MAT1072 / Matematik 2

İki Katlı İntegraller

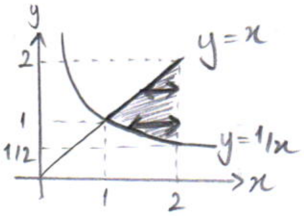
1) $\displaystyle\int_1^2\int_{1/x}^{x} \frac{x^2}{y^2}\,dy\,dx$

a) İntegrali verildiği şekilde hesaplayınız.

b) İntegrasyon sırasını değiştirerek yeni integrali yazınız.

a) $\displaystyle\int_1^2\int_{1/x}^{x} \frac{x^2}{y^2}\,dy\,dx = \int_1^2 \left[-\frac{x^2}{y}\right]_{1/x}^{x} dx = \int_1^2 \left[-\frac{x^2}{x} + \frac{x^2}{1/x}\right] dx = -\frac{x^2}{2} + \frac{x^4}{4}\Big|_1^2 = 2 + \frac{1}{4} = \frac{9}{4}$

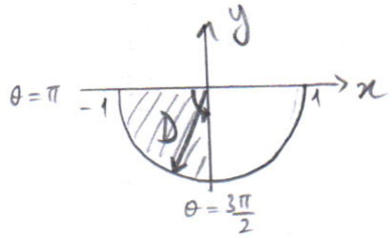
b) $\left.\begin{matrix} x=1 & y=1/x \\ x=2 & y=x \end{matrix}\right\}$



$\displaystyle I = \int_{1/2}^{1}\int_{1/y}^{2} \frac{x^2}{y^2}\,dx\,dy + \int_1^2\int_y^2 \frac{x^2}{y^2}\,dx\,dy$

2) $\displaystyle\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dy\,dx$ integralini hesaplayınız.

$D: \left.\begin{matrix} x=0 & y=-\sqrt{1-x^2} \\ x=-1 & y=0 \end{matrix}\right\}$

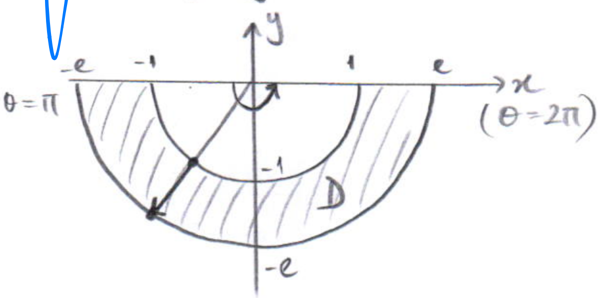


$\left.\begin{matrix} x=r\cos\theta \\ y=r\sin\theta \end{matrix}\right\}\ dy\,dx = r\,dr\,d\theta$

$$I = \int_{\pi}^{3\pi/2}\int_0^1 \frac{2}{1+\sqrt{r^2}}\,r\,dr\,d\theta = 2\int_{\pi}^{3\pi/2}\int_0^1 \frac{r}{r+1}\,dr\,d\theta = 2\int_{\pi}^{3\pi/2}\int_0^1 \left(1-\frac{1}{1+r}\right) dr\,d\theta$$

$$= 2\int_{\pi}^{3\pi/2} (r - \ln(1+r))\Big|_0^1 d\theta = 2(1-\ln 2)\left(\frac{3\pi}{2}-\pi\right) = (1-\ln 2)\pi$$

3) D bölgesi, $D = \{(x,y)\in\mathbb{R}^2 \mid 1 \le x^2+y^2 \le e^2,\ y\le 0\}$ ile verilsin. Üstten $z = \dfrac{\ln(x^2+y^2)}{x^2+y^2}$ yüzeyi ve alttan xy düzleminde D ile sınırlı cismin hacmini hesaplayınız.



$\left.\begin{matrix} x=r\cos\theta \\ y=r\sin\theta \end{matrix}\right\}\ dx\,dy = r\,dr\,d\theta$

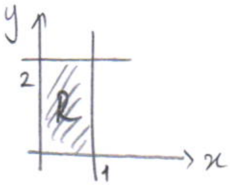
$$V = \iint_D \frac{\ln(x^2+y^2)}{x^2+y^2}\,dx\,dy = \int_{\pi}^{2\pi}\int_1^e \frac{\ln(r^2)}{r^2}\,r\,dr\,d\theta$$

$$= \int_{\pi}^{2\pi}\int_1^e \frac{2\ln r}{r}\,dr\,d\theta = \int_{\pi}^{2\pi}\int_1^e 2u\,du\,d\theta = \int_{\pi}^{2\pi} \ln^2 r\Big|_1^e d\theta$$

$\left[\begin{matrix} \ln r = u \\ \frac{dr}{r} = du \end{matrix}\right]$

$$= \int_{\pi}^{2\pi} d\theta = \theta\Big|_{\pi}^{2\pi} = 2\pi - \pi = \pi$$

4) Üstten $z=10+x^2+3y^2$ paraboloidi ve alttan $z=0$ da $R:\begin{cases}0\le x\le 1\\ 0\le y\le 2\end{cases}$ dikdörtgeni ile sınırlı bölgenin hacmini hesaplayınız.

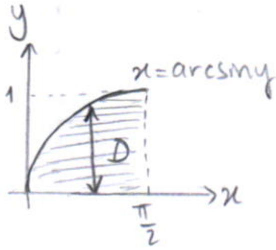


$$V=\iint_R (10+x^2+3y^2)\,dy\,dx=\int_0^1\int_0^2 (10+x^2+3y^2)\,dy\,dx$$

$$=\int_0^1 \left(10y+x^2y+y^3\Big|_0^2\right)dx=\int_0^1 (20+2x^2+8)\,dx=28x+\frac{2x^3}{3}\Big|_0^1=\frac{86}{3}$$

5) $\displaystyle\int_0^1\int_{\arcsin y}^{\pi/2} e^{\cos x}\,dx\,dy$ integralini hesaplayınız.

$$D:\begin{cases} y=0\,,\ y=1\\ x=\arcsin y\,,\ x=\frac{\pi}{2}\end{cases}$$

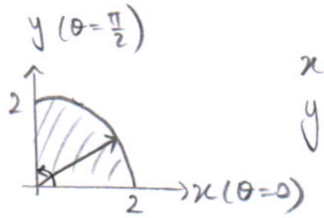


$$\int_0^1\int_{\arcsin y}^{\pi/2} e^{\cos x}\,dx\,dy=\int_0^{\frac{\pi}{2}}\int_0^{\sin x} e^{\cos x}\,dy\,dx=\int_0^{\pi/2} e^{\cos x}\cdot y\Big|_0^{\sin x}dx=\int_0^{\frac{\pi}{2}} e^{\cos x}\sin x\,dx$$

$$=-e^{\cos x}\Big|_0^{\frac{\pi}{2}}=e-1$$

6) $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}} \frac{6xy}{(1+2x^2+2y^2)^{3/2}}\,dy\,dx$ integralini hesaplayınız.

$$\left.\begin{matrix} x=0 & y=\sqrt{4-x^2}\\ x=2 & y=0\end{matrix}\right\}$$



$$\left.\begin{matrix} x=r\cos\theta\\ y=r\sin\theta\end{matrix}\right\}\ dy\,dx=r\,dr\,d\theta$$

$$I=\int_0^{\pi/2}\int_0^2 \underbrace{\frac{3r^3\sin 2\theta}{(1+2r^2)^{3/2}}\,dr}\,d\theta=\int_0^{\pi/2} 3\cdot\frac{1}{3}\sin 2\theta\,d\theta=-\frac{1}{2}\cos 2\theta\Big|_0^{\pi/2}=-\frac{1}{2}(\cos\pi-\cos 0)=1$$

$$\int_0^2 \frac{r^3}{(1+2r^2)^{3/2}}\,dr=\frac{1}{4}\cdot\frac{1}{2}\int_0^2 \frac{u-1}{u^{3/2}}\,du=\frac{1}{8}\int_0^2 (u^{-1/2}-u^{-3/2})\,du=\frac{1}{4}\left(u^{1/2}+u^{-1/2}\right)\Big|_0^2=\frac{1}{4}\left((1+2r^2)^{1/2}+(1+2r^2)^{-1/2}\right)\Big|_0^2$$

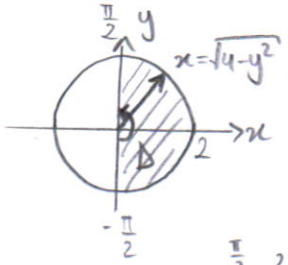
$$=\frac{1}{3}$$

$$\left[\begin{matrix} u=1+2r^2 \Rightarrow r^2=\frac{u-1}{2}\\ du=4r\,dr\end{matrix}\right]$$

7) D bölgesi, $x=\sqrt{4-y^2}$ ve y-ekseni ile sınırlı bir bölge olmak üzere

$\iint_D e^{-x^2-y^2}\,dA$ integralini hesaplayınız

$$D:\left.\begin{array}{l} x=\sqrt{4-y^2} \Rightarrow x^2+y^2=4 \\ x=0 \end{array}\right\}$$



$$\left.\begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \end{array}\right\} \begin{array}{l} x^2+y^2=r^2 \\ dA=r\,dr\,d\theta \end{array}$$

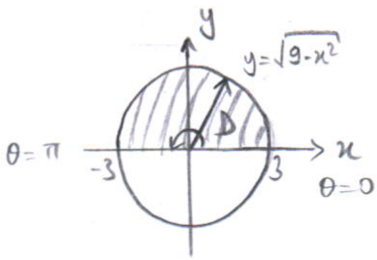
$$\iint_D e^{-x^2-y^2}\,dA=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 e^{-r^2}r\,dr\,d\theta \xrightarrow[2rdr=du]{r^2=u} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 \frac{e^{-u}}{2}\,du\,d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\frac{1}{2}e^{-r^2}\Big|_0^2\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(-\frac{1}{2}e^{-4}+\frac{1}{2}\right)d\theta=-\frac{1}{2}e^{-4}\theta+\frac{\theta}{2}\Big|_{-\frac{\pi}{2}}^{\pi/2}$$

$$=\frac{\pi}{2}\left(1-e^{-4}\right)$$

8) $\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\sin(x^2+y^2)\,dy\,dx$ integralini hesaplayınız.

$$D:\left.\begin{array}{l} y=\sqrt{9-x^2} \Rightarrow x^2+y^2=9 \\ y=0 \\ x=3,\ x=-3 \end{array}\right\}$$



$$\left.\begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \end{array}\right\} \begin{array}{l} x^2+y^2=r^2 \\ dy\,dx=r\,dr\,d\theta \end{array}$$

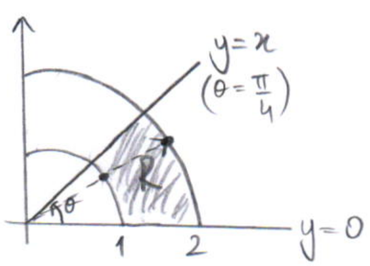
$$\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\sin(x^2+y^2)\,dy\,dx=\int_0^{\pi}\int_0^3 \sin(r^2)r\,dr\,d\theta \xrightarrow[du=2rdr]{u=r^2} \int_0^{\pi}\int_0^3 \frac{\sin u}{2}\,du\,d\theta$$

$$=\int_0^{\pi} -\frac{1}{2}\cos r^2\Big|_0^3\,d\theta=\int_0^{\pi}\left(-\frac{1}{2}\cos 9+\frac{1}{2}\right)d\theta=\left(-\frac{1}{2}\cos 9\right)\theta+\frac{1}{2}\theta\Big|_0^{\pi}$$

$$=\frac{\pi}{2}(1-\cos 9)$$

9) R bölgesi $\{(x,y)\in\mathbb{R}^2 \mid 1\le x^2+y^2\le 4,\ 0\le y\le x\}$ olmak üzere,

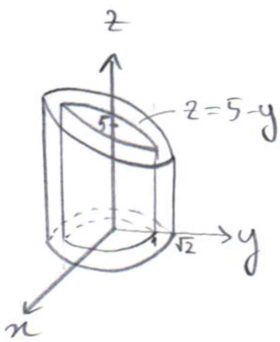
$\iint_R \arctan\left(\frac{y}{x}\right)dA$ integralini hesaplayınız.



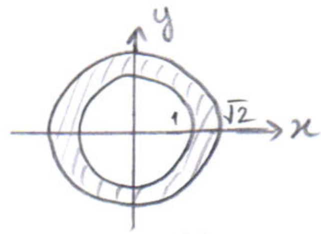
$$\iint_R \arctan\left(\frac{y}{x}\right)dA=\int_0^{\frac{\pi}{4}}\int_1^2 \underbrace{\arctan(\tan\theta)}_{=\theta}\,r\,dr\,d\theta$$

$$=\int_0^{\frac{\pi}{4}}\theta\,\frac{r^2}{2}\Big|_1^2\,d\theta=\int_0^{\pi/4}\frac{3\theta}{2}\,d\theta=\frac{3\theta^2}{4}\Big|_0^{\frac{\pi}{4}}=\frac{3\pi^2}{64}$$

10) $z+y=5$ düzleminin altında, xy-düzleminin üstünde $x^2+y^2=2$ ile $x^2+y^2=1$ silindirleri arasında kalan cismin hacmini bulunuz.



İzdüşüm bölgesi :



$x=r\cos\theta$

$y=r\sin\theta$

$dxdy=r\,dr\,d\theta$

$$V=\int_0^{2\pi}\int_1^{\sqrt{2}}(5-r\sin\theta)\,r\,dr\,d\theta=\int_0^{2\pi}\left(\frac{5}{2}+\frac{1-2\sqrt{2}}{3}\sin\theta\right)d\theta$$

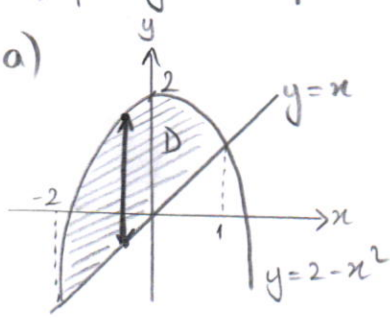
$$=\frac{5}{2}\theta-\frac{1-2\sqrt{2}}{3}\cos\theta\,\Big|_0^{2\pi}=5\pi$$

11) D bölgesi $y=2-x^2$ parabolü ve $y=x$ doğrusu ile sınırlı bölge olsun.

a) Üstten $z=x^2$ yüzeyi ve alttan xy-düzleminde D bölgesi tarafından sınırlanan cismin hacmini bulunuz.

b) $f(x,y)=x^2$ fonksiyonunun D bölgesi üzerinde ortalama değerini bulunuz.

a)



$$2-x^2=x \Rightarrow x^2+x-2=0 \Rightarrow x=-2,\ x=1$$

$$V=\iint_D x^2\,dx\,dy=\int_{-2}^{1}\int_{x}^{2-x^2}x^2\,dy\,dx=\int_{-2}^{1}x^2y\,\Big|_x^{2-x^2}dx$$

$$=\int_{-2}^{1}x^2(2-x^2-x)\,dx=\int_{-2}^{1}(-x^4-x^3+2x^2)\,dx$$

$$=-\frac{x^5}{5}-\frac{x^4}{4}+\frac{2}{3}x^3\,\Big|_{-2}^{1}=\frac{63}{20}$$

b) D üzerinde f nin ortalama değeri $=\dfrac{1}{D\text{'nin alanı}}\cdot\iint_D f(x,y)\,dA$

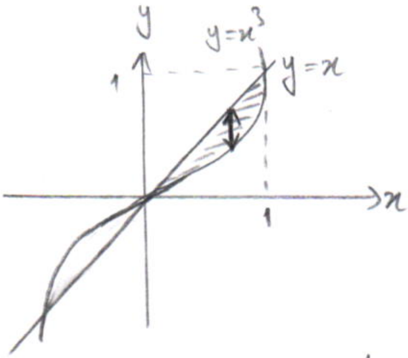
D bölgesinin alanı : $A=\iint_D dy\,dx=\int_{-2}^{1}\int_{x}^{2-x^2}dy\,dx=\int_{-2}^{1}(2-x^2-x)\,dx$

$$=2x-\frac{x^3}{3}-\frac{x^2}{2}\,\Big|_{-2}^{1}=\frac{9}{2}$$

Ort. değer $=\dfrac{V}{A}=\dfrac{63}{20}\cdot\dfrac{2}{9}=\dfrac{7}{10}$

12) $\displaystyle\int_0^1\int_y^{\sqrt[3]{y}} \frac{1}{1+x^4}\,dx\,dy$ integralini hesaplayınız.

$$\left.\begin{matrix} x=y & x=\sqrt[3]{y} \\ y=0 & y=1 \end{matrix}\right\}$$



$$I=\int_0^1\int_{x^3}^{x} \frac{1}{1+x^4}\,dy\,dx = \int_0^1 \frac{x-x^3}{1+x^4}\,dx$$

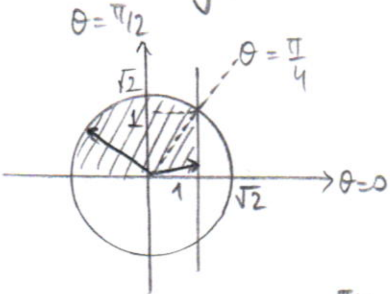
$$= \int_0^1 \frac{x(1-x^2)}{1+x^4}\,dx$$

$x^2=t$
$2x\,dx=dt$
$x\to 0 \quad t\to 0$
$x\to 1 \quad t\to 1$

$$I=\frac{1}{2}\int_0^1 \frac{1-t}{1+t^2}\,dt = \frac{1}{2}\int_0^1 \frac{1}{1+t^2}\,dt - \frac{1}{2}\int_0^1 \frac{t}{1+t^2}\,dt = \frac{1}{2}\arctan t\Big|_0^1 - \frac{1}{4}\ln(1+t^2)\Big|_0^1$$

$$= \frac{1}{2}\left(\frac{\pi}{4}-0\right) - \frac{1}{4}\ln 2 = \frac{\pi-\ln 4}{8}$$

13) $D:\begin{cases} x^2+y^2\le 2 \\ x\le 1 \\ y\ge 0\end{cases}$ bölgesinde $I=\iint_D x\,dA$ integralinin sınırlarını kutupsal dönüşüm ile yazınız.

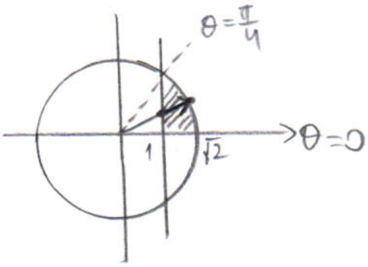


$x=r\cos\theta$
$y=r\sin\theta$
$dx\,dy=r\,dr\,d\theta$

$$\left\{\begin{matrix} x=1 \Rightarrow r\cos\theta=1 \Rightarrow r=\dfrac{1}{\cos\theta}=\sec\theta \\ x^2+y^2=2 \Rightarrow r=\sqrt{2}\end{matrix}\right.$$

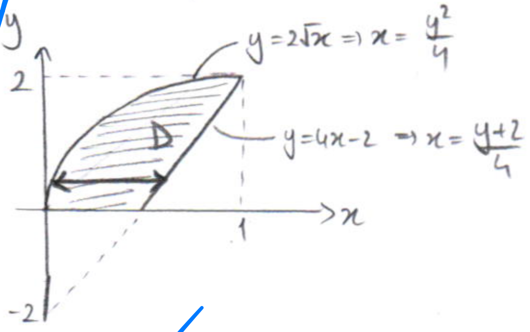
$$I=\iint_D x\,dA = \int_0^{\pi/4}\int_0^{\sec\theta} r\cos\theta\, r\,dr\,d\theta + \int_{\pi/4}^{\pi}\int_0^{\sqrt{2}} r\cos\theta\, r\,dr\,d\theta$$

$D:\begin{cases} x^2+y^2\le 2 \\ x\ge 1 \\ y\ge 0\end{cases}$ bölgesinde aynı integral;



$$I=\iint_D x\,dA = \int_0^{\pi/4}\int_{\sec\theta}^{\sqrt{2}} r\cos\theta\, r\,dr\,d\theta$$

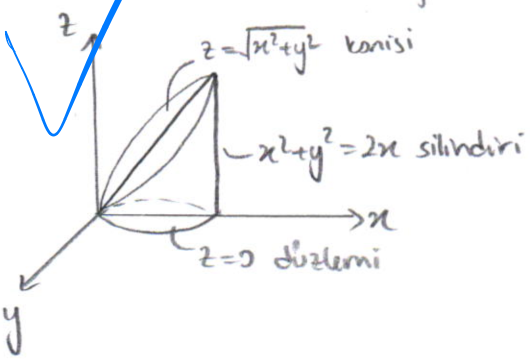
14) $z=16-x^2-y^2$ yüzeyinin altında ve $z=0$'da, $y=2\sqrt{x}$, $y=4x-2$ ve x-ekseni ile sınırlı D bölgesinin üstünde bulunan cismin hacmini veren integrali yazınız.



$$V=\iint_D (16-x^2-y^2)\,dx\,dy$$

$$=\int_0^2\int_{y^2/4}^{\frac{y+2}{4}} (16-x^2-y^2)\,dx\,dy$$

15) $x^2+y^2=2x$, $z=0$, $z=\sqrt{x^2+y^2}$ yüzeyleri ile sınırlandırılmış cismin hacmini veren integrali yazınız.



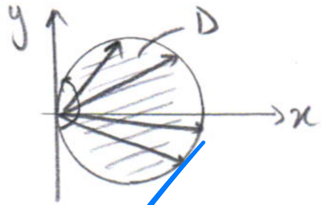
$$x^2+y^2-2x+1-1=0 \Rightarrow (x-1)^2+y^2=1$$

$$V=\iint_D f(x,y)\,dx\,dy=\iint_D \sqrt{x^2+y^2}\,dx\,dy$$

yükseklik konisinden, taban alanı silindirden

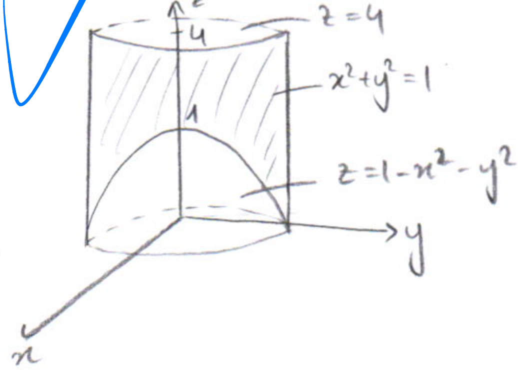
$$\left.\begin{aligned} x&=r\cos\theta \\ y&=r\sin\theta \end{aligned}\right\} \quad \begin{aligned} dx\,dy&=r\,dr\,d\theta \\ \sqrt{x^2+y^2}&=r \end{aligned}$$

$$\left.\begin{aligned} x^2+y^2&=2x \\ r^2&=2r\cos\theta \end{aligned}\right\} \quad r=2\cos\theta$$

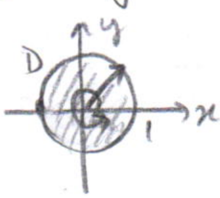


$$V=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r\,r\,dr\,d\theta$$

16) $x^2+y^2=1$ silindirinin içinde, $z=4$ düzleminin altında, $z=1-x^2-y^2$ parabolünün üzerinde kalan cismin hacmini bulunuz.



$D: x^2+y^2=1$



$x=r\cos\theta$
$y=r\sin\theta$
$dx\,dy=r\,dr\,d\theta$
$x^2+y^2=r^2$

$$V=\iint_D \left(4-(1-x^2-y^2)\right)dx\,dy=\int_0^{2\pi}\int_0^1 (3+r^2)\,r\,dr\,d\theta=\int_0^{2\pi} \left.\frac{3}{2}r^2+\frac{r^4}{4}\right|_0^1 d\theta=\int_0^{2\pi}\frac{7}{4}\,d\theta$$

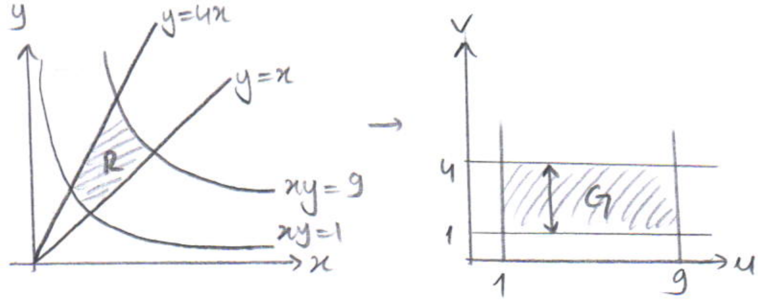
$$=\frac{7}{2}\pi$$

17) $R: \begin{cases} xy=1 \\ xy=9 \\ y=x \\ y=4x \end{cases}$ eğrilerinin 1. bölgede sınırladığı bölge ise

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy = ?$$

$\left.\begin{matrix} xy=u \\ \frac{y}{x}=v \end{matrix}\right\}$

R		G
$xy=1$	$\rightarrow$	$u=1$
$xy=9$	$\rightarrow$	$u=9$
$y=x$	$\rightarrow$	$v=1$
$y=4x$	$\rightarrow$	$v=4$



$$J(u,v) = \frac{1}{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}} = \frac{1}{\begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix}} = \frac{1}{\frac{y}{x}+\frac{y}{x}} = \frac{1}{\frac{2y}{x}} = \frac{1}{2v}$$

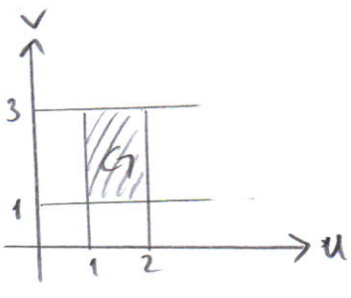
$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy = \int_1^9\int_1^4 (\sqrt{v}+\sqrt{u}) \frac{1}{2v} dvdu = \int_1^9 \left(\sqrt{v} + \frac{1}{2}\sqrt{u}\ln v\right)\Big|_1^4 du$$

$$= \int_1^9 (1+\ln 2\sqrt{u})\, du = u + \ln 2 \cdot \frac{2}{3} u^{3/2}\Big|_1^9 = 8 + \frac{52}{3}\ln 2$$

18) $y=x^2$, $y=2x^2$, $x=y^2$, $x=3y^2$ parabolleri ile sınırlandırılmış bölgenin alanını iki katlı integral yardımıyla hesaplayınız.

$\left.\begin{matrix} \frac{y}{x^2}=u \\ \frac{x}{y^2}=v \end{matrix}\right\}$

D		G
$\frac{y}{x^2}=1$	$\rightarrow$	$u=1$
$\frac{y}{x^2}=2$	$\rightarrow$	$u=2$
$\frac{x}{y^2}=1$	$\rightarrow$	$v=1$
$\frac{x}{y^2}=3$	$\rightarrow$	$v=3$



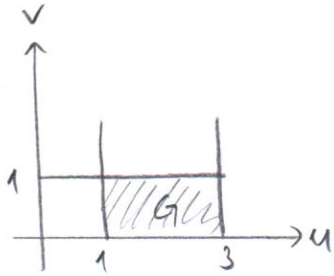
$$J(u,v) = \frac{1}{\begin{vmatrix} -\frac{2y}{x^3} & \frac{1}{x^2} \\ \frac{1}{y^2} & -\frac{2x}{y^3} \end{vmatrix}} = \frac{x^2y^2}{3} = \frac{1}{3u^2v^2} \Rightarrow A = \iint_G \frac{1}{3u^2v^2} dudv = \frac{1}{3}\int_1^2\int_1^3 \frac{1}{u^2v^2} dvdu$$

$$= \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{3} = \frac{1}{9}$$

19) Eğer D, $x+y=1$, $x+y=3$, $x-y=0$, $x-y=1$ doğruları ile oluşturulmuş bölge ise $\iint_D \frac{x^2-y^2}{\sqrt{1+3(x-y)^2}}\,dA$ integralini hesaplayınız.

$\left.\begin{array}{l} u=x+y \\ v=x-y \end{array}\right\}$

D		G
$x+y=1$	$\longrightarrow$	$u=1$
$x+y=3$	$\longrightarrow$	$u=3$
$x-y=0$	$\longrightarrow$	$v=0$
$x-y=1$	$\longrightarrow$	$v=1$



$$J(u,v)=\frac{1}{\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}}=-\frac{1}{2}$$

$$\iint_D \frac{x^2-y^2}{\sqrt{1+3(x-y)^2}}\,dA=\iint_G \frac{uv}{\sqrt{1+3v^2}}\,\frac{1}{2}\,du\,dv=\frac{1}{2}\int_0^1\int_1^3 \frac{uv}{\sqrt{1+3v^2}}\,du\,dv=\frac{1}{2}\int_0^1 4\cdot\frac{v}{\sqrt{1+3v^2}}\,dv$$

$$=2\cdot\frac{1}{3}\sqrt{1+3v^2}\,\Big|_0^1=\frac{2}{3}$$

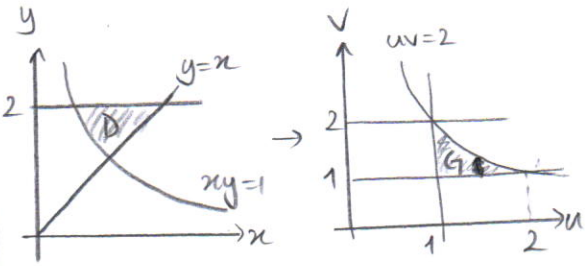
20) $\displaystyle\int_1^2\int_{1/y}^{y}\sqrt{\frac{y}{x}}\,e^{\sqrt{xy}}\,dx\,dy=?$

$\left.\begin{array}{l} u=\sqrt{xy} \\ v=\sqrt{\frac{y}{x}} \end{array}\right\}$ $\frac{u}{\sqrt2}=\frac{\sqrt2}{v}$

D		G
$x=\frac{1}{y}\Rightarrow xy=1$	$\longrightarrow$	$u=1$
$x=y$	$\longrightarrow$	$v=1$
$y=2$	$\longrightarrow$	$uv=2$ $\left(\begin{array}{l}u=\sqrt{2x}\\ v=\sqrt{\frac{2}{x}}\end{array}\right)$
$y=1$ (nokta)		

$u=\sqrt{2x}$ $\quad\sqrt{x}=\frac{u}{\sqrt2}$

$v=\sqrt{\frac{2}{x}}$ $\quad\sqrt{x}=\frac{\sqrt2}{v}$



$u^2=xy$, $v^2=\frac{y}{x}$ $\Rightarrow$ $u^2v^2=y^2$, $\frac{u^2}{v^2}=x^2$ $\Rightarrow$ $y=uv$, $x=\frac{u}{v}$

$$J(u,v)=\begin{vmatrix}\frac{1}{v} & -\frac{u}{v^2}\\ v & u\end{vmatrix}=\frac{2u}{v}$$

$$\iint_D \sqrt{\frac{y}{x}}\,e^{\sqrt{xy}}\,dx\,dy=\iint_G v e^{u}\,\frac{2u}{v}\,du\,dv=\int_1^2\int_1^{2/u} 2ue^{u}\,dv\,du=2\int_1^2 vue^{u}\Big|_1^{2/u}\,du$$

$$=2\int_1^2 (2e^{u}-ue^{u})\,du=2\left((2-u)e^{u}+e^{u}\right)\Big|_1^2=2e(e-2)$$