

Regular Expressions

Regular Expressions

Regular expressions
describe regular languages

Example: $(a + b \cdot c)^*$

describes the language

$$\{a, bc\}^* = \{\varepsilon, a, bc, aa, abc, bca, \dots\}$$

Recursive Definition

Primitive regular expressions: \emptyset , ε , a

Given regular expressions r_1 and r_2

$$r_1 + r_2$$

$$r_1 \cdot r_2$$

$$r_1^*$$

$$(r_1)$$

Are regular expressions

Examples

A regular expression: $(a + b \cdot c)^* \cdot (c + \emptyset)$

Not a regular expression: $(a + b +)$

Languages of Regular Expressions

$L(r)$: language of regular expression r

Example

$$L((a + b \cdot c)^*) = \{\varepsilon, a, bc, aa, abc, bca, \dots\}$$

Definition

For primitive regular expressions:

$$L(\emptyset) = \emptyset$$

$$L(\varepsilon) = \{\varepsilon\}$$

$$L(a) = \{a\}$$

Definition (continued)

For regular expressions r_1 and r_2

$$L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) = L(r_1) L(r_2)$$

$$L(r_1^*) = (L(r_1))^*$$

$$L((r_1)) = L(r_1)$$

Example

Regular expression: $(a + b) \cdot a^*$

$$\begin{aligned} L((a + b) \cdot a^*) &= L((a + b)) L(a^*) \\ &= L(a + b) L(a^*) \\ &= (L(a) \cup L(b)) (L(a))^* \\ &= (\{a\} \cup \{b\}) (\{a\})^* \\ &= \{a, b\} \{\varepsilon, a, aa, aaa, \dots\} \\ &= \{a, aa, aaa, \dots, b, ba, baa, \dots\} \end{aligned}$$

Example

Regular expression $r = (a + b)^*(a + bb)$

$$L(r) = \{a, bb, aa, abb, ba, bbb, \dots\}$$

Example

Regular expression $r = (aa)^*(bb)^*b$

$$L(r) = \{a^{2n}b^{2m}b : n, m \geq 0\}$$

Example

Regular expression

$$r = (0 + 1)^* 00 (0 + 1)^*$$

$$L(r) = \{ \text{all strings containing substring } 00 \}$$

Example

Regular expression $r = (1 + 01)^* (0 + \varepsilon)$

$L(r) = \{ \text{all strings without substring } 00 \}$

Equivalent Regular Expressions

Definition:

Regular expressions r_1 and r_2

are **equivalent** if $L(r_1) = L(r_2)$

Example

$L = \{ \text{all strings without substring } 00 \}$

$$r_1 = (1 + 01)^* (0 + \varepsilon)$$

$$r_2 = (1^* 0 1 1^*)^* (0 + \varepsilon) + 1^* (0 + \varepsilon)$$

$L(r_1) = L(r_2) = L \rightarrow r_1 \text{ and } r_2$
are equivalent
regular expressions

Regular Expressions and Regular Languages

Theorem

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Expressions} \end{array} \right\} = \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Proof:

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Expressions} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Expressions} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Proof - Part 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Expressions} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

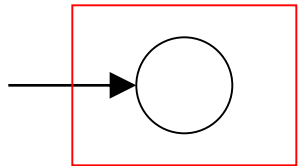
For any regular expression r
the language $L(r)$ is regular

Proof by induction on the size of r

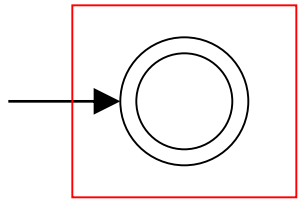
Induction Basis

Primitive Regular Expressions: \emptyset , ε , a

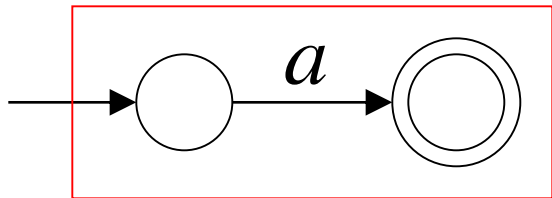
Corresponding
NFAs



$$L(M_1) = \emptyset = L(\emptyset)$$



$$L(M_2) = \{\varepsilon\} = L(\varepsilon)$$



$$L(M_3) = \{a\} = L(a)$$

regular
languages

Inductive Hypothesis

Suppose

that for regular expressions r_1 and r_2 ,
 $L(r_1)$ and $L(r_2)$ are regular languages

Inductive Step

We will prove:

$$L(r_1 + r_2)$$

$$L(r_1 \cdot r_2)$$

$$L(r_1^*)$$

$$L((r_1))$$

Are regular
Languages

By definition of regular expressions:

$$L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) = L(r_1) L(r_2)$$

$$L(r_1^*) = (L(r_1))^*$$

$$L((r_1)) = L(r_1)$$

By inductive hypothesis we know:

$L(r_1)$ and $L(r_2)$ are regular languages

We also know:

Regular languages are closed under:

Union

$$L(r_1) \cup L(r_2)$$

Concatenation

$$L(r_1) L(r_2)$$

Star

$$(L(r_1))^*$$

Therefore:

$$L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) = L(r_1) L(r_2)$$

$$L(r_1^*) = (L(r_1))^*$$

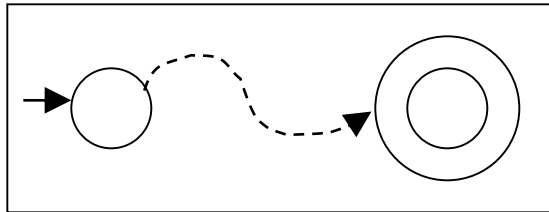
Are regular
languages

$L((r_1)) = L(r_1)$ is trivially a regular language
(by induction hypothesis)

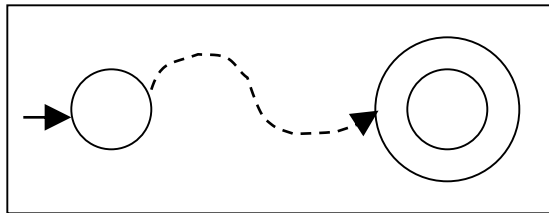
Using the regular closure of operations,
we can construct recursively the NFA M
that accepts $L(M) = L(r)$

Example: $r = r_1 + r_2$

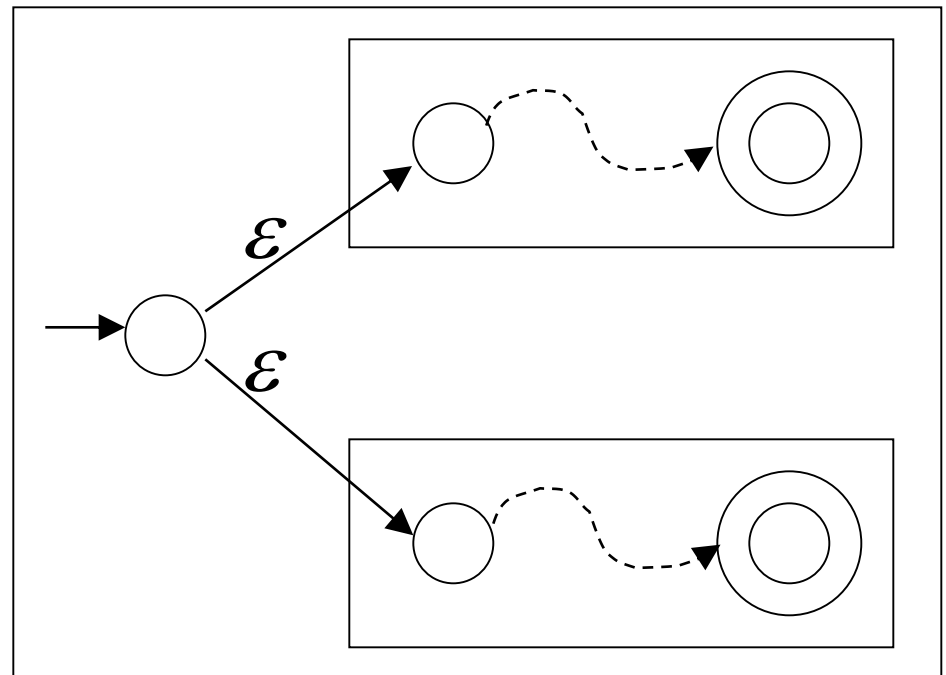
$$L(M_1) = L(r_1)$$



$$L(M_2) = L(r_2)$$



$$L(M) = L(r)$$



Proof - Part 2

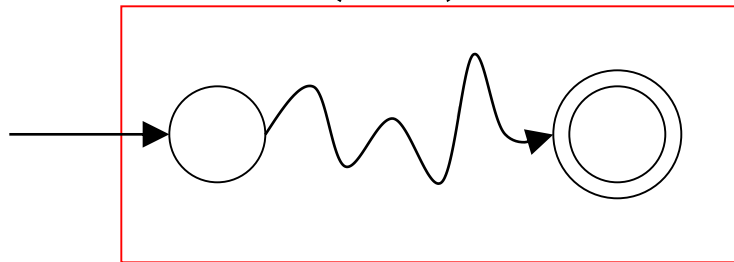
$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Expressions} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

For any regular language L there is
a regular expression r with $L(r) = L$

We will convert an NFA that accepts L
to a regular expression

Since L is regular, there is a NFA M that accepts it

$$L(M) = L$$



Take it with a single accept state

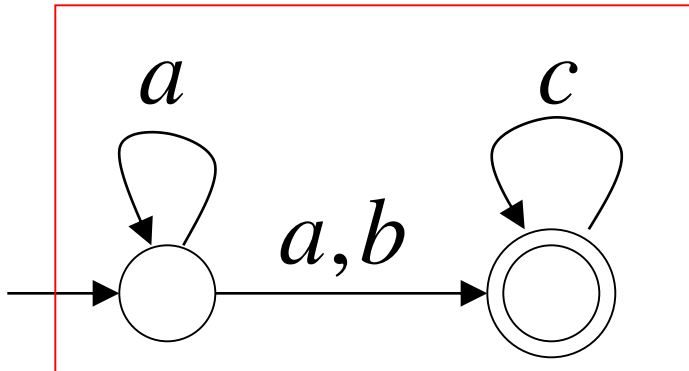
From M construct the equivalent

Generalized Transition Graph

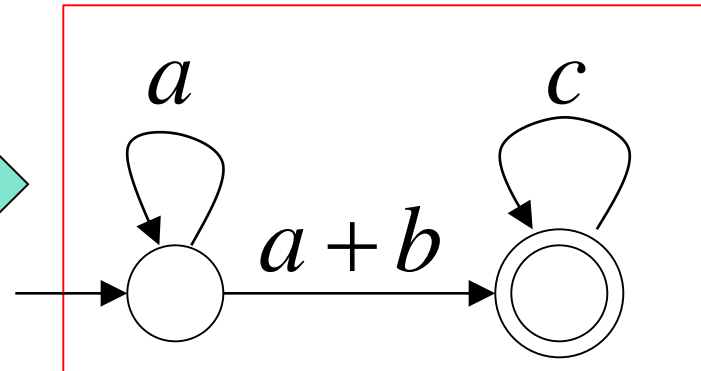
in which transition labels are regular expressions

Example:

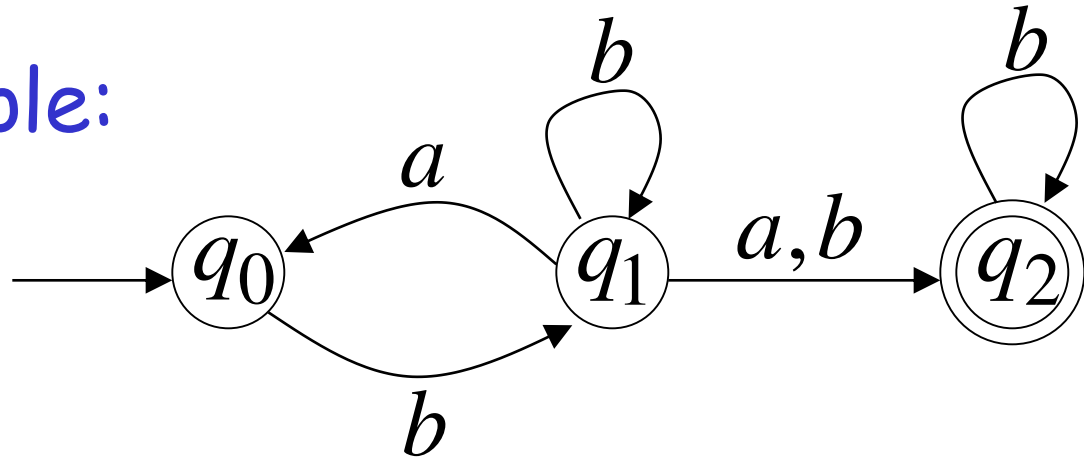
M



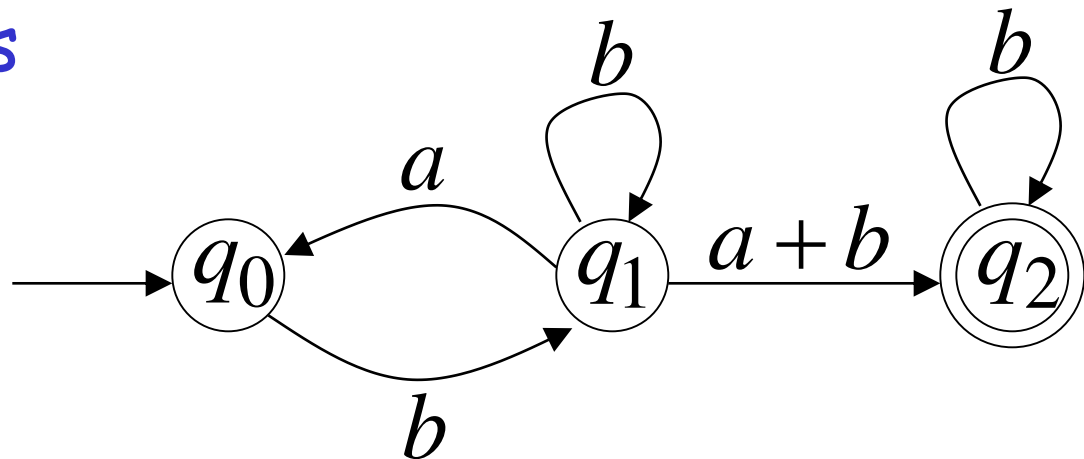
Corresponding
Generalized transition graph



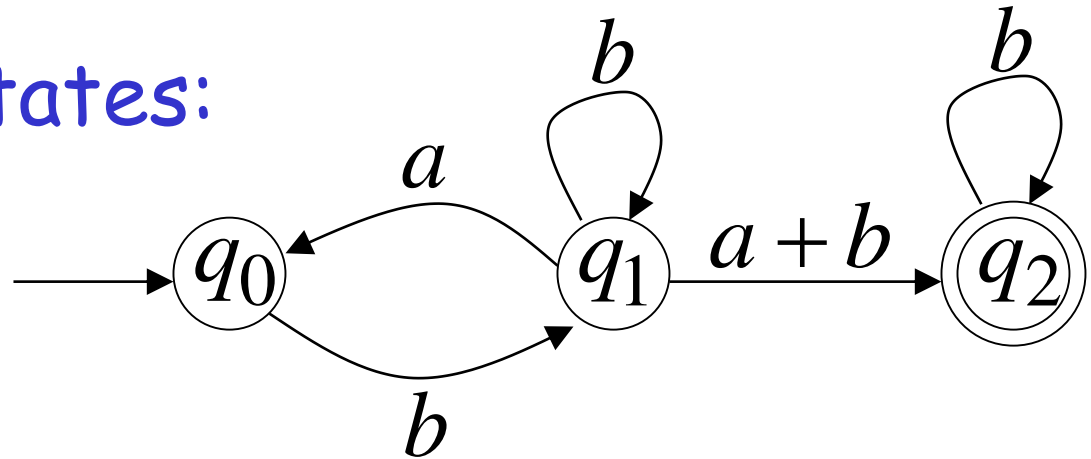
Another Example:



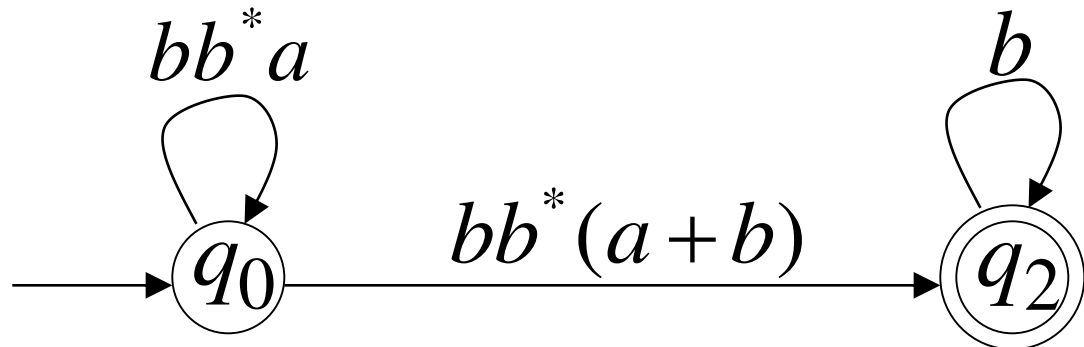
Transition labels
are regular
expressions



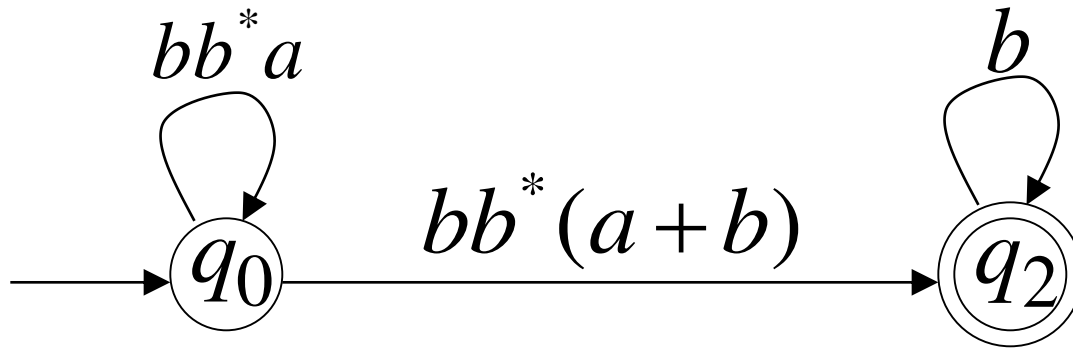
Reducing the states:



Transition labels
are regular
expressions



Resulting Regular Expression:

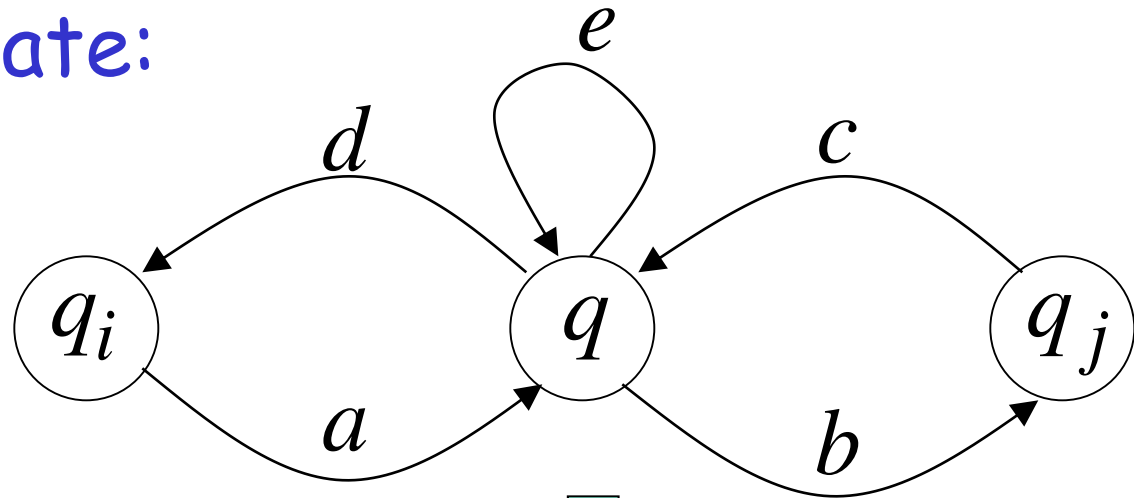


$$r = (bb^*a)^*bb^*(a+b)b^*$$

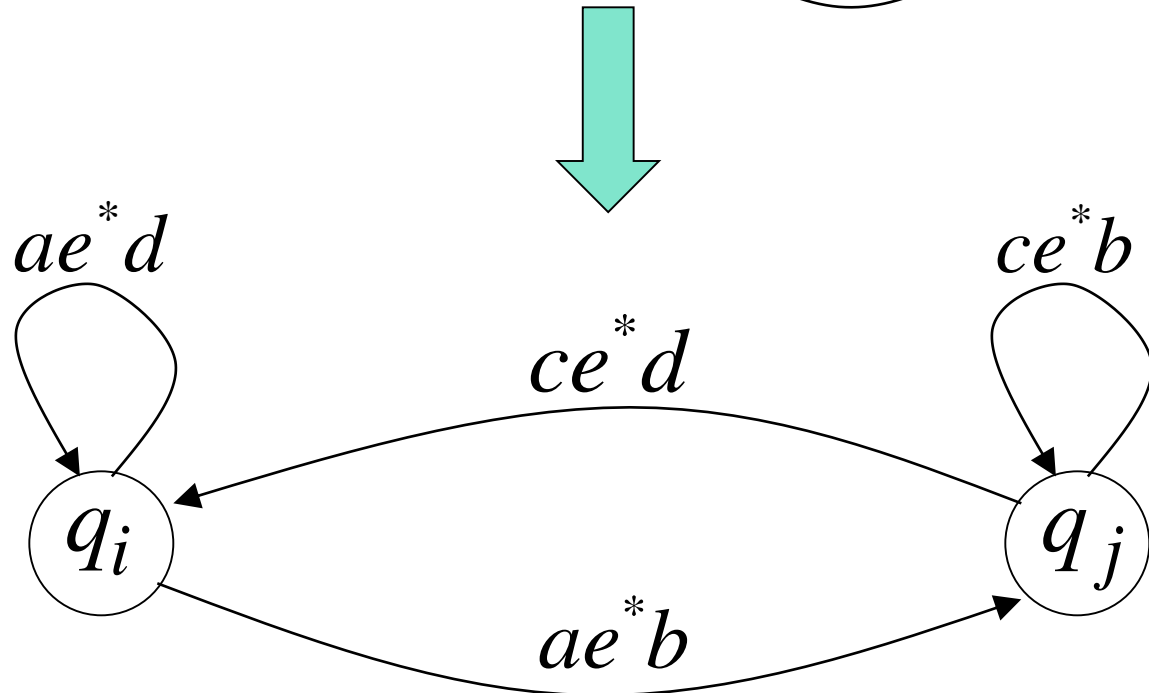
$$L(r) = L(M) = L$$

In General

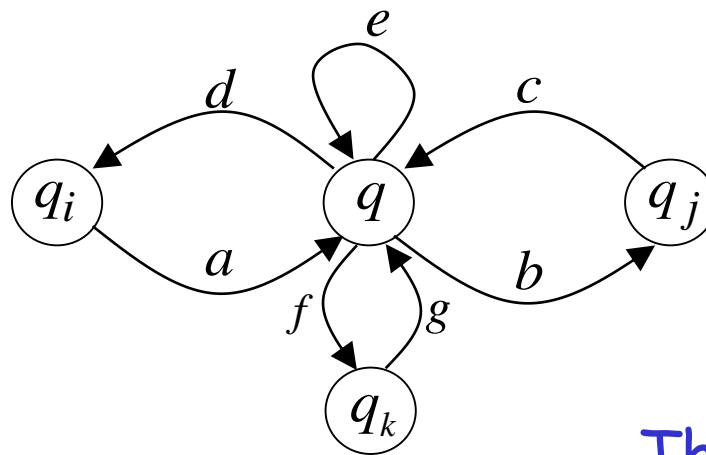
Removing a state:



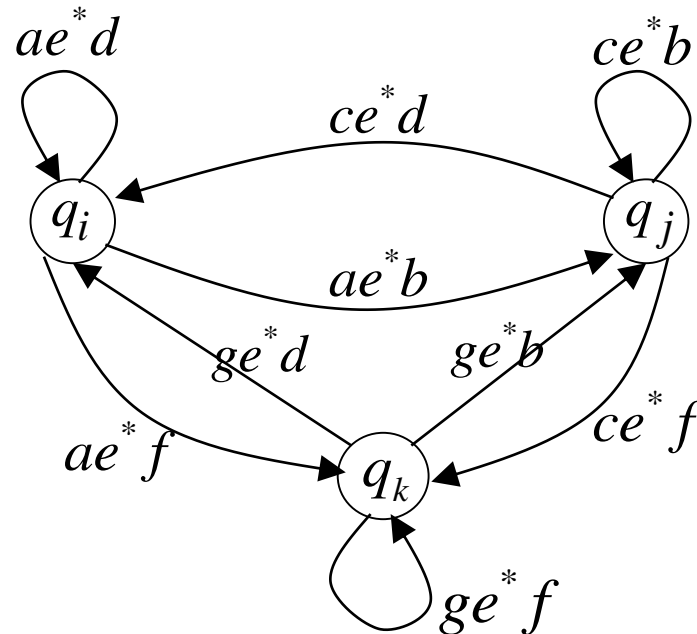
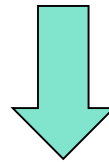
2-neighbors



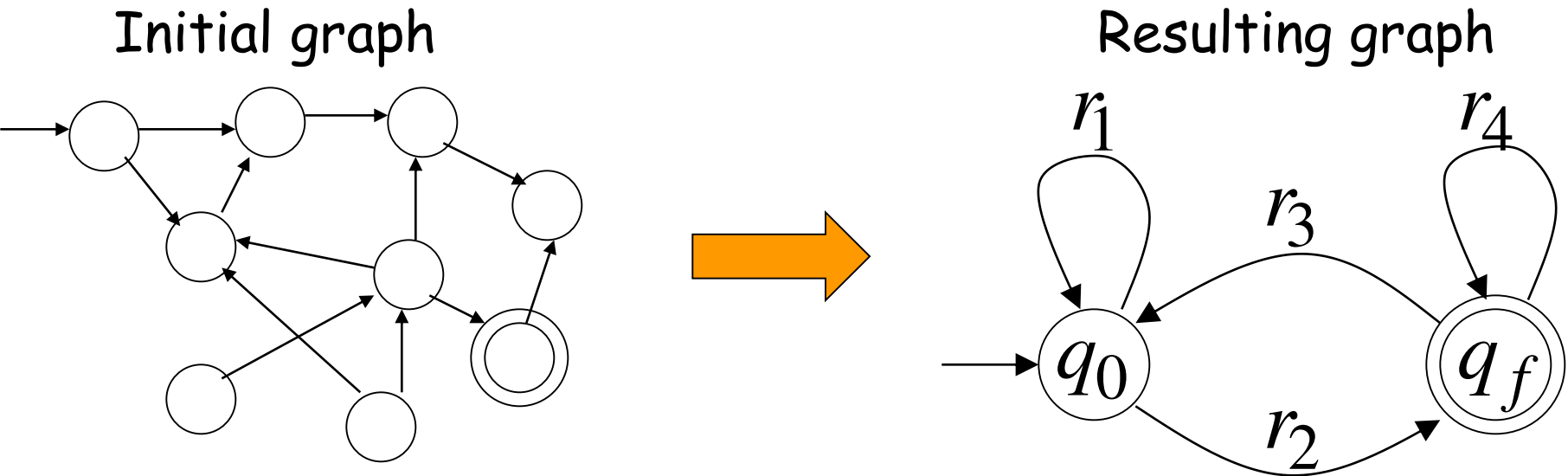
3-neighbors



This can be generalized
to arbitrary number
of neighbors to q



By repeating the process until two states are left, the resulting graph is

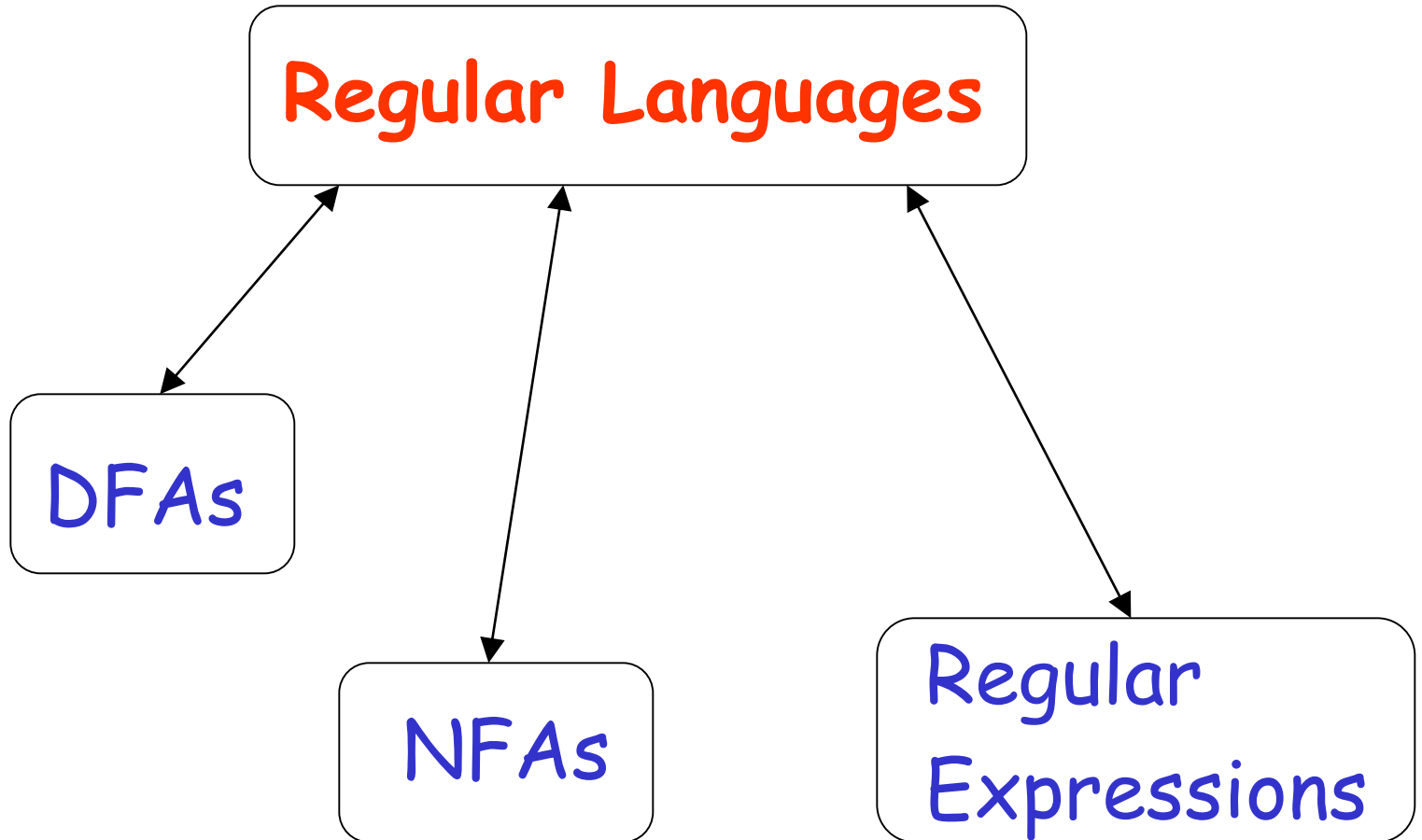


The resulting regular expression:

$$r = r_1^* r_2 (r_4 + r_3 r_1^* r_2)^*$$

$$L(r) = L(M) = L$$

Standard Representations of Regular Languages



When we say: We are given
a Regular Language L

We mean: Language L is in a standard
representation

(DFA, NFA, or Regular Expression)