NP-complete Languages

Polynomial Time Reductions

Polynomial Computable function f :

There is a deterministic Turing machine M such that for any string w computes $f(w)$ in polynomial time: $O(|w|^k)$

Observation:

the length of $f(w)$ is bounded

$$|f(w)| = O(|w|^k)$$

since, M cannot use more than $O(|w|^k)$ tape space in time $O(|w|^k)$

Definition:

Language A
is polynomial time reducible to
language B

if there is a polynomial computable function f such that:

$$w \in A \iff f(w) \in B$$

Theorem:

Suppose that A is polynomial reducible to B.
If $B \in P$ then $A \in P$.

Proof:

Let M be the machine that decides B in polynomial time

Machine M' to decide A in polynomial time:

On input string w:

1. Compute $f(w)$
2. Run M on input $f(w)$
3. If $f(w) \in B$ acccept w

Example of a polynomial-time reduction:

We will reduce the

3CNF-satisfiability problem

to the

CLIQUE problem

3CNF formula:

literal variable or its complement

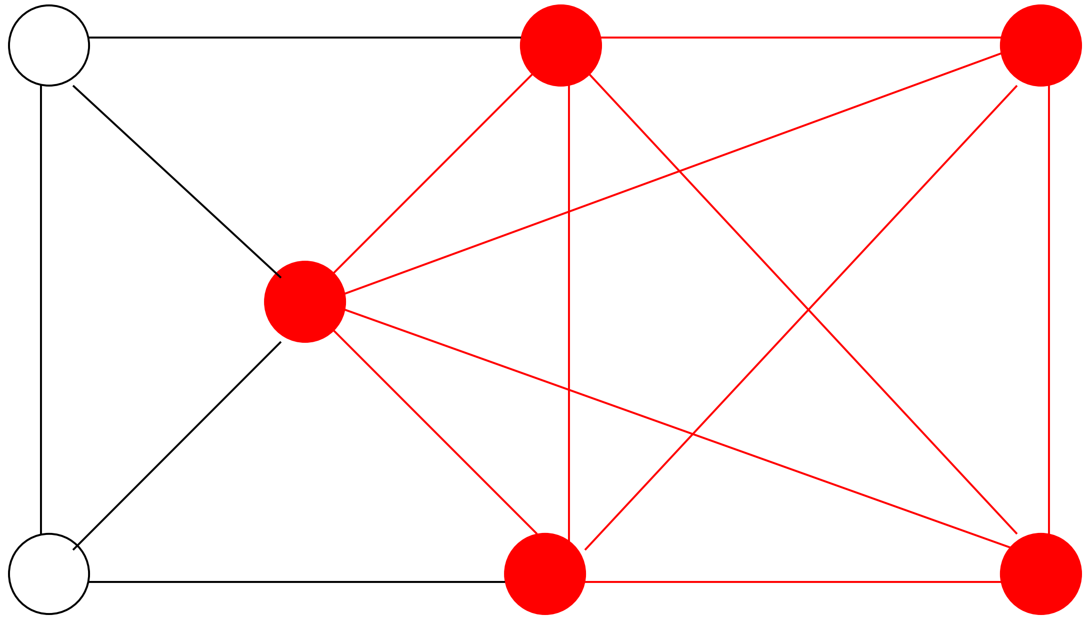
$$(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge \underbrace{(x_3 \vee \overline{x_5} \vee x_6)}_{\text{clause}} \wedge (x_3 \vee \overline{x_6} \vee x_4) \wedge (x_4 \vee x_5 \vee x_6)$$

Each clause has three literals

Language:

3CNF-SAT $= \{ w : w$ is a satisfiable 3CNF formula$\}$

A 5-clique in graph G



Language:

$$\text{CLIQUE} = \{<G, k>: \text{ graph } G \text{ contains a } k\text{-clique}\}$$

Theorem: 3CNF-SAT is polynomial time reducible to CLIQUE

Proof: give a polynomial time reduction of one problem to the other

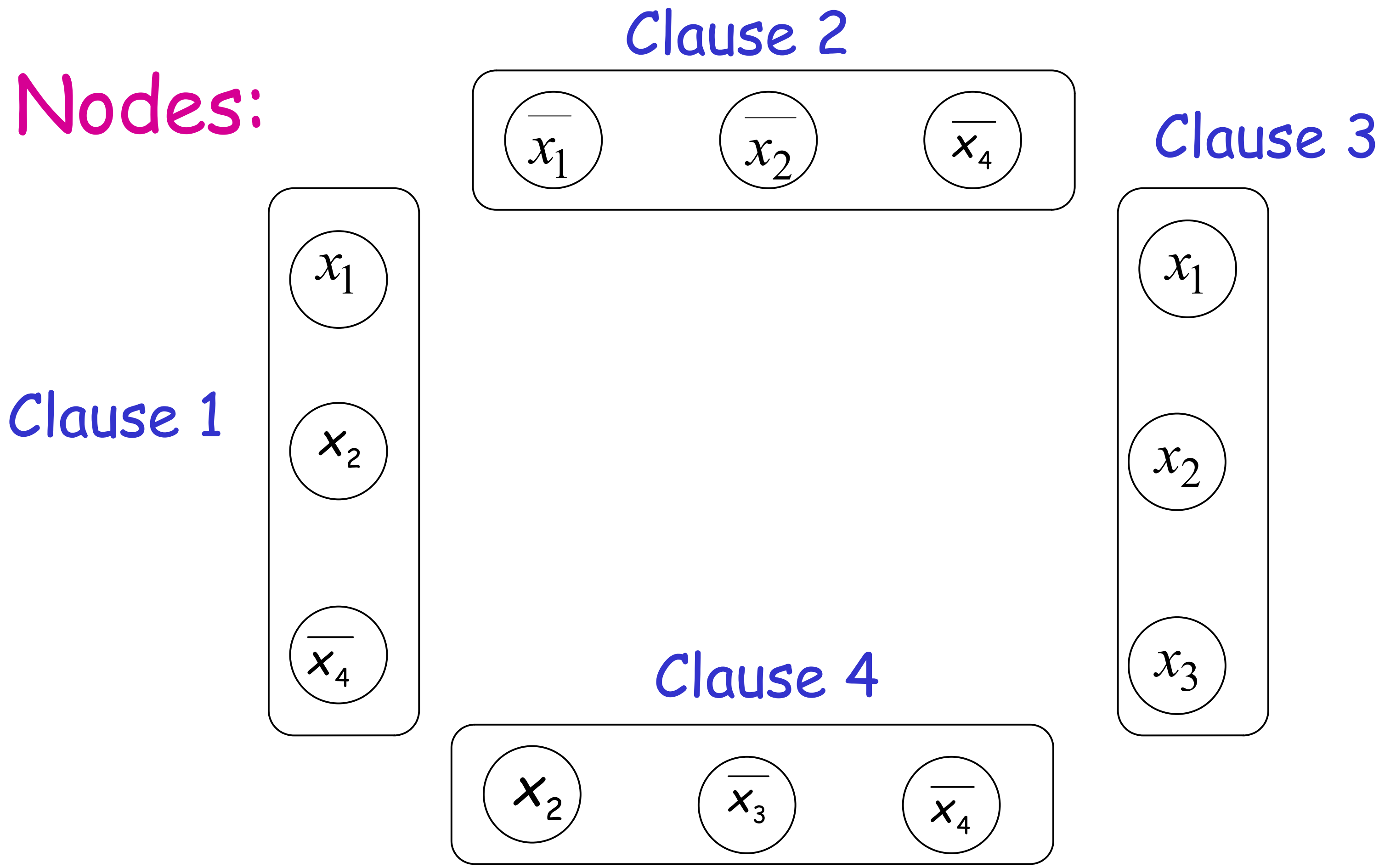
Transform formula to graph

Transform formula to graph.

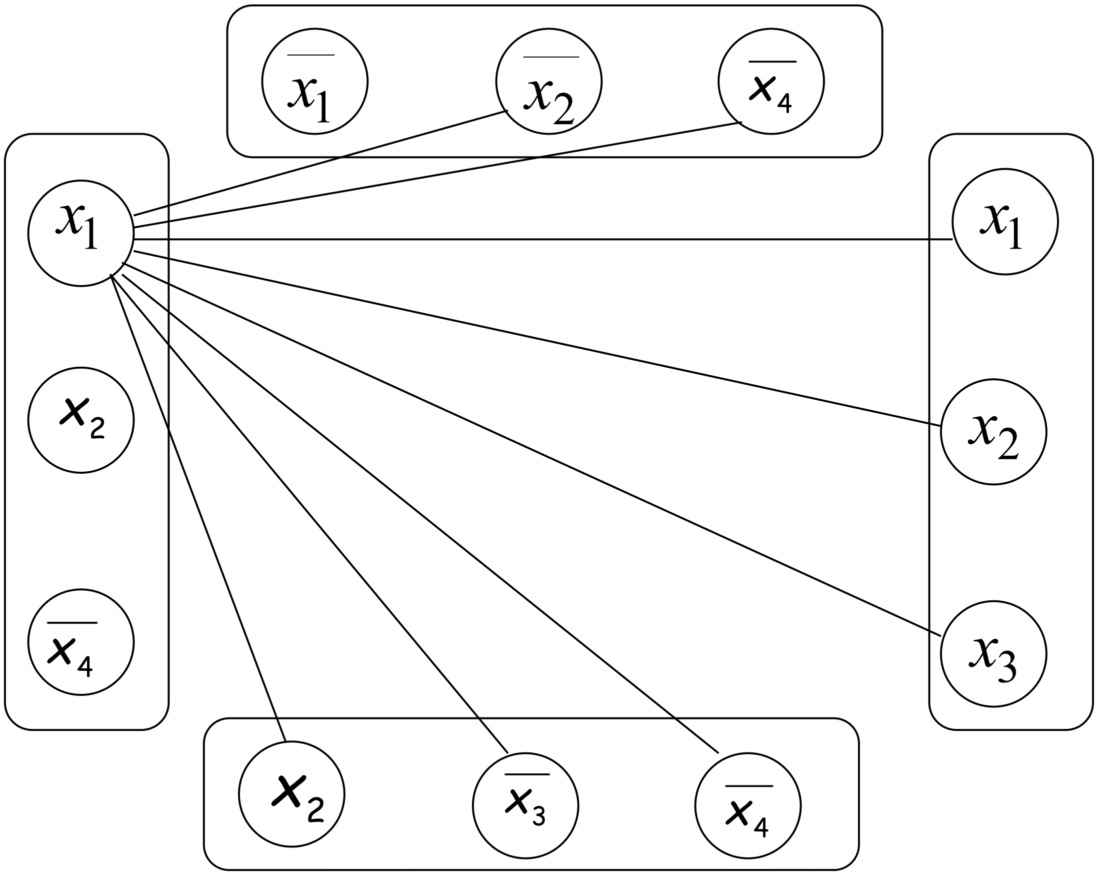
Example:

$$(x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$

Create Nodes:

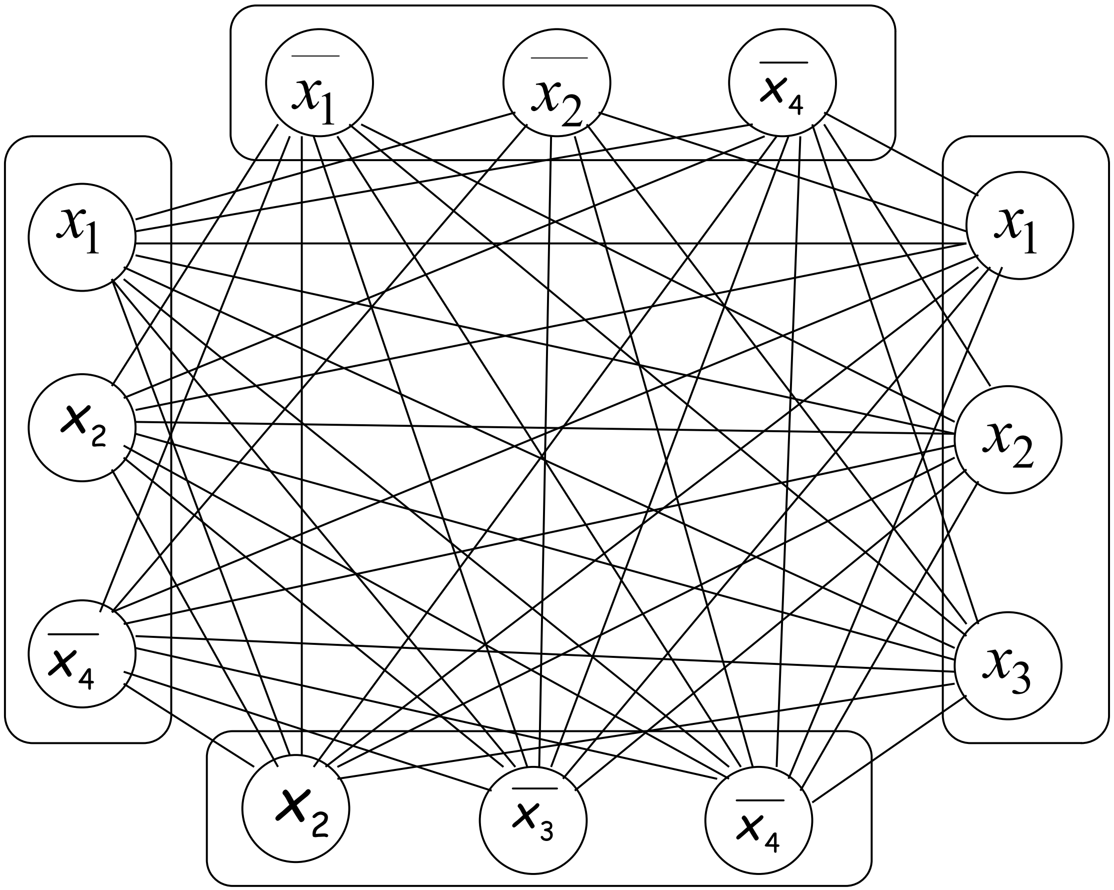


$$(x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$



Add link from a literal ξ to a literal in every other clause, except the complement $\overline{\xi}$

$$(x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$



Resulting Graph

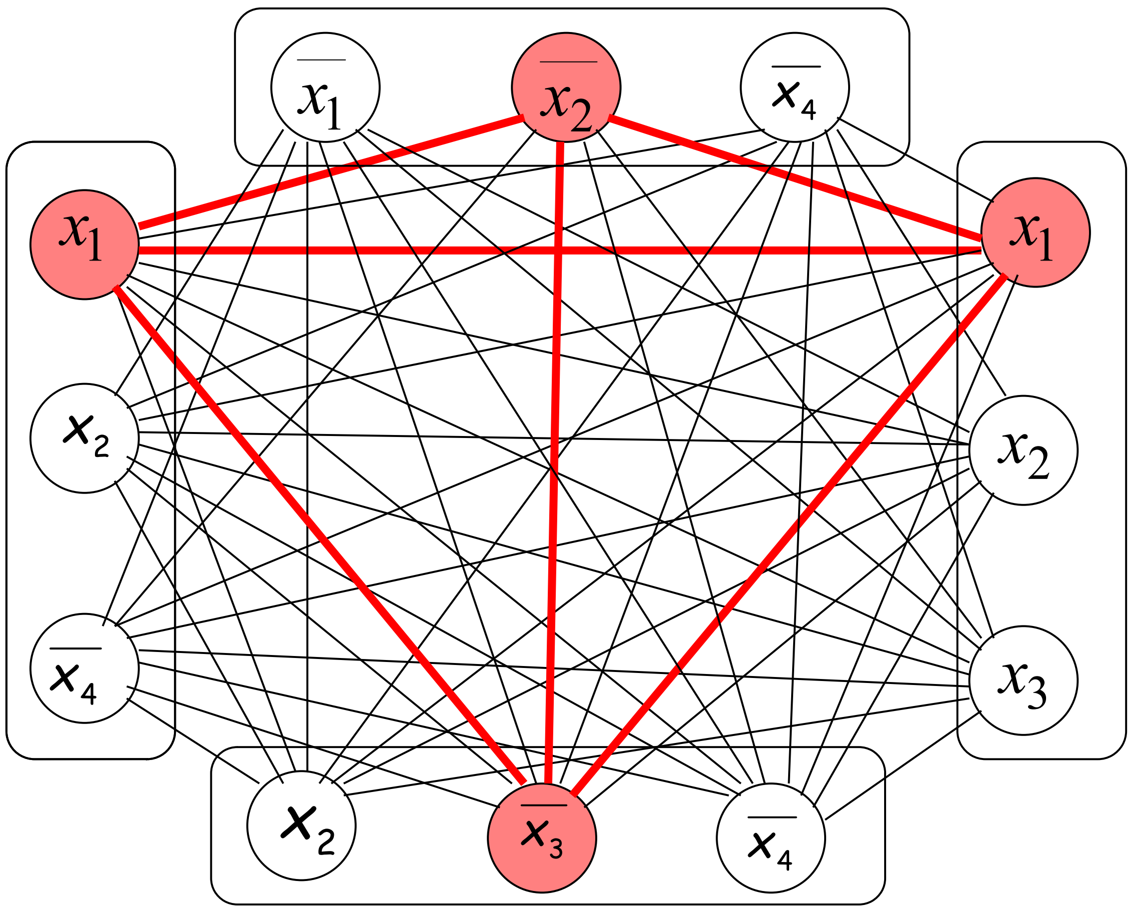
$$(x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4}) = 1$$

$x_1 = 1$

$x_2 = 0$

$x_3 = 0$

$x_4 = 1$

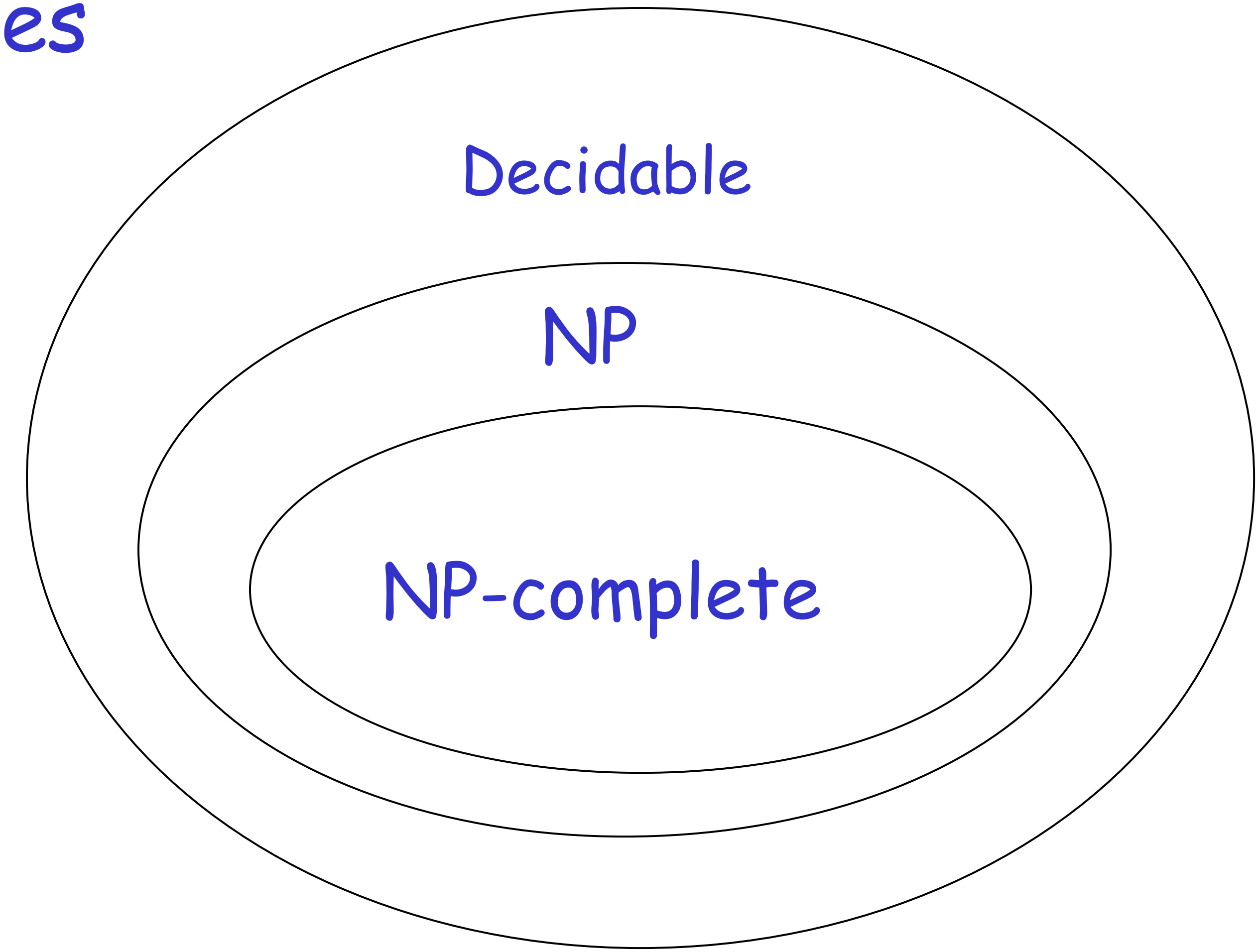


The formula is satisfied if and only if the Graph has a 4-clique

End of Proof

NP-complete Languages

We define the class of NP-complete languages



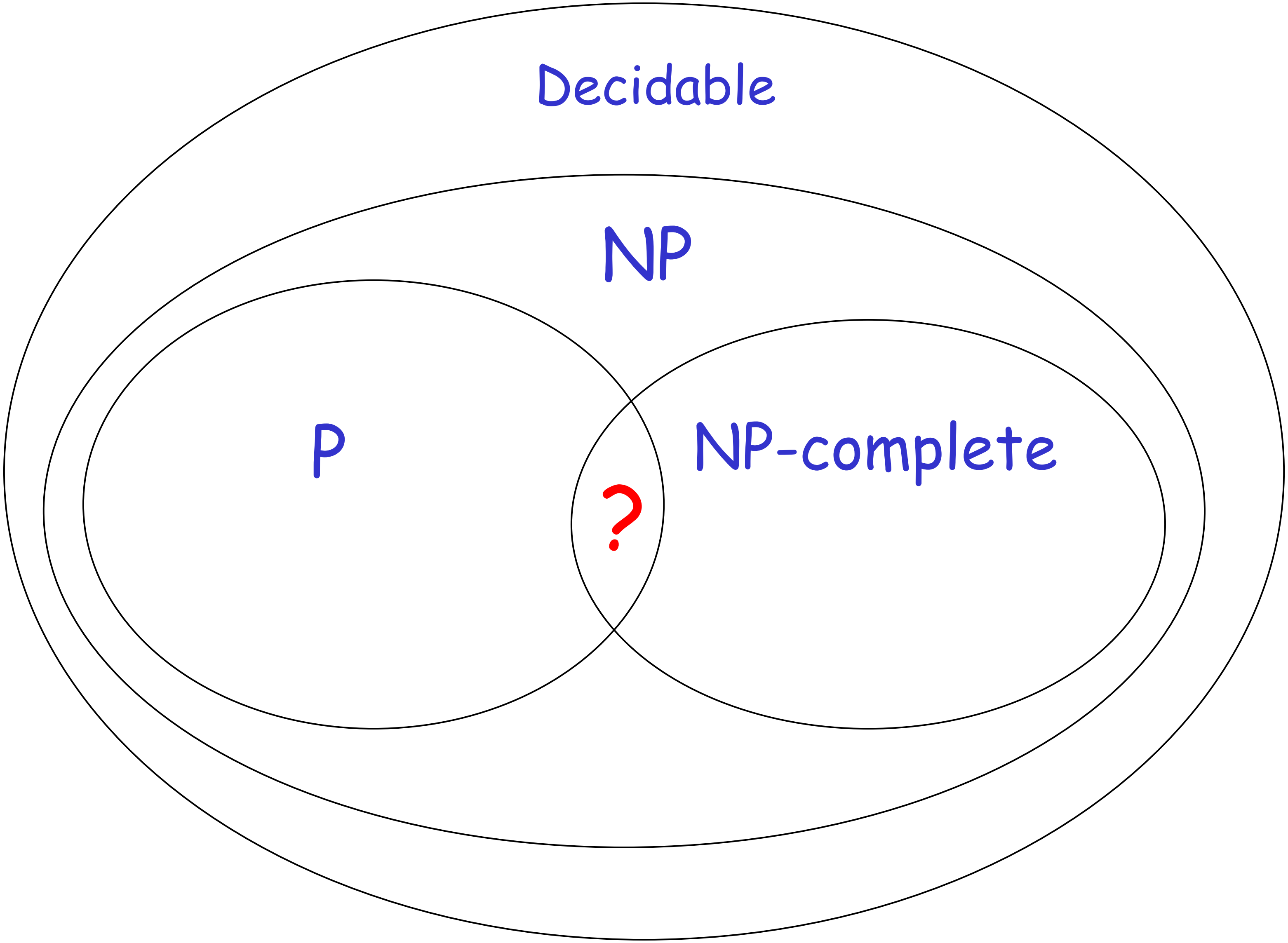
A language L is NP-complete if:

- L is in NP, and
- Every language in NP is reduced to L in polynomial time

Observation:

If a NP-complete language is proven to be in P then:

$$P = NP$$


Decidable
NP
P
NP-complete
?

An NP-complete Language

Cook-Levin Theorem:

Language SAT (satisfiability problem) is NP-complete

Proof:

Part1: SAT is in NP

(we have proven this in previous class)

Part2: reduce all NP languages to the SAT problem in polynomial time

Take an arbitrary language $L \in NP$

We will give a polynomial reduction of L to SAT

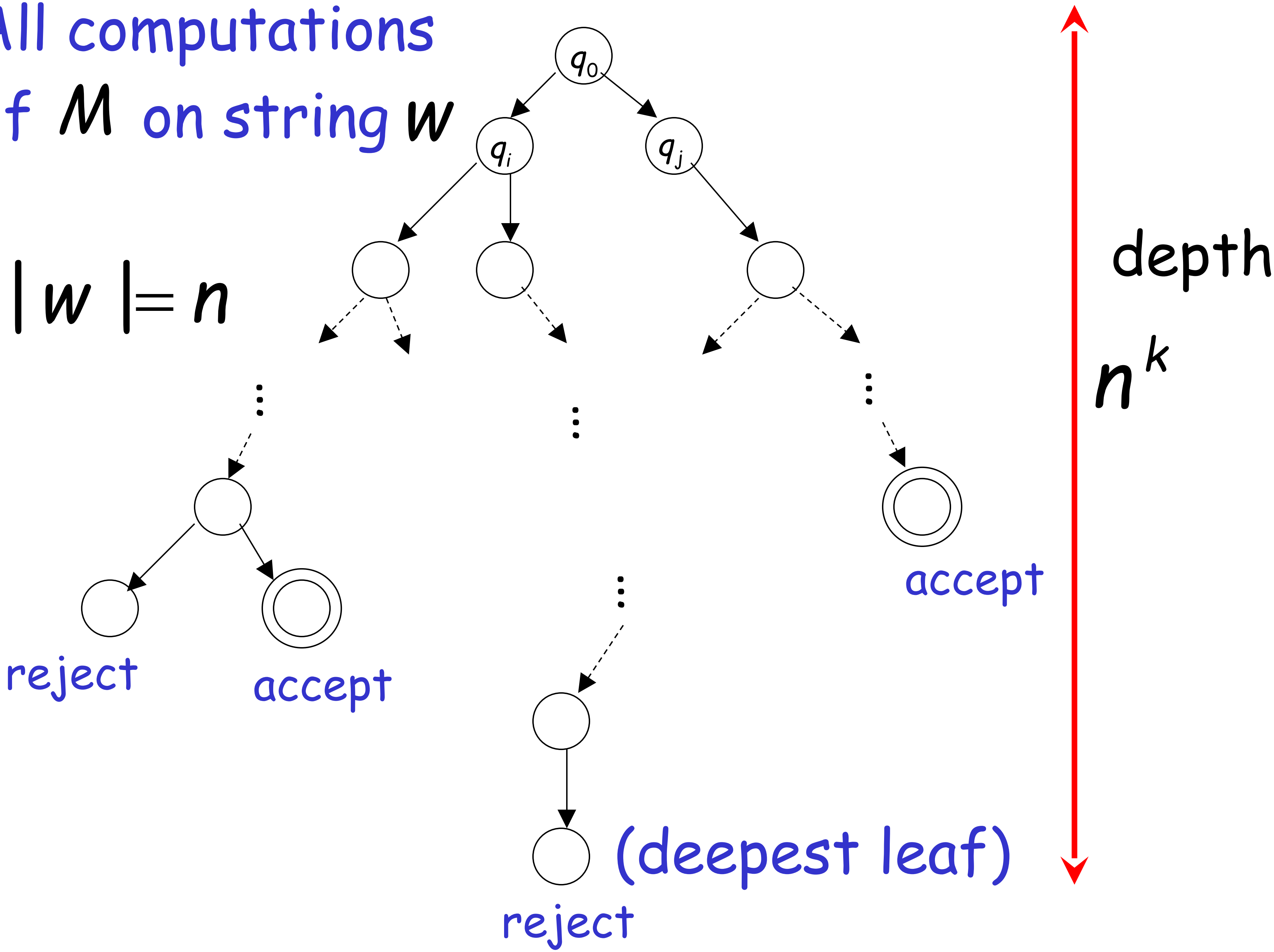
Let M be the NonDeterministic Turing Machine that decides L in polyn. time

For any string w we will construct in polynomial time a Boolean expression $\varphi(M,w)$

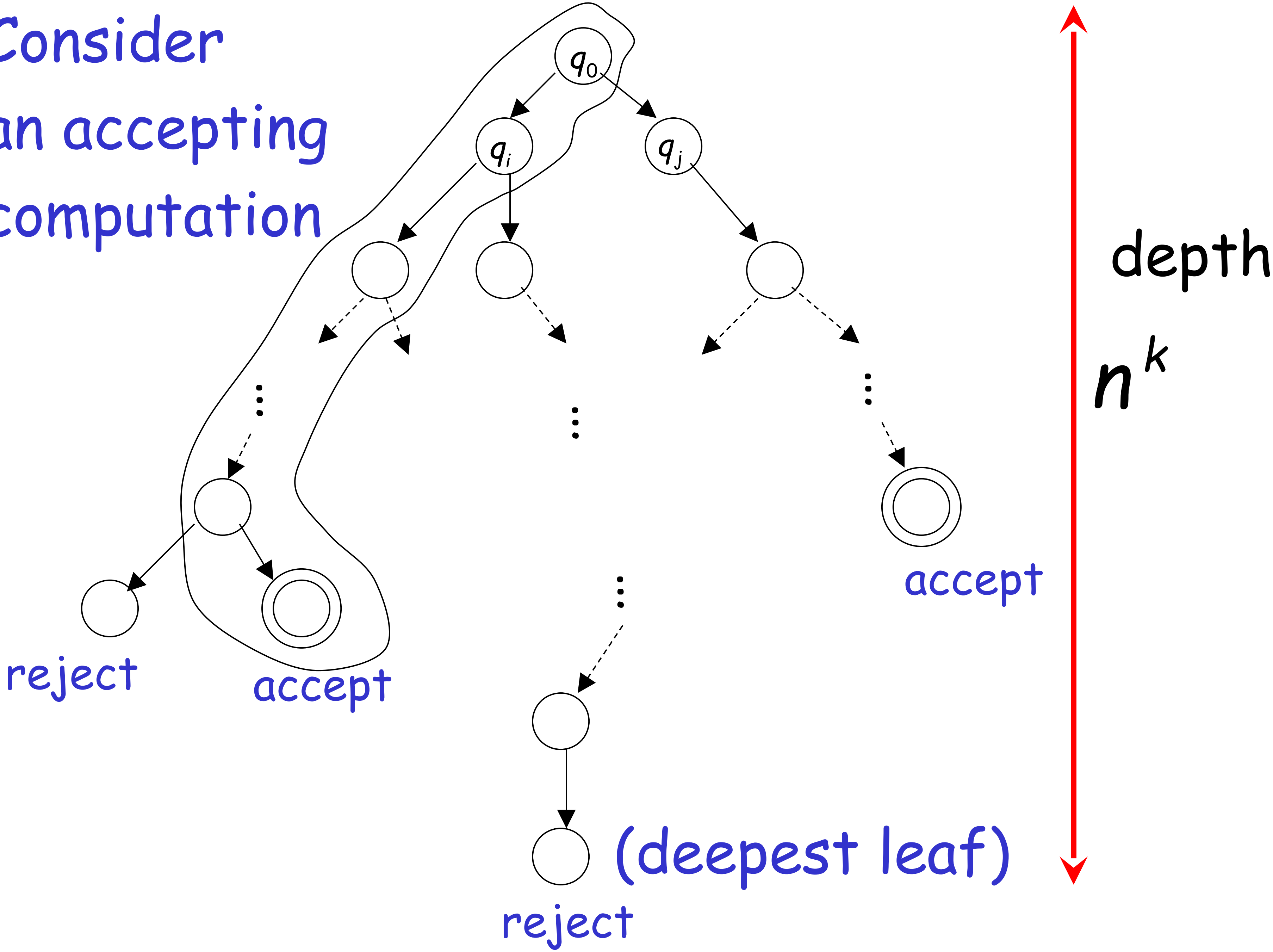
such that: $w \in L \iff \varphi(M,w)$ is satisfiable

All computations of M on string w

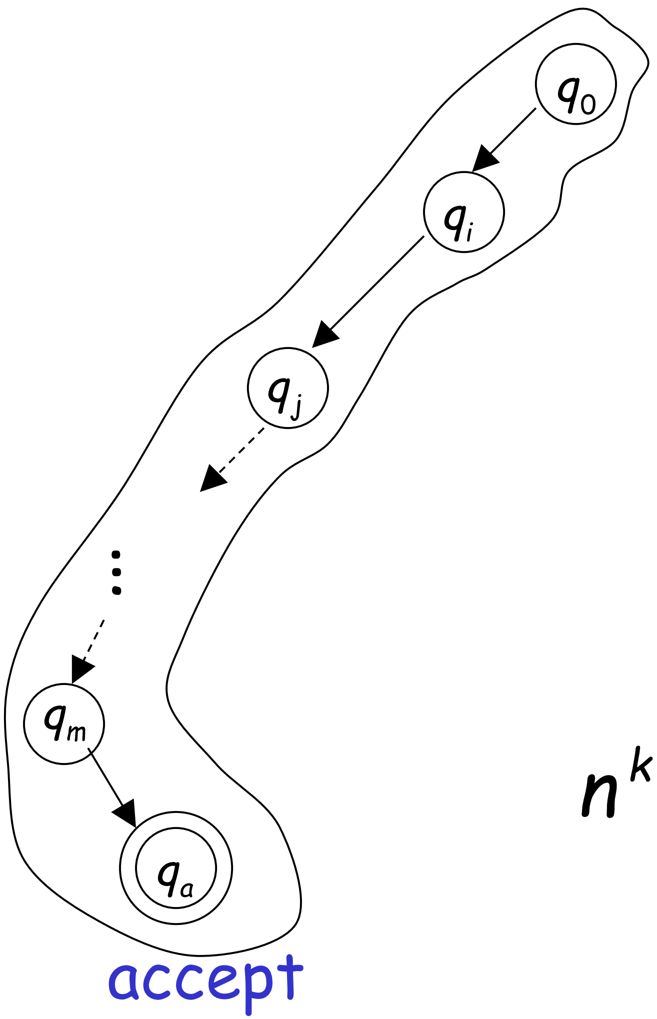
$|w| = n$



Consider an accepting computation



Computation path



Sequence of Configurations

initial state

$$1: \quad q_0\, \sigma_1 \sigma_2 \cdots \sigma_n$$

$$2: \quad \succ \sigma_1' \, q_i \;\; \sigma_2 \cdots \sigma_n$$

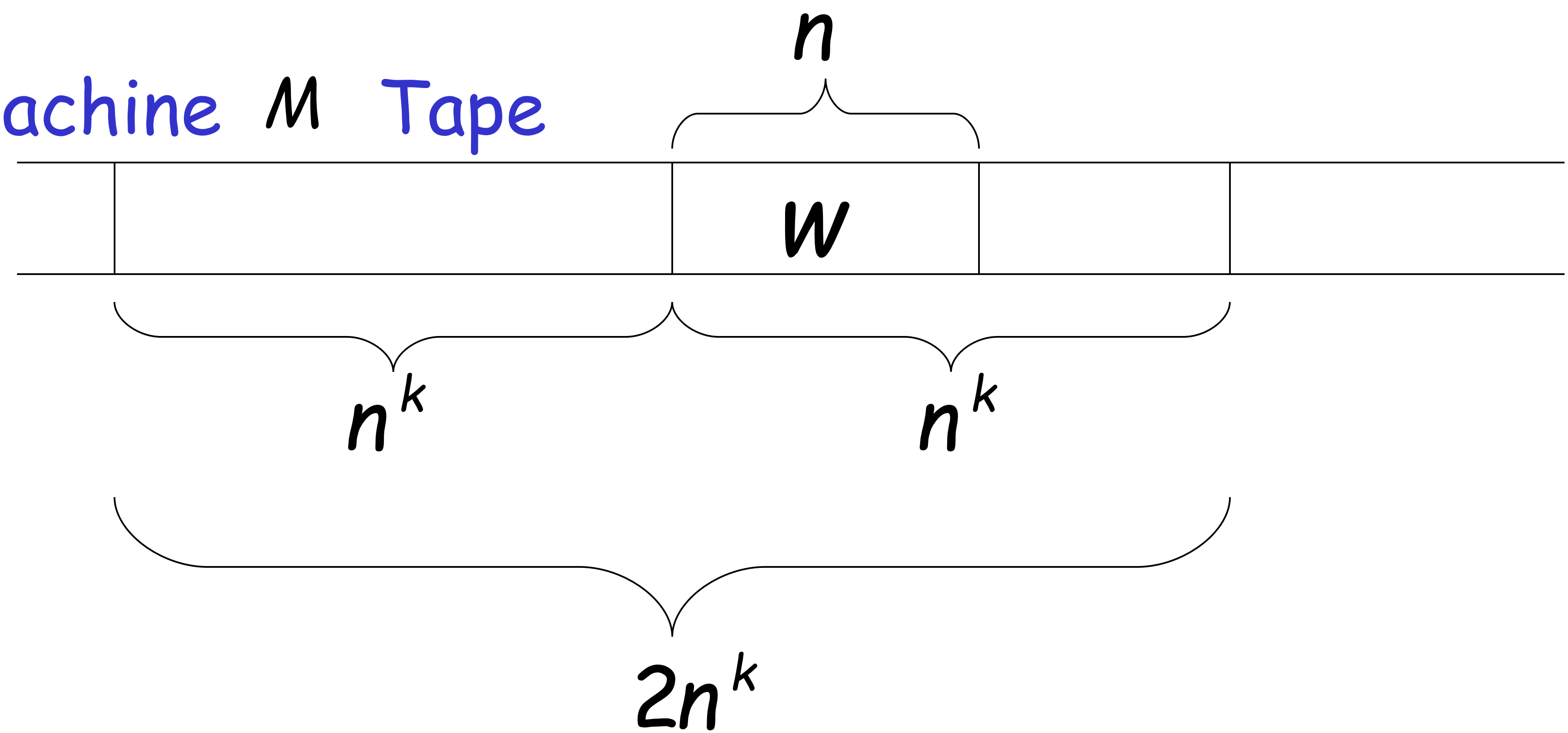
$$\vdots$$

$$n^k \geq x : \quad \succ \sigma_1' \cdots \sigma_l' \, q_a \, \sigma_{l+1}' \cdots \sigma_{n^k}'$$

accept state

$$w = \sigma_1 \sigma_2 \cdots \sigma_n$$

Machine M Tape



Maximum working space area on tape during n^k time steps

Tableau of Configurations

$1:$	#	$\Diamond$	$\cdots$	$\Diamond$	q_0	σ_1	σ_2	$\cdots$	σ_n	$\Diamond$	$\cdots$	$\Diamond$	#
$2:$	#	$\Diamond$	$\cdots$	$\Diamond$	σ_1'	q_i	σ_2	$\cdots$	σ_n	$\Diamond$	$\cdots$	$\Diamond$	#
$\vdots$													
$x:$	#	$\Diamond$	$\cdots$	σ''	σ_1'	σ_2'	σ_3'	$\cdots$	q_a	σ_{l+1}'	$\cdots$	σ_{n^k}	#
$n^k:$	#	$\Diamond$	$\cdots$	σ''	σ_1'	σ_2'	σ_3'	$\cdots$	q_a	σ_{l+1}'	$\cdots$	σ_{n^k}	#

Accept configuration

indentical rows

n^k $\quad$ n^k

$2n^k + 3$

Tableau Alphabet

$$C = \{\#\} \cup \{\text{tape alphabet}\} \cup \{\text{set of states}\}$$

$$= \{\#\} \cup \{\alpha_1, \ldots, \alpha_r\} \cup \{q_1, \ldots, q_t\}$$

Finite size (constant)

$$|C| = O(1)$$

For every cell position i, j

and

for every symbol in tableau alphabet $s \in C$

Define variable $x_{i,j,s}$

Such that if cell i, j contains symbol s

Then $x_{i,j,s} = 1$

Else $x_{i,j,s} = 0$

Examples:

$n^k + 3$

1:	#	$\Diamond$	$\cdots$	$\Diamond$	q_0	σ_1	σ_2	$\cdots$	σ_n	$\Diamond$	$\cdots$	$\Diamond$	#
2:	#	$\Diamond$	$\cdots$	$\Diamond$	σ_1'	q_i	σ_2	$\cdots$	σ_n	$\Diamond$	$\cdots$	$\Diamond$	#

$$x_{1,1,\#} = 1 \qquad x_{2,n^k+3,q_i} = 1$$

$$x_{1,1,\Diamond} = 0 \qquad x_{2,n^k+3,\#} = 0$$

$\varphi(M,w)$ is built from variables $x_{i,j,s}$

$$\varphi(M,w) = \varphi_{cell} \wedge \varphi_{start} \wedge \varphi_{accept} \wedge \varphi_{move}$$

When the formula is satisfied,
it describes an accepting computation
in the tableau
of machine M on input w

φ_{cell} makes sure that every cell in the tableau contains exactly one symbol

$$\varphi_{cell} = \bigwedge_{\text{all } i,j} \left[\left(\bigvee_{s \in C} x_{i,j,s} \right) \wedge \left(\bigwedge_{\substack{s,t \in C \\ s \neq t}} \left(\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}} \right) \right) \right]$$

Every cell contains at least one symbol

Every cell contains at most one symbol

Size of φ_{cell} :

$$\varphi_{cell} = \bigwedge_{\text{all } i,j} \left[\left(\bigvee_{s \in C} x_{i,j,s} \right) \wedge \left(\bigwedge_{\substack{s,t \in C \\ s \neq t}} \left(\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}} \right) \right) \right]$$

$$(2n^k + 3)n^k \times (\; |C| \quad + \quad |C|^2 \;)$$

$$= O(n^{2k})$$

φ_{start} makes sure that the tableau starts with the initial configuration

$$\varphi_{start} = x_{1,1,\#} \wedge x_{1,2,\Diamond} \wedge \cdots \wedge x_{1,n^k+1,\Diamond}$$
$$\wedge\, x_{1,n^k+2,q_0} \wedge x_{1,n^k+3,\sigma_1} \wedge \cdots \wedge x_{1,n^k+n+2,\sigma_n}$$
$$\wedge\, x_{1,n^k+n+3,\Diamond} \wedge x_{1,2n^k+2,\Diamond} \wedge \cdots \wedge x_{1,2n^k+2,\#}$$

Describes the initial configuration in row 1 of tableau

Size of φ_{start}:

$$\varphi_{start} = x_{1,1,\#} \wedge x_{1,2,\Diamond} \wedge \cdots$$
$$\wedge x_{1,n^k+1,\Diamond} \wedge x_{1,n^k+2,q_0} \wedge x_{1,n^k+3,\sigma_1} \wedge \cdots$$
$$\wedge x_{1,2n^k+2,\Diamond} \wedge x_{1,2n^k+3,\#}$$

$$2n^k + 3 = O(n^k)$$

φ_{accept} makes sure that the computation leads to acceptance

$$\varphi_{accept} = \bigvee_{\substack{\text{all } i,j \\ \text{all } q \in F}} x_{i,j,q}$$

Accepting states

An accept state should appear somewhere in the tableau

Size of φ_{accept} :

$$\varphi_{accept} = \bigvee_{\substack{\text{all } i,j \\ \text{all } q \in F}} x_{i,j,q}$$

$$(2n^k + 3)n^k = O(n^{2k})$$

φ_{move} makes sure that the tableau gives a valid sequence of configurations

φ_{move} is expressed in terms of legal windows

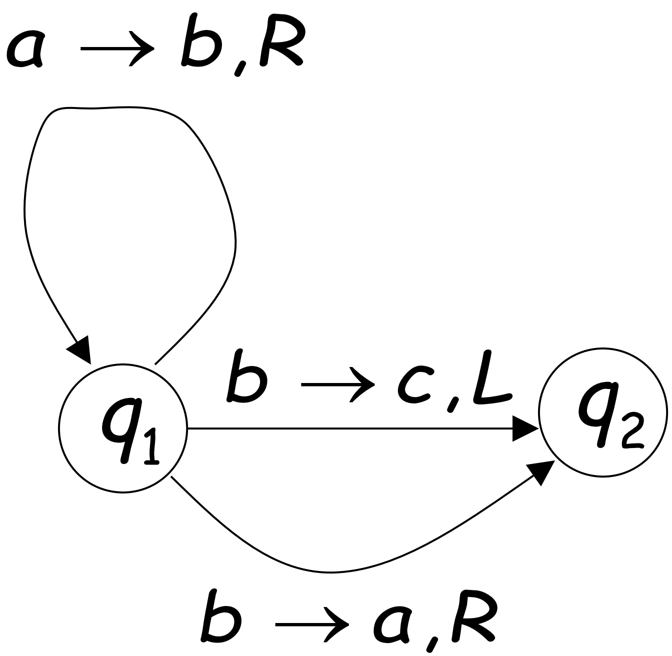
Tableau

Window

a	q_1	b
q_2	a	c

2x6 area of cells

Possible Legal windows



a	q_1	b
q_2	a	c

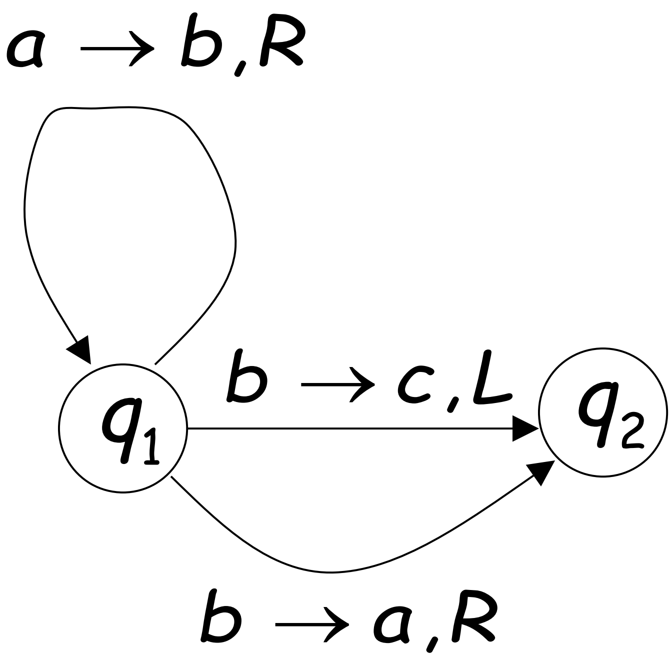
a	q_1	b
a	a	q_2

a	a	q_1
a	a	b

a	b	a
a	b	q_2

Legal windows obey the transitions

Possible illegal windows



a	b	a
a	a	a

a	q_1	b
q_1	a	a

b	q_1	b
q_2	b	q_2

$$\varphi_{move} = \bigwedge_{\text{all } i,j} (\text{window } (i,j) \text{ is legal})$$

window (i,j) is legal:

$$\left(\left(\begin{array}{c|ccc} & j & & \\ i & a & q_1 & b \\ & q_2 & a & c \end{array} \text{ is legal} \right) \vee \left(\begin{array}{c|ccc} & j & & \\ i & a & q_1 & b \\ & a & a & q_2 \end{array} \text{ is legal} \right) \vee \left(\begin{array}{c|ccc} & j & & \\ i & a & a & q_1 \\ & a & a & b \end{array} \text{ is legal} \right) \right) \cdots$$

all possible legal windows in position (i,j)

$$\begin{array}{c|c|c|c|}
 & j & & \\
\hline
i & a & q_1 & b \\
\hline
 & q_2 & a & c \\
\hline
\end{array}$$

(is legal)

Formula:

$$x_{i,j,a} \wedge x_{i,j+1,q_1} \wedge x_{i,j+2,b}$$

$$\wedge\, x_{i+1,j,q_2} \wedge x_{i+1,j+1,a} \wedge x_{i+1,j+2,c}$$

Size of φ_{move} :

Size of formula for a legal window
in a cell i,j: 6

Number of possible legal windows
in a cell i,j: at most $|C|^6$

Number of possible cells: $(2n^k+3)n^k$

$$\leq \; |C|^6 \cdot (2n^k+3)n^k = O(n^{2k})$$

Size of $\varphi(M,w)$:

$$\varphi(M,w) = \varphi_{cell} \wedge \varphi_{start} \wedge \varphi_{accept} \wedge \varphi_{move}$$

$$O(n^{2k}) + O(n^{k}) + O(n^{2k}) + O(n^{2k})$$

$$= O(n^{2k})$$

it can also be constructed in time $O(n^{2k})$

polynomial in n

$$\varphi(M,w) = \varphi_{\text{cell}} \wedge \varphi_{\text{start}} \wedge \varphi_{\text{accept}} \wedge \varphi_{\text{move}}$$

we have that:

$$w \in L \iff \varphi(M,w) \text{ is satisfiable}$$

Since,

$w \in L \iff \varphi(M,w)$ is satisfiable

and

$\varphi(M,w)$ is constructed in polynomial time

$\Downarrow$

L is polynomial-time reducible to SAT

END OF PROOF

Observation 1:

The $\varphi(M,w)$ formula can be converted to CNF (conjunctive normal form) formula in polynomial time

$$\varphi(M,w) = \varphi_{cell} \wedge \varphi_{start} \wedge \varphi_{accept} \wedge \varphi_{move}$$

Already CNF

NOT CNF

But can be converted to CNF using distributive laws

Distributive Laws:

$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$$

Observation 2:

The $\varphi(M,w)$ formula can also be converted to a 3CNF formula in polynomial time

$$(a_1 \vee a_2 \vee \cdots \vee a_l)$$

convert

$$(a_1 \vee a_2 \vee z_1) \wedge (\overline{z_1} \vee a_3 \vee z_2) \wedge (\overline{z_2} \vee a_4 \vee z_3) \wedge \cdots \wedge (\overline{z_{l-3}} \vee a_{l-1} \vee z_l)$$

From Observations 1 and 2:

CNF-SAT and
3CNF-SAT are
NP-complete languages

(they are known NP languages)

Theorem:

If: a. Language A is NP-complete

b. Language B is in NP

c. A is polynomial time reducible to B

Then: B is NP-complete

Proof:

Any language L in NP

is polynomial time reducible to A.

Thus, L is polynomial time reducible to B

(sum of two polynomial reductions,

gives a polynomial reduction)

Corollary: CLIQUE is NP-complete

Proof:

a. 3CNF-SAT is NP-complete

b. CLIQUE is in NP (shown in last class)

c. 3CNF-SAT is polynomial reducible to CLIQUE (shown earlier)

Apply previous theorem with

A=3CNF-SAT and B=CLIQUE