BLM1612 - Circuit Theory

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Filters and Bode Plots

Filters

- Objective of Lecture
  - Describe the filter types and functions.

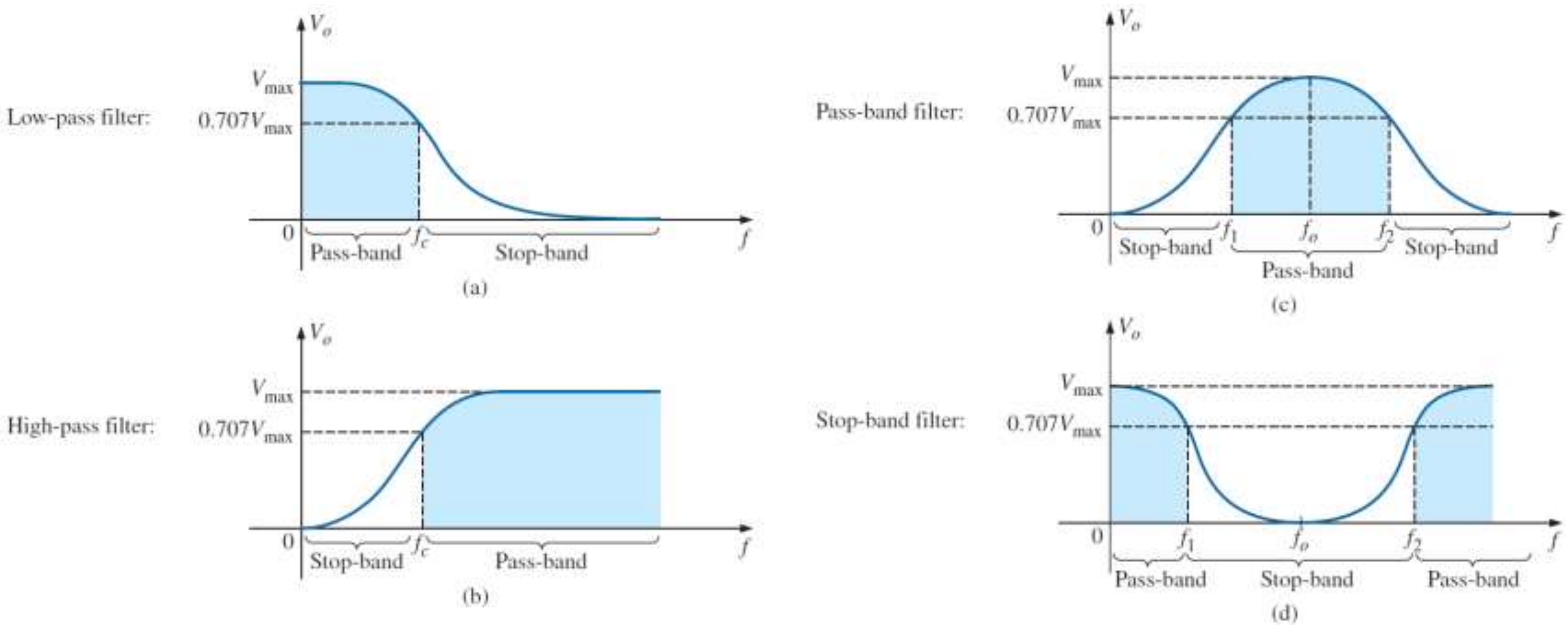
Filters

- Any combination of passive (R, L, and C) and/or active (transistors or operational amplifiers) elements designed to select or reject a band of frequencies is called a filter.
- In general, there are two classifications of filters:
  - Passive filters
    - series or parallel combinations of R, L, and C elements.
  - Active filters
    - transistors and operational amplifiers in combination with R, L, and C elements.

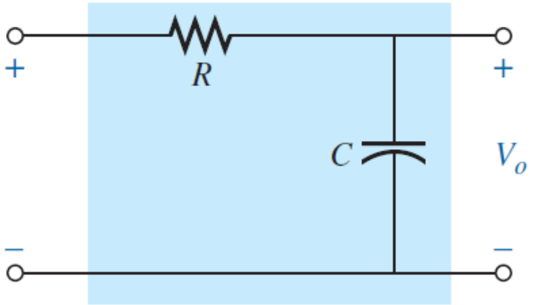
Filters

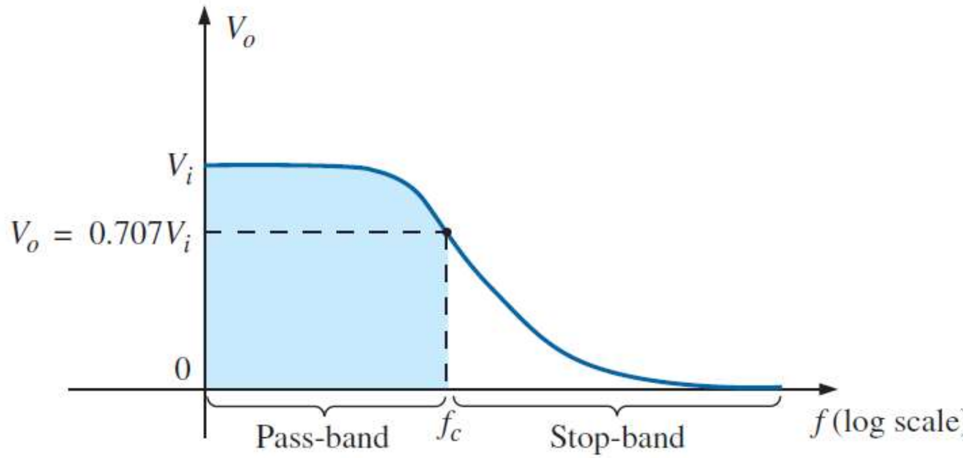
- In general, all filters belong to the four broad categories:

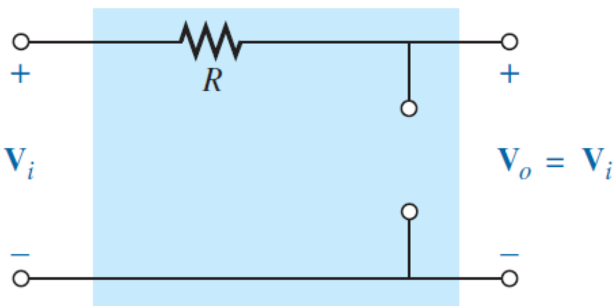
  (a) low-pass, (b) high-pass, (c) pass-band, (d) stop-band



R-C LOW-PASS FILTER

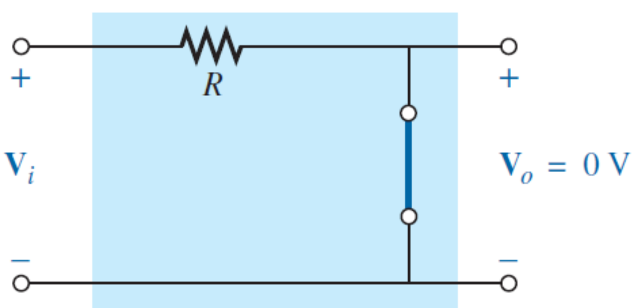






At $f = 0$ Hz,

$$X_C = \frac{1}{2\pi fC} = \infty\ \Omega$$

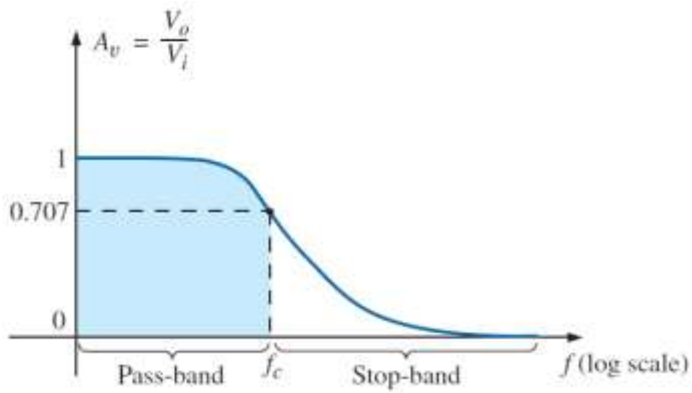


At very high frequencies, the reactance is

$$X_C = \frac{1}{2\pi fC} \cong 0\ \Omega$$

R-C LOW-PASS FILTER

- For filters, a normalized plot is used



$$\mathbf{V}_o = \frac{X_C \angle -90^\circ \mathbf{V}_i}{R - j X_C}$$

$$\mathbf{A}_v = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{X_C \angle -90^\circ}{R - j X_C} = \frac{X_C \angle -90^\circ}{\sqrt{R^2 + X_C^2} \angle -\tan^{-1}(X_C/R)}$$

$$\mathbf{A}_v = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{X_C}{\sqrt{R^2 + X_C^2}} \angle -90^\circ + \tan^{-1}\left(\frac{X_C}{R}\right)$$

R-C LOW-PASS FILTER

The magnitude of the ratio V_o/V_i is therefore determined by

$$\boxed{A_v = \frac{V_o}{V_i} = \frac{X_C}{\sqrt{R^2 + X_C^2}} = \frac{1}{\sqrt{\left(\frac{R}{X_C}\right)^2 + 1}}}$$

and the phase angle is determined by

$$\boxed{\theta = -90^\circ + \tan^{-1}\frac{X_C}{R} = -\tan^{-1}\frac{R}{X_C}}$$

R-C LOW-PASS FILTER

For the special frequency at which $X_C = R$, the magnitude becomes

$$A_v = \frac{V_o}{V_i} = \frac{1}{\sqrt{\left(\frac{R}{X_C}\right)^2 + 1}} = \frac{1}{\sqrt{1 + 1}} = \frac{1}{\sqrt{2}} = 0.707$$

which defines the critical or cutoff frequency in Fig. in Slt. 5

The frequency at which $X_C = R$ is determined by

$$\frac{1}{2\pi f_c C} = R$$

and

$$\boxed{f_c = \frac{1}{2\pi RC}}$$

R-C LOW-PASS FILTER

Solving for $\mathbf{V}_o$ and substituting $\mathbf{V}_i = V_i \angle 0°$ gives

$$\mathbf{V}_o = \left[\frac{X_C}{\sqrt{R^2 + X_C^2}} \angle\theta\right]\mathbf{V}_i = \left[\frac{X_C}{\sqrt{R^2 + X_C^2}} \angle\theta\right] V_i \angle 0°$$

and

$$\mathbf{V}_o = \frac{X_C V_i}{\sqrt{R^2 + X_C^2}} \angle\theta$$

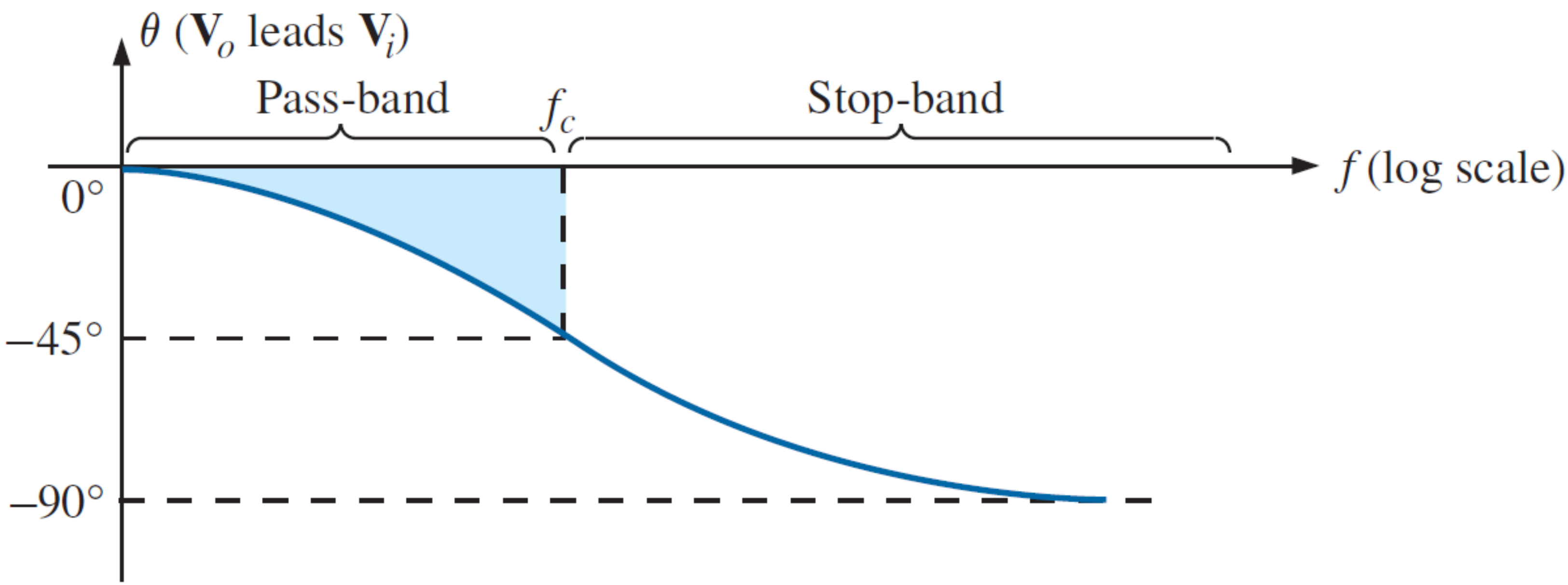
The angle θ is, therefore, the angle by which $\mathbf{V}_o$ leads $\mathbf{V}_i$. Since $\theta = -\tan^{-1} R/X_C$ is always negative (except at $f = 0$ Hz), it is clear that $\mathbf{V}_o$ will always lag $\mathbf{V}_i$, leading to the label *lagging network* for the network in Fig. in Slt. 5

R-C LOW-PASS FILTER

At high frequencies, X_C is very small and R/X_C is quite large, resulting in $\theta = -\tan^{-1} R/X_C$ approaching $-90°$.

At low frequencies, X_C is quite large and R/X_C is very small, resulting in θ approaching $0°$.

At $X_C = R$, or $f = f_c$, $-\tan^{-1} R/X_C = -\tan^{-1} 1 = -45°$.



R-C LOW-PASS FILTER

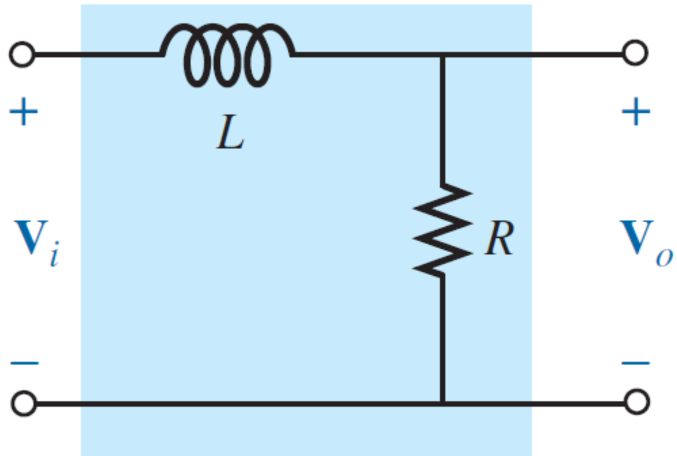
- In summary, for the low-pass *R-C* filter:

$$f_c = \frac{1}{2\pi RC}$$

For	$f < f_c,$	$V_o > 0.707V_i$
whereas for	$f > f_c,$	$V_o < 0.707V_i$
At f_c,	$\mathbf{V}_o$ lags $\mathbf{V}_i$ by 45°	

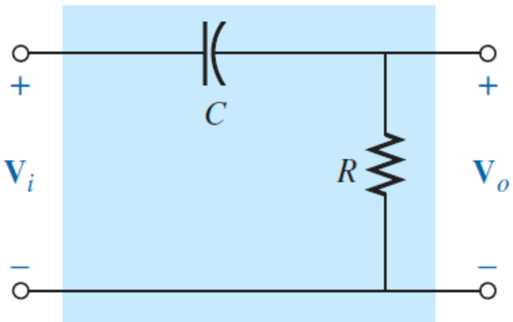
R-L LOW-PASS FILTER

- The low-pass filter response can also be obtained using the R-L combination

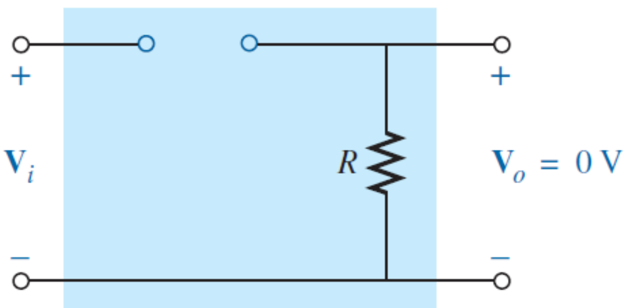


$$\boxed{f_c = \frac{R}{2\pi L}}$$

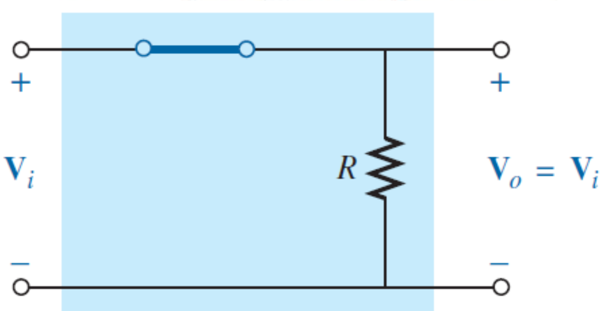
R-C HIGH-PASS FILTER

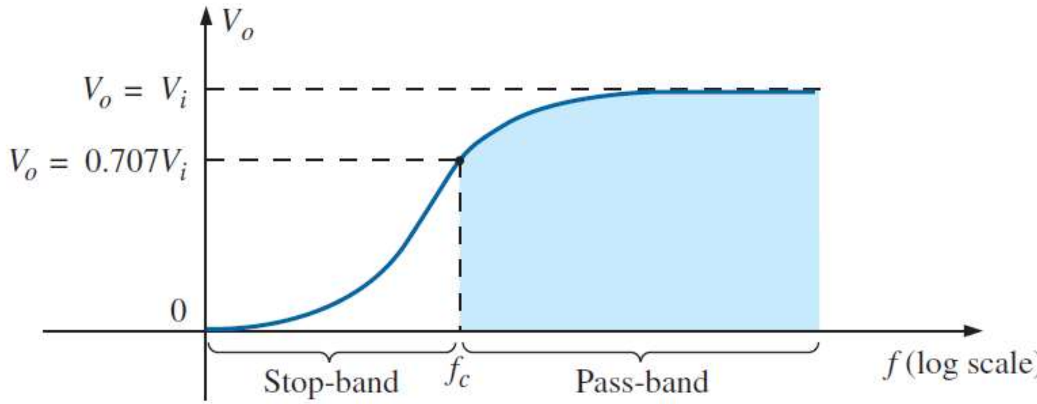


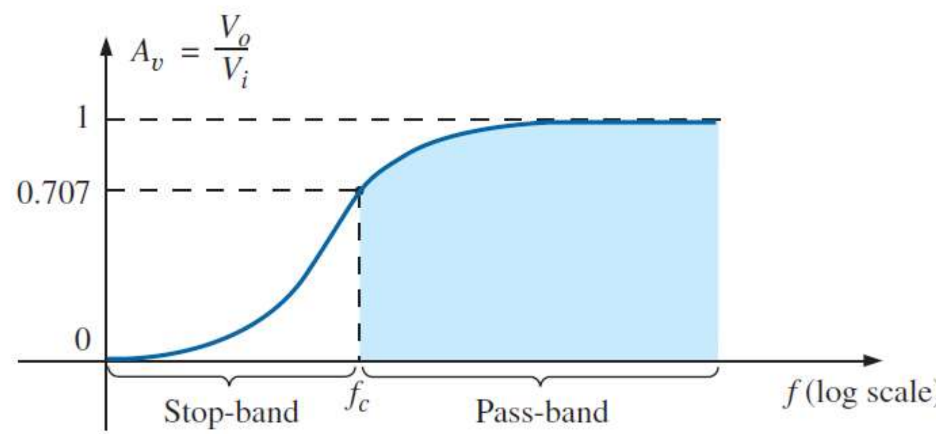
At $f = 0$ Hz,



At very high frequencies,







R-C HIGH-PASS FILTER

At any intermediate frequency, the output voltage can be determined using the voltage divider rule:

$$\mathbf{V}_o = \frac{R \angle 0^\circ \, \mathbf{V}_i}{R - j X_C}$$

or

$$\frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{R \angle 0^\circ}{R - j X_C} = \frac{R \angle 0^\circ}{\sqrt{R^2 + X_C^2} \angle -\tan^{-1}(X_C/R)}$$

and

$$\frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{R}{\sqrt{R^2 + X_C^2}} \angle \tan^{-1}(X_C/R)$$

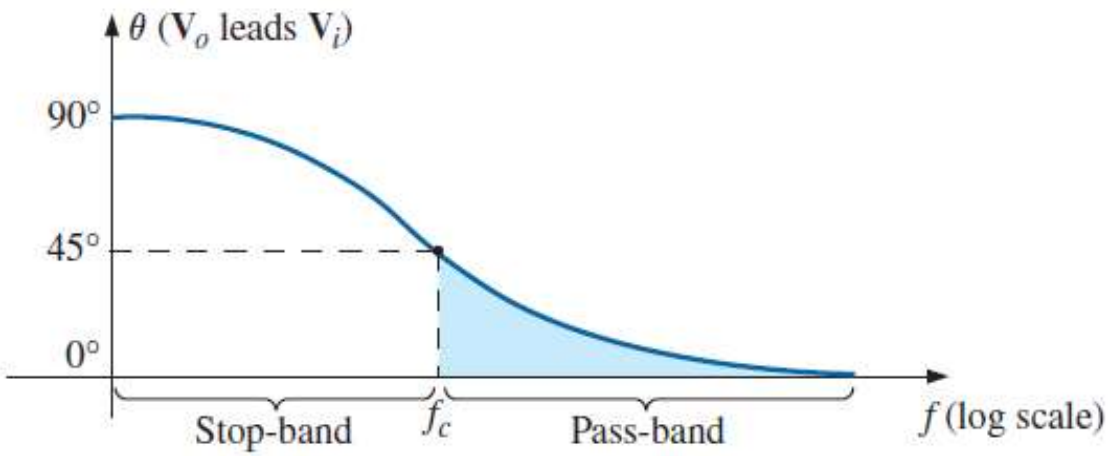
R-C HIGH-PASS FILTER

The magnitude of the ratio $\mathbf{V}_o/\mathbf{V}_i$ is therefore determined by

$$\boxed{A_v = \frac{V_o}{V_i} = \frac{R}{\sqrt{R^2 + X_C^2}} = \frac{1}{\sqrt{1 + \left(\frac{X_C}{R}\right)^2}}}$$

and the phase angle θ by

$$\boxed{\theta = \tan^{-1}\frac{X_C}{R}}$$



$$\boxed{f_c = \frac{1}{2\pi RC}}$$

R-C HIGH-PASS FILTER

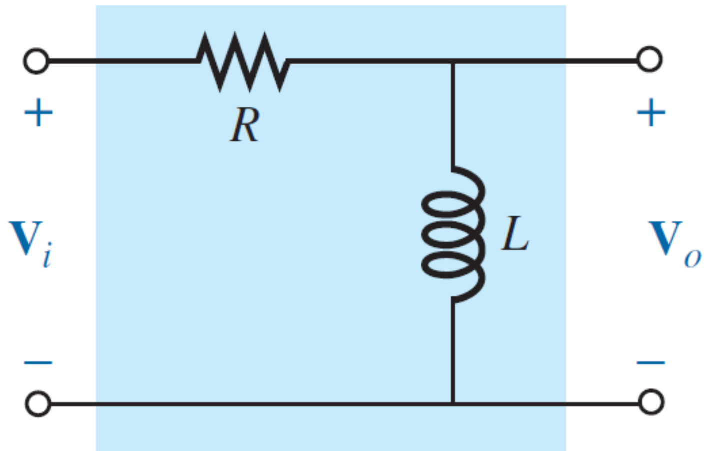
- In summary, for the high-pass *R-C* filter:

$$f_c = \frac{1}{2\pi RC}$$

For	$f < f_c$,	$V_o < 0.707V_i$
whereas for	$f > f_c$,	$V_o > 0.707V_i$
At f_c,	$\mathbf{V}_o$ leads $\mathbf{V}_i$ by 45°	

R-L HIGH-PASS FILTER

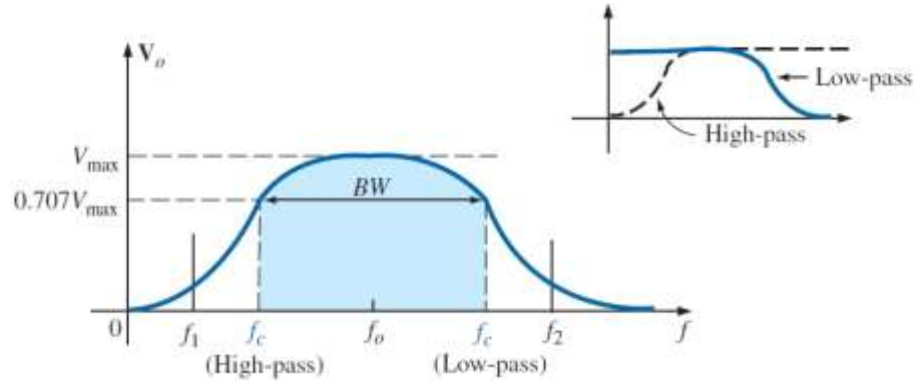
- The highpass filter response can also be obtained using the R-L combination



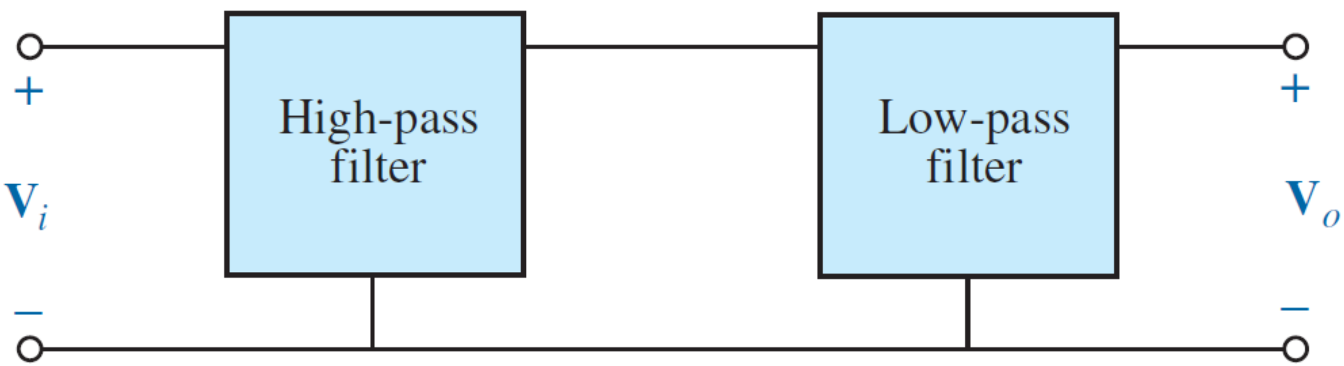
$$\boxed{f_c = \frac{R}{2\pi L}}$$

PASS-BAND FILTERS

- A number of methods are used to establish the pass-band characteristic.

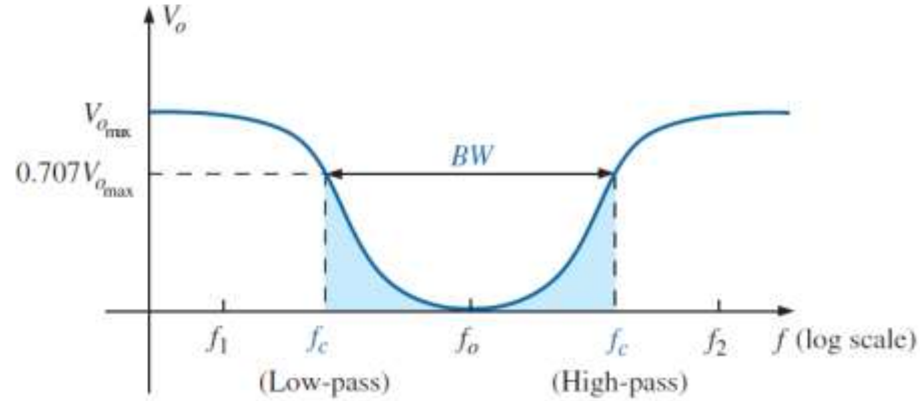


  - One method uses both a low-pass and a high-pass filter in cascade.

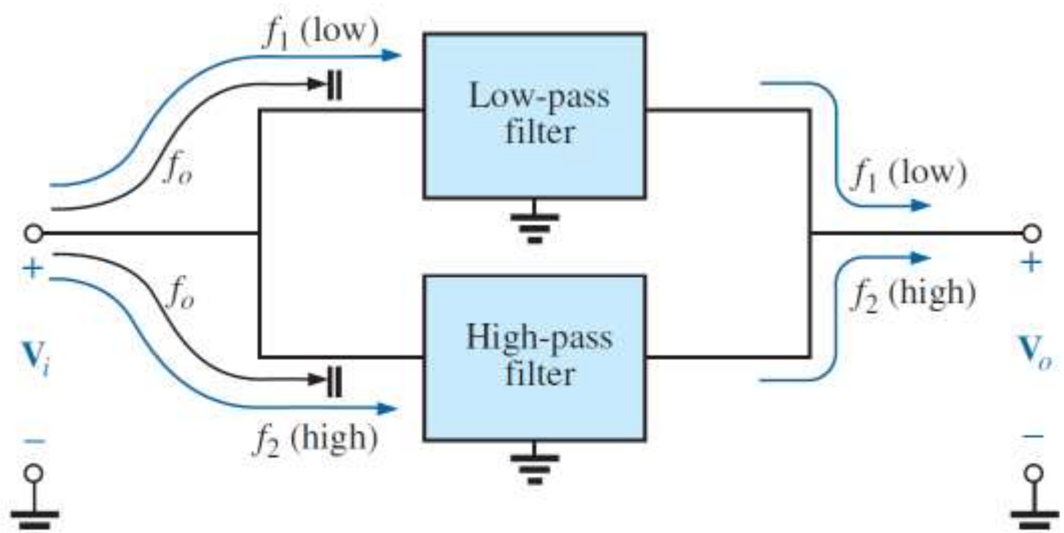


STOP-BAND FILTERS

- A number of methods are used to establish the stop-band characteristic.



  – Stop-band filters can also be constructed using a low-pass and a high-pass filter.



Bode Plots

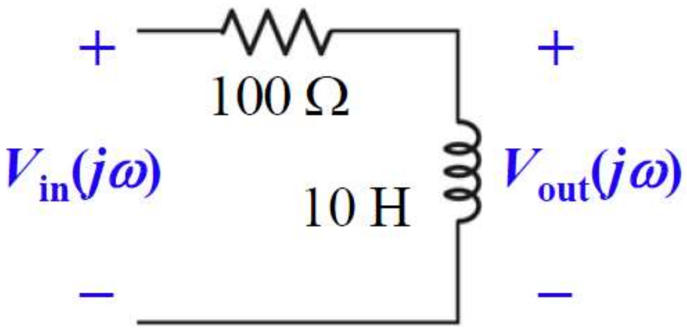
- The frequency range required in frequency response is often so wide that it is inconvenient to use a linear scale for the frequency axis.
- Standard practice to plot the transfer function on a pair of semilogarithmic plots:
  - The magnitude in decibels is plotted against the logarithm of the frequency;
  - The phase in degrees is plotted against the logarithm of the frequency.
- Bode plots are semilog plots of the magnitude (in decibels) and phase (in degrees) of a transfer function versus frequency.

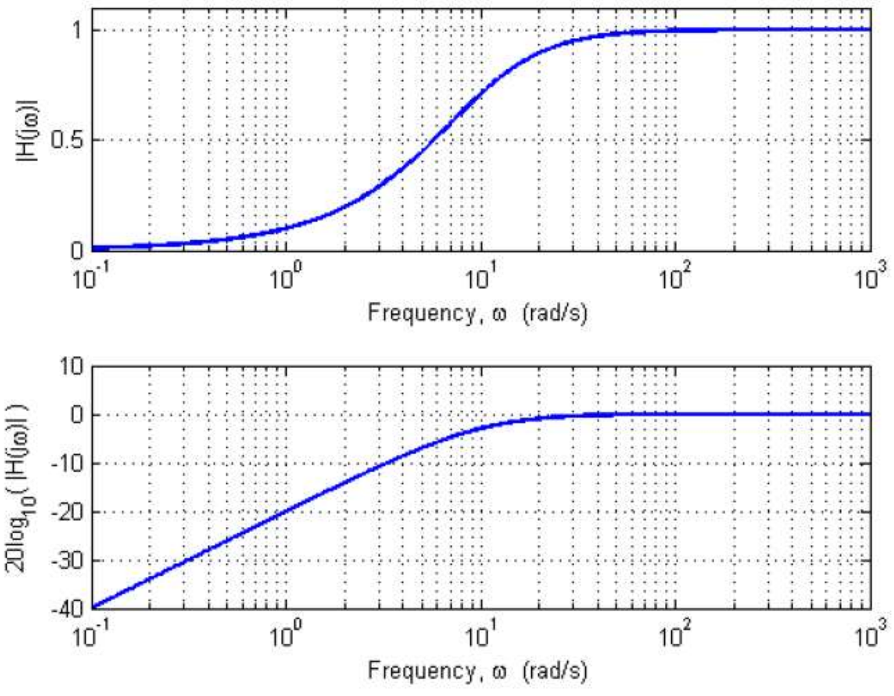
Example 01…

- $|H(j\omega)|$ vs. Frequency

Determine the transfer function, $\mathbf{H}(j\omega)$ and plot $|\mathbf{H}(j\omega)|$ and $20\log_{10}|\mathbf{H}(j\omega)|$ vs. ω for

$$0.1\,\text{rad/s} < \omega < 1\,\text{krad/s}$$





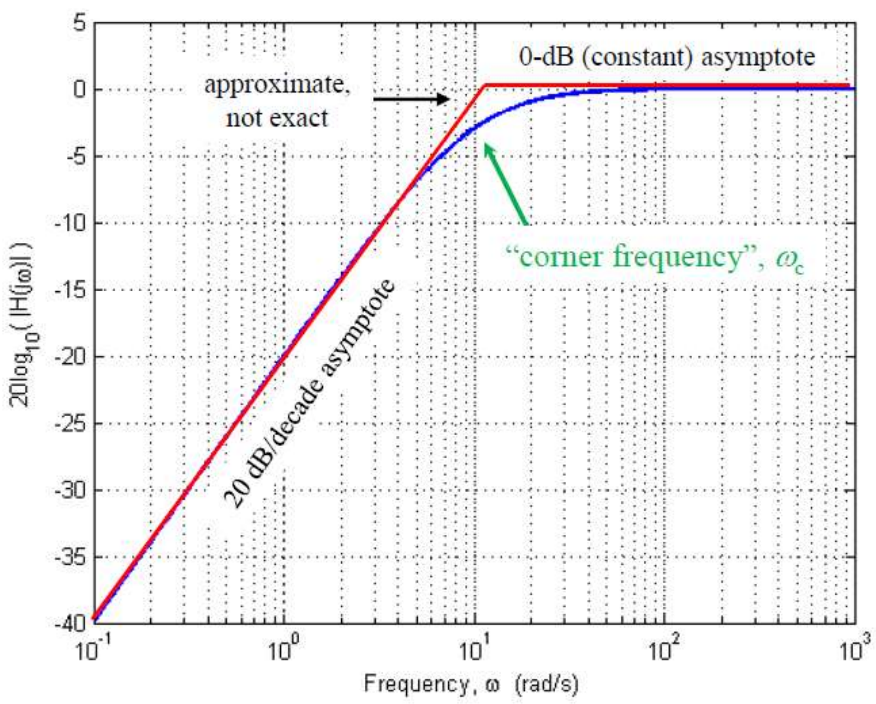
$$\mathbf{H}(j\omega) = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} = \frac{j\omega}{j\omega+10}$$

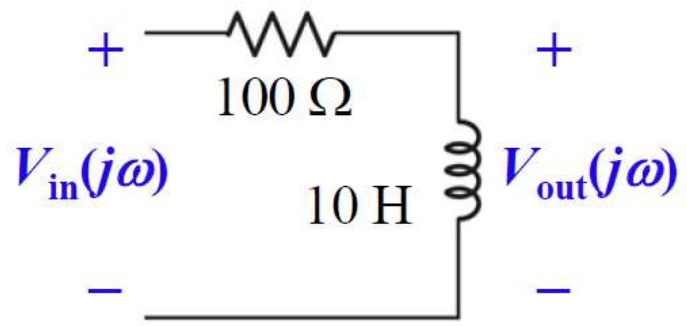
$$|\mathbf{H}(j\omega)| = \frac{\omega}{\sqrt{\omega^2+10^2}}$$

Example 01…

- Bode Amplitude Plot

A **Bode amplitude plot** is an <u>approximation</u> to the logarithmic plot $20\log_{10}|\mathbf{H}(j\omega)|$ vs. ω drawn using the *asymptotes* of the exact transfer function.

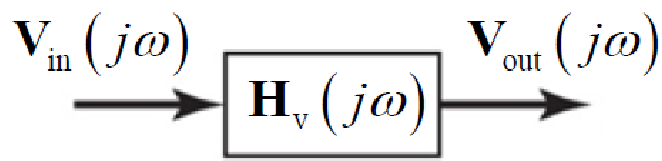




(Voltage) Transfer Function

Let $\mathbf{V}_{in}(j\omega)$ be the phasor form of the <u>voltage input</u> to a circuit, and let $\mathbf{V}_{out}(j\omega)$ be the phasor form of the <u>voltage output</u> from a circuit, expressed as a ratio:

$$\mathbf{H}_v(j\omega) = \frac{\mathbf{V}_{out}(j\omega)}{\mathbf{V}_{in}(j\omega)}$$



$\mathbf{H}_v(j\omega)$ may generally be written as a function of one (factored) polynomial $\mathbf{N}(j\omega)$ divided by another (factored) polynomial $\mathbf{D}(j\omega)$.

$$\mathbf{H}_v(j\omega) = \frac{\mathbf{N}(j\omega)}{\mathbf{D}(j\omega)} = j\omega_0 \frac{(1+j\omega_1)(1+j\omega_2)(1+j\omega_3)\ldots(1+j\omega_N)}{(1+j\omega_\alpha)(1+j\omega_\beta)(1+j\omega_\chi)\ldots(1+j\omega_Z)}$$

zeros of $\mathbf{H}(j\omega)$ are values of $j\omega$ for which $\mathbf{N}(j\omega) = 0$ $\rightarrow$ $-j\omega_1, -j\omega_2, -j\omega_3, \ldots -j\omega_N$

poles of $\mathbf{H}(j\omega)$ are values of $j\omega$ for which $\mathbf{D}(j\omega) = 0$ $\rightarrow$ $-j\omega_\alpha, -j\omega_\beta, -j\omega_\chi, \ldots -j\omega_Z$

Standard Form

➢ A transfer function may be written in terms of factors that have real and imaginary parts.

$$H(\omega)=\frac{N(\omega)}{D(\omega)}=\frac{K(j\omega)^{\pm 1}\left(1+{}^{j\omega}\!/_{z_1}\right)\left[1+{}^{j2\varsigma_1\omega}\!/_{\omega_k}+\left({}^{j\omega}\!/_{\omega_k}\right)^2\right]\cdots}{\left(1+{}^{j\omega}\!/_{p_1}\right)\left[1+{}^{j2\varsigma_1\omega}\!/_{\omega_n}+\left({}^{j\omega}\!/_{\omega_n}\right)^2\right]\cdots}$$

➢ This representation is called the STANDARD FORM. It has several different factors. We can draw the Bode plots by plotting each of the terms of the transfer function separately and then adding them.

The different factors of the transfer function are

1.) Gain term K.

2.) A pole $(j\omega)^{-1}$ or a zero $(j\omega)$ at the origin.

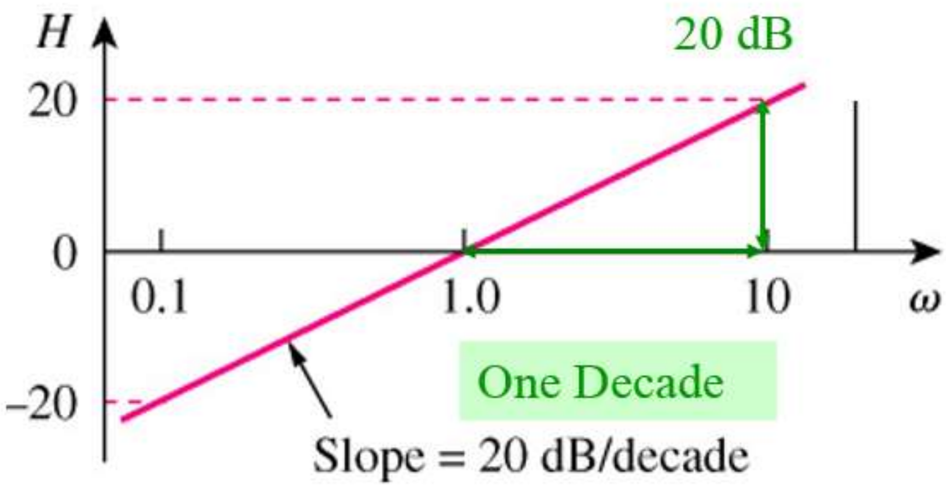
3.) A simple pole $\left(1+{}^{j\omega}\!/_{p_1}\right)$ or zero $\left(1+{}^{j\omega}\!/_{z_1}\right)$

4.) A quadratic pole $\left[1+{}^{j2\varsigma_1\omega}\!/_{\omega_n}+\left({}^{j\omega}\!/_{\omega_n}\right)^2\right]$ or zero $\left[1+{}^{j2\varsigma_1\omega}\!/_{\omega_k}+\left({}^{j\omega}\!/_{\omega_k}\right)^2\right]$

Decade

➤ A DECADE is an interval between 2 frequencies with a ratio of 10 (between 10 Hz and 100 Hz or between 500 Hz and 5000 Hz). 20 dB/decade means that magnitude changes 20 dB whenever the frequency changes tenfold or one decade.

➤ Slopes are expressed in dB/decade.



Bode Plot Procedure

➢ To plot the Bode plots of a given transfer function.

1.) Put the transfer function in STANDARD FORM.
2.) Write the Magnitude and phase equations from the STANDARD FORM.
3.) Plot the magnitude of each term separately.
4.) Add all magnitude terms to obtain the magnitude transfer function.
5.) Repeat 2-4 for the phase response.
6.) The total magnitude response in Decibel units is the summation and subtraction of the responses of different terms.
7.) The total phase response in degrees is the summation and subtraction of the phase responses of different terms.

STANDARD FORM

$$H(\omega) = \frac{0.4(1 + j\omega/10)}{j\omega(1 + j\omega/5)^2}$$

$$\begin{aligned} H_{db} &= 20\log_{10} 0.4 + \\ &\quad 20\log_{10}\left|1 + j\omega/10\right| - \\ &\quad 20\log_{10}\left|j\omega\right| - \\ &\quad 40\log_{10}\left|1 + j\omega/5\right| \end{aligned}$$

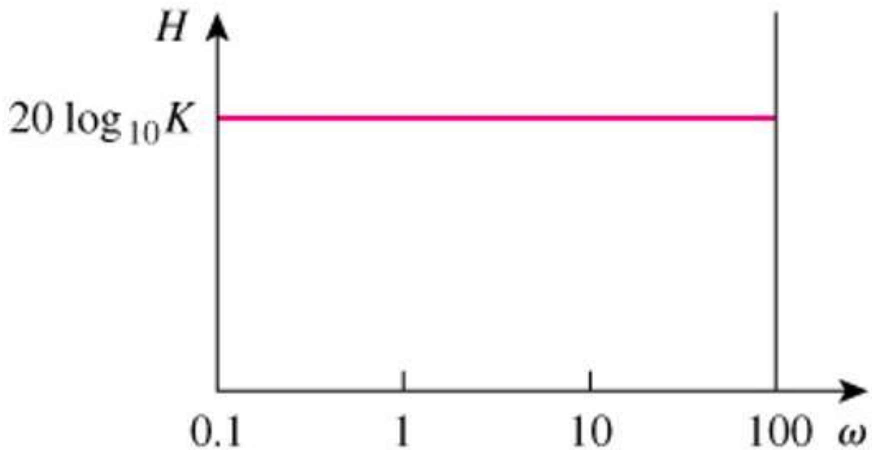
$$\begin{aligned} \phi &= 0° + \tan^{-1}(\omega/10) \\ &\quad - 90° - 2\tan^{-1}(\omega/5) \end{aligned}$$

Term Types

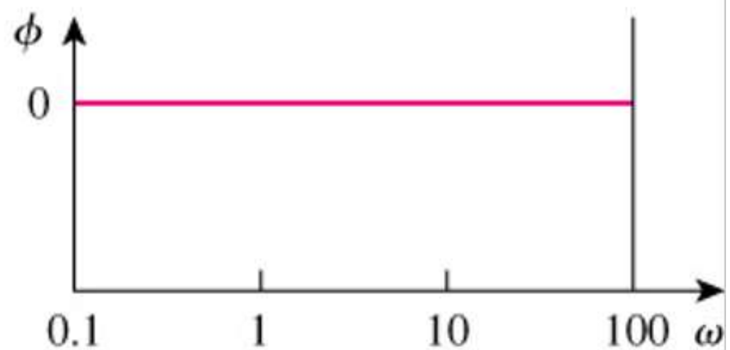
- We examine how to plot different terms that may appear in a transfer function. The total response will be obtained by adding all the responses.

$$H(\omega)=\frac{N(\omega)}{D(\omega)}=\frac{K(j\omega)^{\pm 1}\left(1+{j\omega}/{z_1}\right)\left[1+{j2\varsigma_1\omega}/{\omega_k}+\left({j\omega}/{\omega_k}\right)^2\right]\cdots}{\left(1+{j\omega}/{p_1}\right)\left[1+{j2\varsigma_1\omega}/{\omega_n}+\left({j\omega}/{\omega_n}\right)^2\right]\cdots}$$

CONSTANT TERM $20\log_{10}K$



(a)

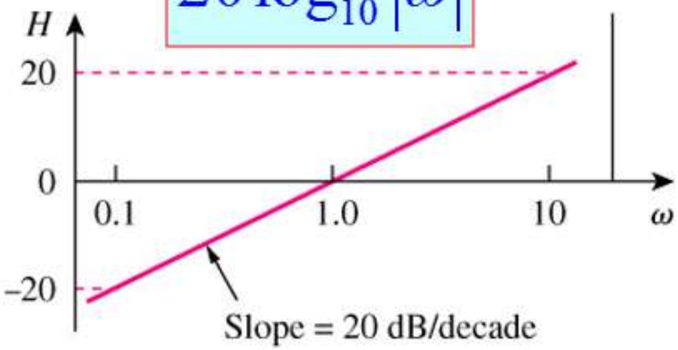


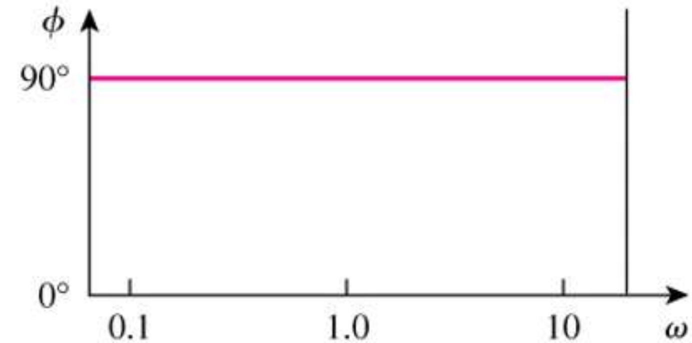
(b)

Term Types

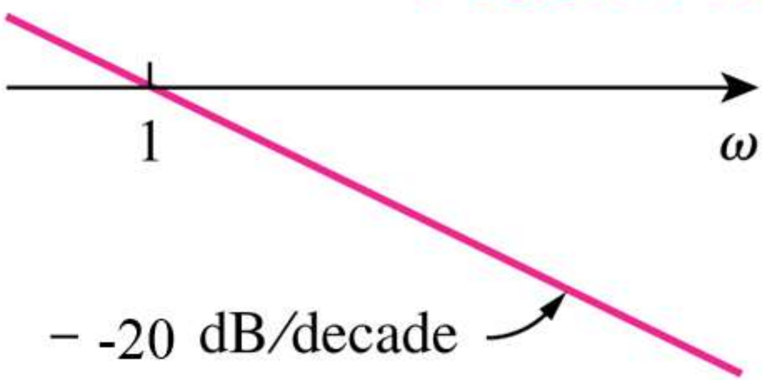
ZERO AT THE ORIGIN $(j\omega)$

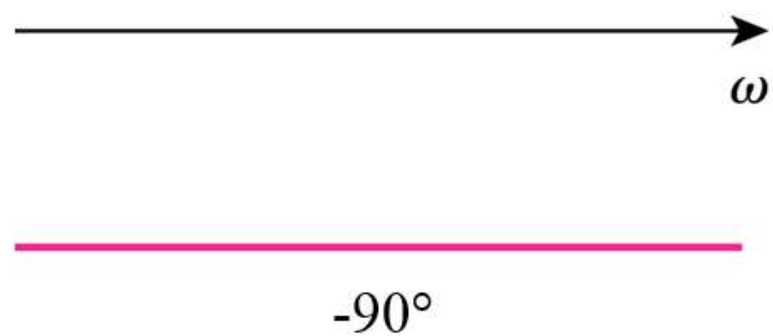
$$\boxed{20\log_{10}|\omega|}$$





POLE AT THE ORIGIN $(j\omega)^{-1}$





Term Types

Let the simplest transfer function be the **single-zero** function given by

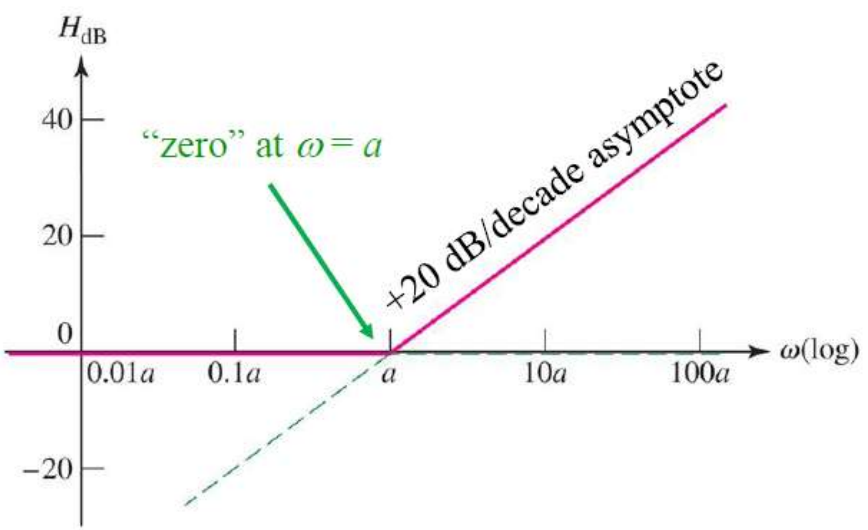
$$\mathbf{H}(j\omega) = 1 + \frac{j\omega}{a}$$

The amplitude may be written as $\left|\mathbf{H}(j\omega)\right| = \sqrt{1 + \frac{\omega^2}{a^2}}$ $H_{\mathrm{dB}} = 20\log_{10}\sqrt{1 + \frac{\omega^2}{a^2}}$

Plotted against frequency ω,
the single-zero $|\mathbf{H}(j\omega)|$ looks like:

$$\omega << a, \quad H_{\mathrm{dB}} \approx 0$$

$$\omega >> a, \quad H_{\mathrm{dB}} \approx 20\log_{10}(\omega/a)$$



Term Types

Let the simplest transfer function be the **single-zero** function given by

$$\mathbf{H}(j\omega) = 1 + \frac{j\omega}{a}$$

a = "zero" of $\mathbf{H}(j\omega)$

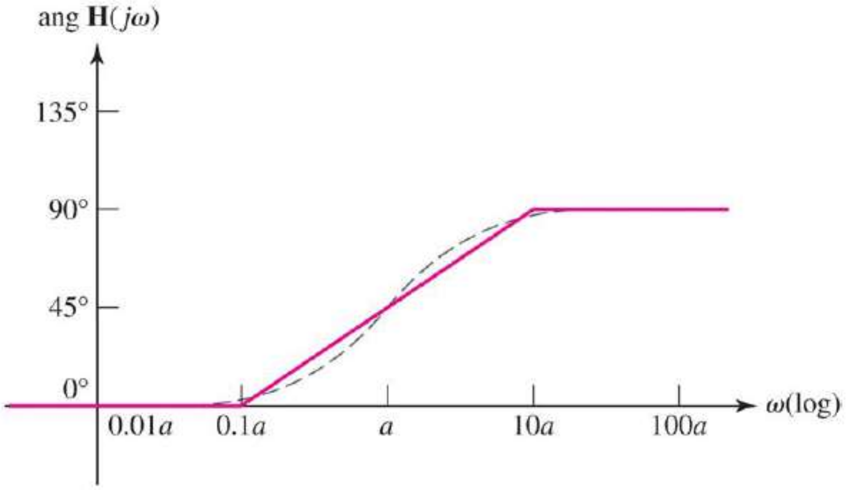
The **phase** may be written as $$\text{ang}\{\mathbf{H}(j\omega)\} = \tan^{-1}\frac{\omega}{a}$$

Plotted against frequency ω,
the <u>single-zero</u> ang$\{\mathbf{H}(j\omega)\}$ looks like:

$$\omega < a/10, \quad \text{ang}\{\mathbf{H}(j\omega)\} \approx 0$$

$$\omega \approx a, \quad \text{ang}\{\mathbf{H}(j\omega)\} \approx +45^\circ$$

$$\omega > 10a, \quad \text{ang}\{\mathbf{H}(j\omega)\} \approx +90^\circ$$



Term Types

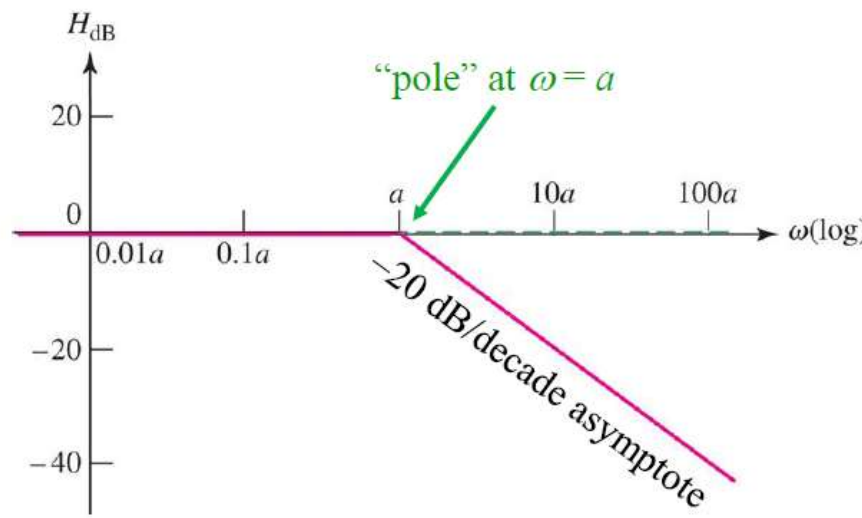
Let another simple transfer function be the **single-pole** function given by

$$\mathbf{H}(j\omega) = \frac{1}{1 + j\omega/a}$$

The amplitude may be written as $\quad |\mathbf{H}(j\omega)| = \dfrac{1}{\sqrt{1+\dfrac{\omega^2}{a^2}}} \qquad H_{dB} = -20\log_{10}\sqrt{1+\dfrac{\omega^2}{a^2}}$

Plotted against frequency ω,
the single-pole $|\mathbf{H}(j\omega)|$ looks like:

$$\omega << a, \quad H_{dB} \approx 0$$
$$\omega >> a, \quad H_{dB} \approx -20\log_{10}(\omega/a)$$



Term Types

Let another simple transfer function be the **single-pole** function given by

$$\mathbf{H}(j\omega) = \frac{1}{1 + j\omega/a}$$

a = "pole" of $\mathbf{H}(j\omega)$

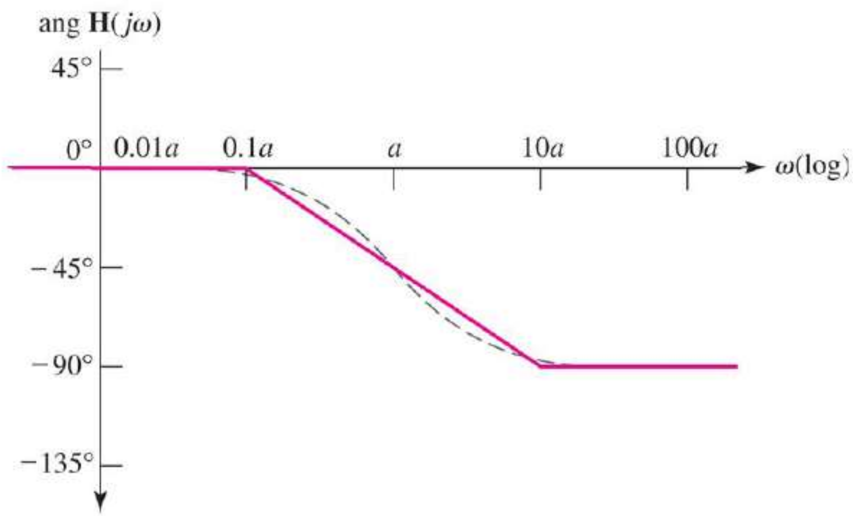
The **phase** may be written as $$\text{ang}\{\mathbf{H}(j\omega)\} = -\tan^{-1}\frac{\omega}{a}$$

Plotted against frequency ω,
the single-pole $\text{ang}\{\mathbf{H}(j\omega)\}$ looks like:

$$\omega < a/10, \quad \text{ang}\{\mathbf{H}(j\omega)\} \approx 0$$

$$\omega \approx a, \quad \text{ang}\{\mathbf{H}(j\omega)\} \approx -45°$$

$$\omega > 10a, \quad \text{ang}\{\mathbf{H}(j\omega)\} \approx -90°$$



Example 02…

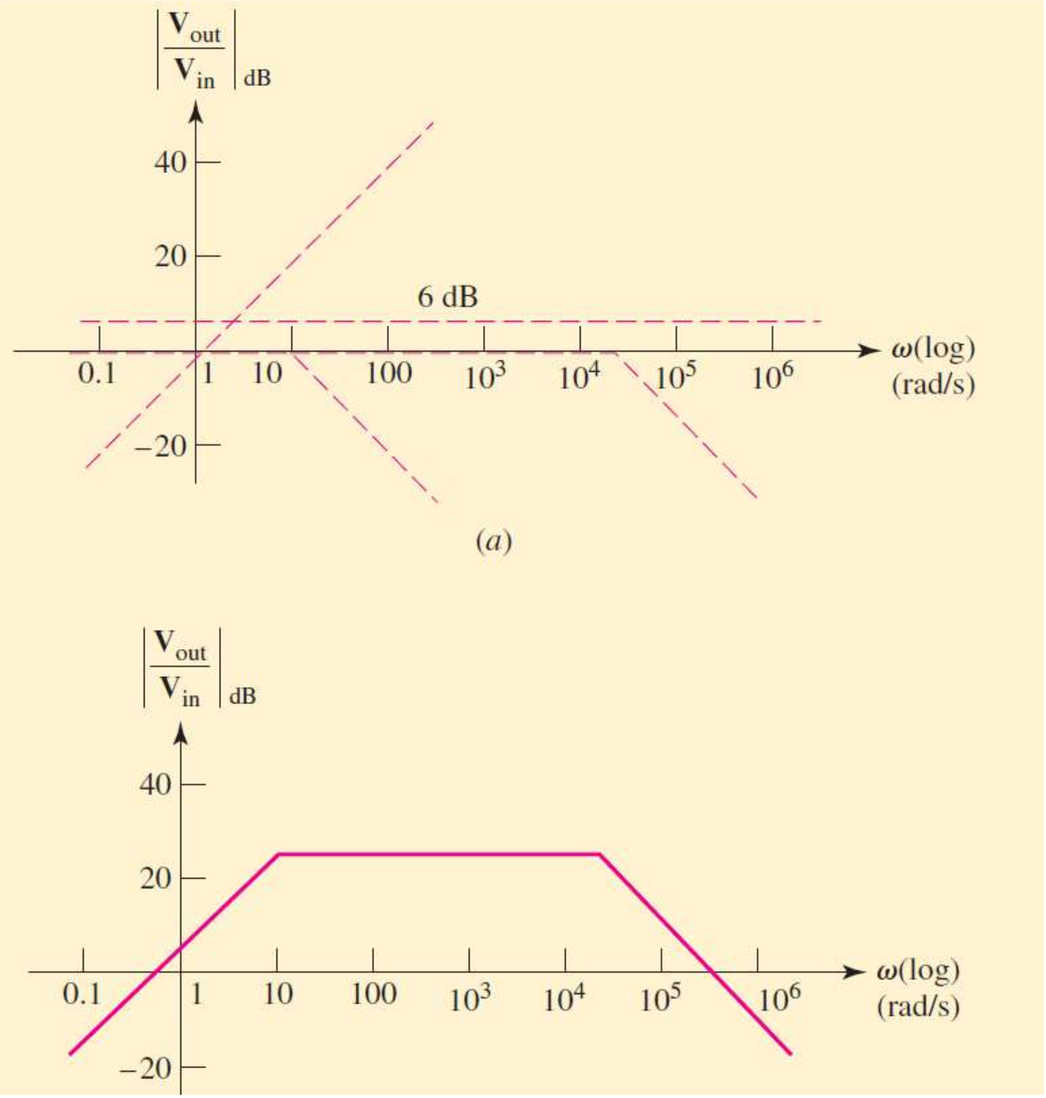
- Construct a Bode plot for the amplitude and phase of the transfer function that is given

$$\mathbf{H}(j\omega) = \frac{-2j\omega}{(1 + j\omega/10)(1 + j\omega/20{,}000)}$$

…Example 02…

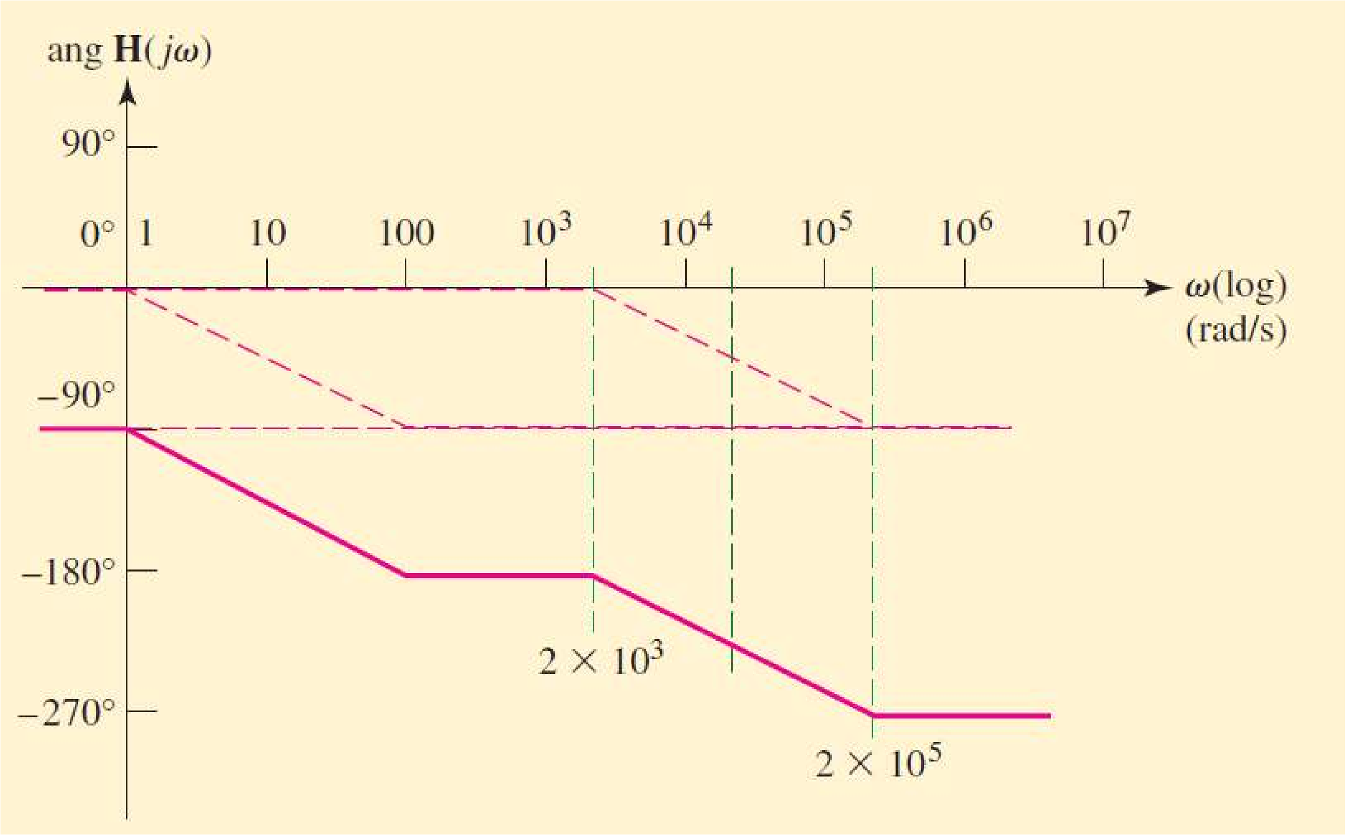
- Amplitude plot:

$$\mathbf{H}(j\omega) = \frac{-2j\omega}{(1 + j\omega/10)(1 + j\omega/20{,}000)}$$



…Example 02…

- Phase plot: $$\mathbf{H}(j\omega)=\frac{-2j\omega}{(1+j\omega/10)(1+j\omega/20{,}000)}$$



…Example 02…

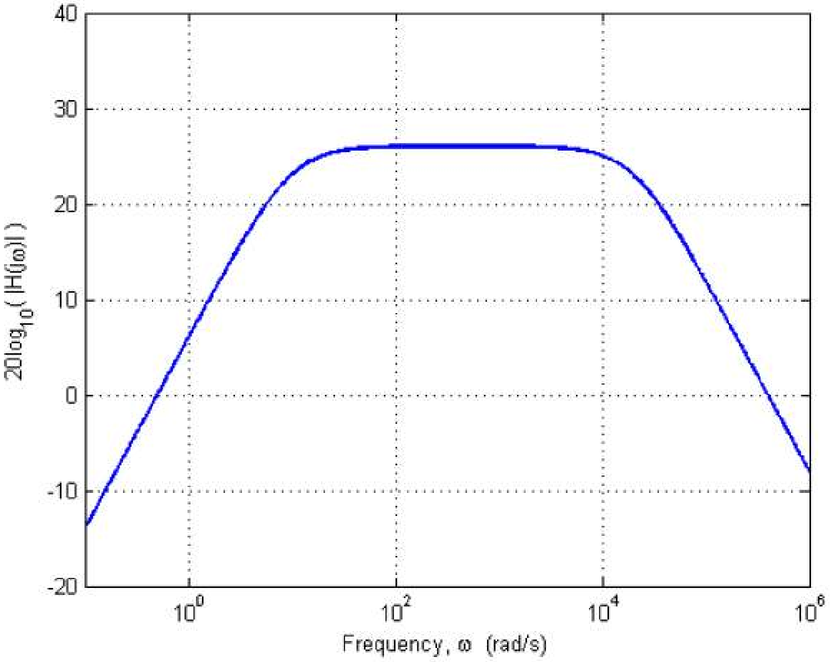
- Amplitude plot - Matlab

The **exact** amplitude, plotted using Matlab:

```
w = logspace(-1,6,1001);

H = -2*j*w ./ ( (1 + j*w/10) ...
         .*(1 + j*w/20000) );

semilogx(w,20*log10(abs(H)),...
           'Linewidth',2)
axis([10^-1 10^6 -20 40]);
grid
ylabel('20log_{10}( |H(j\omega)| )');
xlabel('Frequency, \omega  (rad/s)')
```



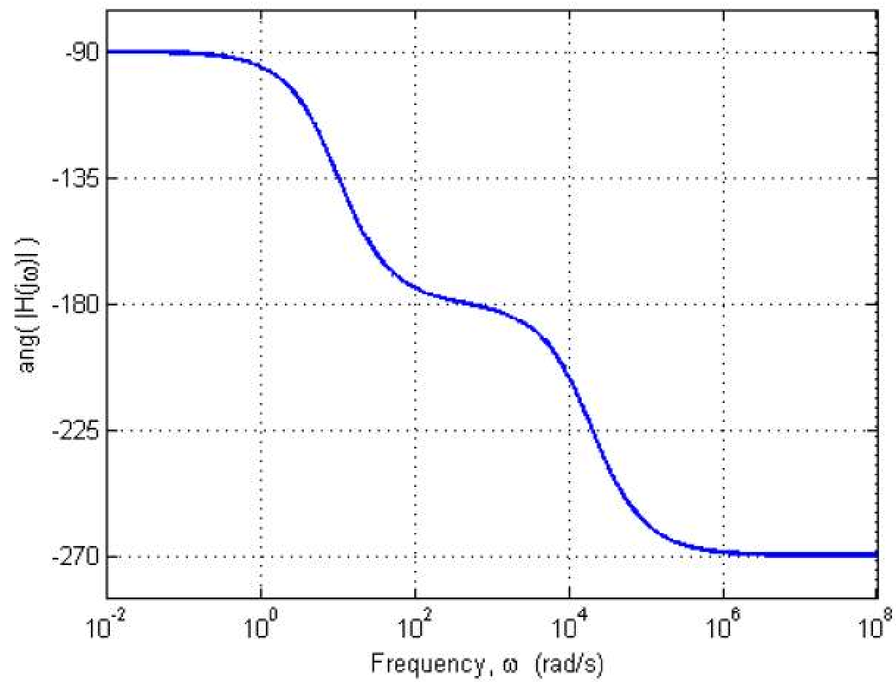
…Example 02

- Phase plot - Matlab

```
w = logspace(-2,8,1001);

H = -2*j*w ./ ( (1 + j*w/10) ...
                .*(1 + j*w/20000) );

semilogx(w,phase(H)*180/pi,...
         'Linewidth',2)
axis([10^-2 10^8 -285 -75]);
set(gca,'Ytick',[-360:45:360])
grid
ylabel('ang( |H(j\omega)| )');
xlabel('Frequency, \omega  (rad/s)')
```



Example 03…

Construct Bode plots for

$$H(\omega) = \frac{200\,j\omega}{(j\omega+2)(j\omega+10)}$$

- Express transfer function in Standard form.

$$\text{STANDARD FORM} \quad H(\omega) = \frac{10\,j\omega}{(1+j\omega/2)(1+j\omega/10)}$$

- Express the magnitude and phase responses.

$$H_{db} = 20\log_{10}10 + 20\log_{10}|\,j\omega\,| - 20\log_{10}|1+j\omega/2| - 20\log_{10}|1+j\omega/10|$$

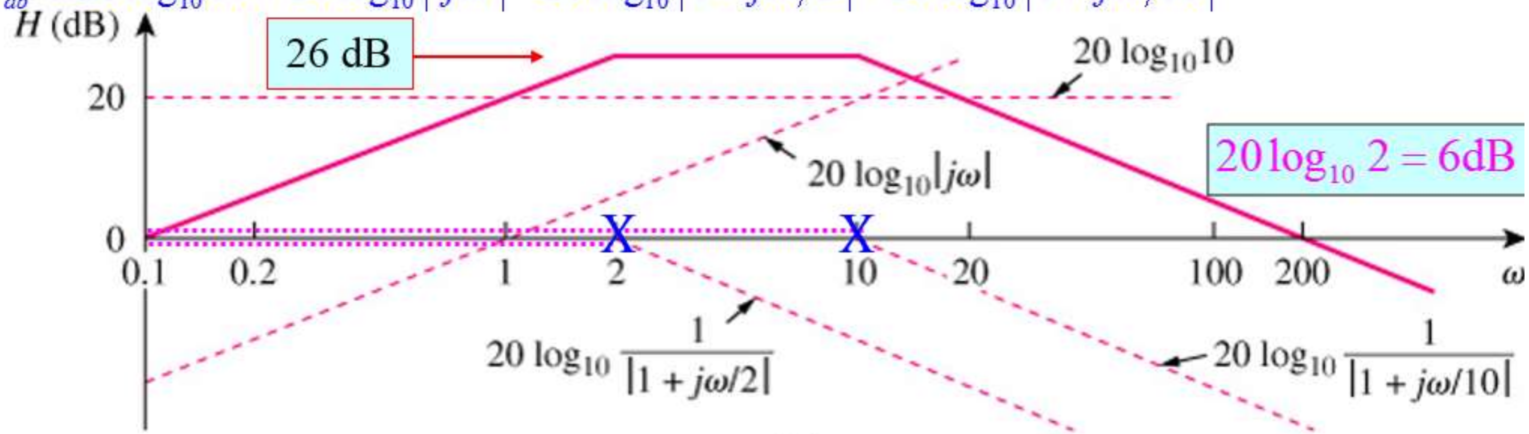
$$\phi = 90° - \tan^{-1}(\omega/2) - \tan^{-1}(\omega/10)$$

- Two corner frequencies at ω=2, 10 and a zero at the origin ω=0.
- Sketch each term and add to find the total response.

…Example 03

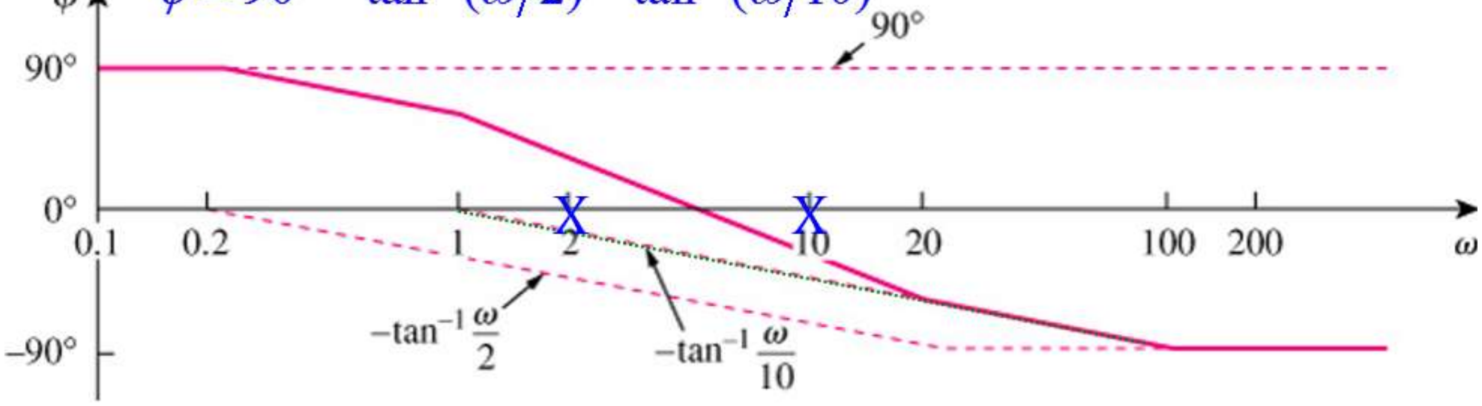
Construct Bode plots for $H(\omega)=\dfrac{200\,j\omega}{(j\omega+2)(j\omega+10)}$

$$H_{db}=20\log_{10}10+20\log_{10}|j\omega|-20\log_{10}|1+j\omega/2|-20\log_{10}|1+j\omega/10|$$



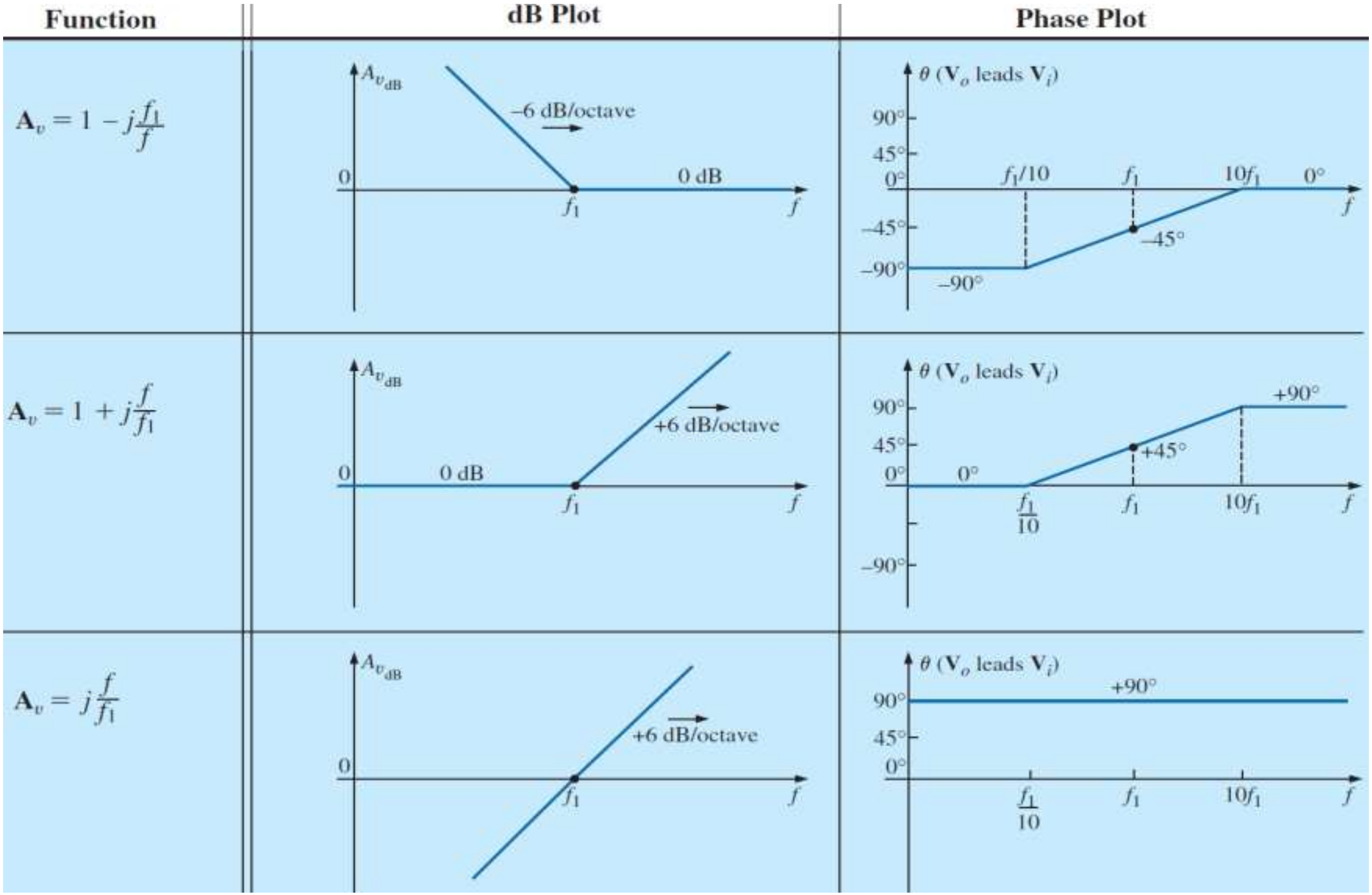
(a)

$$\phi=90^{\circ}-\tan^{-1}(\omega/2)-\tan^{-1}(\omega/10)$$



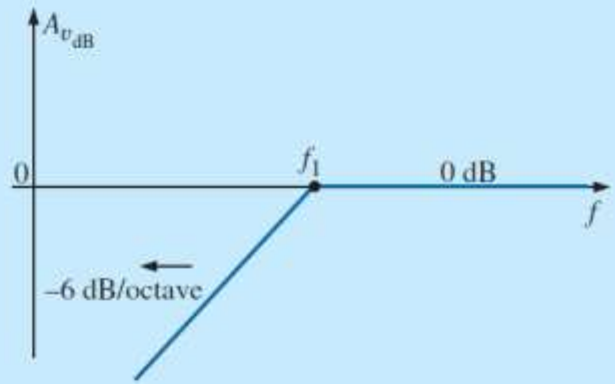
(b)

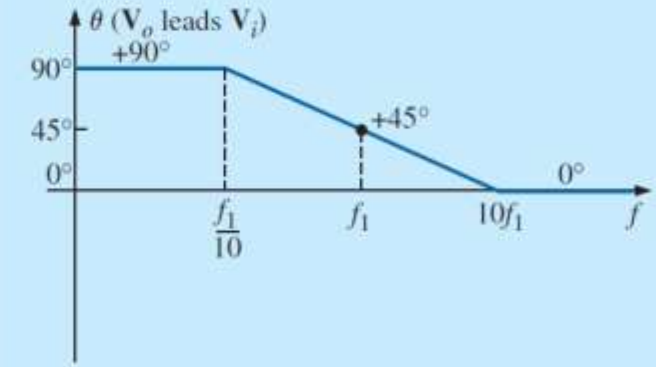
Idealized Bode plots for various functions



Idealized Bode plots for various functions

$$\mathbf{A}_v = \frac{1}{1 - j\dfrac{f_1}{f}}$$





$$\mathbf{A}_v = \frac{1}{1 + j\dfrac{f}{f_1}}$$