 5)$ of 5-colorings of G is 5.4.9 = 180. More generally, $P(G, \lambda) = \lambda(\lambda - 1)(\lambda - 2)^2$ for every integer λ.

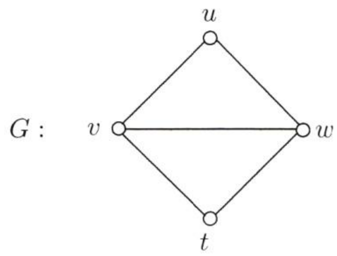


Figure 1. *A graph G with* $P(G,\lambda) = \lambda(\lambda-1)(\lambda-2)^2$

There are some classes of graphs G for which $P(G, \lambda)$ can be easily computed.

3. Chromatic Polynomials

3.1.Theorem: For every positive integer λ

(a) $P(K_n, \lambda) = \lambda(\lambda-1)(\lambda-2)....(\lambda-n+1) = \lambda^{(n)}$

(b) $P(K_n, \lambda) = \lambda^n$

Proof: In particular, if $\lambda \geq n$ in Theorem 2(a), then

$$P(K_n, \lambda) = \lambda^{(n)} = \frac{\lambda!}{(\lambda-n)!}$$

We now determine the chromatic polynomial of C_4 in Figure 2. There are λ choices for the color of v_1. The vertices v_2 and v_4 must be assigned colors different from the assigned to v_1. The vertices v_2 and v_4 may be assigned the same color or may be assigned different colors. If v_2 and v_4 are assigned the same color, then there are $\lambda - 1$ choices for that color. The vertex v_3 can then be assigned any color except the color assigned to v_2 and v_4. Hence the number of distinct λ-colorings of C_4 in which v_2 and v_4 are colored the same is $\lambda(\lambda - 1)^2$.

If, on the other hand, v_2 and v_4 are colored differently, then there are $\lambda - 1$ choices for v_2 and $\lambda - 2$ choices for v_4. Since v_3 can be assigned any color except the two colors assigned to v_2 and v_4, the number of λ-colorings of C_4 in which v_2

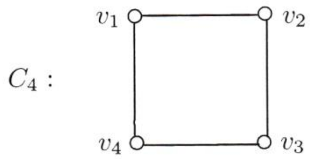


Figure 2. *The chromatic polynomial of* C_4

and v_4 are colored differently is $\lambda(\lambda - 1)(\lambda - 2)^2$. Hence the number of distinct λ-colorings of C_4 is

$$\begin{aligned} P(C_4, \lambda) &= \lambda(\lambda-1)^2 + \lambda(\lambda-1)(\lambda-2)^2 \\ &= \lambda(\lambda-1)(\lambda^2 - 3\lambda + 3) \\ &= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda \\ &= (\lambda-1)^4 + (\lambda-1) \end{aligned}$$

The preceding example illustrates an important observation. Suppose that u and v are nonadjacent vertices in a graph G. The number of λ-colorings of G equals the number of λ-colorings of G in which u and v are colored differently plus the number of λ-colorings of G in which u and v are colored the same. Since the number of λ-colorings of G in which u and v are colored differently is the number of λ-colorings of $G + uv$ while the number of λ-colorings of G in which u and v are colored the same is the number of λ-colorings of the graph H obtained by identifying u and v (an elementary homomorphism), it follows that

$$P(G, \lambda) = P(G + uv, \lambda) + P(H, \lambda)$$

This observation is summarized below by Erdos, R.J.Wilson **[8]**, chromatic index of graphs.

3.2. Theorem: Let G be a graph containing nonadjacent vertices u and v and let H be the graph obtained from G by identifying u and v. Then

$$P(G, \lambda) = P(G + uv, \lambda) + P(H, \lambda)$$

Proof: Note that if G is a graph of order $n \geq 2$ and size $m \geq 1$, then $G + uv$ has order n and size $m+1$ while H has order $n - 1$ and size at most m. The equation stated in Theorem 2.1 can also be expressed as

$$P(G + uv, \lambda) = P(G, \lambda) - P(H, \lambda)$$

In this context, Theorem 2.1 can be rephrased in terms of an edge deletion and an elementary contraction.

3.3. Corollary: Let G be a graph containing adjacent vertices u and v and let F be the graph obtained from G by identifying u and v. Then

$$P(G, \lambda) = P(G - uv, \lambda) - P(F, \lambda)$$

Proof: By systematically applying Theorem 2.1 to pairs of nonadjacent vertices in a graph G, we eventually arrive at a collection of complete graphs. We now illustrate this. Suppose that we wish to compute the chromatic polynomial of the graph G of Figure 3. For the nonadjacent vertices u and v of G and the graph H obtained by identifying u and v, it follows that by Theorem 2.1 that the chromatic polynomial of G is the sum of the chromatic polynomials of $G + uv$ and H.

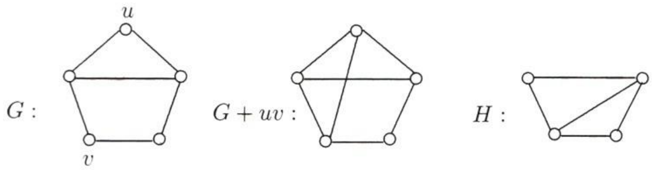
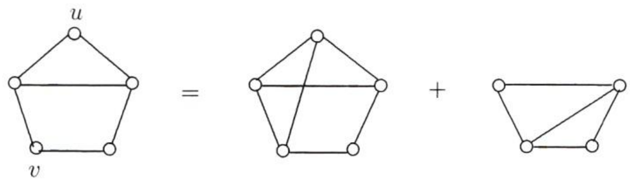


Figure 3

At this point it is useful to adopt a convention introduced by Alexander Zykov **[15]** and utilized later by Ronald Read **[9]**. Rather than repeatedly writing the equation that appears in the statement of Theorem 2.1, we represent the chromatic polynomial of a graph by a drawing of the graph and indicate on the drawing which pair u, v of nonadjacent vertices will be separately joined by an edge and identified. So, for the graph G of **Figure 3**, we have Continuing in this manner, as shown in **Figure 4**, we obtain



$$P(G, \lambda) = \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda$$

Using this approach, we see that the chromatic polynomial of every graph is the sum of chromatic polynomials of complete graphs. A consequence of this observation is the following.

3.4. Theorem: The chromatic polynomial $P(G, \lambda)$ of a graph G is a polynomial in λ.

Proof. There are some interesting properties possessed by the chromatic polynomial of every graph. In fact, if G is a graph of order n and size m, then the chromatic polynomial

$P(G, \lambda)$ of G can be expressed as

$P(G, \lambda) = c_o\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n,$

$= \lambda^{(5)} + 4\lambda^{(4)} + 3\lambda^{(3)} = \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda$

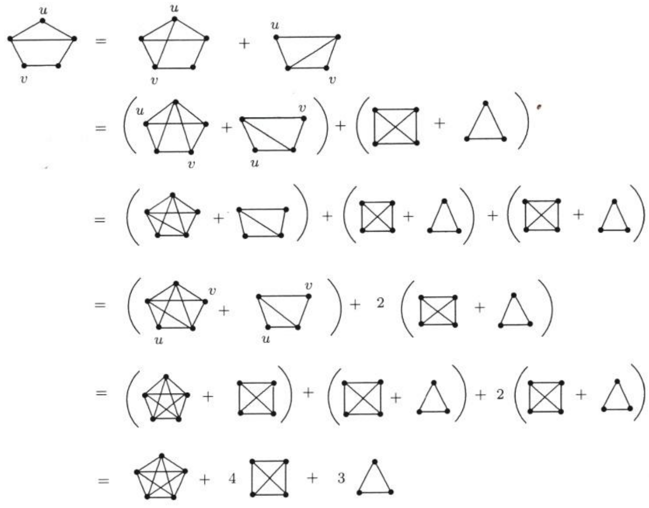


Figure 5. $P(G, \lambda) = P(G + uv, \lambda) + P(H, \lambda)$

Where $c_0 = 1(P(G, \lambda)$ is a polynomial of degree n with leading coefficient 1), $c_1 = -m$, $c_i \geq 0$ if i is even with $0 \leq i \leq n$, and $c_i \leq 0$ if i is odd with $1 \leq i \leq n$. Since $P(G, 0) = 0$, it follows that $c_n = 0$.

3.5. Theorem: Let G be a graph of order n and size m. Then $P(G, \lambda)$ is a polynomial of degree n with leading coefficient 1 such that the coefficient of λ^{n-1} is $-m$, and whose coefficients alternate in sign.

Proof: We proceed by induction on m. If $m = 0$, then $G = \overline{K_n}$ and $P(G, \lambda)\lambda^n = \lambda^n$, as we have seen. Then $P(\overline{K_n}, \lambda) = \lambda^n$ has the desired properties.

Assume that the result holds for all graphs whose size is less than m, where $m \geq 1$. Let G be a graph of order m and let $e = uv$ an edge of G. By Corollary 2.2,

$$P(G, \lambda) = P(G - e, \lambda) - P(F, \lambda),$$

Where F is the graph obtained from G by identifying u and v. Since $G - e$ has order n and size $m - 1$, it follows by the induction hypothesis that

$$P(G - e, \lambda) = a_o\lambda^n + a_1\lambda^{n-l} + a_2\lambda^{n-2} + \ldots + a_{n-1}\lambda + a_n,$$

Where $a_0 = 1$, $a_1 = -(m-1)$, $a_i \geq 0$ if i is even with $0 \leq i \leq n$, $a_i \leq 0$ if i is odd with $1 \leq i \leq n$. Furthermore, since F has order $n-1$ and size m', where $m' \leq$ m-1, it follows that

$$P(F, \lambda) = b_o\lambda^{n-1} + b_1\lambda^{n-2} + b_2\lambda^{n-3} + \ldots + b_{n-2}\lambda + b_{n-1},$$

Where $b_0 = 1$, $b_1 = -m'$, $b_i \geq 0$ if i is even with $0 \leq i \leq n-1$, and $b_i \leq 0$ if i is odd with $1 \leq i \leq n-1$. By Corollary

$$P(G,\lambda) = P(G-e, \lambda) - P(F,\lambda)$$

$$= (a_o\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_{n-1}\lambda + a_n) - (b_o\lambda^{n-1} + b_1\lambda^{n-2} + b_2\lambda^{n-3} + \ldots + b_{n-2}\lambda + b_{n-1})$$

$$= a_o\lambda^n + (a_1 - b_0)\lambda^{n-l} + (a_2 - b_1)\lambda^{n-2} + \ldots + (a_{n-1} - b_{n-2})\lambda + (a_n - b_{n-1})$$

Since $a_0 = 1$, $a_1 - b_0 = -(m-1) - 1 = -m$, $a_i - b_{i-1} \geq 0$ if i is even with $2 \leq i \leq n$, and $a_i - b_{i-1} \leq 0$ if i is odd with $0 \leq i \leq n$, $P(G,\lambda)$ has the desired properties and the theorem follows by mathematical induction.

Suppose that a graph G contains an end-vertex v whose only neighbor is u. Then, of course, $P(G\text{-}v,\lambda)$ is the number of λ-colorings of G-v. The vertex v can then be assigned any of the λ colors except the color assigned to u. This observation gives the following.

3.6. Theorem: If G is a graph containing an end-vertex v, then

$$P(G,\lambda) = (\lambda - 1)\, P(G - v,\ \lambda)$$

One consequence of this result is the following.

3.7. Corollary: If T is a tree of order $n \geq 1$, then

$$P(T,\lambda) = \lambda\,(\lambda - 1)^{n-1}$$

Proof: We proceed by induction on n. For $n = 1$, $T = K_1$ and certainly $P(K_1,\lambda) = \lambda$. Thus the basis step of the induction is true. Suppose that $P(T', \lambda) = \lambda(\lambda\text{-}1)^{n-2}$ for every tree T' of order $n\text{-}1 \geq 1$ and let T be a tree of order n. Let v be an end-vertex of T. Thus T-v is a tree of order $n\text{-}1$. By Theorem 3.2 and the induction hypothesis

$P(T,\lambda) = (\lambda - 1)P(T - v,\lambda) = (\lambda - 1)[\lambda(\lambda - 1)^{n-2}] = \lambda(\lambda - 1)^{n-1}$ as desired.

Two graphs are chromatically equivalent if they have the same chromatic polynomial.

In this section we'll summarize some well-known facts about the chromatic polynomial's coefficients, roots and substitutions as well as their relations to some graph-theoretic properties of G.

4. Coefficients of the Chromatic Polynomials

Claim 3.1: The lead coefficient of chr(G, k) is always 1.

Proof: Use the partitioning argument from theorem 2.5. The only partition that contributes to the lead coefficient is the one with n parts, giving an addend of $k(k - 1)\ldots(k - n+1)$ where the coefficient of k^n is 1.

Claim 3.2: The coefficient of k^{n-1} in chr(G,k) is the negative of the number of edges.

Proof: Apply the deletion-contraction argument. By contracting an edge we obtain a graph on $n - 1$ vertices, whose chromatic polynomial has 1 as the coefficient of k^{n-1} according to our previous claim. Therefore deleting the edge should augment the coefficient of k^{n-1} by 1, finally reaching in zero as all edges are removed. So initially it had to be the negative of the number of edges.

Claim 3.3: *The constant term, i.e. the coefficient of 1 in* chr*(G, k) is always zero.*

Proof: Substituting $k = 0$ into the chromatic polynomial yields 0 since G cannot be colored using 0 colors.

Claim 3.4: The coefficient of k in chr(G, k) is non-zero if and only if G is connected.

Proof: We know that the chromatic polynomial of a disconnected graph is the product of that of its components. If we have at least two terms, each being divisible by k, then their product is divisible by k^2, thus its coefficient of k is zero.

For connected graphs we'll prove the slightly stronger result that the coefficient of k is positive if n is odd and negative if n is even. This works by induction based on the deletion-contraction argument. We can always select an edge that is not a bridge unless the graph is a tree. otherwise both deletion and contraction gives a connected graph so we may continue the induction.

Lemma 3.5: Let c(F) denote the number of components in a spanning subgraph F. Then

$$\operatorname{chr}(G,k) = \sum_{F\subseteq E(G)} (-1)^{|F|}\, k^{c(F)}$$

Proof: The number of ways we may assign colors to $V(G)$ so that vertices connected by F-edges do share the same color is $k^{c(F)}$. This is the number of colorings that *violate* the vertex coloring condition for all edges in F. Since the chromatic polynomial counts the colorings that violate this condition for no edges in $E(G)$, the result follows from the principle of inclusion-exclusion.

Claim 3.6: The coefficients of the chromatic polynomial alternate in sign. That is, for the coefficient a_m of k^m we have $a_m \geq 0$ if $n \equiv m(2)$ and $a_m \leq 0$ otherwise.

Proof: The claim holds for the empty graph and is preserved during a deletion-contraction step.

Claim 3.7: For a connected graph G the coefficients satisfy.

$$1 = |a_n| < |a_{n-1}| < \cdots < \left|a_{\left[\frac{n}{2}\right]+1}\right|$$

Proof: We would like to show $|a_{m+1}| < |a_m|$ for m $> \frac{n}{2}$.

For trees we have chr$(T_n, k) = k(k-1)^{n-1}$ from section 1.4, so $a_m = (-1)^{n-m}\binom{n-1}{m-1}$ and thus the claim is $\binom{n-1}{m} < \binom{n-1}{m-1}$. Rearranging transforms this to $n - m < m$ which we have assumed.

Otherwise we may select a non-bridge edge as in the proof of claim 3.4 and apply deletion-contraction. Our previous claim tells us that the corresponding coefficients of chr$(G\backslash e, k)$ and chr$(G/e, k)$ have opposite signs, and therefore their absolute values add up. For the contracted graph we have

$$0 = |a'_n| < 1 = |a'_{n-1}| < |a'_{n-2}| < \cdots \left|a'_{\left[\frac{n-1}{2}\right]+1}\right|,$$

where the last index is no more than $\left[\frac{n}{2}\right] + 1$, so both $G\backslash e$ and G/e satisfy the inequalities in the claim and the final addition also preserves them.

This claim is suspected to be possibly strengthenend:

Conjecture 3.8: (unimodal conjecture). There exists some k such that

$|a_n| \leq |a_{n-1}| \leq \cdots \leq |a_{k+1}| \leq |a_k| \geq |a_{k-1}| \geq \cdots \geq |a_2| \geq |a_1|$

This claim has been verified for a few classes of graphs, but remains generally unknown. For some related results see **[12].**

5. ROOTS OF CHROMATIC POLYNOMIALS

A nonnegative integer root k of the chromatic polynomial means noncolorability with k colors by definition. It follows that $k \in \mathbb{N}$ is a root if and only if $k < \chi(G)$.

Lemma 4.1: The derivative of the chromatic polynomial satisfies $(-1)^n$ chr$'(G,1) > 0$ for any biconnected graph and ≥ 0 for any connected graph G.

Proof: For connected graphs that are not biconnected there exists a cut vertex v and we may write the chromatic polynomial chr(G, q) as a product $\frac{\mathrm{chr}(G_{1,q})\mathrm{chr}(G_{2,q})}{q}$ according to known claim. Neither G_1 nor G_2 is empty, thus both terms have a root at 1, so 1 is at least a double root of chr(G,q) and therefore its derivative also has a root at 1. It follows that the claim is satisfied with an equality.

At this point it is enough to consider biconnected graphs. For K_2 we have chr$'(K_2, 1) = 1$. Otherwise both $G\backslash e$ and G/e are connected for any edge e, so we can obtain the weaker claim by using the deletion-contraction argument. To prove the stronger one, we'll show that there exists an edge 2 for which G/e is also biconnected.

The only possible cut vertex of G/e is the contracted one, since any other vertex would also separate G. So we are looking for such an $e = ij$ that removing i and j from $V(G)$ doesn't cut G apart.

There exists a longest path in G: let i be one of its endpoints and j its neighbor. Suppose that the removal of i and j leaves a disconnected graph and pick two between the two, and they have to traverse i and j respectively. The one going through i can be used to extend the selected longest path, implying a contradiction.

Therefore if we pick e as the final segment of a longest path, G/e is also biconnected and thus the claim holds.

Claim 4.2: The chromatic polynomial has no real root greater than n – 1.

Proof: the chromatic polynomial is a sum of terms having the form $q(q-1)\dots(q-p+1)$ where $1 \le p \le n$, each of them possibly occurring multiple times. It is easy to see that such a term increases strictly mootonicalls for $q > n-1 \ge p-1$, and so does their sum as well.
Since chr(G,q) is nonnegative for $q=n-1$ and strictly increasing afterwards, it can have no root > $n-1$.

Claim 4.3: The chromatic polynomial of a graph has no negative real roots.

Proof: By claim 3.6. we have

$$chr(G,q) = \sum_{m=1}^{n} a_m q^m$$

where $a_m \ge 0$ if $n \equiv m(2)$ and $a_m \le 0$ otherwise. Thus $(-1)^n a_m q^m \ge 0$ for any $q < 0$. We also know that $a_n = 1 > 0$, and therefore $(-1)^n chr(G,q) > 0$ which implies that q cannot be a root.

Claim 4.4: The chromatic polynomial has no real roots between 0 and 1.

Proof: It suffices to deal with connected graphs . We show that $(-1)^n$ chr(G,q) < 0 for any 0 < q < 1. This statement can be easily checked for trees where chr(G,q) = $q(q-1)^{n-1}$ and otherwise it follows from the deletion-contraction property.

An extension of these claims has been proved by Jackson [13]:

Conjecture 4.5: Planar graphs have no real roots in [4, ∞).

They have already solved the weaker version that [5, ∞)contains no real roots. Since then, Appel and Haken have proved the four-color theorem which states that 4 is neither a root **[3][4].** For the remaining interval (4,5), however, the question is still wide open.

Tutte has shown that for planar graphs chr$(G, \varphi + 2) > 0$ where $\varphi = \frac{\sqrt{5}+1}{2}$ is the golden ratio **[12]**. He hoped that it takes us closer to the four-color theorem since $\varphi + 2 \approx 3.618$ is close to 4, but unfortunately there are some planar graphs having a root between the two. In fact, Royle has shown that there are roots arbitrarily close to 4 from below **[9].**

Theorem 4.6: The roots of all chromatics polynomials are dense in $\mathbb{C}$.

Despite this claim for general graphs, there exist root-free zones if we make some restrictions. These are particularly important from the point of view of statistical mechanics.

Theorem 4.7: There exists a universal constant C such that if G has maximum degree D, then all complex roots of chr(G, q) satisfy $|q| < CD$.

A similar bound $|q| < CD + 1$ exists for the second largest degre. For the third largest degree, no such claim can be made: there are arbitrarily large chromatic roots even when all except two vertices have degree 2.

6. SUBSTITUTIONS

For k ∈ $\mathbb{N}$ chr(G,k) means the number of k-coloring of G by definition. But there are some further locations where the evaluation of the chromatic polynomial is interesting.

Claim 5.1: $|\mathrm{chr}(G,-1)|$ gives the number of acyclic orientations of G.

Proof. Denote the number of acyclic orientations of G by $f(G)=(-1)^n \operatorname{chr}(G, -1)$. For the empty graph we have $f(\overline{Kn})=1$ and thus the proposition holds. Now consider a nonempty graph G with an edge e selected. Suppose we have an orientation $\overrightarrow{G_e}$ on all edges except e and we would like to find out how many ways (i.e. 0,1 or 2) we can extend it to an acyclic orientation $\vec{G}$ of G.

Notice that if there exists such an acyclic extension at all, removing the edge e won't break it. On the other hand, if we can't add e in either direction because both would close a directed path in $\overrightarrow{G_e}$ into a cycle, then these two paths make up a cycle in $\overrightarrow{G_e}$ by themselves. Therefore $\overrightarrow{G_e}$ is an acyclic orientation of $G\backslash e$ if and only if e can be added in at least one direction.

$\overrightarrow{G_e}$ specifies an orientation on G/e too. If it contains a cycle passing through the contracted point, one of the two possible orientations of e will extend this cycle into a larger one. And a cycle avoiding the contracted point will be kept intact. Thus if $\overrightarrow{G_e}$ is cyclic, pointing e in least one of the two directions will create a cyclic. If $\overrightarrow{G_e}$ however, both directions of e will result in an acyclic graph, since any cycle in $\vec{G}$ would have been preserved during the contraction.

Thus if $\overrightarrow{G_e}$ specifies an cyclic orientation on 0, 1 or 2 of the graphs $G\backslash e$ and G/e, then it can be extended to G in 0, 1 or 2 ways respectively. This argument shows that $f(G)=f(G\backslash e) + f(G/e)$.

Since the same recursion holds for $f(G)$ and $(-1)^n \operatorname{chr}(G, -1)$and they are equal for empty graphs, they have to be always equal.

Claim 5.2: $\operatorname{chr}'(G,0)$ returns the chromatic invariant $\beta(G)$.

Proof: The derivative of a polynomial at zero equals the linear coefficient. So the clam follows and the properties proved for $\beta(G)$ in section 3.1 apply.

The chromatic polynomial also exhibits interesting behavior at the so-called *Beraha numbers* $B_n = 2 + 2\cos\left(\frac{2\pi}{n}\right)$, but they are outside the scope of this study.

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