Yıldız Technical University (YTU)
Faculty of Electrical and Electronics Engineering.
Computer Engineering Department

BLM2401 Signals and Systems

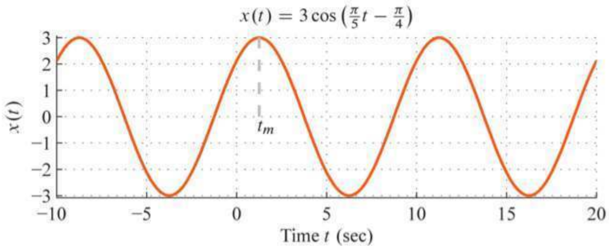
Midterm 1 – Review Questions A

Question. Given $x(t) = 2\cos\left(\frac{\pi}{5}t - \frac{\pi}{4}\right)$, make a plot over time interval $-10 \le t \le 20$ secs.

Solution. Determine the amplitude (A), frequency (f), period (T), phase (ϕ), and time of a maximum (t_m):

$$
\begin{aligned}
\text{By inspection:}\quad & A = 3 \\
& f = \frac{\omega_0}{2\pi} = \frac{(\pi/5)}{2\pi} = \frac{1}{10}\ \text{Hz} \\
& T = \frac{1}{f} = \frac{1}{0.1} = 10\ \text{secs} \\
\text{By inspection:}\quad & \phi = -\frac{\pi}{4} \\
& t_m = -\frac{\phi}{\omega_0} = -\frac{\left(-\frac{\pi}{4}\right)}{\left(\frac{2\pi}{10}\right)} = 1.25
\end{aligned}
$$

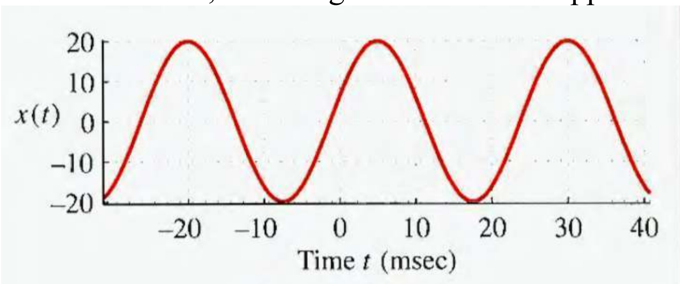
At $t = 0$, the signal value is $x(0) = 2\cos\left(-\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} \approx 2.1$



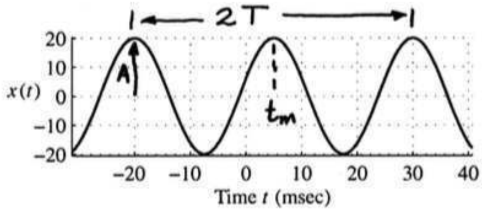
Question. The figure given below is a plot of a sinusoidal wave. From the plot, determine values for the amplitude (A), phase (ϕ),and frequency (ω_0) needed in the representation:

$$x(t) = Acos(\omega_0 t + \phi)$$

Give the answer as numerical values, including the units where applicable.



Solution.



Period: $2T = (30-(-20))$ msec $=$ 50 msec

$\Rightarrow T = 25$ msec

Frequency: $\omega_0 = 2\pi/T = 2\pi\left(\frac{1}{25\times10^{-3}}\right) = 2\pi(40)$ rad/s

$f = 40$ Hz

Amplitude: $A = 20$

Time-Shift: $t_m = +5$ msec

Phase: $\varphi = -\omega_0 t_m = -2\pi(40)\times5\times10^{-3}$

$\varphi = -2\pi(0.2) = -0.4\pi$

$x(t) = 20\cos(80\pi t - 0.4\pi)$

Question. Use the series expansions for e^x, $cos(\theta)$, and $sin(\theta)$ given here to verify Euler's formula.

Solution.

$$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \ldots$$

$$= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} + \ldots$$

Separate the real and imaginary parts:

$$e^{j\theta} = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right)}_{\cos\theta} + j\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)}_{\sin\theta}$$

$$\therefore e^{j\theta} = \cos\theta + j\sin\theta$$

which proves Euler's formula.

Question. Use Euler's formula for the complex exponential to prove DeMoivre's formula,

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

where $i = \sqrt{-1}$. Use it to evaluate $\left(\frac{3}{5} + i\frac{4}{5}\right)^{100}$.

Solution.

$$(\cos\theta + j\sin\theta)^n = (e^{j\theta})^n = e^{jn\theta}$$

$$e^{jn\theta} = \cos(n\theta) + j\sin(n\theta)$$

Thus,

$$(\cos\theta + j\sin\theta)^n = \cos(n\theta) + j\sin(n\theta)$$

$$\left(\frac{3}{5} + j\frac{4}{5}\right)^{100} = \left(e^{j0.927}\right)^{100} = \left(e^{j0.295167\pi}\right)^{100}$$

$$= e^{j29.5167\pi}$$

$$= e^{j1.5167\pi}$$

BECAUSE $e^{j28\pi} = 1$

$$= \cos(1.5167\pi) + j\sin(1.5167\pi)$$

$$= \cos(273^\circ) + j\sin(273^\circ)$$

$$= 0.0525 - j0.9986$$

Question. Simplify the following expressions:

(a) $3e^{j\pi/3} + 3e^{-j\pi/6}$

(b) $\left(\sqrt{3} - j3\right)^{10}$

(c) $\left(\sqrt{3} - j3\right)^{-1}$

(d) $\left(\sqrt{3} - j3\right)^{1/3}$

(e) $\Re\{je^{-j\pi/3}\}$

Solution.

(a) $3e^{j\pi/3} + 4e^{-j\pi/6} = \left(\frac{3}{2} + j\frac{3\sqrt{3}}{2}\right) + \left(4\frac{\sqrt{3}}{2} - j\frac{4}{2}\right)$

$= 4.9641 + j\,0.5981$

$= 5e^{j0.12}$ NOTE: 0.12 rad $= 6.87°$

(b) $\sqrt{3} - j3 = \sqrt{3+3^2}\,e^{-j\pi/3} = \sqrt{12}\,e^{-j\pi/3}.$

$\Rightarrow \left(\sqrt{3} - j3\right)^{10} = \left(\sqrt{12}\,e^{-j\pi/3}\right)^{10}$

$= 2^{10}3^5\,e^{-j10\pi/3}$ $\quad -\frac{10\pi}{3} + 4\pi = \frac{-10\pi + 12\pi}{3} = 2\pi/3$

$= 248{,}832\,e^{+j2\pi/3} = -124{,}416 + j\,215{,}494.83$

(c) $\frac{1}{\sqrt{3} - j3} = \frac{1}{\sqrt{12}\,e^{-j\pi/3}} = \frac{1}{\sqrt{12}}e^{+j\pi/3} = 0.2887e^{+j\pi/3}$

$= 0.14434 + j0.25$

(d) $\left(\sqrt{3} - j3\right)^{1/3} = \left(\sqrt{12}\,e^{-j\pi/3}\right)^{1/3} = \left(\sqrt{12}\,e^{-j(\pi/3 + 2\pi\ell)}\right)^{1/3}$

$= 12^{1/6}\,e^{-j(\pi/9 + \frac{2\pi}{3}\ell)}$ $\quad \ell =$ integer

Need $\ell = 0, 1, 2$

There are 3 answers:

$1.513\,e^{-j\pi/9} = 1.422 - j0.5175$

$1.513\,e^{-j7\pi/9} = -1.159 - j\,0.9726$

$1.513\,e^{-j13\pi/9} = 1.513e^{+j5\pi/9} = -0.2627 + j1.49$

(e) $\Re e\{je^{-j\pi/3}\} = \Re e\{e^{j\pi/2}e^{-j\pi/3}\}.$

$= \Re e\{e^{j\pi/6}\} = \cos(\pi/6) = \frac{\sqrt{3}}{2} = 0.866$

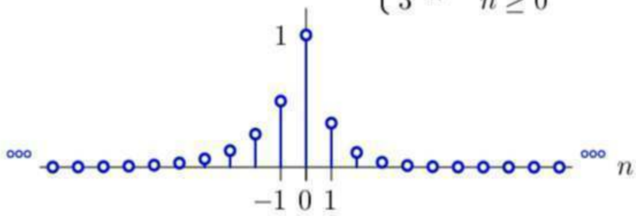
Question.

Find constants c_1, c_2, and c_3 so that the solution to the difference equation

$$c_1 y[n-1] + c_2 y[n] + c_3 y[n+1] = \delta[n]$$

is equal to the signal $y[n]$ shown below.

$$y[n] = \begin{cases} 2^n & n < 0 \\ 3^{-n} & n \geq 0 \end{cases}$$



Enter numerical values (or closed-form numerical expressions) for the constants below. You need only show one valid answer, even if multiple answers exist. If no solution exists, enter **none**, and briefly explain why no solution exists.

$c_1 =$ $c_2 =$ $c_3 =$

Solution.

If $n < 0$ then $y[n] = 2^n$ and $c_1 2^{n-1} + c_2 2^n + c_3 2^{n+1} = 0$, which is true iff $c_1 + 2c_2 + 4c_3 = 0$.

If $n > 0$ then $y[n] = 3^{-n}$ and $c_1 3^{-n-1} + c_2 3^{-n} + c_3 3^{-n+1} = 0$, which is true iff $9c_1 + 3c_2 + c_3 = 0$.

If $n = 0$ then $\frac{1}{2}c_1 + c_2 + \frac{1}{3}c_3 = 1$.

Solving these equations yields a single unique solution, $c_1 = -2/5$, $c_2 = 7/5$, and $c_3 = -3/5$.

Question.

1. *Complex numbers - polar.* Compute the magnitude and phase of the complex numbers:
   a. $3 + 2i$
   b. $1 + i$
   c. $e^{2-i\pi}$

Question.

2. *Complex numbers - rectangular.* Compute the real and imaginary parts of the complex numbers:
   a. e^i
   b. $e^{it}(\cos(3t) + sin(2t))$ (t is real)
   c. $1/(1+i)$

Question.

3. *Energy.* What is the energy of these signals (where t is the independent variable):

   a. $x(t) = Ae^{-at}u(t)$ with $a > 0$.

   b. The unit area rectangular pulse of width a, $\Delta_a(t)$.

Question.

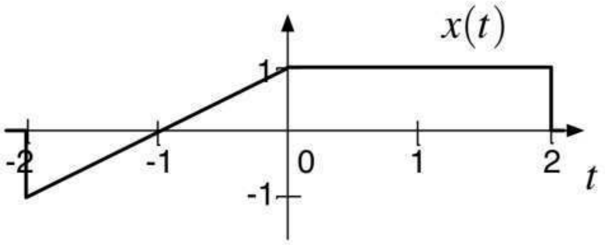
4. *Power.* What is the power of these signals:

   a. $x(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$.

   b. $x(t) = \sum_{k=-N}^{N} A_k e^{i\omega_0 kt}$.

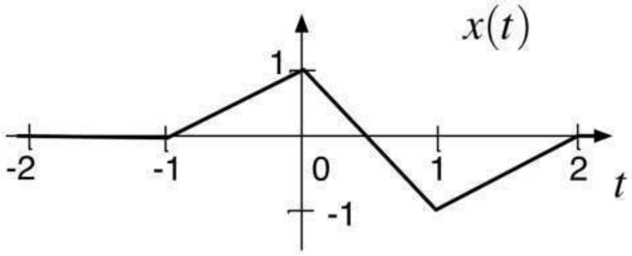
Question.

5. *Even and odd parts.* Find the even and odd decomposition of this signal:



Question.

6. *Time shifting and scaling.* Given the signal $x(t)$ shown below



   draw the following signals:

   a. $x(-2(t-1))$

   b. $x(t/2 + 1/2)$

Question.

7. If $y(t)$ is an even function, and $y(t-1)$ is also even, is $y(t)$ periodic?

Question.

8. A signal $y(t)$ is periodic with fundamental period T_0, and is the sum of two other signals

$$y(t) = x_1(t) + x_2(t).$$

Must $x_1(t)$ and $x_2(t)$ both be periodic?

Question.

9. Assume that the signal $x(t)$ is periodic with period T_0, and that $x(t)$ is odd ($i.e.$ $x(t) = -x(-t)$). What is the value of $x(T_0)$?

Question.

10. Two continuous-time sequences $x_1(t)$ and $x_2(t)$ are periodic with periods T_1 and T_2. Find values of T_1 and T_2 such that $x_1(t)+x_2(t)$ is aperiodic.

Question.

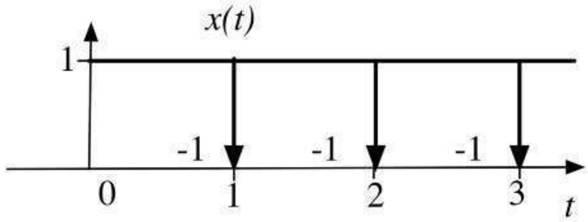
11. Sketch

$$x(t) = \frac{1}{\sqrt{t}}u(t-1)$$

and classify it as an energy or power signal or neither.

Question.

12. For the waveform $x(t)$ plotted below,



evaluate and draw the function

$$y(t) = \int_0^t x(\tau)d\tau.$$

The impulses are negative, and have strength 1. What would you name this waveform?

Question.

13. Evaluate these integrals

(a) $\displaystyle\int_{-\infty}^{\infty} f(t+1)\delta(t+1)\ dt$

(b) $\displaystyle\int_{-\infty}^{\infty} e^{j\omega T}\delta(t)\ dt$

(c) $\displaystyle\int_{0}^{\infty} f(t)\left(\delta(t-1)+\delta(t+1)\right)\ dt$

(d) $\displaystyle\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\delta(t-2)d\tau.$

Question.

Prob. 1. [50 pts] Consider the LTI system characterized by the I/O relationship:

$$\text{System 1:} \quad y(t) = \int_{t-4}^{t} e^{-3(t-\tau)} x(\tau) d\tau$$

Hint:

$$\int_{t-4}^{t} e^{-3(t-\tau)} x(\tau) d\tau = \int_{-\infty}^{t} e^{-3(t-\tau)} x(\tau) d\tau - \int_{-\infty}^{t-4} e^{-3(t-\tau)} x(\tau) d\tau$$

(a) Write the impulse response of the system, $h_1(t)$.

$$h(t) = e^{-3t} u(t) - e^{-12} e^{-3(t-4)} u(t-4)$$

$$= e^{-3t} (u(t) - u(t-4))$$

$$\boxed{\int_{-\infty}^{t-4} e^{-3(t-\tau)} x(\tau) d\tau = \int_{-\infty}^{t-4} e^{-12} e^{-3(t-4-\tau)} x(\tau) d\tau}$$

(b) Is the system causal? Justify your answer using the impulse response.

$h(t) = 0$ For $t < 0$, so system is causal

(c) Is the system stable? Justify your answer using the impulse response.

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{0}^{4} \underbrace{e^{-3t}}_{>0} dt = -\frac{1}{3} e^{-3t} \Big]_{0}^{4}$$

$$= -\frac{1}{3}\left(e^{-12} - 1\right) = \frac{1 - e^{-12}}{3} > 0$$

System is stable!.

(d) **You should solve each part of this problem using known convolution results in conjunction with linearity (homogeneity & superposition) & time-invariance. You do not need to simplify your answers.** Determine & write a closed-form expression for the output ($y(t)$) of System 1when the input is the unit step

$$x(t) = u(t)$$

$$y_d(t) = u(t) * \{e^{-3t}(u(t) - u(t-4))\}$$

$$= u(t) * e^{-3t} u(t) - u(t) * e^{-3(t-4)} u(t-4) e^{-12}$$

$$= z_d(t) - z_d(t-4) e^{-12}$$

$z_d(t)$: Formula A with $a = 0$ and $b = -3$

Formula A: $$e^{at}u(t) * e^{bt}u(t) = \frac{1}{a-b}e^{at}u(t) + \frac{1}{b-a}e^{bt}u(t)$$

(e) Determine and write a closed-form expression for the output ($y(t)$) of System 1 for the input

$$x(t) = e^{-2t}u(t)$$

$$y_e(t) = e^{-2t}u(t) * \{e^{-3t}(u(t) - u(t-4))\}$$

$$= z_e(t) - z_e(t-4)\,e^{-12}$$

$z_e(t)$ = Formula A with $a=-2$ and $b=-3$

Formula A: $$e^{at}u(t) * e^{bt}u(t) = \frac{1}{a-b}e^{at}u(t) + \frac{1}{b-a}e^{bt}u(t)$$

Question.

Express each of the following complex numbers in Cartesian form ($x + jy$): $\frac{1}{2}e^{j\pi}$, $\frac{1}{2}e^{-j\pi}$, $e^{j\pi/2}$, $e^{-j\pi/2}$, $e^{j5\pi/2}$, $\sqrt{2}e^{j\pi/4}$, $\sqrt{2}e^{j9\pi/4}$, $\sqrt{2}e^{-j9\pi/4}$, $\sqrt{2}e^{-j\pi/4}$.

Solution.

Converting from polar to Cartesian coordinates:

$\frac{1}{2}e^{j\pi} = \frac{1}{2}\cos\pi = -\frac{1}{2}$, $\quad \frac{1}{2}e^{-j\pi} = \frac{1}{2}\cos(-\pi) = -\frac{1}{2}$

$e^{j\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right) = j$, $\quad e^{-j\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right) = -j$

$e^{j5\frac{\pi}{2}} = e^{j\frac{\pi}{2}} = j$, $\quad \sqrt{2}e^{j\frac{\pi}{4}} = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right)\right) = 1 + j$

$\sqrt{2}e^{\frac{9j\pi}{4}} = \sqrt{2}e^{\frac{j\pi}{4}} = 1 + j$, $\quad \sqrt{2}e^{\frac{-9j\pi}{4}} = \sqrt{2}e^{\frac{-j\pi}{4}} = 1 - j$

$\sqrt{2}e^{\frac{-j\pi}{4}} = 1 - j$

Question.

Express each of the following complex numbers in polar form ($re^{j\theta}$, with $-\pi < \theta \le \pi$): 5, -2, $-3j$, $\frac{1}{2} - j\frac{\sqrt{3}}{2}$, $1 + j$, $(1-j)^2$, $j(1-j)$, $(1+j)/(1-j)$, $(\sqrt{2} + j\sqrt{2})/(1 + j\sqrt{3})$.

Solution.

Converting from Cartesian to polar coordinates:

$5 = 5e^{j0}$, $\quad -2 = 2e^{j\pi}$, $\quad -3j = 3e^{-j\frac{\pi}{2}}$

$\frac{1}{2} - j\frac{\sqrt{3}}{2} = e^{-j\frac{\pi}{3}}$, $\quad 1 + j = \sqrt{2}e^{j\frac{\pi}{4}}$, $\quad (1-j)^2 = 2e^{-j\frac{\pi}{2}}$

$j(1-j) = e^{j\frac{\pi}{4}}$, $\quad \frac{1+j}{1-j} = e^{j\frac{\pi}{2}}$, $\quad \frac{\sqrt{2}+j\sqrt{2}}{1+j\sqrt{3}} = e^{-j\frac{\pi}{12}}$

Question. Determine the values of power P_∞ and energy E_∞ for each of the following signals.

(a) $x_1(t) = e^{-2t}u(t)$ **(b)** $x_2(t) = e^{j(2t+\pi/4)}$ **(c)** $x_3(t) = \cos(t)$

(d) $x_1[n] = (\frac{1}{2})^n u[n]$ **(e)** $x_2[n] = e^{j(\pi/2n+\pi/8)}$ **(f)** $x_3[n] = \cos(\frac{\pi}{4}n)$

Solution.

(a) $E_\infty = \int_0^\infty e^{-4t}dt = \frac{1}{4}$, $P_\infty = 0$, because $E_\infty < \infty$

(b) $x_2(t) = e^{j(2t+\frac{\pi}{4})}$, $|x_2(t)| = 1$. Therefore, $E_\infty = \int_{-\infty}^{\infty} |x_2(t)|^2 dt = \int_{-\infty}^{\infty} dt = \infty$, $P_\infty =$
$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x_2(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt = \lim_{T\to\infty} 1 = 1$

(c) $x_3(t) = \cos(t)$. Therefore, $E_\infty = \int_{-\infty}^{\infty} |x_3(t)|^2 dt = \int_{-\infty}^{\infty} \cos^2(t)dt = \infty$,

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos^2(t)dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left(\frac{1+\cos(2t)}{2}\right) dt = \frac{1}{2}$$

(d) $x_1[n] = (\frac{1}{2})^n u[n]$, $|x_1[n]|^2 = (\frac{1}{4})^n u[n]$. Therefore, $E_\infty = \sum_{n=-\infty}^{\infty} |x_1[n]|^2 = \sum_{n=0}^{\infty} (\frac{1}{4})^n = \frac{4}{3}$.

$P_\infty = 0$, because $E_\infty < \infty$.

(e) $x_2[n] = e^{j(\frac{\pi n}{2}+\frac{\pi}{8})}$, $|x_2[n]|^2 = 1$. Therefore, $E_\infty = \sum_{n=-\infty}^{\infty} |x_2[n]|^2 = \infty$,

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_2[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1 = 1.$$

(f) $x_3[n] = \cos(\frac{\pi}{4}n)$. Therefore, $E_\infty = \sum_{n=-\infty}^{\infty} |x_3[n]|^2 = \sum_{n=-\infty}^{\infty} \cos^2(\frac{\pi}{4}n) = \infty$,

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \cos^2(\frac{\pi}{4}n) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left(\frac{1+\cos(\frac{\pi}{2}n)}{2}\right) = \frac{1}{2}$$

Question.

Let $x[n]$ be a signal with $x[n] = 0$ for $n < -2$ and $n > 4$. For each signal given below, determine the values of n for which it is guaranteed to be zero.

(a) $x[n-3]$ **(b)** $x[n+4]$ **(c)** $x[-n]$

(d) $x[-n+2]$ **(e)** $x[-n-2]$

Solution.

(a) The signal $x[n]$ is shifted by 3 to the right. The shifted signal will be zero for $n < 1$ and $n > 7$.

(b) The signal $x[n]$ is shifted by 4 to the left. The shifted signal will be zero for $n < -6$ and $n > 0$.

(c) The signal $x[n]$ is flipped. The flipped signal will be zero for $n < -4$ and $n > 2$.

(d) The signal $x[n]$ is flipped and the flipped signal is shifted by 2 to the right. This new signal will be zero for $n < -2$ and $n > 4$.

(e) The signal $x[n]$ is flipped and the flipped signal is shifted by 2 to the left. This new signal will be zero for $n < -6$ and $n > 0$.

Question.

Determine whether or not each of the following signals is periodic:

(a) $x_1(t) = 2e^{j(t+\pi/4)}u(t)$ **(b)** $x_2[n] = u[n] + u[-n]$

(c) $x_3[n] = \sum_{k=-\infty}^{\infty}\{\delta[n-4k] - \delta[n-1-4k]\}$

Solution.

(a) $x_1(t)$ is not periodic because it is zero for $t < 0$.

(b) $x_2[n] = 1$ for all n. Therefore, it is periodic with a fundamental period of 1.

(c) $x_3[n]$ is as shown in the Figure S1.6.

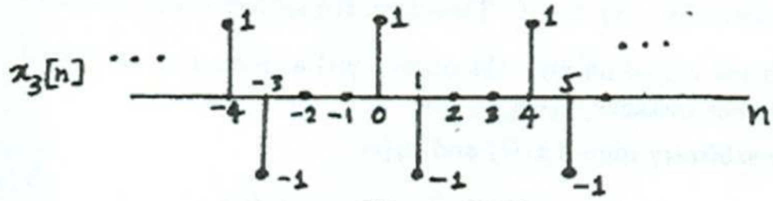


Figure S1.6

Therefore, it is periodic with a fundamental period of 4.

Question.

For each signal given below, determine all the values of the independent variable at which the even part of the signal is guaranteed to be zero.

(a) $x_1[n] = u[n] - u[n-4]$ **(b)** $x_2(t) = \sin(\frac{1}{2}t)$

(c) $x_3[n] = (\frac{1}{2})^n u[n-3]$ **(d)** $x_4(t) = e^{-5t}u(t+2)$

Solution.

(a)

$$\mathcal{E}v\{x_1[n]\} = \frac{1}{2}(x_1[n] + x_1[-n]) = \frac{1}{2}(u[n] - u[n-4] + u[-n] - u[-n-4])$$

Therefore, $\mathcal{E}v\{x_1[n]\}$ is zero for $|n| > 3$.

(b) Since $x_2(t)$ is an odd signal, $\mathcal{E}v\{x_2(t)\}$ is zero for all values of t.

(c)

$$\mathcal{E}v\{x_3[n]\} = \frac{1}{2}(x_1[n] + x_1[-n]) = \frac{1}{2}[(\frac{1}{2})^n u[n-3] - (\frac{1}{2})^{-n} u[-n-3]]$$

Therefore, $\mathcal{E}v\{x_3[n]\}$ is zero when $|n| < 3$ and when $|n| \to \infty$.

(d)

$$\mathcal{E}v\{x_4(t)\} = \frac{1}{2}(x_4(t) + x_4(-t)) = \frac{1}{2}[e^{-5t}u(t+2) - e^{5t}u(-t+2)$$

Therefore, $\mathcal{E}v\{x_4(t)\}$ is zero only when $|t| \to \infty$.

Question.

Express the real part of each of the following signals in the form $Ae^{-at}\cos(\omega t + \phi)$, where A, a, ω, and ϕ are real numbers with $A > 0$ and $-\pi < \phi \leq \pi$:

(a) $x_1(t) = -2$ **(b)** $x_2(t) = \sqrt{2}e^{j\pi/4}\cos(3t + 2\pi)$

(c) $x_3(t) = e^{-t}\sin(3t + \pi)$ **(d)** $x_4(t) = je^{(-2+j100)t}$

Solution.

(a) $\mathcal{R}e\{x_1(t)\} = -2 = 2e^{0t}\cos(0t + \pi)$

(b) $\mathcal{R}e\{x_2(t)\} = \sqrt{2}\cos(\frac{\pi}{4})\cos(3t + 2\pi) = \cos(3t) = e^{0t}\cos(3t + 0)$

(c) $\mathcal{R}e\{x_3(t)\} = e^{-t}\sin(3t + \pi) = e^{-t}\cos(3t + \frac{\pi}{2})$

(d) $\mathcal{R}e\{x_4(t)\} = -e^{-2t}\sin(100t) = e^{-2t}\sin(100t + \pi) = e^{-2t}\cos(100t + \frac{\pi}{2})$

Question.

Determine whether or not each of the following signals is periodic. If a signal is periodic, specify its fundamental period.

(a) $x_1(t) = je^{j10t}$ **(b)** $x_2(t) = e^{(-1+j)t}$ **(c)** $x_3[n] = e^{j7\pi n}$

(d) $x_4[n] = 3e^{j3\pi(n+1/2)/5}$ **(e)** $x_5[n] = 3e^{j3/5(n+1/2)}$

Solution.

(a) $x_1(t)$ is a periodic complex exponential.

$$x_1(t) = je^{j10t} = e^{j(10t+\frac{\pi}{2})}$$

The fundamental period of $x_1(t)$ is $\frac{2\pi}{10} = \frac{\pi}{5}$.

(b) $x_2(t)$ is a complex exponential multiplied by a decaying exponential. Therefore, $x_2(t)$ is not periodic.

(c) $x_3[n]$ is a periodic signal.

$$x_3[n] = e^{j7\pi n} = e^{j\pi n}$$

$x_3[n]$ is a complex exponential with a fundamental period of $\frac{2\pi}{\pi} = 2$.

(d) $x_4[n]$ is a periodic signal. The fundamental period is given by $N = m(\frac{2\pi}{3\pi/5}) = m(\frac{10}{3})$. By choosing $m = 3$, we obtain the fundamental period to be 10.

(e) $x_5[n]$ is not periodic. $x_5[n]$ is a complex exponential with $\omega_0 = 3/5$. We cannot find any integer m such that $m(\frac{2\pi}{\omega_0})$ is also an integer. Therefore, $x_5[n]$ is not periodic.

Question.

Determine the fundamental period of the signal $x(t) = 2\cos(10t + 1) - \sin(4t - 1)$.

Solution.

$$x(t) = 2\cos(10t + 1) - \sin(4t - 1)$$

Period of first term in RHS $= \frac{2\pi}{10} = \frac{\pi}{5}$
Period of second term in RHS $= \frac{2\pi}{4} = \frac{\pi}{2}$
Therefore, the overall signal is periodic with a period which is the least common multiple of the periods of the first and second terms. This is equal to π.

Question.

Determine the fundamental period of the signal $x[n] = 1 + e^{j4\pi n/7} - e^{j2\pi n/5}$.

Solution.

$$x[n] = 1 + e^{j\frac{4\pi}{7}n} - e^{j\frac{2\pi}{5}n}$$

Period of the first term in the RHS = 1
Period of the second term in the RHS $= m(\frac{2\pi}{4\pi/7}) = 7$ (when $m = 2$)
Period of the third term in the RHS $= m(\frac{2\pi}{2\pi/5}) = 5$ (when $m = 1$)
Therefore, the overall signal $x[n]$ is periodic with a period which is the least common multiple of the periods of the three terms in $x[n]$. This is equal to 35.

Question.

Consider the discrete-time signal

$$x[n] = 1 - \sum_{k=3}^{\infty} \delta[n - 1 - k].$$

Determine the values of the integers M and n_0 so that $x[n]$ may be expressed as

$$x[n] = u[Mn - n_0].$$

Solution.

The signal $x[n]$ is as shown in Figure S1.12. $x[n]$ can be obtained by flipping $u[n]$ and then shifting the flipped signal by 3 to the right. Therefore, $x[n] = u[-n + 3]$. This implies that $M = -1$ and $n_0 = -3$.

Question.

Consider the continuous-time signal

$$x(t) = \delta(t+2) - \delta(t-2).$$

Calculate the value of E_∞ for the signal

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Solution.

$$y(t) = \int_{-\infty}^{t} x(\tau)dt = \int_{-\infty}^{t} (\delta(\tau+2) - \delta(\tau-2))dt = \begin{cases} 0, & t < -2 \\ 1, & -2 \le t \le 2 \\ 0, & t > 2 \end{cases}$$

Therefore,

$$E_\infty = \int_{-2}^{2} dt = 4$$

Question.

Consider a periodic signal

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ -2, & 1 < t < 2 \end{cases}$$

with period $T = 2$. The derivative of this signal is related to the "impulse train"

$$g(t) = \sum_{k=-\infty}^{\infty} \delta(t-2k)$$

with period $T = 2$. It can be shown that

$$\frac{dx(t)}{dt} = A_1 g(t-t_1) + A_2 g(t-t_2).$$

Determine the values of A_1, t_1, A_2, and t_2.

Solution.

The signal $x(t)$ and its derivative $g(t)$ are shown in Figure S1.14.

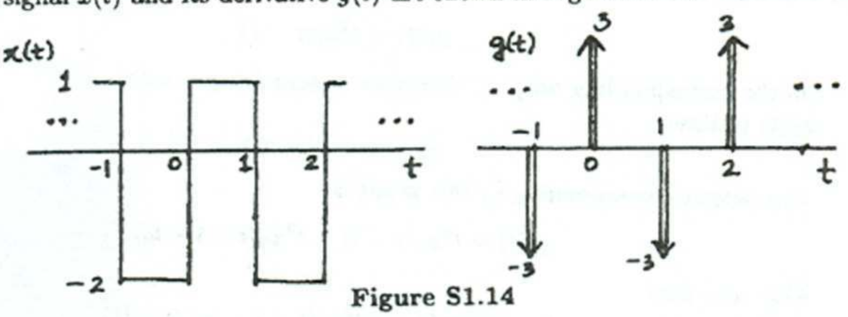


Figure S1.14

Therefore,

$$g(t) = 3\sum_{k=-\infty}^{\infty} \delta(t-2k) - 3\sum_{k=-\infty}^{\infty} \delta(t-2k-1)$$

This implies that $A_1 = 3$, $t_1 = 0$, $A_2 = -3$, and $t_2 = 1$.

Question.

Consider a system S with input $x[n]$ and output $y[n]$. This system is obtained through a series interconnection of a system S_1 followed by a system S_2. The input-output relationships for S_1 and S_2 are

$$\begin{aligned} S_1: \quad & y_1[n] = 2x_1[n] + 4x_1[n-1], \\ S_2: \quad & y_2[n] = x_2[n-2] + \frac{1}{2}x_2[n-3], \end{aligned}$$

where $x_1[n]$ and $x_2[n]$ denote input signals.

(a) Determine the input-output relationship for system S.

(b) Does the input-output relationship of system S change if the order in which S_1 and S_2 are connected in series is reversed (i.e., if S_2 follows S_1)?

Solution.

(a) The signal $x_2[n]$, which is the input to S_2, is the same as $y_1[n]$. Therefore,

$$\begin{aligned} y_2[n] &= x_2[n-2] + \frac{1}{2}x_2[n-3] \\ &= y_1[n-2] + \frac{1}{2}y_1[n-3] \\ &= 2x_1[n-2] + 4x_1[n-3] + \frac{1}{2}(2x_1[n-3] + 4x_1[n-4]) \\ &= 2x_1[n-2] + 5x_1[n-3] + 2x_1[n-4] \end{aligned}$$

The input-output relationship for S is

$$y[n] = 2x[n-2] + 5x[n-3] + 2x[n-4]$$

(b) The input-output relationship does not change if the order in which S_1 and S_2 are connected in series is reversed. We can easily prove this by assuming that S_1 follows S_2. In this case, the signal $x_1[n]$, which is the input to S_1, is the same as $y_2[n]$. Therefore,

$$\begin{aligned} y_1[n] &= 2x_1[n] + 4x_1[n-1] \\ &= 2y_2[n] + 4y_2[n-1] \\ &= 2(x_2[n-2] + \frac{1}{2}x_2[n-3]) + 4(x_2[n-3] + \frac{1}{2}x_2[n-4]) \\ &= 2x_2[n-2] + 5x_2[n-3] + 2x_2[n-4] \end{aligned}$$

The input-output relationship for S is once again

$$y[n] = 2x[n-2] + 5x[n-3] + 2x[n-4]$$

Question.

Consider a discrete-time system with input $x[n]$ and output $y[n]$. The input-output relationship for this system is

$$y[n] = x[n]x[n-2].$$

(a) Is the system memoryless?
(b) Determine the output of the system when the input is $A\delta[n]$, where A is any real or complex number.
(c) Is the system invertible?

Solution.

(a) The system is not memoryless because $y[n]$ depends on past values of $x[n]$.

(b) The output of the system will be $y[n] = \delta[n]\delta[n-2] = 0$.

(c) From the result of part (b), we may conclude that the system output is always zero for inputs of the form $\delta[n-k]$, $k \in \mathcal{I}$. Therefore, the system is not invertible.

Question.

Consider a continuous-time system with input $x(t)$ and output $y(t)$ related by

$$y(t) = x(\sin(t)).$$

(a) Is this system causal?
(b) Is this system linear?

Solution.

(a) The system is not causal because the output $y(t)$ at some time may depend on future values of $x(t)$. For instance, $y(-\pi) = x(0)$.

(b) Consider two arbitrary inputs $x_1(t)$ and $x_2(t)$.

$$x_1(t) \longrightarrow y_1(t) = x_1(\sin(t))$$

$$x_2(t) \longrightarrow y_2(t) = x_2(\sin(t))$$

Let $x_3(t)$ be a linear combination of $x_1(t)$ and $x_2(t)$. That is,

$$x_3(t) = ax_1(t) + bx_2(t)$$

where a and b are arbitrary scalars. If $x_3(t)$ is the input to the given system, then the corresponding output $y_3(t)$ is

$$\begin{aligned} y_3(t) &= x_3(\sin(t)) \\ &= ax_1(\sin(t)) + bx_2(\sin(t)) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

Therefore, the system is linear.

Question.

Consider a discrete-time system with input $x[n]$ and output $y[n]$ related by

$$y[n] = \sum_{k=n-n_0}^{n+n_0} x[k],$$

where n_0 is a finite positive integer.

(a) Is this system linear?

(a) Is this system time-invariant?

(c) If $x[n]$ is known to be bounded by a finite integer B (i.e., $|x[n]| < B$ for all n), it can be shown that $y[n]$ is bounded by a finite number C. We conclude that the given system is stable. Express C in terms of B and n_0.

Solution.

(a) Consider two arbitrary inputs $x_1[n]$ and $x_2[n]$.

$$x_1[n] \longrightarrow y_1[n] = \sum_{k=n-n_0}^{n+n_0} x_1[k]$$

$$x_2[n] \longrightarrow y_2[n] = \sum_{k=n-n_0}^{n+n_0} x_2[k]$$

Let $x_3[n]$ be a linear combination of $x_1[n]$ and $x_2[n]$. That is,

$$x_3[n] = ax_1[n] + bx_2[n]$$

where a and b are arbitrary scalars. If $x_3[n]$ is the input to the given system, then the corresponding output $y_3[n]$ is

$$\begin{aligned} y_3[n] &= \sum_{k=n-n_0}^{n+n_0} x_3[k] \\ &= \sum_{k=n-n_0}^{n+n_0} (ax_1[k] + bx_2[k]) = a\sum_{k=n-n_0}^{n+n_0} x_1[k] + b\sum_{k=n-n_0}^{n+n_0} x_2[k] \\ &= ay_1[n] + by_2[n] \end{aligned}$$

Therefore, the system is linear.

(b) Consider an arbitrary input $x_1[n]$. Let

$$y_1[n] = \sum_{k=n-n_0}^{n+n_0} x_1[k]$$

be the corresponding output. Consider a second input $x_2[n]$ obtained by shifting $x_1[n]$ in time:

$$x_2[n] = x_1[n - n_1]$$

The output corresponding to this input is

$$y_2[n] = \sum_{k=n-n_0}^{n+n_0} x_2[k] = \sum_{k=n-n_0}^{n+n_0} x_1[k - n_1] = \sum_{k=n-n_1-n_0}^{n-n_1+n_0} x_1[k]$$

Also note that

$$y_1[n - n_1] = \sum_{k=n-n_1-n_0}^{n-n_1+n_0} x_1[k].$$

Therefore,

$$y_2[n] = y_1[n - n_1]$$

This implies that the system is time-invariant.

(c) If $|x[n]| < B$, then

$$y[n] \le (2n_0 + 1)B$$

Therefore, $C \le (2n_0 + 1)B$.

Question.

Let $x(t)$ be a continuous-time signal, and let

$$y_1(t) = x(2t) \text{ and } y_2(t) = x(t/2).$$

The signal $y_1(t)$ represents a speeded up version of $x(t)$ in the sense that the duration of the signal is cut in half. Similarly, $y_2(t)$ represents a slowed down version of $x(t)$ in the sense that the duration of the signal is doubled. Consider the following statements:

(1) If $x(t)$ is periodic, then $y_1(t)$ is periodic.
(2) If $y_1(t)$ is periodic, then $x(t)$ is periodic.
(3) If $x(t)$ is periodic, then $y_2(t)$ is periodic.
(4) If $y_2(t)$ is periodic, then $x(t)$ is periodic.

For each of these statements, determine whether it is true, and if so, determine the relationship between the fundamental periods of the two signals considered in the statement. If the statement is not true, produce a counterexample to it.

Solution.

All statements are true.
(1) $x(t)$ periodic with period T; $y_1(t)$ periodic, period $T/2$.
(2) $y_1(t)$ periodic, period T; $x(t)$ periodic, period $2T$.
(3) $x(t)$ periodic, period T; $y_2(t)$ periodic, period $2T$.
(4) $y_2(t)$ periodic, period T; $x(t)$ periodic, period $T/2$.

Question.

Let $x[n]$ be a discrete-time signal, and let

$$y_1[n] = x[2n] \text{ and } y_2[n] = \begin{cases} x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}.$$

The signals $y_1[n]$ and $y_2[n]$ respectively represent in some sense the speeded up and slowed down versions of $x[n]$. However, it should be noted that the discrete-time notions of speeded up and slowed down have subtle differences with respect to their continuous-time counterparts. Consider the following statements:

(1) If $x[n]$ is periodic, then $y_1[n]$ is periodic.
(2) If $y_1[n]$ is periodic, then $x[n]$ is periodic.
(3) If $x[n]$ is periodic, then $y_2[n]$ is periodic.
(4) If $y_2[n]$ is periodic, then $x[n]$ is periodic.

For each of these statements, determine whether it is true, and if so, determine the relationship between the fundamental periods of the two signals considered in the statement. If the statement is not true, produce a counterexample to it.

Solution.

(1) True. $x[n] = x[n+N]; y_1[n] = y_1[n+N_0]$. i.e. periodic with $N_0 = N/2$ if N is even, and with period $N_0 = N$ if N is odd.

(2) False. $y_1[n]$ periodic does no imply $x[n]$ is periodic. i.e. let $x[n] = g[n] + h[n]$ where

$$g[n] = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \quad \text{and} \quad h[n] = \begin{cases} 0, & n \text{ even} \\ (1/2)^n, & n \text{ odd} \end{cases}.$$

Then $y_1[n] = x[2n]$ is periodic but $x[n]$ is clearly not periodic.

(3) True. $x[n+N] = x[n]$; $y_2[n+N_0] = y_2[n]$ where $N_0 = 2N$

(4) True. $y_2[n+N] = y_2[n]$; $x[n+N_0] = x[n]$ where $N_0 = N/2$

Question.

Consider a system S with input $x[n]$ and output $y[n]$ related by

$$y[n] = x[n]\{g[n] + g[n-1]\}.$$

(a) If $g[n] = 1$ for all n, show that S is time invariant.
(b) If $g[n] = n$, show that S is not time invariant.
(c) If $g[n] = 1 + (-1)^n$, show that S is time invariant.

Solution.

(a) $y[n] = 2x[n]$. **Therefore, the system is time invariant.**

(b) $y[n] = (2n-1)x[n]$. **This is not time-invariant because** $y[n-N_0] \neq (2n-1)x[n-N_0]$.

(c) $y[n] = x[n]\{1+(-1)^n + 1 + (-1)^{n-1}\} = 2x[n]$. **Therefore, the system is time invariant.**

Question.

Evaluate each of the following integrals, and express your answer in Cartesian (rectangular) form:

(a) $\int_0^4 e^{j\pi t/2}dt$ **(b)** $\int_0^6 e^{j\pi t/2}dt$
(c) $\int_2^8 e^{j\pi t/2}dt$ **(d)** $\int_0^\infty e^{-(1+j)t}dt$
(e) $\int_0^\infty e^{-t}\cos(t)dt$ **(f)** $\int_0^\infty e^{-2t}\sin(3t)dt$

Solution.

(a) The desired integral is

$$\int_0^4 e^{j\pi t/2}dt = \left.\frac{e^{\pi t/2}}{j\pi/2}\right|_0^4 = 0.$$

(b) The desired integral is

$$\int_0^6 e^{j\pi t/2}dt = \left.\frac{e^{\pi t/2}}{j\pi/2}\right|_0^6 = (2/j\pi)[e^{j3\pi} - 1] = \frac{4j}{\pi}.$$

(c) The desired integral is

$$\int_2^8 e^{j\pi t/2}dt = \left.\frac{e^{\pi t/2}}{j\pi/2}\right|_2^8 = (2/j\pi)[e^{j4\pi} - e^{j\pi}] = -\frac{4j}{\pi}.$$

(d) The desired integral is

$$\int_0^\infty e^{-(1+j)t}dt = \left.\frac{e^{-(1+j)t}}{-(1+j)}\right|_0^\infty = \frac{1}{1+j} = \frac{1-j}{2}.$$

(e) The desired integral is

$$\int_0^\infty e^{-t}\cos(t)dt = \int_0^\infty \left[\frac{e^{-(1+j)t} + e^{-(1-j)t}}{2}\right]dt = \frac{1/2}{1+j} + \frac{1/2}{1-j} = \frac{1}{2}.$$

(f) The desired integral is

$$\int_0^\infty e^{-2t}\sin(3t)dt = \int_0^\infty \left[\frac{e^{-(2-3j)t} - e^{-(2+3j)t}}{2j}\right]dt = \frac{1/2j}{2-3j} + \frac{1/2j}{2+3j} = \frac{3}{13}.$$

Question.

Determine and sketch the even and odd parts of the signals depicted in Figure P1.23.
Label your sketches carefully.

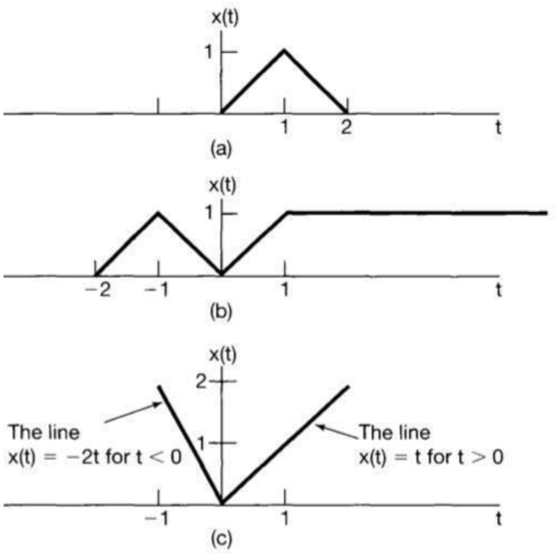


Figure P1.23

Solution.

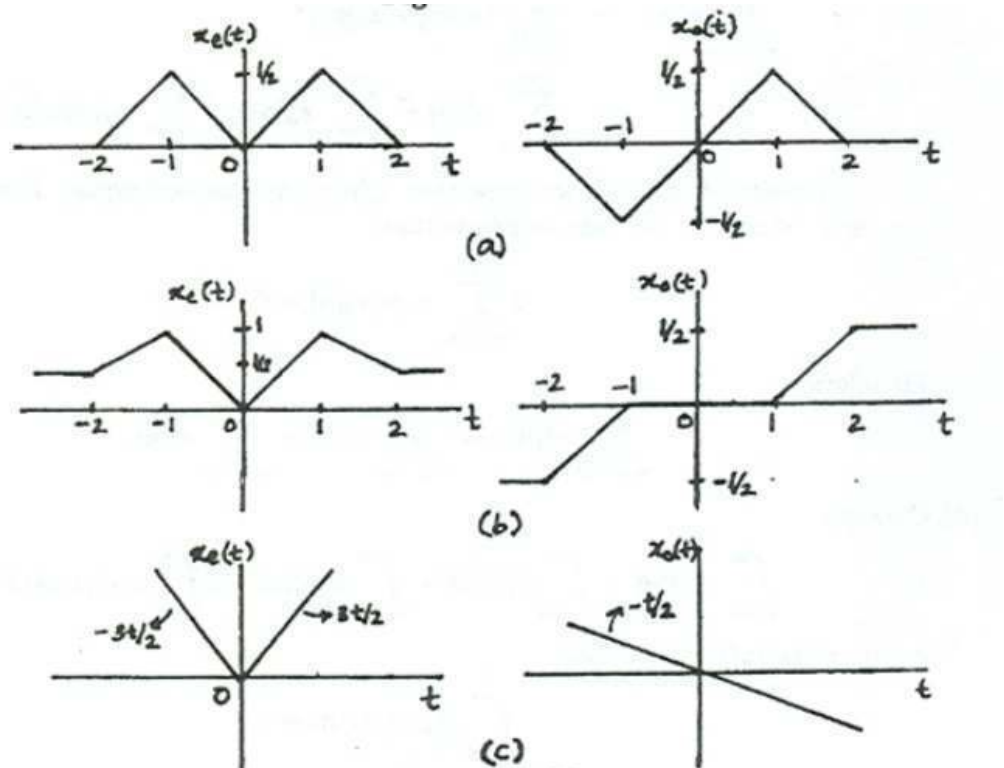


Figure S1.23

Question.

A continuous-time signal $x(t)$ is shown in Figure P1.21. Sketch and label carefully each of the following signals:

(a) $x(t-1)$ **(b)** $x(2-t)$ **(c)** $x(2t+1)$

(d) $x(4-\frac{t}{2})$ **(e)** $[x(t)+x(-t)]u(t)$ **(f)** $x(t)[\delta(t+\frac{3}{2})-\delta(t-\frac{3}{2})]$

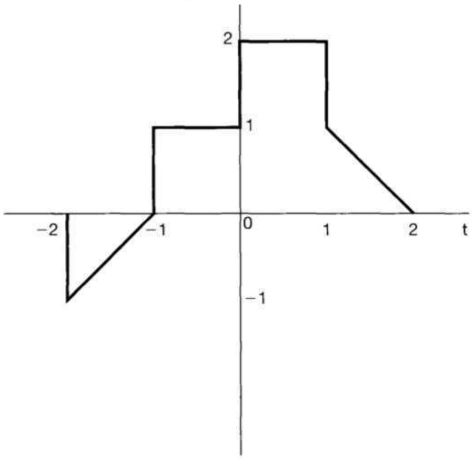
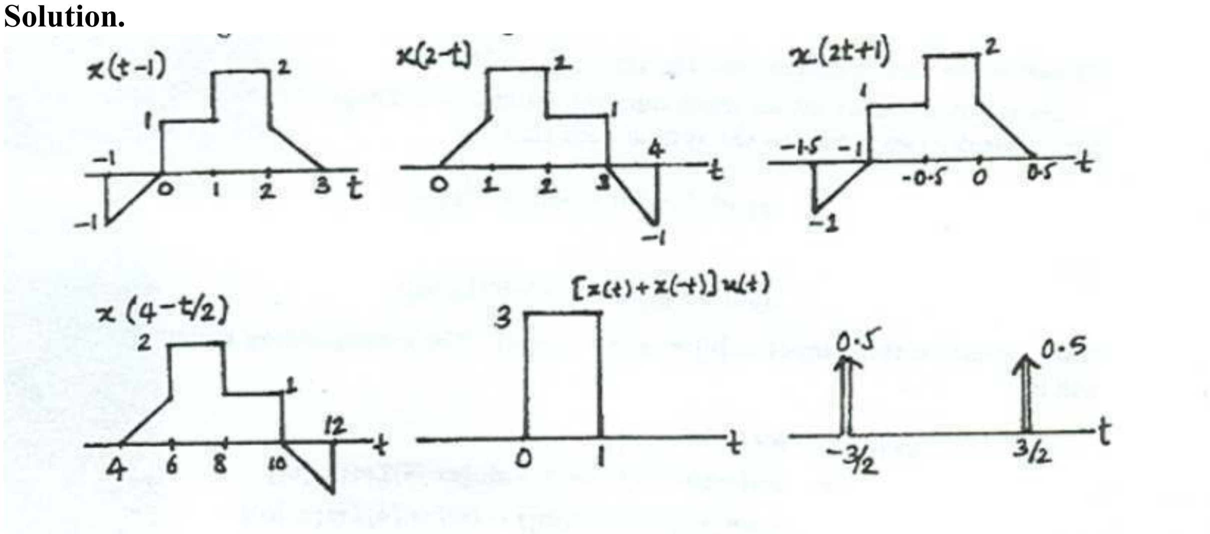


Figure P1.21

Solution.



Question.

A discrete-time signal is shown in Figure P1.22. Sketch and label carefully each of the following signals:

(a) $x[n-4]$ **(b)** $x[3-n]$ **(c)** $x[3n]$

(d) $x[3n+1]$ **(e)** $x[n]u[3-n]$ **(f)** $x[n-2]\delta[n-2]$

(g) $\frac{1}{2}x[n] + \frac{1}{2}(-1)^n x[n]$ **(h)** $x[(n-1)^2]$

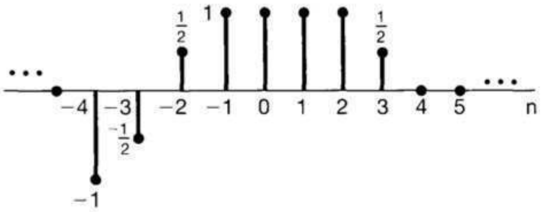


Figure P1.22

Solution.

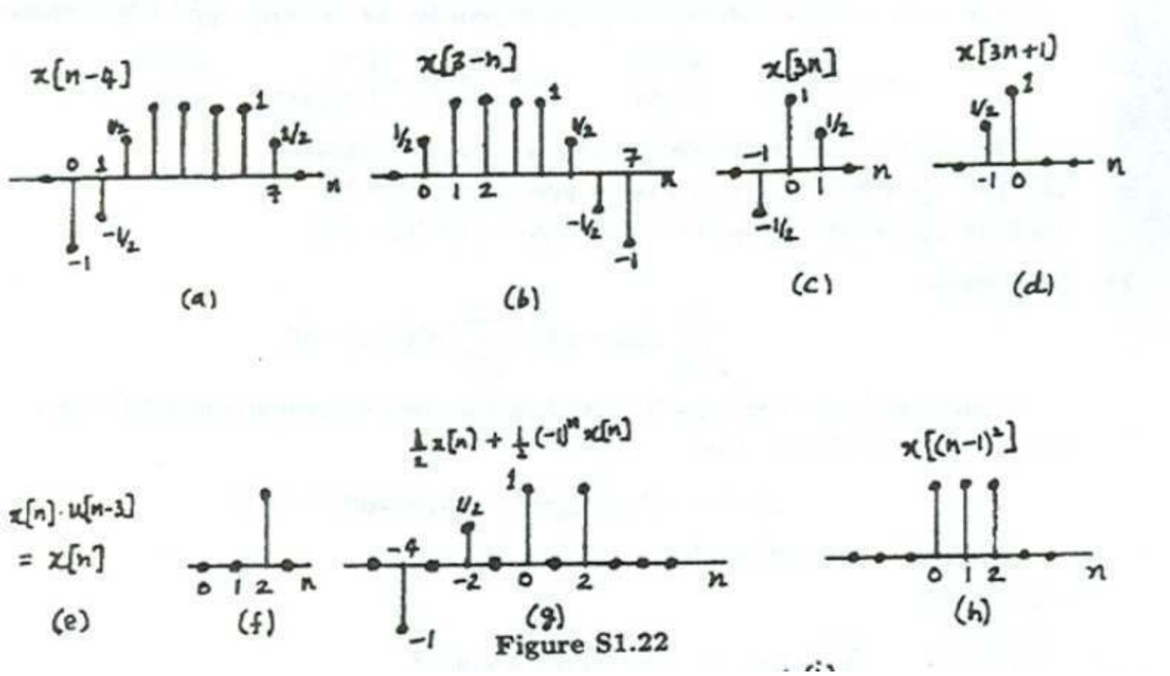


Figure S1.22

Question.

The relations considered in this problem are used on many occasions throughout the book.

(a) Prove the validity of the following expression:

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N, & \alpha = 1 \\ \frac{1-\alpha^N}{1-\alpha}, & \text{for any complex number } \alpha \neq 1 \end{cases}$$

This is often referred to as the *finite sum formula.*

(b) Show that if $|\alpha| < 1$, then

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}.$$

This is often referred to as the *infinite sum formula.*

(c) Show also if $|\alpha| < 1$, then

$$\sum_{n=0}^{\infty} n\alpha^n = \frac{\alpha}{(1-\alpha)^2}.$$

(d) Evaluate

$$\sum_{n=k}^{\infty} \alpha^n,$$

assuming that $|\alpha| < 1$.

Solution.

(a) For $\alpha = 1$, it is fairly obvious that

$$\sum_{n=0}^{N-1} \alpha^n = N.$$

For $\alpha \neq 1$, we may write

$$(1-\alpha)\sum_{n=0}^{N-1} \alpha^n = \sum_{n=0}^{N-1} \alpha^n - \sum_{n=0}^{N-1} \alpha^{n+1} = 1 - \alpha^N.$$

Therefore,

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}.$$

(b) For $|\alpha| < 1$,

$$\lim_{N\to\infty} \alpha^N = 0.$$

Therefore, from the result of the previous part,

$$\lim_{N\to\infty} \sum_{n=0}^{N-1} \alpha^n = \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}.$$

(c) Differentiating both sides of the result of part (b) wrt α, we get

$$\begin{aligned} \frac{d}{d\alpha}\left(\sum_{n=0}^{\infty} \alpha^N\right) &= \frac{d}{d\alpha}\left(\frac{1}{1-\alpha}\right) \\ \sum_{n=0}^{\infty} n\alpha^{n-1} &= \frac{1}{(1-\alpha)^2} \end{aligned}$$

(d) We may write

$$\sum_{n=k}^{\infty} \alpha^n = \alpha^k \sum_{n=0}^{\infty} \alpha^n = \frac{\alpha^k}{1-\alpha} \text{ for } |\alpha| < 1.$$

Question.

Determine and sketch the even and odd parts of the signals depicted in Figure P1.24. Label your sketches carefully.

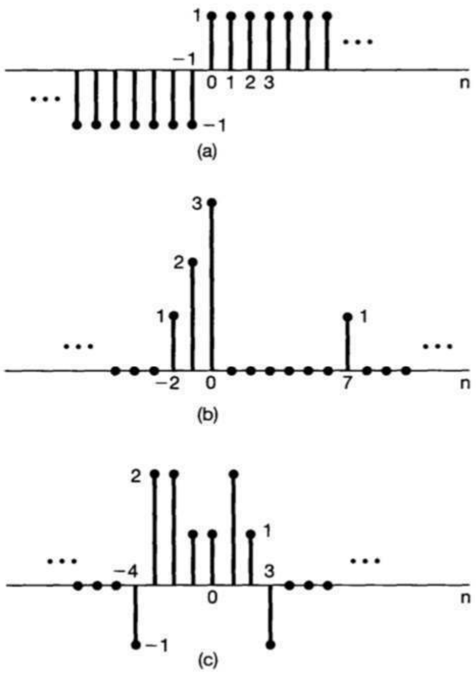
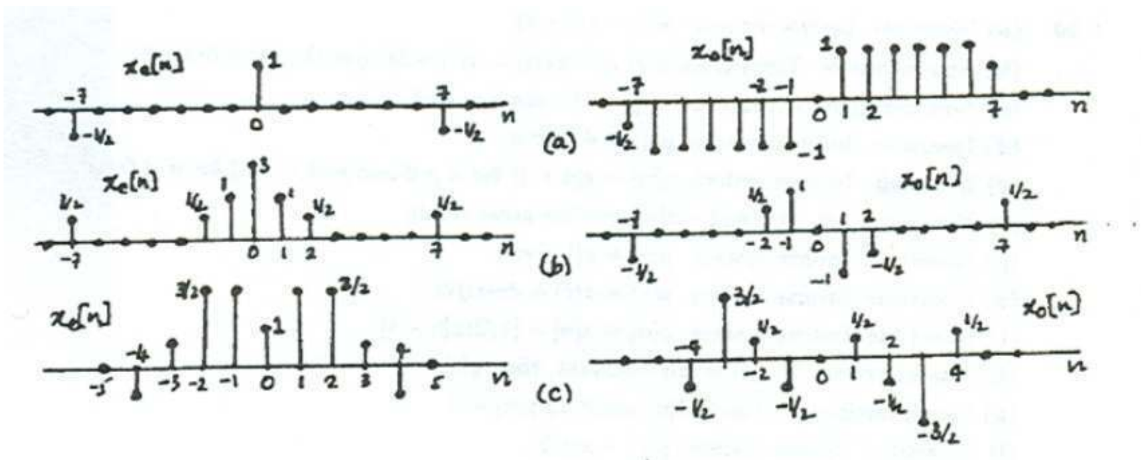


Figure P1.24

Solution.



Question.

Determine whether or not each of the following discrete-time signals is periodic. If the signal is periodic, determine its fundamental period.

(a) $x[n] = \sin(\frac{6\pi}{7}n + 1)$ **(b)** $x[n] = \cos(\frac{n}{8} - \pi)$ **(c)** $x[n] = \cos(\frac{\pi}{8}n^2)$

(d) $x[n] = \cos(\frac{\pi}{2}n)\cos(\frac{\pi}{4}n)$ **(e)** $x[n] = 2\cos(\frac{\pi}{4}n) + \sin(\frac{\pi}{8}n) - 2\cos(\frac{\pi}{2}n + \frac{\pi}{6})$

Solution.

(a) Periodic, period = 7.

(b) Not periodic.

(c) Periodic, period = 8.

(d) $x[n] = (1/2)[\cos(3\pi n/4) + \cos(\pi n/4)]$. Periodic, period = 8.

(e) Periodic, period = 16.

Question.

Determine whether or not each of the following continuous-time signals is periodic. If the signal is periodic, determine its fundamental period.

(a) $x(t) = 3\cos(4t + \frac{\pi}{3})$ **(b)** $x(t) = e^{j(\pi t-1)}$

(c) $x(t) = [\cos(2t - \frac{\pi}{3})]^2$ **(d)** $x(t) = \mathcal{E}v\{\cos(4\pi t)u(t)\}$

(e) $x(t) = \mathcal{E}v\{\sin(4\pi t)u(t)\}$ **(f)** $x(t) = \sum_{n=-\infty}^{\infty} e^{-(2t-n)}u(2t-n)$

Solution.

(a) Periodic, period $= 2\pi/(4) = \pi/2$.

(b) Periodic, period $= 2\pi/(\pi) = 2$.

(c) $x(t) = [1 + \cos(4t - 2\pi/3)]/2$. Periodic, period $= 2\pi/(4) = \pi/2$.

(d) $x(t) = \cos(4\pi t)/2$. Periodic, period $= 2\pi/(4\pi) = 1/2$.

(e) $x(t) = [\sin(4\pi t)u(t) - \sin(4\pi t)u(-t)]/2$. Not periodic.

(f) Not periodic.