



BLM3620 Digital Signal Processing

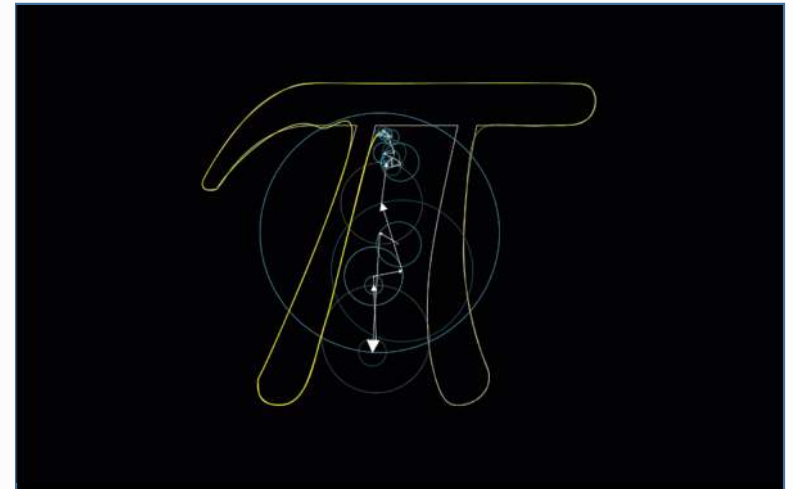
Dr. Ali Can KARACA

ackaraca@yildiz.edu.tr

Yıldız Technical University – Computer Engineering

Lecture #3 – Spectrum Representation (for continuous-time signals)

- Spectrum of a Sum of Sinusoids
- Fourier Series – Analysis and Synthesis
- Example: Amplitude Modulation
- Spectrogram
- MATLAB Applications



Important Materials:

- James H. McClellan, R. W. Schafer, M. A. Yoder, *DSP First Second Edition*, Pearson, 2015.
- Lizhe Tan, Jean Jiang, *Digital Signal Processing: Fundamentals and Applications*, Third Edition, Academic Press, 2019.

Auxiliary Materials:

- Prof. Sarp Ertürk, *Sayısal İşaret İşleme*, Birsen Yayınevi.
- Prof. Nizamettin Aydın, DSP Lecture Notes.
- J. G. Proakis, D. K. Manolakis, *Digital Signal Processing Fourth Edition*, Pearson, 2014.
- J. K. Perin, *Digital Signal Processing, Lecture Notes*, Stanford University, 2018.

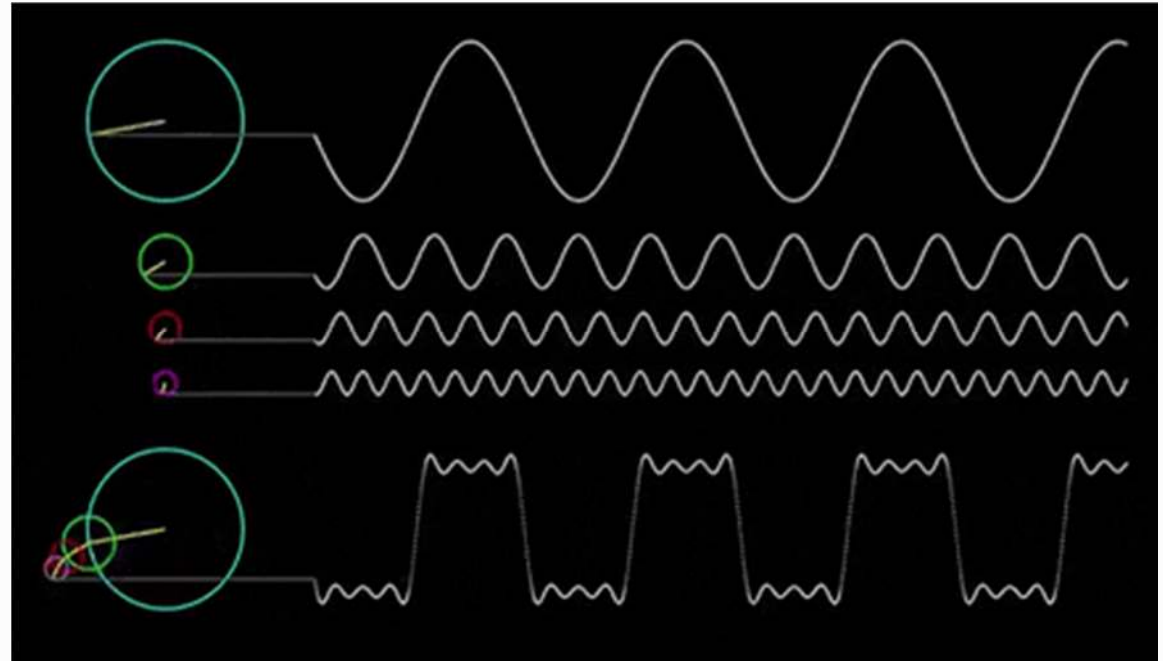
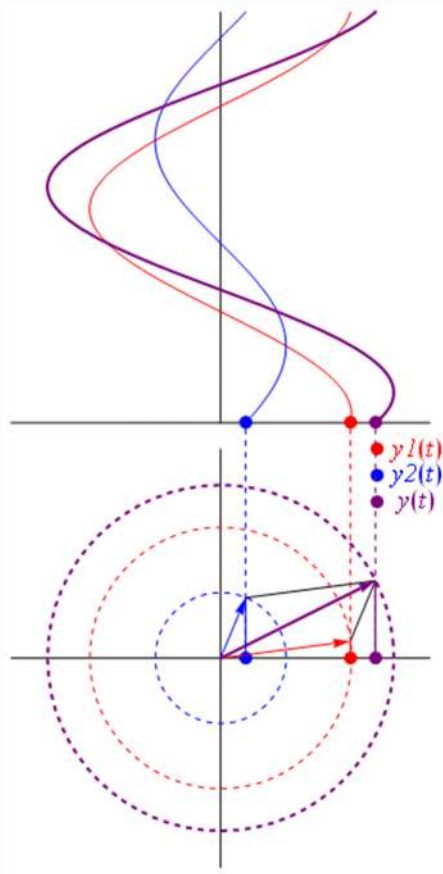
Syllabus



Week	Lectures
1	Introduction to DSP and MATLAB
2	Sinuzoids and Complex Exponentials
3	Spectrum Representation
4	Sampling and Aliasing
5	Discrete Time Signal Properties and Convolution
6	Convolution and FIR Filters
7	Frequency Response of FIR Filters
8	Midterm Exam
9	Discrete Time Fourier Transform and Properties
10	Discrete Fourier Transform and Properties
11	Fast Fourier Transform and Windowing
12	z- Transforms
13	FIR Filter Design and Applications
14	IIR Filter Design and Applications
15	Final Exam

For more details -> Bologna page: <http://www.bologna.yildiz.edu.tr/index.php?r=course/view&id=5730&aid=3>

Recall: Sum of Phasors and Fourier Series



$$x(t) = \sum_{k=-M}^M a_k e^{j2\pi f_k t}$$

Demo Link: <https://dspfirst.gatech.edu/chapters/02sines/demos/phasors/index.html>

Fourier Series

- Sinusoids with **DIFFERENT** Frequencies
 - SYNTHESIZE by Adding Sinusoids

Harmonic freqs : $f_k = k f_0$

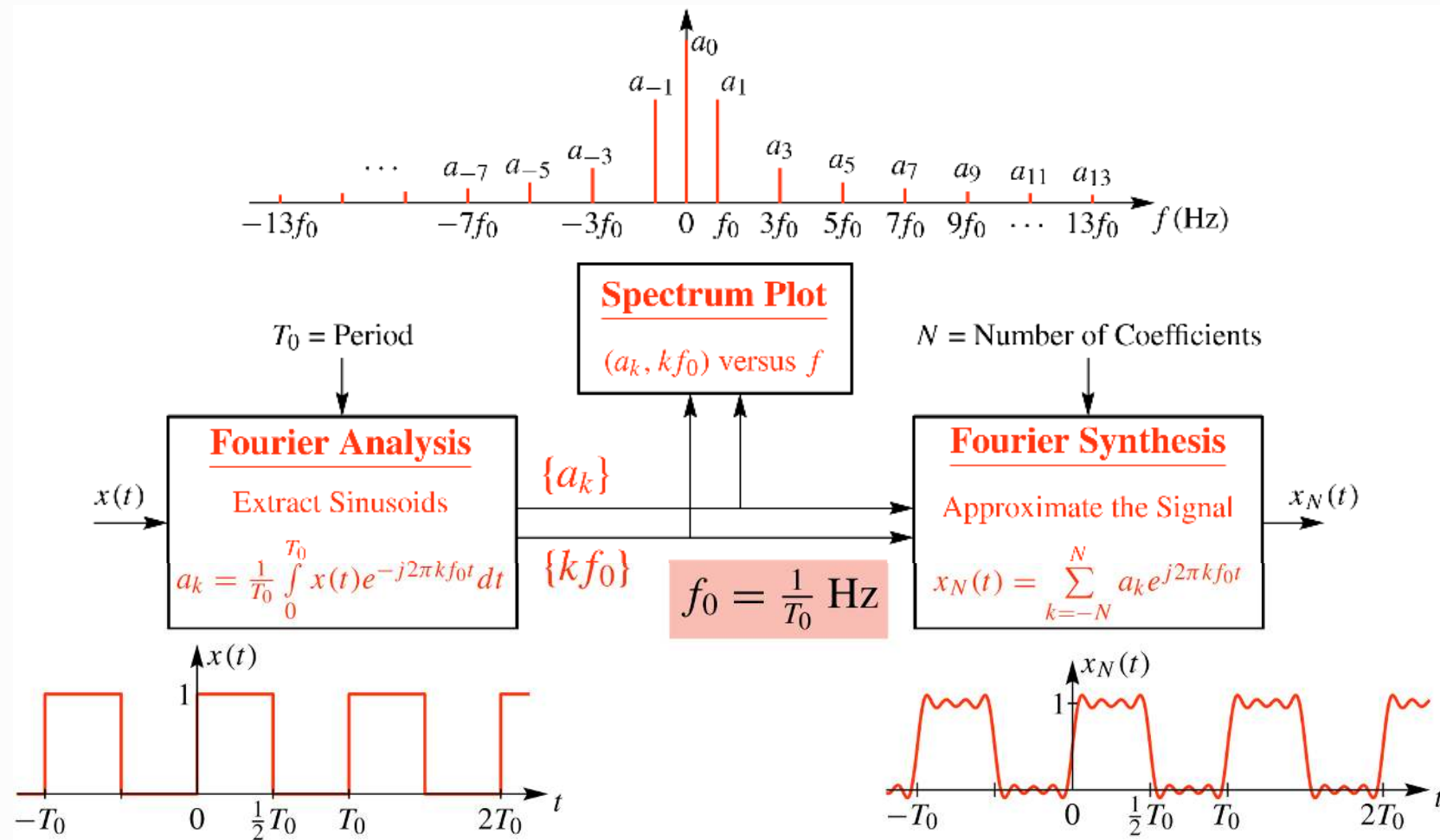
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k F_0 t}$$

$$x(t) = A_0 + \sum_{k=1}^N A_k \cos(2\pi k F_0 t + \varphi_k)$$

- **SPECTRUM** Representation
 - Graphical Form shows **DIFFERENT** Freqs

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi k / T_0)t} dt$$

Fourier Series Summary



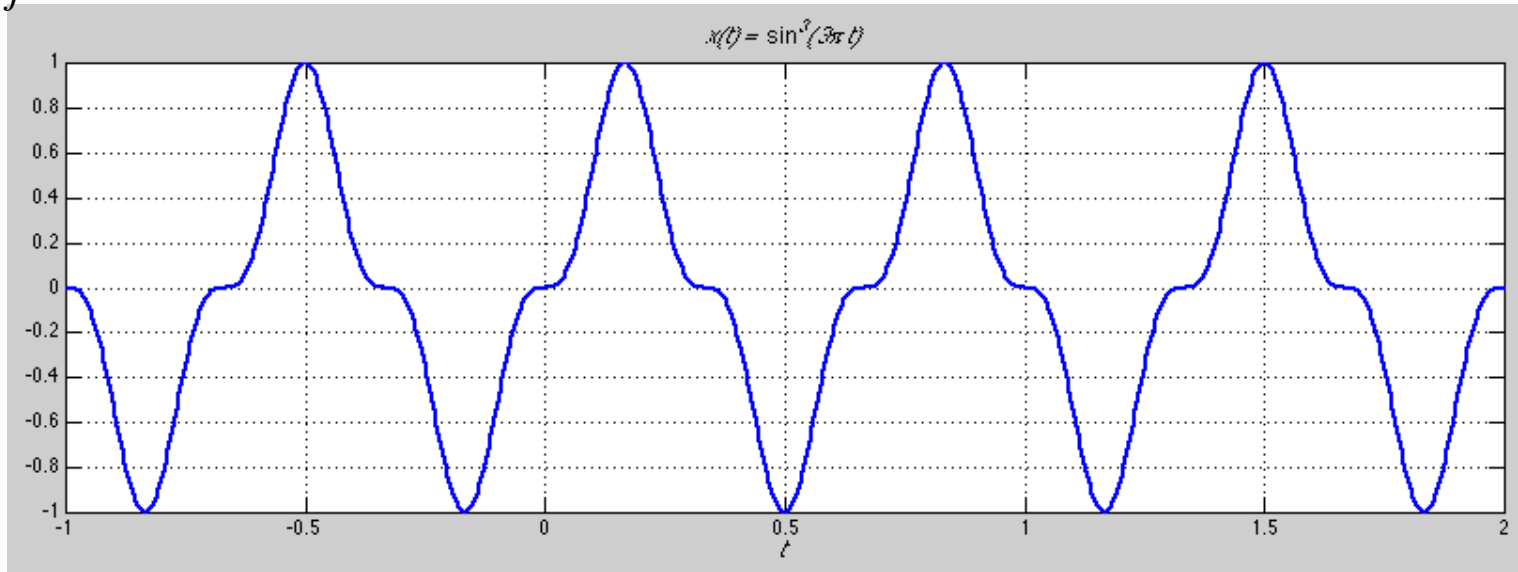
Strategies to Find Fourier Series Coefficients

- Some thoughts:
 - Starting from signal, $x(t)$, which frequencies and complex amplitudes are required?
 - ONLY FOR PERIODIC SIGNALS!
 - Two possible analysis methods:
 - 1. Read off coefficients from inverse Euler's
 - 2. Evaluate Fourier series integral
 - Can plot the spectrum for the Fourier Series
 - Equally spaced lines at kF_0

STRATEGY 1:

$$x(t) = \sin^3(3\pi t)$$

$$\sin(\omega t) = \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t})$$



$$x(t) = \left(\frac{1}{2j} (e^{j\omega t} - e^{-j\omega t}) \right)^3 = \frac{j}{8} (e^{j\omega t} - e^{-j\omega t})^3$$

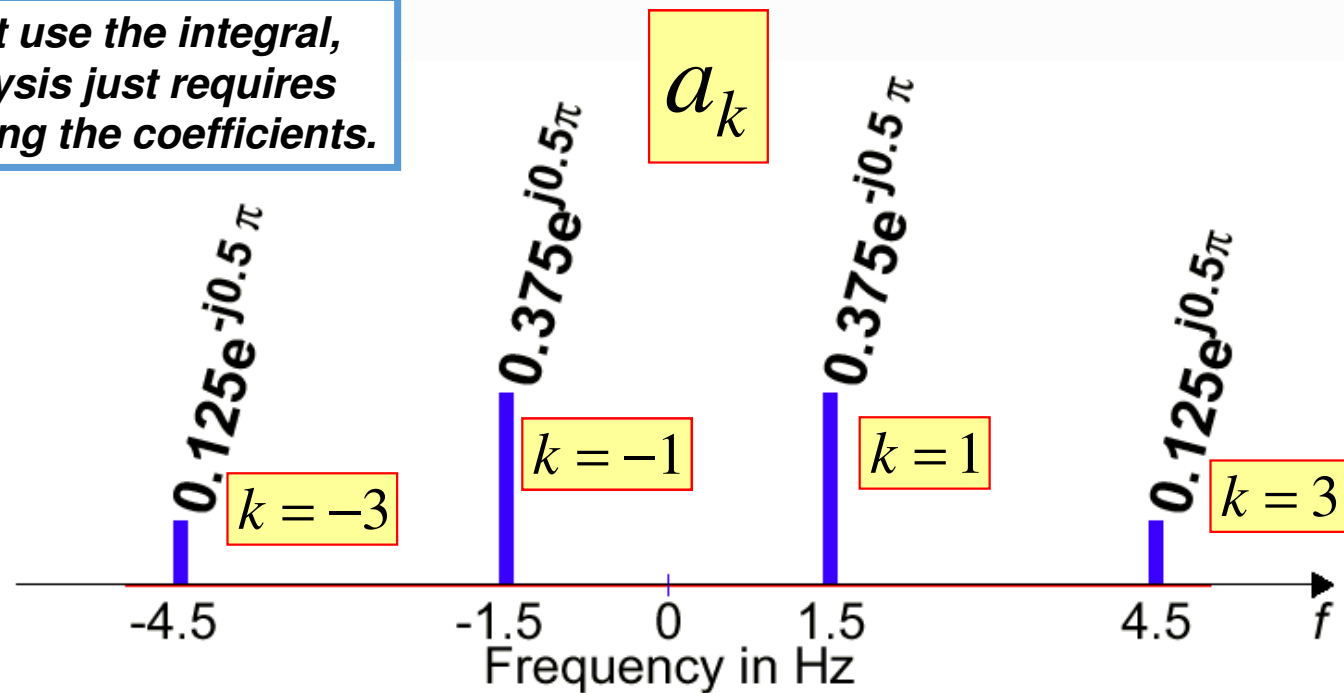
Example



$$x(t) = \sin^3(3\pi t)$$

$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

*Don't use the integral,
Analysis just requires
picking the coefficients.*



STRATEGY 2: $x(t) \rightarrow a_k$

- **ANALYSIS**

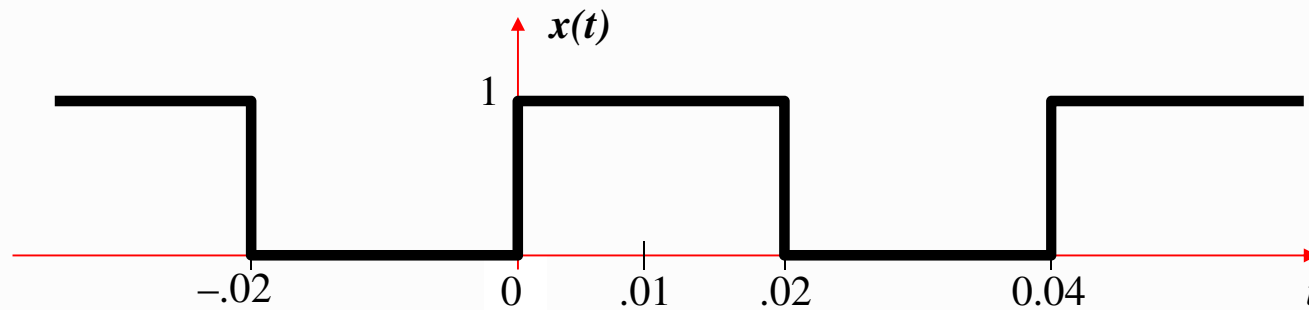
- Get representation from the signal
- Works for PERIODIC Signals
- Fourier Series
 - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 kt} dt$$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2} T_0 \\ 0 & \frac{1}{2} T_0 \leq t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



FS for a SQUARE WAVE $\{a_k\}$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k \neq 0)$$

$$x(t) = \begin{cases} 1 & 0 \leq t < .02 \\ 0 & .02 \leq t < .04 \end{cases}$$

$$a_k = \frac{1}{0.04} \int_0^{\textcircled{.02}} 1 e^{-j(2\pi/.04)kt} dt = \frac{1}{.04(-j2\pi k/.04)} e^{-j(2\pi/.04)kt} \Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)} (e^{-j(\pi)k} - 1) = \frac{1 - (-1)^k}{j2\pi k} \quad (\underline{k \neq 0})$$

Square Wave Coeffs: $\{a_k\}$

- Complex Amplitude a_k for k -th Harmonic
 - Does not depend on the period, T_0
 - DC value is 0.5

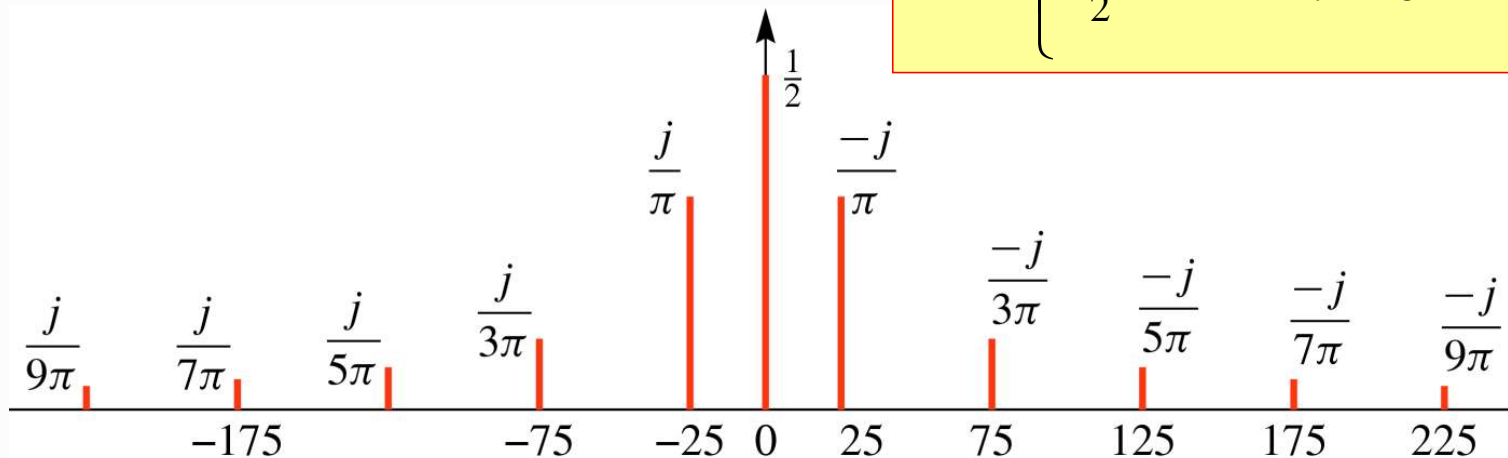
$$a_k = \frac{1 - (-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

$$T_0 = 0.04 \Rightarrow$$

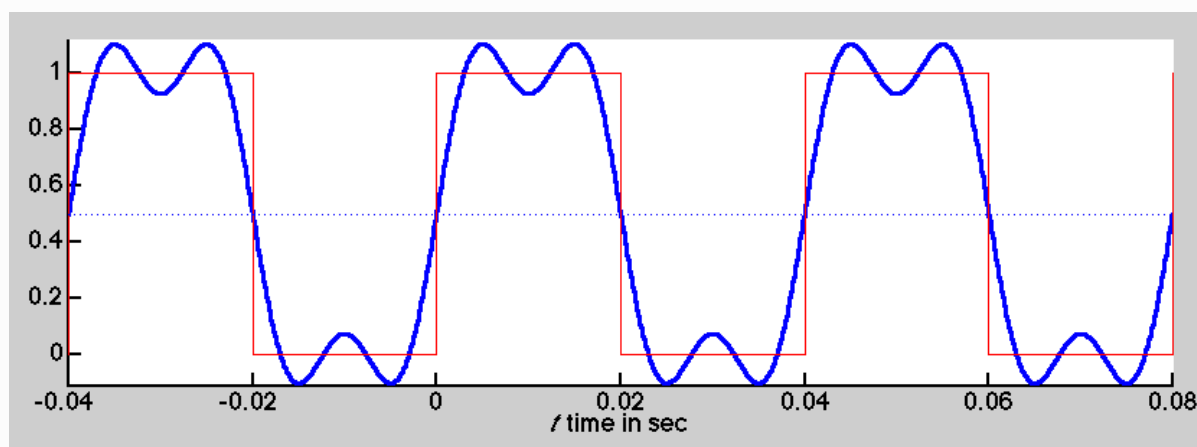
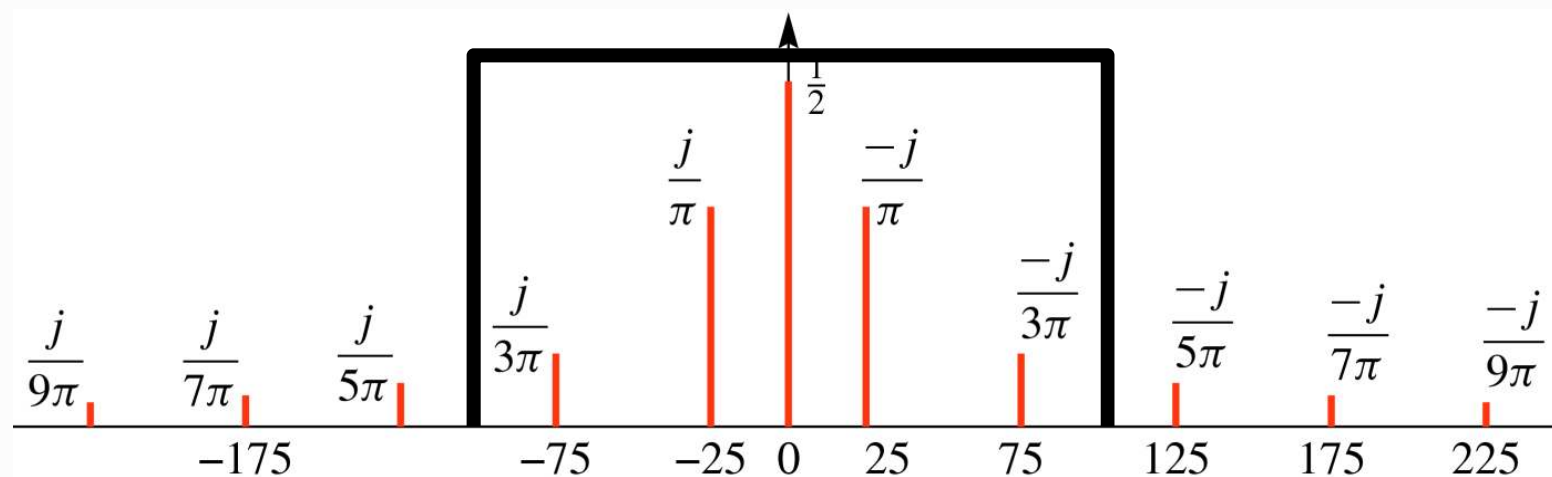
$$\omega_0 = 2\pi / (0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \frac{-j}{\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$



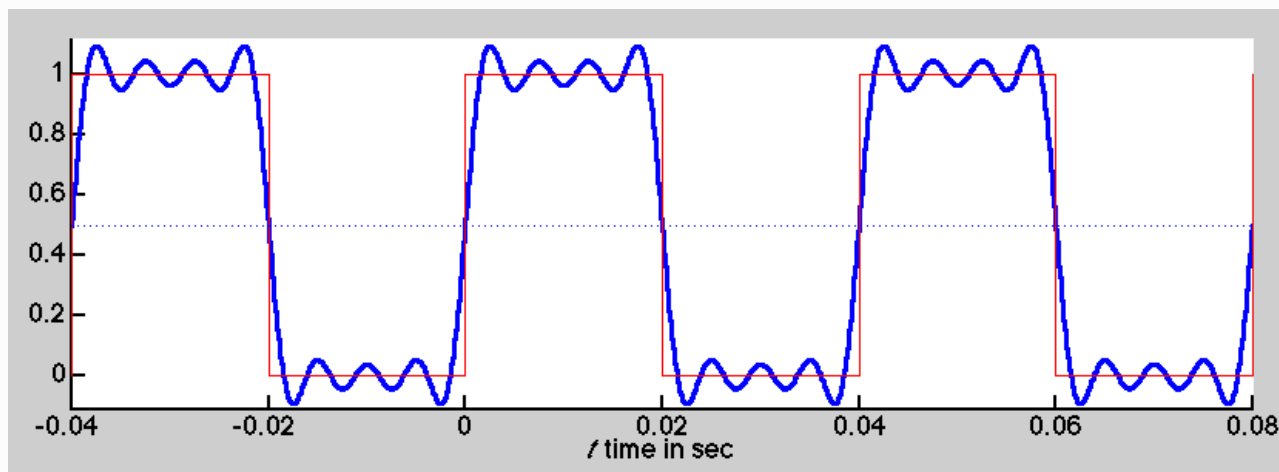
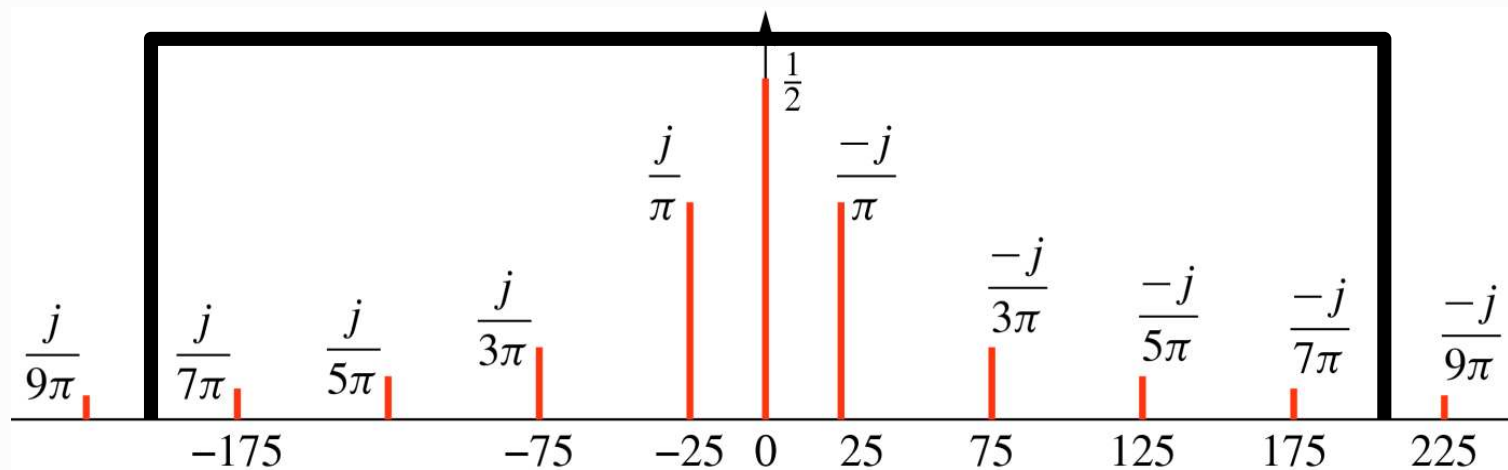
Synthesis: 1st & 3rd Harmonics

$$y(t) = \frac{1}{2} + \frac{2}{\pi} \cos(2\pi(25)t - \frac{\pi}{2}) + \frac{2}{3\pi} \cos(2\pi(75)t - \frac{\pi}{2})$$

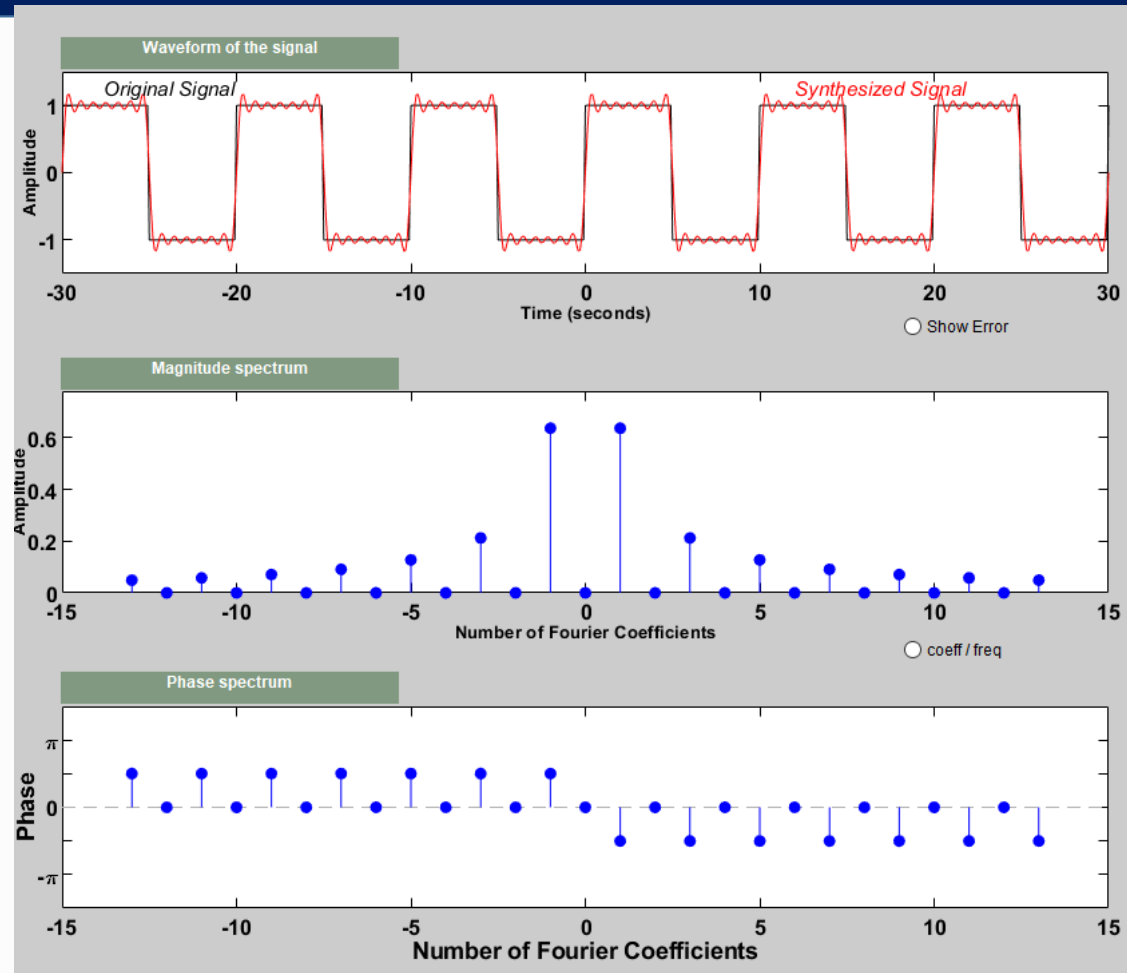


Synthesis: up to 7th Harmonic

$$y(t) = \frac{1}{2} + \frac{2}{\pi} \cos(50\pi t - \frac{\pi}{2}) + \frac{2}{3\pi} \sin(150\pi t) + \frac{2}{5\pi} \sin(250\pi t) + \frac{2}{7\pi} \sin(350\pi t)$$



Fourier Series Demo



<https://dspfirst.gatech.edu/chapters/03spect/demosLV/fseries/index.html>

More Examples for Strategy -1

$$x(t) = 10 + 14 \cos(200\pi t - \pi/3) + 8 \cos(500\pi t + \pi/2)$$

Find spectrum of signal $x(t)$.

Apply inverse Euler formula:

$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} \\ + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

$$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

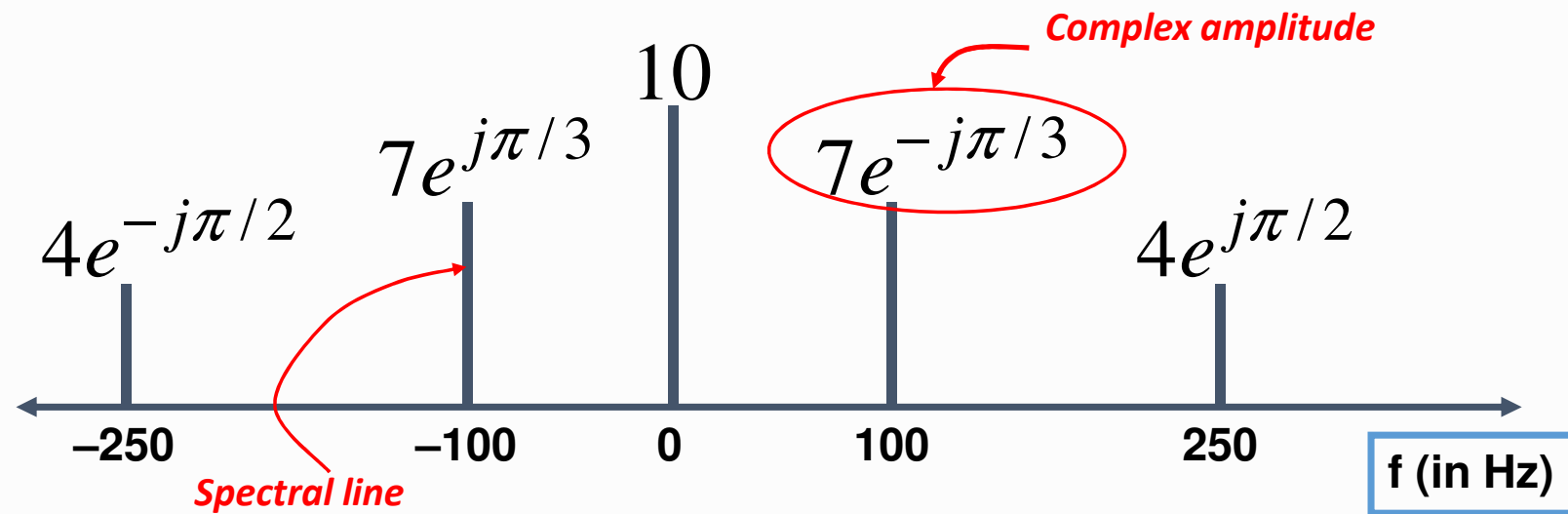
$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

Find the complex amplitude and frequency of these phasors:

$$\{(0, 10), (100, 7e^{-j\pi/3}), (-100, 7e^{j\pi/3}), (250, 4e^{j\pi/2}), (-250, 4e^{-j\pi/2})\}$$

$$\begin{matrix} f_0 & a_0 & & f_2 & a_2 & & f_{-2} & a_{-2} \end{matrix}$$

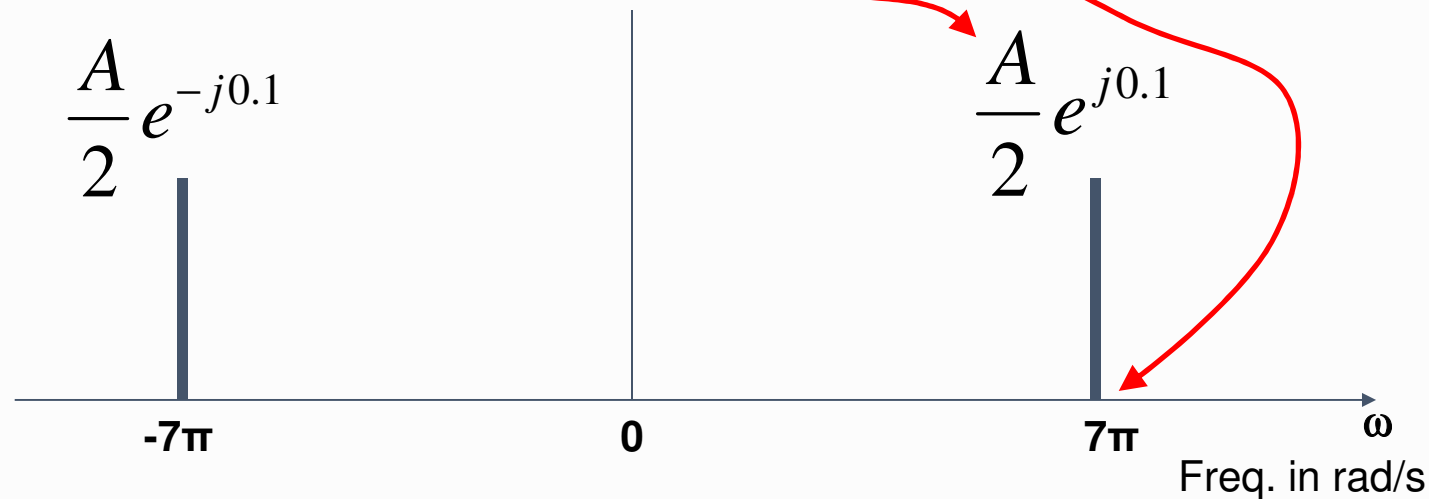
Spectrum Representation



Spectrum Interpretation



$$A \cos(7\pi t + 0.1) = \frac{A}{2} e^{j0.1} e^{j7\pi t} + \frac{A}{2} e^{-j0.1} e^{-j7\pi t}$$



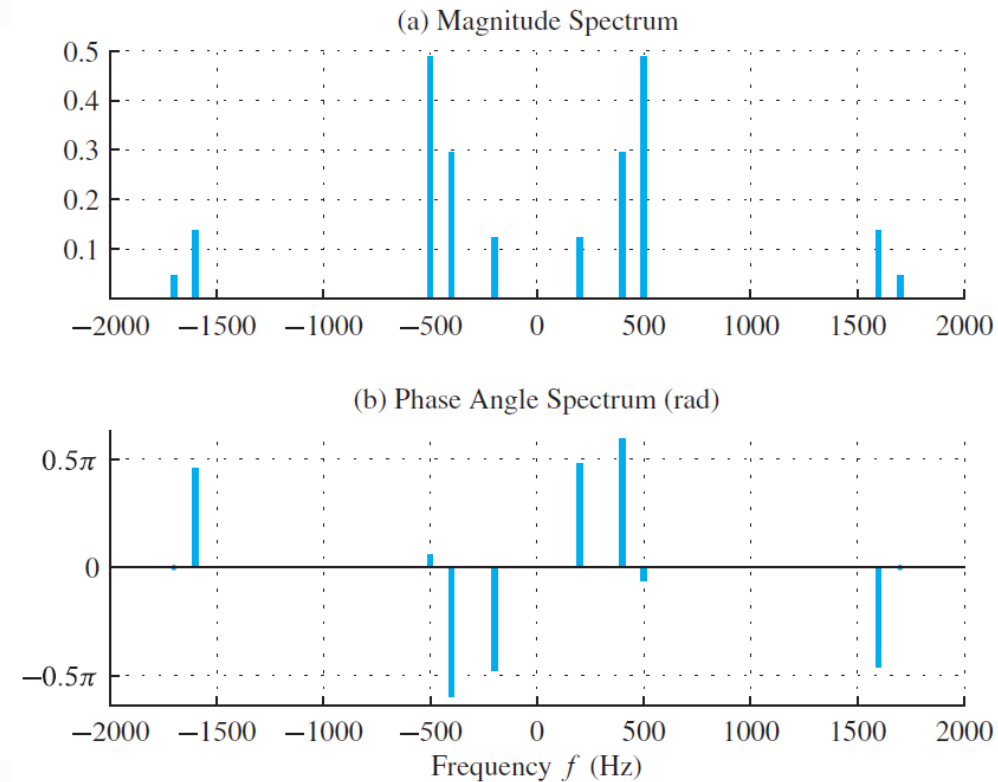
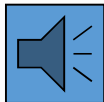
- One has a positive frequency
- The other has **negative** freq.
- Amplitude of each is half as big

$$A \cos(\omega t + \varphi) = \frac{1}{2} A e^{j\varphi} e^{j\omega t} + \frac{1}{2} A e^{-j\varphi} e^{-j\omega t}$$

Example: Sythetic Vowel

Table 3-1 Complex amplitudes for the periodic signal that approximates a complicated waveform like a vowel, such as “ah.” The a_k coefficients are given for positive indices k , but the values for negative k are the conjugates, $a_{-k} = a_k^*$.

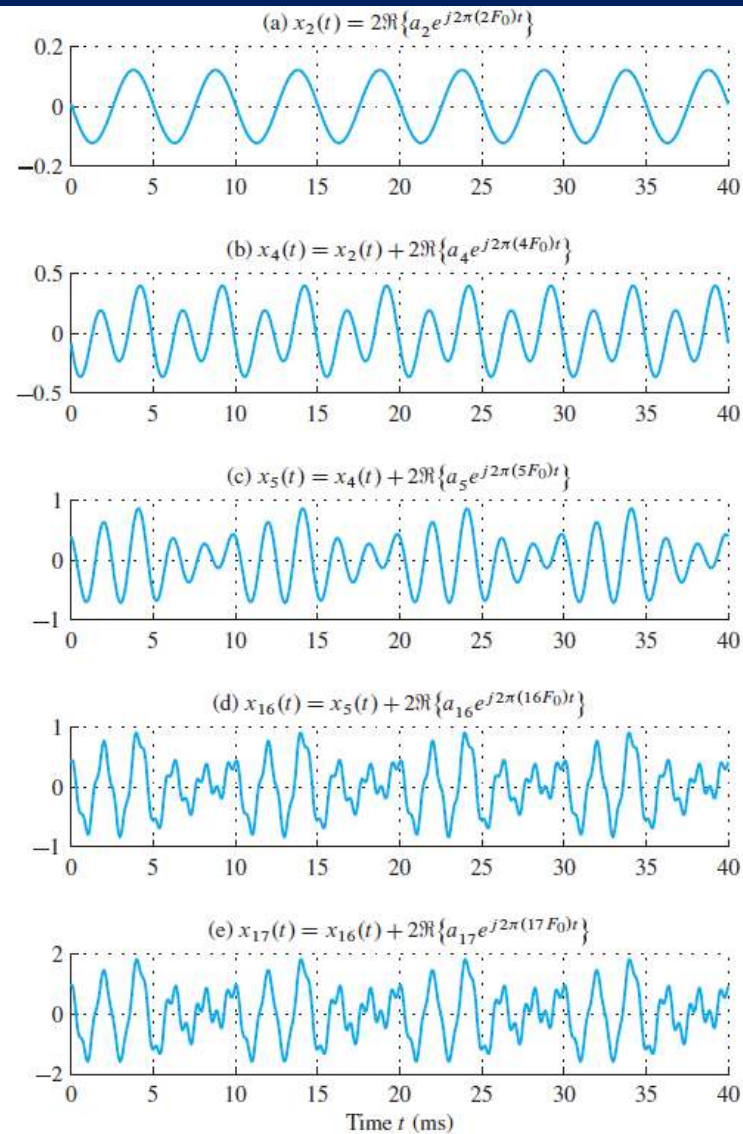
k	f_k (Hz)	a_k	Mag	Phase
1	100	0	0	0
2	200	$0.00772 + j0.122$	0.1223	1.508
3	300	0	0	0
4	400	$-0.08866 + j0.2805$	0.2942	1.877
5	500	$0.48 - j0.08996$	0.4884	-0.185
6	600	0	0	0
\vdots	\vdots	\vdots	\vdots	\vdots
15	1500	0	0	0
16	1600	$0.01656 - j0.1352$	0.1362	-1.449
17	1700	$0.04724 + j0$	0.04724	0



Vowel Waveform

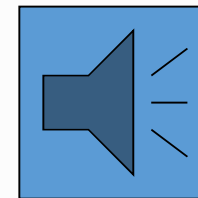


(a) The 200-Hz term alone. (b) Sum of the 400-Hz and 200-Hz terms. Additional terms are added one at a time until the entire synthetic vowel signal is created in (e). (c) Adding the 500-Hz term, which changes the fundamental period, (d) adding the 1600-Hz term, and (e) adding the 1700-Hz term.



- **Now, a much HARDER problem**

- Given a recording of a song, have the computer write the music



- Can a machine extract frequencies?
 - Yes, if we COMPUTE the spectrum for $x(t)$
 - During short intervals

Frequency can change with time ☹️ What can we do?



Time-Varying FREQUENCIES Diagram

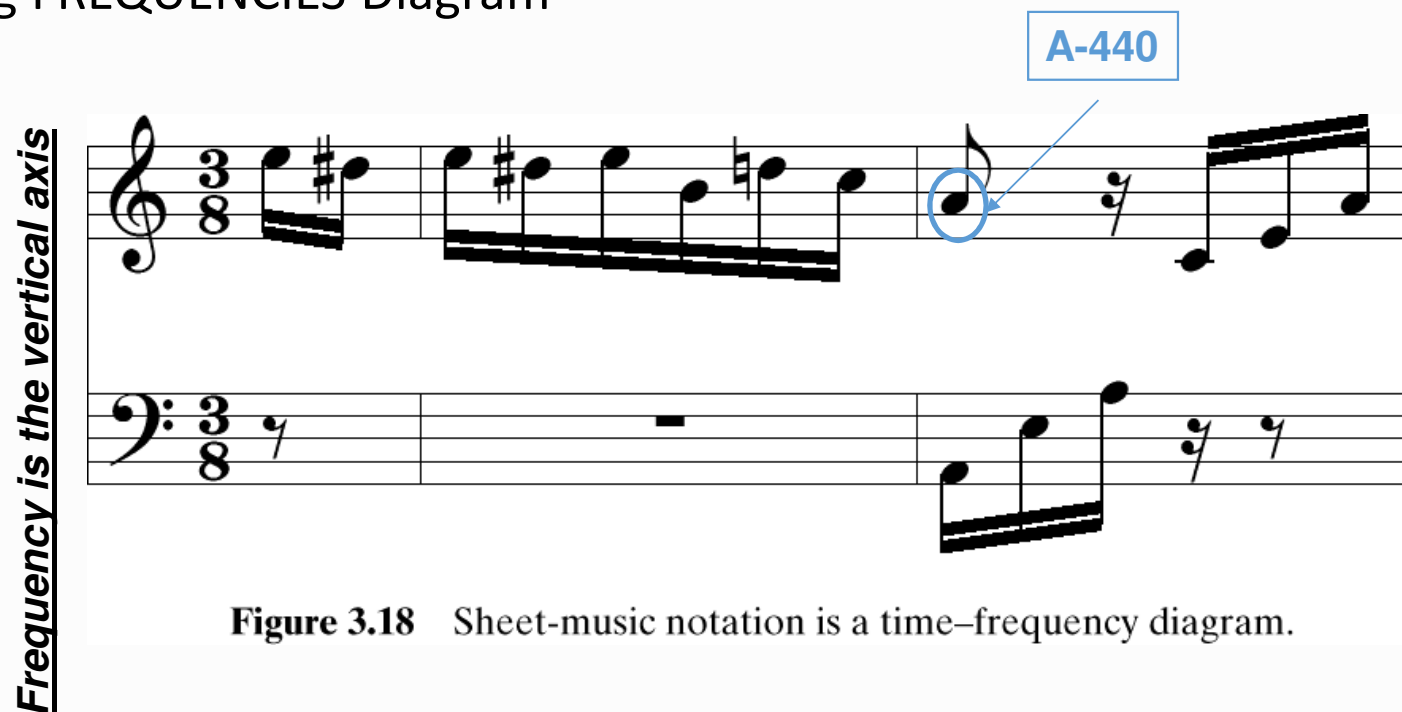


Figure 3.18 Sheet-music notation is a time–frequency diagram.

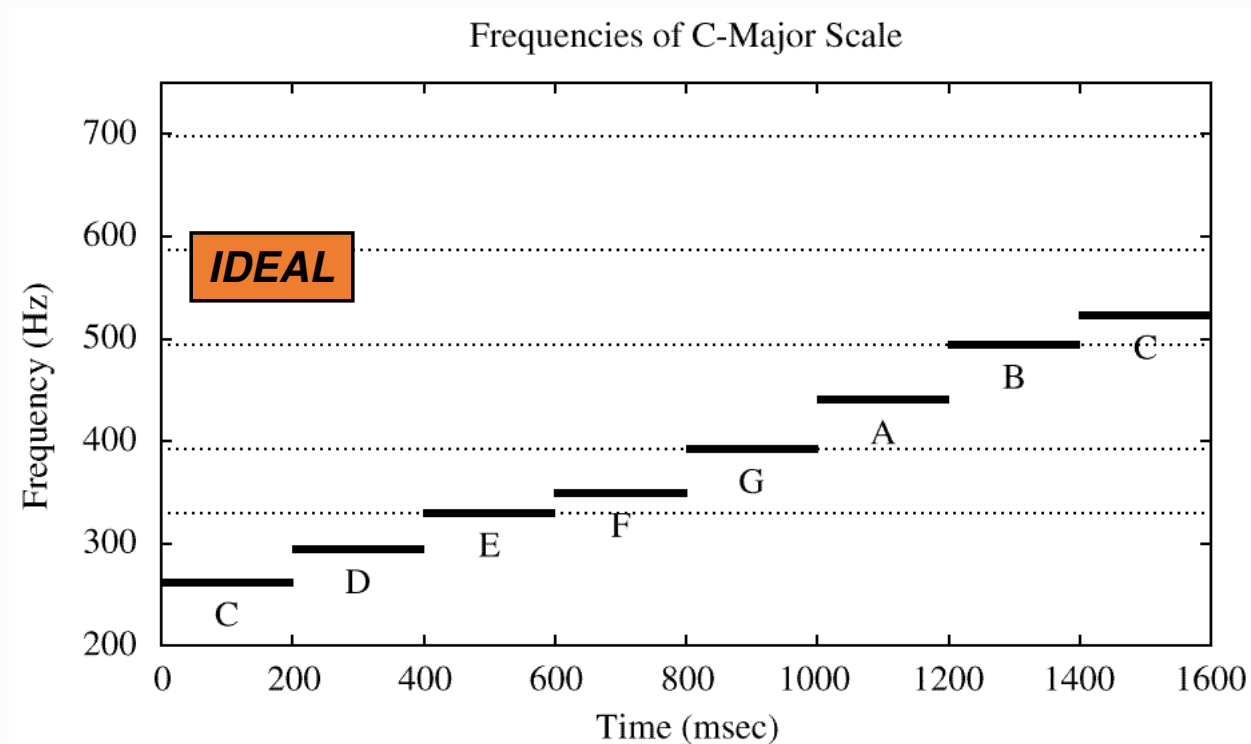
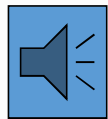
Time is the horizontal axis

SIMPLE TEST SIGNAL



- C-major SCALE: stepped frequencies
 - Frequency is constant for each note

Middle C	D ₄	E ₄	F ₄	G ₄	A ₄	B ₄	C ₅
262 Hz	294	330	349	392	440	494	523

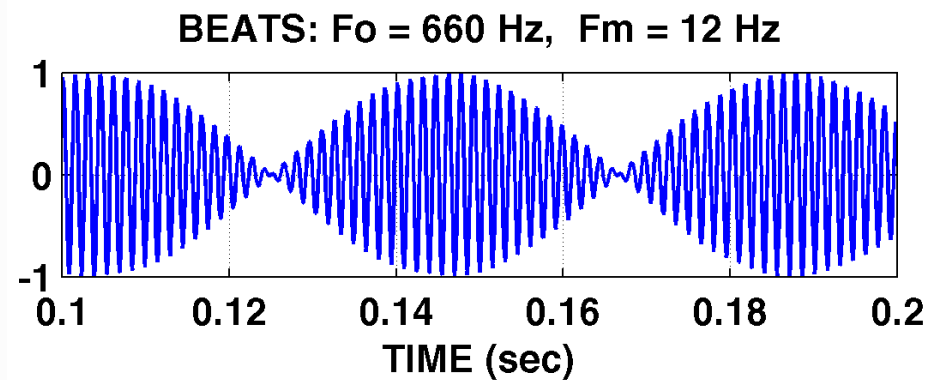
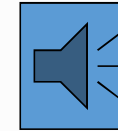
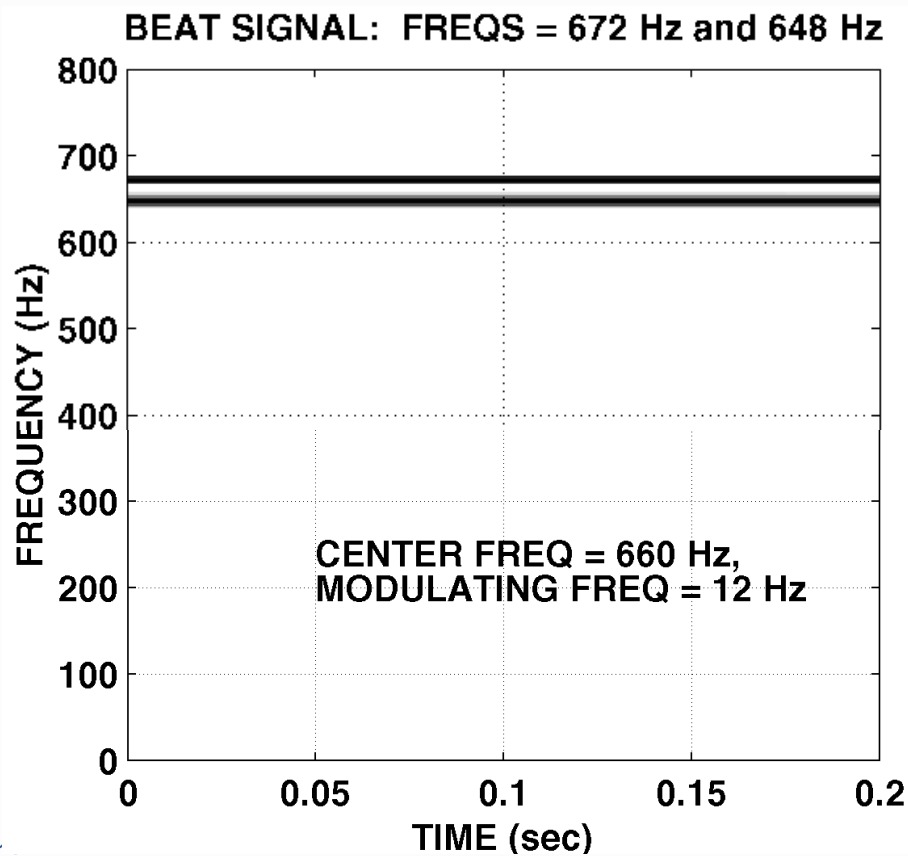


- SPECTROGRAM Tool
 - MATLAB function is `spectrogram.m`
 - SP-First has `plotspec.m` & `spectgr.m`
- ANALYSIS program
 - Takes $x(t)$ as input
 - Produces spectrum values X_k
 - Breaks $x(t)$ into **SHORT TIME SEGMENTS**
 - Then uses the FFT (Fast Fourier Transform)

SPECTROGRAM EXAMPLE



- Two **Constant** Frequencies: Beats



$$\begin{aligned} &\cos(2\pi(672)t) + \cos(2\pi(648)t) \\ &= 2\cos(2\pi(12)t)\cos(2\pi(660)t) \end{aligned}$$

AM Radio Signal

- Same form as BEAT Notes, but higher in freq

$$\cos(2\pi(\underline{660})t) \sin(2\pi(12)t)$$



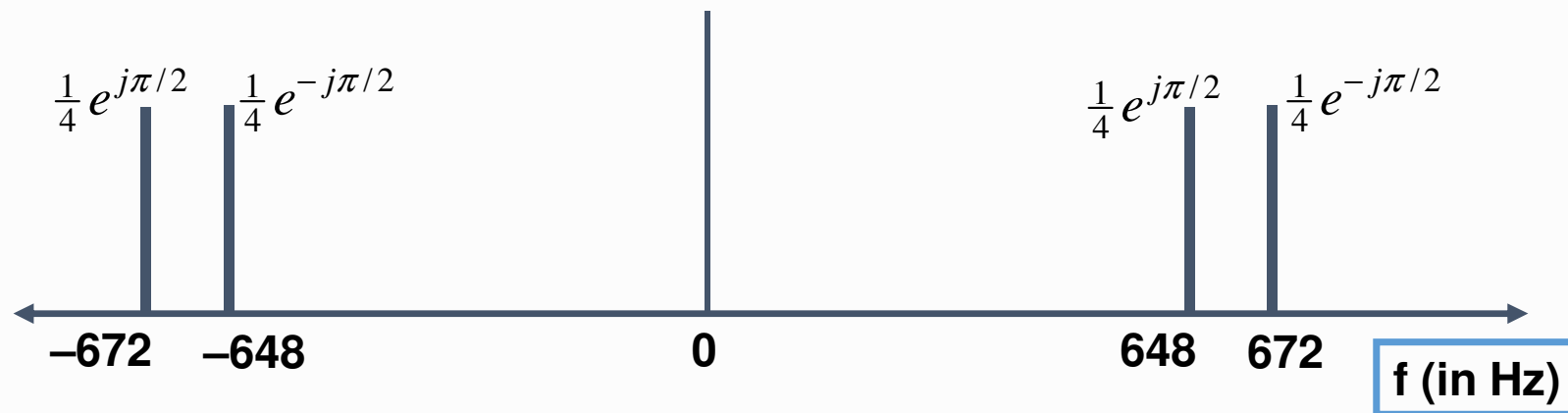
$$\frac{1}{2} \left(e^{j2\pi(660)t} + e^{-j2\pi(660)t} \right) \frac{1}{2j} \left(e^{j2\pi(12)t} - e^{-j2\pi(12)t} \right)$$

$$\frac{1}{4j} \left(e^{j2\pi(672)t} - e^{-j2\pi(672)t} - e^{j2\pi(648)t} + e^{-j2\pi(648)t} \right)$$

$$\frac{1}{2} \cos(2\pi(672)t - \frac{\pi}{2}) + \frac{1}{2} \cos(2\pi(648)t + \frac{\pi}{2})$$

SPECTRUM of AM (Amplitude Modulation)

- **SUM** of 4 complex exponentials:



What is the fundamental frequency?

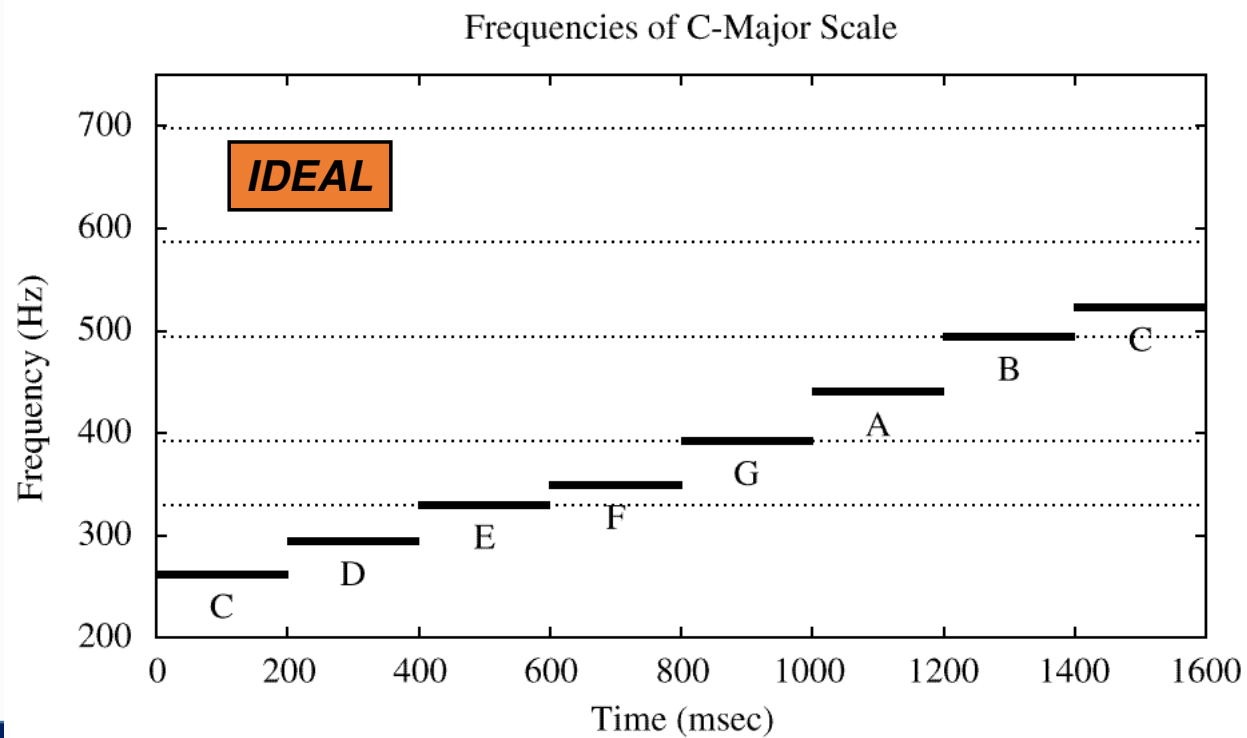
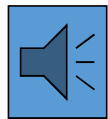
648 Hz ?

24 Hz ?

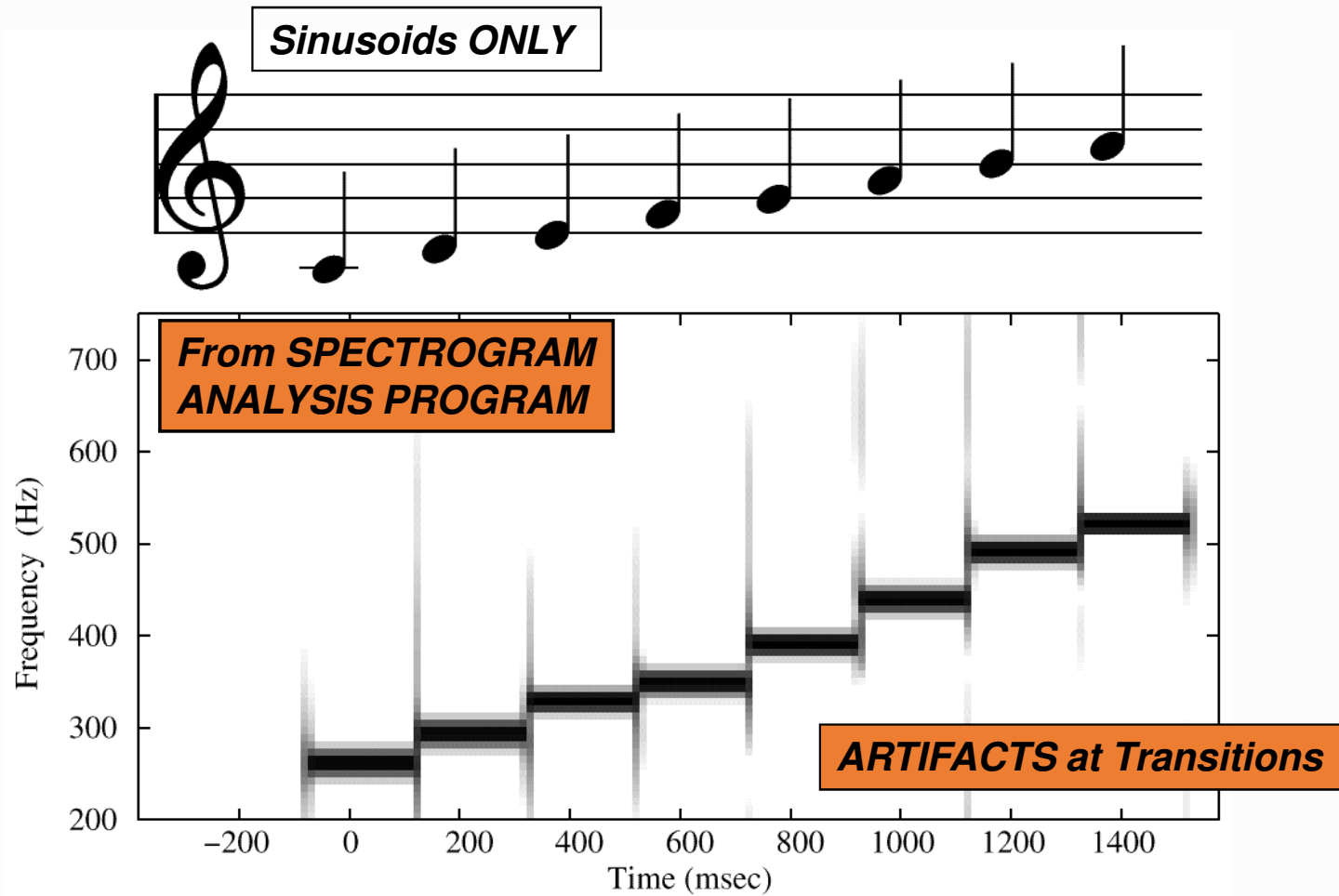
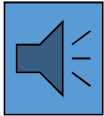
STEPPED FREQUENCIES



- C-major SCALE: successive sinusoids
 - Frequency is constant for each note



SPECTROGRAM of C-Scale

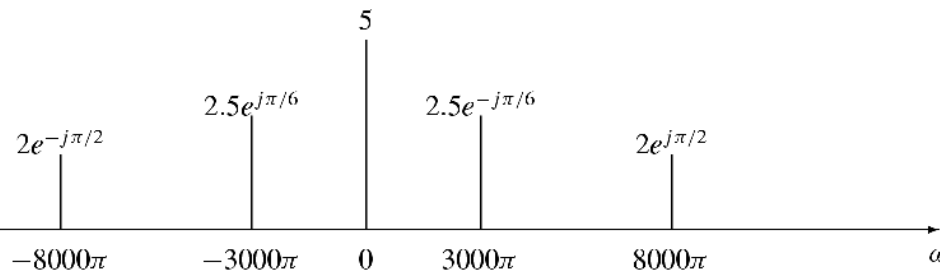


Example 1



PROBLEM:

A real signal $x(t)$ has the following two-sided spectrum:



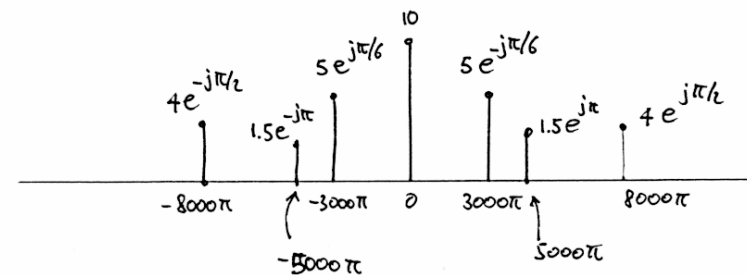
(a) Write an equation for $x(t)$ as a sum of cosines.

(b) Plot the spectrum of the signal $y(t) = 2x(t) - 3 \cos(5000\pi(t - 0.002))$.

$$a) x(t) = 5 + \underbrace{(2.5 \times 2)}_{\text{careful! do not forget this factor 2!}} \cos\left(3000\pi t - \frac{\pi}{6}\right) + \underbrace{(2 \times 2)}_{\text{careful! do not forget this factor 2!}} \cos\left(8000\pi t + \frac{\pi}{2}\right)$$

$$\begin{aligned} b) y(t) &= 2x(t) - 3 \cos(5000\pi(t - 0.002)) \\ &= 2x(t) - 3 \cos(5000\pi t - 10\pi) \\ &= 2x(t) - 3 \cos(5000\pi t) \\ &= 2x(t) + 3 \cos(5000\pi t + \pi) \end{aligned}$$

$$\text{Spec}(y) = 2 \cdot \text{Spec}(x) + \text{Spec}(3 \cos(5000\pi t + \pi))$$



Example 2



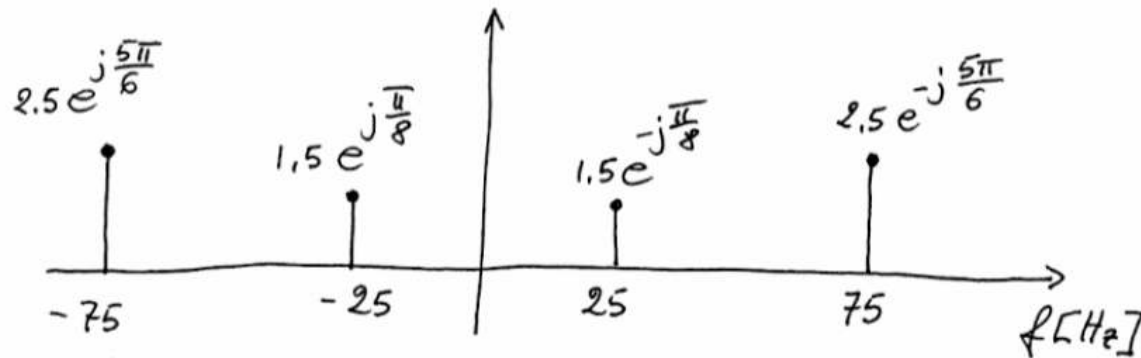
A signal composed of sinusoids is given by the equation

$$x(t) = 3 \cos(50\pi t - \pi/8) - 5 \cos(150\pi t + \pi/6)$$

- (a) Sketch the spectrum of this signal indicating the complex amplitude of each frequency component. You do not have to make separate plots for real/imaginary parts or magnitude/phase. Just indicate the complex amplitude value at the appropriate frequency.
- (b) Is $x(t)$ periodic? If so, what is the period? Which harmonics are present?

Answer

$$\begin{aligned}
 a) \quad x(t) &= \frac{3}{2} e^{-j\frac{\pi}{8}} e^{j50\pi t} + \frac{3}{2} e^{j\frac{\pi}{8}} e^{-j50\pi t} \\
 &\quad - \frac{5}{2} e^{j\frac{\pi}{6}} e^{j150\pi t} - \frac{5}{2} e^{-j\frac{\pi}{6}} e^{-j150\pi t} \\
 &= \frac{5}{2} e^{j\frac{5\pi}{6}} e^{-j150\pi t} + \frac{3}{2} e^{j\frac{\pi}{8}} e^{-j50\pi t} \\
 &\quad + \frac{3}{2} e^{-j\frac{\pi}{8}} e^{j50\pi t} + \frac{5}{2} e^{-j\frac{5\pi}{6}} e^{j150\pi t}
 \end{aligned}$$



b) Yes, $x(t)$ is periodic: $T = \frac{1}{25} = 40 \text{ ms}$
First and third harmonics are present.

Example 3

A periodic signal, $x(t)$, is given by

$$x(t) = 2 + \sin(300\pi t) + 3 \cos(600\pi t + \pi/3)$$

(a) What is the period of $x(t)$?

FUNDAMENTAL FREQ.: $\omega_0 = 300\pi = \frac{2\pi}{T}$

$$\Rightarrow T = \frac{1}{150} \text{ SEC.}$$

(b) Find the Fourier series coefficients of $x(t)$ for $-6 \leq k \leq 6$.

Using Euler's Relation

$$x(t) = 2 + \frac{1}{2} e^{-j\pi/2} \cdot e^{j300\pi t} + \frac{1}{2} e^{j\pi/2} e^{-j300\pi t} + \frac{3}{2} e^{j\pi/3} \cdot e^{j2(300\pi)t} + \frac{3}{2} e^{-j\pi/3} e^{-j2(300\pi)t}$$

$$a_0 = 2$$

$$a_1 = \frac{1}{2} e^{-j\pi/2}$$

$$a_{-1} = \frac{1}{2} e^{j\pi/2}$$

$$a_2 = \frac{3}{2} e^{j\pi/3}$$

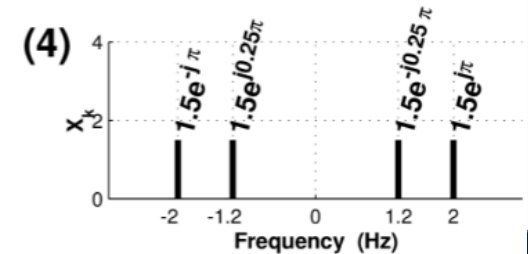
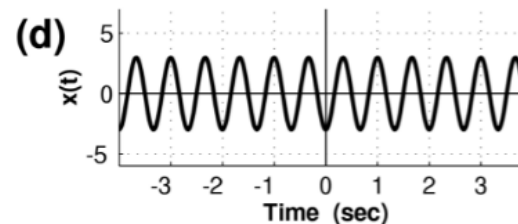
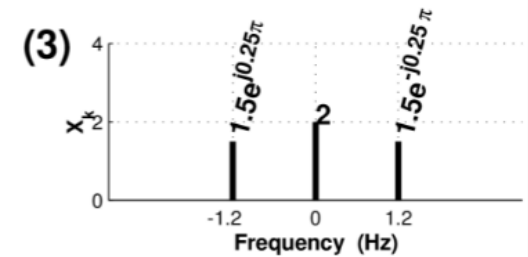
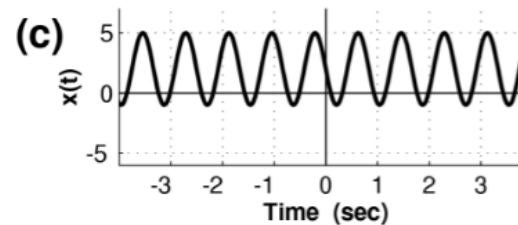
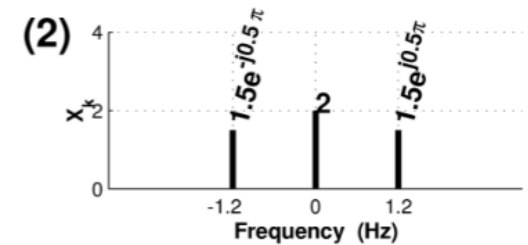
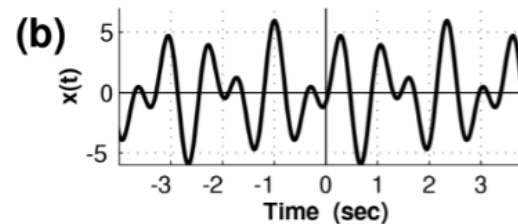
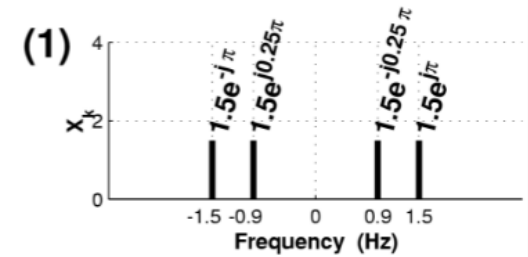
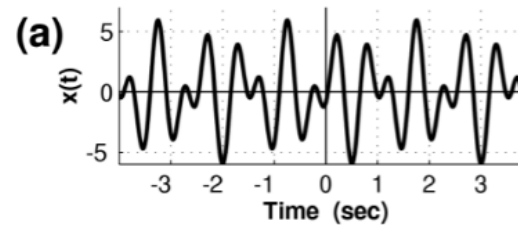
$$a_{-2} = \frac{3}{2} e^{-j\pi/3}$$

$$a_k = 0 \text{ For All Other } k$$

Example 4

Several signals are plotted below along with their corresponding spectra. However, they are in a random order. For each of the signals (a)–(e), determine the correct spectrum (1)–(5). Write your answers in the following table:

(a)	(b)	(c)	(d)	(e)
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(a)	4	(b)	1	(c)	2	(d)	5	(e)	3
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More Examples



Can be found here:

<https://dspfirst.gatech.edu/database/?d=homework&chap=3>

<https://dspfirst.gatech.edu/chapters/03spect/demos/spectrog/index.html>