BLM2041 Signals and Systems

Syllabus

The Instructors:

Doç. Dr. Ali Can Karaca

ackaraca@yildiz.edu.tr

Dr. Ahmet Elbir

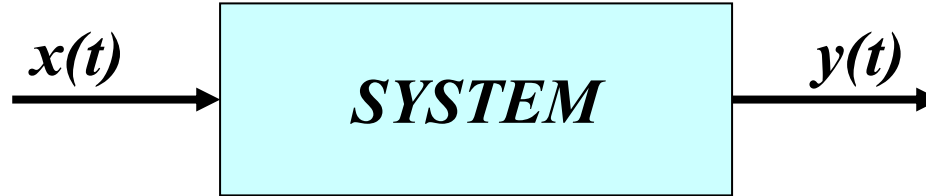aelbir@yildiz.edu.tr

BLM2041 Signals and Systems

Sampling & Aliasing

LECTURE OBJECTIVES

- SAMPLING can cause ALIASING
  - **Sampling Theorem**
    - Sampling Rate > 2(Highest Frequency)
- Spectrum for digital signals, $x[n]$
  - Normalized Frequency

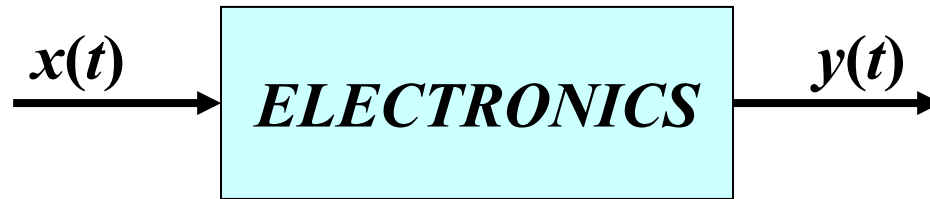$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi \ell$$

ALIASING

SYSTEMS Process Signals

$x(t)$ → [*SYSTEM*] → $y(t)$

- PROCESSING GOALS:
  - Change $x(t)$ into $y(t)$
    - For example, more BASS
  - Improve $x(t)$,
    - e.g., image deblurring
  - Extract information from $x(t)$

System IMPLEMENTATION

- ## ANALOG/ELECTRONIC:

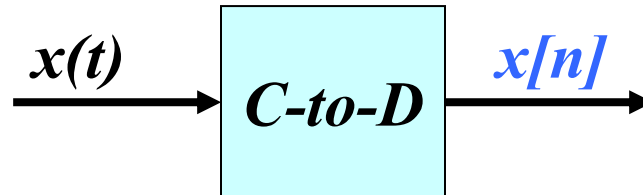 - ### Circuits: resistors, capacitors, op-amps

$$x(t) \rightarrow \boxed{ELECTRONICS} \rightarrow y(t)$$

- ## DIGITAL/MICROPROCESSOR

  - ### Convert $x(t)$ to numbers stored in memory

$$x(t) \rightarrow \boxed{A\text{-}to\text{-}D} \xrightarrow{x[n]} \boxed{COMPUTER} \xrightarrow{y[n]} \boxed{D\text{-}to\text{-}A} \rightarrow y(t)$$
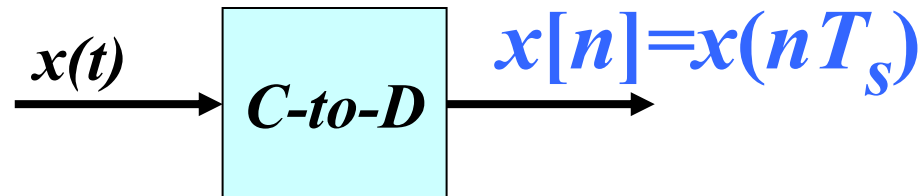
SAMPLING $x(t)$

- ## SAMPLING PROCESS
    - Convert $x(t)$ to numbers $x[n]$
    - "n" is an integer;
    - $x[n]$ is a sequence of values
    - Think of "n" as the storage address in memory

- ## UNIFORM SAMPLING at $t = nT_s$
    - IDEAL: $x[n] = x(nT_s)$

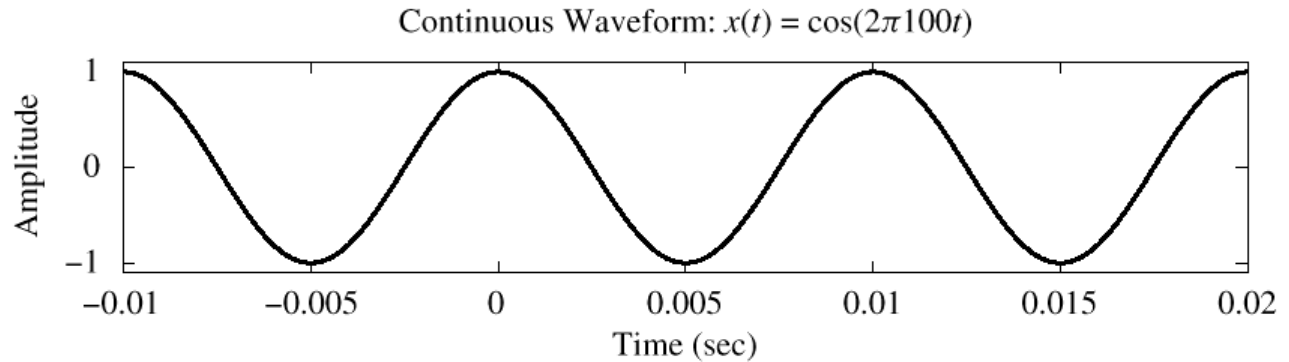$x(t) \longrightarrow$ **C-to-D** $\longrightarrow x[n]$

SAMPLING RATE, f_s

- SAMPLING RATE (f_s)
  - $f_s = 1/T_s$
    - NUMBER of SAMPLES PER SECOND
      - $T_s = 125$ microsec $\rightarrow f_s = 8000$ samples/sec
      - UNITS ARE HERTZ: 8000 Hz

- UNIFORM SAMPLING at $t = nT_s = n/f_s$
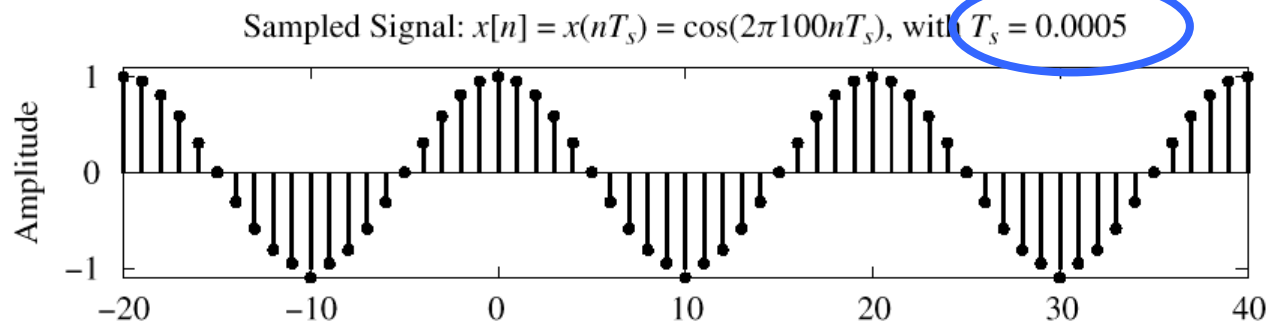 - IDEAL: $x[n] = x(nT_s) = x(n/f_s)$

$$x(t) \rightarrow \boxed{C\text{-}to\text{-}D} \rightarrow x[n] = x(nT_s)$$

SAMPLING RATE, f_s

Continuous Waveform: $x(t) = \cos(2\pi 100t)$

$f = 100\text{Hz}$

Sampled Signal: $x[n] = x(nT_s) = \cos(2\pi 100 nT_s)$, with $T_s = 0.0005$

$f_s = 2\,\text{kHz}$

Sampled Signal: $x[n] = x(nT_s) = \cos(2\pi 100 nT_s)$, with $T_s = 0.002$

$f_s = 500\text{Hz}$

Sample Index (n)
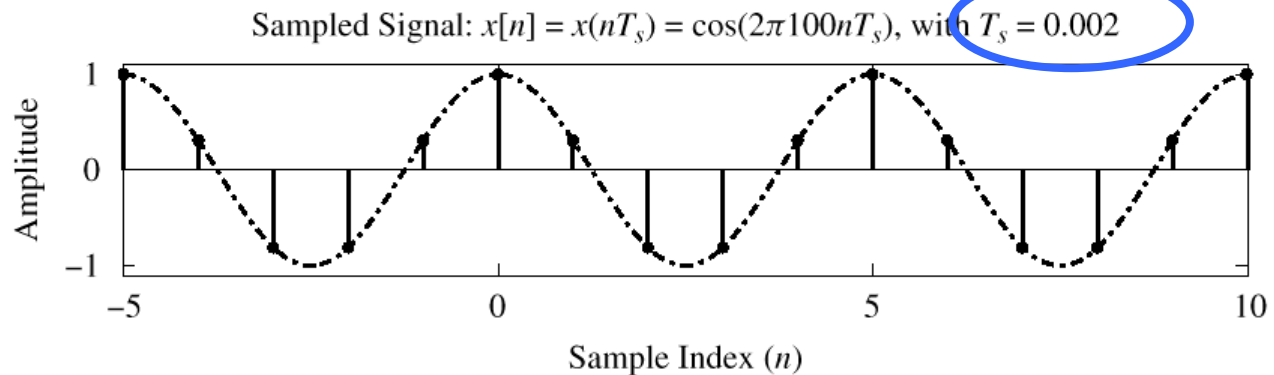
SAMPLING THEOREM

- HOW OFTEN ?
  - DEPENDS on FREQUENCY of SINUSOID
  - ANSWERED by SHANNON/NYQUIST Theorem
  - ALSO DEPENDS on "**RECONSTRUCTION**"

Shannon Sampling Theorem

A continuous-time signal $x(t)$ with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{max}$.

Reconstruction? Which One?

Given the samples, draw a sinusoid through the values

Two continuous cosine functions drawn through the same samples
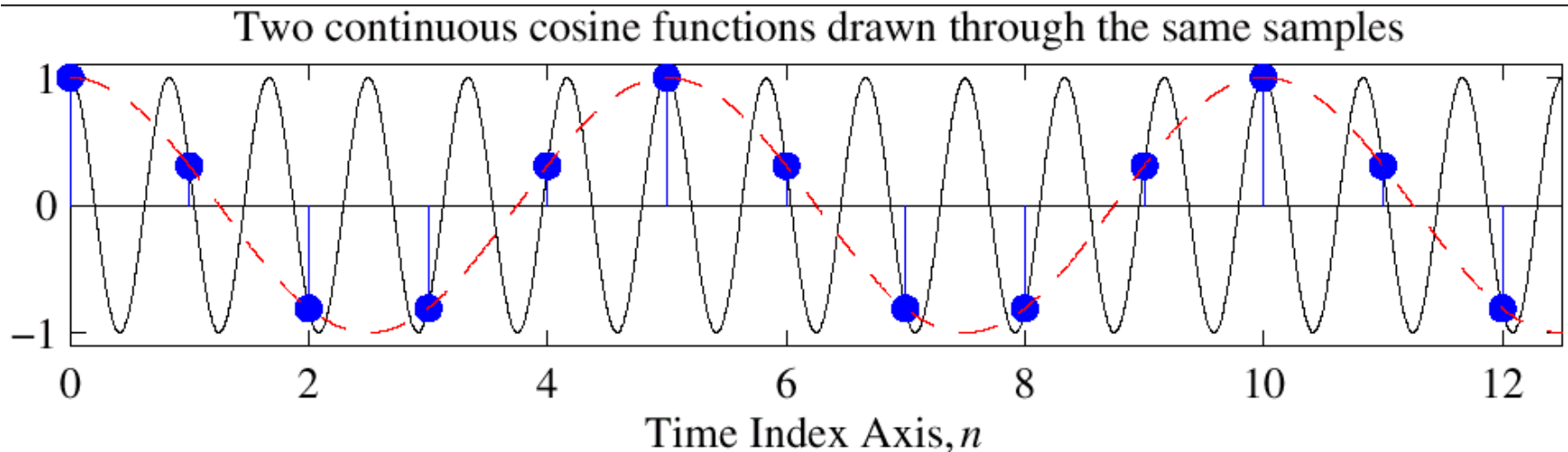
$$x[n] = \cos(0.4\pi n)$$

When n is an integer
$\cos(0.4\pi n) = \cos(2.4\pi n)$

STORING DIGITAL SOUND

- **$x[n]$** is a SAMPLED SINUSOID
  - A list of numbers stored in memory
- EXAMPLE: audio CD
- CD rate is 44,100 samples per second
  - 16-bit samples
  - Stereo uses 2 channels
- Number of bytes for 1 minute is
  - $2 \times (16/8) \times 60 \times 44100 = 10.584$ Mbytes

DISCRETE-TIME SINUSOID

- Change x(t) into x[n]
- **DERIVATION**

$$x(t) = A\cos(\omega t + \varphi)$$

$$x[n] = x(nT_s) = A\cos(\omega n T_s + \varphi)$$
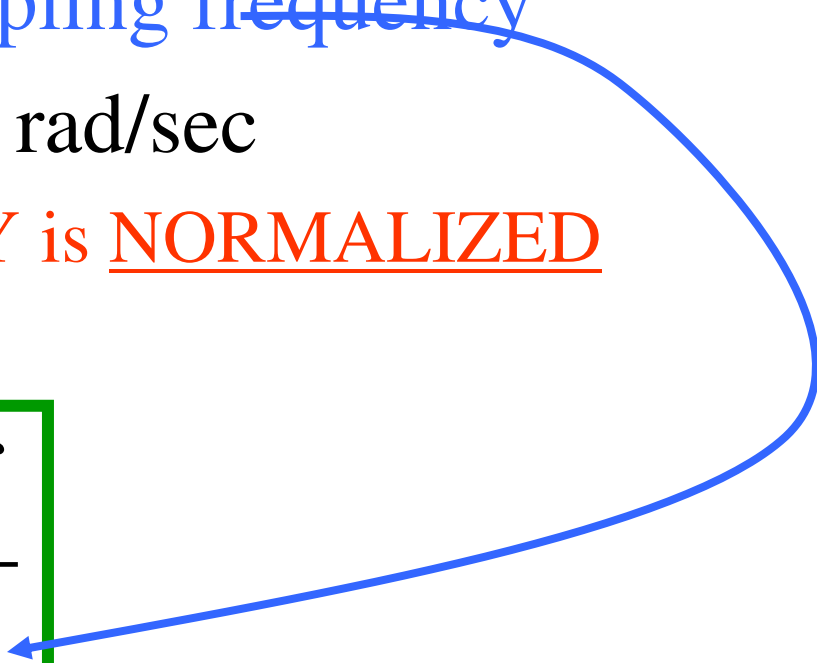
$$x[n] = A\cos((\omega T_s)n + \varphi)$$

$$x[n] = A\cos(\hat{\omega}n + \varphi)$$

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$ ***DEFINE DIGITAL FREQUENCY***
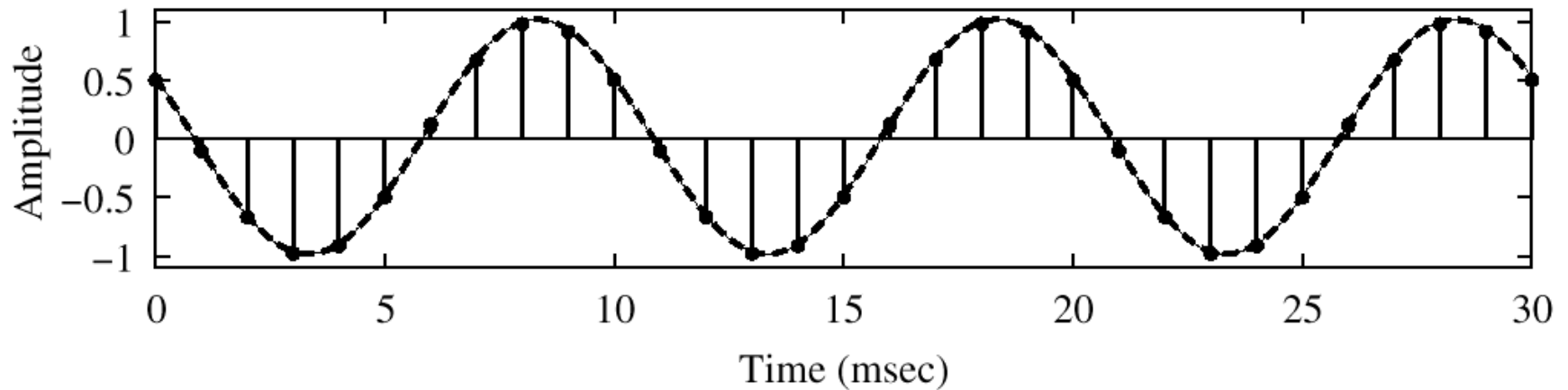
12

DIGITAL FREQUENCY

- Digital frequency $\hat{\omega}$ VARIES from **0** to **2π**, as f varies from 0 to the sampling frequency

- UNITS are radians, **not** rad/sec
  - DIGITAL FREQUENCY is NORMALIZED

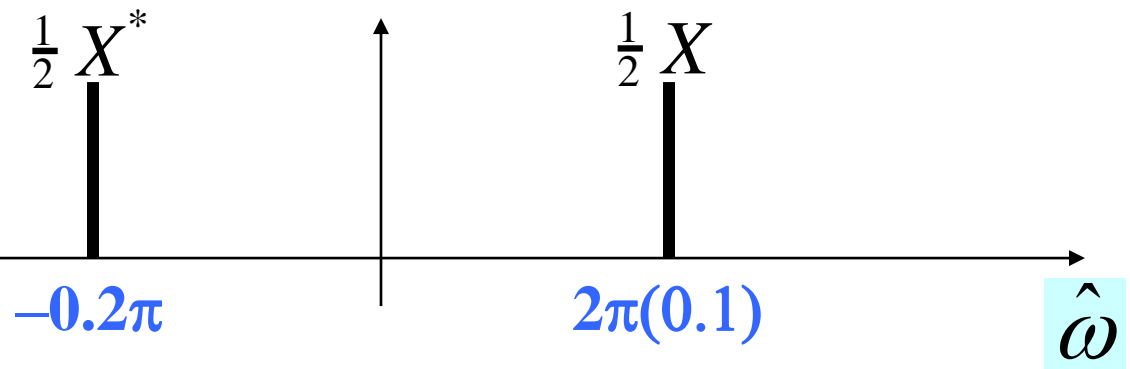$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}$$

SPECTRUM (DIGITAL)

$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$

100-Hz Cosine Wave: Sampled with $T_s = 1$ msec (1000 Hz)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 1\,\text{kHz}$$

$\tfrac{1}{2} X^{*}$

$\tfrac{1}{2} X$

−0.2π

2π(0.1)

$\hat{\omega}$
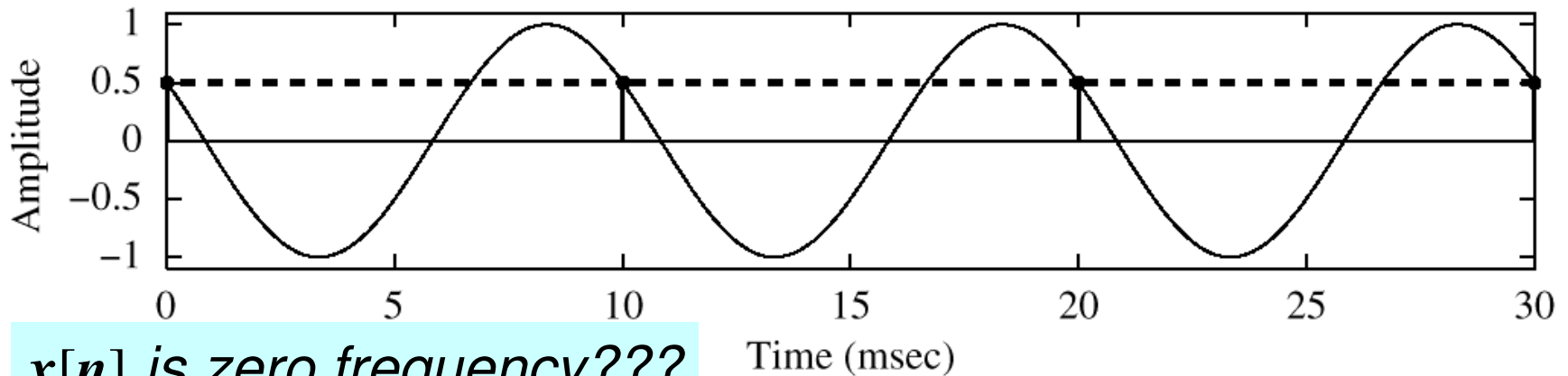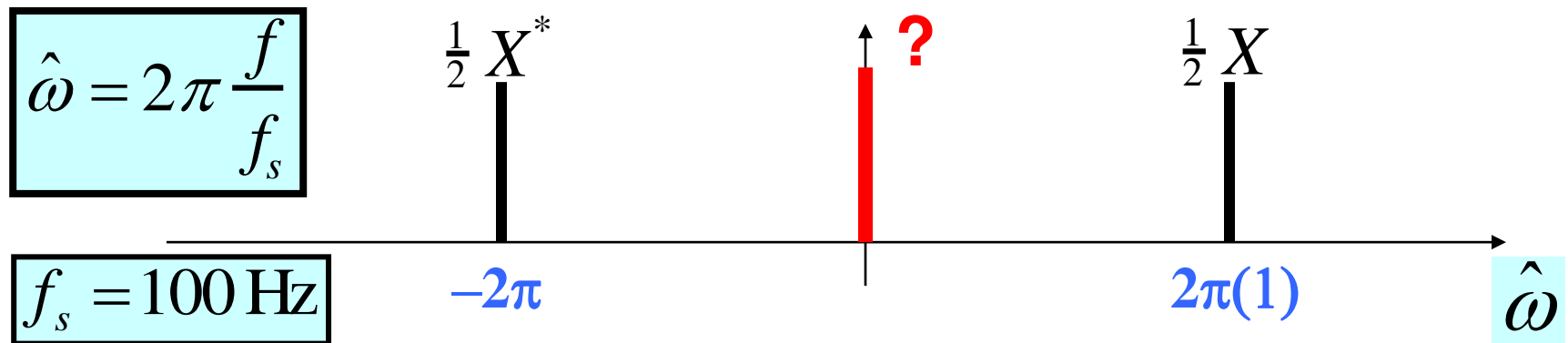
SPECTRUM (DIGITAL) ???

$$x[n] = A\cos(2\pi(100)(n/100) + \varphi)$$

100-Hz Cosine Wave: Sampled with $T_s = 10$ msec (100 Hz)

$x[n]$ *is zero frequency???*

$$\hat{\omega} = 2\pi\frac{f}{f_s}$$

$$f_s = 100\,\text{Hz}$$

$\frac{1}{2}X^*$

?

$\frac{1}{2}X$

-2π

$2\pi(1)$

$\hat{\omega}$

The REST of the STORY

- Spectrum of x[n] has more than one line for each complex exponential
  - Called **ALIASING**
  - **MANY SPECTRAL LINES**

- SPECTRUM is PERIODIC with period = **2π**
  - Because

$$A\cos(\hat{\omega}n + \varphi) = A\cos((\hat{\omega} + 2\pi)n + \varphi)$$

ALIASING DERIVATION

- Other Frequencies give the same $\hat{\omega}$

$$x_1(t) = \cos(400\pi t) \quad \text{sampled at } f_s = 1000\,\text{Hz}$$

$$x_1[n] = \cos(400\pi \tfrac{n}{1000}) = \cos(0.4\pi n)$$

$$x_2(t) = \cos(2400\pi t) \quad \text{sampled at } f_s = 1000\,\text{Hz}$$

$$x_2[n] = \cos(2400\pi \tfrac{n}{1000}) = \cos(2.4\pi n)$$

$$x_2[n] = \cos(2.4\pi n) = \cos(0.4\pi n + 2\pi n) = \cos(0.4\pi n)$$

$$\Rightarrow x_2[n] = x_1[n] \qquad 2400\pi - 400\pi = 2\pi(1000)$$

ALIASING DERIVATION–2

- Other Frequencies give the same $\hat{\boldsymbol{\omega}}$

If $x(t) = A\cos(2\pi(\underline{f + \ell f_s})t + \varphi)$ $\boxed{t \leftarrow \dfrac{n}{f_s}}$

and we want : $x[n] = A\cos(\hat{\omega}n + \varphi)$

then : $\hat{\omega} = \dfrac{2\pi(f + \ell f_s)}{f_s} = \dfrac{2\pi f}{f_s} + \dfrac{2\pi\ell f_s}{f_s}$

$$\boxed{\hat{\omega} = \omega T_s = \dfrac{2\pi f}{f_s} \boxed{+ 2\pi\ell}}$$

ALIASING CONCLUSIONS

- ADDING f_s or $2f_s$ or $-f_s$ to the FREQ of $x(t)$ gives exactly the same $x[n]$
  - The samples, $x[n] = x(n/f_s)$ are EXACTLY THE SAME VALUES

- GIVEN $x[n]$, WE CANNOT DISTINGUISH f_o FROM $(f_o + f_s)$ or $(f_o + 2f_s)$

NORMALIZED FREQUENCY

- DIGITAL FREQUENCY

Normalized Radian Frequency

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} \boxed{+ 2\pi \ell}$$

Normalized Cyclic Frequency

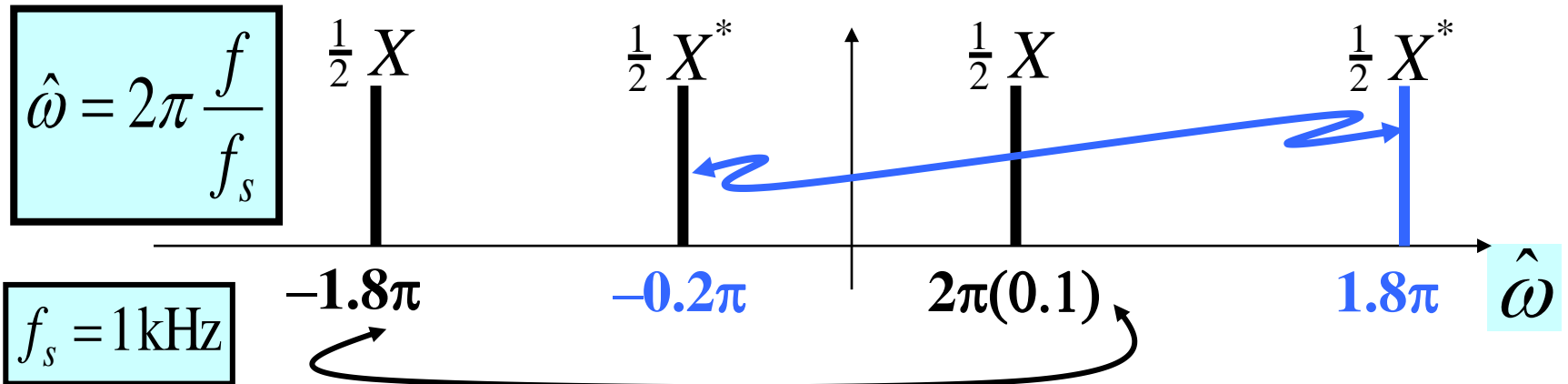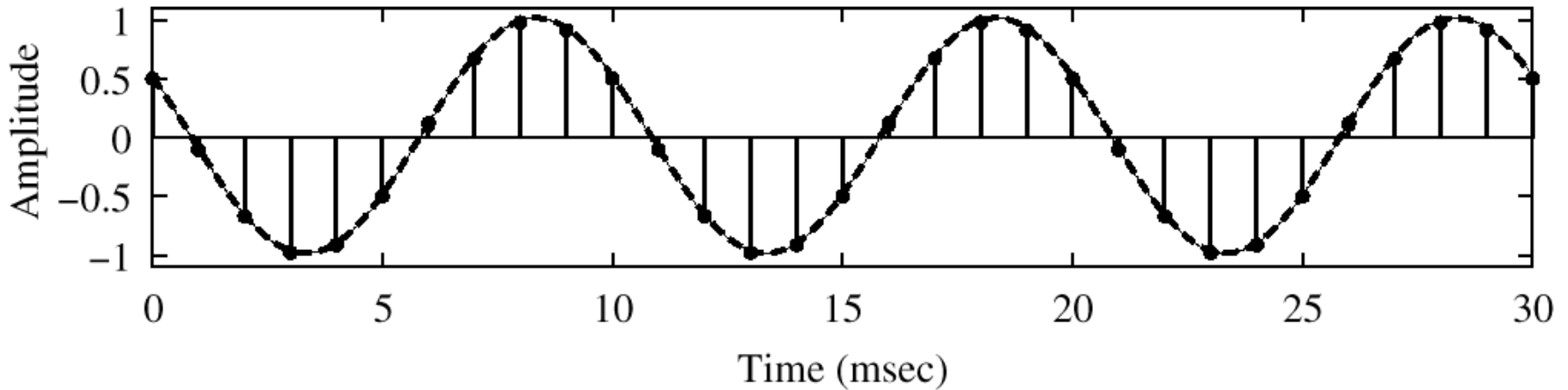$$\hat{f} = \hat{\omega}/(2\pi) = f T_s = f/f_s$$

SPECTRUM for *x*[*n*]

- PLOT versus NORMALIZED FREQUENCY
- INCLUDE **ALL** SPECTRUM LINES
  - ALIASES
    - ADD MULTIPLES of 2π
    - SUBTRACT MULTIPLES of 2π
  - FOLDED ALIASES
    - ALIASES of NEGATIVE FREQS

SPECTRUM (MORE LINES)

$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$

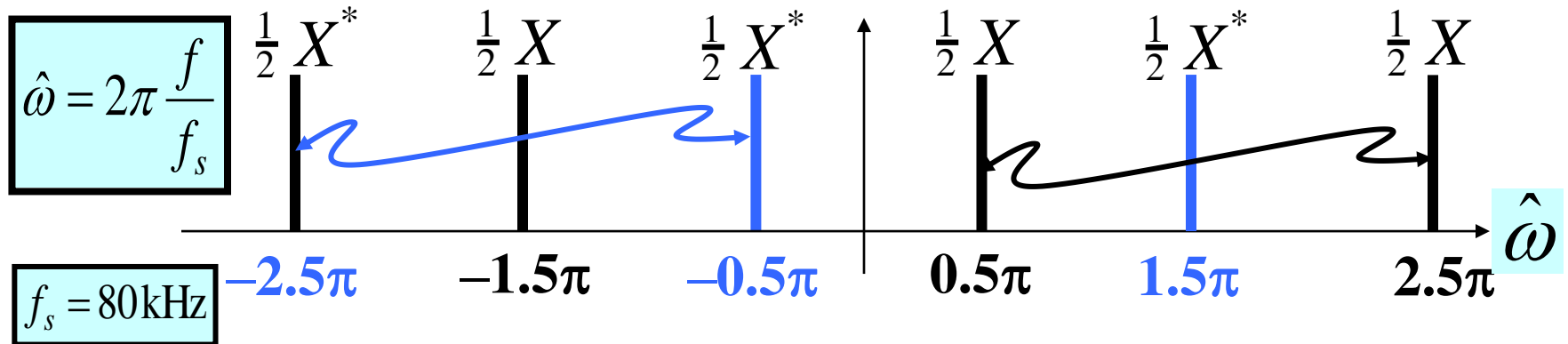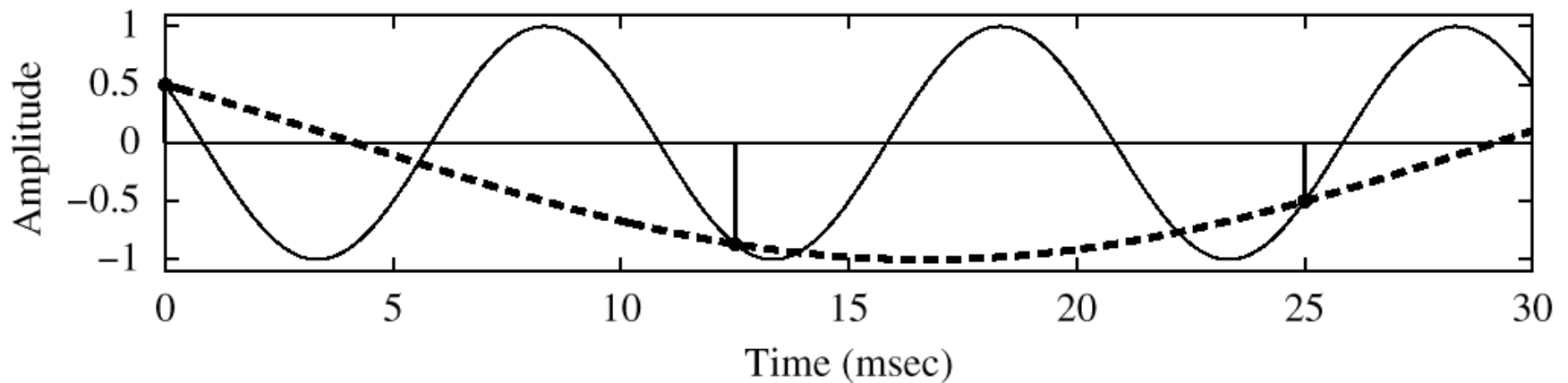100-Hz Cosine Wave: Sampled with $T_s = 1$ msec (1000 Hz)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 1\text{kHz}$$

$\frac{1}{2}X$ $\frac{1}{2}X^*$ $\frac{1}{2}X$ $\frac{1}{2}X^*$

-1.8π -0.2π $2\pi(0.1)$ 1.8π $\hat{\omega}$

SPECTRUM (ALIASING CASE)

$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$

100-Hz Cosine Wave: Sampled with $T_s = 12.5$ msec (80 Hz)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 80\text{kHz}$$

$\frac{1}{2}X^*$ $\frac{1}{2}X$ $\frac{1}{2}X^*$ $\frac{1}{2}X$ $\frac{1}{2}X^*$ $\frac{1}{2}X$

-2.5π -1.5π -0.5π 0.5π 1.5π 2.5π $\hat{\omega}$

SPECTRUM (FOLDING CASE)

$$x[n] = A\cos(2\pi(100)(n/125) + \varphi)$$

100-Hz Cosine Wave: Sampled with $T_s = 8$ msec (125 Hz)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 125\text{Hz}$$

$\frac{1}{2}X^*$ $\frac{1}{2}X$ $\frac{1}{2}X^*$ $\frac{1}{2}X$

-1.6π -0.4π 0.4π 1.6π

$\hat{\omega}$