Discrete Mathematics Problems

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Chapter 0

Preface

This booklet consists of problem sets for a typical undergraduate discrete mathematics course aimed at computer science students. These problem may be used to supplement those in the course textbook. We felt that in order to become proficient, students need to solve many problems on their own, without the temptation of a solutions manual! These problems have been collected from a variety of sources (including the authors themselves), including a few problems from some of the texts cited in the references.

Difficult problems are marked with a •.

References to the bibliography are indicated by [x], where x is the number of a bibliography entry.

Chapter 1

Logic

1.1 Basics

Evaluate each of the following.

1. If 2 is even, then 5=6.

2. If 2 is odd, then 5=6.

3. If 4 is even, then 10 = 7+3.

4. If 4 is odd, then 10= 7+3.

   In the following, assume that p is true, q is false, and r is true.

5. $p \vee q \vee r$ (you may want to add parentheses!)

6. $\neg q \wedge p$

7. $p \rightarrow (q \vee p)$

8. $q \oplus p$

9. $r \oplus p$

10. $q \rightarrow \neg p$

11. $(q \wedge p) \vee (q \vee (r \wedge p))$

1.2 Truth Tables and Logical Equivalences

Give truth tables for each of the following:

1. $p \vee q \wedge r$

2. $\neg p \rightarrow q$

3. $(p \vee q) \oplus p$

4. $(p \wedge q) \vee (q \rightarrow p)$

5. $(p \rightarrow q) \rightarrow \neg q$

6. $(p \vee q) \rightarrow (q \wedge \neg p)$

7. $(p \wedge q) \vee \neg r$

8. $(p \leftrightarrow \neg q) \rightarrow p$

9. $(p \vee q) \leftrightarrow (p \wedge q)$

Which of the following are tautologies?

10. $(p \vee \neg p) \oplus p$

11. $(p \wedge q) \vee (\neg p \wedge \neg q)$

12. $(p \to \neg p) \to \neg q$

13. $(p \vee q) \oplus (\neg p \wedge \neg q)$

Prove or disprove each of the following using (a) truth tables and (b) the rules of logic.

14. $(p \wedge q) \to (p \vee q) \Leftrightarrow \text{True}$

15. $\neg(p \wedge q) \vee q \Leftrightarrow \text{True}$

16. [2] $(p \wedge (p \to q)) \to q \Leftrightarrow \text{True}$

17. $p \to q \Leftrightarrow \neg q \to \neg p$

18. $\neg(\neg p \wedge q) \vee q \Leftrightarrow p \vee q$

19. $\neg(\neg p \wedge q) \vee q \Leftrightarrow q \to p$

20. $p \wedge (q \vee r) \Leftrightarrow p \vee (q \wedge r)$

21. $p \wedge (q \vee r) \Leftrightarrow p \wedge (q \wedge r)$

22. $(p \wedge \neg(q \wedge \neg r)) \vee (p \wedge q) \Leftrightarrow r$

23. $p \rightarrow (p \vee q) \Leftrightarrow$ True

    Re-write the following using only $\neg, \wedge, \vee$

24. $p \rightarrow (q \vee r)$

25. $p \oplus (q \rightarrow r)$

26. $\neg q \rightarrow \neg(p \rightarrow q)$

    Re-write the following CNF formulae into DNF

27. $(p \vee q \vee r) \wedge (\neg p \vee q)$

28. $(p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (p \vee \neg q)$

    Use truth tables to verify the following:

29. [2] $(p \vee (p \wedge q)) \Leftrightarrow p$

30. [2] $(p \wedge (p \vee q)) \Leftrightarrow p$

31. • Show that $(p \vee q \vee r \vee s)$ can be re-written into an equivalent CNF formula such that each clause contains exactly 3 variables or negations of variables.

32. Show that p $\rightarrow$ not (not q and not p) is logically equivalent to True.

1.3 Quantifiers

Evaluate each of the following for the universe Z, the set of integers

1. $P(2)$, where $P(x) = x \leq 10$

2. $P(4)$ where $P(x) = (x = 1) \vee (x > 5)$

3. $P(x)$ where $P(x) = (x < 0) \wedge (x \neq 23)$

4. $\exists x (x = 5) \wedge (x = 6)$

5. $\exists x (x = 5) \wedge (x \leq 5)$

6. $\forall x (x = 5) \wedge (x \leq 5)$

7. $\forall x (x < 0) \vee (x \leq 2x)$

8. $\forall x \; x^2 > 0$

9. $\neg \exists x \; x^2 = 0$

10. $\exists x \forall y \; x < y$

11. $\forall x \exists y \; x < y$

    In the following, let the universe be Z^+

12. $\exists x \exists y (x + y = 0) \vee (x * y = 0)$

13. $\forall x \forall y (x * y \geq x + y)$

14. $\forall x \exists y (x < y)$

15. $\exists x \forall y (x \leq y)$

16. $\exists x \forall y ((x = 3) \vee (y = 4)$

17. $\forall x \exists y \forall z (x^2 - y + z = 0)$

18. $\exists x \forall y ((x > \frac{1}{y}))$

19. $\forall x \exists y (x^2 = y - 1)$

20. $\exists y \forall x \exists z ((y = x + z) \wedge (z \leq x))$

Re-write the following without any negations on quantifiers

21. $\neg \exists x P(x)$

22. $\neg \exists x \neg \exists y P(x, y)$

23. $\neg \forall x P(x)$

24. $\neg \exists x \forall y P(x, y)$

25. $\forall x \neg \exists y P(x, y)$

26. Argue that $\exists x \forall y P(x, y) \rightarrow \forall x \exists y P(x, y)$ is (or is not) a tautology.

27. Argue that $(\forall x (P(x) \vee \exists y P(y)))$ is equivalent to $\exists x P(x)$.

28. [2] Argue that $\forall x(P(x) \vee y)$ is equivalent to $(\forall x P(x)) \vee y$

1.4 Circuits

Design logic circuits, using AND, OR, and NOT gates to solve the following problems.

1. Input two bits, x, y and output two bits representing $x - y$ ($1 - 1 = 00$, $1 - 0 = 01$, $0 - 0 = 00$, $0 - 1 = 11$).
2. Input two bits x, y and output two bits representing the absolute value of $x - y$
3. Input three bits x, y, z and output one bit which is the majority of the three input bits

Chapter 2

Sets

List the elements of the following sets. Assume the universe is Z.(Note: 2^X denotes the power set of X)

1. $\{x | x^2 = 6\}$

2. $\{x | x^2 = 9\}$

3. $\{x | (x \bmod 2 = 1) \wedge (x < 10)\}$ (Assume universe is Z^+ for this problem)

4. [2] $\{x | x = x^2\}$

5. $\{x | \forall k \in \{2, 3, \ldots x - 1\} x \bmod k = 1\}$ (Assume universe is Z^+ for this problem)

6. $\{a, b, c\} \times \{1, 2\}$

7. $\{1, 2\} \times \{a, b, c\}$

8. $\{a, b, c\} \times \emptyset$

9. $\{a, b\} \times \{1\} \times \{x, y\}$

10. $2^{\{a,\{a\}\}}$

11. [2] Is $x \in \{x\}$? Is $x \subseteq \{x\}$? Is $\emptyset \in \{x\}$? Is $\emptyset \subset \{x\}$? Is $\{x\} \in \{x\}$?

What is the cardinality of each of the following sets?

12. $\{\{x\}\}$

13. $\emptyset$

14. $\{\emptyset\}$

15. $\{\{\emptyset\}\}$

16. $\{x, \{x\}, \emptyset\}$

17. Z

18. 2^Z

19. $2^\emptyset$

20. $2^{2^\emptyset}$

Let $A = \{1, 2, 4, 5, 7, 8\}, B = \{x|(x \in Z^+) \wedge (x < 10)\}, C = \{x|(x \in Z^+) \wedge (x \bmod 3 < 2\}$. List/describe the elements of the following sets.

21. $A \cap B$

22. $A \cap C$

23. $A \cup B$

24. $A - B$

25. $B - A$

26. $A \oplus B$

27. $B - C$

28. $A \cup \emptyset$

29. $B \cup \{\emptyset\}$

30. $\overline{C}$

31. $\overline{A \cap B \cap C}$

32. $A - A$

33. $A \cap \overline{B}$

34. In general, when are two sets D, E such that $D \cap E = D \cup E$?

35. If $A \subset B$, then what is $|A \cap B|$?

Prove the following set identities, using either Venn Diagrams or the rules of sets.

36. [2] $A \cap (B - A) = \emptyset$

37. [2] $(A \cap B) \cup (A \cap \overline{B}) = A$

38. [2] $(A - B) - C \subseteq A - C$

39. [2] $(A - C) \cap (C - B) = \emptyset$

40. Argue that the symmetric difference operator does, or does not, always satisfy the associative property.

List the elements in the following sets. Assume the universe is Z^+:

41. $\{x | x < 8\}$

42. $\{x | x = 6 \vee x \geq 4\}$

43. List the elements of $\{1, 2, 3, 4\} \cup \{2, 3, 5, 7\} \cup \{1, 5, 9\}$

44. List the elements of $\{1, 2, 3, 4\} \cap \{2, 3, 5, 7\}$

Chapter 3

Functions

What is the value of each of the following:

1. $\lfloor 1.5 \rfloor$
2. $\lfloor -2.6 \rfloor$
3. $\lceil 1.1 \rceil$
4. $\lceil -1.3 \rceil$
5. When is x such that $\lceil x \rceil x = \lfloor x \rfloor x$
6. [2] Show that $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$
7. $\log_2 64$
8. $\lceil \log_2 27 \rceil$
9. $\lfloor \log_2 85 \rfloor$

10. 7!

11. $2^6 * 2^5$

Convert the following to binary:

12. 13

13. 35

14. 57

Convert the following binary numbers to base 10:

15. 101110

16. 10001

17. 11100

In each of the following, assume that $f : Z \to Z$. Then identify whether each is a function, onto function, one-to-one function, bijection.

18. $f(x) = x^2 - 1$

19. $f(x) = x^2 + 1$

20. $f(x) = \sqrt{x}$

21. $f(x) = \lceil \sqrt{x} \rceil$

22. $f(x) = 5$

23. $f(x) = 2^x$

24. $f(x) = (i)\frac{x}{2}$ if x is even; $(ii) 2x - 1$ if x is odd

25. $f(x) = \log^* x$, where $\log^* x = 1 + \log^*(\lfloor \log_2 x \rfloor)$ and $\log^* x = 1$ for $x \leq 2$.

    Also compute $\log^* 64$.

26. Define a recursive function such that $f(n) = 5^{(2^n)}$.

27. Let $f(n) = 1$ for $n \leq 1$ and $f(n) = 2f(n-1) + 3$. Compute the values of $f(n)$ for $n \leq 5$.

28. Let $f(n) = 1$ for $n \leq 2$ and $f(n) = 2f(n-1) + f(n-1)f(n-2)$. Compute the values of $f(n)$ for $n \leq 5$.

29. Show that $\log(n!) \leq n \log n$ for all $n \geq 1$.

30. Let $f(n) = 2$ for $n \leq 2$ and $f(n) = f(n-1) + f(n-2) + 1$. Compute the values of $f(n)$ for $n \leq 6$.

31. Let $f(n) = 1$ for $n \leq 2$ and $f(n) = 2f(n-1) - 1$. Compute the values of $f(n)$ for $n \leq 5$.

Chapter 4

Integers and Matrices

1. What is the prime factorization of 90? of 8100?

2. What is 57 mod x for each $x = 2, 3, 4, 5$?

3. How many integers less than 45 are relatively prime to 45?

4. [2] Show that if a, b, m are positive integers, then $a \bmod m = b \bmod m$ if $a \cong b (\text{mod } m)$.

5. [2] Show that if a, b, m are positive integers, then $a \cong b (\text{mod } m)$ if $a \bmod m = b \bmod m$.

6. What is the gcd of 200 and 88?

7. What is the gcd of 17 and 42?

8. Show that if $a|b$ and $b|c$ then $a|c$.

9. What is the value of $\sum_{i=2}^{i=6} 2^{i+1}$?

10. Is $2^n - 1$ prime for all values of n?

11. What is the value of $\sum_{i=0}^{i=5} i + 1$?

12. What is the value of $\sum_{i=1}^{i=8} \lfloor \frac{i}{2} \rfloor$?

13. What is 17 mod 5?

14. What is 81 mod 7?

15. What is 81^2 mod 7?

16. What is the value of $\sum_{i=1}^{i=6} i - 2$?

17. Are 7 and 49 relatively prime?

18. What is the value of $(19 * 3001)$ mod 3?

19. Find a closed form formula that is equal to of $\sum_{i=2}^{i=n} 3i - 1$?

20. What is 10! mod 2?

21. What is 487! mod 67?

22. Prove that if every even integer $n \geq 4$ is the sum of two (not necessarily distinct) primes, then every odd integer $m \geq 7$ is the sum of three (not necessarily distinct) primes.

23. What is 7^{272} mod 3?

24. What is $8^{33} \bmod 3$?

25. What is the largest positive integer k such that $(324 \bmod k) = (374 \bmod k) = (549 \bmod k)$? (What principle can you use to solve this problem quickly?)

26. For what positive values of n is $2^n + 1$ divisible by 3? Prove your answer is correct.

Chapter 5

Proofs

5.1 Direct Proofs

1. Prove that if n is odd, then n^2 is odd.

2. Use the solution to the previous problem to prove that if n is odd, then n^3 is odd. Also, find a direct proof that does not rely on the solution to the previous problem.

3. Prove that n is even if and only if n^2 is even.

4. Prove that the power set of an infinite set is also infinite.

5. Let $P(A)$ denote the power set of set A. Let A and B be sets such that $A \neq B$. That is, there exists an element x such that $x \in A$ and $x \notin B$. Argue that $P(A) \neq P(B)$ by showing there exists a specific element in $P(A)$ that is not in $P(B)$ or a specific element in $P(B)$ that is not in $P(A)$.

6. Prove that if n and m are positive, even integers, then nm is divisible by 4.

7. A perfect number is a positive integer n such that the sum of the factors of n is equal to $2n$ (1 and n are considered factors of n). So 6 is

a perfect number since $1+2+3+6=12=2*6$. Prove that a prime number cannot be a perfect number.

8. Prove that there does not exist an integer $n>3$ such that $n, n+2, n+4$ are each prime.

9. Prove that $(a \bmod 2)(b \bmod 2) = (ab) \bmod 2$.

10. Prove or disprove that if $a^2 \equiv b^2 \pmod{c}$ then $a \equiv b \pmod{c}$.

11. [2] Prove that there are no integer solutions to $x^2-3=4y$. (Hint: Do a proof by cases, the cases being the value of x modulo 4).

12. Prove that if p, q are positive integers such that $p|q$ and $q|p$, then $p=q$.

13. Prove that for any integer x, the integer $x(x+1)$ is even.

14. Prove that the product of any two odd integers is odd.

15. Let $x \geq y \geq 1$. Prove that if $gcd(x,y)=y$ then $lcm(x,y)=x$.

16. Let x, y be positive integers. Prove that $lcm(x,y)=xy/gcd(x,y)$.

5.2 Proofs by Contradiction

1. Let A, B be sets. Prove that if $|A \cup B| = |A| + |B|$, then $A \cap B = \emptyset$.

2. Let $f(x)$ and $g(x)$ be functions. Prove, using contradiction method, that if $f(g(x))$ is one-to-one, then $g(x)$ is one-to-one. That is, suppose that $g(x)$ were not one-to-one and derive that $f(g(x))$ cannot be one-to-one.

3. Let A, B be sets. Prove that $(A - B) \cap (B - A) = \emptyset$.

4. Let A, B be non-empty sets. Prove that if $A \times B = B \times A$, then $A = B$.

5. Prove that in any set of n numbers, there is one number whose value is at least the average of the n numbers.

6. Let A, B be finite sets. Prove that if $A - B = 0$ and there is a bijection between A and B, then $A = B$.

7. This problem is taken from Maryland Math Olympiad problem, and was posted on the Computational Complexity Web Log. Suppose we color each of the natural numbers with a color from $\{red, blue, green\}$. Prove that there exist distinct x, y such that $|x - y|$ is a perfect square. (Hint: it suffices to consider the integers between 0 and 225).

8. Prove that $\sqrt{3}$ is irrational. One way to do this is similar to the proof done in class that $\sqrt{2}$ is irrational, but consider two cases depending on whether a^2 is even or odd.

5.3 Proofs by Induction

Where unspecified, f_n refers to the Fibonacci sequence.

1. Prove that $\sum_{i=0}^{n} 2i = n(n+1)$.

2. Let $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ when $n > 1$ (the Fibonacci sequence). Prove using induction that $f_n > 2n$ when $n \geq 7$ (note that $f_8 = 21$).

3. • Prove that $f_n | f_{2n}$, where f_n is the n^{th} Fibonacci number ($f_0 = 0, f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}$ for $n > 2$).
Hint: Use induction to show that $f_{2n} = f_n(f_{n+1} + f_{n-1}) = f_n f_{n+1} + f_n f_{n-1}$. In so doing, you will need the fact that $f_{2n-1} = (f_n)^2 + (f_{n-1})^2$.

4. Use induction to prove that each positive integer can be written as the sum of a number of distinct, nonconsecutive Fibonacci numbers. (Note: such a sum may consist of only one number, such as $3 = 3$).

5. Prove that $f_n \leq (1+\sqrt{5})^n$.

6. • (hard, lots of algebra) Prove that $f_{n-1}f_{n+1} = (f_n)^2 - (-1)^n$.

7. Prove that $\sum_{i=0}^{n}(f_i)^2 = f_n f_{n+1}$.

8. Prove that f_n, f_{n+1} are relatively prime (i.e., have no common factors except 1).

9. Define function $f(n)$ as follows. $f(1) = 3$ and $f(n) = n * f(n-1)$ when $n > 1$. Use induction to prove that $f(n) > 2^n$ for all $n \geq 1$.

10. Prove that for any integer $k \geq 0$, $(ab)^k = a^k b^k$.

11. (a) Prove that $2^n < n!$ for $n \geq 4$. (b) Prove that $n! < n^n$ for all $n > 0$.

12. Prove that $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$.

13. Prove that $\sum_{i=1}^{n}(2i-1) = n^2$.

14. Prove that $n^3 - n$ is divisible by 3 for any integer $n \geq 0$.

15. Prove that if A is a set with n elements, then the power set of A contains 2^n elements.

16. Prove that $\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$.

17. Prove that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

18. Prove that $\sum_{i=0}^{n} \frac{1}{2^i} = 2 - \frac{1}{2^n}$.

19. Prove that $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

20. Prove that if $k \neq 1$ then $\sum_{i=0}^{n} ar^i = \frac{a(k^{n+1}-1)}{k-1}$.

21. Prove or disprove that $\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$.

22. Prove that $\sum_{i=0}^{n} 3^i = \frac{3^{n+1}-1}{2}$.

23. Suppose $x + \frac{1}{x}$ is an integer. Prove that $x^2 + \frac{1}{x^2}, x^3 + \frac{1}{x^3}, \ldots, x^n + \frac{1}{x^n}$ are also integers.

24. Prove that for $n \geq 1$, $\sum_{i=1}^{2n} (-1)^{i+1} \frac{1}{i} = \sum_{i=n+1}^{2n} \frac{1}{i}$.

25. Prove that the number of ways to order n items is $n!$.

26. (hard) [2] Prove that any set of $n+1$ numbers taken from $\{1, 2, \ldots, 2n\}$ contains a pair a, b such that $a|b$.

27. Prove that any positive integer can be factored into primes. (Note it is more difficult to prove this factorization is unique, don't worry about that for this problem).

28. [2] Prove that $n^3 + 2n$ is divisible by 3 for all positive integers n.

29. [2] Prove that any amount of postage greater than seven cents can be formed using only three and five cent stamps. Based on your proof, write a recursive algorithm that prints the actual stamps used.

30. Prove that $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geq 2(\sqrt{n+1}-1)$, for all $n \geq 1$.

31. Prove that $8|9^n - 1$ for all $n \geq 1$.

32. What is the largest number you cannot write as the sum of 6, 9, or 20? That is, what is the largest x such that $x \neq 6u + 9v + 20w$, where $u, v, w \geq 0$.

33. • [1] Prove that $f_1 + f_2 \ldots + f_n = f_{n+2} - 1$, where f_n is the n^{th} Fibonacci number ($f_0 = 0, f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}$ for $n > 2$).

Chapter 6

Graphs

6.1 Basic Problems

6.1.1 Graphs

1. How many edges do the following graphs have: C_n, P_n, K_n, $K_{n,m}$?

2. Does there exist a graph with degree sequence $0, 1, 3, 5, 5$? (degree sequence is the ordered list of vertex degrees in a graph)

3. Does there exist a graph with degree sequence $1, 1, 2, 2, 3, 3$?

4. Does there exist a graph with degree sequence $1, 1, 2, 2, 3, 4, 4, 4$?

5. Does there exist a graph with degree sequence $1, 1, 1, 1, 2, 2, 2$?

6. Does there exist a connected graph with degree sequence $1, 1, 1, 1, 2, 2, 2$?

7. Let $K_{n,m}$ be the bipartite graph with n vertices in one part and m in the other part and having nm edges. Draw $K_{2,3}$ so that no edges cross (i.e., give a planar embedding).

8. Draw a 5×5 grid graph. How many edges does the $n \times n$ grid graph have?

9. Let $G = (V, E)$ be a graph. Define $G^k = (V, E')$ to be the simple graph formed from G by adding an edge between any two vertices whose distance in G is at most k (the *distance* between two vertices is the length of a shortest path between them). Draw C_5^2. How many edges does C_n^2 have?

10. Write the adjacency matrix for C_5^2.

11. Write the adjacency matrix for P_5^2.

12. Is C_5^2 isomorphic to P_5^2? (G and H are isomorphic if they have the same number of vertices and there is a function f such that uv is an edge in G if and only of $f(u)f(v)$ is an edge in H, for all edges uv in G, where u and v are vertices in G. In other words, G and H can be drawn so as to look exactly the same).

13. If $G = (V, E)$ is a graph, then $\overline{G}$ is the graph with vertex set V and all edges of the form uv where $uv \notin E$. Are C_{10} and $\overline{C_{10}^4}$ isomorphic?

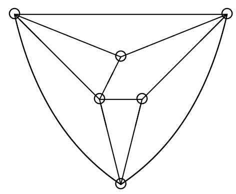
14. What is the smallest k such that C_n^k isomorphic to K_n?

15. Let G have n vertices. If $\overline{G}$ is a connected graph, what is the maximum number of edges that G can have?

16. Let G have 17 vertices. Can G possibly be isomorphic to $\overline{G}$? What if G has 13 vertices? $\overline{G}$ is the graph formed from G by deleting all edges in G and adding all edges between all vertices that are not adjacent in G.

17. Explain why a graph with n vertices, where each vertex has degree at least $n - 2$, cannot be bipartite when $n \geq 5$.

18. What is the vertex connectivity and edge connectivity of the following graph:



19. Suppose a graph has two internally-disjoint $x-y$ paths (i.e., two paths that have nothing in common except x and y). Is it necessarily 2-connected? Does this violate Menger's Theorem? Why or why not?

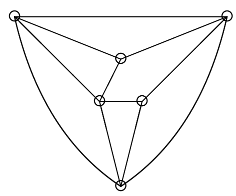
20. Is it true that every degree one vertex in a graph has a neighbor that is a cut-vertex? (a cut-vertex in a connected graph is a vertex whose deletion results in a disconnected graph).

21. Consider K_n. Let us call the graph formed by taking K_n and removing one edge $K_n - e$. What is the chromatic number of $K_8 - e$?

22. Draw a graph G that is such that $\chi(G) > \chi(G - v)$ for all vertices v. (Such a graph is called *critical*).

23. Draw a graph G that is such that $\chi(G) > \chi(G - e)$ for all edges v. (Such a graph is called *edge-critical*).

24. What is the chromatic number of each of the following graphs?



25. What is a largest clique in each of the graphs above?

26. A $\frac{k}{q}$ *circular coloring* of graph $G = (V, E)$ is a function $f : V \to \{0, 1, \ldots, k-1\}$ such that if $uv \in E$ then $q \leq |f(u) - f(v)| \leq k - q$. Show that C_5 has a $\frac{5}{2}$ coloring. Show that C_7 has a $\frac{7}{3}$ coloring. Show that if $\chi(G) = c$, then G has a $\frac{c}{1}$ coloring.

27. The *edge chromatic number* of a graph is the minimum number of colors needed to color the edges of a graph so that edges sharing a vertex have different colors. Argue that the edge chromatic number of G is at least the maximum degree of G. Find a graph whose edge chromatic number exceeds the maximum degree.

28. Suppose we wish to color the vertices of a bipartite graph $G = (A, B, E)$ with positive integers, so that adjacent vertices receive different colors. Let $c(v)$ denote the color assigned to vertex v. We wish to minimize $\Sigma\ c(v)$, that is, the sum of all the colors. Prove or disprove the following statement. An minimum "sum" coloring exists for all G using only colors 1 and 2.

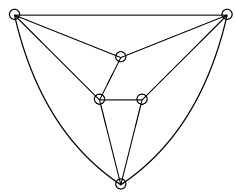
29. It is NP-hard to even approximate the chromatic number of a graph in polynomial time. That is, it is NP-complete to decide if G's chromatic number is at most $c * k$, for any constant c. Give a polynomial time algorithm (or argue that one does not exist) to approximate the chromatic number of a graph G having maximum degree three. That is, your algorithm should run in polynomial time and use no more than twice as many colors as necessary.

30. For which m, n does $K_{m,n}$ contain an Euler Circuit (a closed walk containing all the edges of the graph)? A Hamiltonian Cycle (a cycle containing all the vertices of the graph)? For which is it planar?

31. True or False: all $m \times n$ grid graphs contain a Hamiltonian cycle (assume $m \geq n \geq 2$). Justify your answer.

32. A *perfect code* is a subset of vertices of the graph, D, such that each vertex in $G - D$ is adjacent to exactly one element of D and each vertex in D has no neighbors that are in D. Give an example of a tree with 10 vertices that has a perfect code and an example with 10 vertices that does not have a perfect code.

33. Find a maximum sized matching in the graphs below.



34. Is it true that every tree has a perfect matching? Does any tree have more than one perfect matching?

6.1.2 Directed Graphs

1. Show that the edges of a k-chromatic graph can be oriented so that the resulting graph has a longest directed path of length $k - 1$.

2. The *converse* of a directed graph is obtained by reversing the orientation of each arc. Find a directed graph this is isomorphic to its converse. Find one that is not isomorphic to its converse.

3. Draw an acyclic tournament on six vertices.

6.2 Problems Requiring Proofs

1. Show that if u and v are the only odd-degree vertices in G, then there is a uv path in G.

2. • Let G be a 2-connected graph and u, v be two non-adjacent vertices in G. Show there must be at least 2 distinct paths between u and v.

3. • Show that in a 2-connected graph with at least 3 vertices, each pair of vertices u, v must lie on a common cycle.
(In fact, one can prove something much stronger than the previous two problems: that G is a k-connected graph, $k \geq 1$, if and only if, for all u, v, there are k different uv paths that have nothing in common other than the vertices u, v. This is known as Menger's Theorem.)

4. Use the contradiction method to prove that every simple undirected graph contains two vertices having the same degree.

5. Argue that every cycle in a bipartite graph contains an even number of edges.

6. Prove that a graph with n vertices and $\delta(G) = n - 1$ is connected. ($\delta(G)$ is the minimum degree).

7. Prove that every graph G contains an independent set (a set of vertices that induce a subgraph with no edges) with at least $\frac{n}{\Delta(G)+1}$ vertices, where $\Delta(G)$ is the largest vertex degree in G.

8. Prove that a tree with n vertices has $n - 1$ edges.

9. Let T be a tree in which the average vertex degree is k, that is, $\frac{\sum_{v \in V} deg(v)}{n} = k$. From the value k, can you deduce what n is?

10. Let T be a tree. Suppose we add two edges to T forming a graph T'. How many cycles can T' have?

11. Prove that an outerplanar graph (i.e., a planar graph that can be embedded in the plan with all the vertices bordering the exterior face)

can be colored with three colors.

12. Prove that any graph with n vertices and m edges has at least $m-n+1$ cycles.

13. • Let T be a tree. Prove that T^3 contains a Hamiltonian cycle (use induction).

14. Prove that if $G \cong \overline{G}$ and G has n vertices, then n is equal to either 0 or 1 modulo 4. Hint: Consider the number of edges G must have in this case and determine when this number can be an integer.

15. Use induction to prove that a tournament contains a Hamiltonian path.

16. [3] Prove (use contradiction) that a tournament contains a vertex v that is a "king," that is, there is a directed path of length at most 2 from v to each other vertex in the tournament.

17. Let f, e, v denote the number of faces, edges, and vertices, respectively, in a planar graph. Use induction on e to prove that $f = e - v + 2$ (this is known as Euler's formula).

18. Use induction to prove that K_n has $n(n-1)/2$ edges.

19. Prove that a graph with minimum vertex degree at least two must contain a cycle. (Hint: Consider a longest path in the graph and look at the ends of the path).

20. Prove that every graph with at least two vertices has at least two vertices that are not cut-vertices. (Hint: consider a spanning tree of the graph).

Chapter 7

Counting

Let $|A| = 12, |B| = 7, |C| = 10$.

1. If $|A \cap B| = 0$, how many ways can we choose two elements, one from A and one from B.

2. If $|A \cap B| = 4$, what is $|A \cup B|$?

3. If $|A \cap B| = 0$, $|A \cap C| = 0$, $|B \cap C| = 1$, how many ways can we choose three distinct elements, one from A and one from B and one from C?

4. If $|A \cap B| = 1$, how many ways can we choose three distinct elements from $A \cup B$?

5. Prove or disprove that $|A \cup B| + |A \cap B| = |A| + |B|$

6. How many bits are needed to express the integer n?

7. How many bits are needed to express the integer 2^n?

8. How many bit strings are there of length 10?

9. How many bit strings are there of length 10 that do not end in “111”

10. How many bit strings are there of length 6 are there that do not contain “1111” as a substring?

11. How many different SSN’s are there that do not contain any even digit?

12. [2] How many positive integers less that 1000 are (a) divisible by 7; (b) divisible by 7 but not by 11; (c) divisible by 7 and 11; (d) divisible by 7 or 11; (e) divisible by exactly one of 7, 11 (f) divisible by neither 7 nor 11; (g) have distinct digits; (h) have distinct digits and are even.

13. Repeat the previous question, but only consider three digit numbers.

14. How many different functions $f : \{0, 1, \ldots, n\} \to \{0, 1\}$ are there?

15. How many different one-to-one functions $f : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n+1\}$ are there?

16. Repeat the previous question, but require that $f(x) < f(x + 1)$ for all $0 \leq x < n$.

17. How many three digit numbers contain distinct digits? Have a digit repeated? Have consecutive digits that are the same?

18. How many 10 digit numbers have no two digits the same? How many 10 digit numbers have no two digits the same and do not start with 0 or 1?

19. On a multiple choice test with 100 questions and 5 answers per question, how many different ways can the test be completed?

20. On a multiple choice test with 100 questions and 5 answers per question, how many different ways can the test be completed if every answer is wrong?

21. On a multiple choice test with 10 questions and 5 answers per question, how many different ways can the test be completed if exactly 5 of the answers are wrong?

22. On a multiple choice test with 100 questions and 5 answers per question, how many different ways can the test be completed if no two consecutive answers are ever the same?

23. On a multiple choice test with 98 questions and 5 answers per question, explain why some answer must occur at least 20 times on the answer key.

24. How many people must be in a room to ensure at least two were born on the same day of the week?

25. How many people must be in a room to ensure at least three were born on the same day of the week?

26. How many people must be in a room to ensure at least two were born on a Monday?

27. How many people must be in a room to ensure that either (i) at least two were born on a Monday or (ii) at least three were born on a day other than Monday.

28. Suppose four disjoint sets contain 15 items in total. Enumerate the possible cardinalities of these sets provided that no single set contains more than five items.

29. What are the permutations of the letters a, b, c, d? How many of these permutations have a preceding b? How many end with ab?

30. How many ways can we choose 5 items from a box containing 10 items? 2 items from a box containing 10 items? 8 items from a box containing 10 items?

31. How many ways can we choose 3 numbers from the set $\{1, 2, 3, 4, 5, 6, 7\}$ (and the order we choose matters: so 1, 2, 3 is different from 2, 3, 1).

32. Repeat the previous question, but this time the order does not matter.

33. How many ways can we choose 3 numbers from the set $\{1, 2, 3, 4, 5, 6, 7\}$ so that the numbers are chosen in increasing order.

34. How many ways can we choose 3 numbers from the set $\{1, 2, 3, 4, 5, 6, 7\}$ so that 7 is chosen.

35. (a) How many ways can we choose 3 numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ so that more odd numbers are chosen than even?
(b)How many ways can we choose 3 numbers from the set $\{1, 2, 3, 4, 5, 6, 7\}$ so that more odd numbers are chosen than even?

36. How many bit strings of length 8 contain at least three 0's?

37. How many bit strings of length 8 contain at least three 0's and at least two 1's?

38. How many bit strings of length 8 contain an equal number of 0's and 1's?

39. How many bit strings of length 8 contain more 0's than 1's?

40. How many paths are there of length 5 are there between two distinct vertices in the complete graph K_{10}?

41. How many paths are there between two distinct vertices in the complete graph K_n?

42. Let K_n be such that the vertices are labelled $1, 2, 3, \ldots, n$. How many paths are there between v_1 and v_n such that the labels on the path are strictly increasing?

43. How many different induced subgraphs are there of K_n that contain at least one edge?

44. If $|A \cap B| = 2$, $|A| = 8, |B| = 7$, how many ways can we one from A and one from B so that we do not choose both elements contained in $A \cap B$? (Hint: the answer is not 55).

45. Suppose we have 10 different men and 2 different women. How many ways can we seat them on a row of seats so that the two women sit next to each other?

46. Suppose we have 10 different men and 13 different women. How many ways can we seat them on a row of seats so that no two women sit next to each other?

47. • Suppose we have 10 different men and 3 different women. How many ways can we seat them on a row of seats so that no two women sit next to each other?

48. • Suppose we have 12 different men and 7 different women. How many ways can we seat them around a circular table so that no two women sit next to each other? (Note: it may help to assume the seats are numbered)

49. On a multiple choice test with 100 questions and 2 answers per question, how many different ways can the test be completed if no two consecutive answers are ever the same?

50. • How many ways can 3 indistinguishable balls be placed into 3 boxes if (let $x - y - z$ denote the number of balls in each of the four bins) :
(a) For example, 2-1-0 is different from 0-1-2
(b) For example, 2-1-0 is the same as 0-1-2, i.e, any two arrangements are the same if they have the same numbers, in any order.
(c) Generalize your answer to n balls and n boxes for both parts (a) and (b). [For part a), work out the first few terms in the sequence and consider looking in the Online Encyclopedia of Integer Sequences.] [For part b, the answer is equivalent to the number of "integer partitions" of n, why?].

51. • How many ways can 4 indistinguishable balls be placed into 3 boxes if (let $x - y - z$ denote the number of balls in each of the four bins) :
(a) For example, 2-1-1 is different from 1-2-1
(b) For example, 2-1-1 is the same as 1-1-2, i.e, any two arrangements are the same if they have the same numbers, in any order.
(c) Generalize your answer to $n+1$ balls and n boxes for both parts (a) and part (b) [For part a), work out the first few terms in the sequence and consider looking in the Online Encyclopedia of Integer Sequences.] [For part b, answer is related to number the number of "integer partitions" of $n + 1$.

52. • Suppose cards come in four varieties: Spaces, Clubs, Hearts, and Diamonds. We assume cards are not numbered. A *hand* consists of 5 cards dealt from a deck containing these cards. The order of the cards in a hand does not matter.

(a) Suppose 5 cards are dealt from an infinite deck. How many different hands are there?

(b) Continuing part (a), how many of these hands have exactly 3 spades?

(c) Continuing part (a), how many of these hands have at least 3 spades?

(d) Suppose 5 cards are dealt from a 52 card deck. How many different hands are there?

(d) Continuing part (b), how many of these hands have exactly 3 spades?

(e) Continuing part (b), how many of these hands have at least 3 spades?

(f) Continuing part (b), how many of these hands have at least 3 cards of the same variety?

53. How many ways can a $2 \times n$ board be tiled with 1×2 tiles?

54. Consider a rectangular table with n chairs on each side. How many ways can n married couples sit at the table to that each couple sits either beside each other or directly across from each other?

55. Let $f : A \to B$, with $|B| = 2$. How many different f's are there? How many different f's are there that are onto functions? How many are onto if $|B| = 3$?

56. How many different ways can a team win a best 4 out of 7 series of games? (A team must win 4 games; they might win 4 games to 0 or 4 games to 3; the order of wins matters in this problem).

57. We want to make flags with horizontal stripes. How many different flags can we make if:

    a) We have 3 colors and a flag has 3 stripes, all must be different colors
    b) We have 6 colors and a flag has 6 stripes, all must be different colors
    c) Repeat a and b, but assume two flags are the same if they have the

same colors in reverse order (so ABC = CBA, for example)
d) We have 6 stripes, but only 3 colors.
e) Same as d), but assume each color must be used exactly twice.
f) We have 3 colors and allow a color to be used any number of times. How many different flags can we make (and assume two flags are the same if they have the same colors in reverse order).
g) Same as f) but with 4 colors.

58. Suppose we have six 3 cent stamps and 7 five cent stamps. How many different amounts of postage can we make?

59. Suppose we have three 3 cent stamps and 7 nine cent stamps. How many different amounts of postage can we make?

60. Suppose we have six 3 cent stamps and 7 nine cent stamps. How many different amounts of postage can we make?

61. Suppose we have six 9 cent stamps and 7 twelve cent stamps. How many different amounts of postage can we make?

62. How many length 8 bit strings have consecutive 0's? (Hint: is probably easier to count those without consecutive 0's).

63. How many two-letter strings (lowercase letters only) contain at least one of $\{a, b, c, d\}$?

64. How many different ways can you distribute 4 different cookies to 4 different people: (a) so that each person gets at least one cookie (b) so that each person may get any number of cookies?

Chapter 8

Other Topics

8.1 Relations

8.2 Algorithm Analysis

8.3 Recurrence Relations

8.4 Generating Functions

8.5 Boolean Algebra

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