



## BLM3620 Digital Signal Processing

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# Syllabus



Week	Lectures
1	Introduction to DSP and MATLAB
2	Sinusoids and Complex Exponentials
3	Spectrum Representation
4	Sampling and Aliasing
5	Discrete Time Signal Properties and Convolution
6	Convolution and FIR Filters
7	Frequency Response of FIR Filters
8	Midterm Exam
9	Discrete Time Fourier Transform and Properties
10	Discrete Fourier Transform and Properties
11	Fast Fourier Transform and Windowing
12	z- Transforms
13	FIR Filter Design and Applications
14	IIR Filter Design and Applications
15	Final Exam

For more details -> Bologna page: <http://www.bologna.yildiz.edu.tr/index.php?r=course/view&id=5730&aid=3>

## Lecture #2 – Sinusoids and Complex Exponentials

- Sinusoidal Signals
- Frequency, Period, Phase and Amplitude
- Complex Exponential Signals
- Phasor Addition
- MATLAB Applications

## ***Important Materials:***

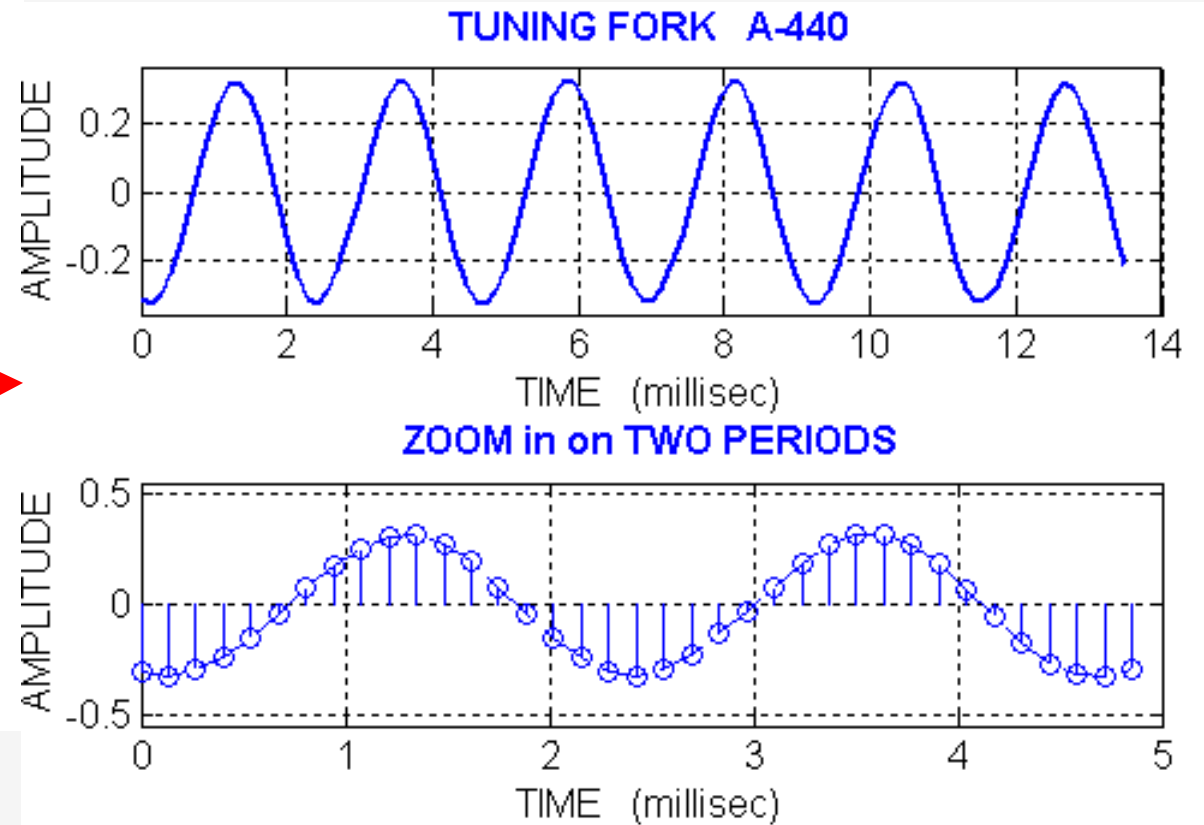
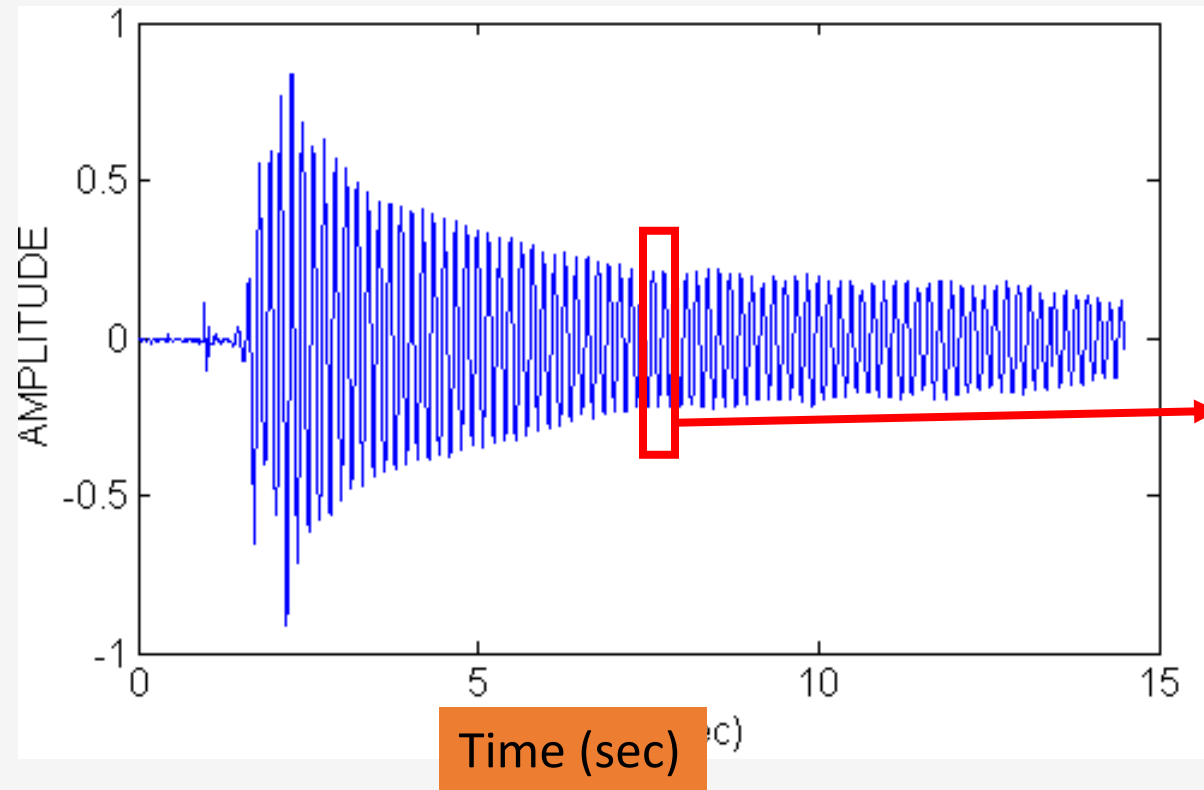
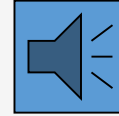
- James H. McClellan, R. W. Schafer, M. A. Yoder, *DSP First Second Edition*, Pearson, 2015.
- Lizhe Tan, Jean Jiang, *Digital Signal Processing: Fundamentals and Applications*, Third Edition, Academic Press, 2019.

## ***Auxiliary Materials:***

- Prof. Sarp Ertürk, *Sayısal İşaret İşleme*, Birsen Yayınevi.
- Prof. Nizamettin Aydın, DSP Lecture Notes.
- J. G. Proakis, D. K. Manolakis, *Digital Signal Processing Fourth Edition*, Pearson, 2014.
- J. K. Perin, *Digital Signal Processing, Lecture Notes*, Stanford University, 2018.

# Recall: Tuning Fork

Sinusoids are important part of our world.



# SINES and COSINES



- Always use the COSINE FORM

$$A \cos(2\pi(440)t + \varphi)$$

- Sine is a special case:

$$\sin(\omega t) = \cos(\omega t - \frac{\pi}{2})$$
A blue double-headed arrow points from the phase term  $\varphi$  in the cosine formula above to the  $-\frac{\pi}{2}$  term in the sine formula below, illustrating the phase shift relationship between sine and cosine waves.

$$A \cos(\omega t + \varphi)$$

- **FREQUENCY**

- Radians/sec
- Hertz (cycles/sec)

$$\omega$$

$$\omega = (2\pi)f$$

- **PERIOD** (in sec)

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

- **AMPLITUDE**

- Magnitude

$$A$$

- **PHASE**

$$\varphi$$

# Some Trigonometric Identities



Number	Equation
1	$\sin^2 \theta + \cos^2 \theta = 1$
2	$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
3	$\sin 2\theta = 2 \sin \theta \cos \theta$
4	$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$
5	$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$



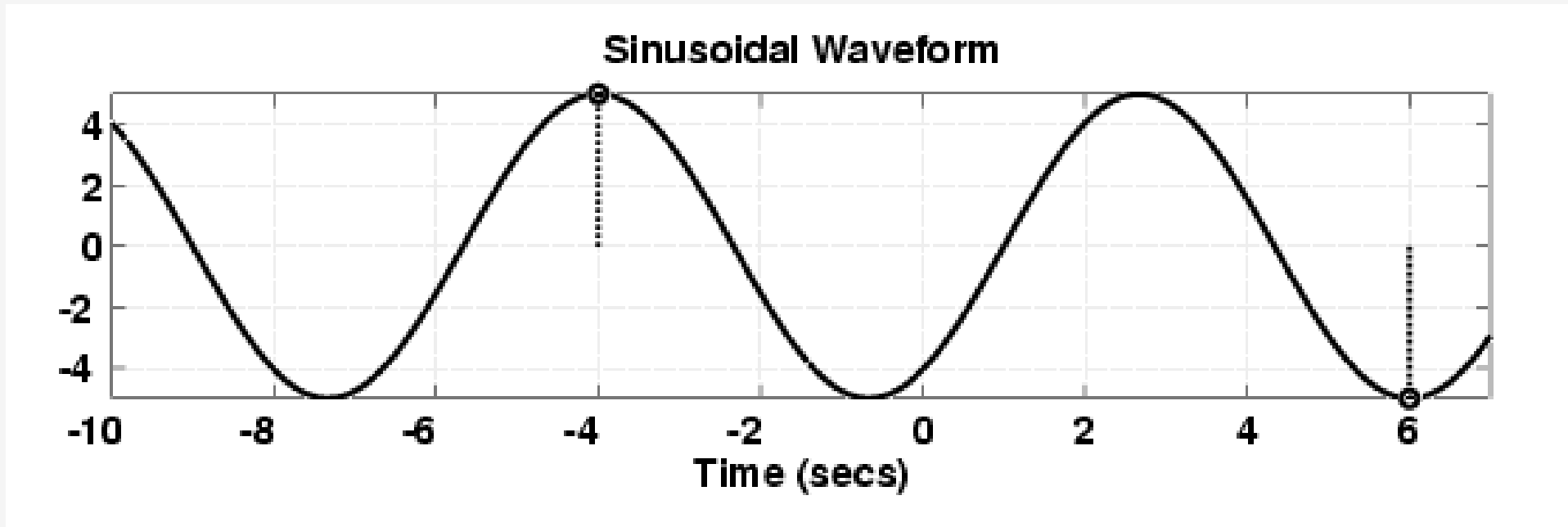
# EXAMPLE of SINUSOID



- Given the Formula

$$5\cos(0.3\pi t + 1.2\pi)$$

- Make a plot



# PLOT COSINE SIGNAL



$$5\cos(0.3\pi t + 1.2\pi)$$

- Formula defines  $A$ ,  $\omega$ , and  $\phi$

$$\begin{aligned} A &= 5 \\ \omega &= 0.3\pi \\ \phi &= 1.2\pi \end{aligned}$$

# PLOTTING COSINE SIGNAL from the FORMULA



$$5 \cos(0.3\pi t + 1.2\pi)$$

- Determine period:

$$T = 2\pi / \omega = 2\pi / 0.3\pi = 20 / 3$$

- Determine a peak location by solving

$$(\omega t + \varphi) = 0 \Rightarrow (0.3\pi t + 1.2\pi) = 0$$

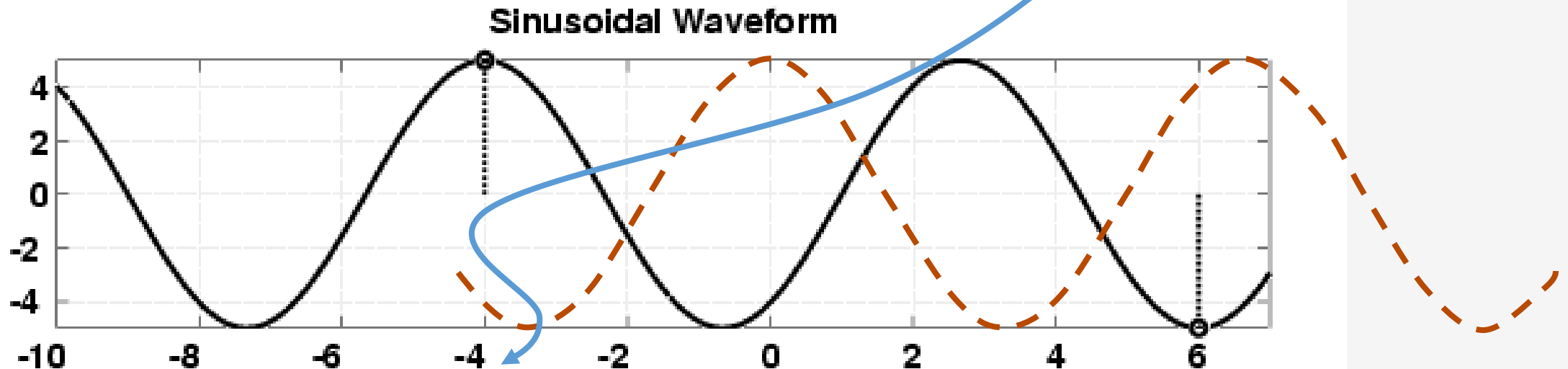
- Zero crossing is  $T/4$  before or after
- Positive & Negative peaks spaced by  $T/2$

# Time-shifted Sinusoid



$$x(t) = 5 \cos(0.3\pi t) \quad \text{One peak at } t = 0$$

$$x(t+4) = 5 \cos(0.3\pi(t+4)) = 5 \cos(0.3\pi(t - (-4)))$$



Peak shifts from  $t=0$  to  $t = -4$

# How to determine Amplitude, Phase and Period from a plot

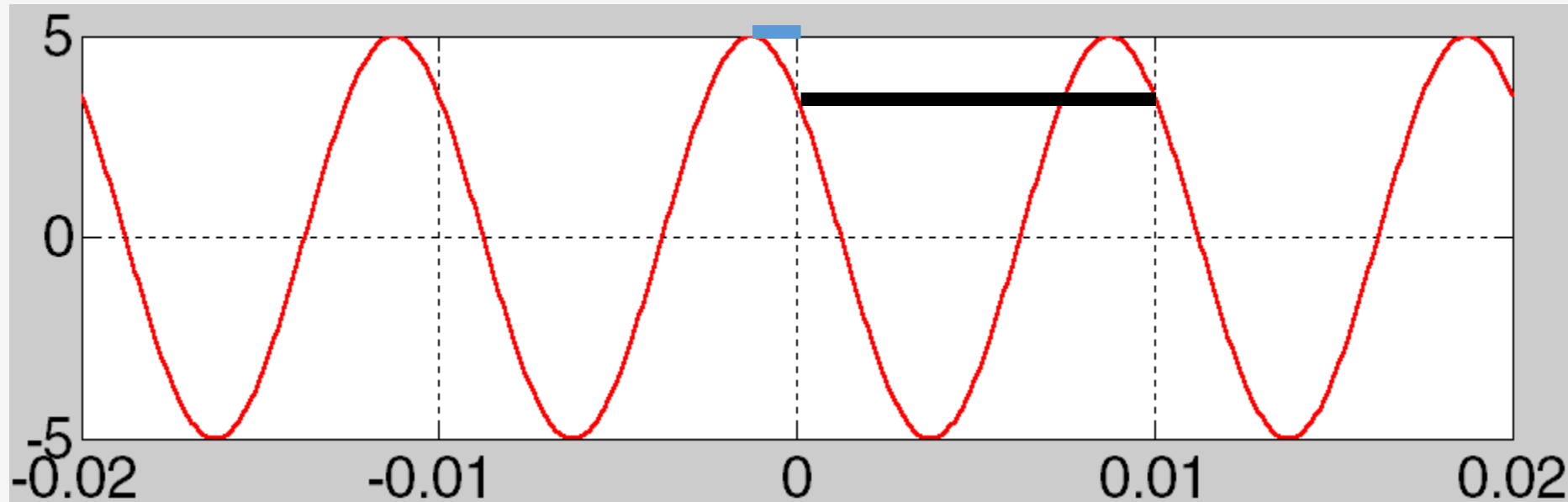


- Measure the period,  $T$ 
  - Between peaks or zero crossings
- Compute frequency:  $\omega = 2\pi/T$
- Measure time of a peak:  $t_m$ 
  - Compute phase:  $\phi = -\omega t_m$
- Measure height of positive peak:  $A$

3 steps

A blue box containing the text '3 steps' is connected by three blue curved lines to the three main steps of the process: 'Compute frequency', 'Compute phase', and 'Measure height of positive peak'.

# $(A, \omega, \phi)$ from a PLOT

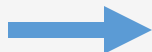


$$T = \frac{0.01 \text{ sec}}{1 \text{ period}} = \frac{1}{100}$$



$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.01} = 200\pi$$

$$t_m = -0.00125 \text{ sec}$$



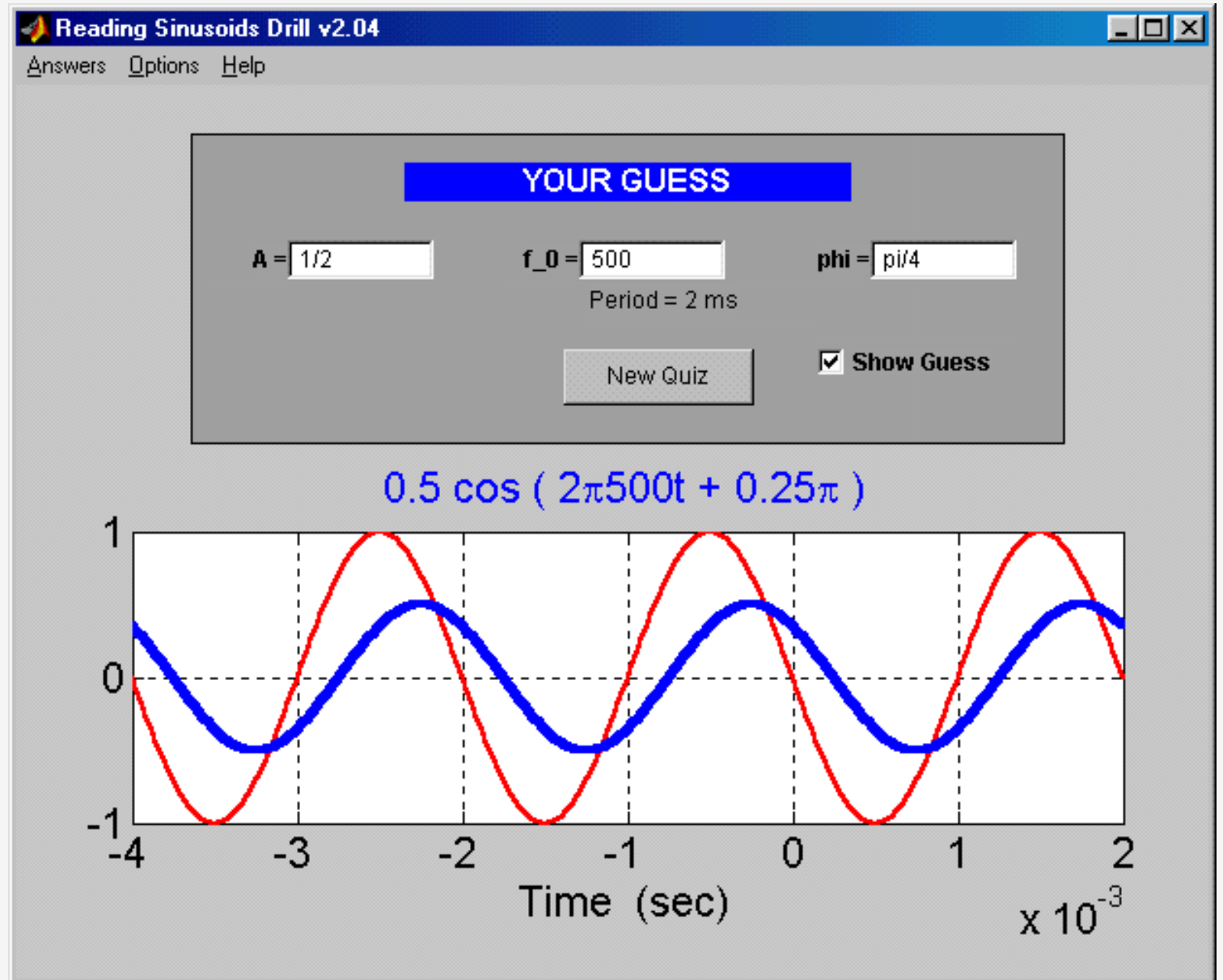
$$\phi = -\omega t_m = -(200\pi)(t_m) = 0.25\pi$$

# SINE DRILL (MATLAB GUI)

<https://dspfirst.gatech.edu/matlab/#sindrill>

**SinDrill** is a program that tests the users ability to determine basic parameters of a sinusoid.

After a plot of a sinusoid is displayed, the user must correctly guess its amplitude, frequency, and phase.



# Phase is Ambiguous

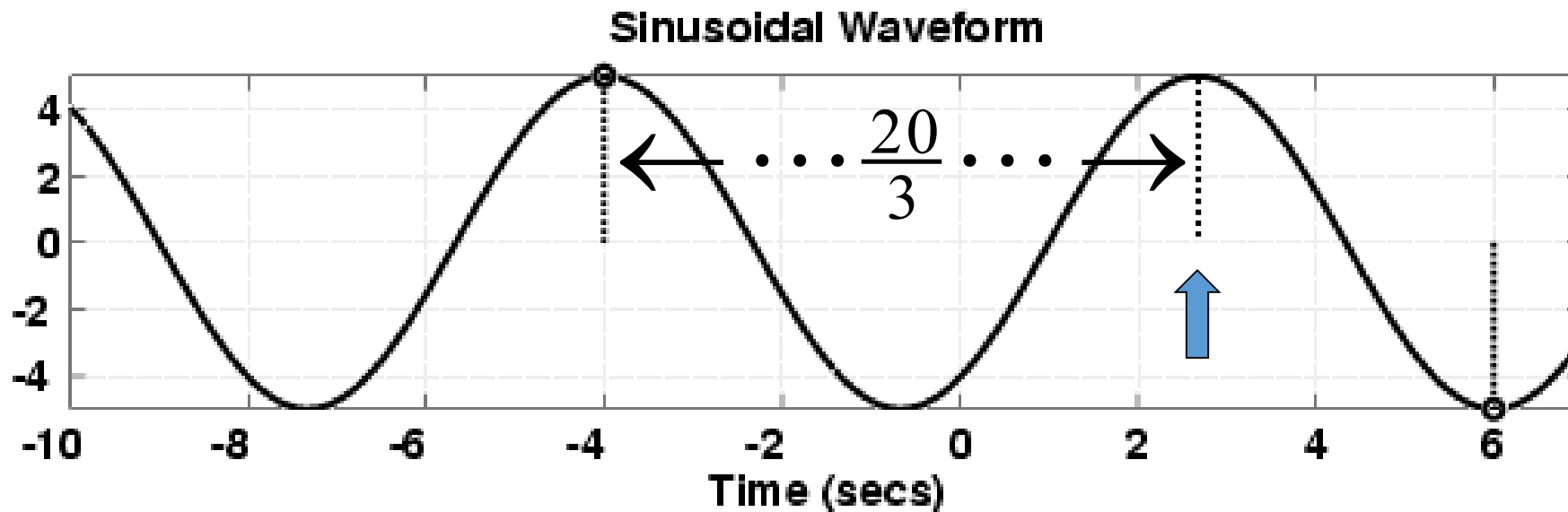


The cosine signal is periodic

– Period is  $2\pi$

$$A \cos(\omega t + \varphi + 2\pi) = A \cos(\omega t + \varphi)$$

$$5 \cos(0.3\pi t + 1.2\pi) = 5 \cos(0.3\pi t - 0.8\pi)$$

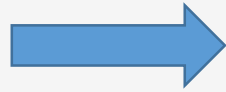




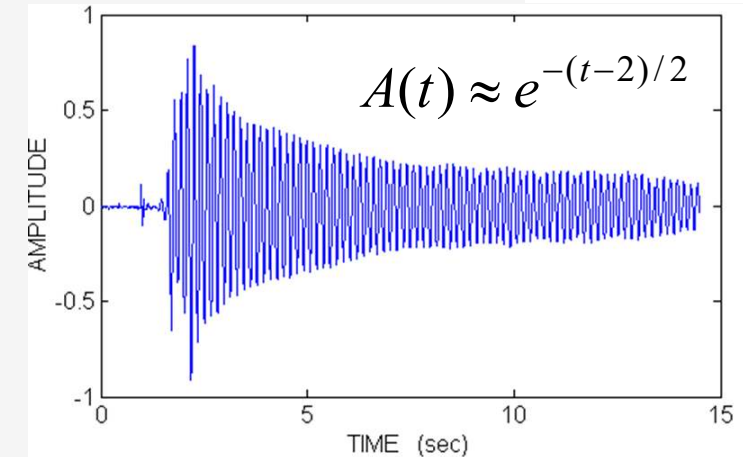
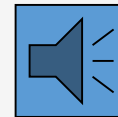
# Attenuation: Amplitude Varies with Time (Fade Out?)



$$x(t) = A \cos(\omega t + \varphi)$$



$$A(t) = A e^{-t/\alpha}$$



```
fs = 8000;  
% define array tt for time  
% time runs from -1s to +3.2s  
% sampled at an interval of 1/fs  
tt = 0: 1/fs : 3.2;  
xx = 2.1 * cos(2*pi*440*tt + 0.4*pi);
```

```
soundsc (xx,fs)
```

$$x(t) = 2.1 \cos(880 \pi t + 0.4 \pi)$$

```
fs = 8000;  
tt = 0: 1/fs : 3.2;  
yy = exp(-tt*1.2); % exponential decay  
yy = xx.*yy;
```

```
soundsc (yy,fs)
```

$$y(t) = 2.1 e^{-1.2t} \cos(880 \pi t + 0.4 \pi)$$

# Growing Sinuzoid? (Exponential Sinuzoid)

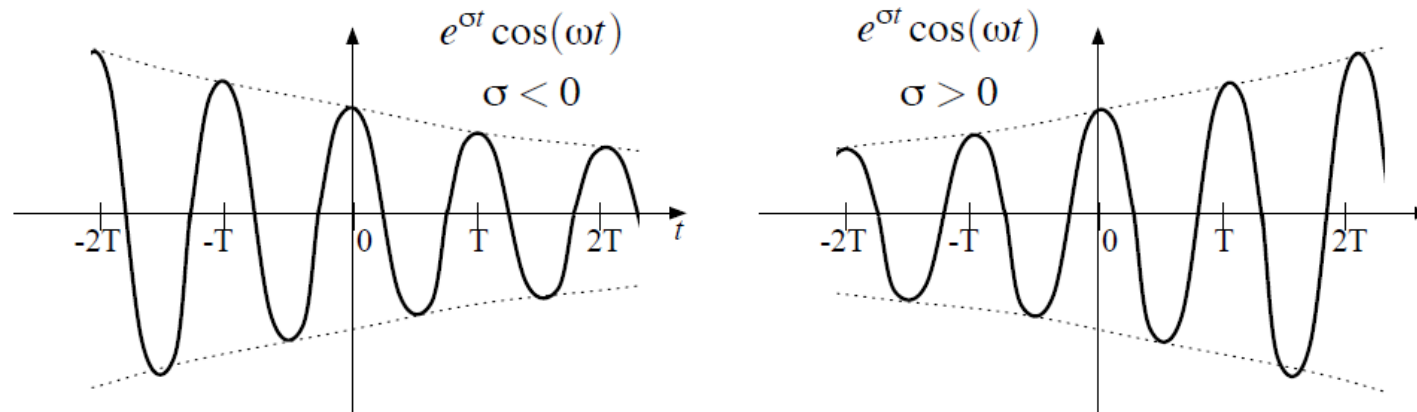


## Damped or Growing Sinusoids

- A damped or growing sinusoid is given by

$$x(t) = e^{\sigma t} \cos(\omega t + \theta)$$

- Exponential growth ( $\sigma > 0$ ) or decay ( $\sigma < 0$ ), modulated by a sinusoid.

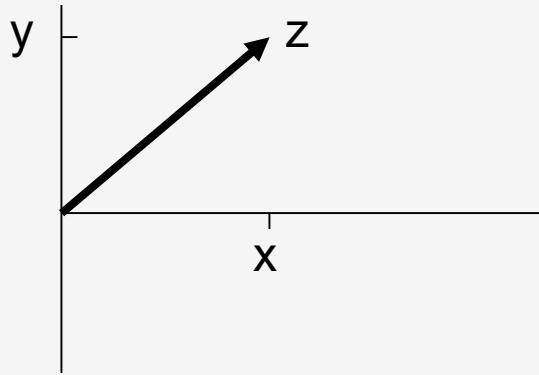


# Remember: Complex Numbers



## Cartesian Coordinate System

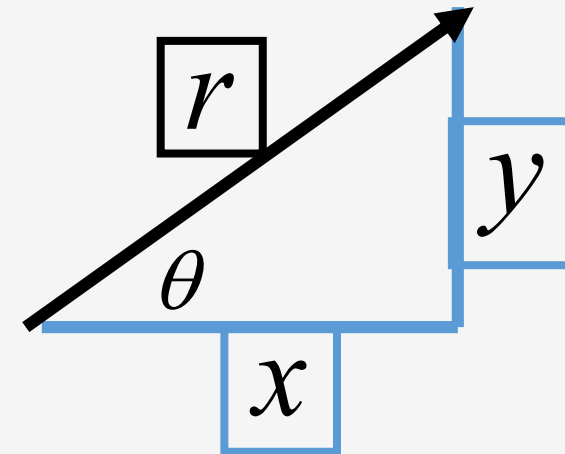
- To solve:  $z^2 = -1$ 
  - $z = j$
  - Math and Physics use  $z = i$
- Complex number:  $z = x + jy$



## Polar Coordinate System

$$r^2 = x^2 + y^2$$
$$\theta = \text{Tan}^{-1}\left(\frac{y}{x}\right)$$

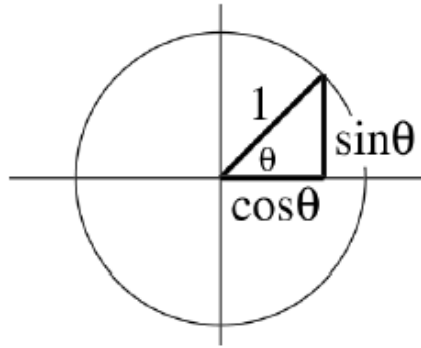
$$x = r \cos \theta$$
$$y = r \sin \theta$$



# Euler's Formula (Important!!)

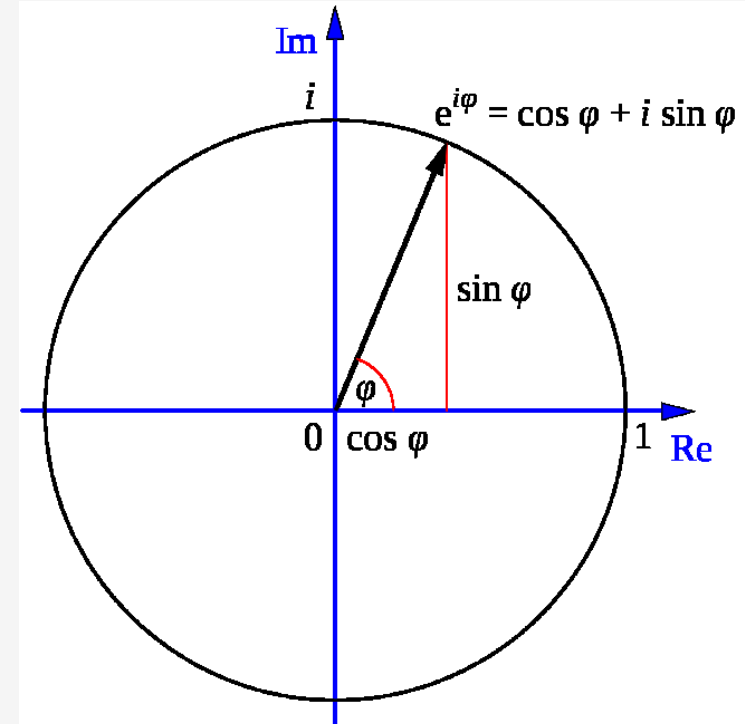
- **Complex Exponential**

- Real part is cosine
- Imaginary part is sine
- Magnitude is one

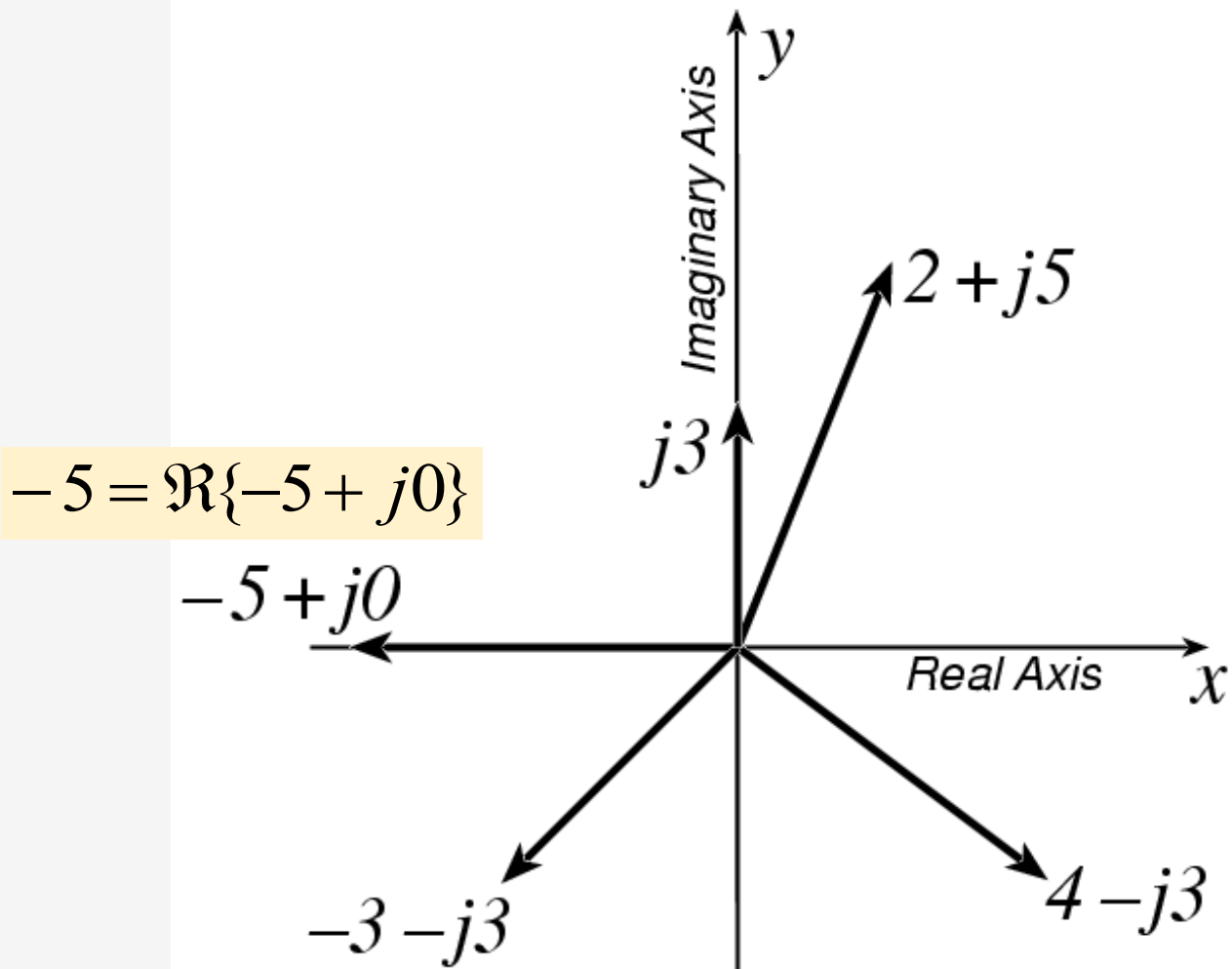


$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

$$re^{j\theta} = r \cos(\theta) + jr \sin(\theta)$$



# Remember: Complex Numbers



Real part:

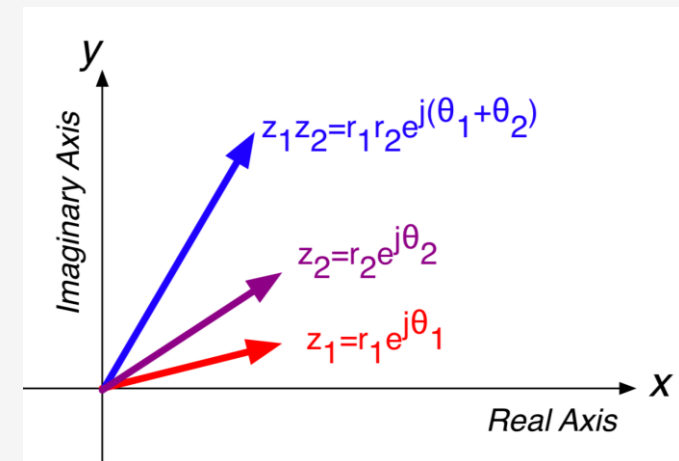
$$x = \Re\{z\}$$

Imaginary part:

$$y = \Im\{z\}$$

Complex addition?

Complex multiplication?



**Zdrill tool**

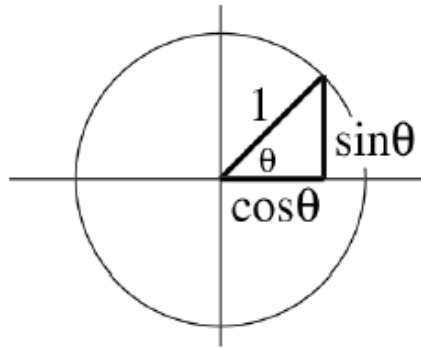
<https://dspfirst.gatech.edu/matlab/#zdrill>

# Euler's Formula (Important!!)



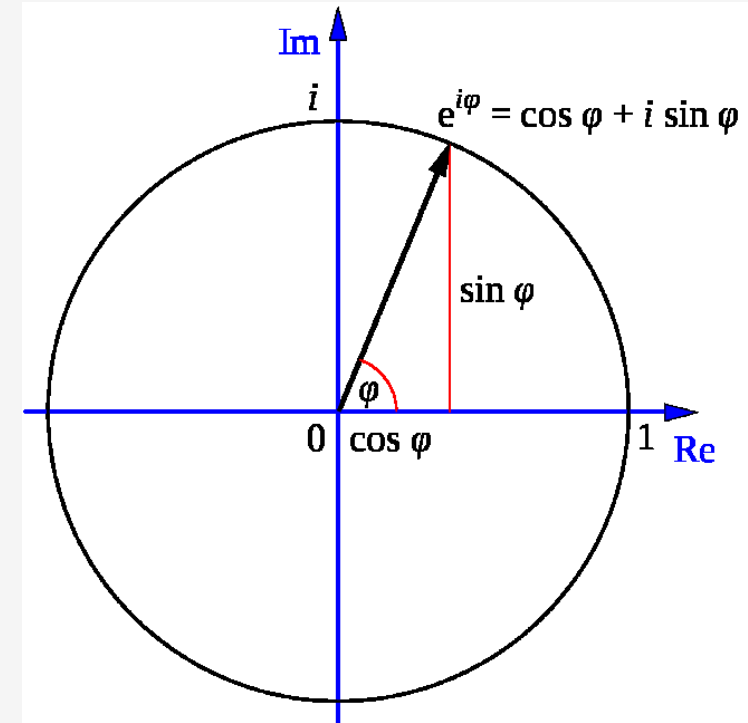
- **Complex Exponential**

- Real part is cosine
- Imaginary part is sine
- Magnitude is one



$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

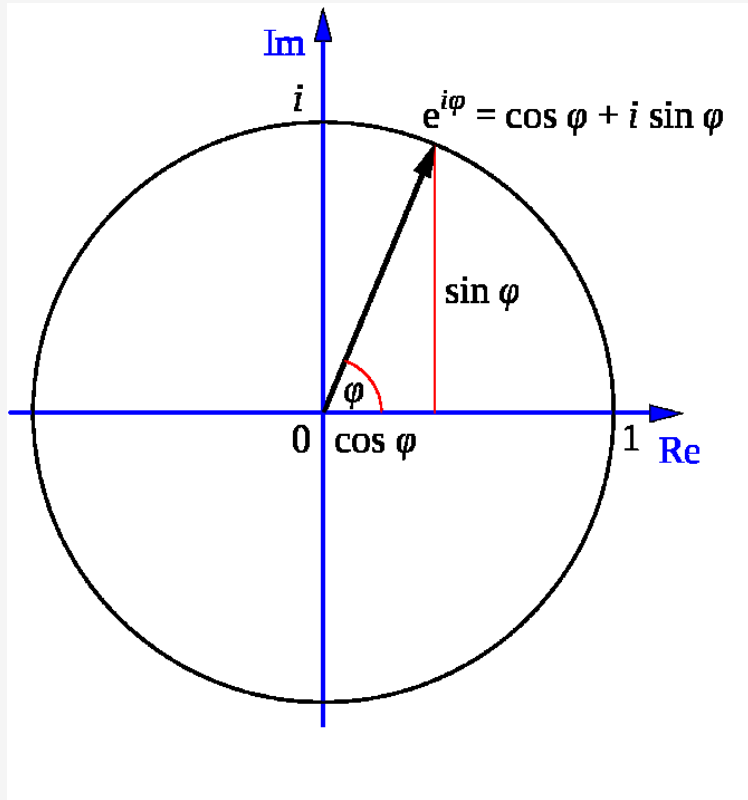
$$re^{j\theta} = r \cos(\theta) + jr \sin(\theta)$$



What happens if we write variable instead of Theta?

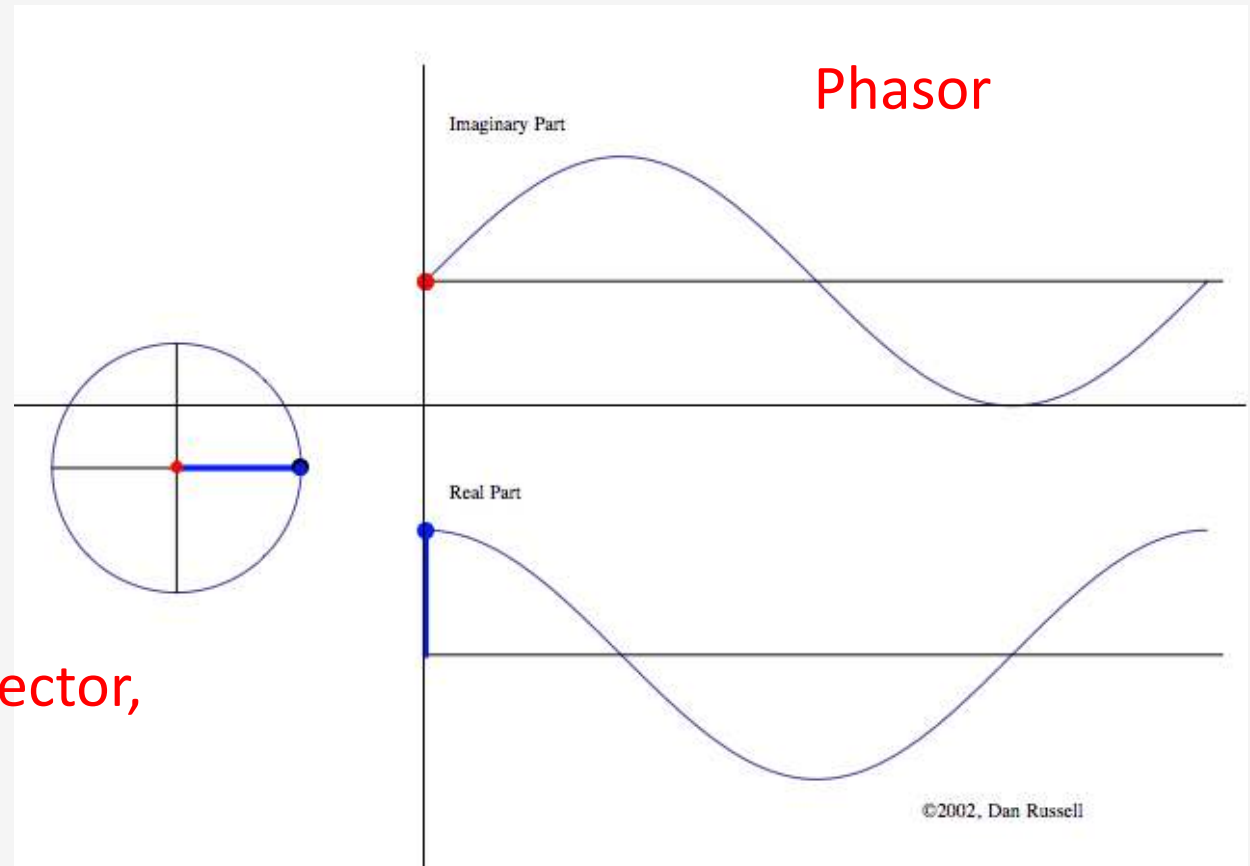
$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

# Euler's Formula (Important!!)



What happens if we write variable instead of Theta?

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$



Complex Exponential includes a rotating vector,  
= complex summation of sinuzoids

# Euler's Formula Reversed



- Solve for **cosine** (or sine)

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j \sin(-\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j \sin(\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2 \cos(\omega t)$$

$$\cos(\omega t) = \frac{1}{2} (e^{j\omega t} + e^{-j\omega t})$$



# INVERSE Euler's Formula



- Solve Euler's formula for **cosine** (or sine)

$$\cos(\omega t) = \frac{1}{2} (e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t})$$

# Phasor Form of A Cosine



$$A \cos(\omega t + \varphi) = \Re\{(Ae^{j\varphi})e^{j\omega t}\}$$

Complex Amplitude: Constant

Varies with time

- Find the COMPLEX AMPLITUDE for:

$$x(t) = \sqrt{3} \cos(77\pi t + 0.5\pi)$$

- Use EULER'S FORMULA:

$$\begin{aligned} x(t) &= \Re\{\sqrt{3}e^{j(77\pi t + 0.5\pi)}\} \\ &= \Re\{\sqrt{3}e^{j0.5\pi} e^{j77\pi t}\} \end{aligned}$$

$$X = \sqrt{3}e^{j0.5\pi}$$

- Determine the 60-Hz sinusoid whose COMPLEX AMPLITUDE is:

$$X = \sqrt{3} + j3$$

- Convert ***X*** to **POLAR**:

$$\begin{aligned} x(t) &= \Re\{(\sqrt{3} + j3)e^{j(120\pi t)}\} \\ &= \Re\{\sqrt{12}e^{j\pi/3}e^{j120\pi t}\} \end{aligned}$$

$$\Rightarrow x(t) = \sqrt{12} \cos(120\pi t + \pi / 3)$$

# Want to Add Sinusoids with same frequency

Adding sinusoids of common frequency results in sinusoid with SAME frequency

$$x(t) = \sum_{k=1}^N A_k \cos(\omega_0 t + \varphi_k)$$
$$= A \cos(\omega_0 t + \varphi)$$

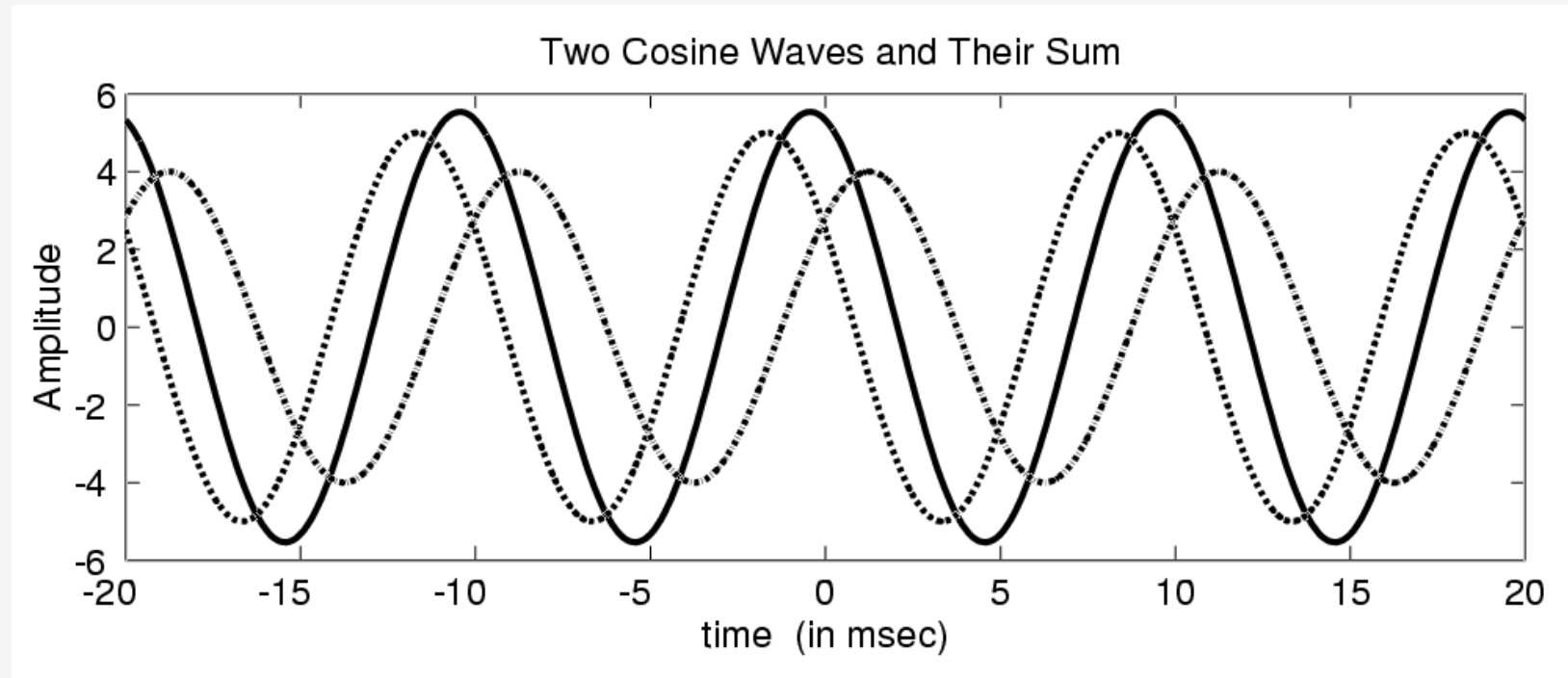
Get the new complex amplitude by complex addition

$$\sum_{k=1}^N A_k e^{j\varphi_k} = A e^{j\varphi}$$

# Want to Add Sinusoids with same frequency



Adding sinusoids of common frequency results in sinusoid with SAME frequency



# Want to Add Sinusoids with same frequency



- ADD THESE 2 SINUSOIDS:

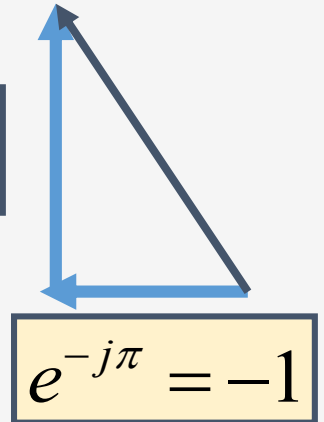
$$x_1(t) = \cos(77\pi t - \pi)$$

$$x_2(t) = \sqrt{3} \cos(77\pi t + 0.5\pi)$$

- COMPLEX (PHASOR) ADDITION:

$$1e^{-j\pi} + \sqrt{3}e^{j0.5\pi}$$

$$\sqrt{3}e^{j\pi/2} = j\sqrt{3}$$



$$-1 + j\sqrt{3} = 2e^{j2\pi/3}$$

$$x_3(t) = 2 \cos(77\pi t + \frac{2\pi}{3})$$

# Phasor Addition

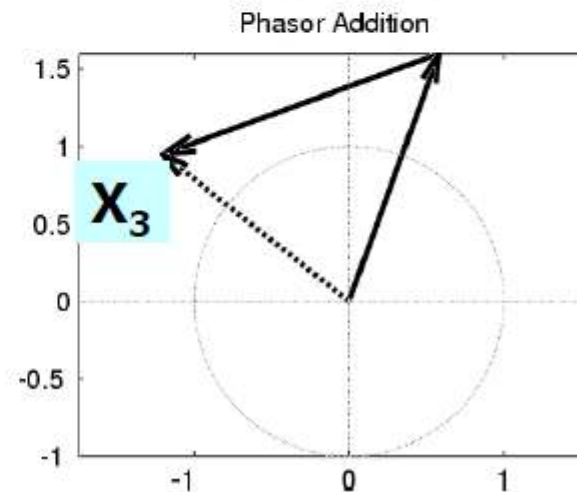
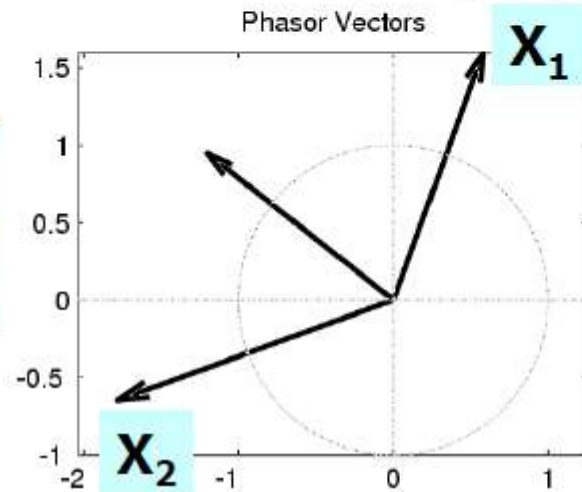


$$x_1(t) = 1.7 \cos(2\pi(10)t + 70\pi/180)$$

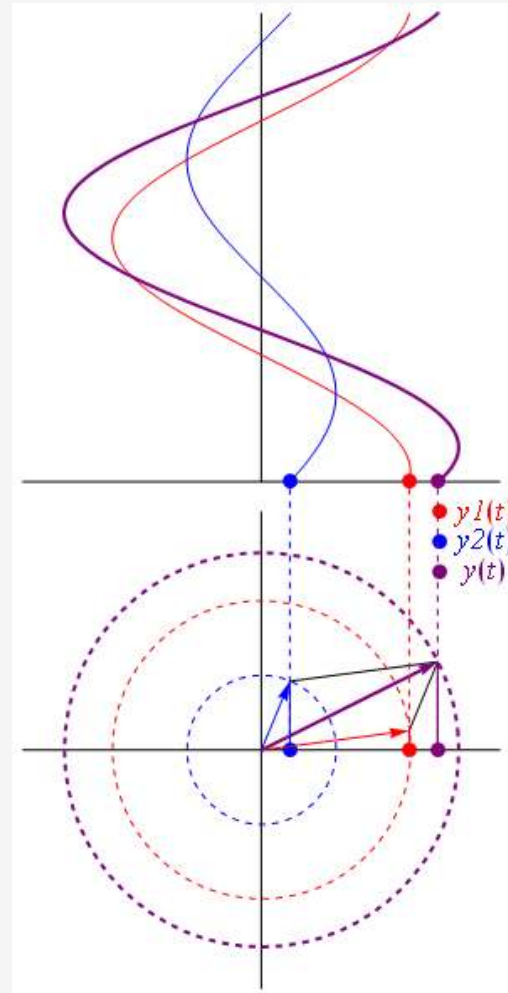
$$x_2(t) = 1.9 \cos(2\pi(10)t + 200\pi/180)$$

$$\begin{aligned} x_3(t) &= x_1(t) + x_2(t) \\ &= 1.532 \cos(2\pi(10)t + 141.79\pi/180) \end{aligned}$$

VECTOR  
(PHASOR)  
ADD



# Sum of Phasors and Fourier Series





# Plotting A Complex Exponential in MATLAB



```
% Plot signal
tt = 0: 1/10000 : 3.2;
xx = 2.1*exp(2*pi*10*tt*1j);
xx2 = 0.5*exp(2*pi*10*tt*1j);

figure(1); plot (tt,real(xx)); xlim([0 0.01]);
figure(2); plot (tt,imag(xx)); xlim([0 0.01]);

% Simulate Phasor
close all;
figure(1);

for i = 1:length(tt)

    x = real(xx(i));    y = imag(xx(i));

    plot([0 x],[0 y]);
    xlim([-4 4]);      ylim([-4 4]);    drawnow;

end
```

```
% Simulate sum of Phasor-2
close all;
figure(1);

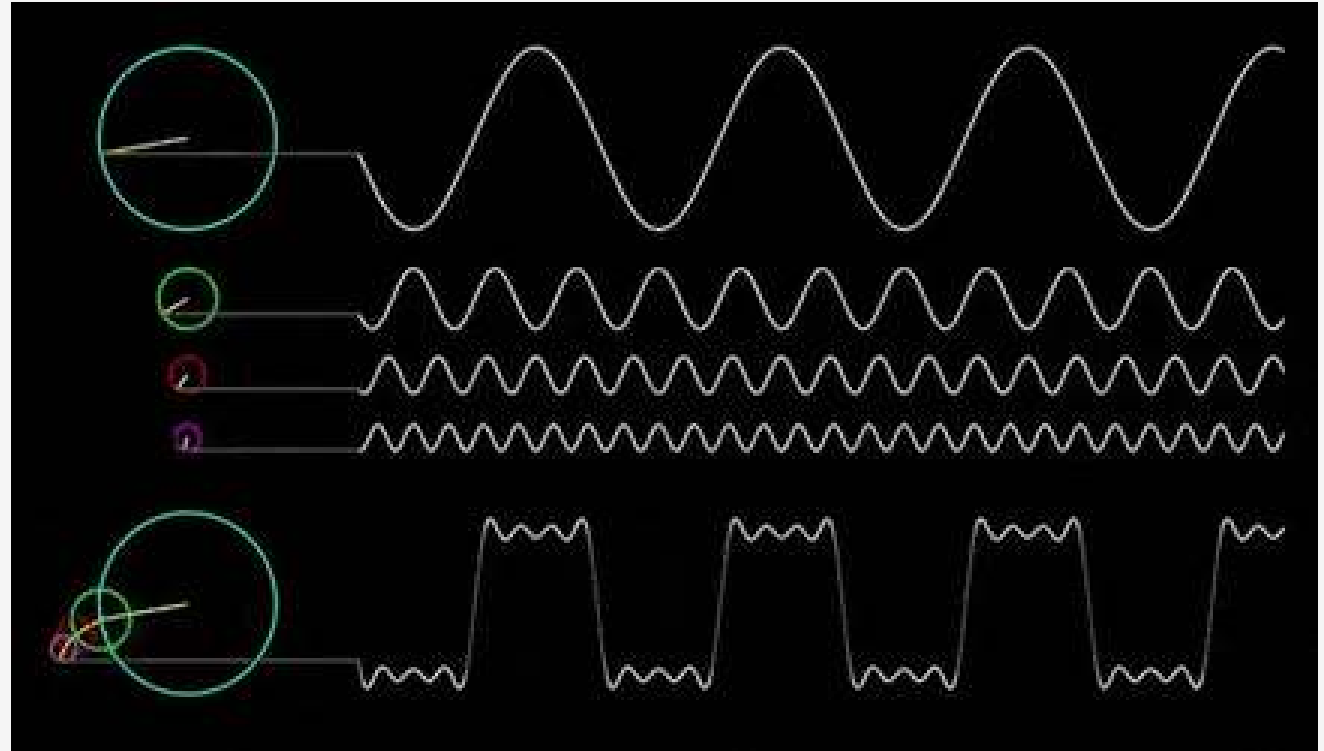
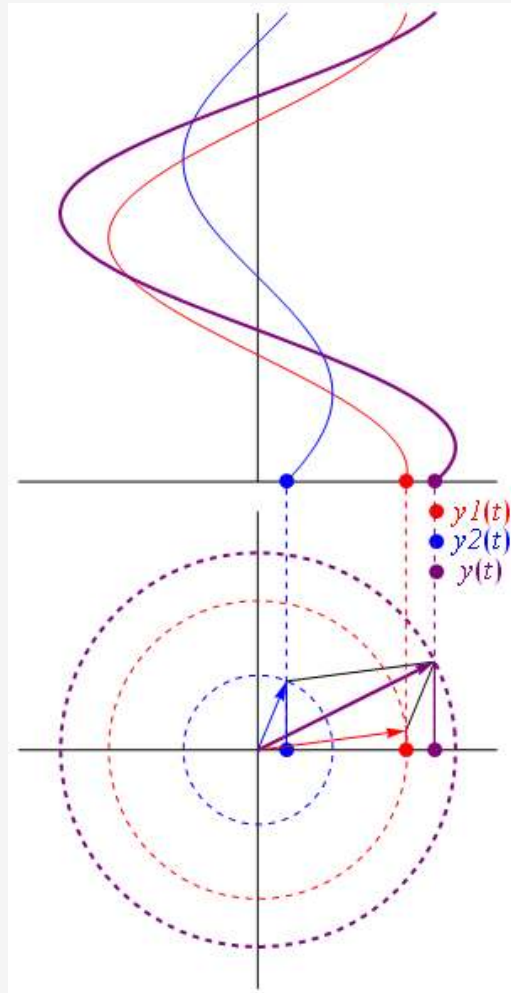
for i = 1:length(tt)

    x = real(xx(i));
    y = imag(xx(i));
    x2 = real(xx2(i));
    y2 = imag(xx2(i));

    plot([0 x],[0 y],'r'); hold on;
    plot([x x+x2],[y y+y2],'b');
    plot([0 x+x2],[0 y+y2],'k');
    xlim([-4 4]);      ylim([-4 4]);
    drawnow; hold off;

end
```

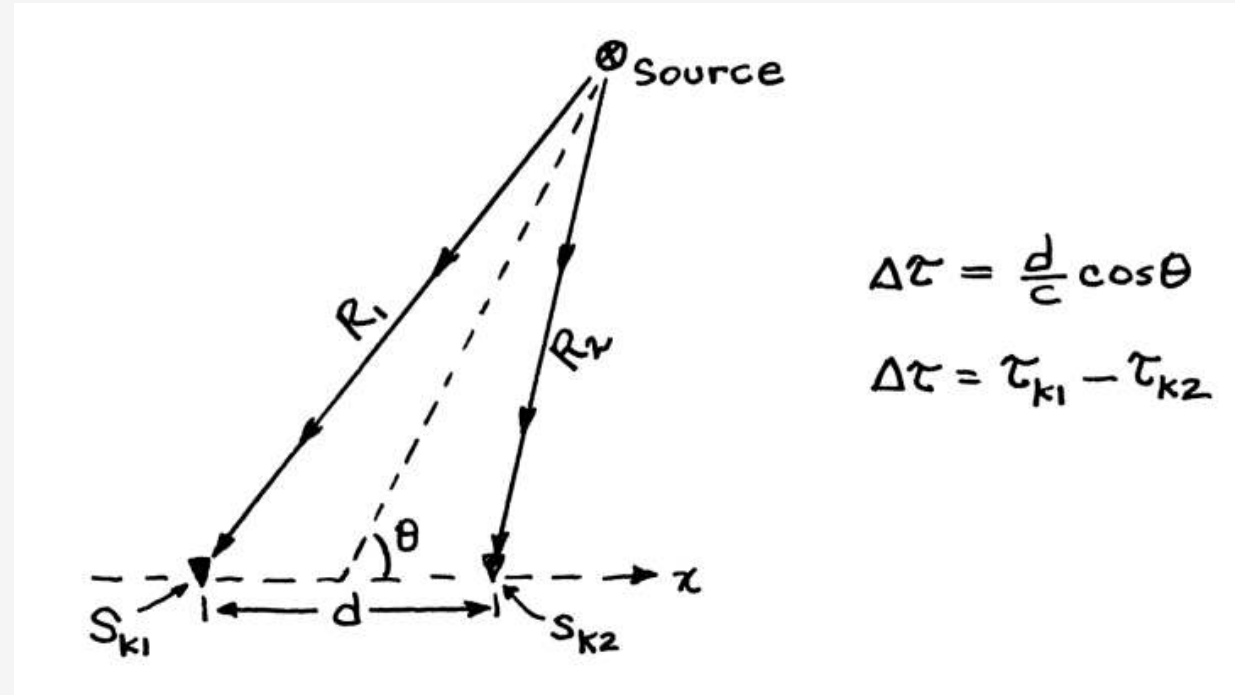
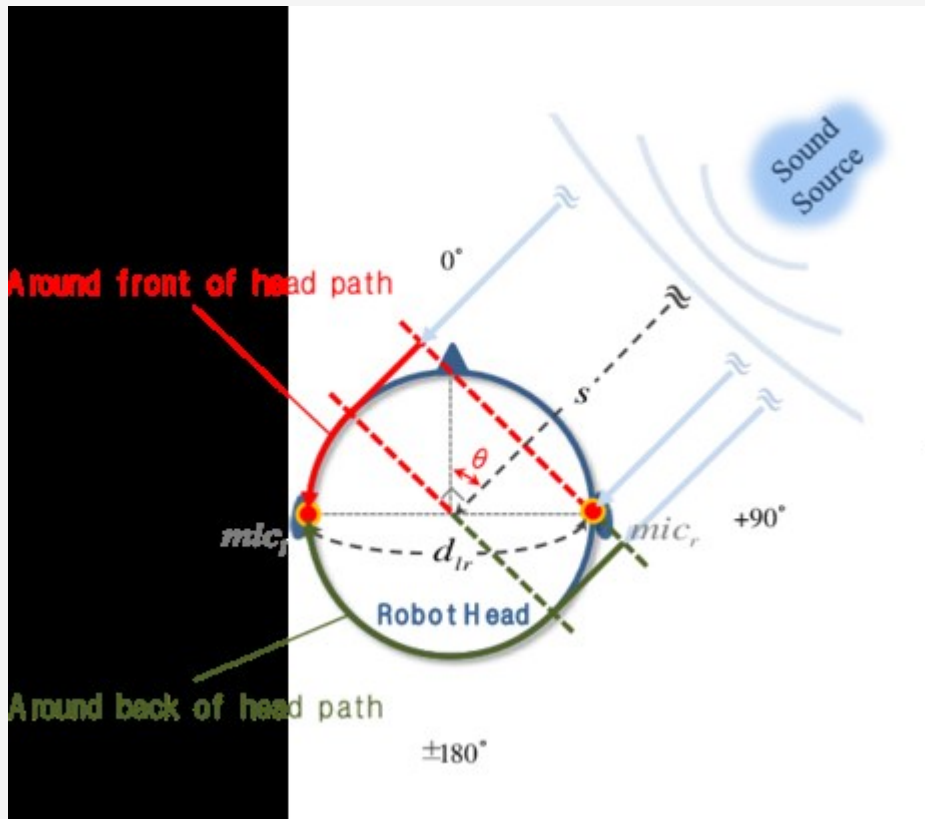
# Sum of Phasors and Fourier Series



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Demo Link: <https://dspfirst.gatech.edu/chapters/02sines/demos/phasors/index.html>

# Where Can We Use Phase Info: Binaural Sound Localization



Sensor  $S_{k1}$ :  $r_{k1}(t) = s(t - \tau_{k1})$

Sensor  $S_{k2}$ :  $r_{k2}(t) = s(t - \tau_{k2})$

# Exercise - 1

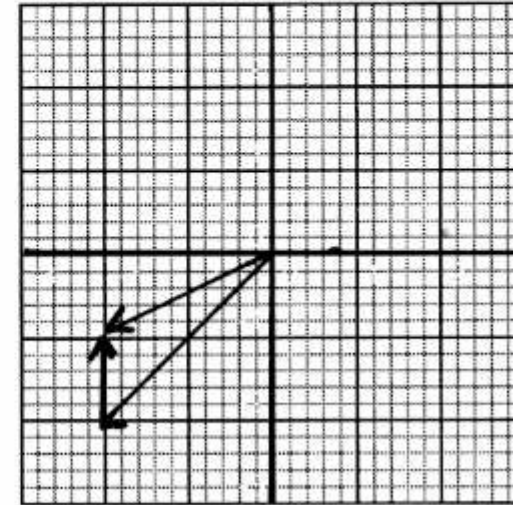
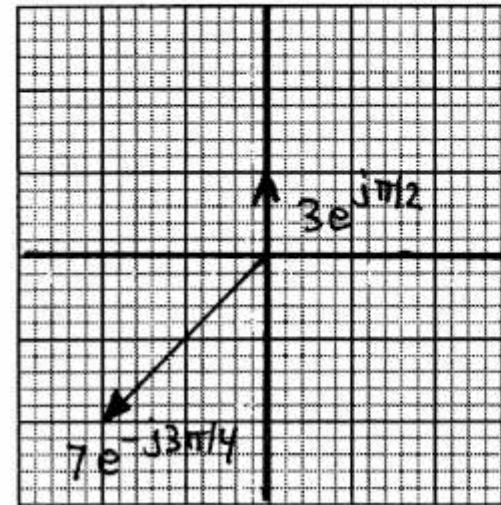
Define  $x(t)$  as

$$x(t) = 7 \cos(100\pi t - 3\pi/4) + 3 \cos(100\pi(t + 0.005))$$

- (a) Use phasor addition to express  $x(t)$  in the form  $x(t) = A \cos(\omega_0 t + \phi)$  by finding the numerical values of  $A$  and  $\phi$ , as well as  $\omega_0$ .

$$\begin{aligned} x(t) &= 7 \cos(100\pi t - 3\pi/4) + 3 \cos(100\pi t + \pi/2) \\ &= \operatorname{Re} \left\{ 7e^{-j3\pi/4} e^{j100\pi t} + 3e^{j\pi/2} e^{j100\pi t} \right\} \\ &= \operatorname{Re} \left\{ \underbrace{\left( 7e^{-j3\pi/4} + 3e^{j\pi/2} \right)}_{5.3199 e^{-j0.8806\pi}} e^{j100\pi t} \right\} \\ &= \operatorname{Re} \left\{ 5.3199 e^{-j0.8806\pi} \cdot e^{j100\pi t} \right\} \\ &= 5.3199 \cos(100\pi t - 0.8806\pi) \end{aligned}$$

- (b) Make two complex plane plots to illustrate how complex amplitudes (phasors) were used to solve part (a). On the first plot, show the two complex amplitudes being added; on the second plot, show your solution as a vector and the addition of the two complex amplitudes as vectors (head-to-tail).



## Exercise - 2

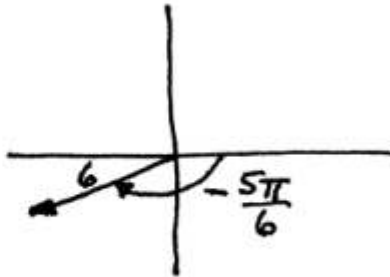


Simplify the following complex-valued expressions. In each case reduce the answers to a **simple** numerical form. Let

$$V = -3 + j3\sqrt{3}.$$

(a) Express  $jV$  in polar form. In addition plot  $jV$  as a vector.

$$\begin{aligned} jV &= -3j - 3\sqrt{3} \\ &= 6e^{-j\frac{5\pi}{6}} \end{aligned}$$



(d) Express  $\Re\{j^3 V e^{j15t}\}$  in the standard “cosine” form.

$$\begin{aligned} \Re\{j^3 V e^{j15t}\} &= \Re\{e^{-j\frac{\pi}{2}} \cdot 6e^{j\frac{2\pi}{3}} e^{j15t}\} = \Re\{6e^{j\frac{\pi}{6}} e^{j15t}\} \\ &= \boxed{6 \cos(15t + \frac{\pi}{6})} \end{aligned}$$

## Exercise - 3



The phase of a sinusoid can be related to time shift:  $x(t) = A \cos(2\pi f_0 t + \phi) = A \cos(2\pi f_0 (t - t_1))$

In the following parts, assume that the period of the sinusoidal wave is  $T = 20$  sec.

- (a) "When  $t_1 = 5$  sec, the value of the phase is  $\phi = 3\pi/2$ ."

Explain whether this is TRUE or FALSE.

$$\phi = -2\pi(t_1/T)$$

$$t_1 = 5 \Rightarrow \phi = -2\pi(5/20) = -\pi/2$$

BUT YOU CAN ADD  $2\pi$ , SO  $\phi = -\pi/2 + 2\pi = 3\pi/2$

TRUE

- (b) "When  $t_1 = -5$  sec, the value of the phase is  $\phi = \pi/4$ ."

Explain whether this is TRUE or FALSE.

$$\phi = -2\pi(-5/20) = +\pi/2$$

FALSE

$\pi/2 - \pi/4 = \pi/4$  IS NOT MULTIPLE of  $2\pi$



# Homework - 1



**P-2.10** Define  $x(t)$  as

$$x(t) = 2 \sin(\omega_0 t + \pi/4) + \cos(\omega_0 t)$$

- (a) Express  $x(t)$  in the form  $x(t) = A \cos(\omega_0 t + \phi)$ .
- (b) Find a complex-valued signal  $z(t)$  such that  $x(t) = \Re\{z(t)\}$ .

**P-2.7** Simplify the following expressions:

- (a)  $3e^{j\pi/3} + 4e^{-j\pi/6}$
- (b)  $(\sqrt{3} - j3)^{10}$
- (c)  $(\sqrt{3} - j3)^{-1}$
- (d)  $(\sqrt{3} - j3)^{1/3}$
- (e)  $\Re\{je^{-j\pi/3}\}$

Give the answers in *both* Cartesian form ( $x + jy$ ) and polar form ( $re^{j\theta}$ ).

**P-2.11** Define  $x(t)$  as

$$x(t) = 5 \cos(\omega t) + 5 \cos(\omega t + 120^\circ) + 5 \cos(\omega t - 120^\circ)$$

Simplify  $x(t)$  into the standard sinusoidal form:  $x(t) = A \cos(\omega t + \phi)$ . Use phasors to do the algebra, but also provide a plot of the vectors representing each of the three phasors.