BLM2041 Signals and Systems

Week 10

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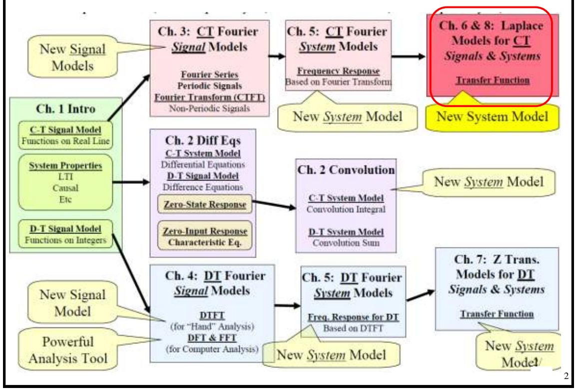
Where are we now?



What we have seen so far….

- Diff. Equations describe systems
  - Differential Eq. for CT
  - Difference Eq. for DT
- Convolution with the Impulse Response can be used to analyze the system
  - An integral for CT
  - A summation for DT
- Fourier Transform (and Series) describe what frequencies are in a signal
  - CTFT for CT has an integral form
- The Frequency Response of a system gives a multiplicative method of analysis
  - Freq. Response = CTFT of impulse response for CT system

We now look at one "power tool" for system analysis:

Laplace Transform for CT Systems

Extension of CTFT

Laplace Transform

We've seen that the FT is a useful tool for

-signal analysis (understanding signal structure)

-systems analysis/design

But only if:

1. System is in zero state
2. Impulse response satisfies $\int_{-\infty}^{\infty} |h(t)| dt < \infty$
3. Input satisfies $\int_{-\infty}^{\infty} |x(t)| dt < \infty$

Called "Absolutely Integrable"

Well… there *are* a few signals that we *can* handle with FT that *do not* satisfy this: Sinusoids and unit step are two of them

So…frequency response is a tool that can only be used under these three conditions!

The Laplace Transform is a generalization of the CTFT…
it *can* handle cases when these three conditions are *not* met.

Laplace Transform

There are two analysis methods that the Laplace Transform enables:

Zero state

LT & Transfer Function

$x(t)$ and $h(t)$ may or may not be absolutely integrable

So… this just allows us to do the same thing that the FT does… but for a larger class of signals/systems

Non zero-state

LT-based solution of differential equations

$x(t)$ and $h(t)$ may or may not be absolutely integrable

This not only admits a larger class of signals/systems… it also gives a powerful tool for solving for both the zero-state ***AND*** the zero-input solutions…
ALL AT ONCE

Laplace Transform Definition

Given a C-T signal $x(t)$ $-\infty < t < \infty$ we've already seen how to use the CTFT:

$$CTFT: X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Unfortunately the CTFT doesn't "converge" for some signals… the LT mitigates this problem by including decay in the transform:

$$e^{-j\omega t} \quad vs. \quad e^{-st} = e^{-(\sigma + j\omega)t} = e^{-\sigma t} e^{-j\omega t}$$

Controls decay of integrand

For the Laplace Transform we use: $s = \sigma + j\omega$. So… s is just a complex variable that we almost always view in rectangular form

$$CTFT: X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \Rightarrow \quad \boxed{LT: X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt}$$

Laplace Transform Definition

There are 2 types of LT: Two-sided (bilateral) / One-sided (unilateral)

Two-Sided LT

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \qquad \text{with} \quad s = \underbrace{\sigma + j\omega}_{\text{complex variable}}$$

One-Sided LT

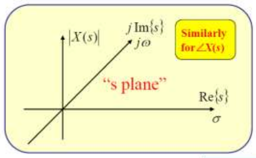
$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt \qquad \text{with} \quad s = \overbrace{\sigma + j\omega}^{\text{complex variable}}$$

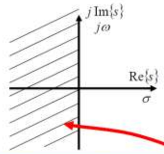
One-sided LT defined this way → even if $x(t) \neq 0,\ t < 0$

Laplace Transform Definition

Note that $X(s)$ is: $\begin{cases} \text{a complex valued function} \\ \text{of a complex variable } s = \sigma + j\omega \end{cases}$

Must plot on a plane… the "s-plane"





For the LT we will often need to keep track if s values are in the Left Half Plane (LHP)… so rectangular form of s is convenient. In LHP if $\text{Re}\{s\} < 0$.

Region of Convergence (ROC)

Set of all s values for which the integral in the LT definition converges.

Each signal has its own region of convergence.

Laplace Transform Definition

Example of Finding a LT

Consider the signal $x(t) = e^{-bt}u(t) \quad b \in \Re$

Back when we studied the FT we had to limit b to being $b > 0$… with the LT we don't need to restrict that!!!

This is a causal signal.

By definition of the LT:

$$X(s) = \int_0^{\infty} e^{-bt}e^{-st}dt = \int_0^{\infty} e^{-(s+b)t}dt$$

This is an easy integral to do!!

The limit is here by the definition of the integral

$$X(s) = \frac{-1}{s+b}\left[e^{-(s+b)t}\right]_{t=0}^{t=\infty} = \frac{-1}{s+b}\Big[\underbrace{\lim_{t\to\infty} e^{-(s+b)t}}_{\text{look at this}} - 1\Big]$$

If this limit does not converge… then we say that the integral "does not exist"

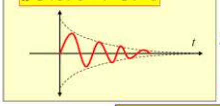
So… we need to find out under what conditions this integral exists.

So… let's look at the function inside this limit…

Laplace Transform Definition

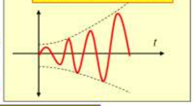
$$e^{-(s+b)t} = e^{-[(\sigma+b)+j\omega]t}$$

if $\sigma + b > 0 \Rightarrow \sigma > -b$



Has Two Main Behaviors

if $\sigma + b < 0 \Rightarrow \sigma < -b$



Thus, $\lim_{t\to\infty} e^{-(s+b)t}$ "exists" only for $\sigma > -b$

So, we can't "find" this $X(s)$ for values of s such that $\text{Re}\{s\} \le -b$

But for s with $\text{Re}\{s\} > -b$ we have no trouble. This set of s is ROC for this transform. Don't worry too much about ROC… at this level it kind of takes care of itself

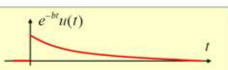
So for $x(t) = e^{-bt}u(t)$ We have $X(s) = \dfrac{1}{s+b} \quad \text{Re}\{s\} > -b$

This result… and many others… is on the Table of Laplace Transforms

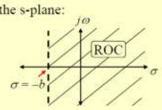
Laplace Transform Definition

If $b > 0$ then $x(t)$ itself decays:

For $b > 0$, $-b$ is negative



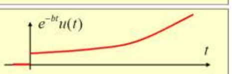
And we have on the s-plane:



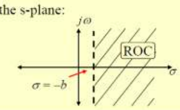
This case can be handled by the FT… and can also be handled by the LT

If $b < 0$ then $x(t)$ itself "explodes":

For $b < 0$, $-b$ is positive



And we have on the s-plane:



This case can't be handled by the FT… but by restricting our focus to values of s in the ROC, the LT can handle it!!!

Laplace Transform Definition

Connection between CTFT & LT

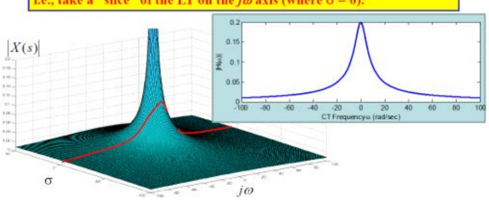
CTFT: $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ LT: $X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$

It appears that letting $\sigma = 0$ gives LT = FT…

But this is only true if ROC includes the "$j\omega$ axis"!!!

If ROC includes "$j\omega$ axis" Then the FT is "embedded" in the LT

Get the FT by taking the LT and evaluating it only on the $j\omega$ axis… i.e., take a "slice" of the LT on the $j\omega$ axis (where $\sigma = 0$).

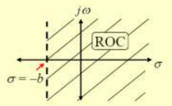


Laplace Transform Definition

Let's Revisit the Example Above

$x(t) = e^{-bt}u(t) \quad \leftrightarrow \quad X(s) = \frac{1}{s+b} \quad \mathrm{Re}\{s\} > -b$

If $b > 0$, then ROC includes the "$j\omega$ axis":



$\Rightarrow X(s)\big|_{s=j\omega} = \left[\frac{1}{s+b}\right]_{s=j\omega} = \underbrace{\left[\frac{1}{j\omega+b}\right]}_{\text{Same as on FT table}}$

Inverse Laplace Transform

Like the FT…once you know $X(s)$ you can use the inverse LT to get $x(t)$

The definition of the inverse LT is:

$\boxed{x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds}$ with c chosen such that $s = c + j\omega$ is in ROC

This is a "complex line integral" in complex s-plane… **HARD TO DO!!**

But…if $X(s) = \frac{b_M s^M + b_{M-1}s^{M-1} + \ldots + b_1 s + b_0}{a_N s^N + a_{N-1}s^{N-1} + \ldots + a_1 s + a_0}$ Ratio of polynomials in s "Rational Function"

Then its easy to find $x(t)$ using partial fraction expansion and a table of LT pairs

Properties of Bilateral Laplace Transform

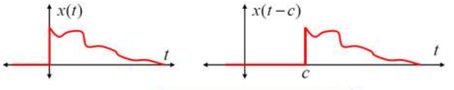
Because of the connection between FT & LT we expect these to be similar to the FT properties we already know!

There are several other properties… they are listed on the Table of Laplace Transform Properties.

Linearity: $ax(t) + by(t) \quad \leftrightarrow \quad aX(s) + bY(s)$

Time Shift :

Figures here for show causal signal (but result is general case)



$x(t-c) \leftrightarrow e^{-cs}X(s)$

Compare to time shift for FT: $e^{-j\omega c}$ vs. e^{-cs} Recall: $s = \sigma + j\omega$

Properties of Bilateral Laplace Transform

Time Differentiation: $\dot{x}(t) \leftrightarrow sX(s)$

Integration: $\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{s}X(s)$

These two properties have a nice "opposite" relationship:

Note: Differentiation $\Rightarrow$ Multiply by s

Integration $\Rightarrow$ Divide by s

"opposites" "opposites"

These two properties are crucial for linking the LT to the solution of Diff. Eq.

They are also crucial for thinking about "system block diagrams"

Properties of Bilateral Laplace Transform

Time Scaling: $x(at) \leftrightarrow \frac{1}{a}X\left(\frac{s}{a}\right)$ $a > 0$

Compare to FT

Note: $a < 0$ makes $x(at)$ non-causal So we limit to $a > 0$

Multiply by t^n: $t^n x(t) \leftrightarrow (-1)^N \frac{d^N}{ds^N}X(s)$

Multiply by Exponential: $e^{at}x(t) \leftrightarrow X(s-a)$ with a real or complex

Shift in s-plane

Multiply by sinusoid:

$x(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}[X(s+j\omega_0) - X(s-j\omega_0)]$

$x(t)\cos(\omega_0 t) \leftrightarrow \frac{j}{2}[X(s+j\omega_0) + X(s-j\omega_0)]$

Properties of Bilateral Laplace Transform

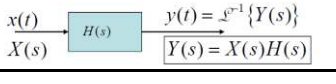
Convolution: $x(t) * h(t) \quad \leftrightarrow \quad X(s)H(s)$ **Same as for FT!**

Recall: If $h(t)$ is system impulse response then $H(\omega)$ is system Frequency Response

We'll see that $H(s)$ is system "Transfer Function"

"Transfer Function" is a generalization of **"Frequency Response"**

So we have:



Using LT to Solve Differential Equations

$$y_{total}(t) = \underbrace{y_{zs}(t)}_{} + \underbrace{y_{zi}(t)}_{}$$

We've seen how to find this using: "convolution w/ impulse response" or using "multiplication w/ frequency response"

We've seen how to find this using the characteristic equation, its roots, and the so-called "characteristic modes"

Here we'll see how to get $y_{total}(t)$ using LT… … get both parts with one tool!!!

19

First-order case: Let's see this for a **1st-order Diff. Eq.** with a **causal input** and a **non-zero initial condition** just before the causal input is applied.

The 1st-order Diff. Eq. describes: a simple RC or RL circuit.

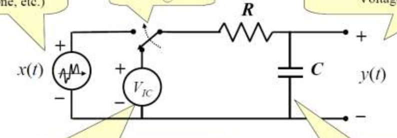
The causal input means: we switch on some input at time $t = 0$.

The initial condition means: just before we switch on the input the capacitor has a specified voltage on it (i.e., it holds some charge).

Input: Time-Varying Voltage (e.g., guitar, microphone, etc.)

switch @ t = 0

Output: Time-Varying Voltage



Assume that for $t<0$ this has been switched on for "a long time"

Thus… the cap is fully charged to V_{IC} volts

20

This circuit is then described by this Diff. Eq.: $\dfrac{dy(t)}{dt} + \dfrac{1}{RC}y(t) = \dfrac{1}{RC}x(t)$

Cap voltage… just before $x(t)$ "turns on" → With IC $y(0^-) = V_{IC}$

$x(t) = 0, \ t < 0$

For this ex. we'll solve the general 1st-order Diff. Eq.: $\dfrac{dy(t)}{dt} + ay(t) = bx(t)$

Now the **key steps** in using the LT are:

- take the LT of both sides of the Differential Equation…
- use the LT properties where appropriate…
- solve the resulting Algebraic Equation for $Y(s)$
- find the inverse LT of the resulting $Y(s)$

Laplace Transform:

Differential Equation… — Hard to solve

turns into an…

Algebraic Equation — Easy to solve

21

We now apply these steps to the 1st-order Diff. Eq.:

$$\mathcal{L}\left\{\frac{dy(t)}{dt} + ay(t)\right\} = \mathcal{L}\{bx(t)\}$$

Apply LT to both sides

$$\mathcal{L}\left\{\frac{dy(t)}{dt}\right\} + a\mathcal{L}\{y(t)\} = b\mathcal{L}\{x(t)\}$$

Use Linearity of LT

$$\left[sY(s) - y(0^-)\right] + aY(s) = bX(s)$$

Use Property for LT of Derivative… accounting for the IC

$$Y(s) = \frac{y(0^-)}{s+a} + \frac{b}{s+a}X(s)$$

Solve algebraic equation for $Y(s)$

Part of sol'n driven by IC "Zero-Input Sol'n"

Part of sol'n driven by input "Zero-State Sol'n"

Note that 1/(s+a) plays a role in both parts…

Hey! s+a is the Characteristic Polynomial!!

Now… the "hard" part is to find the inverse LT of $Y(s)$

22

Now we apply these general ideas to solving for the output of the previous RC circuit with a unit step input…. $x(t) = u(t)$

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t) \quad \Rightarrow \quad Y(s) = \frac{y(0^-)}{s+1/RC} + \left[\frac{1/RC}{s+1/RC}\right]X(s)$$

This "transfers" the input $X(s)$ to the output $Y(s)$

We'll see this later as "The Transfer Function"

Now… we need the LT of the input…

From the LT table we have: $x(t) = u(t) \quad \leftrightarrow \quad X(s) = \dfrac{1}{s}$

$$Y(s) = \frac{y(0^-)}{s+1/RC} + \left[\frac{1/RC}{(s+1/RC)}\right]\frac{1}{s}$$

Now we have "just a function of s" to which we apply the ILT…

23

So now applying the ILT we have:

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{y(0^-)}{s+1/RC} + \left[\frac{1/RC}{(s+1/RC)s}\right]\right\}$$

Apply LT to both sides

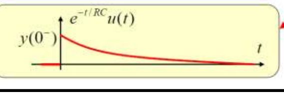
$$y(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{y(0^-)}{s+1/RC}\right\}}_{} + \underbrace{\mathcal{L}^{-1}\left\{\left[\frac{1/RC}{(s+1/RC)s}\right]\right\}}_{}$$

Linearity of LT

This part (zero-input sol'n) is easy… Just look it up on the LT Table!!

This part (zero-state sol'n) is harder… It is **NOT** on the LT Table!!

$$\mathcal{L}^{-1}\left\{\frac{y(0^-)}{s+1/RC}\right\} = y(0^-)e^{-(t/RC)}u(t)$$



So… the part of the sol'n due to the IC (zero-input sol'n) decays down from the IC voltage

24

Now let's find the other part of the solution… the zero-state sol'n… the part that is driven by the input:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{y(0^-)}{s+1/RC}\right\} + \mathcal{L}^{-1}\left\{\left[\frac{1/RC}{(s+1/RC)s}\right]\right\}$$

We can *factor* this function of s as follows:

$$\mathcal{L}^{-1}\left\{\left[\frac{1/RC}{(s+1/RC)s}\right]\right\} = \mathcal{L}^{-1}\left\{\left[\frac{1}{s} - \frac{1}{s+1/RC}\right]\right\}$$

Can do this with "Partial Fraction Expansion", which is just a "fool-proof" way to factor

$$= \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}}_{=u(t)} - \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+1/RC}\right\}}_{=e^{-(t/RC)}u(t)}$$

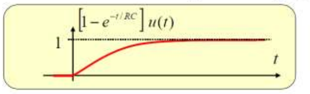
Linearity of LT

Now… each of these terms ***is*** on the LT table:

$$= \left[1 - e^{-(t/RC)}\right]u(t)$$

25

So the zero-state response of this system is: $\left[1 - e^{-(t/RC)}\right]u(t)$



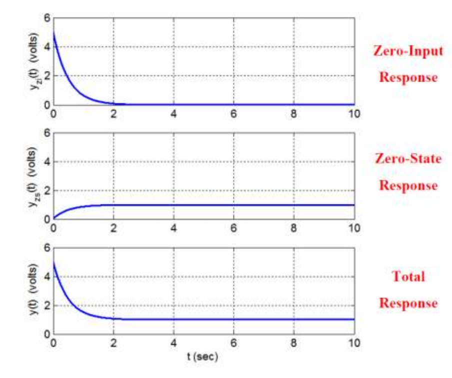
Now putting this zero-state response together with the zero-input response we found gives:

$$y(t) = \underbrace{y(0^-)e^{-(t/RC)}u(t)}_{\text{IC Part}} + \underbrace{\left[1 - e^{-(t/RC)}\right]u(t)}_{\text{Input Part}}$$

Notice that:
The IC Part "Decays Away"
but…
The Input Part "Persists"

26

Here is an example for ***RC* = 0.5 sec** and the initial V_{IC} **= 5 volts**:



27

Second-order case

Circuits with two energy-storing devices (C & L, or 2 Cs or 2 Ls) are described by a second-order Differential Equation…

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = b_1\frac{dx(t)}{dt} + b_0x(t)$$

w/ ICs $\dot y(0^-)$ & $y(0^-)$

Assume Causal Input

$x(t) = 0 \quad t<0$

$\downarrow$

$x(0^-) = 0$

We solve the 2nd-order case using the same steps:

Take LT of Diff. Equation:

$$\underbrace{\left[s^2Y(s) - y(0^-)s - \dot y(0^-)\right]}_{\text{From 2nd derivative property, accounting for ICs}} + a_1\underbrace{\left[sY(s) - y(0^-)\right]}_{\text{From 1st derivative property, accounting for ICs}} + a_0Y(s) = \underbrace{b_1sX(s)}_{\text{From 1st derivative property, causal signal}} + b_0X(s)$$

28

Solve for $Y(s)$: $$Y(s) = \frac{y(0^-)s + \dot y(0^-) + a_1y(0^-)}{s^2 + a_1s + a_0} + \left[\frac{b_1s + b_0}{s^2 + a_1s + a_0}\right]X(s)$$

Part of sol'n driven by IC "Zero-Input Sol'n"

Note this shows up in both places… it is the Characteristic Equation

Part of sol'n driven by input "Zero-State Sol'n"

Note: The role the Characteristic Equation plays here!
It just pops up in the LT method!
The same happened for a 1st-order Diff. Eq…
…and it happens for all orders

Like before…
to get the solution in the time domain find the Inverse LT of Y(s)

29

N^{th}-Order Case

Diff. eq of the system

$$\frac{d^Ny(t)}{dt^N} + a_{N-1}\frac{d^{N-1}y(t)}{dt^{N-1}} + \ldots + a_1\frac{dy(t)}{dt} + a_0y(t) = b_M\frac{dx^M(t)}{dt^M} + b_1\frac{dx(t)}{dt} + b_0x(t)$$

For $M \le N$ and $\left.\dfrac{d^ix(t)}{dt^i}\right|_{t=0^-} = 0 \quad i = 0,1,2,\ldots,M-1$

Taking LT and re-arranging gives:

$$Y(s) = \frac{IC(s)}{A(s)} + \frac{B(s)}{A(s)}X(s)$$

LT of the solution (i.e. the LT of the system output)

where
$A(s) = s^N + a_{N-1}s^{N-1} + \ldots + a_1s + a_0$ "output-side" polynomial
$B(s) = b_Ms^M + \ldots + b_1s + b_0$ "input-side" polynomial
$IC(s) = \text{polynomial in } s \text{ that depends on the ICs}$

Recall: For 2nd order case: $IC(s) = y(0^-)s + \left[\dot y(0^-) + a_1y(0^-)\right]$

30

Consider the case where the LT of $x(t)$ is rational: $X(s)=\frac{N_X(s)}{D_X(s)}$

Then… $Y(s)=\frac{IC(s)}{A(s)}+\frac{B(s)}{A(s)}X(s)=\frac{IC(s)}{A(s)}+\frac{B(s)}{A(s)}\frac{N_X(s)}{D_X(s)}$

This can be expanded like this: $Y(s)=\frac{IC(s)}{A(s)}+\frac{E(s)}{A(s)}+\frac{F(s)}{D_X(s)}$

for some resulting polynomials $E(s)$ and $F(s)$

So… for a <u>system</u> with $H(s)=\frac{B(s)}{A(s)}$ and <u>input</u> with $X(s)=\frac{N_X(s)}{D_X(s)}$

and <u>initial conditions</u> you get:

$$Y(s)=\underbrace{\boxed{\frac{IC(s)}{A(s)}}}_{\text{Zero-Input Response}}+\underbrace{\boxed{\underbrace{\frac{E(s)}{A(s)}}_{\text{Transient Response}}+\underbrace{\frac{F(s)}{D_X(s)}}_{\text{Steady-State Response}}}}_{\text{Zero-State Response}}$$

Decays in time domain if roots of system char. poly. $A(s)$ have negative real parts

31

If all IC's are zero (zero state) $IC(s)=0$

Then: $Y(s)=\underbrace{\left[\frac{B(s)}{A(s)}\right]}_{\equiv H(s)}X(s)$

Called "Transfer Function" of the system…

Zero-State Response

$$Y(s)=\boxed{\underbrace{\frac{E(s)}{A(s)}}_{\text{Transient Response}}+\underbrace{\frac{F(s)}{D_X(s)}}_{\text{Steady-State Response}}}$$

32

<u>**Summary Comments**</u>:

1. From the differential equation one can easily write the $H(s)$ by inspection!
2. The denominator of $H(s)$ is the characteristic equation of the differential equation.
3. The roots of the denominator of $H(s)$ determine the <u>form</u> of the solution…

…recall partial fraction expansions

<u>BIG PICTURE:</u> The roots of the characteristic equation drive the nature of the system response… we can now see that via the LT.

We now see that there are three contributions to a system's response:

zero-input resp.
1. **The part driven by the ICs**
   a. **This will decay away if the Ch. Eq. roots have negative real parts**

zero-state resp.
2. **A part driven by the input that will decay away if the Ch. Eq. roots have negative real parts … "Transient Response"**
3. **A part driven by the input that will persist while the input persists… "Steady State Response"**

33

Partial Fractions

When trying to find the inverse Laplace transform it is helpful to be able to break a complicated ratio of two polynomials into forms that are on the Laplace Transform table.

<u>An Example</u>:

$$\mathcal{L}^{-1}\left\{\left[\frac{1/RC}{(s+1/RC)s}\right]\right\}=\mathcal{L}^{-1}\left\{\left[\frac{1}{s}-\frac{1}{s+1/RC}\right]\right\}$$

Factor LT into simpler forms

$$=\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}}_{=u(t)}-\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+1/RC}\right\}}_{=e^{-(t/RC)}u(t)}$$

Linearity of LT

Here the factoring could be done "by inspection"… in general, this factoring can be done using the method of "partial fraction expansion" (PFE)

34

Partial Fractions

<u>Motivation… Going the "Other Way"</u>

Suppose we had this:

$$Y(s)=\boxed{\frac{2}{(s+3)}+\frac{4}{(s+5)}}+2$$

This has poles at $s=-3$ and $s=-5$

A term like this is called a "Direct Term"

We could find the common denominator and combine:

$$Y(s)=\frac{2(s+5)}{(s+3)(s+5)}+\frac{4(s+3)}{(s+3)(s+5)}+2\frac{(s+3)(s+5)}{(s+3)(s+5)}$$

$$=\frac{2s^2+22s+47}{s^2+8s+15}$$

We want to go from here… to here…

35

<u>Ex. #1</u>: <u>No Direct Terms</u>, <u>No Repeated Roots</u>, <u>No Complex Roots</u>

$$Y(s)=\frac{3s-1}{s^2+3s+2}$$

If the highest power in the numerator is less than the highest power in the denominator then there will be no direct terms.

By using the quadratic formula to find the roots of the denominator we can verify that there are no repeated or complex roots.

The roots are: $s=-2$ and $s=-1$… so we can write: $Y(s)=\frac{3s-1}{(s+1)(s+2)}$

If there are no repeated roots and no direct terms we can always write it as

$$Y(s)=\frac{r_1}{(s+1)}+\frac{r_2}{(s+2)}$$

The numbers r_1 and r_2 are called the "***<u>residues</u>***"… we need to find them!

36

Now we exploit what we know:

$$\frac{3s-1}{(s+1)(s+2)} = \frac{r_1}{(s+1)} + \frac{r_2}{(s+2)}$$

Multiply each side by $(s+1)$ gives:

$$\frac{(3s-1)(s+1)}{(s+1)(s+2)} = \frac{r_1(s+1)}{(s+1)} + \frac{r_2(s+1)}{(s+2)}$$

Canceling $(s+1)$ where we can gives:

$$\frac{(3s-1)}{(s+2)} = r_1 + \frac{r_2(s+1)}{(s+2)}$$

Setting $s = -1$ gives:

$$\frac{(-3-1)}{(-1+2)} = r_1 \quad \Rightarrow \quad r_1 = -4$$

All of this is summarized by this

$$\boxed{r_1 = Y(s)(s+1)\big|_{s=-1}}$$

Similarly… we find the other residue using: $\boxed{r_2 = Y(s)(s+2)\big|_{s=-2} = 7}$

Then we have: $$Y(s) = \frac{-4}{(s+1)} + \frac{7}{(s+2)}$$

Each of these terms is on the LT Table, so we get

$$y(t) = \mathcal{L}^{-1}\left\{\frac{-4}{(s+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{7}{(s+2)}\right\}$$

$$= -4e^{-t}u(t) + 7e^{-2t}u(t)$$

Poles and Zeros

it is possible to directly identify the TF $H(s)$ from the Diff. Eq.:

$$\boxed{\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_1\frac{dx(t)}{dt} + b_0 x(t)}$$

$$\boxed{H(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}}$$

Poles and Zeros

So… we know that $H(s)$ is completely described by the Diff. Eq.… Therefore we should expect that we can tell a lot about a system by looking at the structure of the transfer function $H(s)$… This structure is captured in the idea of "Poles" and "Zeros"…

Poles and Zeros of a system

Given a system with Transfer Function:

$$H(s) = \frac{b_M s^M + b_{M-1}s^{M-1} + \ldots + b_1 s + b_0}{s^N + a_{N-1}s^{N-1} + \ldots + a_1 s + a_0} \begin{array}{l} \leftarrow B(s) \\ \leftarrow A(s) \end{array}$$

We can factor $B(s)$ and $A(s)$: (Recall: $A(s)$ = characteristic polynomial)

$$H(s) = \frac{b_M(s-z_1)(s-z_2)\ldots(s-z_M)}{(s-p_1)(s-p_2)\ldots(s-p_N)}$$

Assume any common factors in $B(s)$ and $A(s)$ have been cancelled out

Note: $H(s)\big|_{s=z_i} = 0 \qquad i = 1,2,\ldots,M \qquad \boxed{\{z_i\}\text{ are called "zeros of } H(s)\text{"}}$

$H(s)\big|_{s=p_i} = \infty \qquad i = 1,2,\ldots,N \qquad \boxed{\{p_i\}\text{ are called "poles of } H(s)\text{"}}$

Note: p_i are the roots of the char. polynomial

Poles and Zeros

Note that knowing the sets $\{z_i\}_{i=1}^{M}$ & $\{p_i\}_{i=1}^{N}$

tells us what $H(s)$ is: (up to the multiplicative scale factor b_M)

-b_M is like a gain (i.e. amplification)

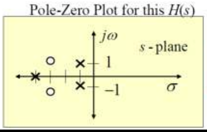
Pole-Zero Plot

This gives us a graphical view of the system's behavior

Example: $$H(s) = \frac{2s^2 + 12s + 20}{s^3 + 6s^2 + 10s + 8} = \frac{2(s+3-j)(s+3+j)}{(s+4)(s+1-j)(s+1+j)}$$

Real coefficients ⇒ complex conjugate pairs

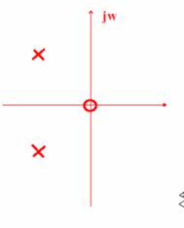
Pole-Zero Plot for this $H(s)$



x denotes a pole

o denotes a zero

From the Pole-Zero Plots we can Visualize the TF function on the s-plane:



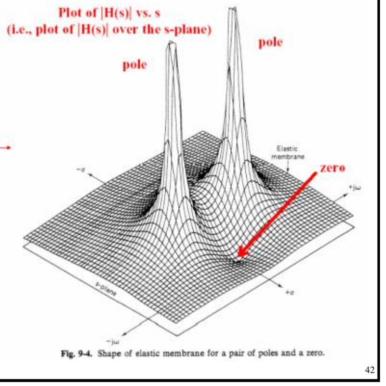


Fig. 9-4. Shape of elastic membrane for a pair of poles and a zero.

From our Visualization of the TF function on the s-plane we can see the Freq. Resp.:

To get the frequency response H(w) from the transfer function H(s) we replace s by jw... this is graphically equivalent to "cutting along the jw axis"

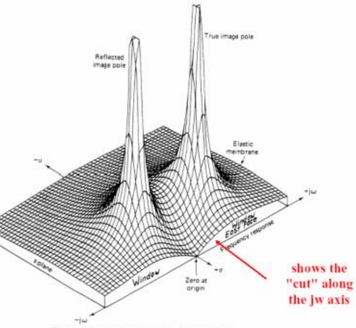


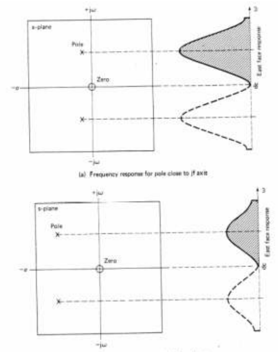
Fig. 9-5. Pole-zero diagram showing the east face.

shows the "cut" along the jw axis

Plots from my favorite book on Op Amps: "Operational Amplifier: Characteristics and Applications" by Robert G. Irvine, Prentice-Hall, 1981

43

Can also look at a pole-zero plot and see the effects on Freq. Resp.



(a) Frequency response for pole close to $j\omega$ axis

(b) Frequency response for pole far from $j\omega$ axis

Fig. 9-6. Frequency response versus pole location.

As the pole moves closer to the $j\omega$ axis it has a stronger effect on the frequency response $H(\omega)$. Poles close to the $j\omega$ axis will yield sharper and taller bumps in the frequency response.

By being able to visualize what $|H(s)|$ will look like based on where the poles and zeros are, an engineer gains the ability to know what kind of transfer function is needed to achieve a desired frequency response… then through accumulated knowledge of electronic circuits (requires experience accumulated AFTER graduation) the engineer can devise a circuit that will achieve the desired effect.

44

Laplace Transform Table

Time Signal	Laplace Transform
$u(t)$	$1/s$
$u(t)-u(t-c),\quad c>0$	$(1-e^{-cs})/s,\quad c>0$
$t^N u(t),\quad N=1,2,3,\ldots$	$\frac{N!}{s^{N+1}},\quad N=1,2,3,\ldots$
$\delta(t)$	1
$\delta(t-c),\quad c$ real	$e^{-cs},\quad c$ real
$e^{-bt}u(t),\quad b$ real or complex	$\frac{1}{s+b},\quad b$ real or complex
$t^N e^{-bt}u(t),\quad N=1,2,3,\ldots$	$\frac{N!}{(s+b)^{N+1}},\quad N=1,2,3,\ldots$
$\cos(\omega_o t)u(t)$	$\frac{s}{s^2+\omega_o^2}$
$\sin(\omega_o t)u(t)$	$\frac{\omega_o}{s^2+\omega_o^2}$
$\cos^2(\omega_o t)u(t)$	$\frac{s^2+2\omega_o^2}{s(s^2+4\omega_o^2)}$
$\sin^2(\omega_o t)u(t)$	$\frac{2\omega_o^2}{s(s^2+4\omega_o^2)}$
$e^{-bt}\cos(\omega_o t)u(t)$	$\frac{s+b}{(s+b)^2+\omega_o^2}$
$e^{-bt}\sin(\omega_o t)u(t)$	$\frac{\omega_o}{(s+b)^2+\omega_o^2}$
$t\cos(\omega_o t)u(t)$	$\frac{s^2-\omega_o^2}{(s^2+\omega_o^2)^2}$

$t\sin(\omega_o t)u(t)$	$\frac{2\omega_o s}{(s^2+\omega_o^2)^2}$
$Ae^{-\zeta\omega_n t}\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t\right]u(t)$ where: $A=\frac{\alpha}{\omega_n\sqrt{1-\zeta^2}}$	$\frac{\alpha}{s^2+2\zeta\omega_n s+\omega_n^2}$
$Ae^{-\zeta\omega_n t}\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t+\phi\right]u(t)$ $A=\beta\sqrt{\frac{(\alpha-\zeta\omega_n)^2}{\omega_n^2(1-\zeta^2)}+1}\quad \phi=\tan^{-1}\left(\frac{\omega_n\sqrt{1-\zeta^2}}{\alpha-\zeta\omega_n}\right)$	$\beta\frac{s+\alpha}{s^2+2\zeta\omega_n s+\omega_n^2}$
$te^{-bt}\cos(\omega_o t)u(t)$	$\frac{(s+b)^2-\omega_o^2}{((s+b)^2+\omega_o^2)^2}$
$te^{-bt}\sin(\omega_o t)u(t)$	$\frac{2\omega_o(s+b)}{((s+b)^2+\omega_o^2)^2}$

45

Laplace Transform Properties

Property Name	Property	
Linearity	$ax(t)+bv(t)$	$aX(s)+bV(s)$
Right Time Shift (**<u>Causal</u>** Signal)	$x(t-c),\quad c>0$	$e^{-cs}X(s)$
Time Scaling	$x(at),\quad a>0$	$\frac{1}{a}X(s/a),\quad a>0$
Multiply by t^n	$t^n x(t),\quad n=1,2,3,\ldots$	$(-1)^n\frac{d^n}{ds^n}X(s),\quad n=1,2,3,\ldots$
Multiply by Exponential	$e^{at}x(t),\quad a$ real or complex	$X(s-a),\quad a$ real or complex
Multiply by Sine	$\sin(\omega_o t)x(t)$	$\frac{j}{2}[X(s+j\omega_o)-X(s-j\omega_o)]$
Multiply by Cosine	$\cos(\omega_o t)x(t)$	$\frac{1}{2}[X(s+j\omega_o)+X(s-j\omega_o)]$
Time Differentiation 2^{nd} Derivative n^{th} Derivative	$\dot{x}(t)$ $\ddot{x}(t)$ $x^{(N)}(t)$	$sX(s)-x(0)$ $s^2X(s)-sx(0)-\dot{x}(0)$ $s^NX(s)-s^{N-1}x(0)-s^{N-2}\dot{x}(0)-$ $\cdots-sx^{(N-2)}(0)-x^{(N-1)}(0)$
Time Integration	$\int_{-\infty}^{t}x(\lambda)d\lambda$	$\frac{1}{s}X(s)$
Convolution in Time	$x(t)*h(t)$	$X(s)H(s)$
Initial-Value Theorem	$x(0)=\lim_{s\to\infty}[sX(s)]$ $\dot{x}(0)=\lim_{s\to\infty}[s^2X(s)-sx(0)]$ $x^{(N)}(0)=\lim_{s\to\infty}[s^{N+1}X(s)-s^Nx(0)-s^{N-1}\dot{x}(0)-\cdots-sx^{(N-1)}(0)]$	
Final-Value Theorem	If $\lim_{t\to\infty}x(t)$ exists, then $\lim_{t\to\infty}x(t)=\lim_{s\to 0}sX(s)$	

46