



# BLM2502

# Theory of

# Computation

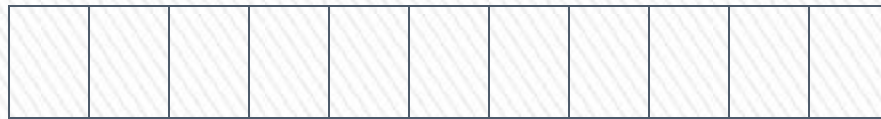


# Pushdown Automata

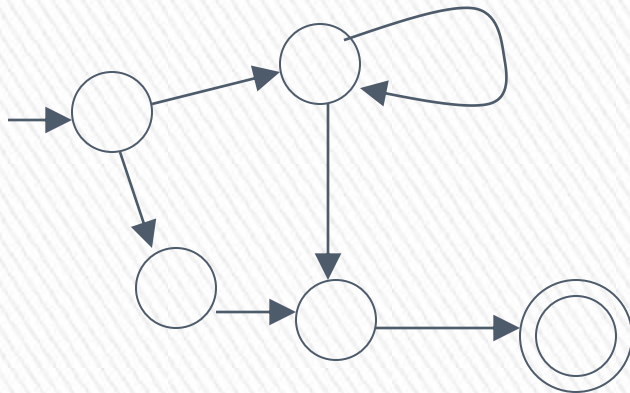
## PDA

# Pushdown Automaton -- PDA

Input String



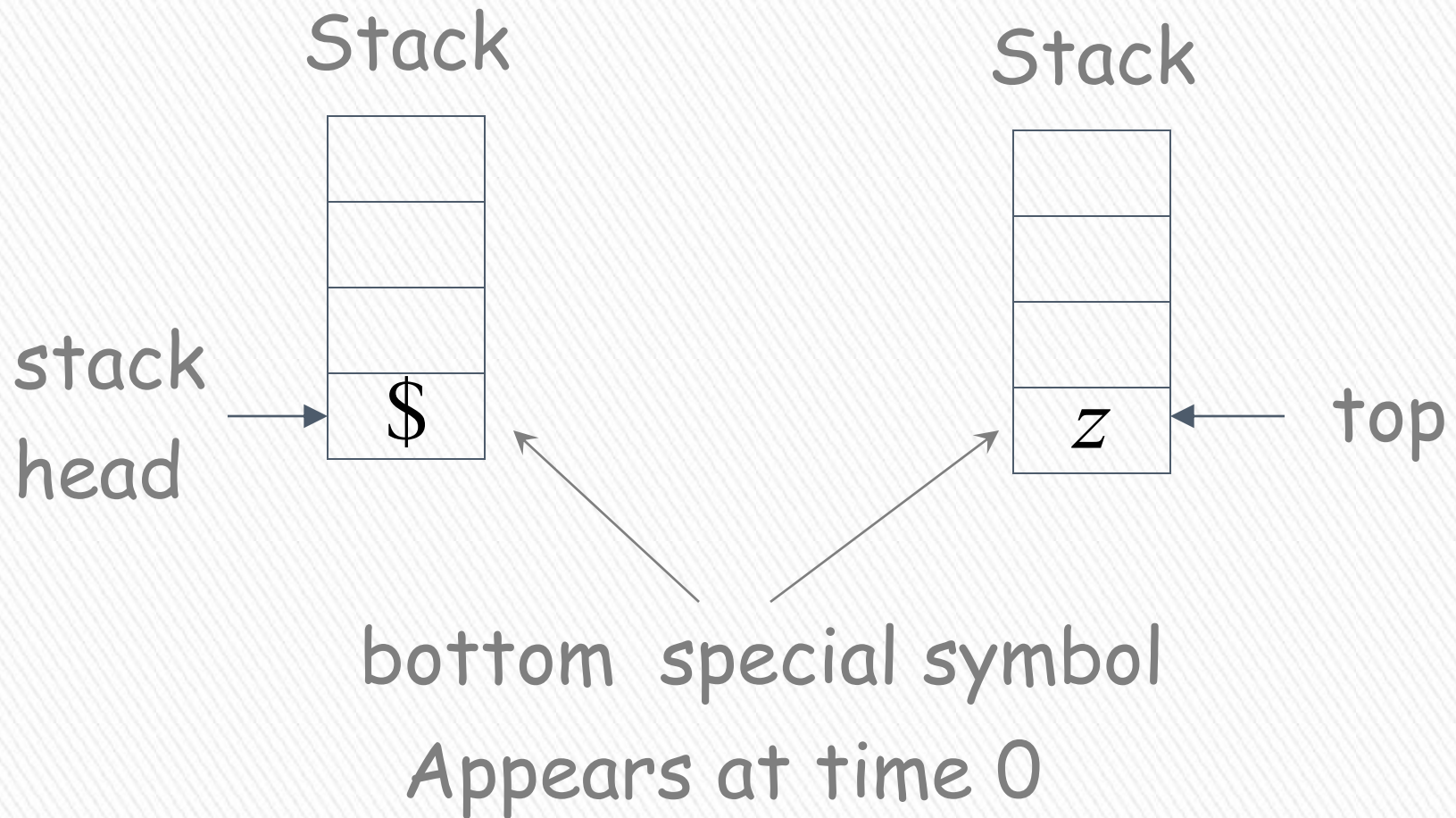
States



Stack



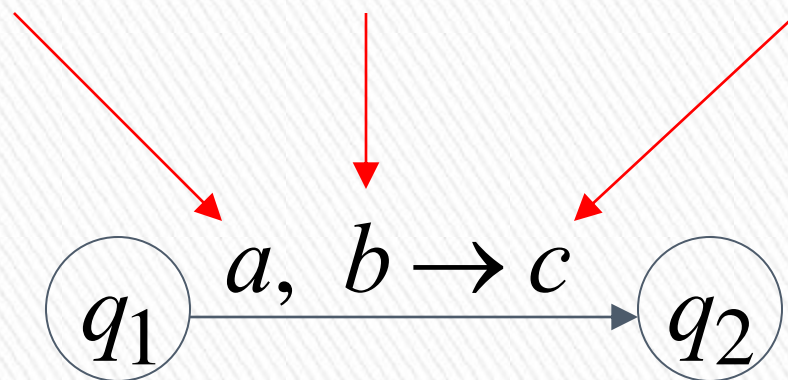
# Initial Stack Symbol



Input  
symbol

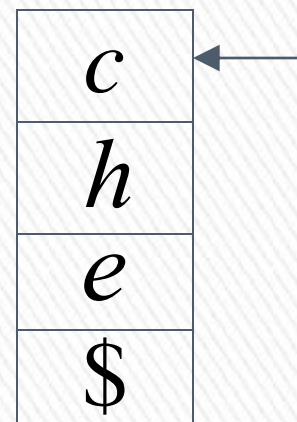
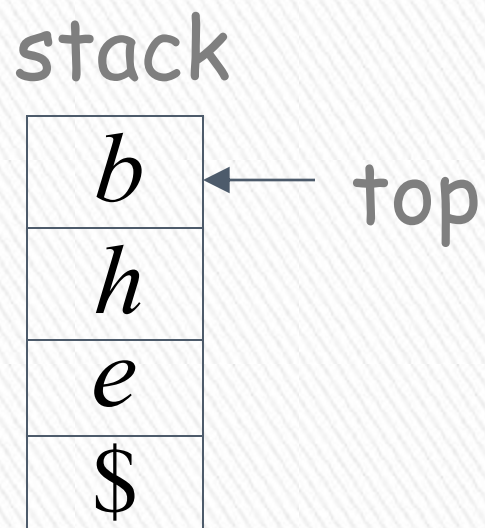
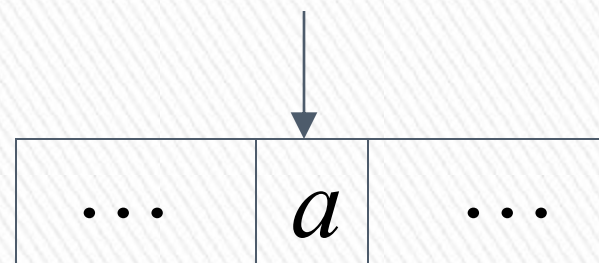
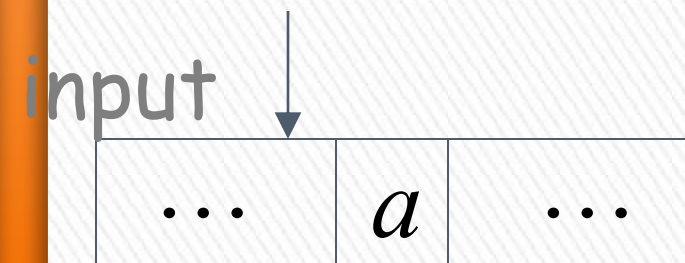
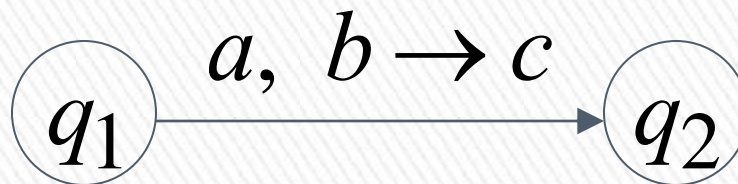
Pop  
symbol

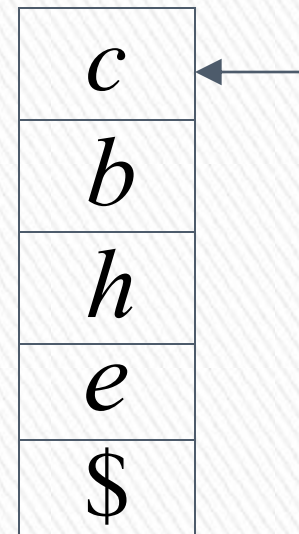
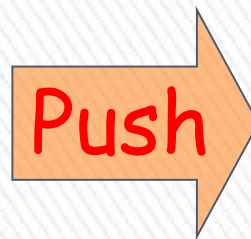
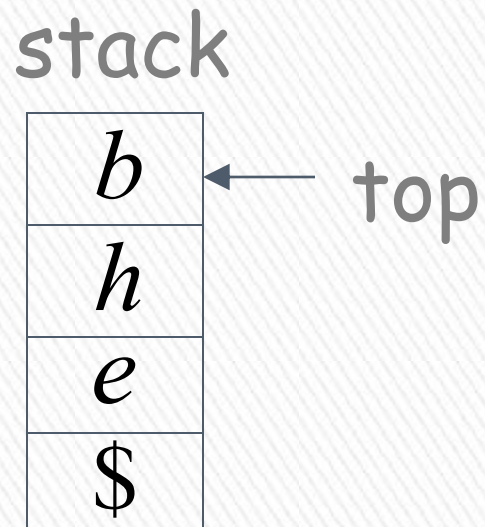
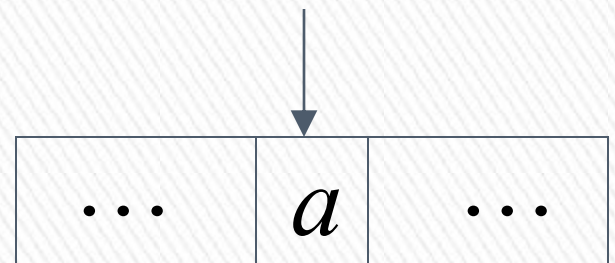
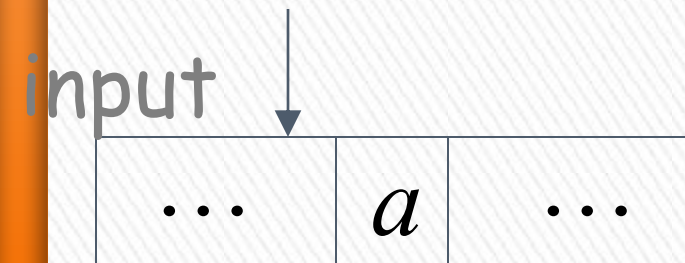
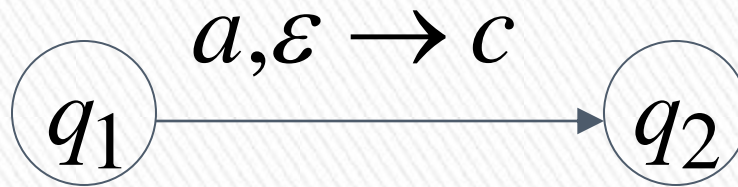
Push  
symbol



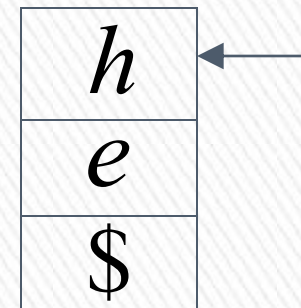
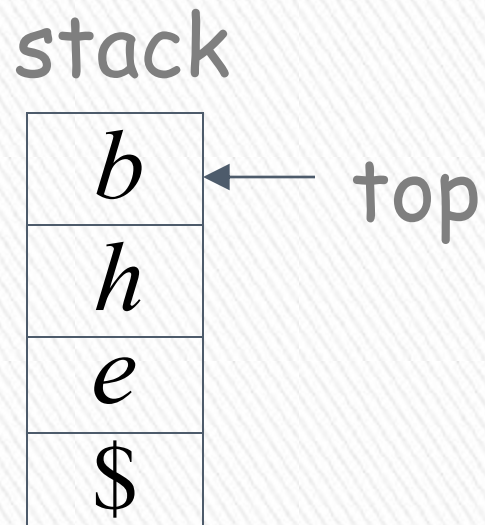
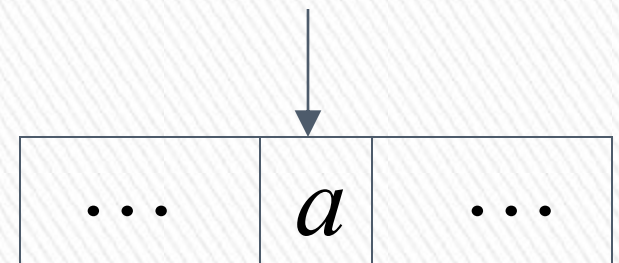
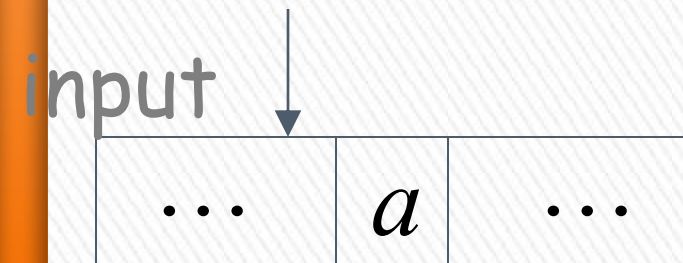
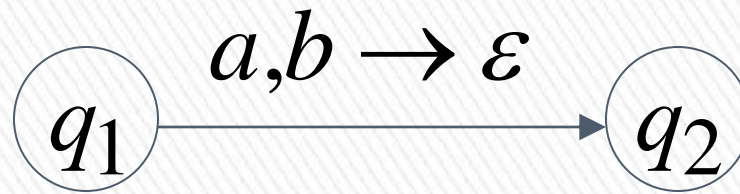
The States



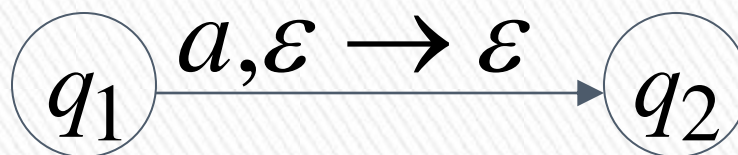




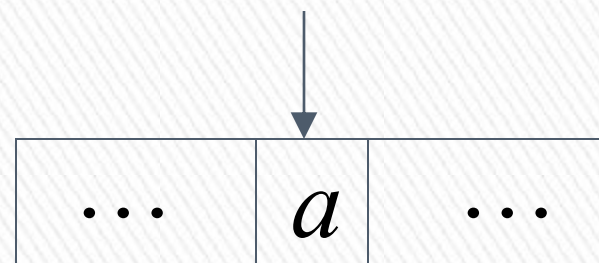
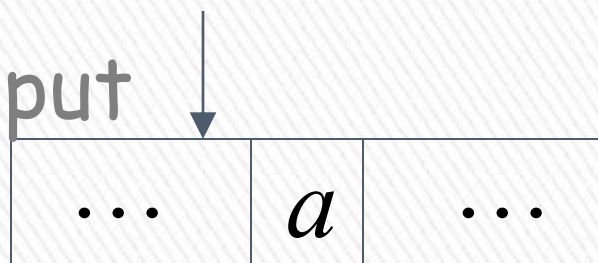




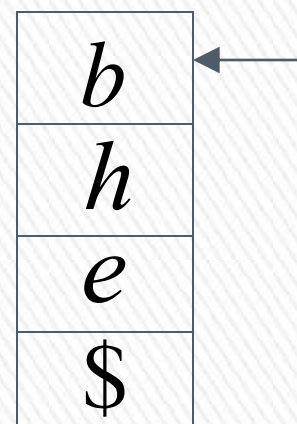
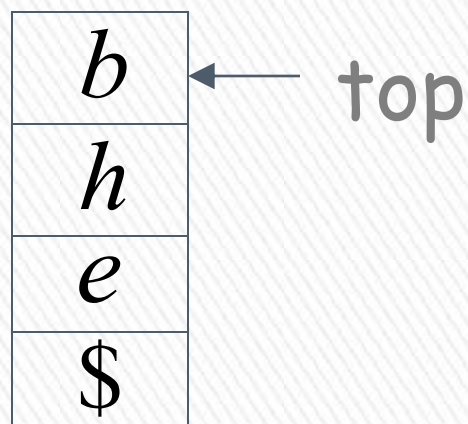




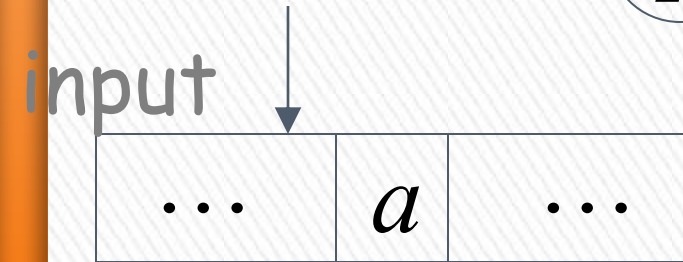
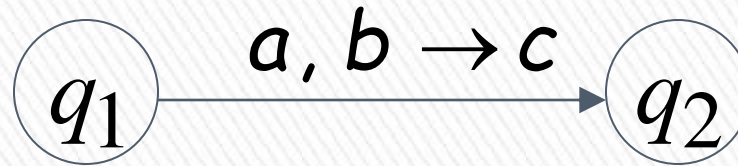
input



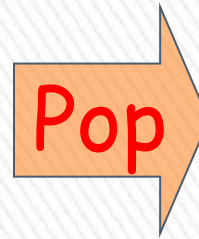
stack



# Pop from Empty Stack



stack



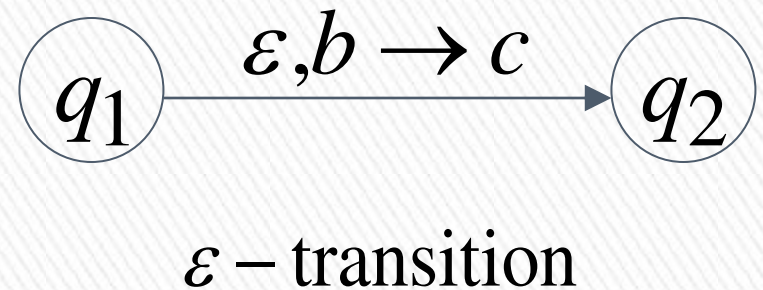
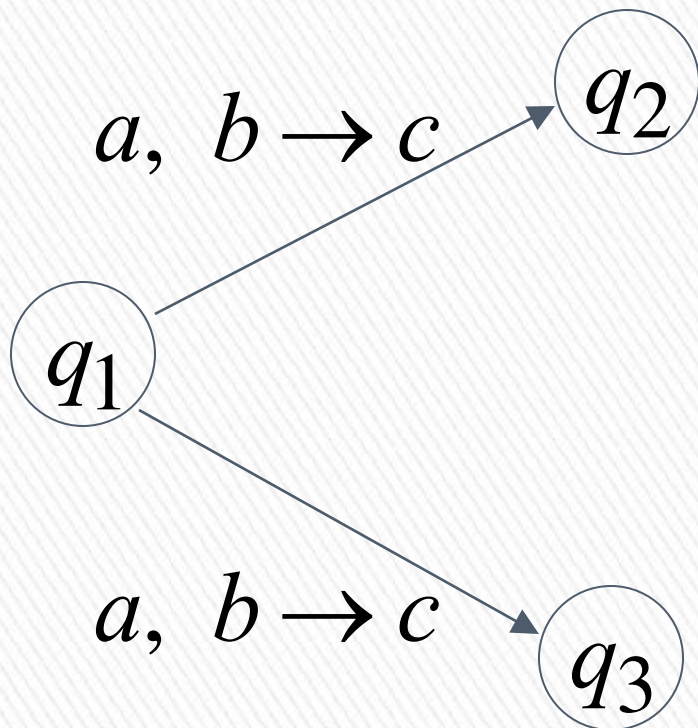
Automaton halts!

If the automaton attempts to pop from empty stack then it halts and rejects input

# Non-Determinism

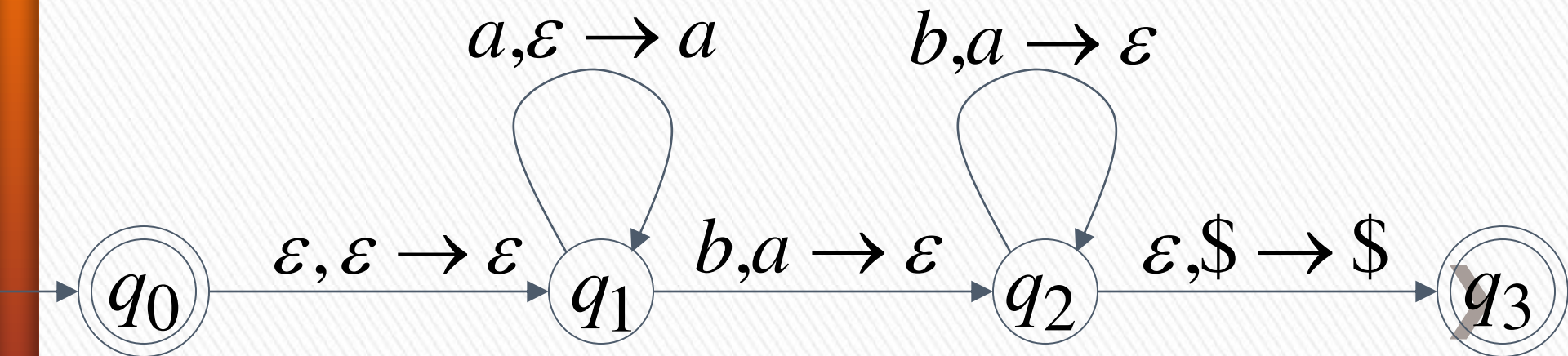
PDAs are non-deterministic

Allowed non-deterministic transitions



# Example PDA

PDA  $M$ :  $L(M) = \{a^n b^n : n \geq 0\}$



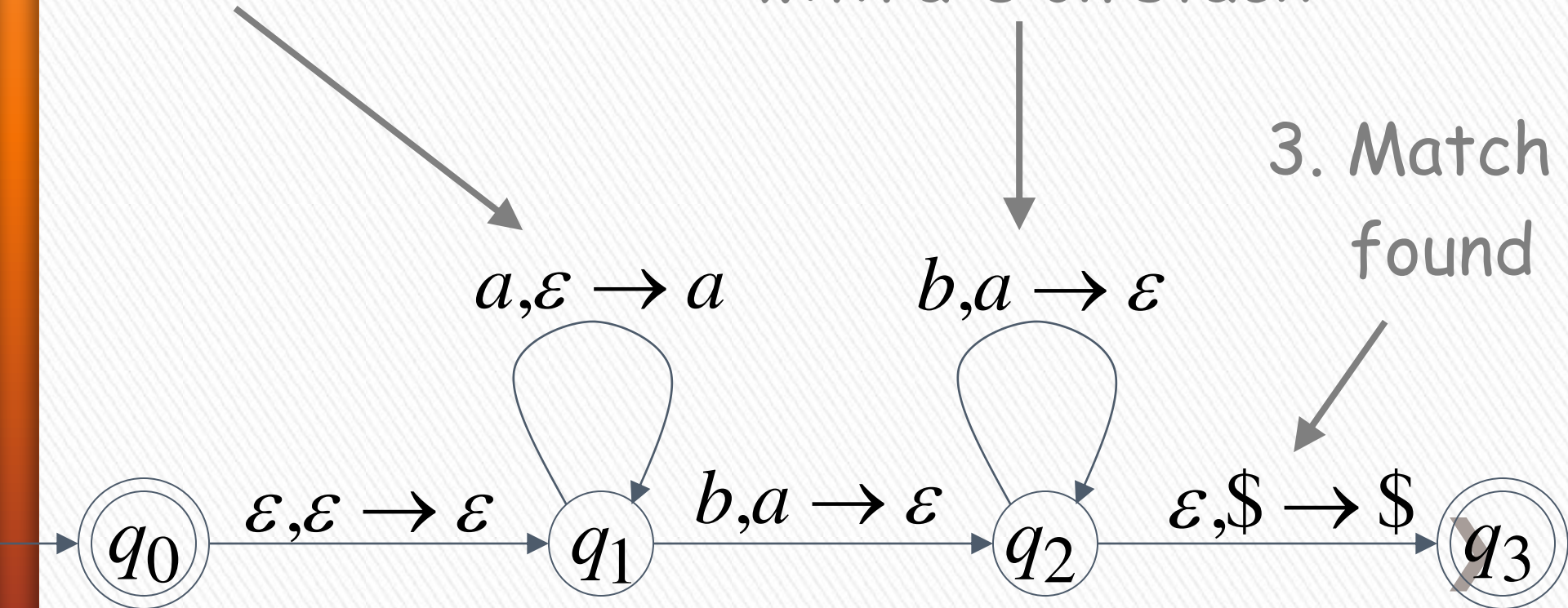
$$L(M) = \{a^n b^n : n \geq 0\}$$

## Basic Idea:

1. Push the a's on the stack

2. Match the b's on input with a's on stack

3. Match found



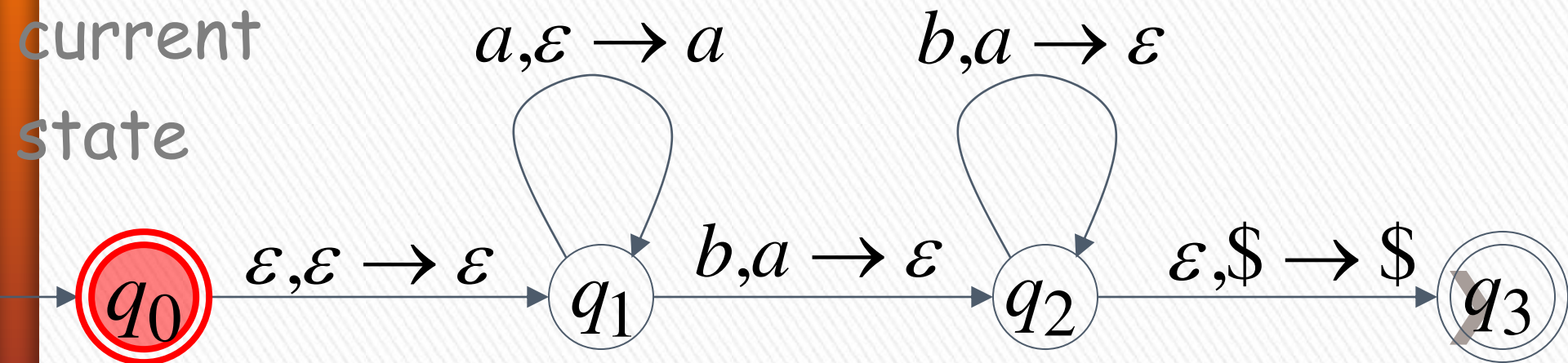
# Execution Example: Time 0

Input



Stack

current  
state



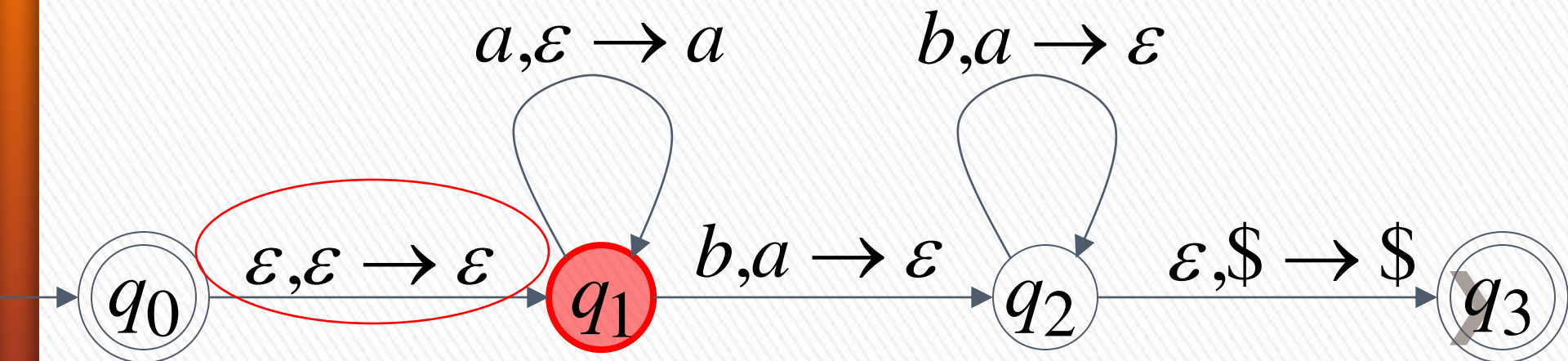


Time 1

Input



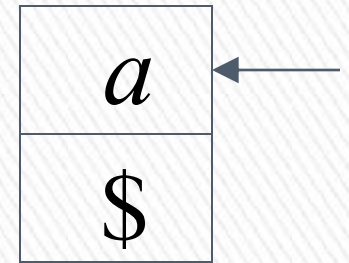
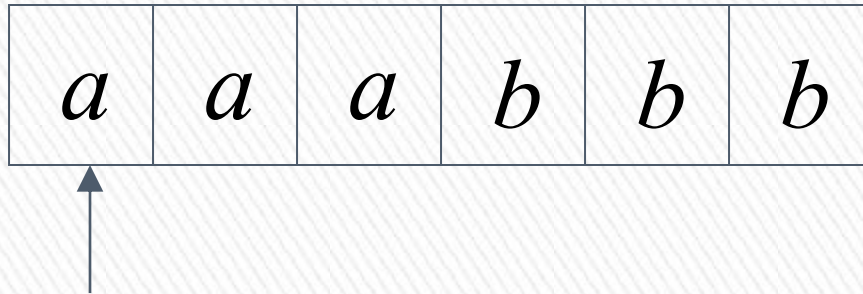
Stack



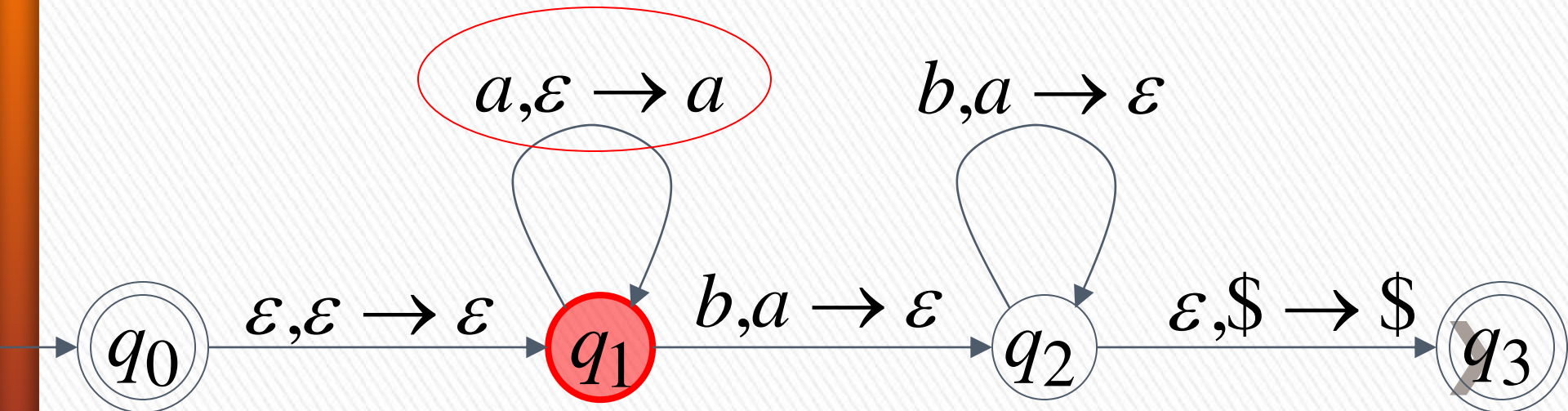


Time 2

Input

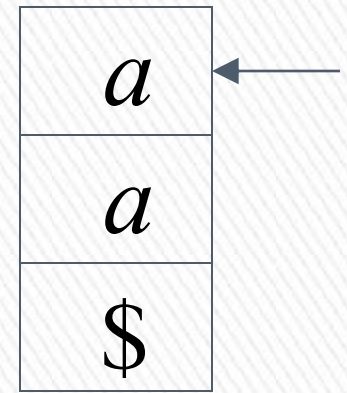
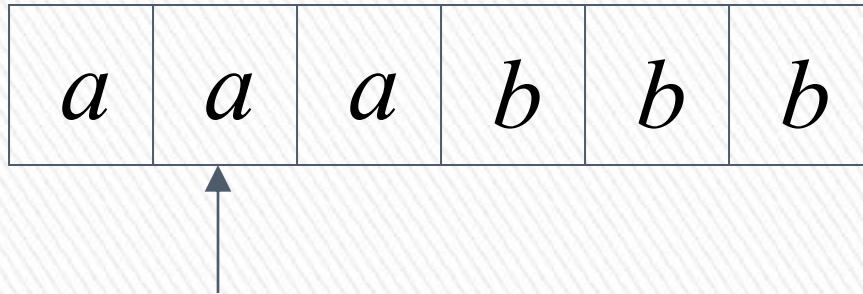


Stack

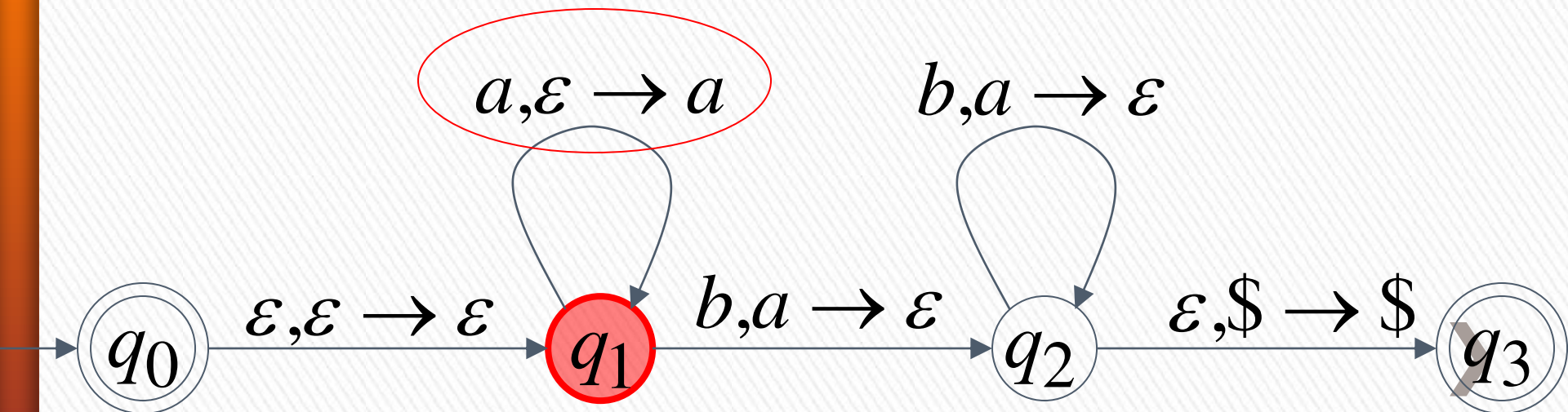


Time 3

Input

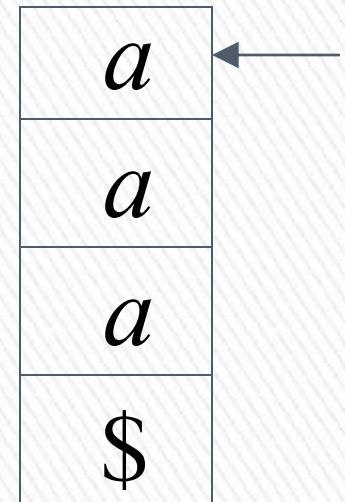


Stack

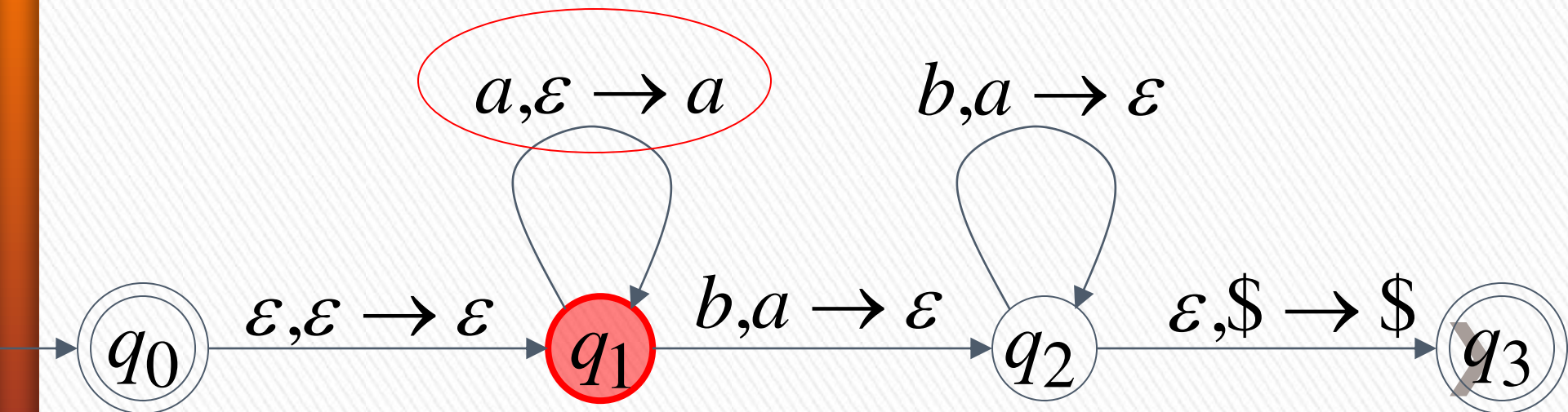


Time 4

Input

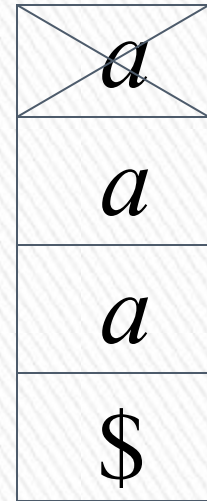


Stack

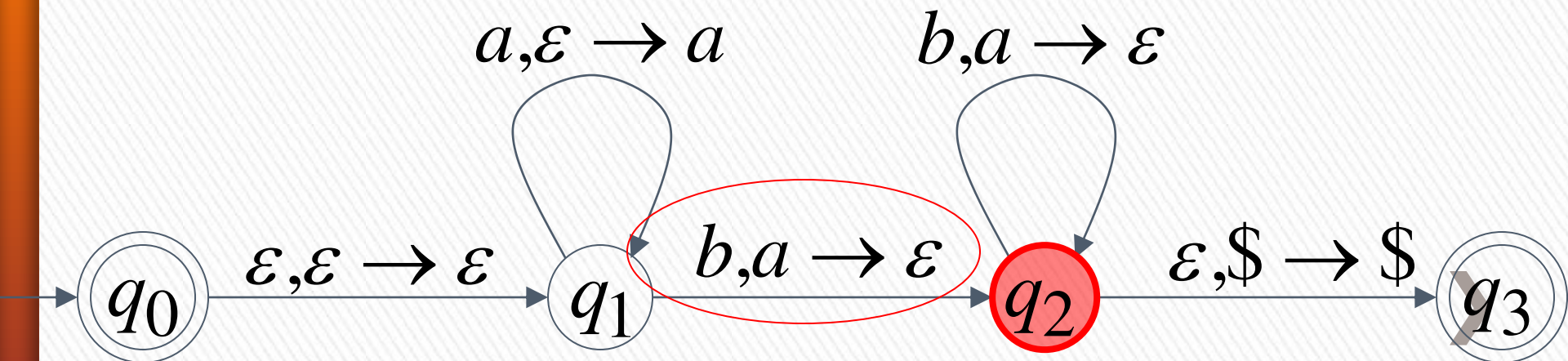


Time 5

Input

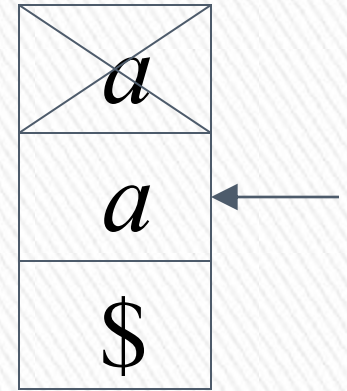


Stack

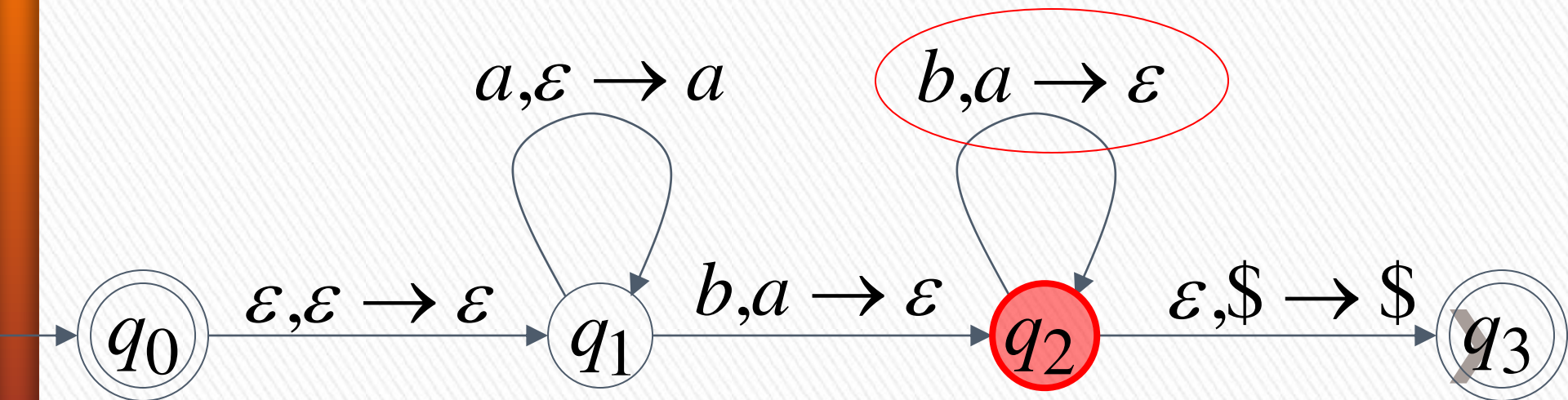


Time 6

Input

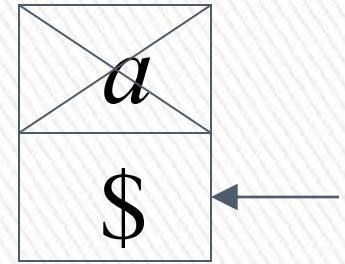
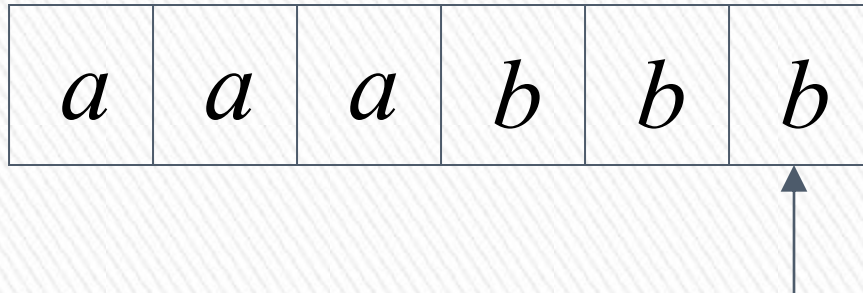


Stack

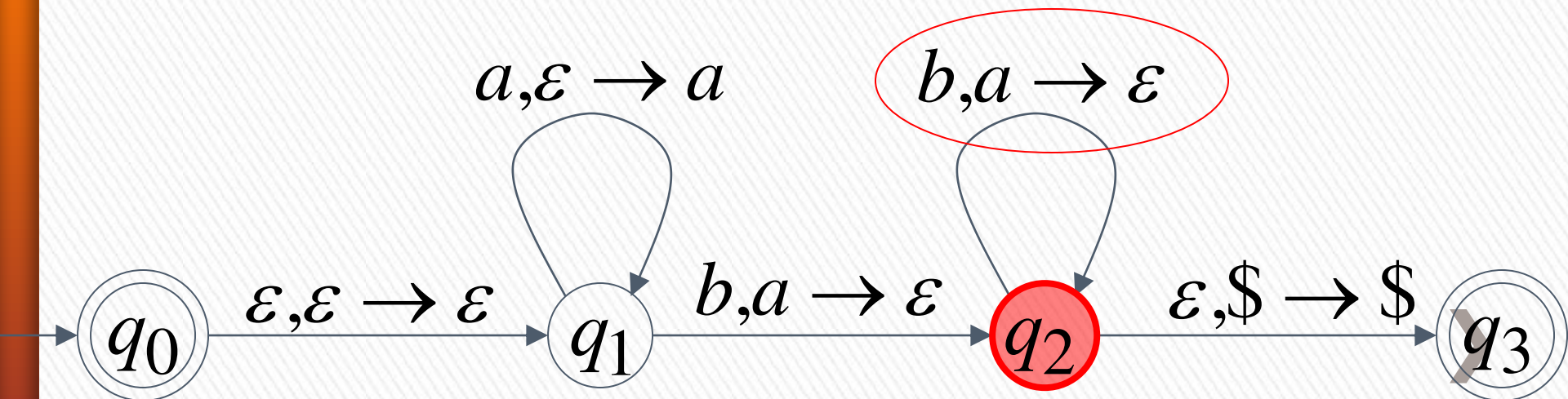


Time 7

Input

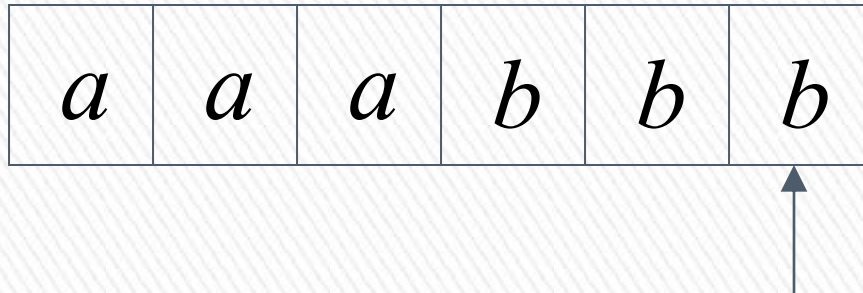


Stack

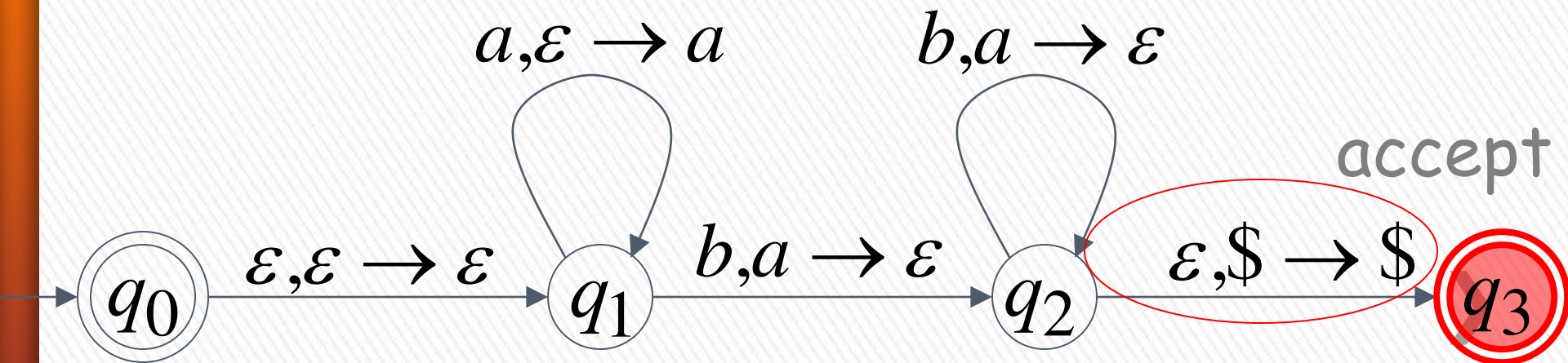


Time 8

Input



Stack





A string is accepted if there is  
a computation such that:

All the input is consumed

**AND**

The last state is an accepting state

we do not care about the stack contents  
at the end of the accepting computation ➤

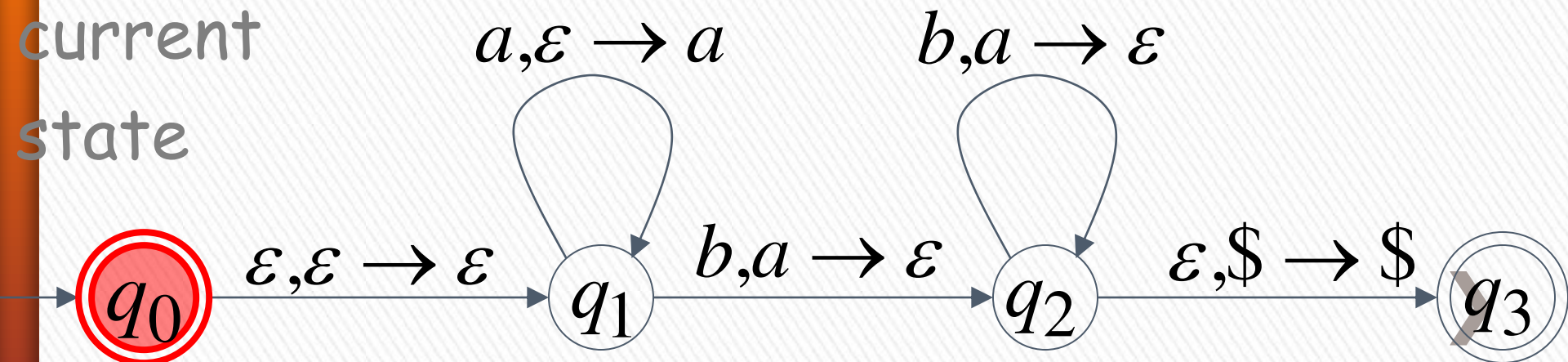
# Rejection Example: Time 0

Input



Stack

current  
state



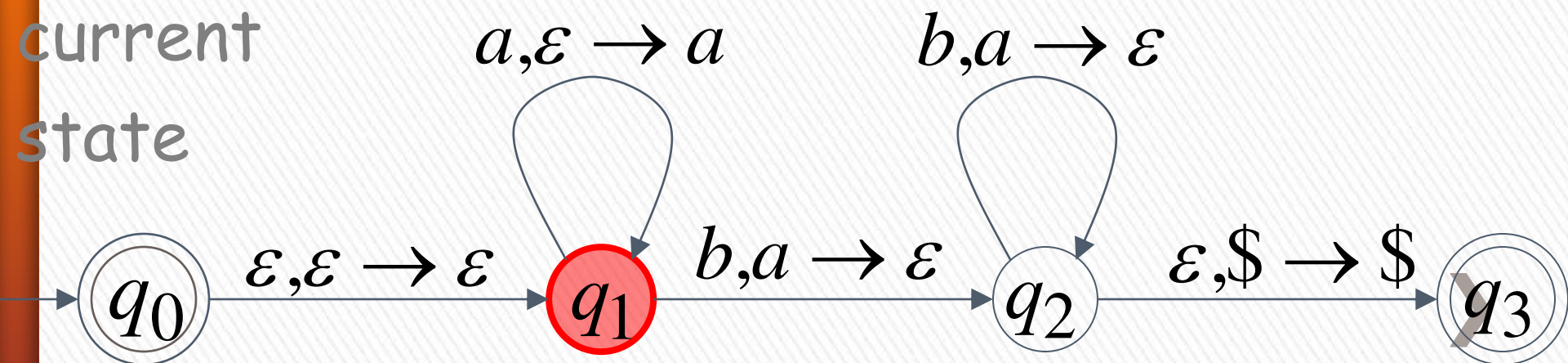
# Rejection Example: Time 1

Input



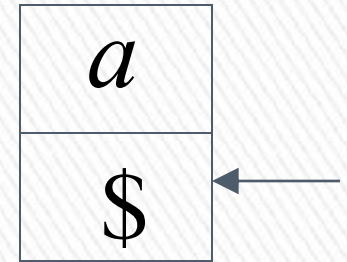
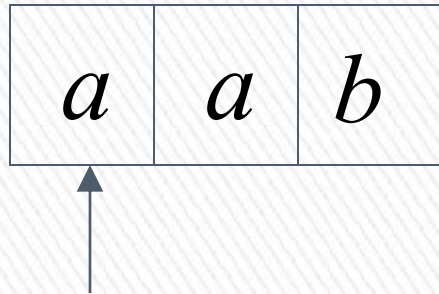
Stack

current  
state



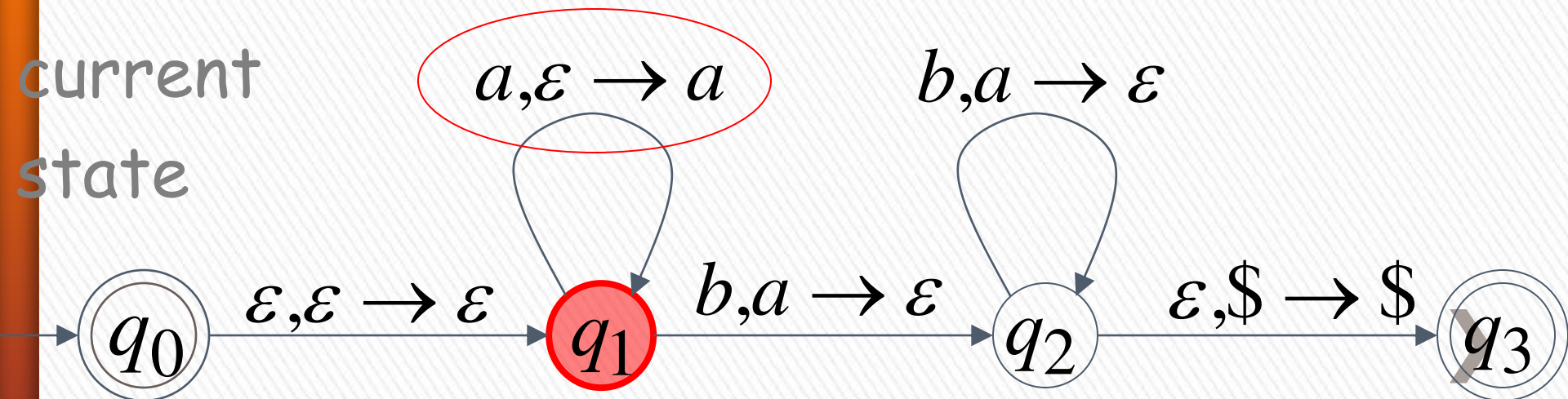
# Rejection Example: Time 2

Input



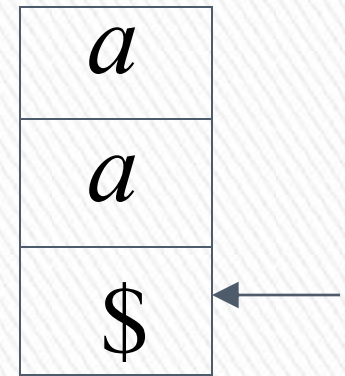
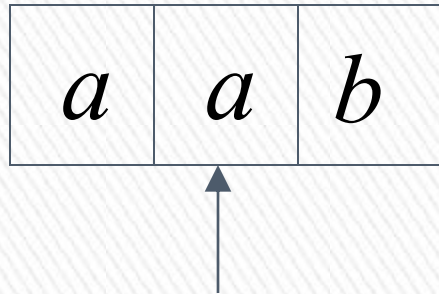
Stack

current  
state



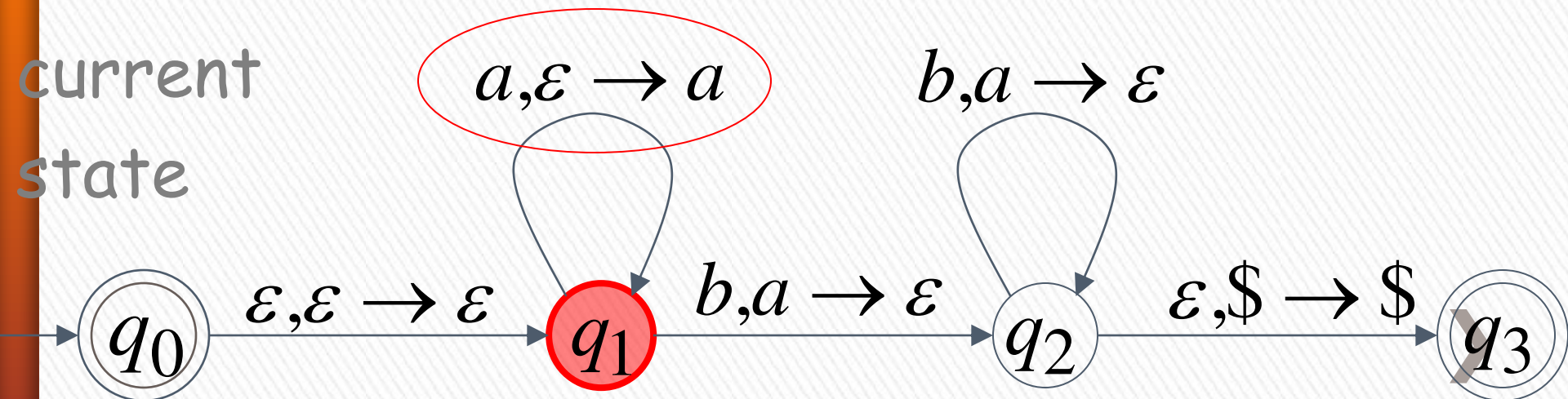
# Rejection Example: Time 3

Input



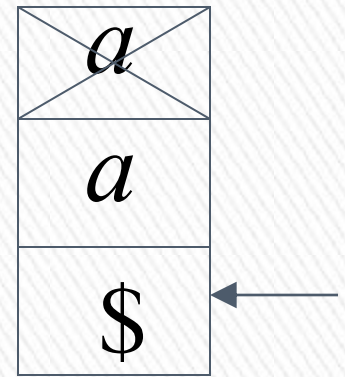
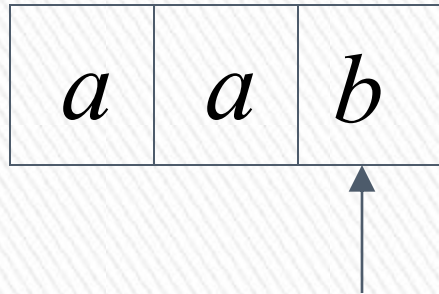
Stack

current  
state



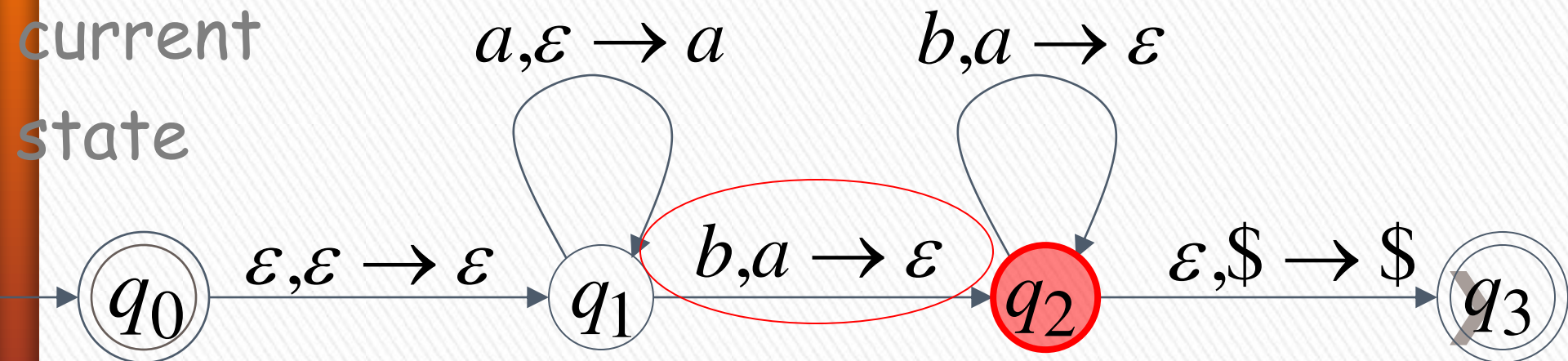
# Rejection Example: Time 4

Input



Stack

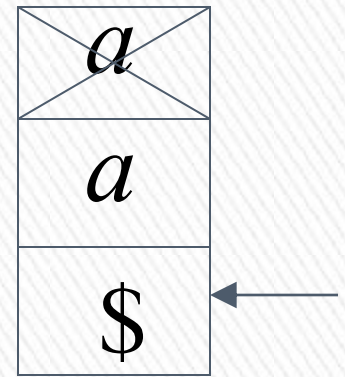
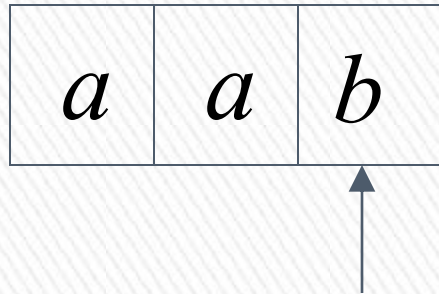
current  
state





# Rejection Example: Time 4

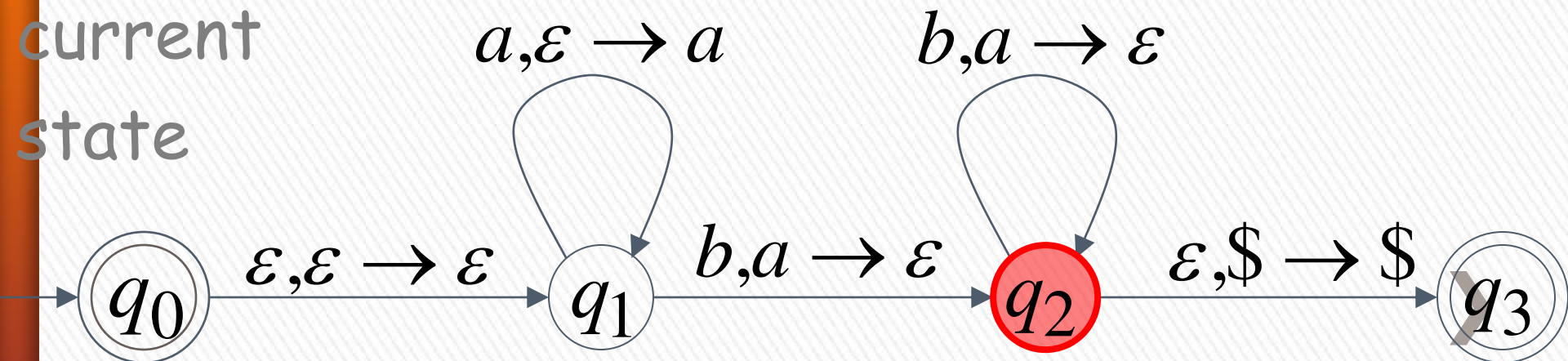
Input



Stack

reject

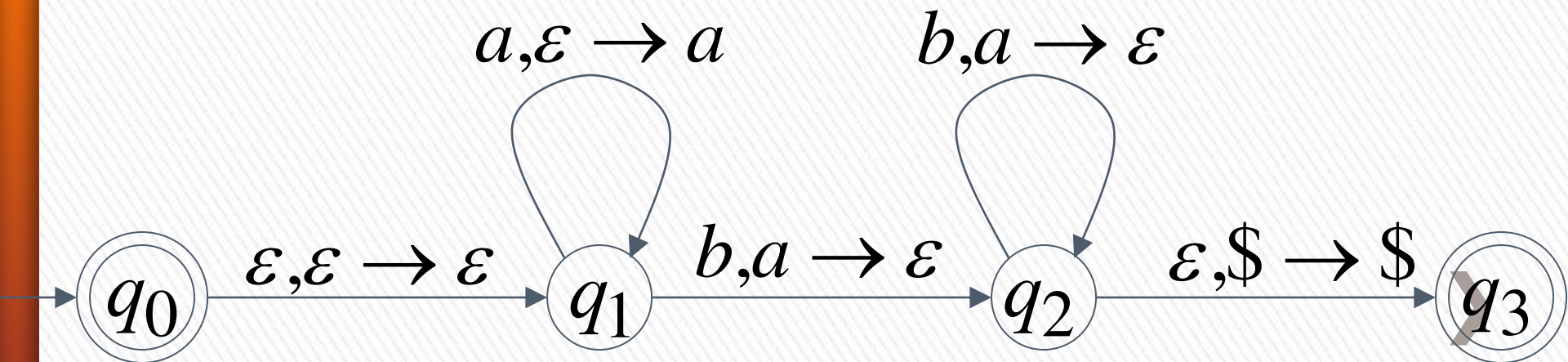
current  
state





There is no accepting computation for  $aab$

The string  $aab$  is rejected by the PDA



# Another PDA example

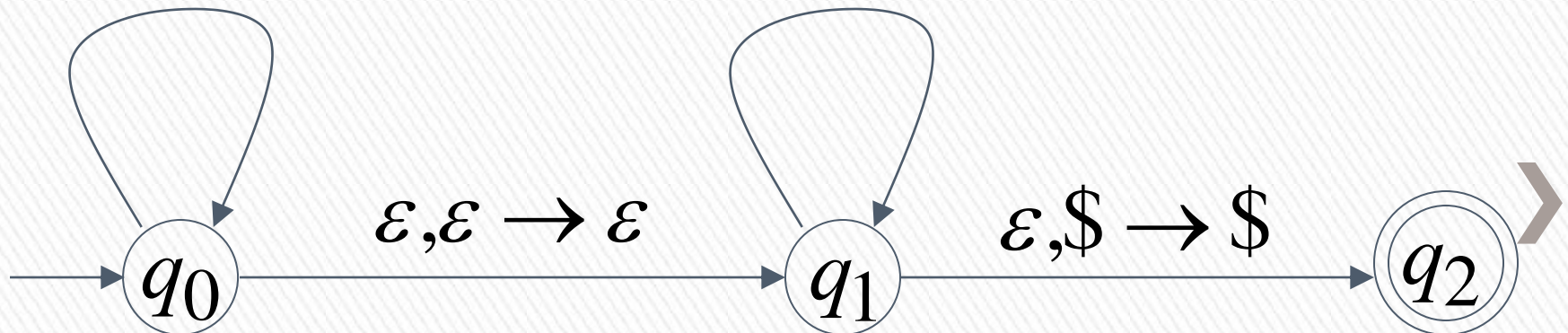
PDA  $M$ :  $L(M) = \{vv^R : v \in \{a,b\}^*\}$

$a, \varepsilon \rightarrow a$

$a, a \rightarrow \varepsilon$

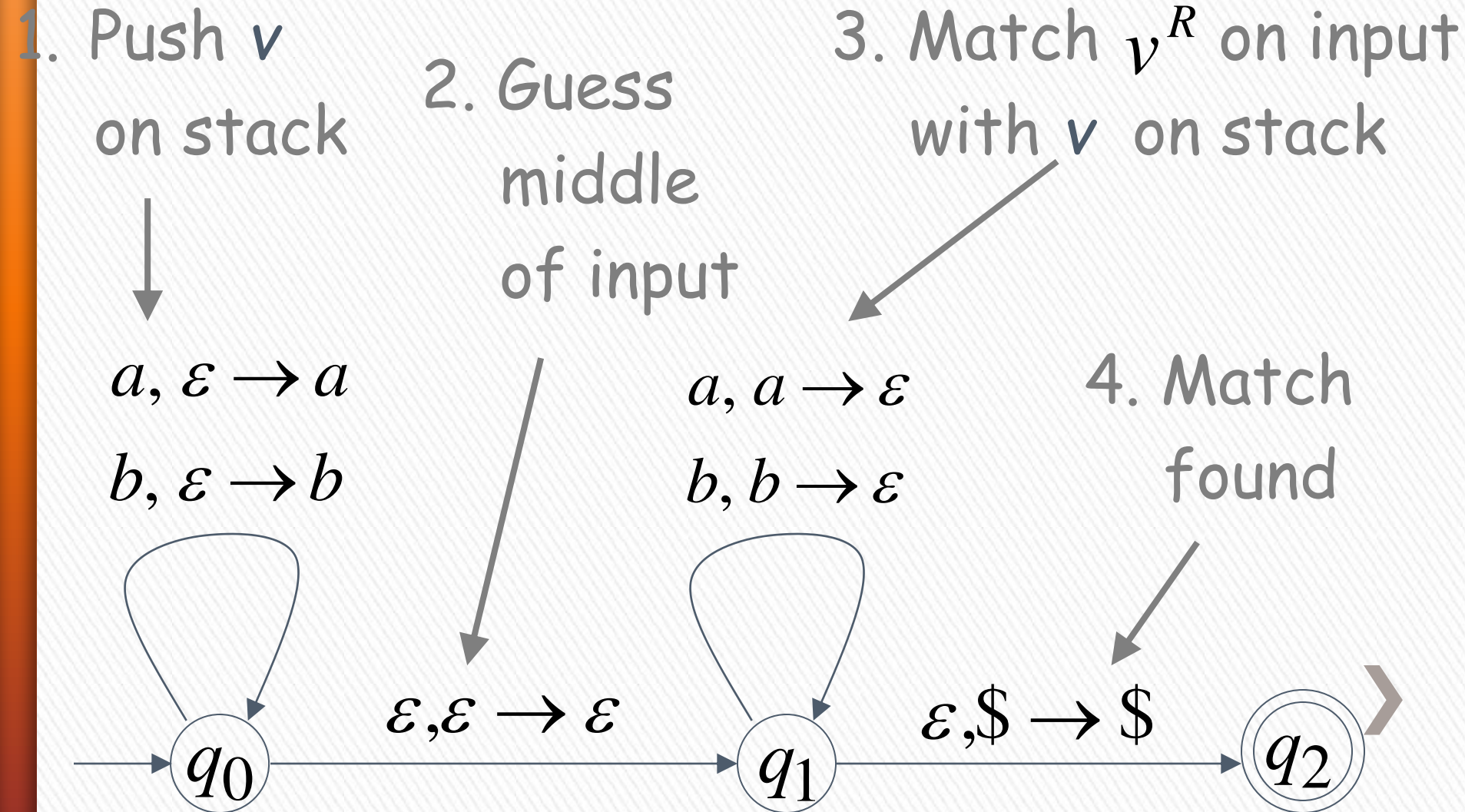
$b, \varepsilon \rightarrow b$

$b, b \rightarrow \varepsilon$



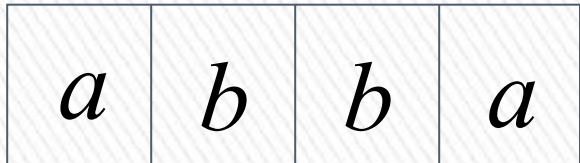
## Basic Idea:

$$L(M) = \{vv^R : v \in \{a,b\}^*\}$$



# Execution Example: Time 0

Input



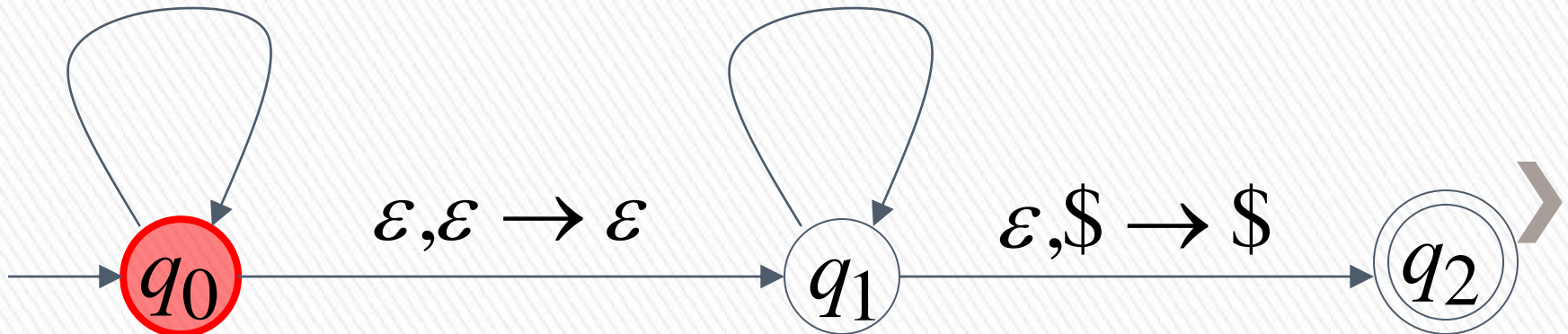
Stack

$a, \varepsilon \rightarrow a$

$a, a \rightarrow \varepsilon$

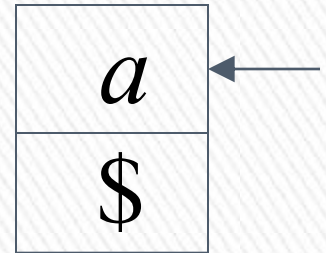
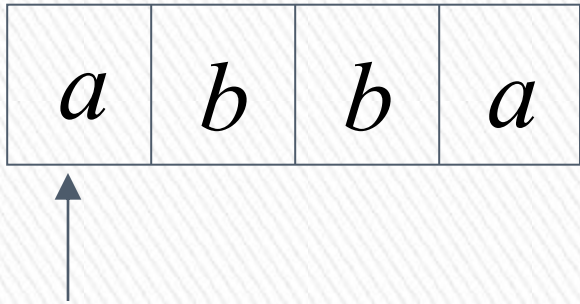
$b, \varepsilon \rightarrow b$

$b, b \rightarrow \varepsilon$



Time 1

Input



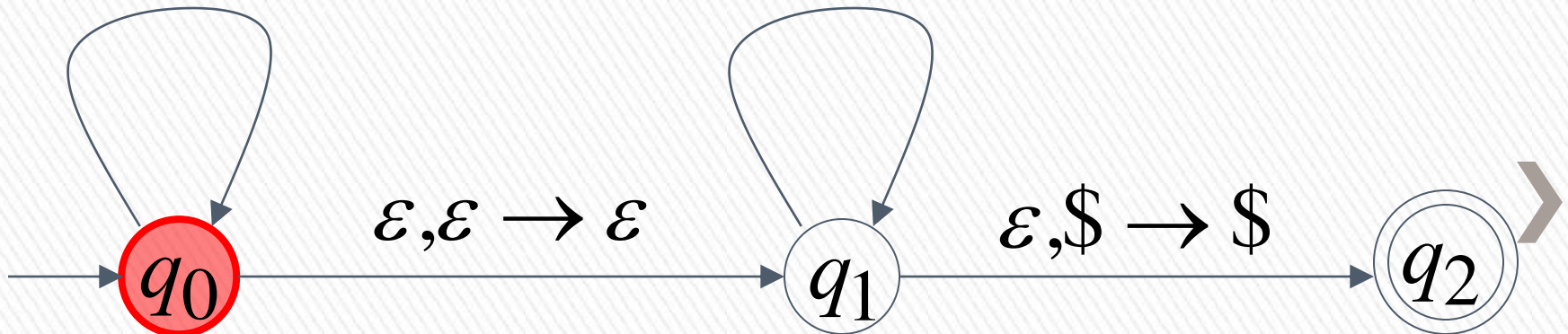
Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

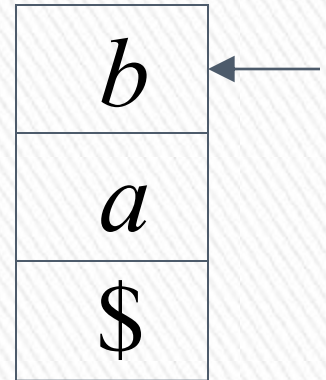
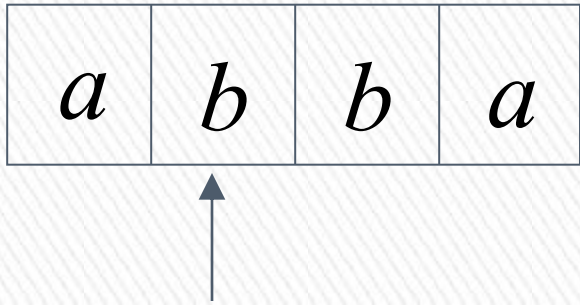
$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$



## Time 2

Input



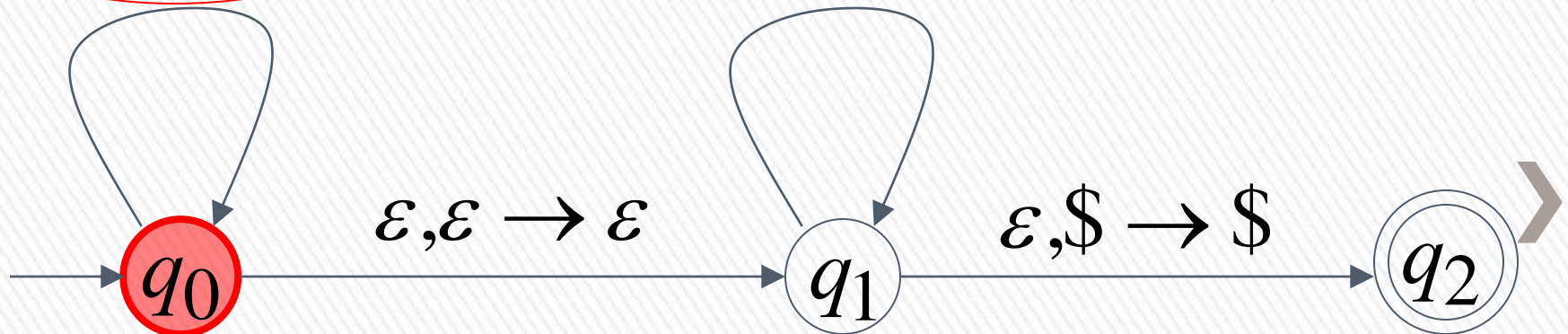
Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

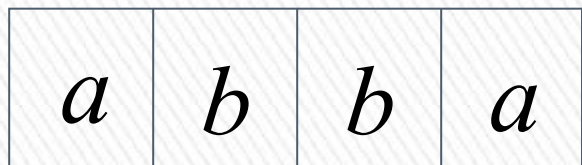
$b, b \rightarrow \varepsilon$



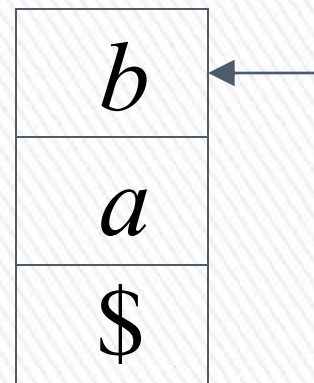


# Time 3

Input



Guess the middle  
of string



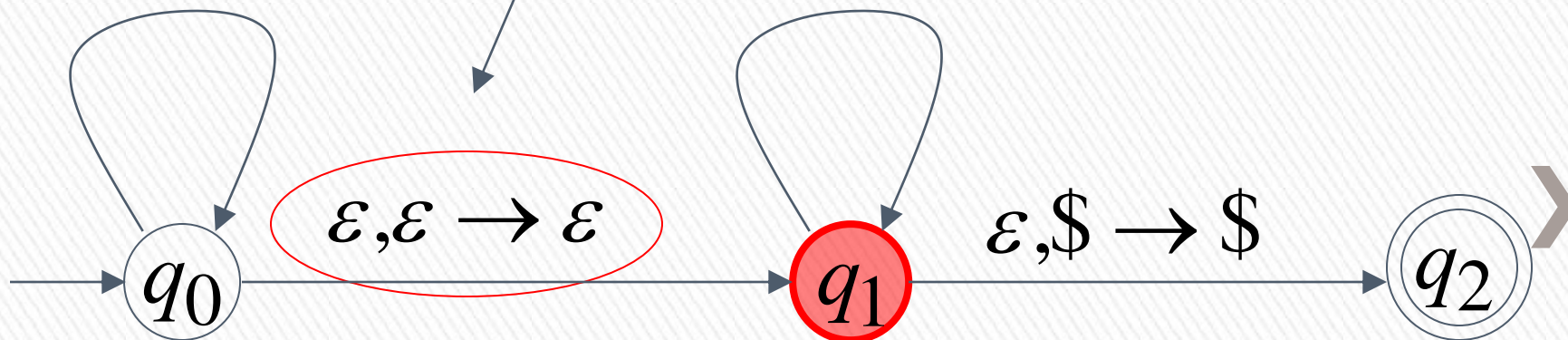
Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

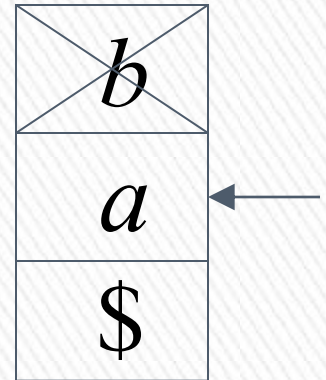
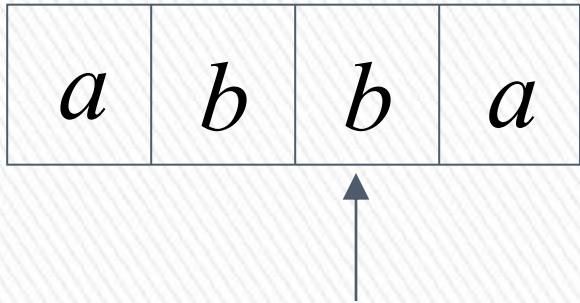
$b, b \rightarrow \varepsilon$





# Time 4

Input



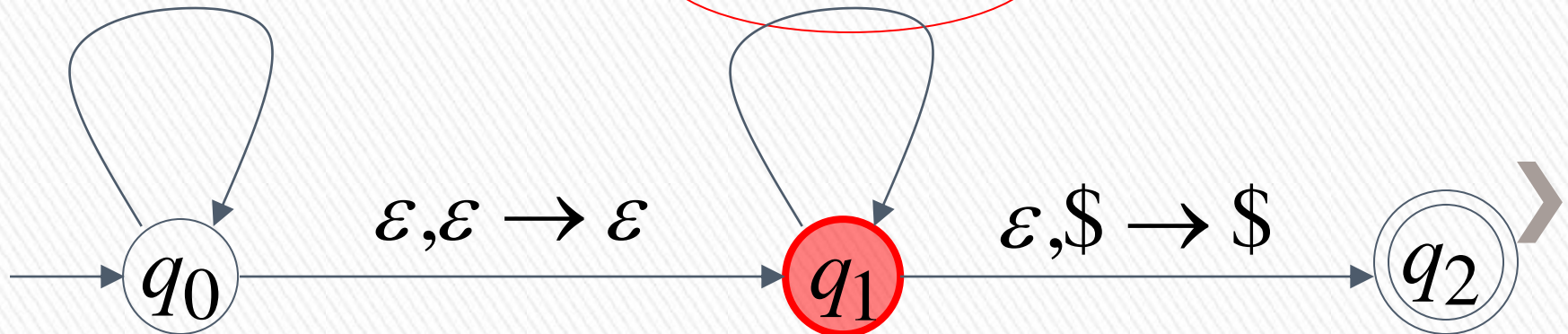
Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

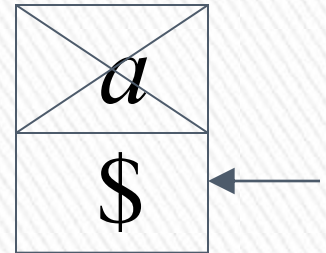
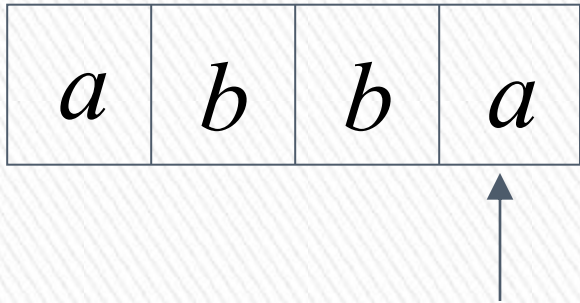
$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$



Time 5

Input



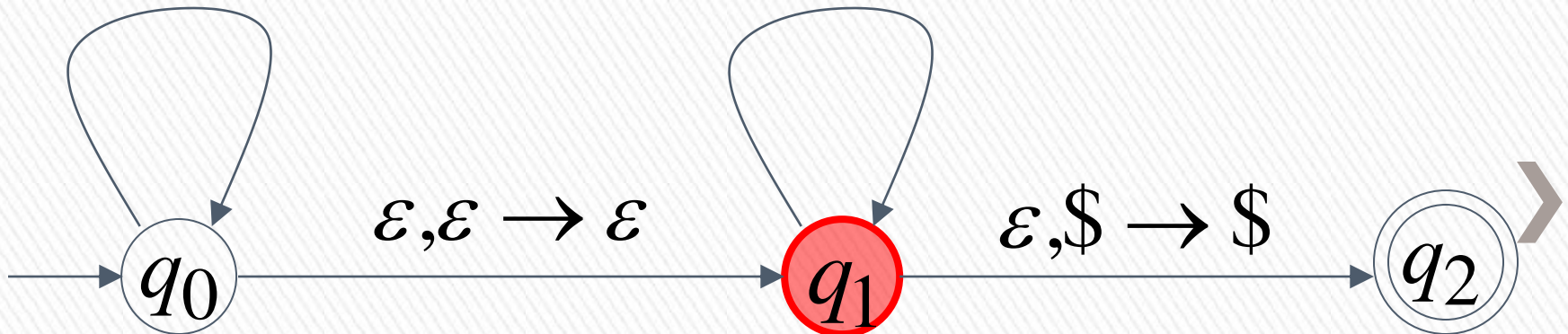
Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

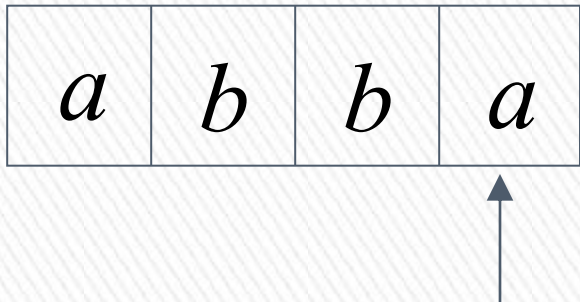
$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$



## Time 6

Input



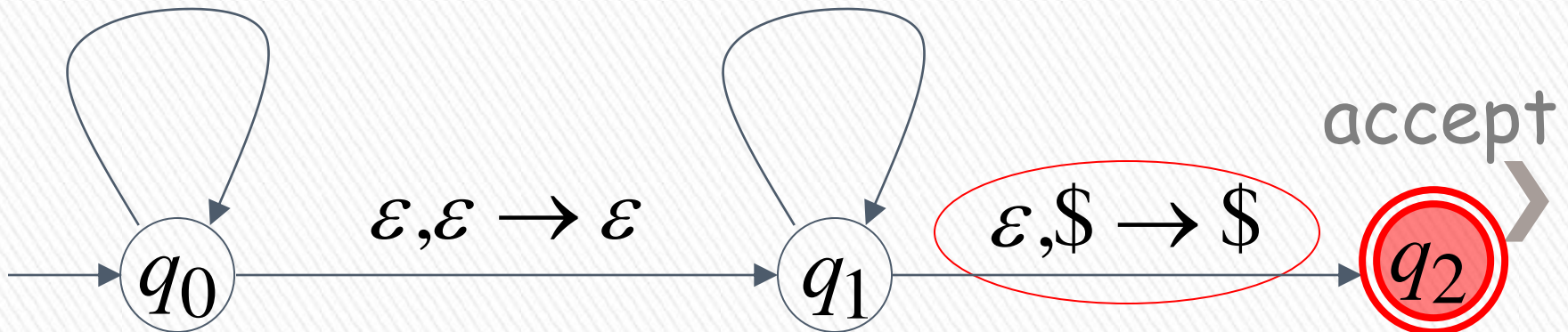
Stack

$a, \varepsilon \rightarrow a$

$a, a \rightarrow \varepsilon$

$b, \varepsilon \rightarrow b$

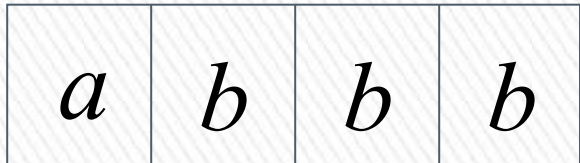
$b, b \rightarrow \varepsilon$



# Rejection Example:

Time 0

Input



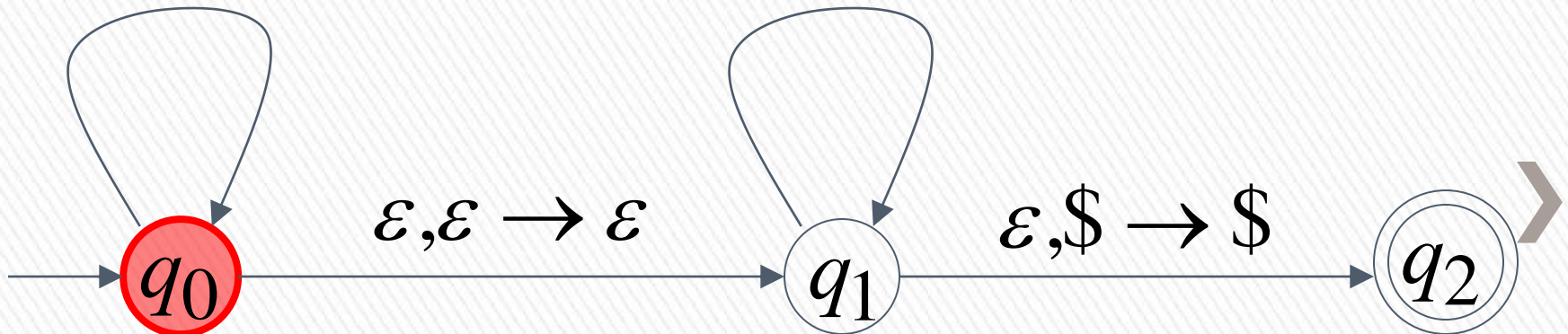
Stack

$a, \varepsilon \rightarrow a$

$a, a \rightarrow \varepsilon$

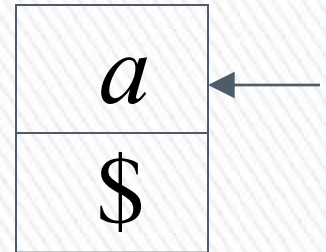
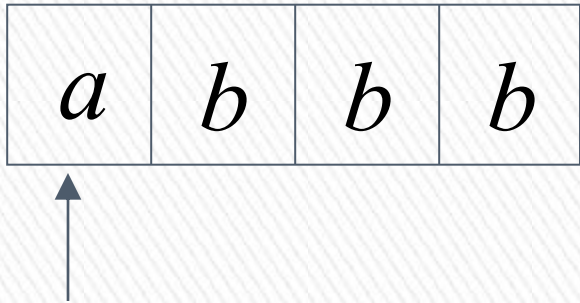
$b, \varepsilon \rightarrow b$

$b, b \rightarrow \varepsilon$



Time 1

Input



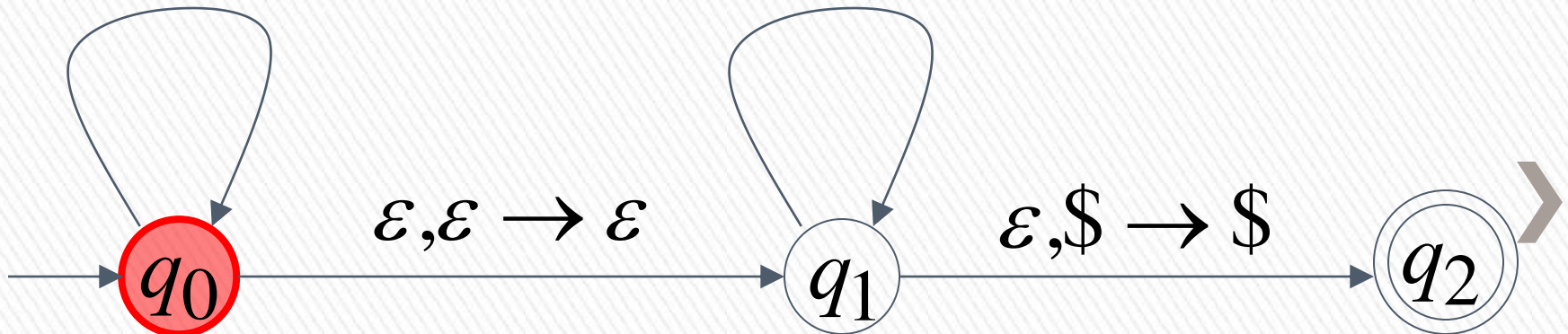
Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

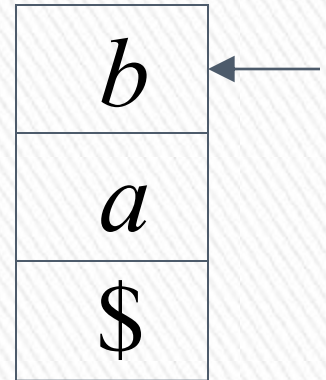
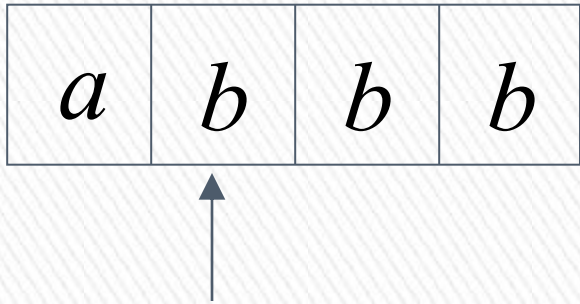
$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$



## Time 2

Input



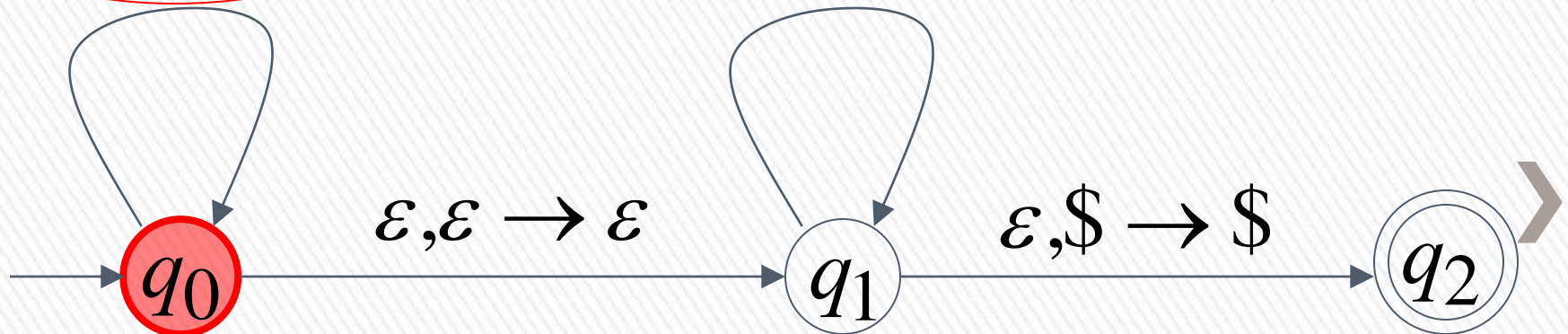
Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

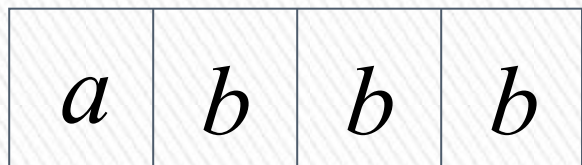
$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

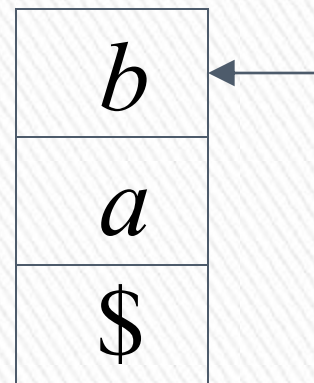


# Time 3

Input



Guess the middle  
of string



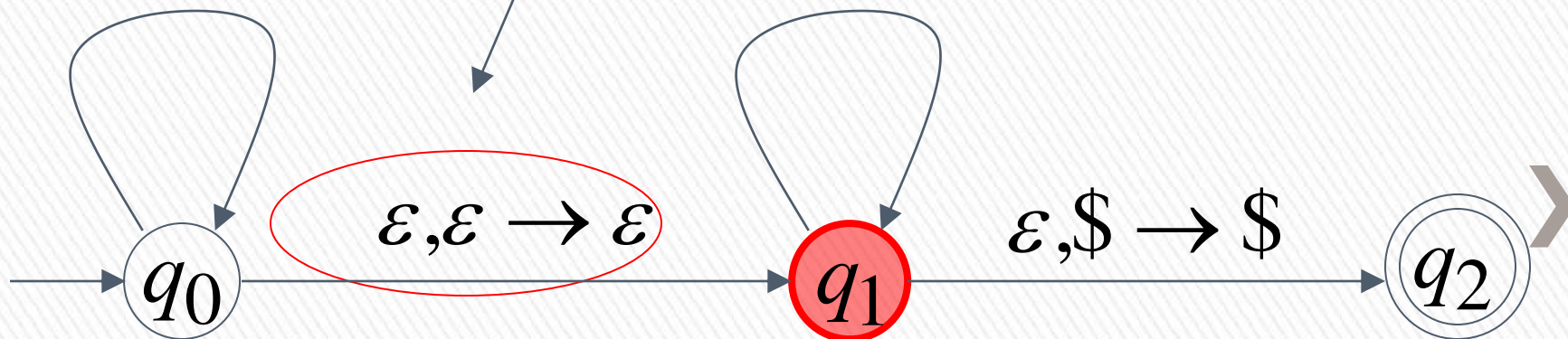
Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

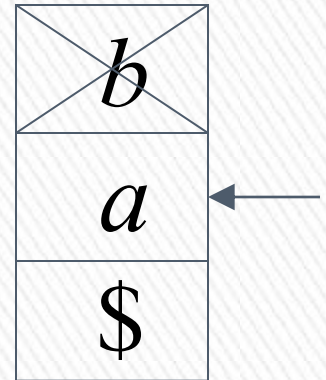
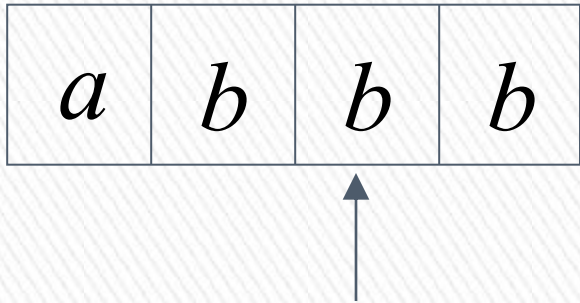
$b, b \rightarrow \varepsilon$





# Time 4

Input



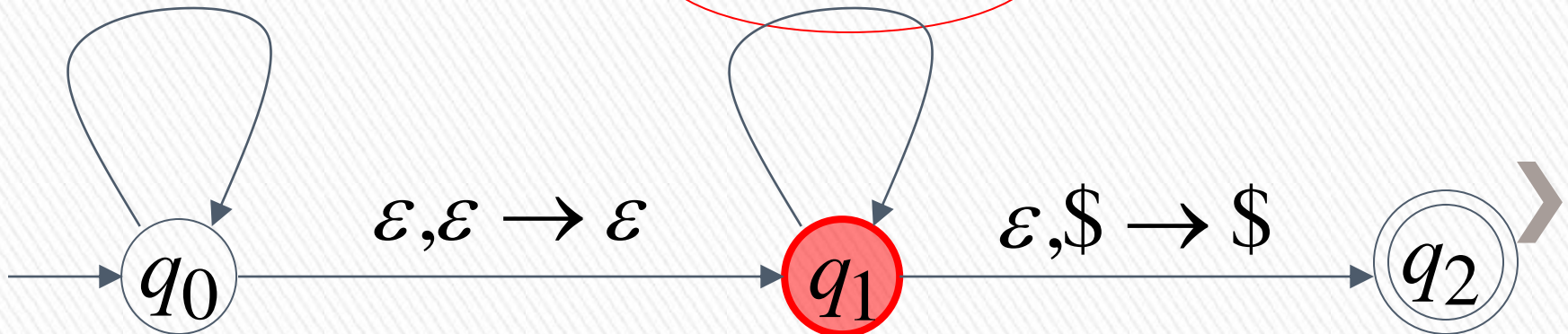
Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

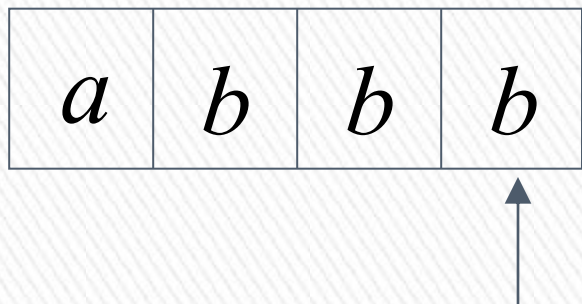
$b, b \rightarrow \varepsilon$



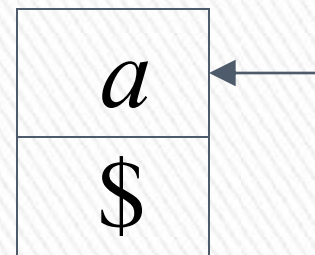
## Time 5

Input

There is no possible transition.



Input is not consumed



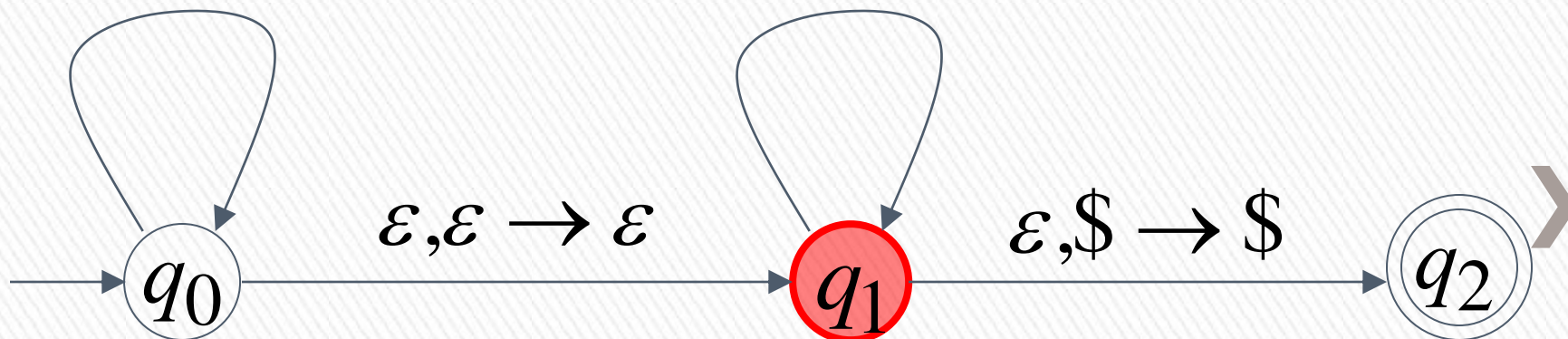
Stack

$a, \varepsilon \rightarrow a$

$a, a \rightarrow \varepsilon$

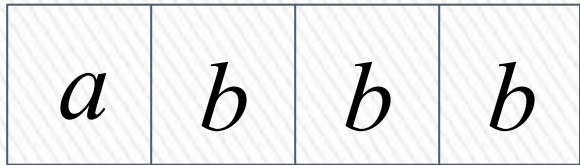
$b, \varepsilon \rightarrow b$

$b, b \rightarrow \varepsilon$



# Another computation on same string:

Input



Time 0



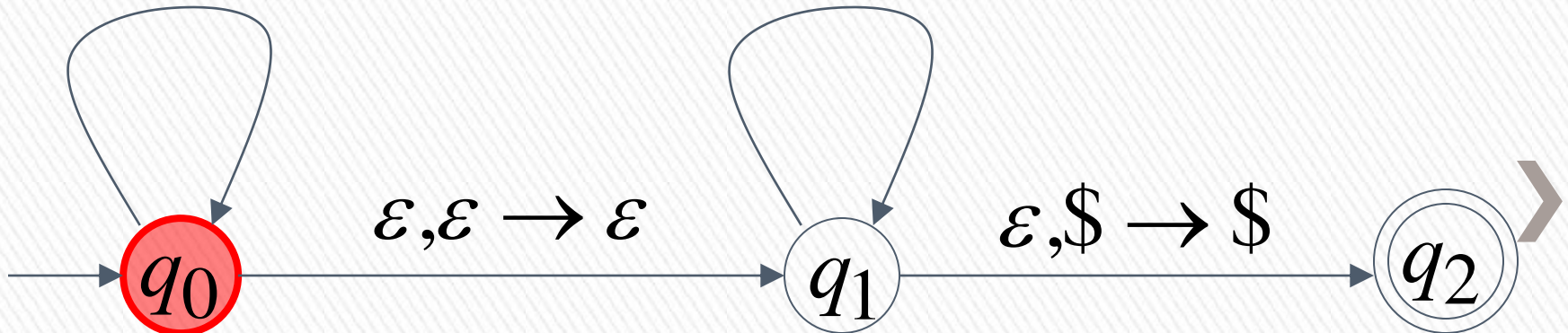
Stack

$a, \varepsilon \rightarrow a$

$a, a \rightarrow \varepsilon$

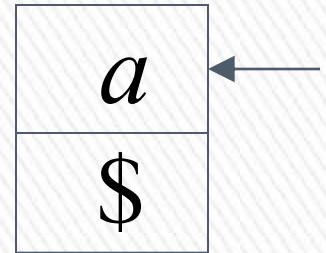
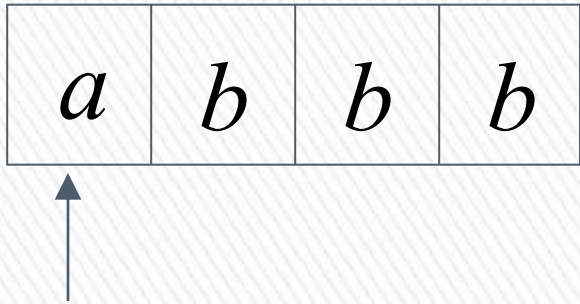
$b, \varepsilon \rightarrow b$

$b, b \rightarrow \varepsilon$



Time 1

Input



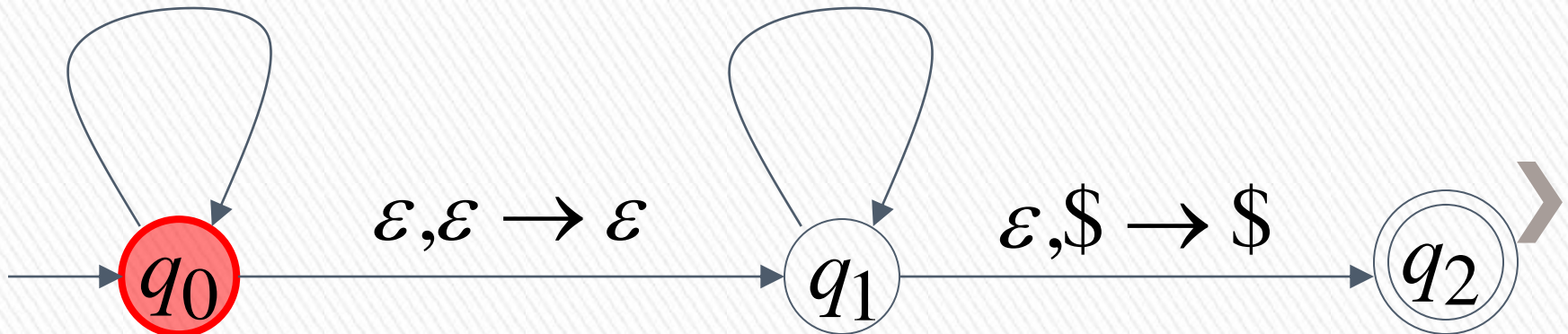
Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

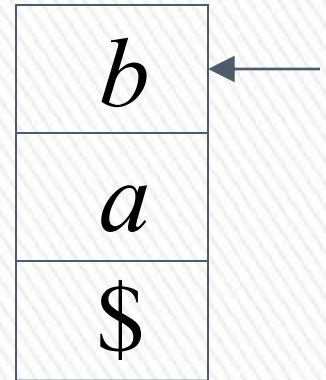
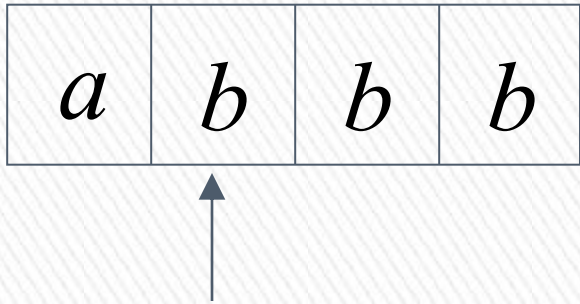
$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$



## Time 2

Input



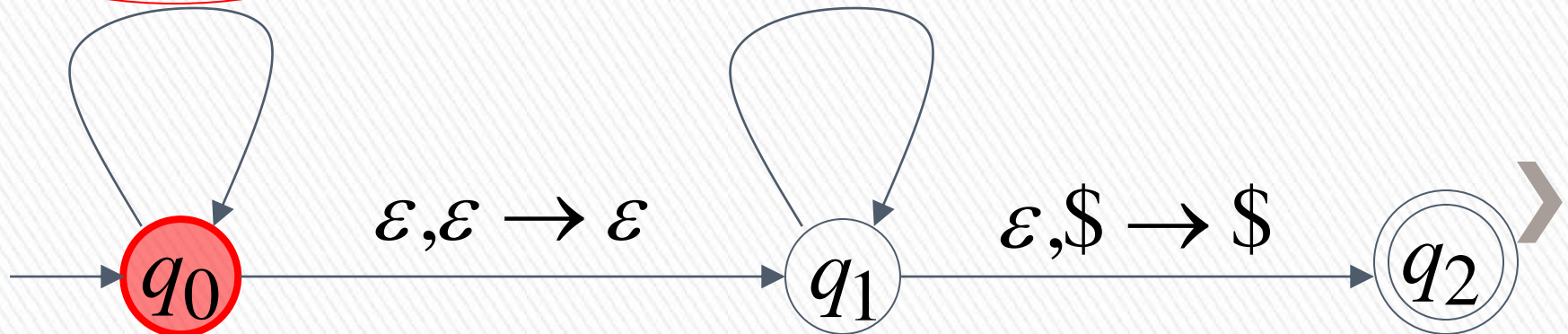
Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

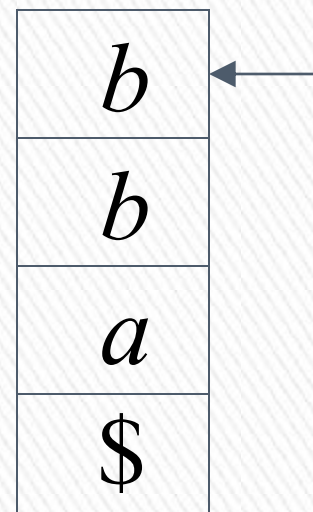
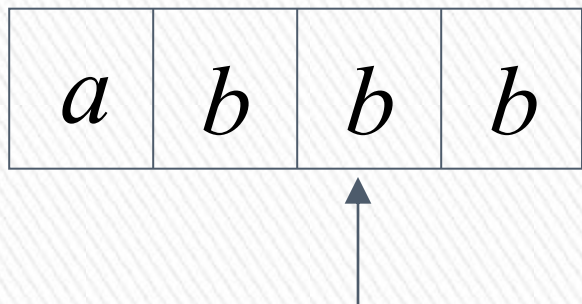
$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$



Time 3

Input



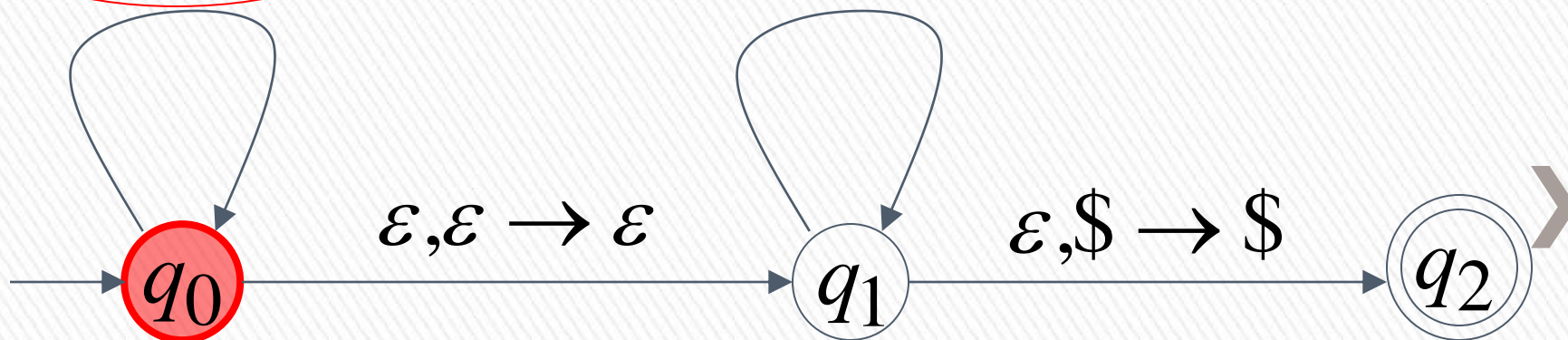
Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

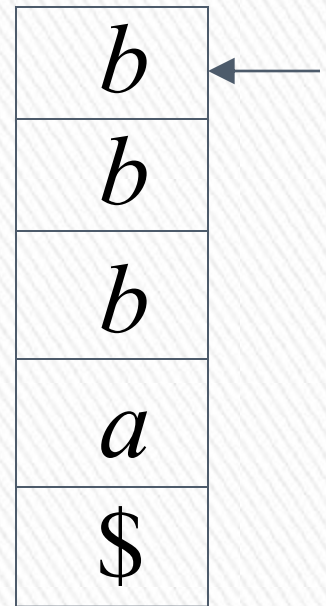
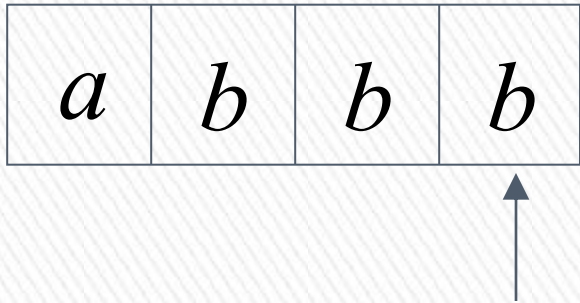
$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$



Time 4

Input



Stack

$a, \varepsilon \rightarrow a$

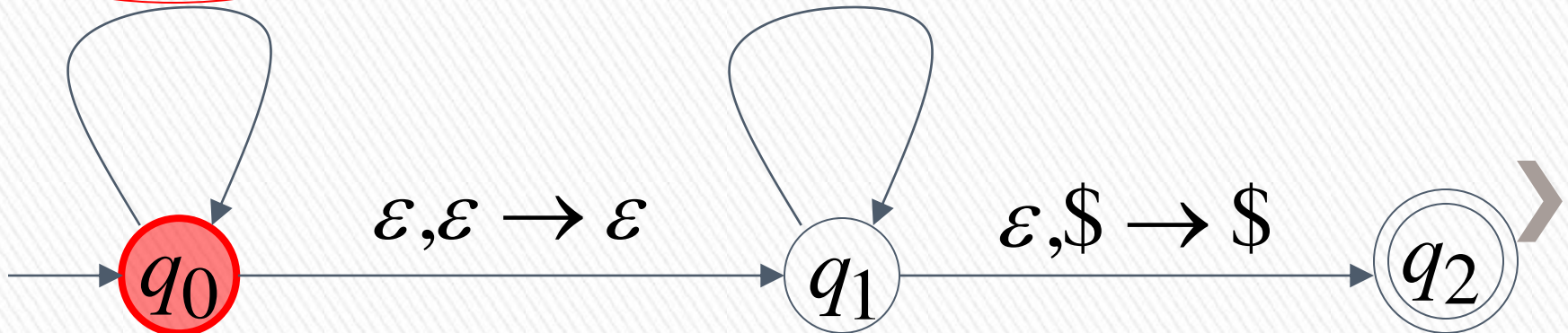
$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

$\varepsilon, \varepsilon \rightarrow \varepsilon$

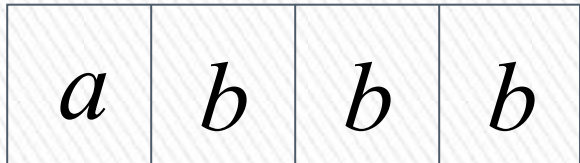
$\varepsilon, \$ \rightarrow \$$



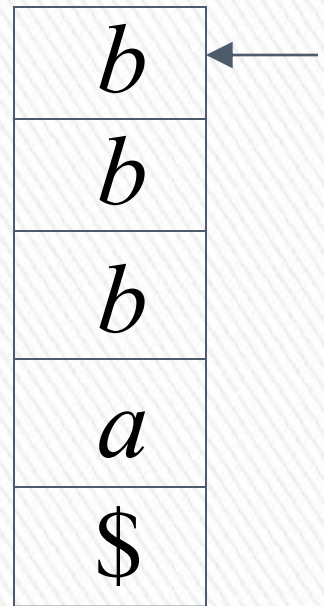


Time 5

Input



No accept state is reached



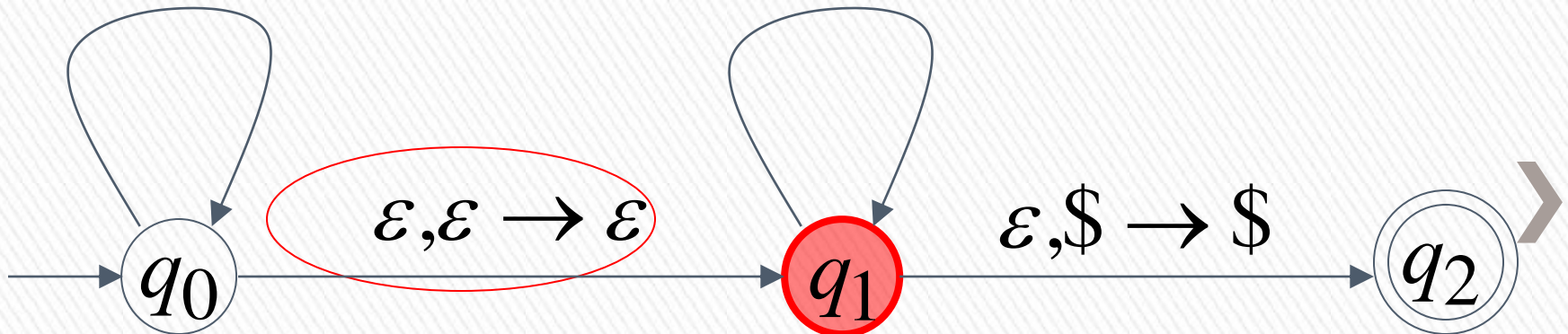
Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

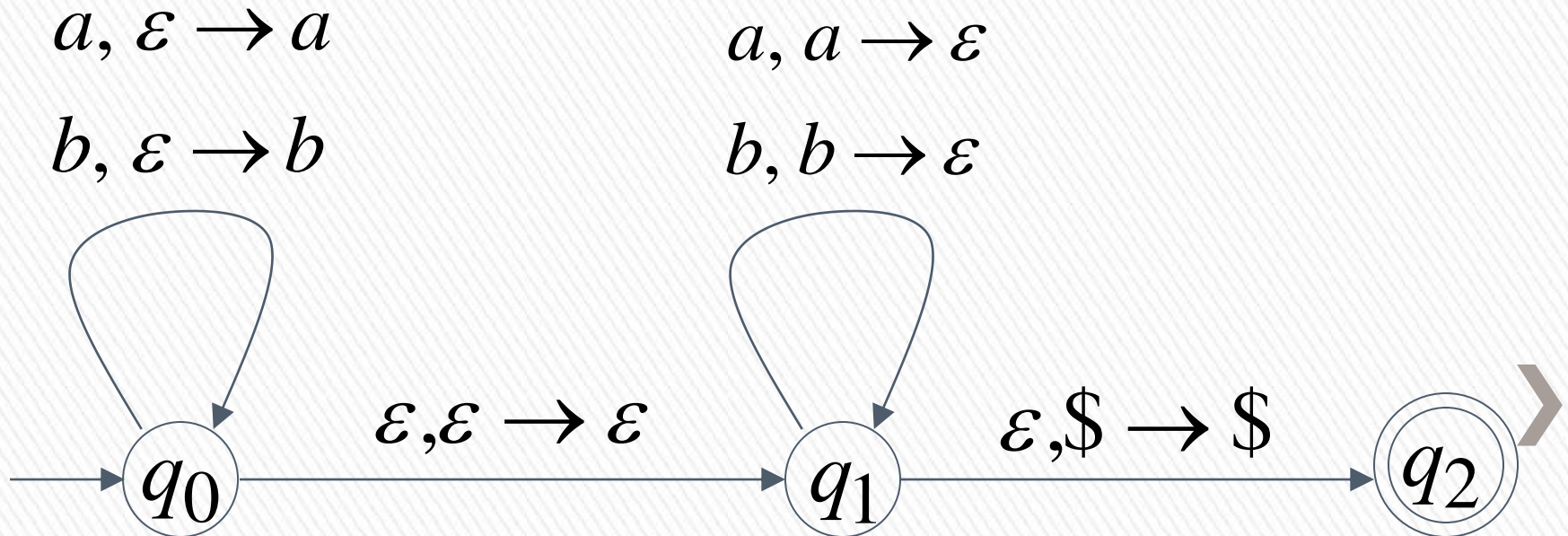
$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

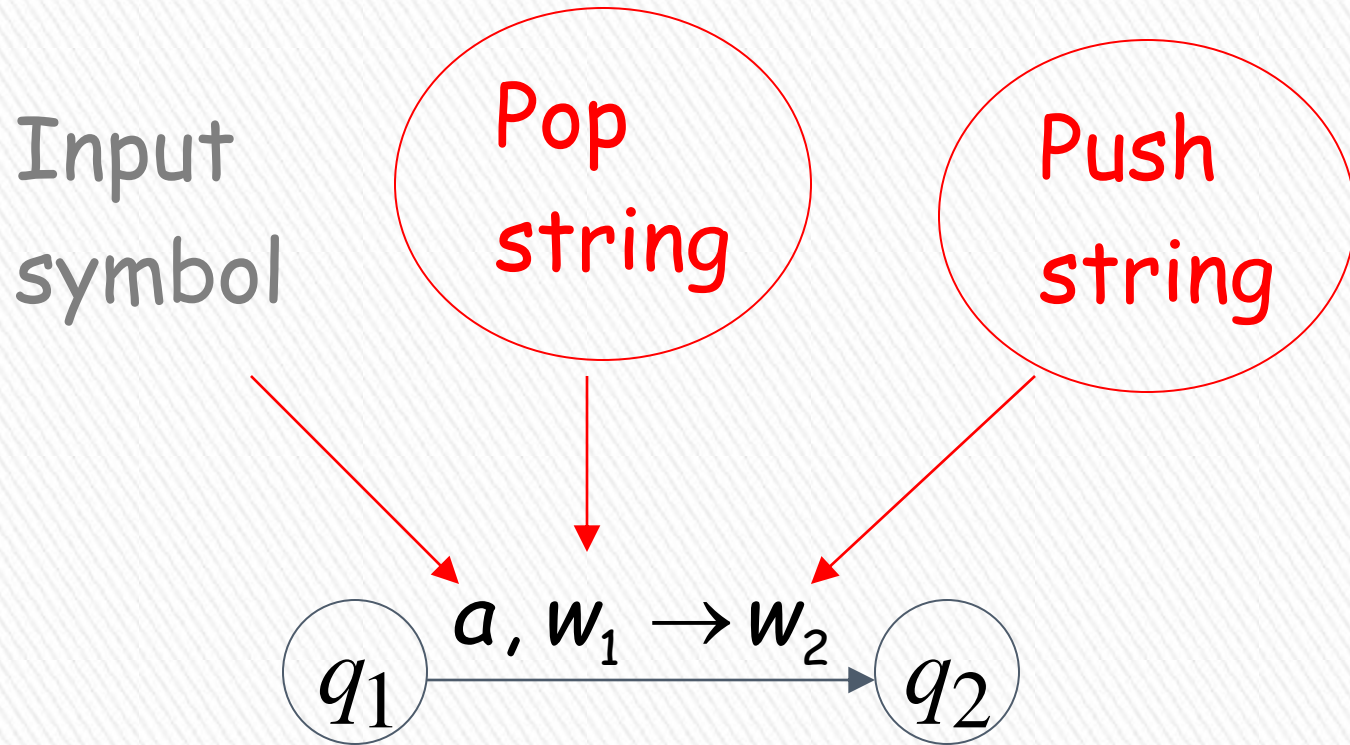


There is no computation  
that accepts string *abbb*

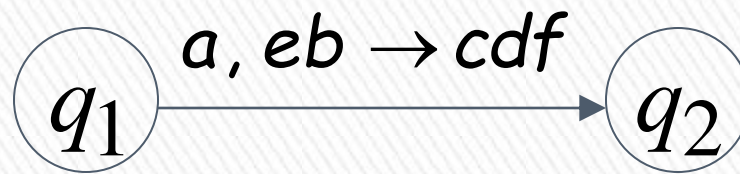
$$abbb \notin L(M)$$



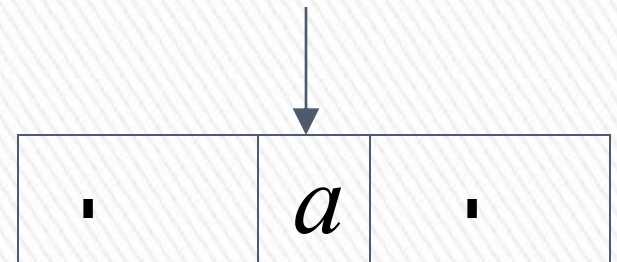
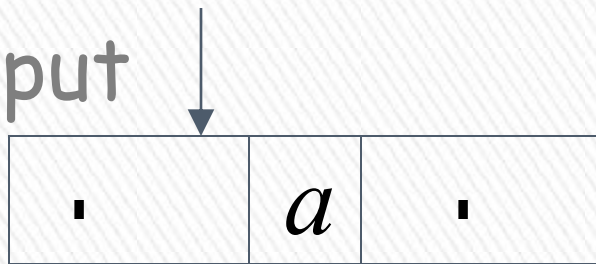
# Pushing & Popping Strings



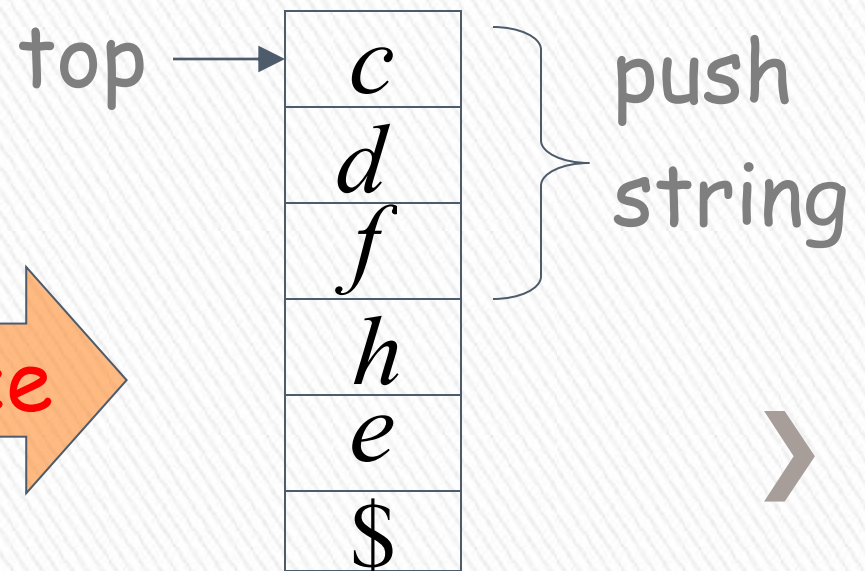
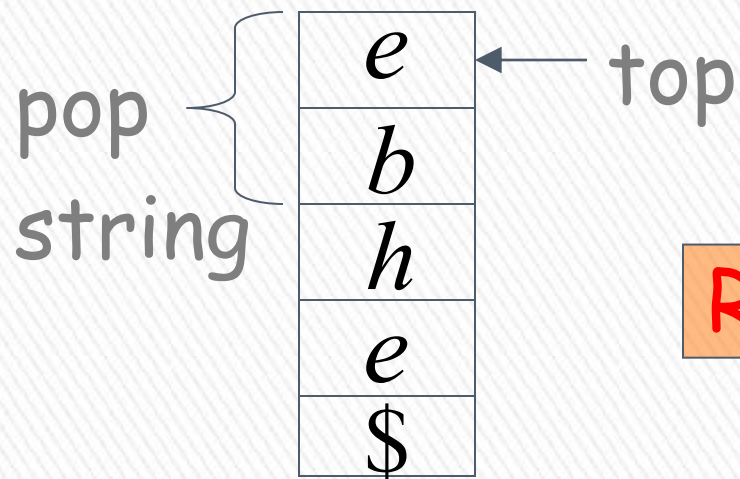
Example:

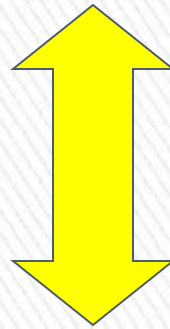
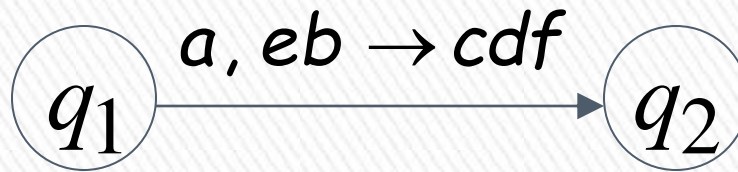


input



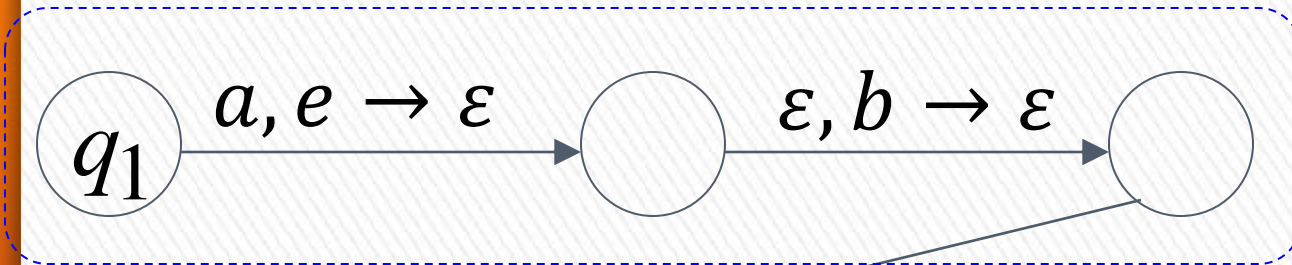
stack





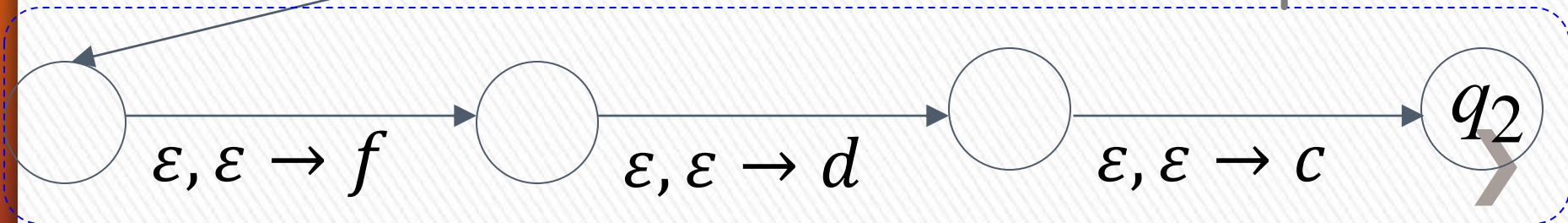
Equivalent  
transitions

pop



$\epsilon, \epsilon \rightarrow \epsilon$

push



# Another PDA example

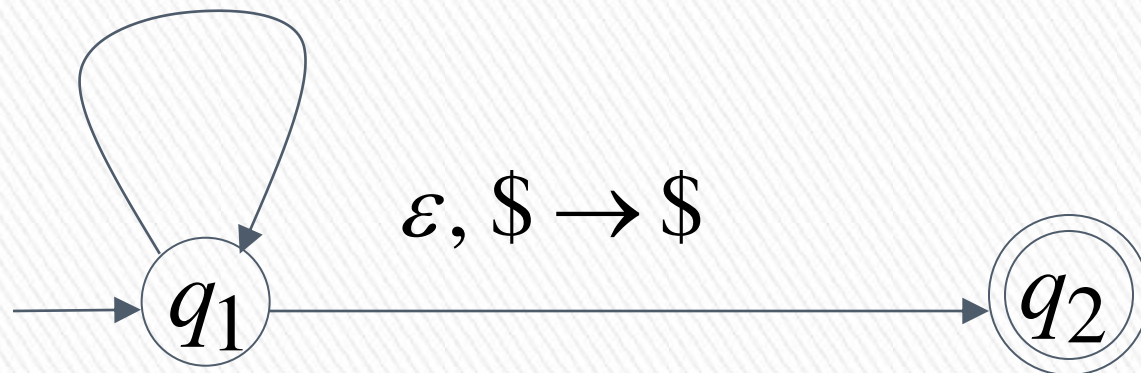
$$L(M) = \{w \in \{a,b\}^* : n_a(w) = n_b(w)\}$$

PDA  $M$

$$a, \$ \rightarrow 0\$ \quad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \quad b, 1 \rightarrow 11$$

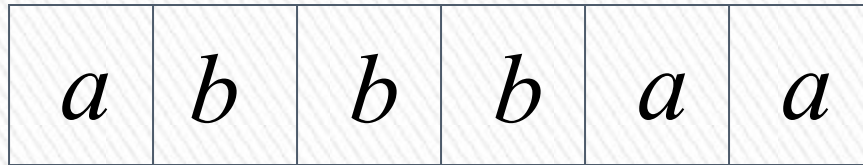
$$a, 1 \rightarrow \varepsilon \quad b, 0 \rightarrow \varepsilon$$





# Execution Example: Time 0

Input



$a, \$ \rightarrow 0\$$        $b, \$ \rightarrow 1\$$

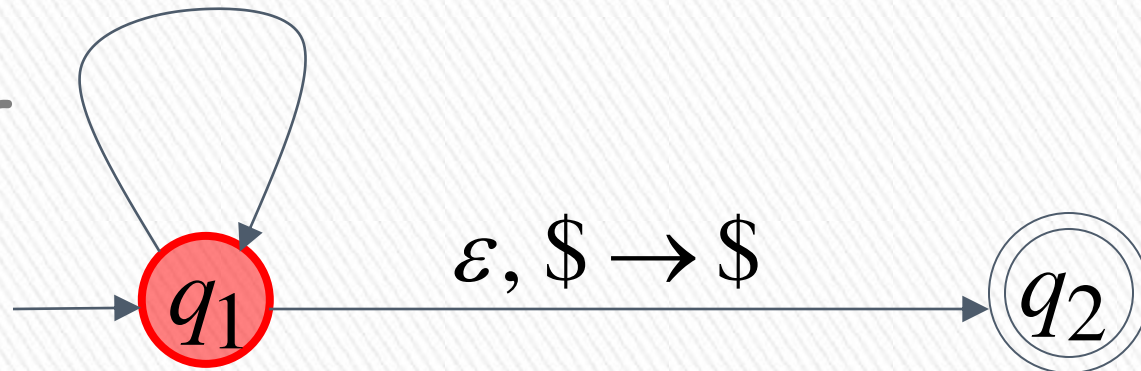
$a, 0 \rightarrow 00$        $b, 1 \rightarrow 11$

$a, 1 \rightarrow \varepsilon$        $b, 0 \rightarrow \varepsilon$



Stack

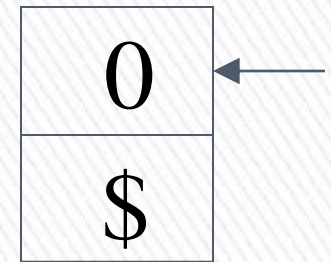
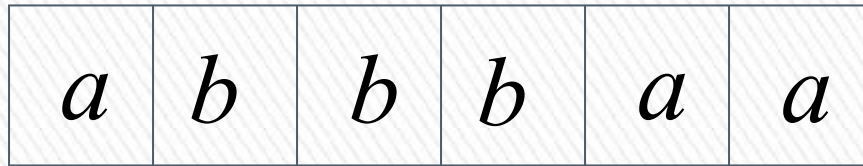
current  
state





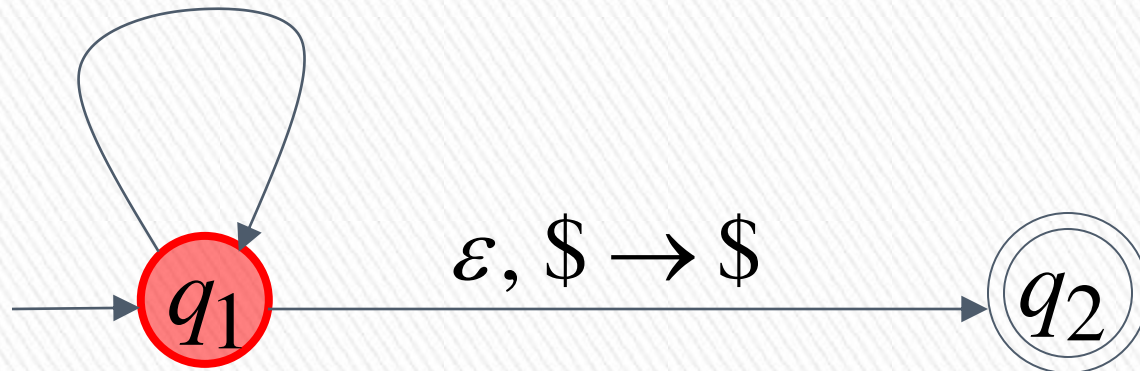
Time 1

Input



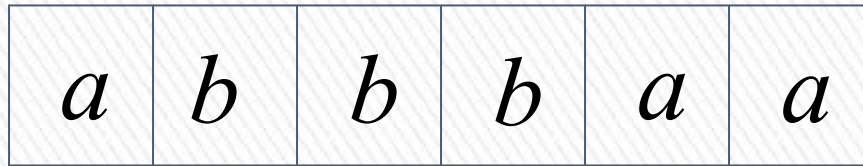
Stack

$a, \$ \rightarrow 0\$$        $b, \$ \rightarrow 1\$$   
 $a, 0 \rightarrow 00$        $b, 1 \rightarrow 11$   
 $a, 1 \rightarrow \varepsilon$        $b, 0 \rightarrow \varepsilon$



## Time 3

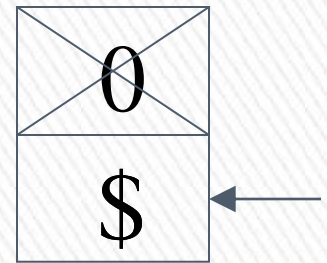
Input



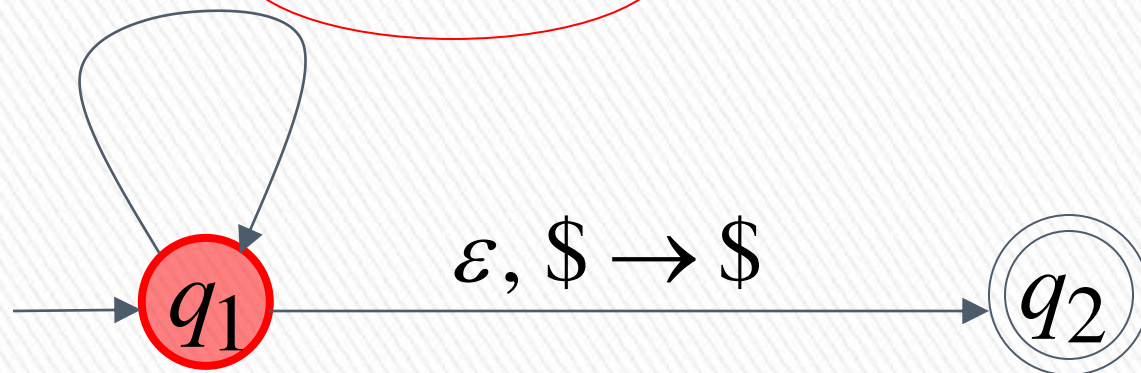
$a, \$ \rightarrow 0\$$        $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$        $b, 1 \rightarrow 11$

$a, 1 \rightarrow \varepsilon$        $b, 0 \rightarrow \varepsilon$

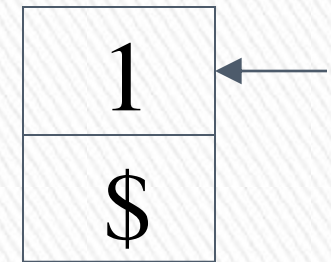
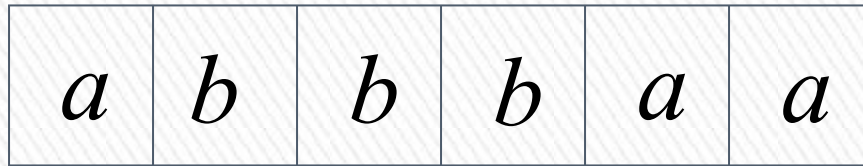


Stack



Time 4

Input

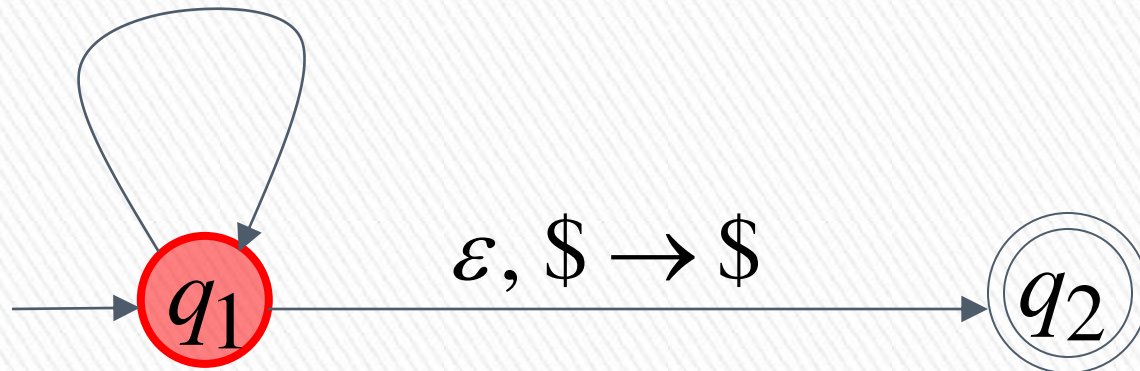


Stack

$a, \$ \rightarrow 0\$$       $b, \$ \rightarrow 1\$$

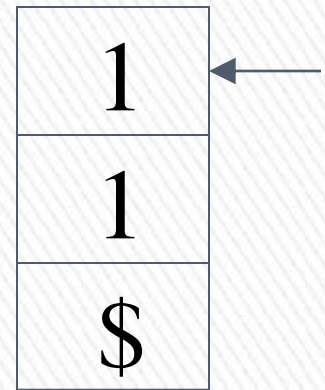
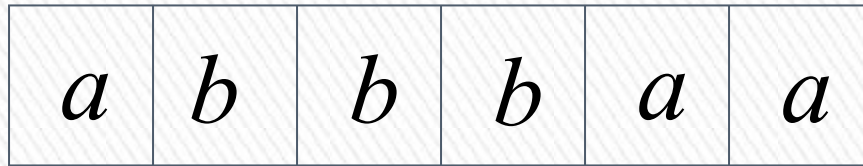
$a, 0 \rightarrow 00$       $b, 1 \rightarrow 11$

$a, 1 \rightarrow \varepsilon$       $b, 0 \rightarrow \varepsilon$



Time 5

Input

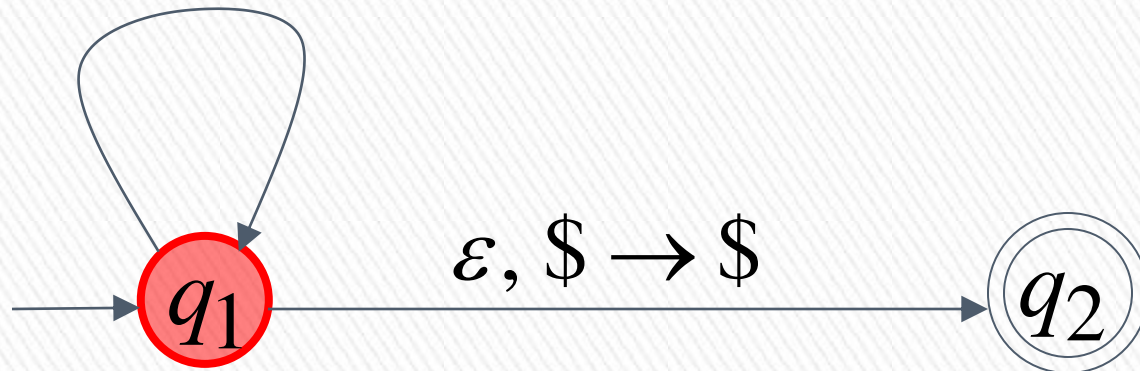


Stack

$a, \$ \rightarrow 0\$$        $b, \$ \rightarrow 1\$$

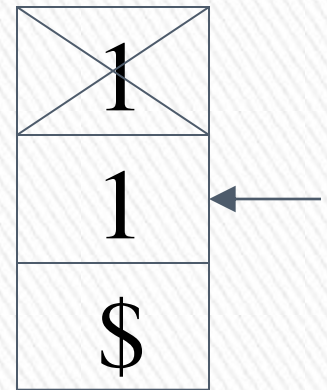
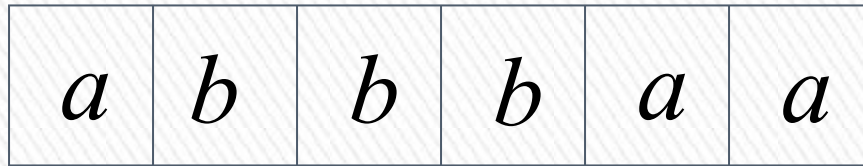
$a, 0 \rightarrow 00$        $b, 1 \rightarrow 11$

$a, 1 \rightarrow \varepsilon$        $b, 0 \rightarrow \varepsilon$



## Time 6

Input

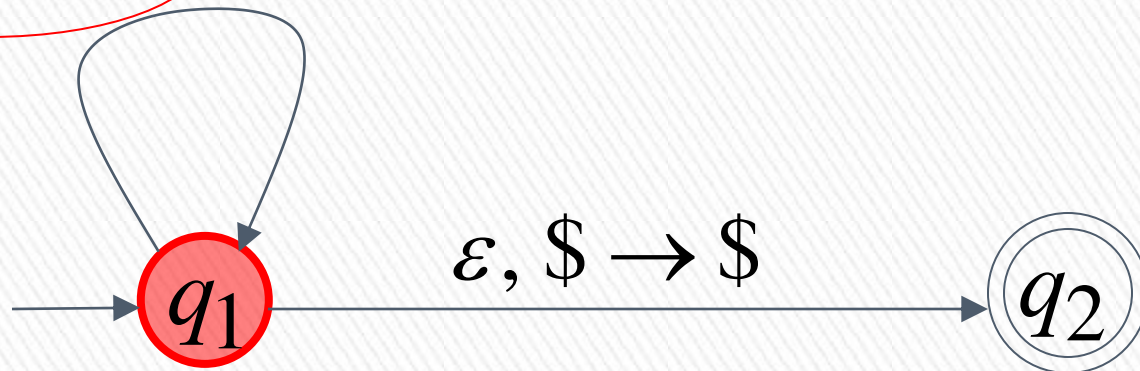


Stack

$a, \$ \rightarrow 0\$$        $b, \$ \rightarrow 1\$$

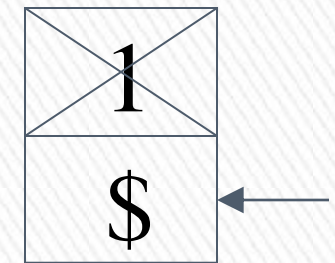
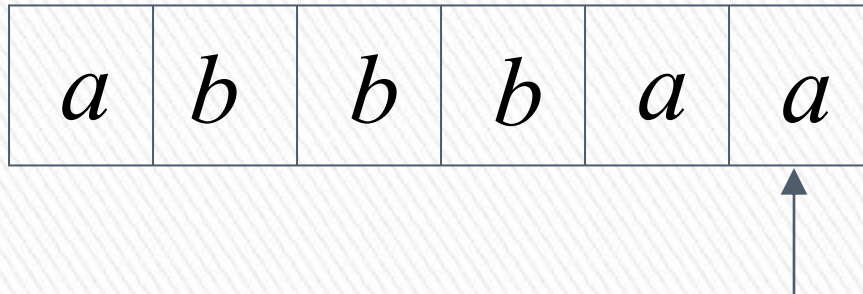
$a, 0 \rightarrow 00$        $b, 1 \rightarrow 11$

$a, 1 \rightarrow \varepsilon$        $b, 0 \rightarrow \varepsilon$



Time 7

Input

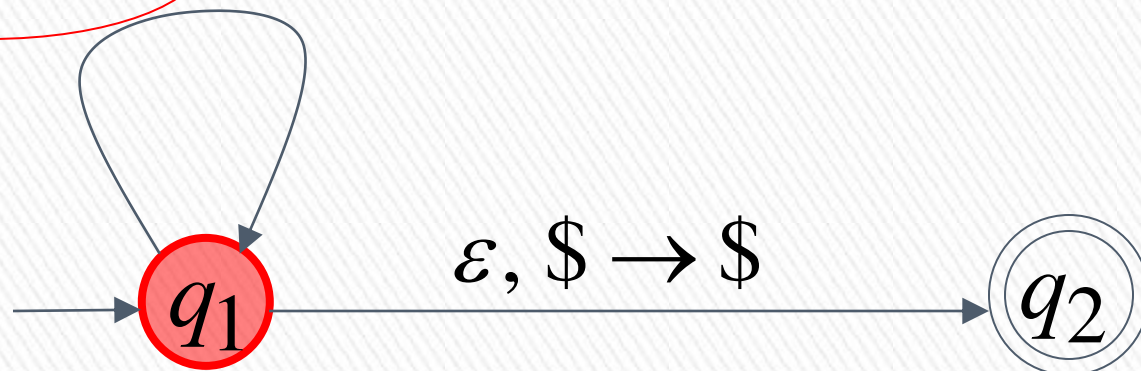


Stack

$a, \$ \rightarrow 0\$$        $b, \$ \rightarrow 1\$$

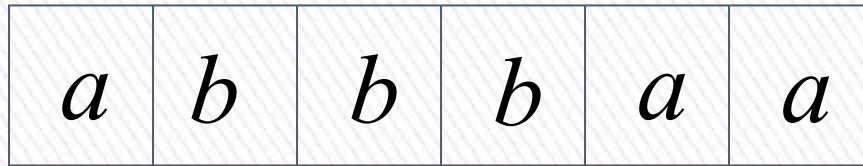
$a, 0 \rightarrow 00$        $b, 1 \rightarrow 11$

$a, 1 \rightarrow \varepsilon$        $b, 0 \rightarrow \varepsilon$



## Time 8

Input



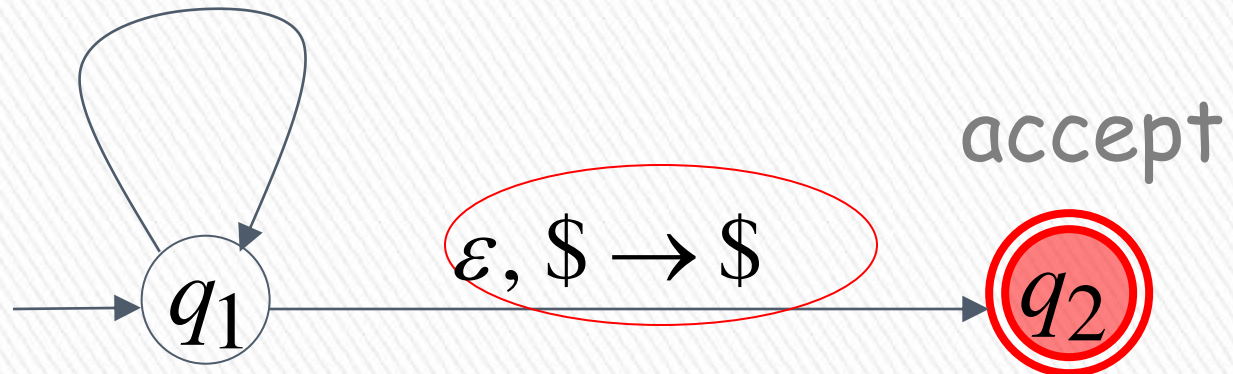
$a, \$ \rightarrow 0\$$        $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$        $b, 1 \rightarrow 11$

$a, 1 \rightarrow \varepsilon$        $b, 0 \rightarrow \varepsilon$

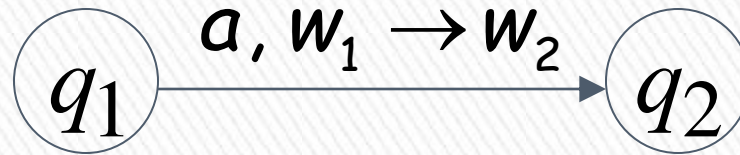


Stack





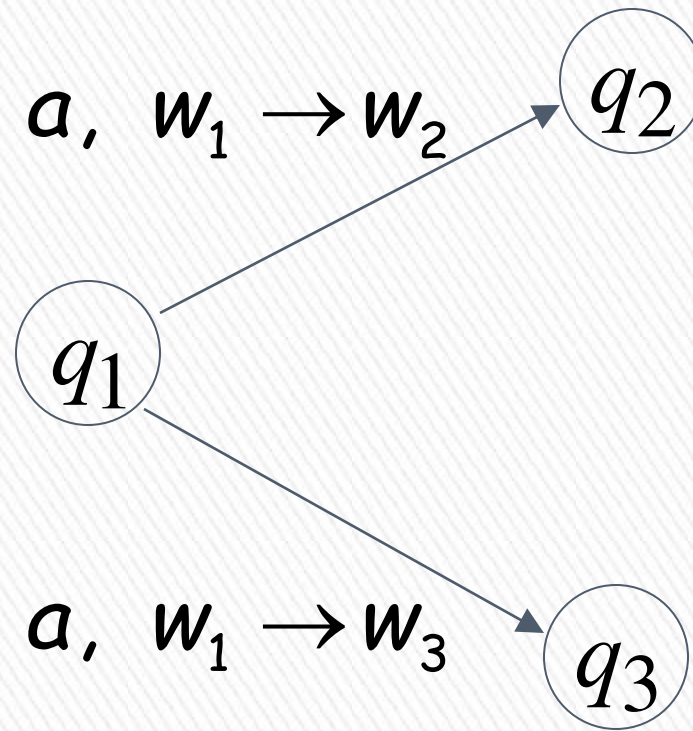
# Formalities for PDAs



Transition function:

$$\delta(q_1, a, w_1) = \{(q_2, w_2)\}$$





Transition function:

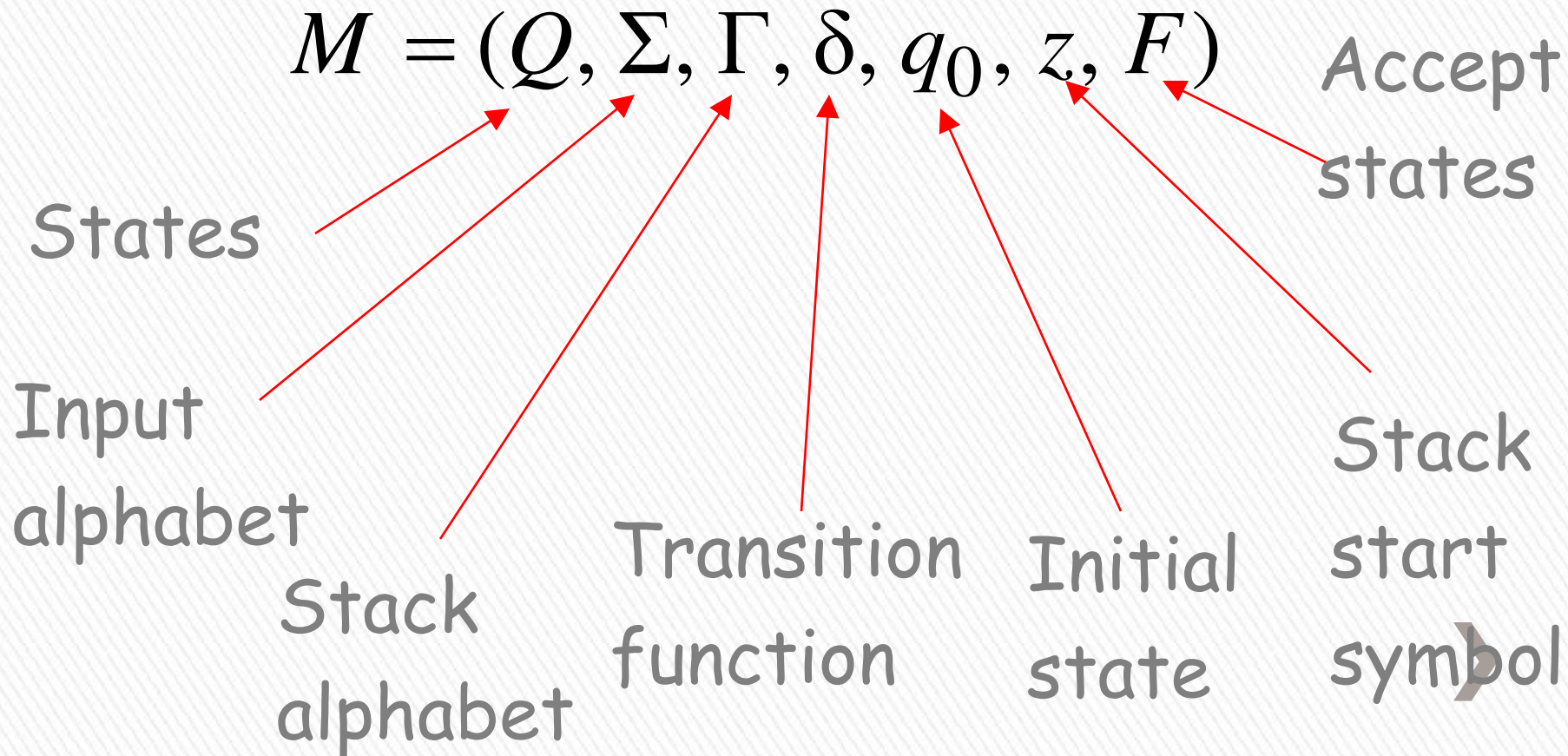
$$\delta(q_1, a, w_1) = \{(q_2, w_2), (q_3, w_3)\}$$



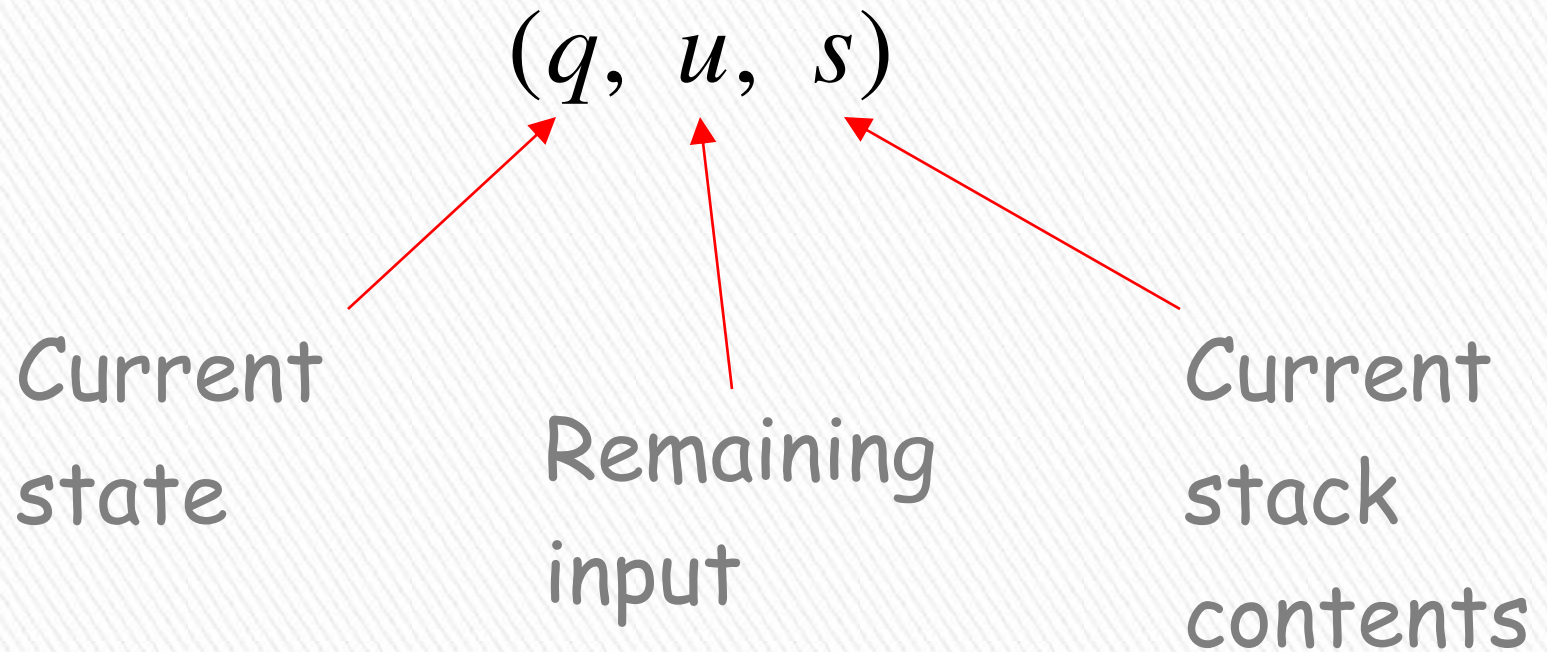
# Formal Definition

## Pushdown Automaton (PDA)

$$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$$



# Instantaneous Description

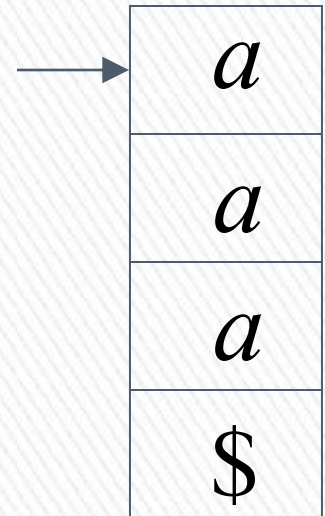


# Example: Instantaneous Description

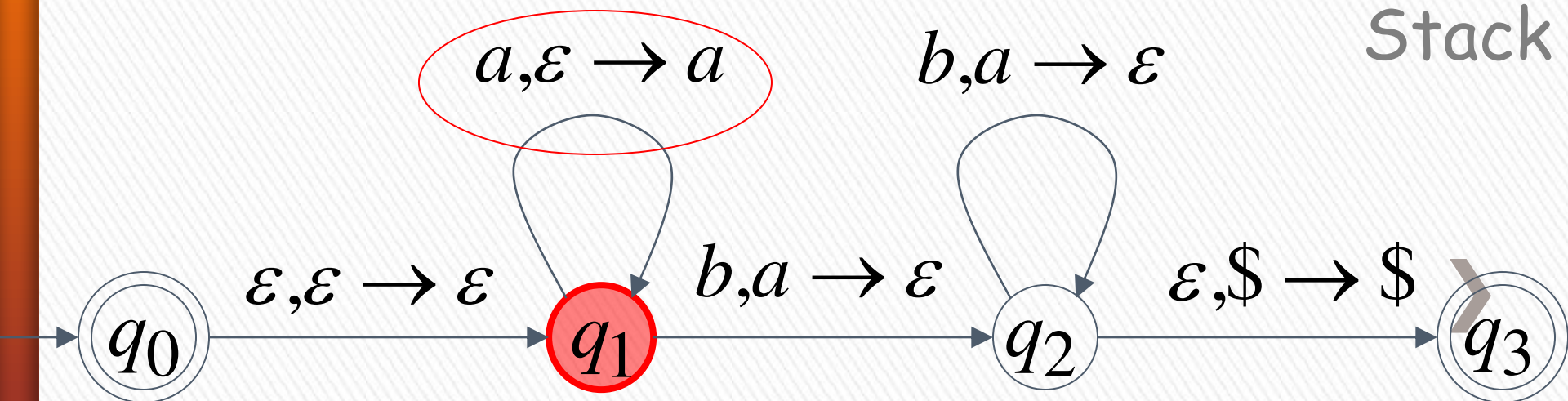
$(q_1, bbb, aaa\$)$

Time 4:

Input



Stack

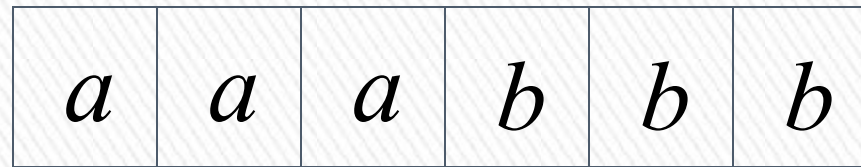


# Example: Instantaneous Description

$(q_2, bb, aa\$)$

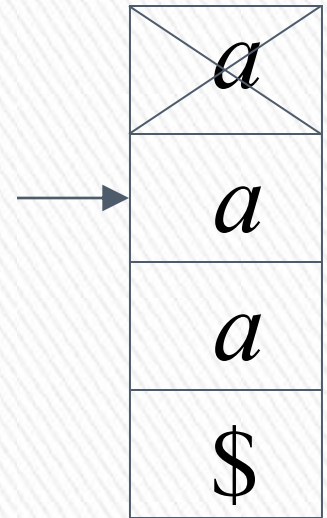
Time 5:

Input

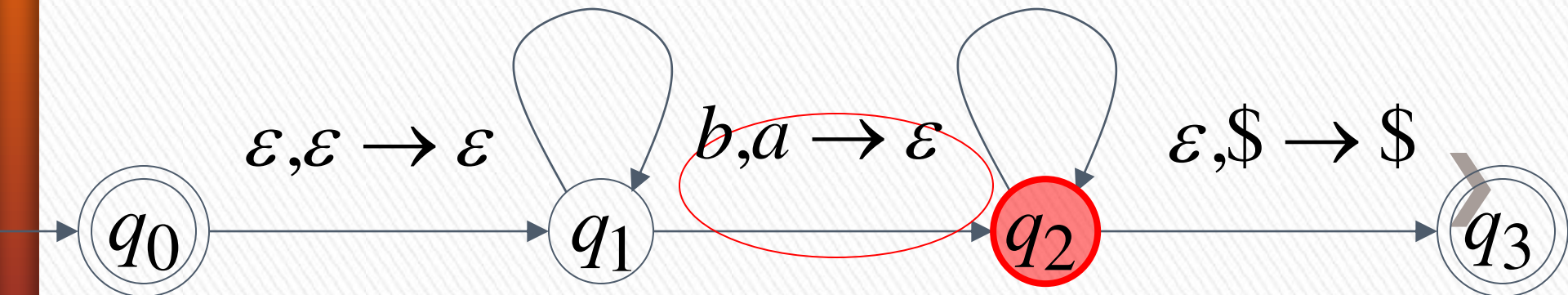


$a, \varepsilon \rightarrow a$

$b, a \rightarrow \varepsilon$



Stack



We write:

$$(q_1, bbb, aaa\$) \succ (q_2, bb, aa\$)$$

Time 4

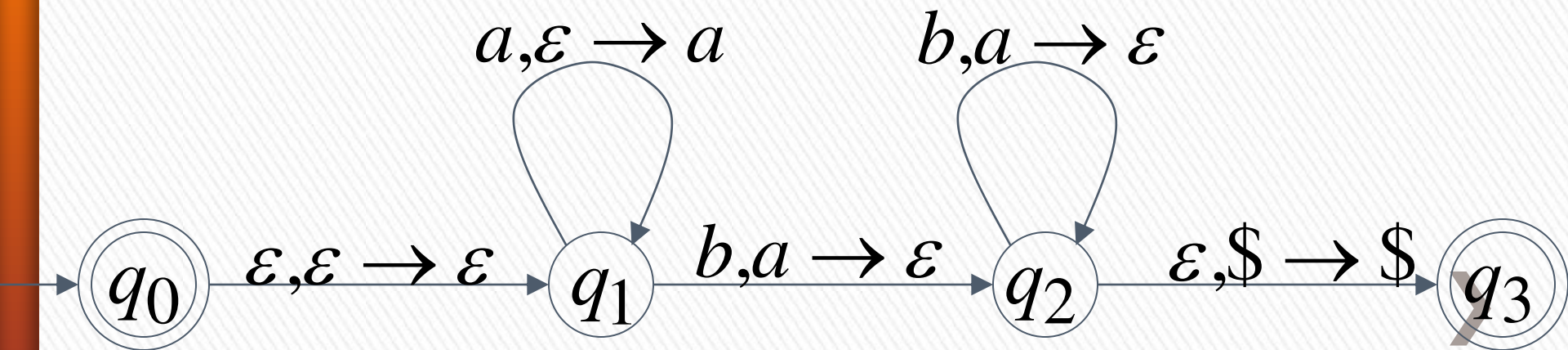
Time 5





# A computation:

$(q_0, aaabbbb, \$) \succ (q_1, aaabbbb, \$) \succ$   
 $(q_1, aabbbb, a\$) \succ (q_1, abbbb, aa\$) \succ (q_1, bbbb, aaa\$) \succ$   
 $(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \varepsilon, \$) \succ (q_3, \varepsilon, \$)$



$$\begin{aligned}
 (q_0, aaabbbb, \$) &\succ (q_1, aaabbbb, \$) \succ \\
 (q_1, aabbbb, a\$) &\succ (q_1, abbbb, aa\$) \succ (q_1, bbbb, aaa\$) \succ \\
 (q_2, bb, aa\$) &\succ (q_2, b, a\$) \succ (q_2, \varepsilon, \$) \succ (q_3, \varepsilon, \$)
 \end{aligned}$$

For convenience we write:

$$(q_0, aaabbbb, \$) \overset{*}{\succ} (q_3, \varepsilon, \$)$$



# Language of PDA

Language  $L(M)$  accepted by PDA  $M$ :

$$L(M) = \{w : (q_0, w, z) \xrightarrow{*} (q_f, \varepsilon, s)\}$$

Initial state

A blue arrow points from the text 'Initial state' to the symbol  $q_0$  in the configuration notation  $(q_0, w, z)$  of the equation above.

Accept state

A blue arrow points from the text 'Accept state' to the symbol  $q_f$  in the configuration notation  $(q_f, \varepsilon, s)$  of the equation above.

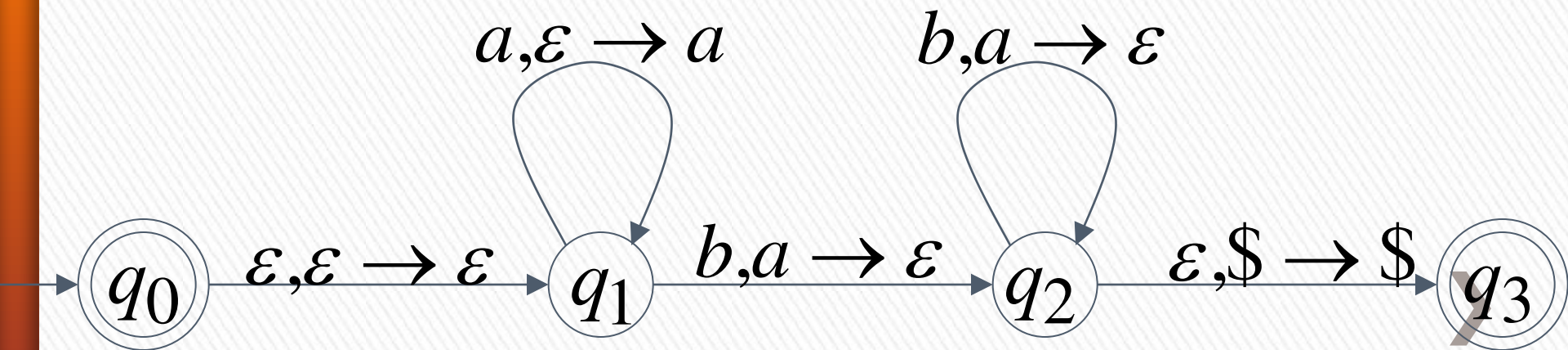
Example:

$$(q_0, aaabbbb, \$) \stackrel{*}{\succ} (q_3, \varepsilon, \$)$$



$$aaabbbb \in L(M)$$

PDA  $M$  :

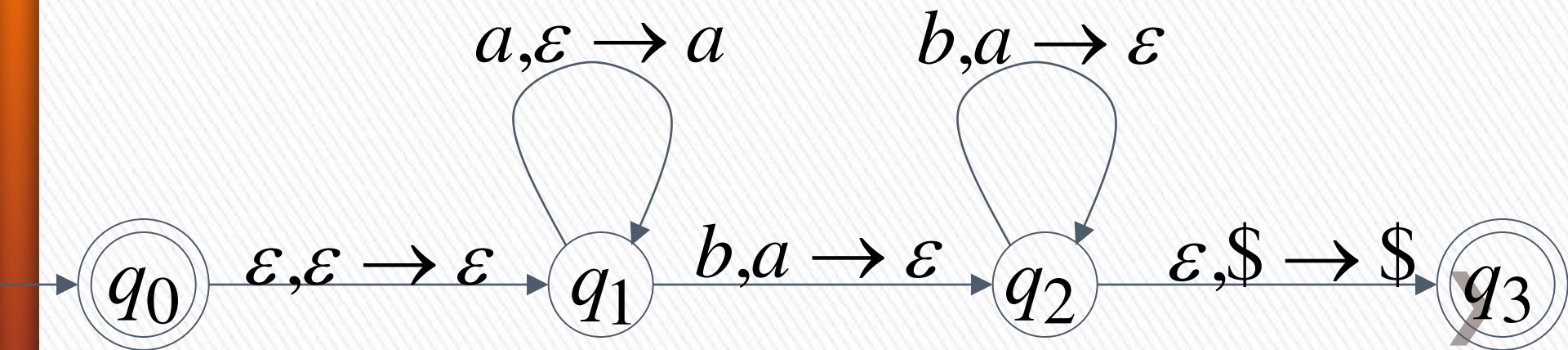


$$(q_0, a^n b^n, \$) \stackrel{*}{\succ} (q_3, \varepsilon, \$)$$



$$a^n b^n \in L(M)$$

PDA  $M$  :



Therefore:

$$L(M) = \{a^n b^n : n \geq 0\}$$

PDA  $M$  :

