



BLM3620 Digital Signal Processing

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Yıldız Technical University – Computer Engineering

Lecture #4 – Sampling and Aliasing

- Sampling
- Principal of Aliasing
- Spectrum of a Discrete-Time Signal
- Over-Sampling & Under-Sampling
- Stroboscopic effect

Important Materials:

- James H. McClellan, R. W. Schafer, M. A. Yoder, *DSP First Second Edition*, Pearson, 2015.
- Lizhe Tan, Jean Jiang, *Digital Signal Processing: Fundamentals and Applications*, Third Edition, Academic Press, 2019.

Auxiliary Materials:

- Prof. Sarp Ertürk, *Sayısal İşaret İşleme*, Birsen Yayınevi.
- Prof. Nizamettin Aydın, DSP Lecture Notes.
- J. G. Proakis, D. K. Manolakis, *Digital Signal Processing Fourth Edition*, Pearson, 2014.
- J. K. Perin, *Digital Signal Processing, Lecture Notes*, Stanford University, 2018.

Syllabus



Week	Lectures
1	Introduction to DSP and MATLAB
2	Sinuzoids and Complex Exponentials
3	Spectrum Representation
4	Sampling and Aliasing
5	Discrete Time Signal Properties and Convolution
6	Convolution and FIR Filters
7	Frequency Response of FIR Filters
8	Midterm Exam
9	Discrete Time Fourier Transform and Properties
10	Discrete Fourier Transform and Properties
11	Fast Fourier Transform and Windowing
12	z- Transforms
13	FIR Filter Design and Applications
14	IIR Filter Design and Applications
15	Final Exam

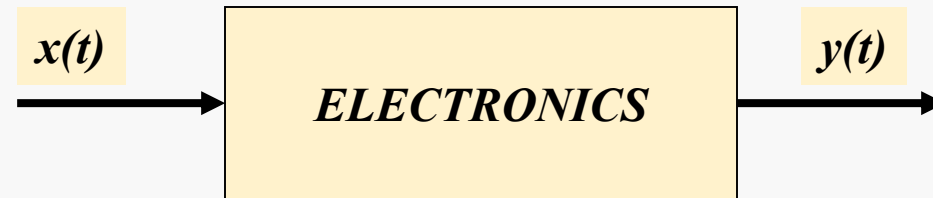
For more details -> Bologna page: <http://www.bologna.yildiz.edu.tr/index.php?r=course/view&id=5730&aid=3>

Remember: Analog & Digital Systems



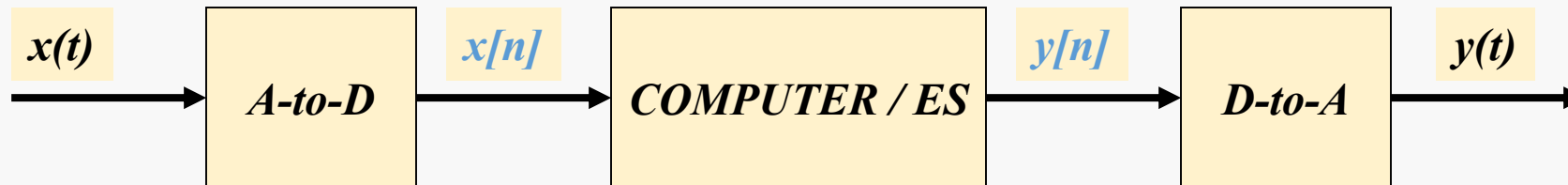
- **ANALOG/ELECTRONIC:**

- Circuits: resistors, capacitors, op-amps



- **DIGITAL/MICROPROCESSOR**

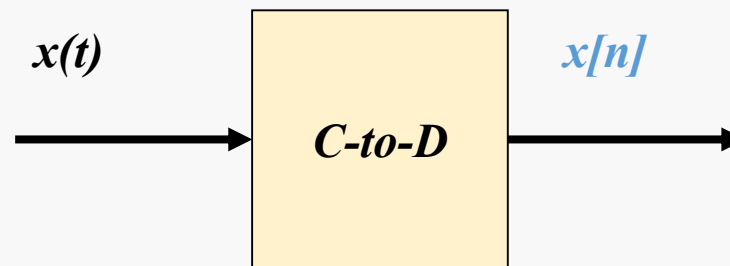
- Convert $x(t)$ to **numbers** stored in memory



Sampling of Analog Signals



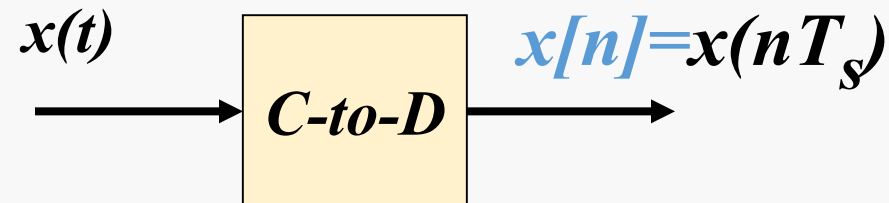
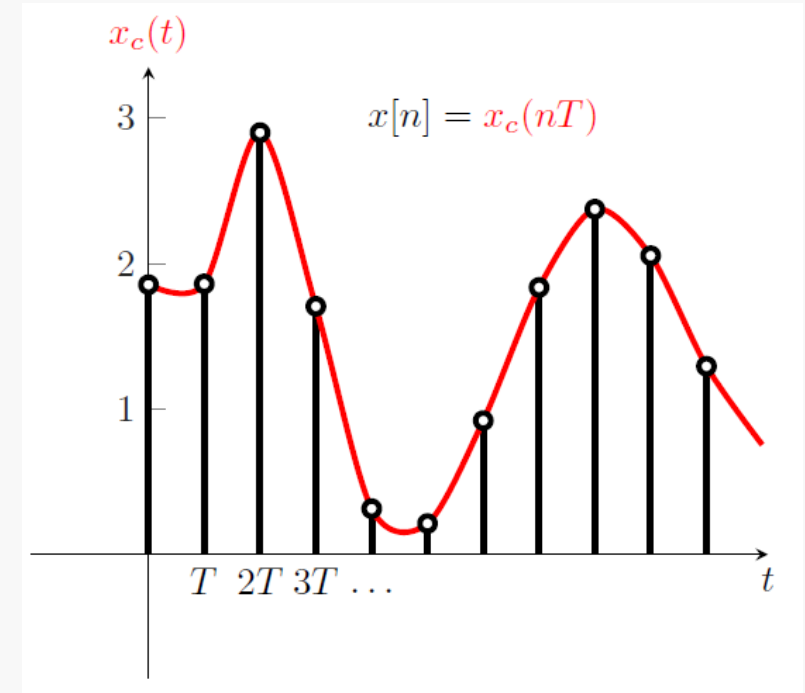
- SAMPLING PROCESS
 - Convert $x(t)$ to **numbers** $x[n]$
 - “n” is an integer index; $x[n]$ is a sequence of values
 - Think of “n” as the storage address in memory
- UNIFORM SAMPLING at $t = nT_s$
 - IDEAL: $x[n] = x(nT_s)$



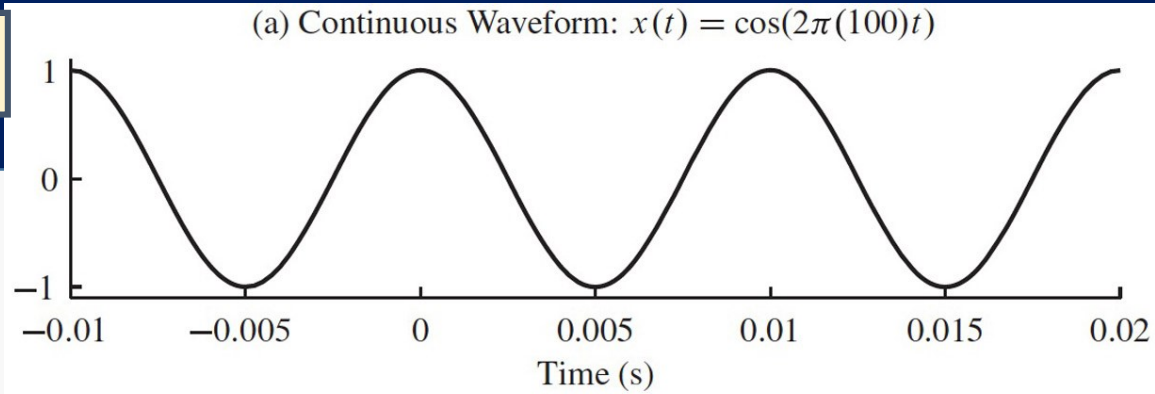
Sampling of Analog Signals



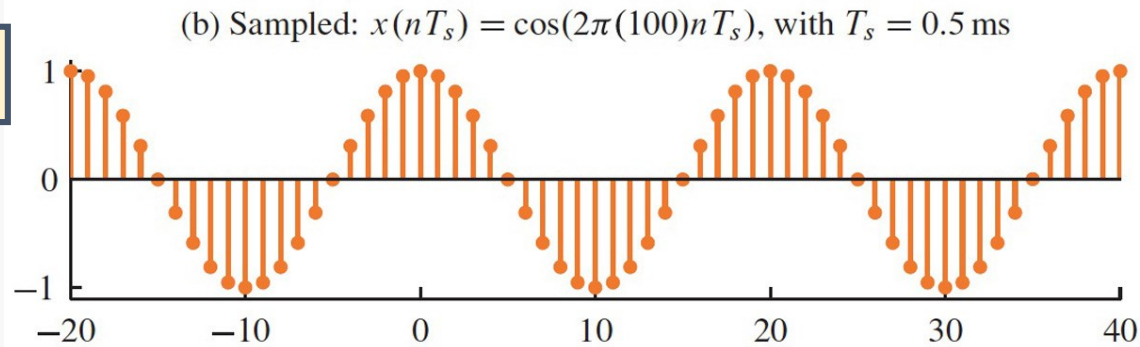
- SAMPLING RATE (f_s)
 - $f_s = 1/T_s$
 - NUMBER of SAMPLES PER SECOND
 - $T_s = 125$ microsec $\rightarrow f_s = 8000$ samples/sec
 - UNITS of f_s ARE HERTZ: 8000 Hz
- UNIFORM SAMPLING at $t = nT_s = n/f_s$
 - IDEAL: $x[n] = x(nT_s) = x(n/f_s)$



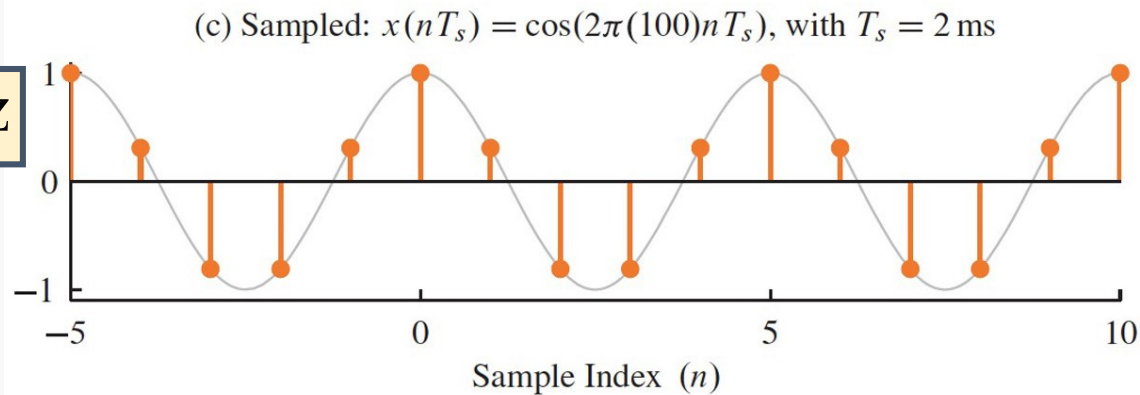
$$f = 100\text{Hz}$$



$$f_s = 2 \text{ kHz}$$



$$f_s = 500\text{Hz}$$



Which one provides the most accurate representation of $x(t)$?

- HOW OFTEN DO WE NEED TO SAMPLE?
 - DEPENDS on FREQUENCY of SINUSOID
 - ANSWERED by SHANNON/NYQUIST Theorem
 - ALSO DEPENDS on “**RECONSTRUCTION**”

Shannon Sampling Theorem

A continuous-time signal $x(t)$ with frequencies no higher than f_{\max} can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{\max}$.

- “Nyquist Rate” Sampling
 - $f_s > \underline{\text{TWICE}}$ the HIGHEST Frequency in $x(t)$
 - “Sampling above the Nyquist rate”
- **BANDLIMITED SIGNALS**
 - DEF: HIGHEST FREQUENCY COMPONENT in SPECTRUM of $x(t)$ is finite
 - NON-BANDLIMITED EXAMPLE
 - TRIANGLE WAVE is **NOT** BANDLIMITED

- Change $x(t)$ into $x[n]$ DERIVATION

$$x(t) = A \cos(\omega t + \varphi)$$

$$x[n] = x(nT_s) = A \cos(\omega nT_s + \varphi)$$

$$x[n] = A \cos((\omega T_s)n + \varphi)$$

$$x[n] = A \cos(\hat{\omega}n + \varphi)$$

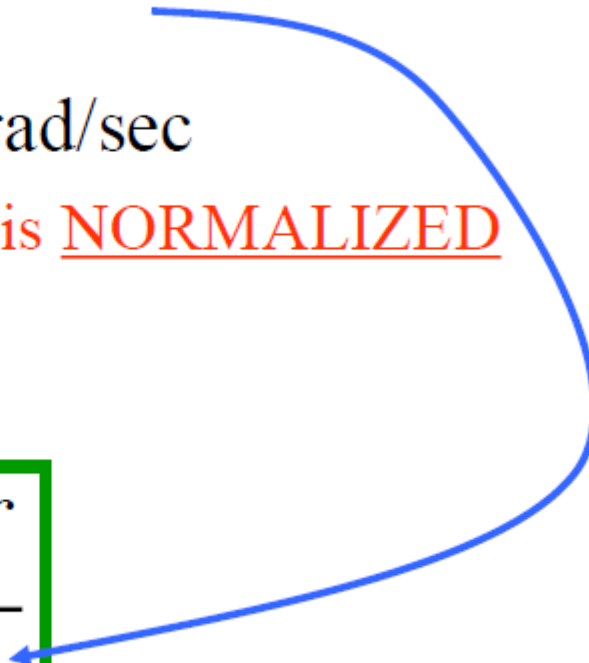
$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

DEFINE DIGITAL FREQUENCY

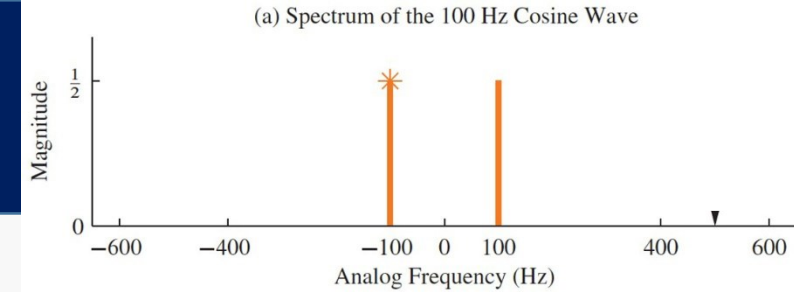
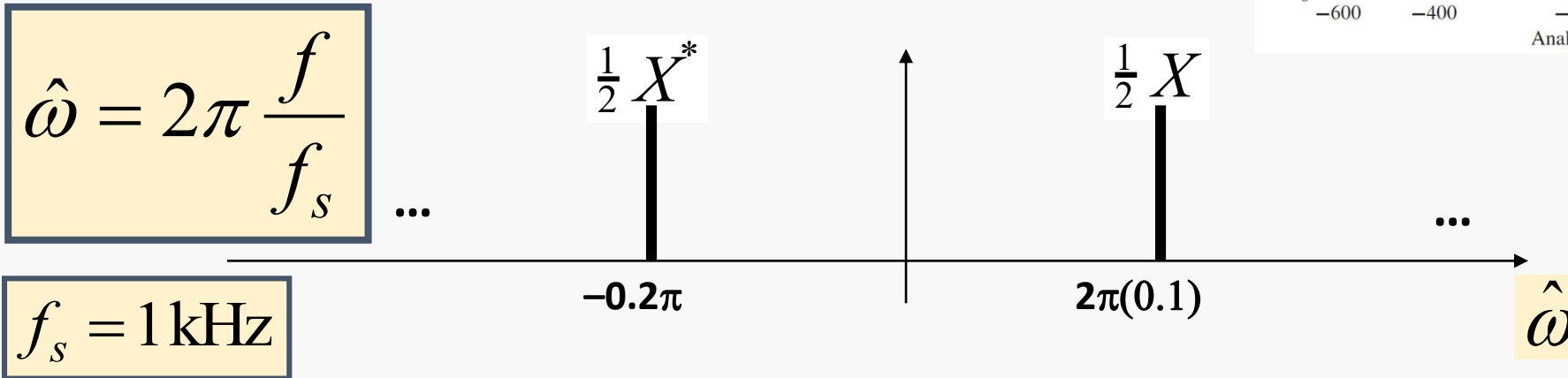
New Notion: Digital Frequency!



- $\hat{\omega}$ VARIES from **0** to **2π** , as f varies from 0 to the sampling frequency
- UNITS are radians, not rad/sec
 - **DIGITAL FREQUENCY** is NORMALIZED

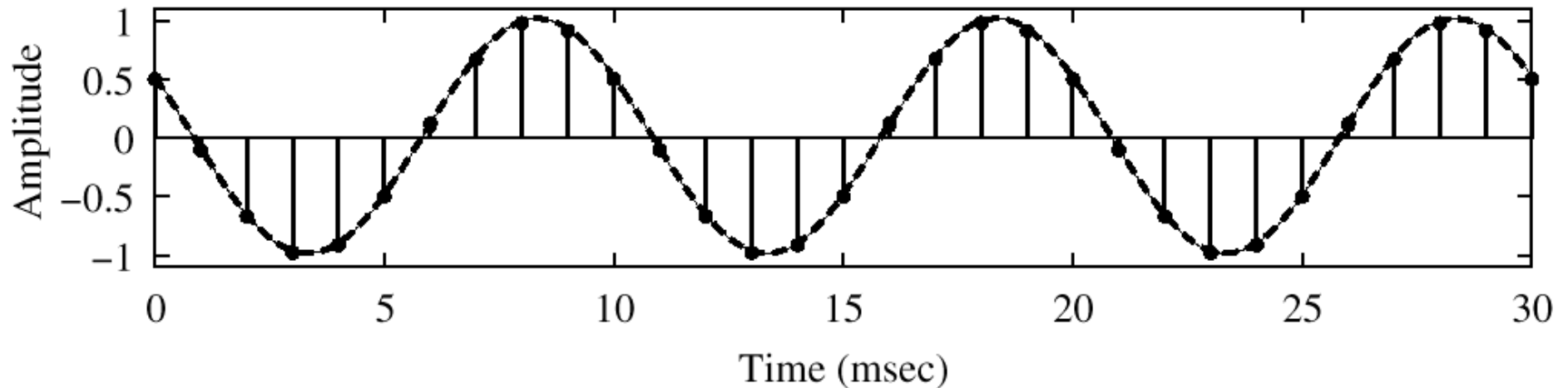
$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}$$
A blue curved arrow originates from the word 'DIGITAL' in the text 'DIGITAL FREQUENCY' and points towards the equation $\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}$.

Spectrum of Digital Signal

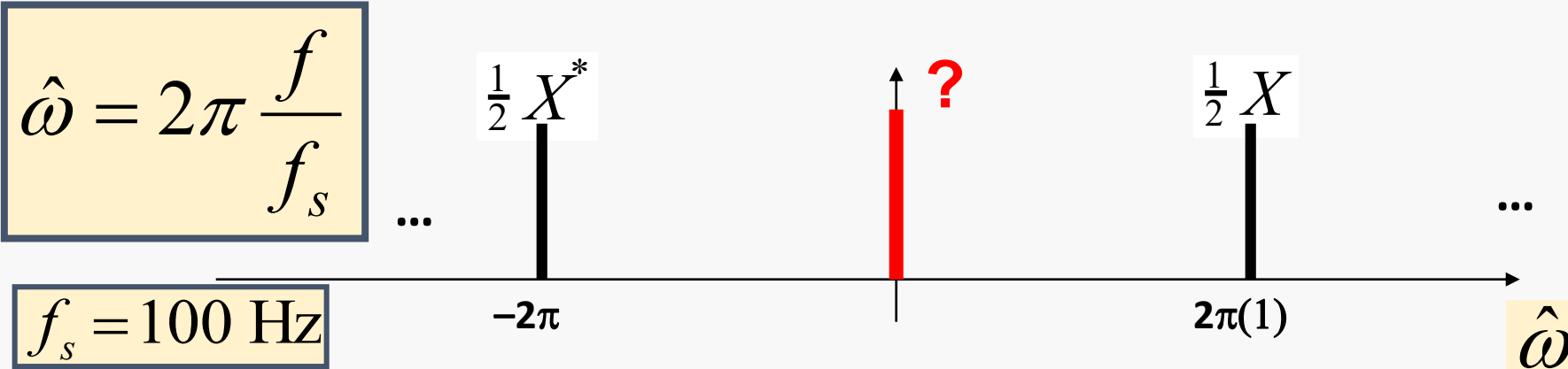


$$x[n] = A \cos(2\pi(100)(n/1000) + \varphi)$$

100-Hz Cosine Wave: Sampled with $T_s = 1 \text{ msec}$ (1000 Hz)

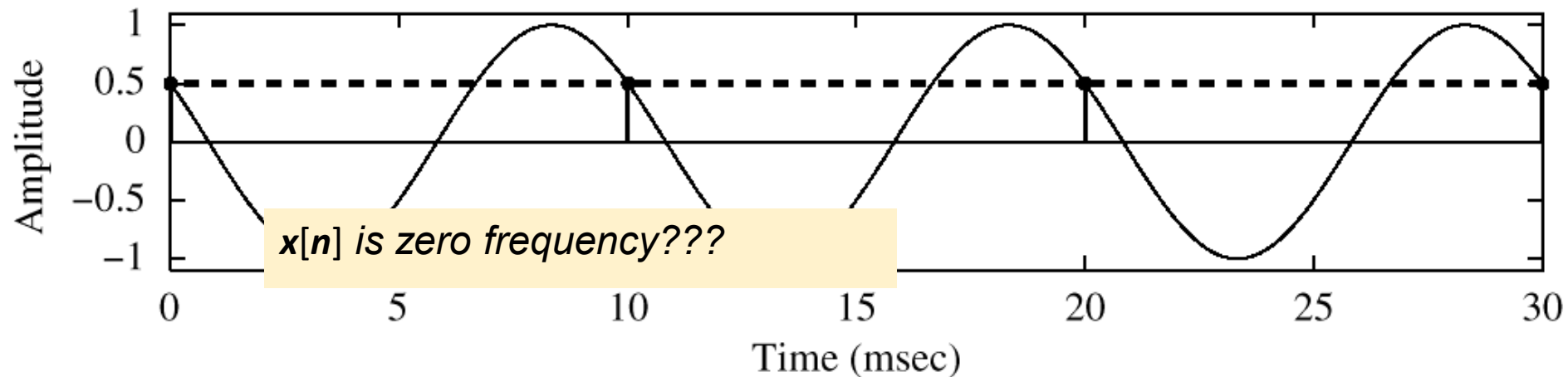


Spectrum of Digital Signal



$$x[n] = A \cos(2\pi(100)(n/100) + \varphi)$$

100-Hz Cosine Wave: Sampled with $T_s = 10 \text{ msec}$ (100 Hz)



The Rest of the Story



- Spectrum of $x[n]$ has more than one line for each complex exponential
 - Called ALIASING
 - MANY SPECTRAL LINES
- SPECTRUM is PERIODIC with period = 2π
 - Because

$$A \cos(\hat{\omega}n + \varphi) = A \cos((\hat{\omega} + 2\pi\ell)n + \varphi)$$

Example for clarification



- Other Frequencies give the same

$\hat{\omega}$

$$x_1(t) = \cos(400\pi t) \quad \text{sampled at } f_s = 1000\text{Hz}$$

$$x_1[n] = \cos(400\pi \frac{n}{1000}) = \cos(0.4\pi n)$$

$$x_2(t) = \cos(2400\pi t) \quad \text{sampled at } f_s = 1000\text{Hz}$$

$$x_2[n] = \cos(2400\pi \frac{n}{1000}) = \cos(2.4\pi n)$$

$$x_2[n] = \cos(2.4\pi n) = \cos(0.4\pi n + 2\pi n) = \cos(0.4\pi n)$$

$$\Rightarrow x_2[n] = x_1[n]$$

$$2400\pi - 400\pi = 2\pi(1000)$$

GIVEN $x[n]$, we CAN'T KNOW whether it came from a sinusoid at f_o or $(f_o + f_s)$ or $(f_o + 2f_s)$...

Digital Frequency Repeats for each Every 2π



- Other Frequencies give the same

$$\hat{\omega}$$

$$\text{If } x(t) = A \cos(2\pi(f + \ell f_s)t + \varphi)$$

$$t \leftarrow \frac{n}{f_s}$$

$$\text{and we want : } x[n] = A \cos(\hat{\omega}n + \varphi)$$

$$\text{then : } \hat{\omega} = \frac{2\pi(f + \ell f_s)}{f_s} = \frac{2\pi f}{f_s} + \frac{2\pi \ell f_s}{f_s}$$

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi \ell$$

Normalized Frequency

$$\hat{\omega}$$



Normalized Radian Frequency

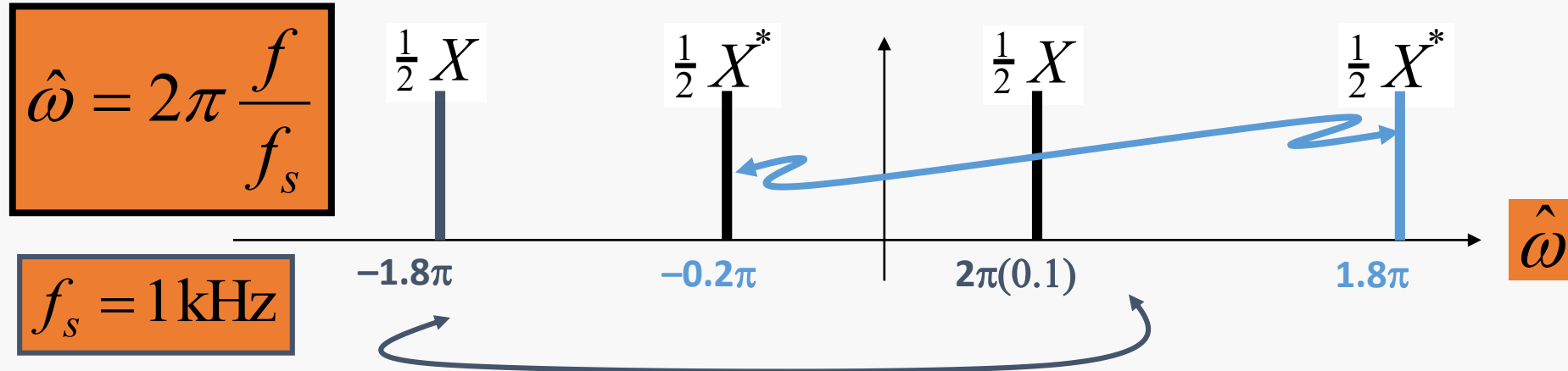
$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi \ell$$

ALIASING

$$\hat{\omega} = \omega T_s = -\frac{2\pi f}{f_s} + 2\pi \ell$$

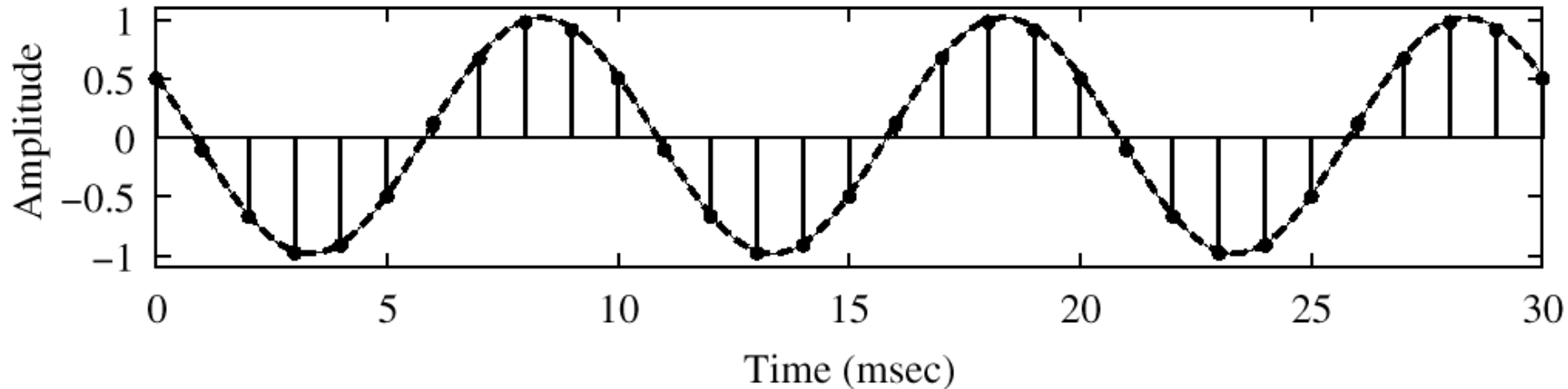
FOLDED ALIAS

Example Spectrum-1



$$x[n] = A \cos(2\pi(100)(n/1000) + \varphi)$$

100-Hz Cosine Wave: Sampled with $T_s = 1 \text{ msec}$ (1000 Hz)



Principal alias :

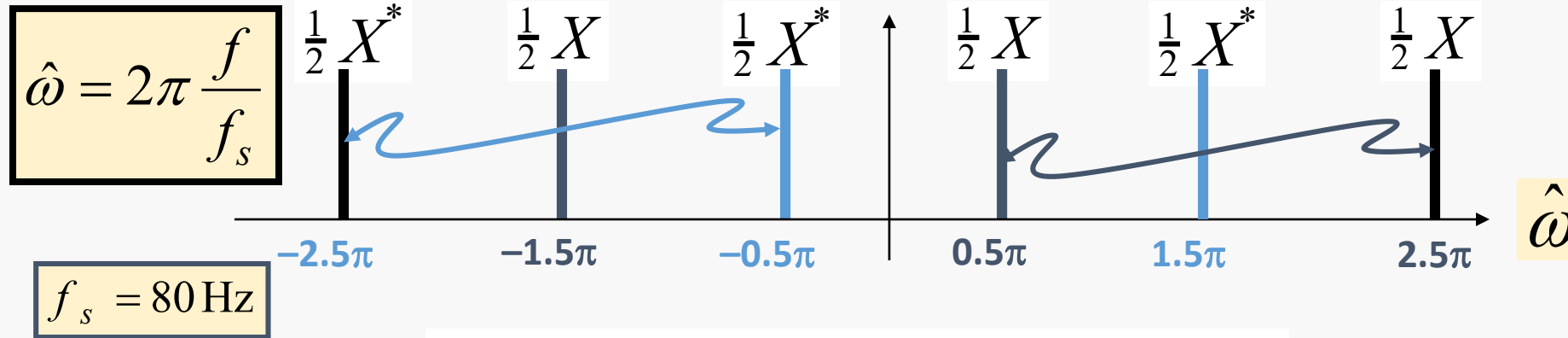
$$f = \frac{\hat{\omega} f_s}{2\pi} = 0.1(1000) = 100 \text{ Hz}$$

$$x(t) = A \cos(2\pi 100t + \varphi)$$

Aliasing Example Spectrum-2

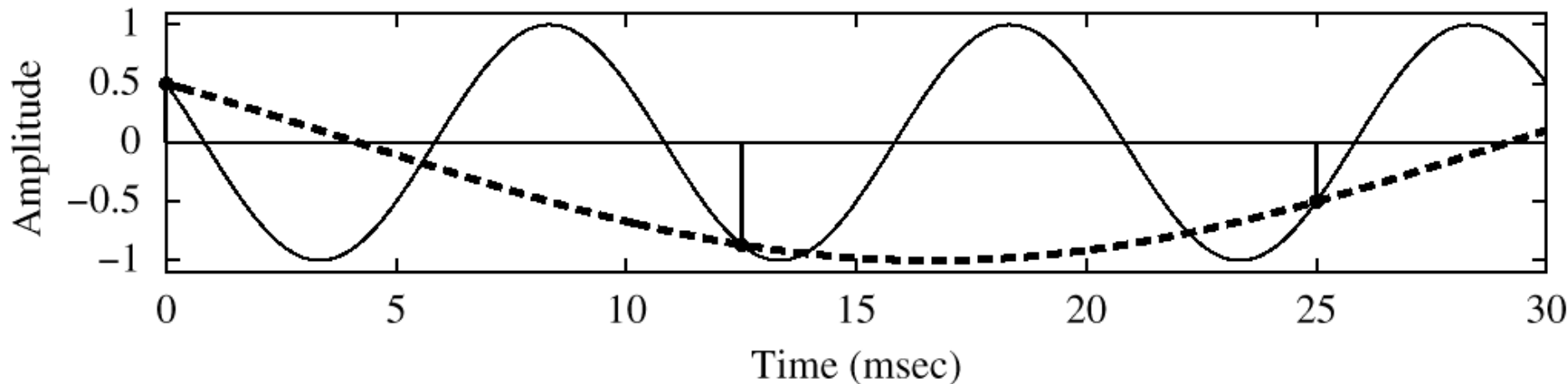


Principal alias is always between $-\pi \leq \hat{\omega} \leq \pi$



$$x[n] = A \cos(2\pi(100)(n/80) + \varphi)$$

100-Hz Cosine Wave: Sampled with $T_s = 12.5$ msec (80 Hz)



Principal alias :

$$f = \frac{\hat{\omega} f_s}{2\pi} = 0.25(80) = 20 \text{ Hz}$$

$$x(t) = A \cos(2\pi 20t + \varphi)$$

From the book (the same example, more clear)

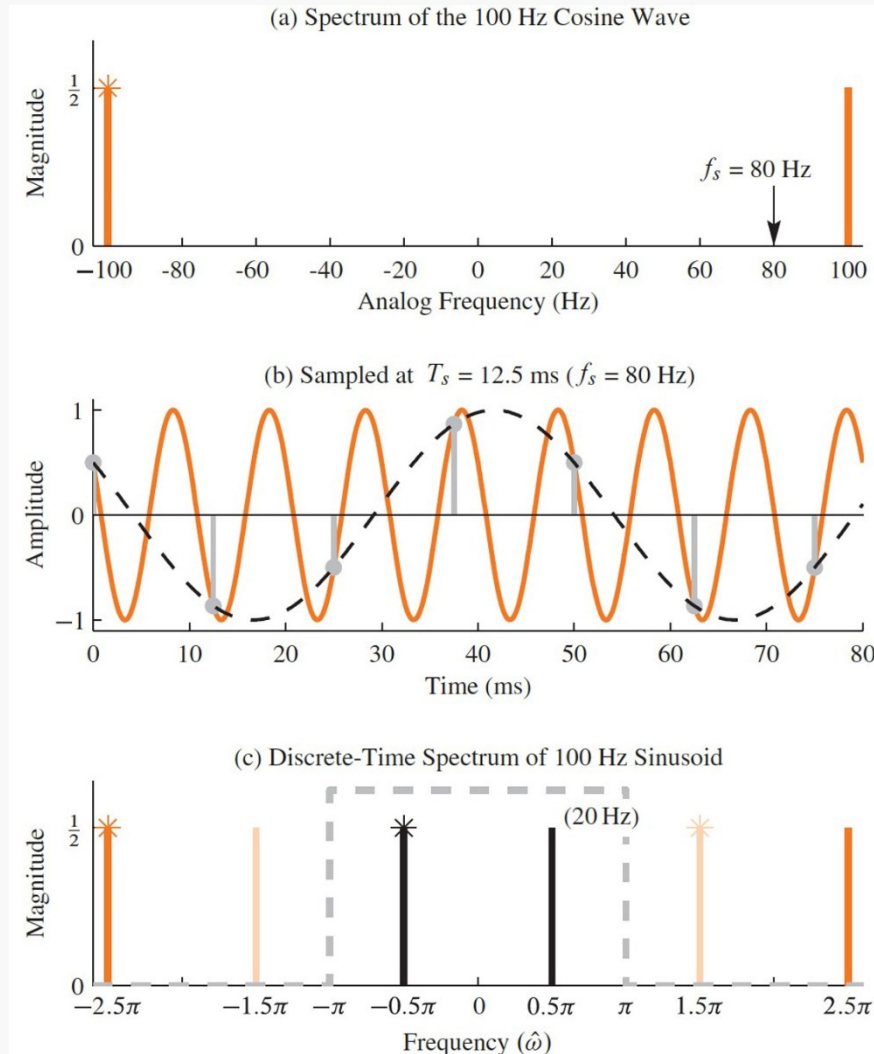


Figure 4-9 Under-sampling a 100 Hz sinusoid at $f_s = 80$ samples/s.

(a) Continuous-time spectrum;

(b) time-domain plot, showing the samples $x[n]$ as gray dots, the original signal $x(t)$ as a continuous **orange** line, and the reconstructed signal $y(t)$ as a dashed black line, which is a 20 Hz sinusoid passing through the same sample points; and

(c) discrete-time spectrum plot, showing the positive and negative frequency components of the original sinusoid at $\pm 2.5\pi$ rad, along with two sets of alias components.

FOLDING (a type of ALIASING)

- EXAMPLE: 3 different $x(t)$; same $x[n]$

$$f_s = 1000$$

$$\cos(2\pi(100)t) \rightarrow \cos[2\pi(0.1)n]$$

$$\cos(2\pi(1100)t) \rightarrow \cos[2\pi(1.1)n] = \cos[2\pi(0.1)n]$$

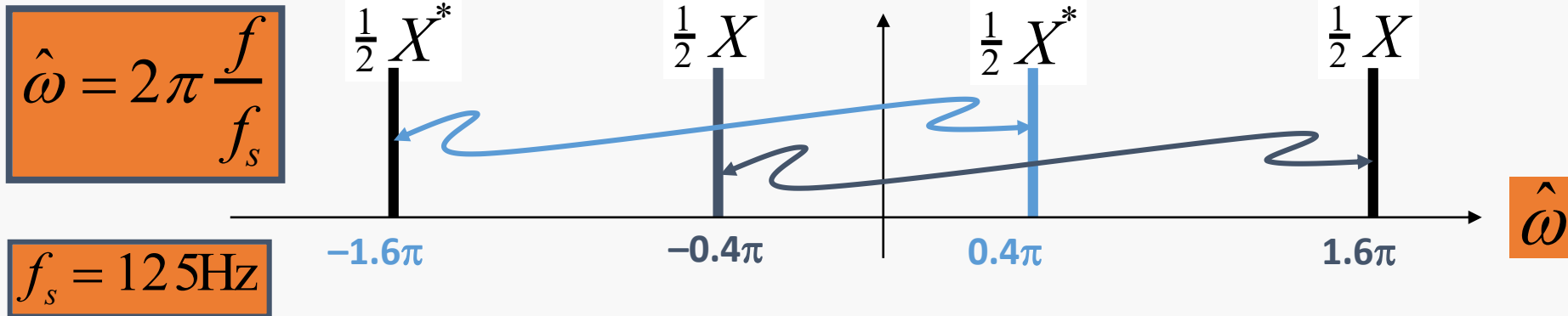
$$\cos(2\pi(900)t) \rightarrow \cos[2\pi(0.9)n]$$

$$= \cos[2\pi(0.9)n - 2\pi n] = \cos[2\pi(-0.1)n] = \cos[2\pi(0.1)n]$$

$$\hat{\omega} = 2\pi \frac{100}{1000} = 2\pi(0.1)$$

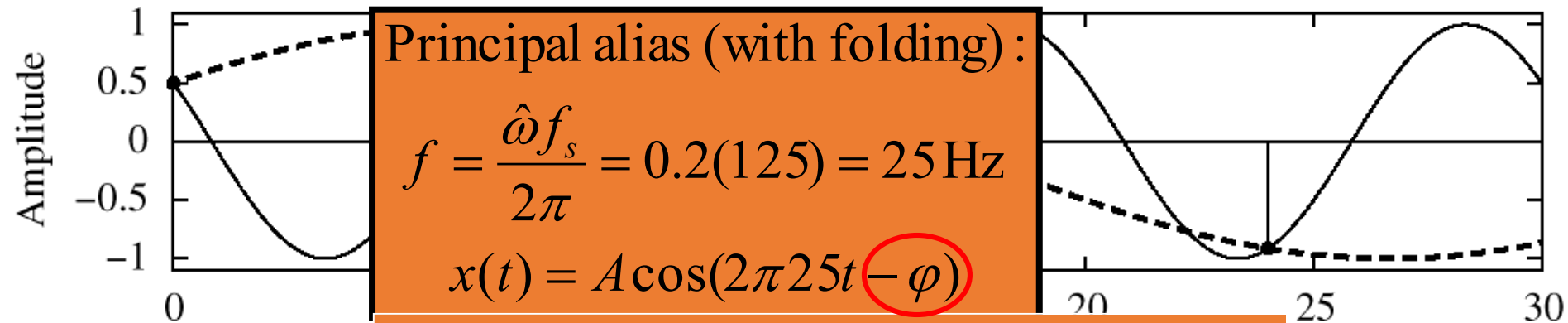
- 900 Hz “folds” to 100 Hz when $f_s = 1\text{kHz}$

Example Folding Case



$$x[n] = A \cos(2\pi(100)(n/125) + \varphi)$$

100-Hz Cosine Wave: Sampled with $T_s = 8$ msec (125 Hz)



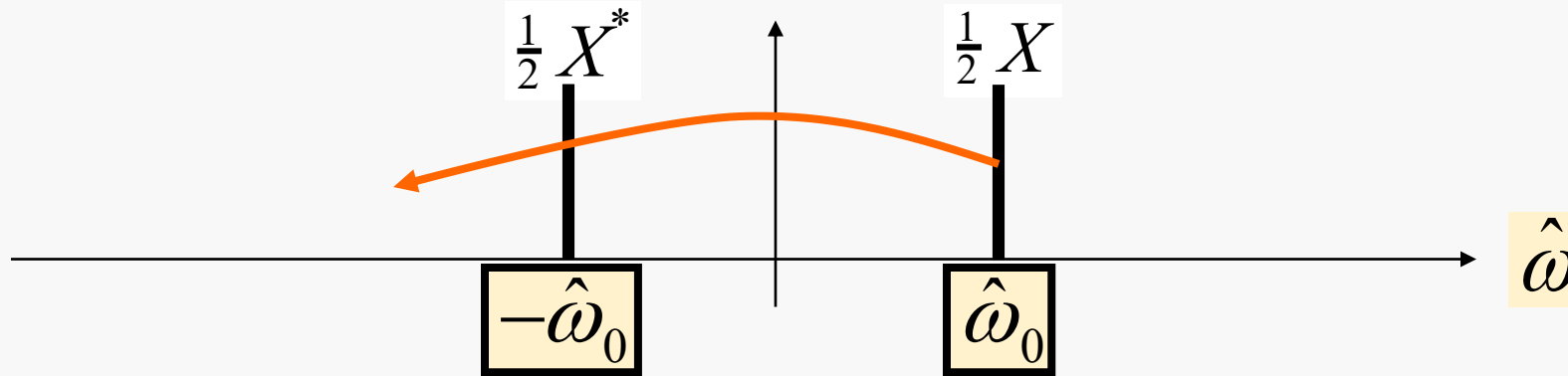
Principal alias is always between $-\pi \leq \hat{\omega} \leq \pi$

Aug 2016

SPECTRUM Explanation of SAMPLING THEOREM



- How do we prevent aliasing?
- Guarantee original signal is principal alias:



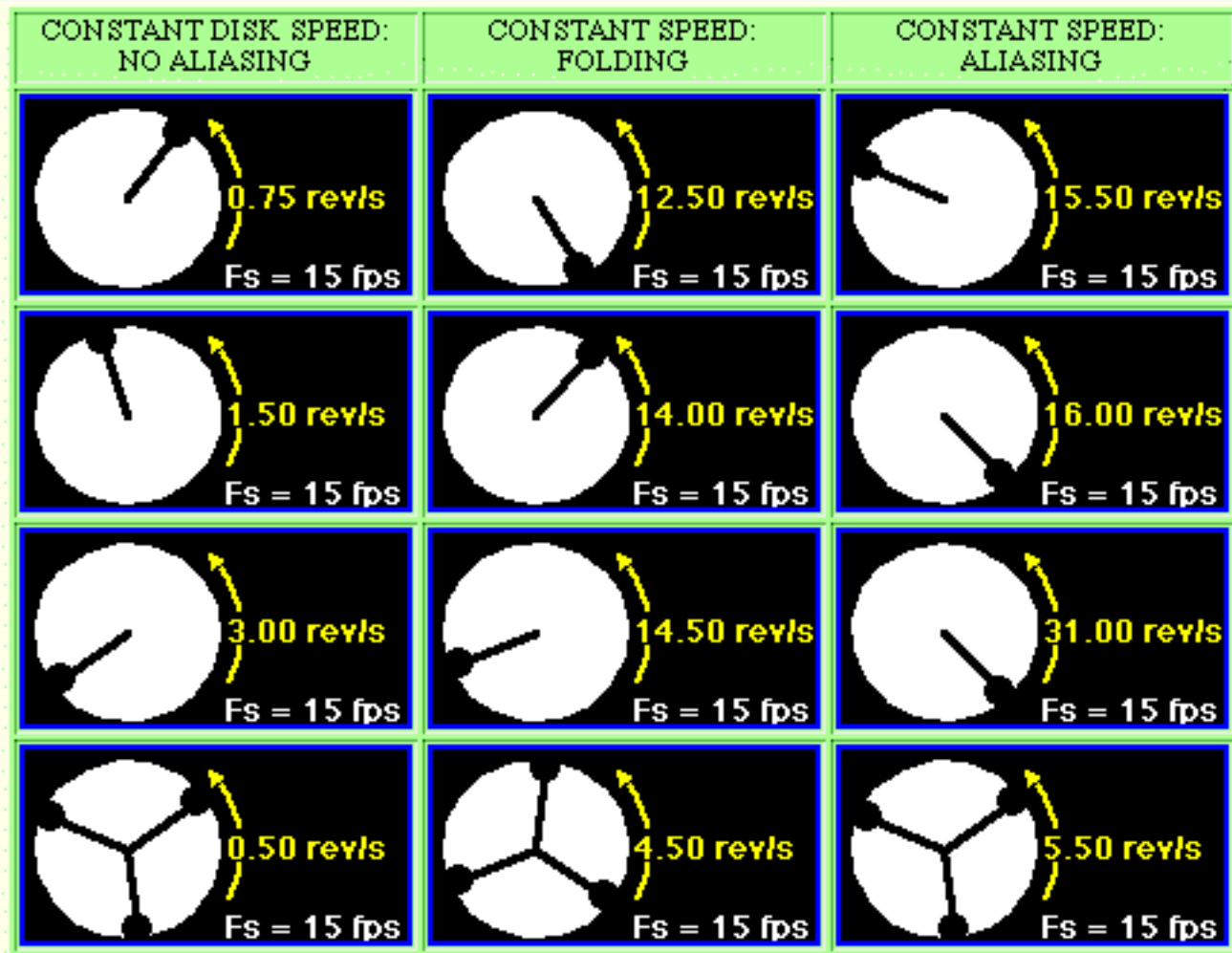
$$\hat{\omega}_0 - 2\pi < -\hat{\omega}_0 \Rightarrow \hat{\omega}_0 < \pi$$
$$\hat{\omega}_0 = \frac{2\pi f_0}{f_s} < \pi \Rightarrow f_0 < \frac{f_s}{2}$$

Be Careful:

<https://www.youtube.com/watch?v=qgvuQGY946g>



Strobe Demo



<https://dspfirst.gatech.edu/chapters/04/samplin/demos/strobe/index.html>

<https://dspfirst.gatech.edu/chapters/04/samplin/demos/synstrob/index.html>

Digital to Analog Reconstruction



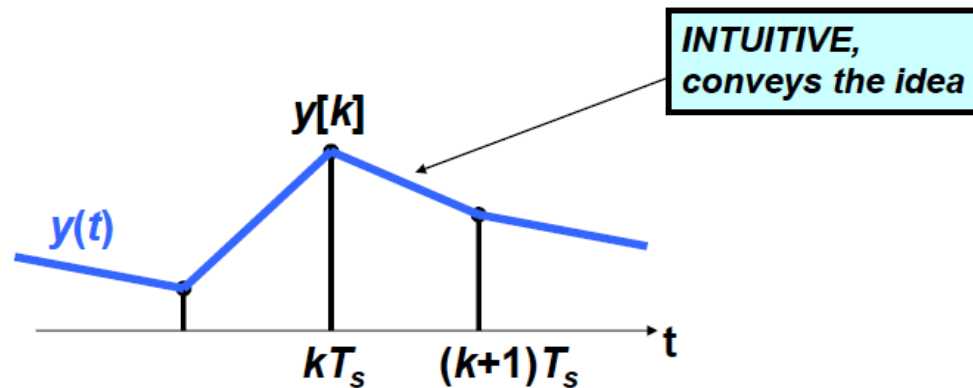
- Create continuous $y(t)$ from $y[n]$

- IDEAL D-to-A:

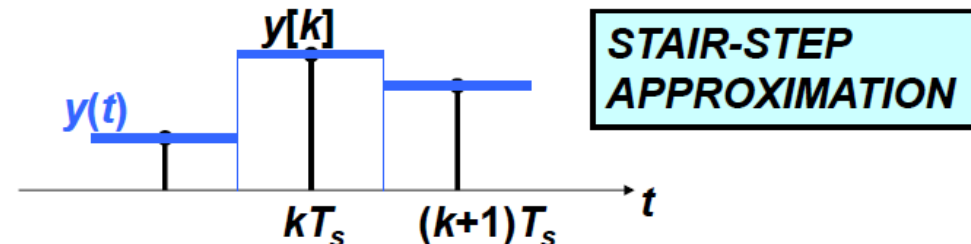
- If you have formula for $y[n]$
 - Invert sampling ($t=nT_s$) by $n=f_s t$
 - $y[n] = A\cos(0.2\pi n + \phi)$ with $f_s = 8000$ Hz
 - $y(t) = A\cos(0.2\pi(8000t) + \phi) = A\cos(2\pi(800)t + \phi)$

- RECONSTRUCT THE SMOOTHEST ONE
 - THE LOWEST FREQ, if $y[n] = \text{sinusoid}$

- CONVERT STREAM of NUMBERS to $x(t)$
- “CONNECT THE DOTS”
- INTERPOLATION

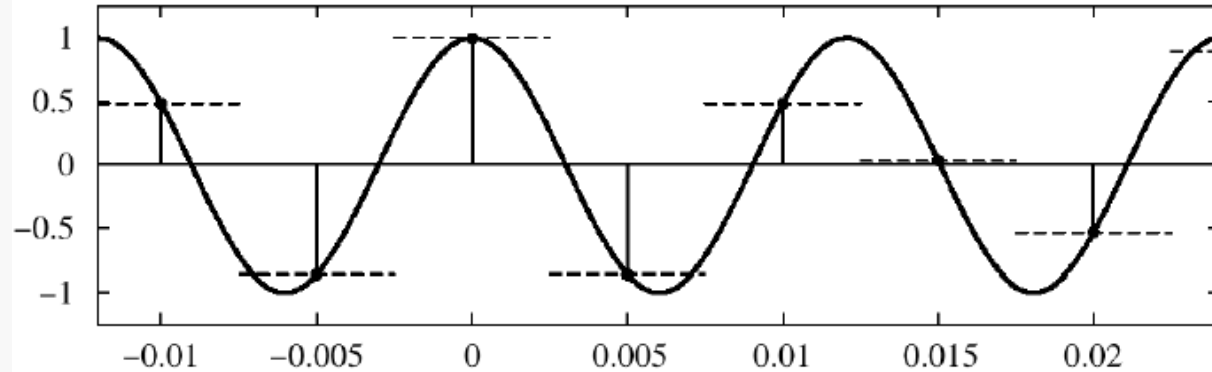


- CONVERT $y[n]$ to $y(t)$
 - $y[k]$ should be the value of $y(t)$ at $t = kT_s$
 - Make $y(t)$ equal to $y[k]$ for
 - $kT_s - 0.5T_s < t < kT_s + 0.5T_s$

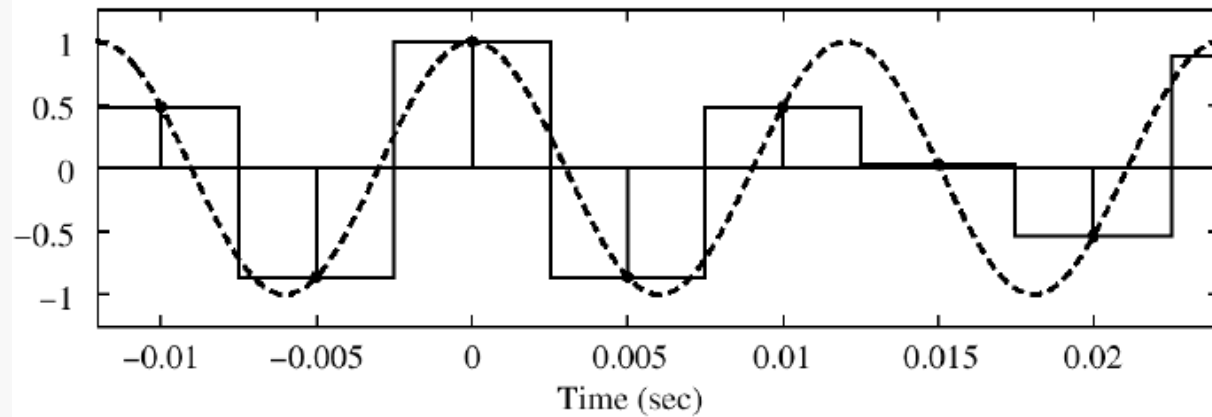


STAIR-STEP APPROXIMATION

Sampling and Zero-Order Reconstruction: $f_0 = 83$ $f_s = 200$

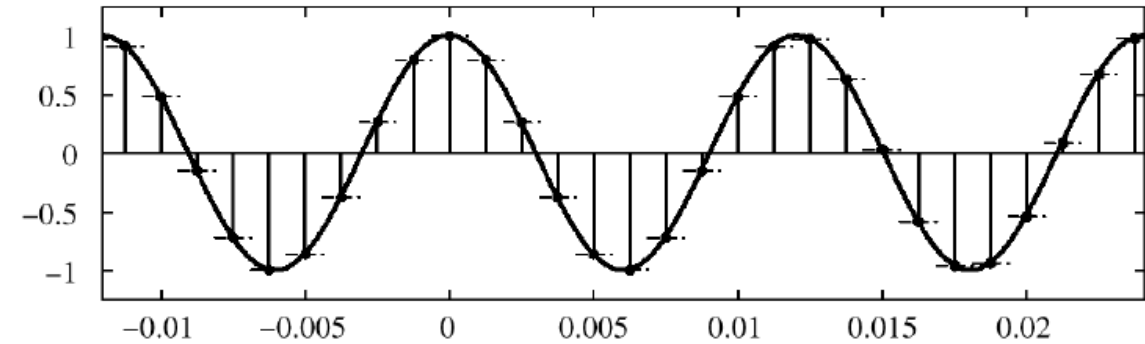


Original and Reconstructed Waveforms



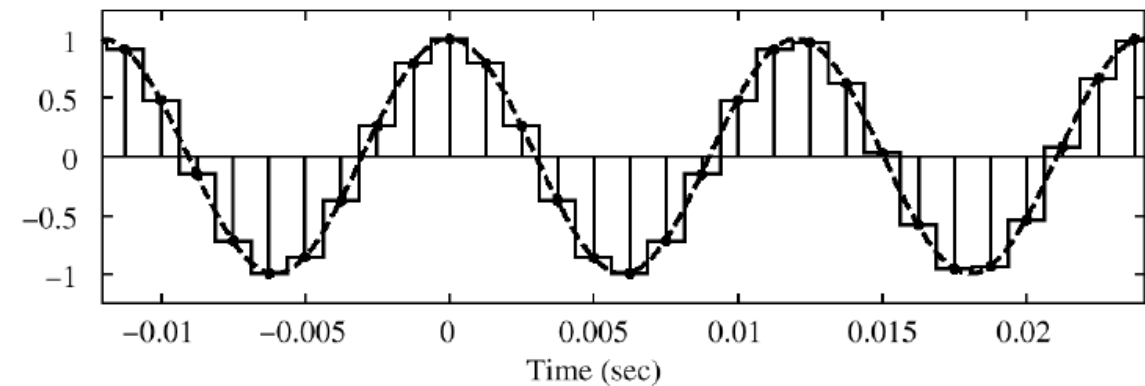
OVER-SAMPLING CASE

Sampling and Zero-Order Reconstruction: $f_0 = 83$ $f_s = 800$

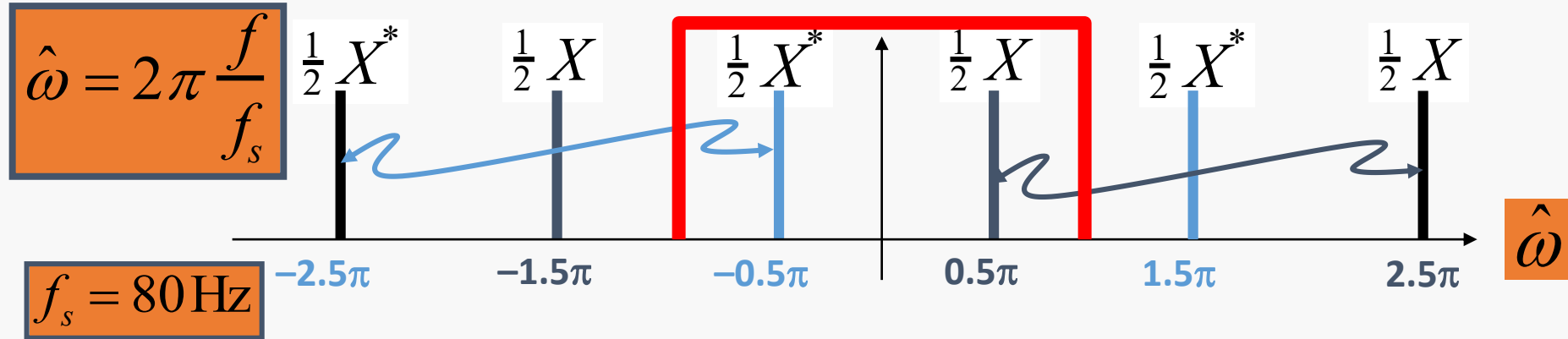


EASIER TO RECONSTRUCT

Original and Reconstructed Waveforms

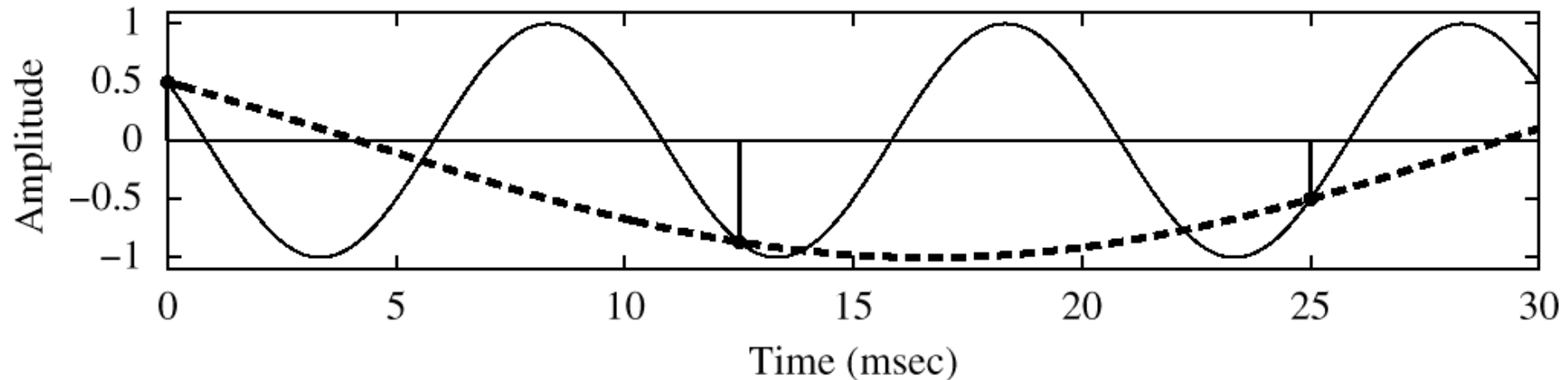


SPECTRUM (ALIASING CASE)

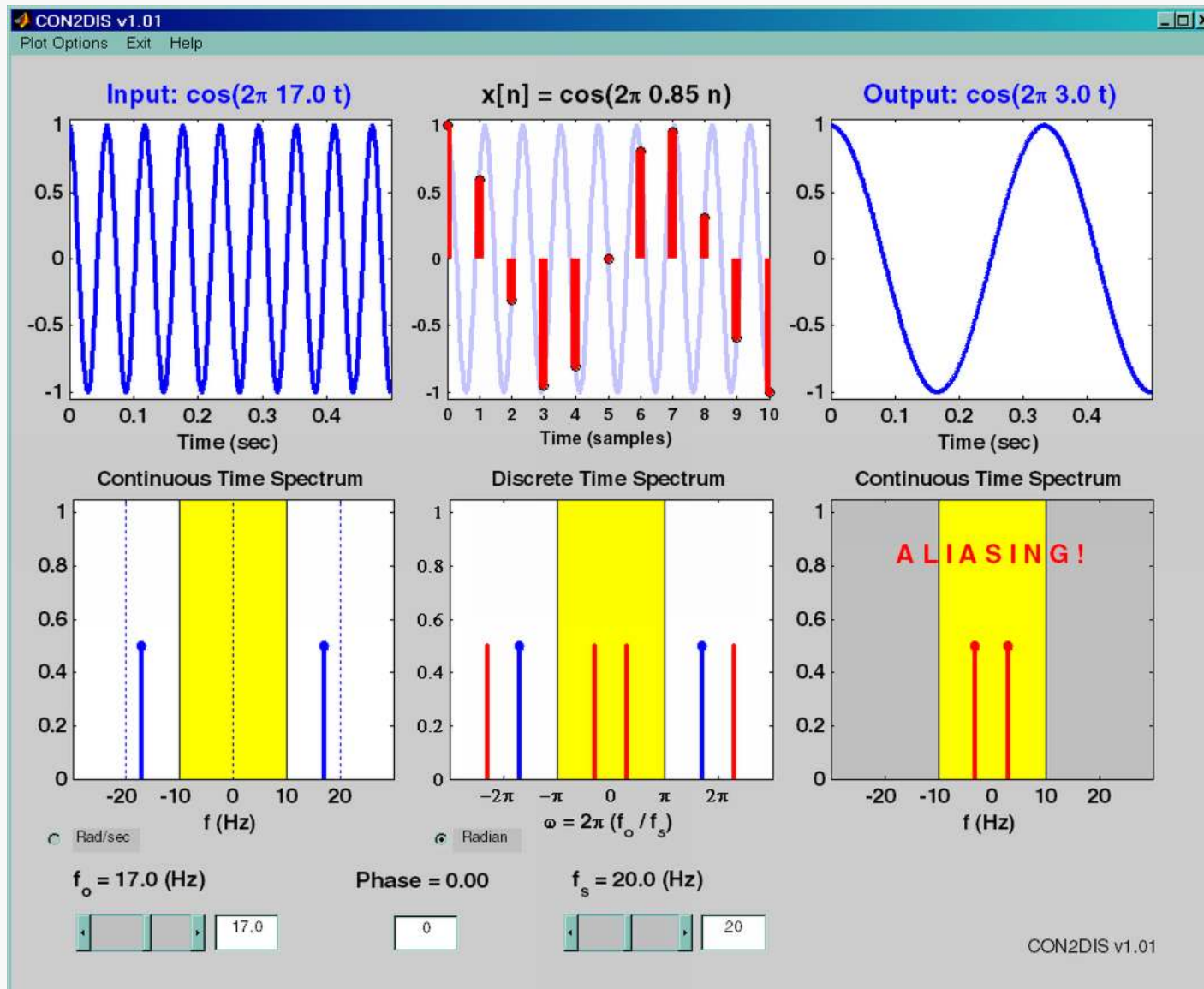


$$x[n] = A \cos(2\pi(100)(n/80) + \varphi)$$

100-Hz Cosine Wave: Sampled with $T_s = 12.5 \text{ msec}$ (80 Hz)

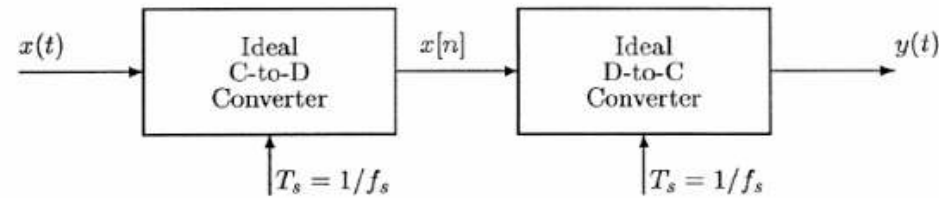


Let's Analyze with MATLAB



<https://dspfirst.gatech.edu/matlab/#con2dis>

Example - 1



Suppose that the output of the D-to-C converter in the system above is found to be

$$y(t) = 2 + 10 \cos(2\pi(150)t + \pi/3)$$

when the sampling rate is $f_s = 1/T_s = 400$ samples/second.

- (a) Give an equation for $x[n]$ in terms of cosine functions. Write your answer on the line below.

With no aliasing, going through a D/C converter is the inverse of going through the C/D. Therefore, we can get $x[n]$ by passing $y(t)$ through a C/D converter.

$$\begin{aligned} \text{Answer: } x[n] &= \frac{2 + 10 \cos(2\pi(150)n/400 + \pi/3)}{1} \\ &= 2 + 10 \cos(3n\pi/4 + \pi/3) \end{aligned}$$

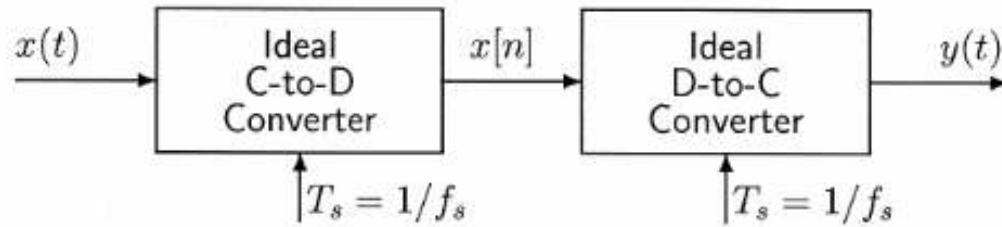
- (b) Determine two *different* input signals $x(t) = x_1(t)$ and $x(t) = x_2(t)$ that could have produced the given output of the D-to-C converter. All of the frequencies in your answers must be positive and less than 400 Hz. Write your answers for both inputs on the lines below.

$$\text{Answer: } x_1(t) = \frac{2 + 10 \cos(2\pi(150)t + \pi/3)}{(no aliasing)}$$

$$\text{Answer: } x_2(t) = \frac{2 + 10 \cos(2\pi(250)t - \pi/3)}{1}$$

We have folding in the second case.

Example-2



Suppose that the continuous-time input $x(t)$ to the above system is given as

$$x(t) = \cos(14000\pi t) + \cos(2000\pi t) + \cos(1000\pi t).$$

(a) What sampling rate is required such that no aliasing occurs for $x(t)$?

$$f_{\max} = 7,000 \text{ Hz}$$
$$f_s = 14,000 \text{ Hz}$$

(c) Given that $x(t) = \cos(25000\pi t)$ and $f_s = 10000$ samples/second, write a simplified expression for the output $y(t)$ in terms of cosine functions.

$$y(t) = \cos(5000\pi t)$$

NOTE THAT ALIASING OCCURS.

$$x[n] = \cos\left(\frac{2500\pi n}{10000}\right) = \cos(0.5\pi n)$$

$$y(t) = \cos(0.5\pi t 10000) = \cos(5000\pi t)$$

Example-3

PROBLEM:

The “spectrum” diagram gives the frequency content of a signal.

- (a) Draw a sketch of the spectrum of $x(t)$ which is “cosine-times-sine”

$$x(t) = \cos(50\pi t) \sin(700\pi t)$$

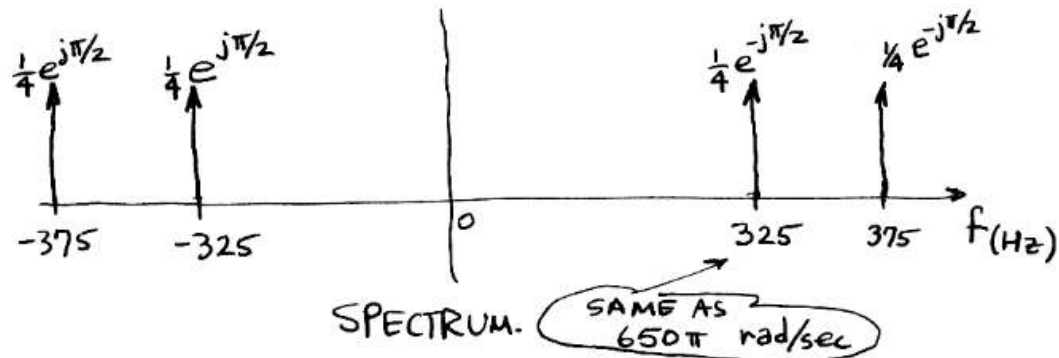
Label the frequencies and complex amplitudes of each component.

- (b) Determine the minimum sampling rate that can be used to sample $x(t)$ without any aliasing.

$$x(t) = \cos(50\pi t) \sin(700\pi t)$$

$$\begin{aligned} \text{(a)} \quad x(t) &= \left(\frac{1}{2} e^{j50\pi t} + \frac{1}{2} e^{-j50\pi t} \right) \left(\frac{1}{2j} e^{j700\pi t} - \frac{1}{2j} e^{-j700\pi t} \right) \\ &= \frac{1}{4j} e^{j750\pi t} + \frac{1}{4j} e^{j650\pi t} - \frac{1}{4j} e^{-j650\pi t} - \frac{1}{4j} e^{-j750\pi t} \end{aligned}$$

\uparrow SAME AS $\frac{1}{4} e^{j\pi/2}$ \uparrow SAME AS $\frac{1}{4} e^{j\pi/2}$



- (b) Sampling Thm says sample at a rate greater than two times the highest freq.

$$\text{HIGHEST FREQ} = 375 \text{ Hz}$$

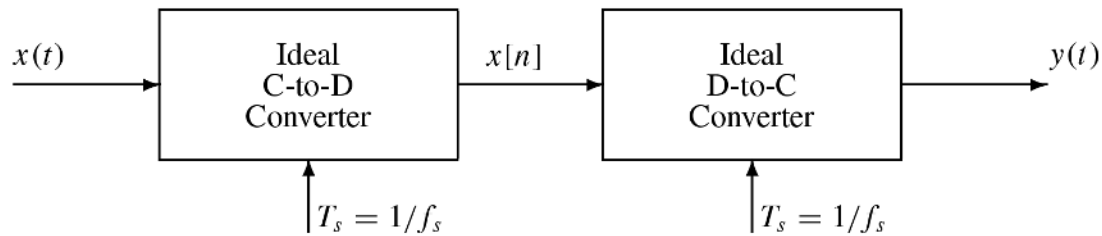
$$\Rightarrow f_s \geq 750 \text{ Hz.}$$

Example-4



PROBLEM:

Consider the following system.

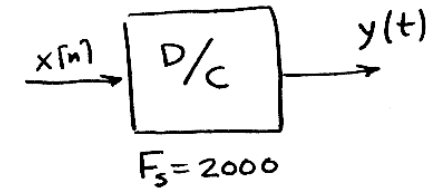


Suppose that the output of the C-to-D converter is

$$x[n] = 5 + 8 \cos(0.4\pi n) + 4 \cos(0.8\pi n + \pi/3)$$

when the sampling rate is $f_s = 1/T_s = 2000$ samples/second. Determine the output $y(t)$ of the ideal D-to-C converter.

$$x[n] = 5 + 8 \cos(0.4\pi n) + 4 \cos(0.8\pi n + \pi/3)$$



For discrete to continuous, we replace "n" with $F_s t$

$$y(t) = x[n] \Big|_{n=F_s t}$$

$$= 5 + 8 \cos(0.4\pi(2000)t) + 4 \cos(0.8\pi(2000)t + \pi/3)$$

$$= 5 + 8 \cos(2\pi(400)t) + 4 \cos(2\pi(800)t + \pi/3)$$