BLM2041 Signals and Systems

Syllabus

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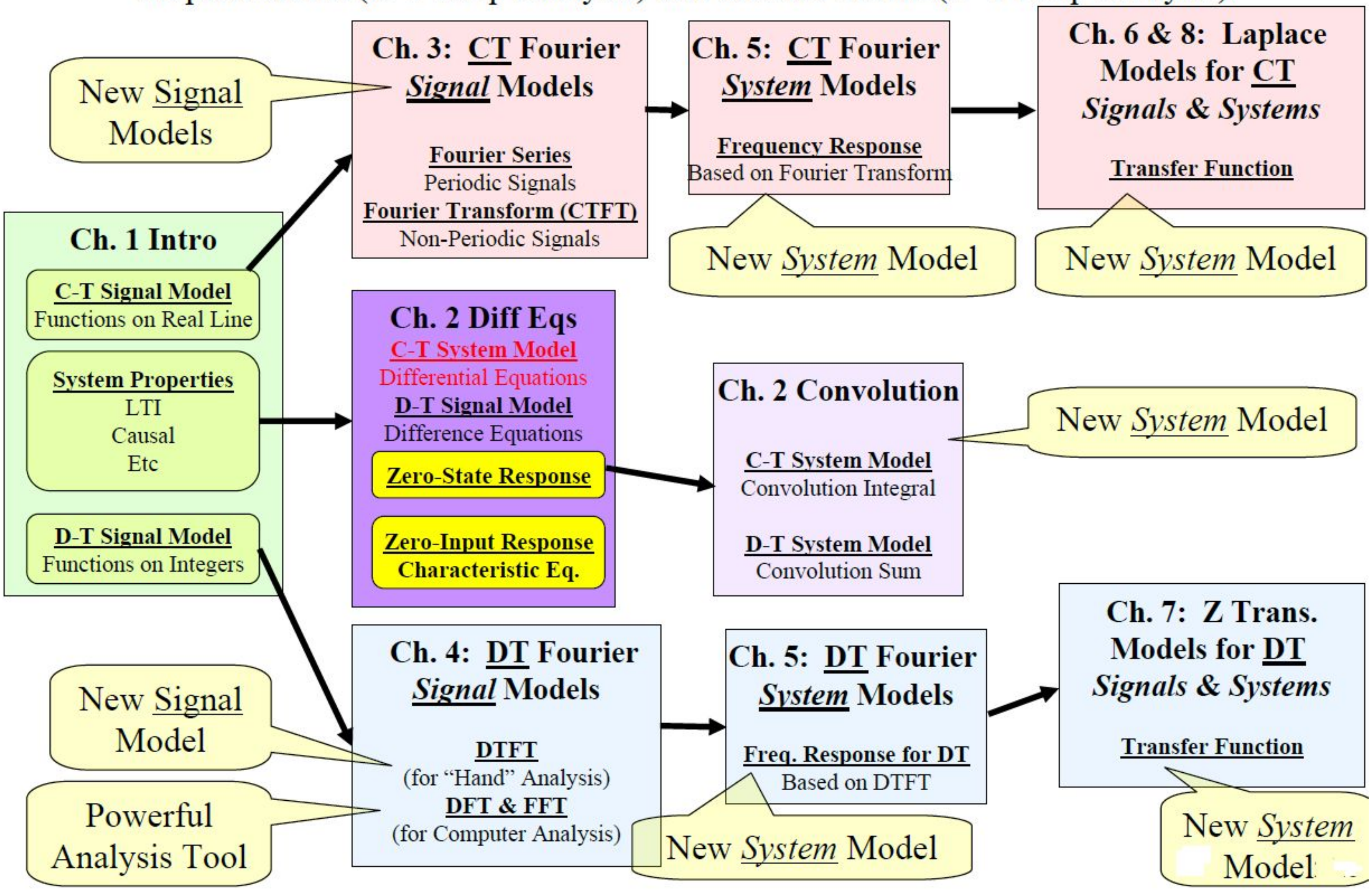
Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



System Modeling

To do engineering design, we must be able to accurately predict the quantitative behavior of a circuit or other system.

This requires math models:

Circuits

Device Rules
- R: $v(t)=Ri(t)$
- L: $v(t)=L(di(t)/di)$
- C: $dv(t)/dt=1/Ci(t)$

Circuit Rules
- -KVL
- -KCL
- -Voltage Divider
- -etc.

Differential Equation

Mechanical

Device Rules
- Mass: $M(d^2p(t)/dt^2)$
- Spring: $k_x p(t)$
- Damping: $k_d(dp(t)/dt)$

System Rules
- -Sum of forces
- -etc.

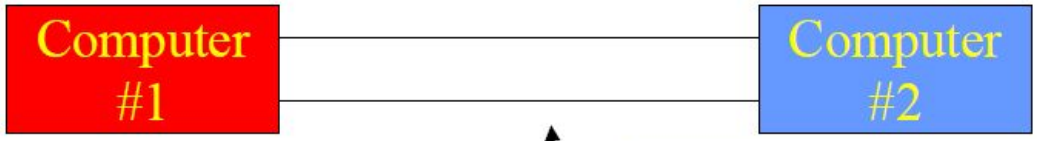
Differential Equation

Similar ideas hold for hydraulic, chemical, etc. systems…

"differential equations rule the world"

Simple Circuit Example

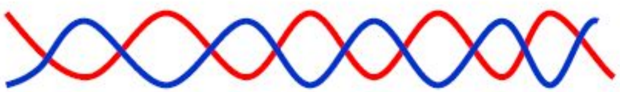
Sending info over a wire cable between two computers



Two conductors separated by an insulator
⇒ capacitance

Two practical examples of the cable

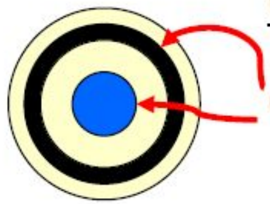
"Twisted Pair" of Insulated Wires



Typical values: 100 Ω/km

50 nF/km

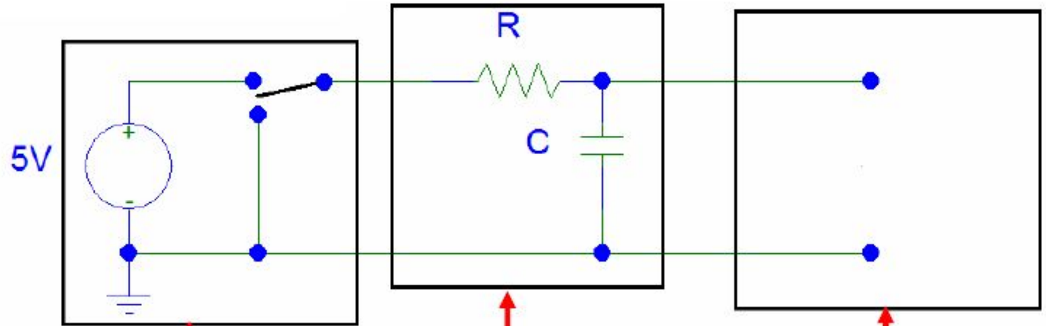
coaxial cable



conductors separated by insulator

Recall: resistance increases with wire length

Simple Model



Driver's Thevenin Equivalent Circuit (Computer #1)

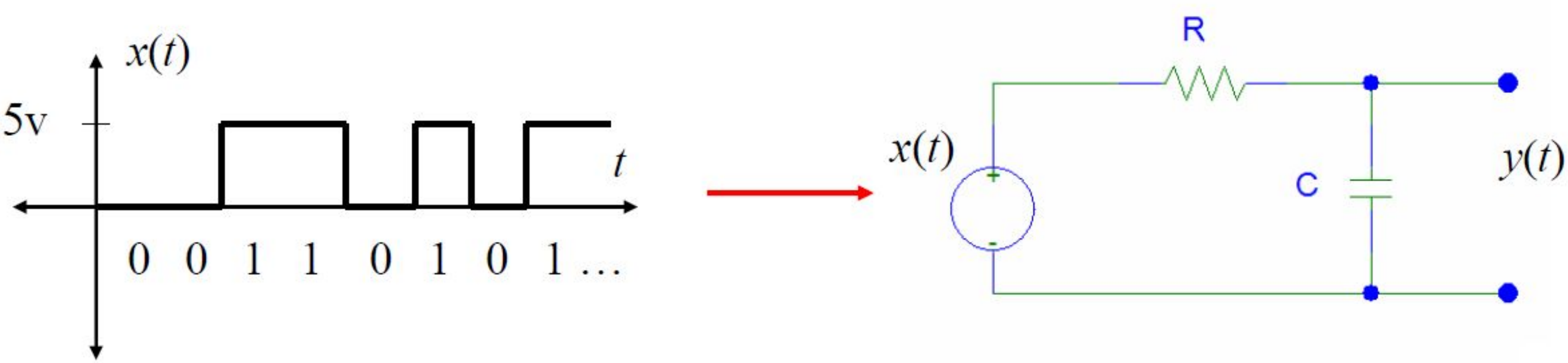
Cable Model

Receiver's Thevenin Equivalent Circuit (Computer #2)

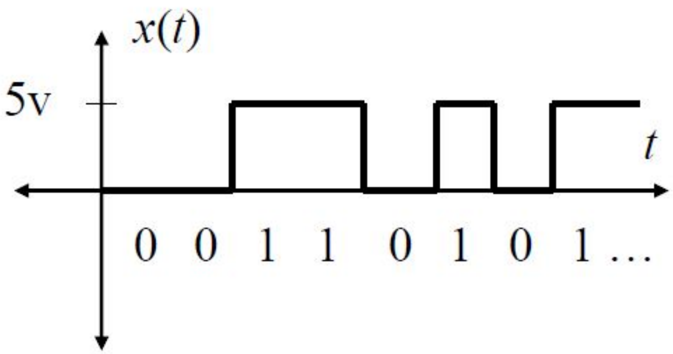
Zero Output Resistance (Ideal)

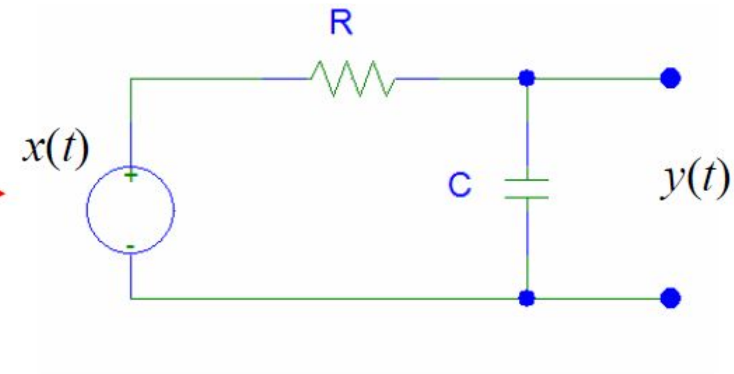
Infinite Input Resistance (Ideal)

Effective Operation:



Simple Model





Use Loop Equation & Device Rules:

$$x(t) = v_R(t) + y(t)$$

$$v_R(t) = Ri(t)$$

$$i(t) = C\frac{dy(t)}{dt}$$

$$\boxed{\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)}$$

This is the Differential Equation to be "Solved":

Given: Input $x(t)$ Find: Solution $y(t)$

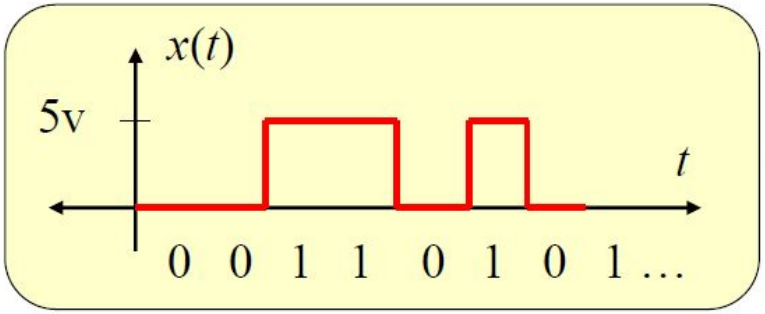
Recall: A "Solution" of the D.E. means… The function that when put into the left side causes it to reduce to the right side

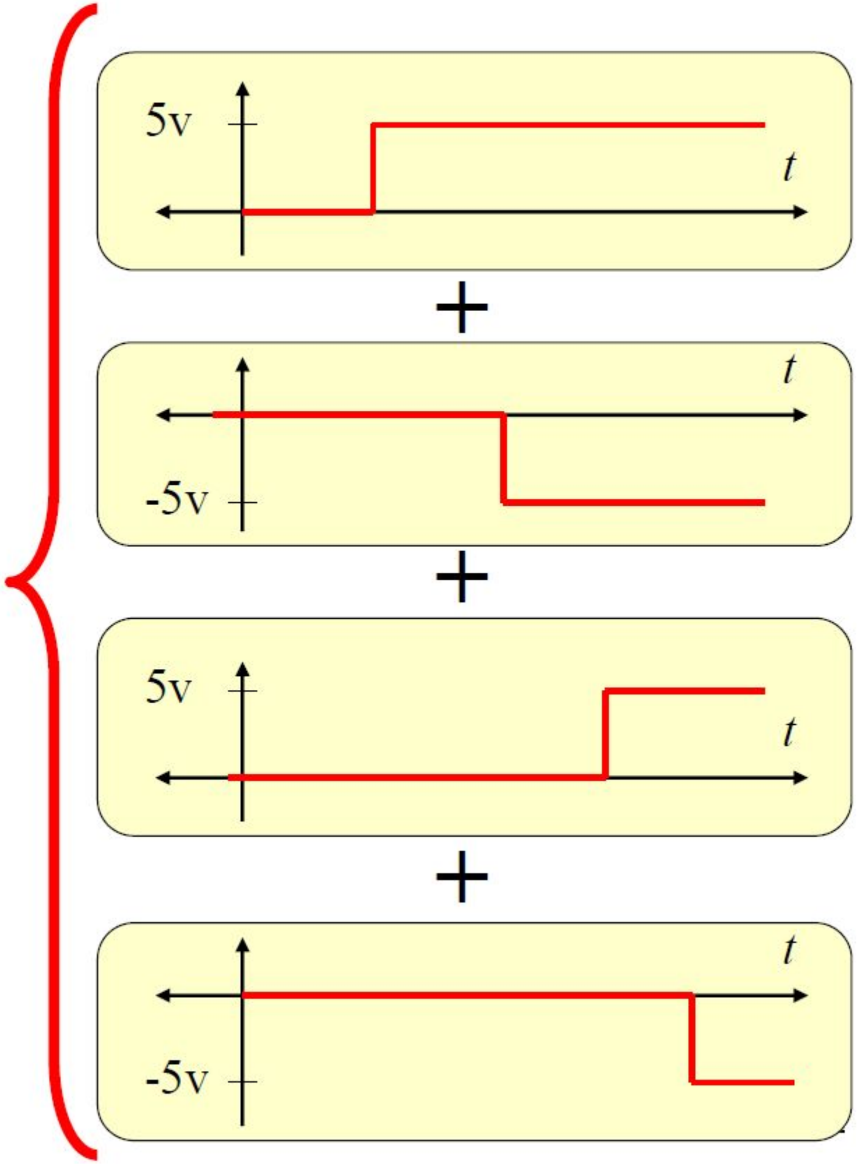
Differential Equation & System … the solution is the output

Simple Model

Now… because this is a **linear** system (it only has *R*, *L*, *C* components!) we can analyze it by **superposition**.

Decompose the input…



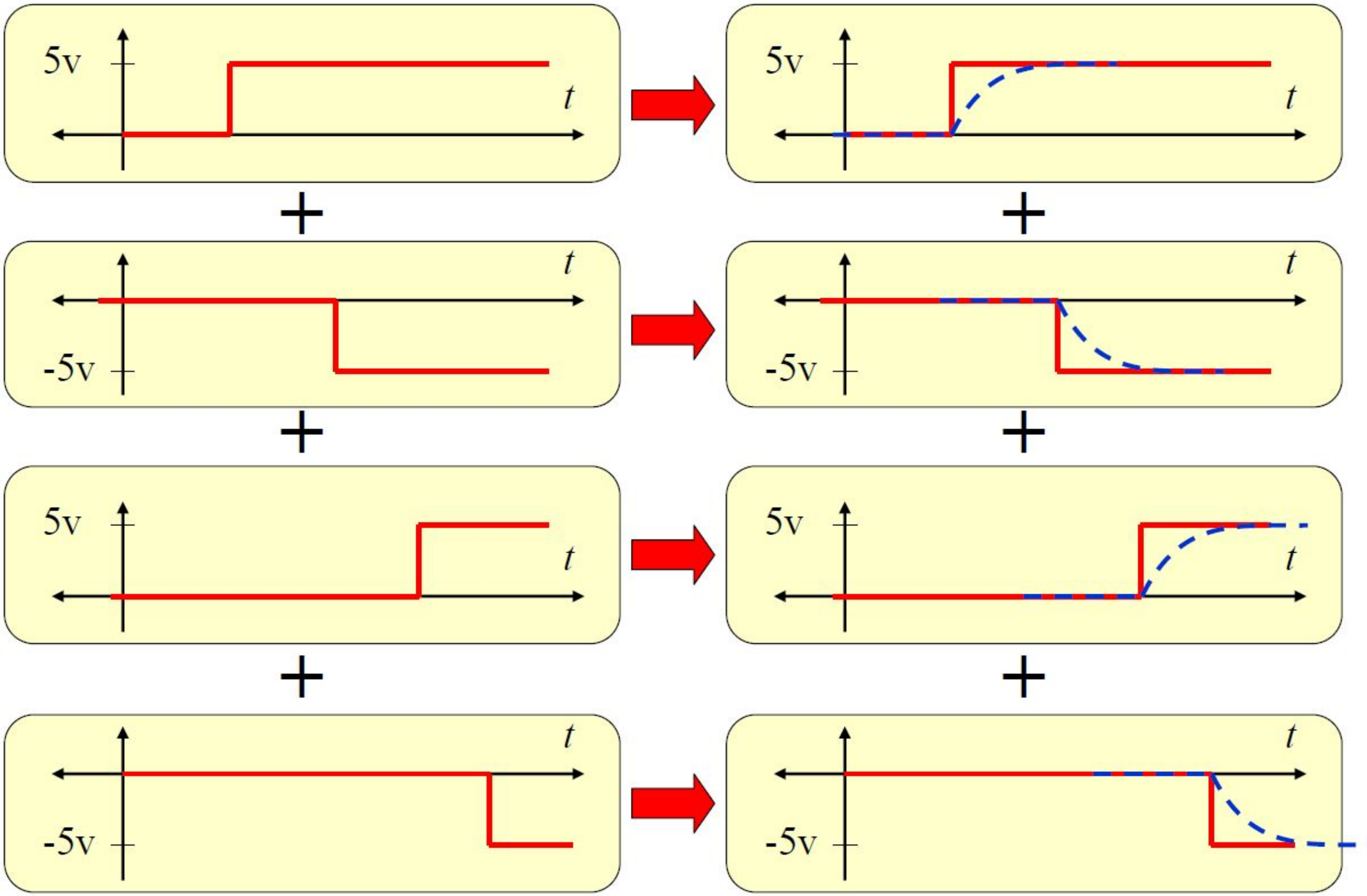


Simple Model

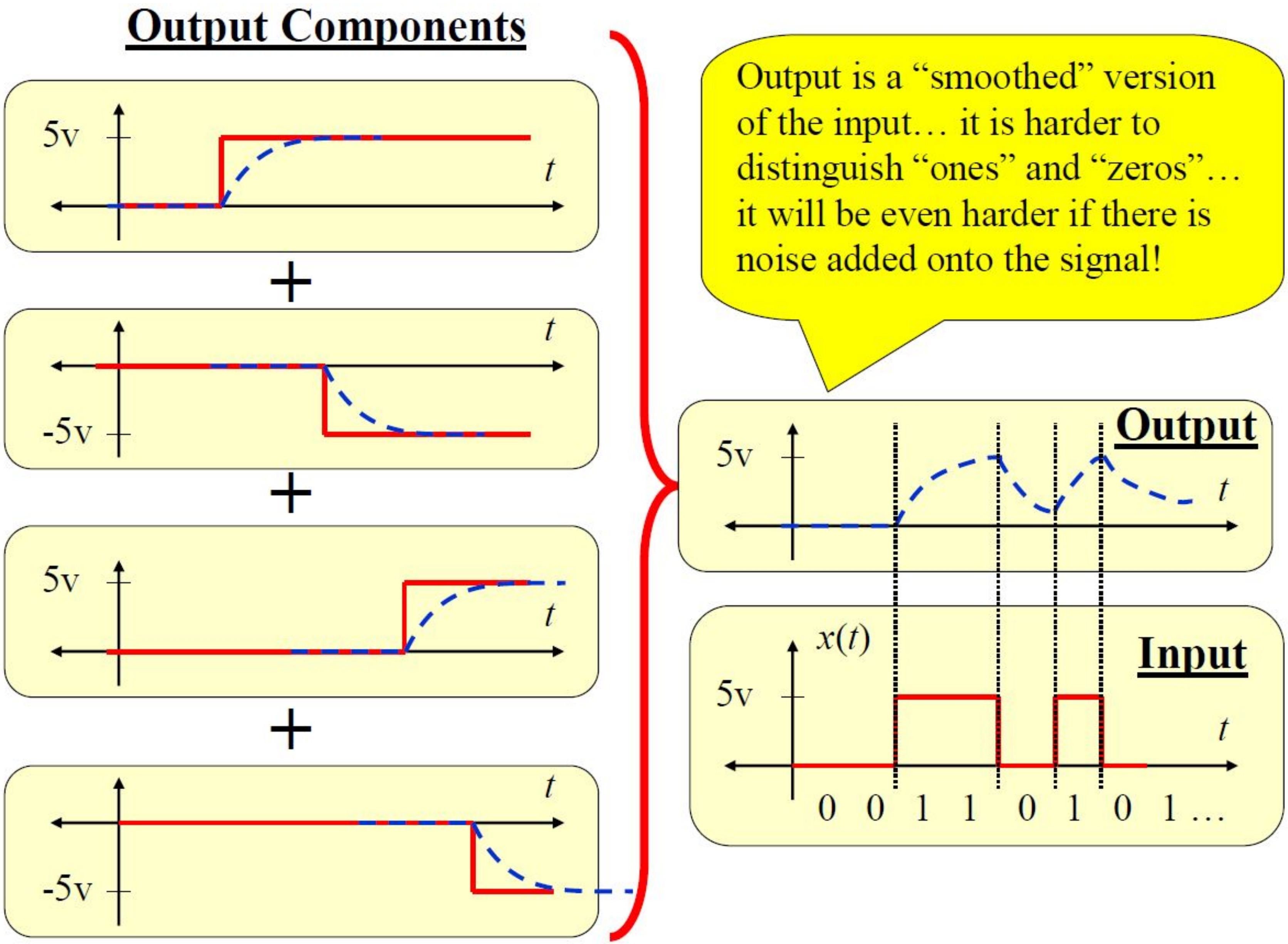
Input Components

Output Components (Blue)

Standard Exponential Response
Learned in "Circuits":

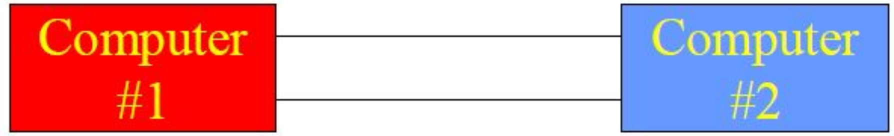


Simple Model

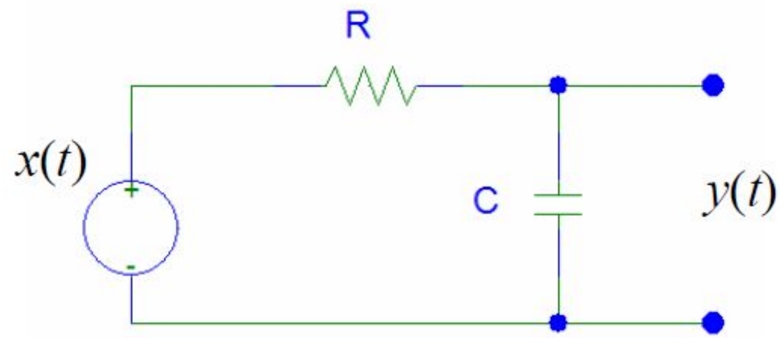


Progression of Ideas an Engineer Might Use for this Problem

Physical System:



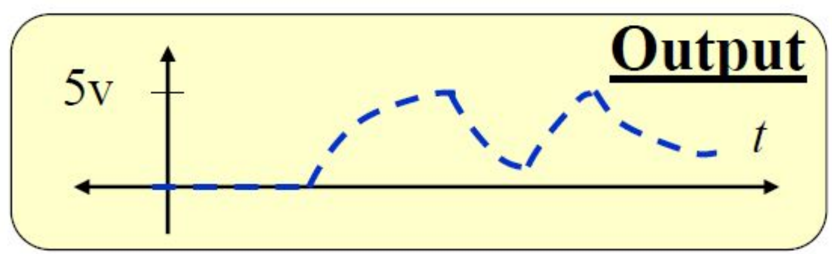
Schematic System:



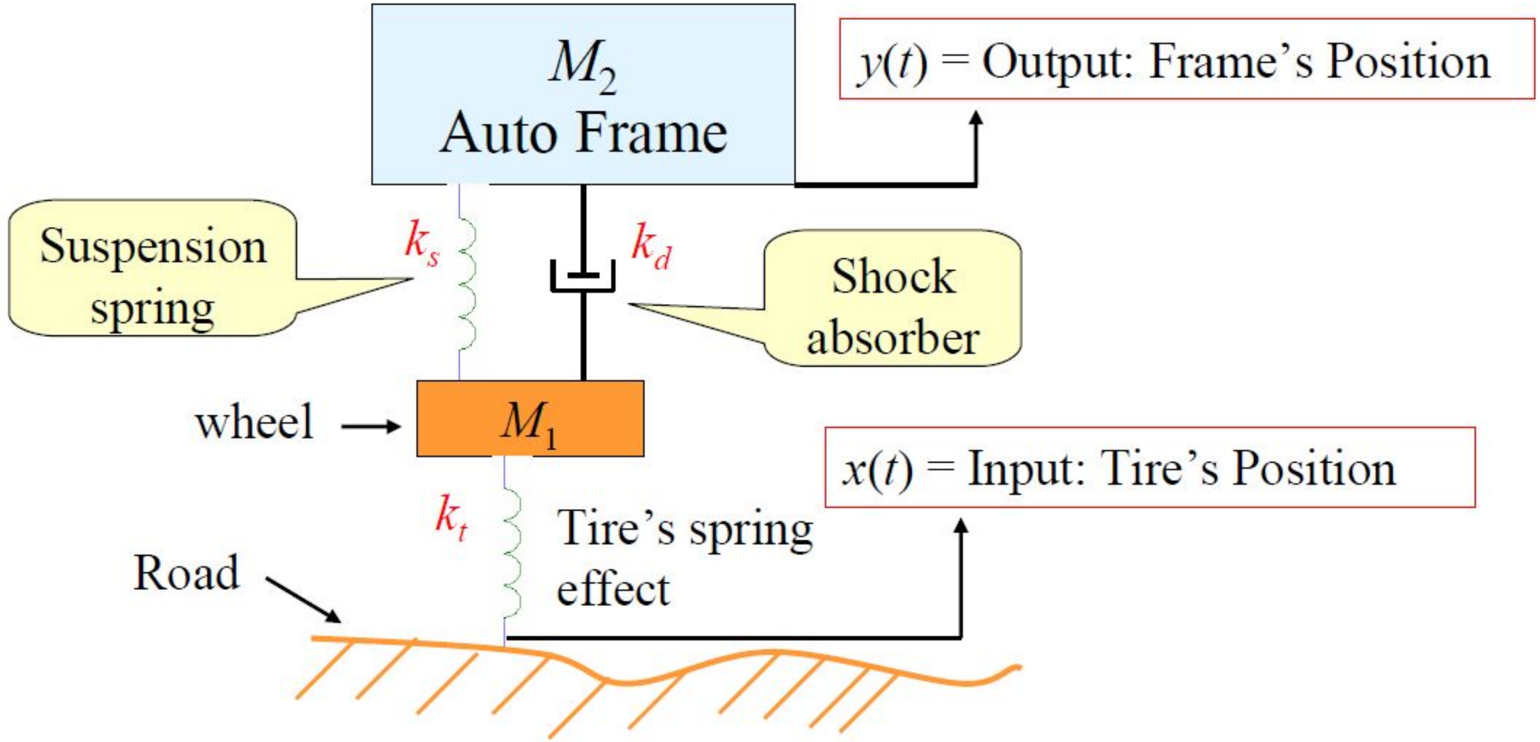
Mathematical System:

$$\boxed{\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)}$$

Mathematical Solution:



Automobile Suspension System Example



Results in 4th order differential equation:

$$\boxed{\frac{d^4 y(t)}{dt^4} + \frac{a_3 d^3 y(t)}{dt^3} + \frac{a_2 d^2 y(t)}{dt^2} + \frac{a_1 dy(t)}{dt} + a_0 y(t) = \underbrace{F[x(t)]}_{\text{Some function of Input } x(t)}}$$

The a_i are functions of system's physical parameters:

$$M_1, M_2, k_s, k_d, k_t$$

Again… to find the output for a given input requires solving the differential equation

Engineers could use this differential equation *model* to theoretically explore:

1. How the car will respond to some typical theoretical test inputs when different possible values of system physical parameters are used
2. Determine what the best set of system physical parameters are for a desired response
3. Then… maybe build a prototype and use it to fine tune the real-world effects that are not captured by this differential equation model

So… What we are seeing is that for an engineer to analyze or design a circuit (or a general physical system) there is almost always an underlying Differential Equation whose solution for a given input tells how the system output behaves

So… engineers need both a qualitative and quantitative understanding of Differential Equations.

The major goal of this course is to provide tools that help gain that qualitative and quantitative understanding!!!

Differential Equations

Differential Equations like this are Linear and Time Invariant:

$$\boxed{a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_0 y(t) = b_m \frac{d^m f(t)}{dt^m} + \ldots + b_1 \frac{df(t)}{dt} + b_0 f(t)}$$

-coefficients are constants $\Rightarrow$ **TI**

-No nonlinear terms $\Rightarrow$ **Linear**

Examples of Nonlinear Terms:

$$f^n(t), \quad \left[\frac{d^k y(t)}{dt^k}\right]\left[\frac{d^p y(t)}{dt^p}\right], \quad y^n(t), \quad \left[\frac{d^k y(t)}{dt^k}\right]\left[\frac{d^p y(t)}{dt^p}\right], \quad etc.$$

Differential Equations

The highest order among all terms becomes the order of the differential equation. In this case, the highest Order is 3. So we call this equation as a '3rd **order** differential equation'

As you see here, the **dependent variable** in differential equation is a 'Function', not a value. This is a key characteristics that defines 'Differential Equation'

Dependent Variable

Independent Variable

Order (=3) Order (=2)

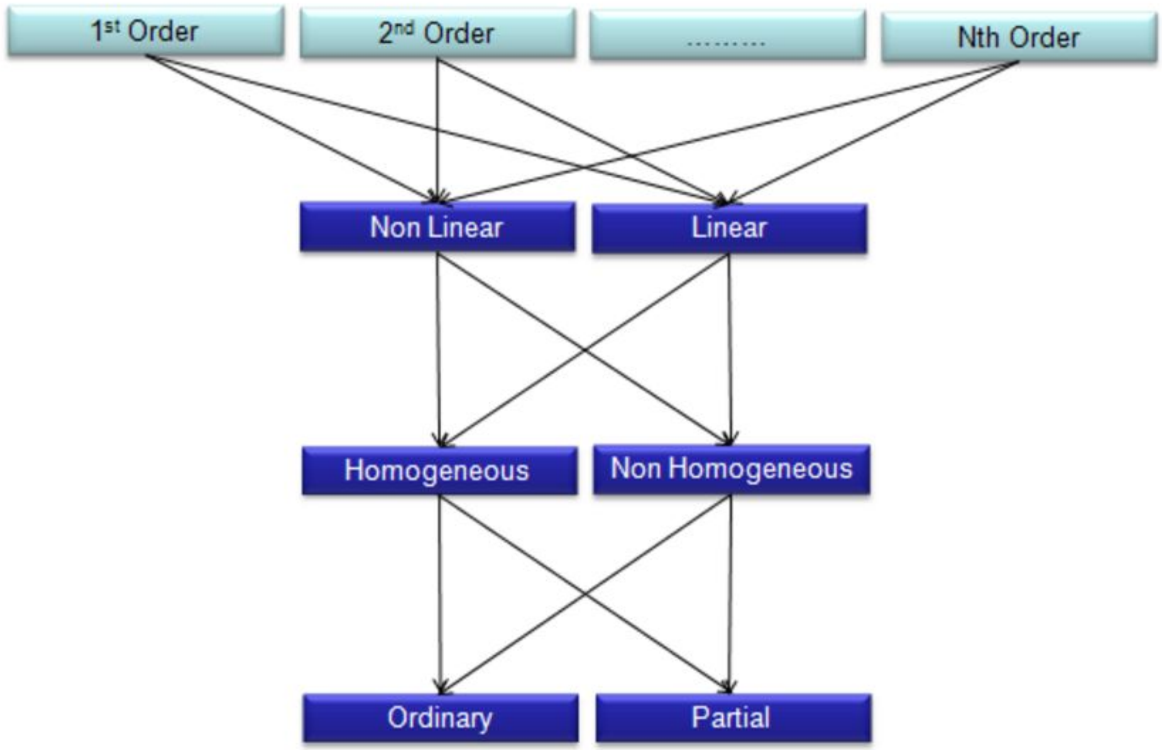
$y(x)$

implies

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 3$$

Independent Variable

Differential Equations

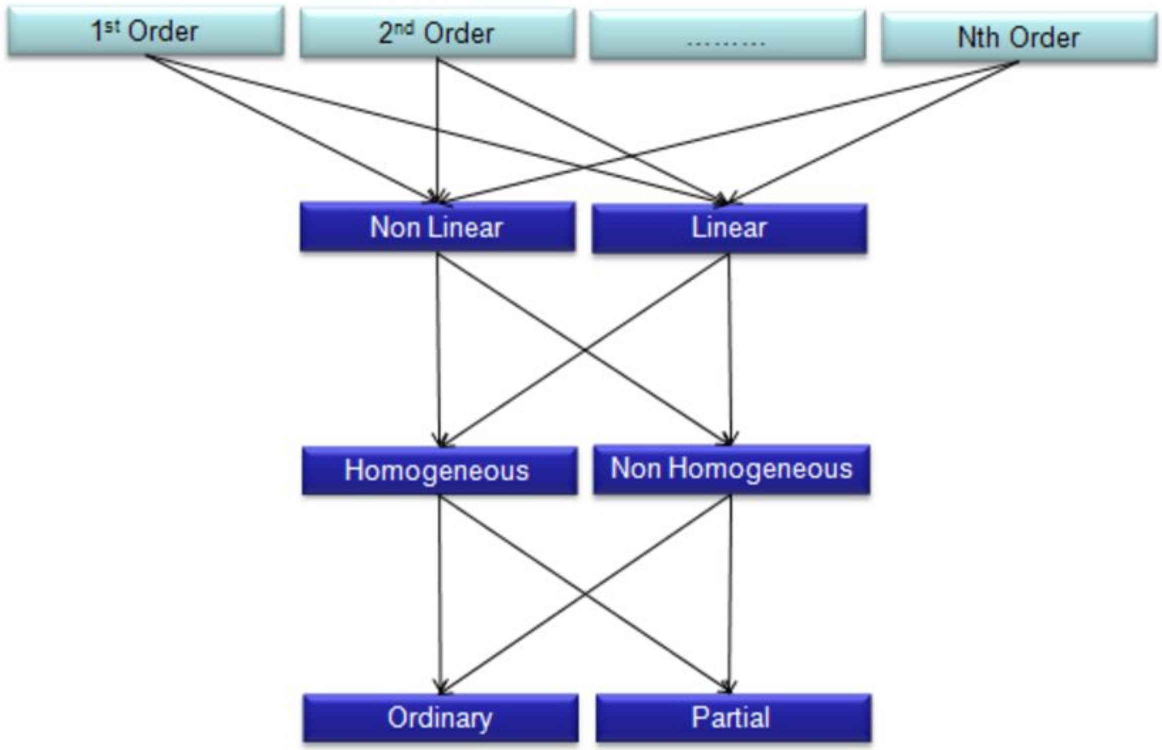


Homogeneous Differential Equation

- All terms has the derivative of y or y itself
- There is no term that is based on function of x itself

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Differential Equations



Dependent Variable

$y(x)$

Independent Variable

implies

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 3$$

Independent Variable

There are only one type of independent variable. This kind of differential equation is called **Ordinary Differential Equation (ODE)**

In the following we will BRIEFLY review the basics of solving Linear, Constant Coefficient Differential Equations under the <u>Homogeneous</u> Condition

"Homogeneous" means the "forcing function" is zero

That means we are finding the "zero-input response" that occurs due to the effect of the initial coniditions.

We will assume: $m \leq n$

m is the highest-order derivative on the "input" side

n is the highest-order derivative on the "output" side

Use "operational notation": $\boxed{\dfrac{d^k y(t)}{dt^k} \equiv D^k y(t)}$

$\Rightarrow$ Write D.E. like this:

$$\underbrace{\left(D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0\right)}_{\triangleq Q(D)} y(t) = \underbrace{\left(b_m D^m + \ldots + b_1 D + b_0\right)}_{\triangleq P(D)} f(t)$$

$$Diff.\ Eq. \Rightarrow \quad \boxed{Q(D)y(t) = P(D)f(t)}$$

Due to linearity: Total Response = Zero-Input Response + Zero-State Response

Z-I Response: found assuming the input $f(t) = 0$ but with given IC's

Z-S Response: found assuming IC's = 0 but with given $f(t)$ applied

Finding the Zero-Input Response **(Homogeneous Solution)**

Assume $f(t) = 0$

$$\Rightarrow D.E.: Q(D)y_{zi}(t) = 0$$

(▲)

$$\Rightarrow \underbrace{\left(D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0\right)y_{zi}(t)} = 0 \qquad \forall t > 0$$

"linear combination" of $y_{zi}(t)$ & its derivatives must be = 0

Consider $y_0(t) = ce^{\lambda t}$

c and λ are possibly complex numbers

Can we find c and λ such that $y_0(t)$ qualifies as a homogeneous solution?

Put $y_0(t)$ into (▲) and use result for the derivative of an exponential: $\boxed{\dfrac{d^n e^{\lambda t}}{dt^n} = \lambda^n e^{\lambda t}}$

$$c\underbrace{(\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0)}_{\text{must } = 0}e^{\lambda t} = 0$$

Characteristic polynomial

$Q(\lambda)$ has at most n unique roots

(can be complex)

$$\Rightarrow Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)$$

$c_1 e^{\lambda_1 t}$ is a solution

$c_2 e^{\lambda_2 t}$ is a solution

$\vdots$

$c_n e^{\lambda_n t}$ is a solution

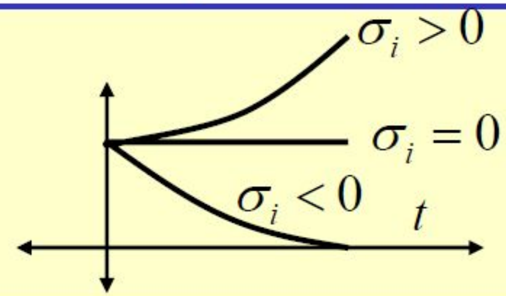
So…any linear combination is also a solution to (▲)

$$\boxed{\text{Z-I Solution}: \; y_{zi}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \ldots + c_n e^{\lambda_n t}}$$

Then, choose $c_1, c_2, \ldots, c_n$ to satisfy the given IC's

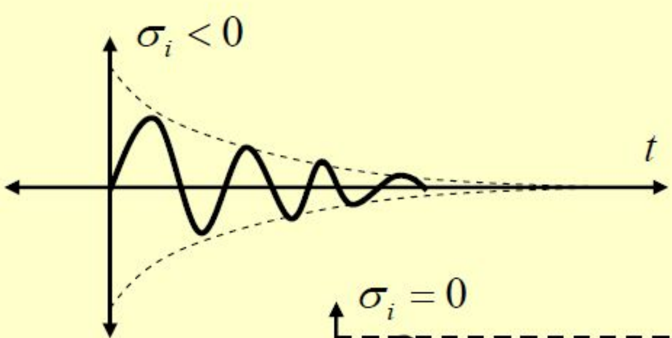
$\left\{e^{\lambda_i t}\right\}_{i=1}^{n}$ ← Set of characteristic modes

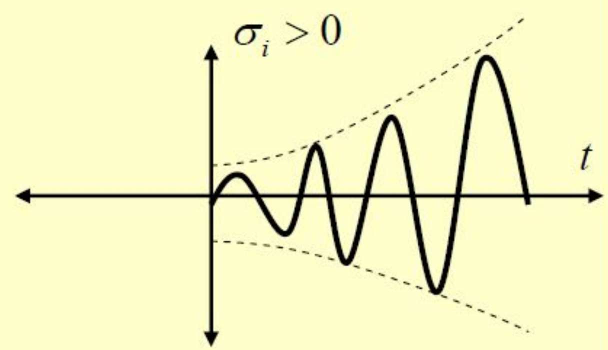
<u>**Real Root**</u>: $\lambda_i = \underset{\text{real}}{\sigma_i} + \underset{\text{real}}{j0} \quad \Rightarrow \quad e^{\sigma_i t}$

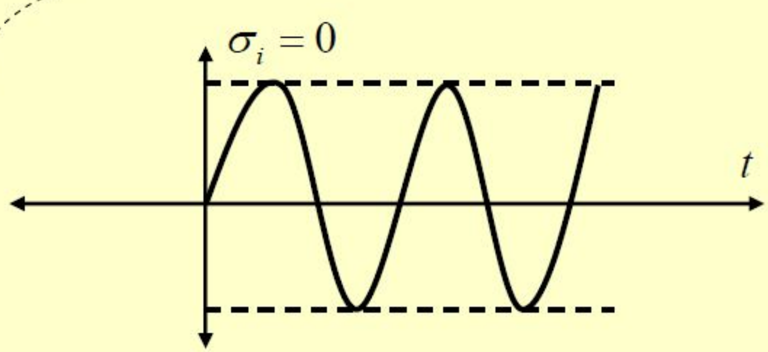


<u>**Complex Root**</u>: $\lambda_i = \sigma_i + j\omega_i$

Mode: $e^{\lambda_i t} = e^{\sigma_i t} + e^{j\omega_i t}$







To get only real-valued solutions requires the system coefficients to be real-valued.

$\Rightarrow$ Complex roots of C.E. will appear in conjugate pairs:

$$\left.\begin{aligned} \lambda_i &= \sigma + j\omega \\ \lambda_k &= \sigma - j\omega \end{aligned}\right\} \quad \text{Conjugate pair}$$

$$c_i e^{\lambda_i t} + c_k e^{\lambda_k t} = \underset{\frac{C}{2}e^{j\theta}}{c_i} e^{\sigma t} e^{j\omega t} + \underset{\frac{C}{2}e^{-j\theta}}{c_k} e^{\sigma t} e^{j\omega t}$$

For some real C

Use Euler! $Ce^{\sigma t}\cos(\omega t + \theta) \quad t > 0$

Repeated Roots

Say there are r repeated roots

$$Q(\lambda) = (\lambda - \lambda_1)^r \underbrace{(\lambda - \lambda_2)(\lambda - \lambda_3)\ldots(\lambda - \lambda_p)} \qquad p = n - r$$

We "can verify" that: $e^{\lambda_i t}, te^{\lambda_1 t}, t^2 e^{\lambda_1 t}, \ldots, t^{r-1} e^{\lambda_1 t}$ satisfy (▲)

ZI Solution:

$$y_{zi}(t) = \underbrace{\left(c_{11} + c_{12}t + \ldots + c_{1r}t^{r-1}\right)}_{\text{effect of } r\text{-repeated roots}} e^{\lambda_i t} + \text{other modes}:$$

Differential Equation Examples

Find the zero-input response (i.e., homogeneous solution) for these three Differential Equations.

Example (a)

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{df(t)}{dt}$$

w/ I.C.'s

y(0)=0, y'(0)=-5

$$D^2y(t) + 3Dy(t) + 2y(t) = Df(t)$$

The zero-input form is:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 0$$

$$D^2y(t) + 3Dy(t) + 2y(t) = 0$$

The Characteristic Equation is:

$$\lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \quad (\lambda+1)(\lambda+2) = 0$$

The Characteristic Equation is:

$$\lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \quad (\lambda + 1)(\lambda + 2) = 0$$

The Characteristic Roots are:

$$\lambda_1 = -1 \quad \& \quad \lambda_2 = -2$$

The Characteristic "Modes" are:

$$e^{\lambda_1 t} = e^{-t} \quad \& \quad e^{\lambda_2 t} = e^{-2t}$$

The zero-input solution is:

$$y_{zi}(t) = C_1 e^{-t} + C_2 e^{-2t}$$

The System forces this form through its Char. Eq.

The IC's determine the specific values of the C_i's

The zero-input solution is:

$$y_{zi}(t) = C_1 e^{-t} + C_2 e^{-2t}$$

and it must satisfy the ICs so:

$$0 = y_{zi}(0) = C_1 e^{-0} + C_2 e^{-0} \quad \Rightarrow \quad C_1 + C_2 = 0$$

The derivative of the z-s soln. must also satisfy the ICs so:

$$-5 = y'_{zi}(0) = -C_1 e^{-0} - 2C_2 e^{-0} \quad \Rightarrow \quad C_1 + 2C_2 = 5$$

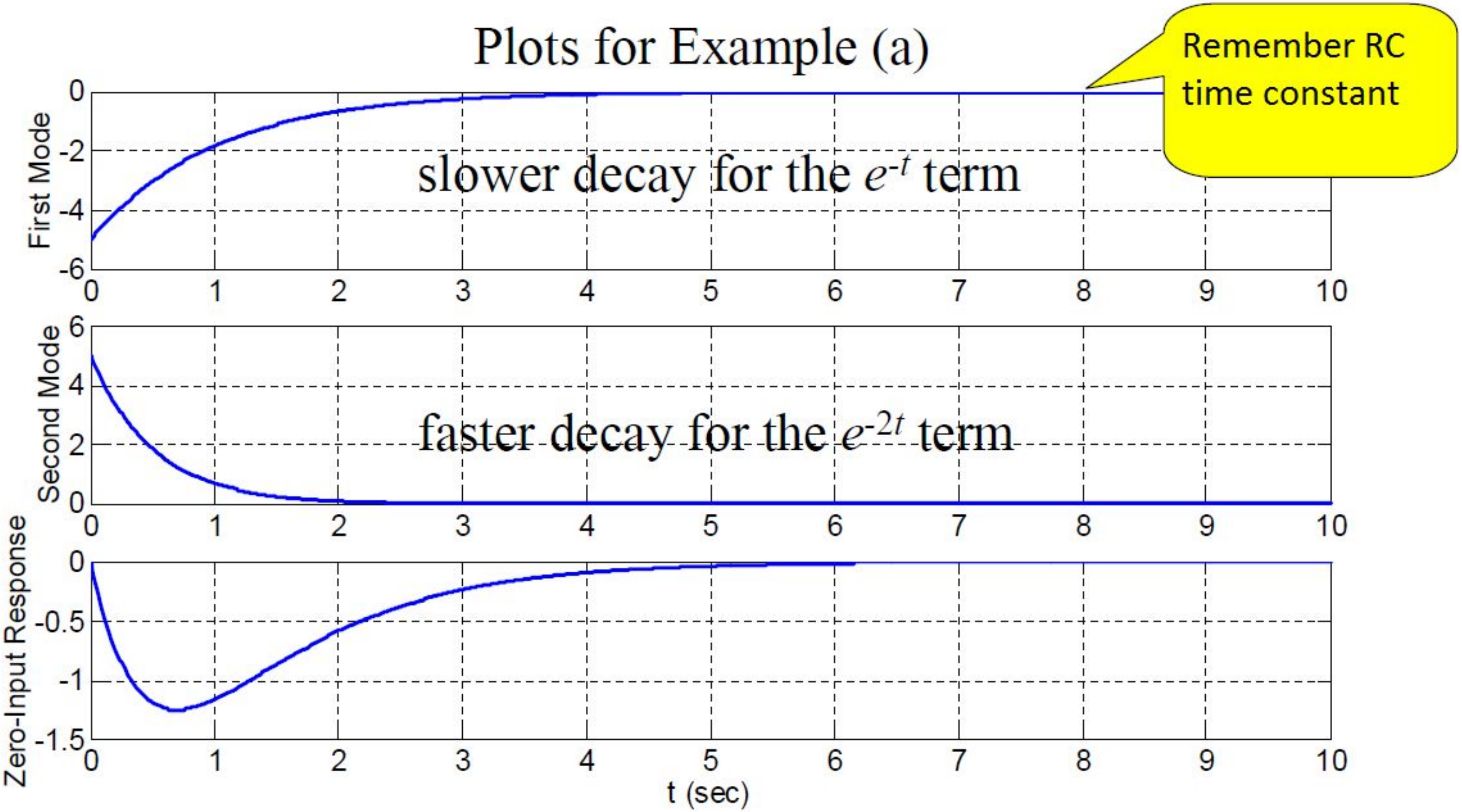
Two Equations in Two Unknowns leads to:

$$C_1 = -5 \quad \& \quad C_2 = 5$$

The "*particular*" zero-input solution is:

$$y_{zi}(t) = \underbrace{-5e^{-t}}_{\text{first mode}} + \underbrace{5e^{-2t}}_{\text{second mode}}$$

Plots for Example (a)



Because the characteristic roots are real and negative…
the modes and the Z-I response all decay to zero w/o oscillations

Example (b):

$$\frac{d^2 y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 9y(t) = 3\frac{df(t)}{dt} + 5f(t)$$

w/ I.C.'s

y(0)=3,y'(0)=-7

$$D^2 y(t) + 6Dy(t) + 9y(t) = 3Df(t) + 5f(t)$$

The zero-input form is:

$$\frac{d^2 y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 9y(t) = 0$$

$$D^2 y(t) + 6Dy(t) + 9y(t) = 0$$

The Characteristic Equation is:

$$\lambda^2 + 6\lambda + 9 = 0 \quad \Rightarrow \quad (\lambda + 3)^2 = 0$$

The Characteristic Equation is:

$$\lambda^2 + 6\lambda + 9 = 0 \quad \Rightarrow \quad (\lambda + 3)^2 = 0$$

The Characteristic Roots are:

$$\lambda_1 = -3 \quad \& \quad \lambda_2 = -3$$

Real, repeated roots

The Characteristic "Modes" are:

$$e^{\lambda_1 t} = e^{-3t} \quad \& \quad te^{\lambda_2 t} = te^{-3t}$$

Using the "rule" to handle repeated roots

The zero-input solution is:

$$y_{zi}(t) = C_1 e^{-3t} + C_2 t e^{-3t}$$

The System forces this form through its Char. Eq.

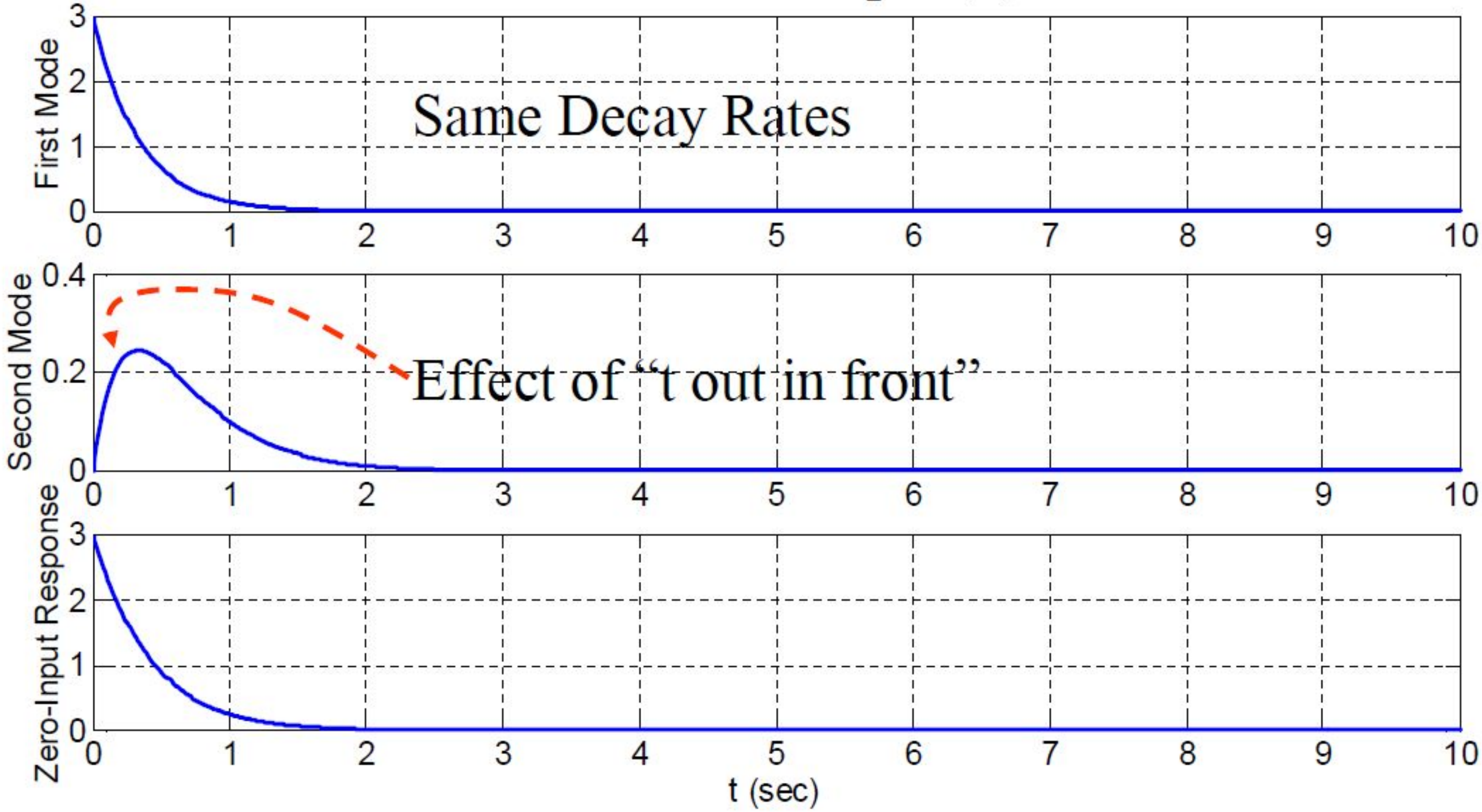
The IC's determine the specific values of the C_i's

Following the same procedure (do it for yourself!!) you get…

The "*particular*" zero-input solution is:

$$y_{zi}(t) = \underbrace{3e^{-3t}}_{\text{first mode}} + \underbrace{2te^{-3t}}_{\text{second mode}} = (3+2t)e^{-3t}$$

Plots for Example (b)



Because the characteristic roots are <u>real and negative</u>…
the modes and the Z-I response all <u>decay to zero w/o oscillations</u> .

Example (c):

$$\frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 40y(t) = \frac{df(t)}{dt} + 2f(t)$$

w/ I.C.'s

y(0)=2,y'(0)=16.78

$$D^2 y(t) + 4Dy(t) + 40y(t) = Df(t) + 2f(t)$$

The zero-input form is:

$$\frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 40y(t) = 0$$

$$D^2 y(t) + 4Dy(t) + 40y(t) = 0$$

The Characteristic Equation is:

$$\lambda^2 + 4\lambda + 40 = 0 \quad \Rightarrow \quad (\lambda + 2 - j6)(\lambda + 2 + j6) = 0$$

The Characteristic Equation is:

$$\lambda^2 + 4\lambda + 40 = 0 \quad \Rightarrow \quad (\lambda + 2 - j6)(\lambda + 2 + j6) = 0$$

The Characteristic Roots are:

$$\lambda_1 = -2 + j6 \quad \& \quad \lambda_2 = -2 - j6$$

complex conjugate roots

The Characteristic "Modes" are:

$$e^{\lambda_1 t} = e^{-2t} e^{+j6t} \quad \& \quad e^{\lambda_2 t} = e^{-2t} e^{-j6t}$$

The zero-input solution is:

$$y_{zi}(t) = C_1 e^{-2t} e^{+j6t} + C_2 e^{-2t} e^{-j6t}$$

The System forces this form through its Char. Eq.

The IC's determine the specific values of the C_i's

Following the same procedure with some manipulation of complex exponentials into a cosine…

The "*particular*" zero-input solution is:

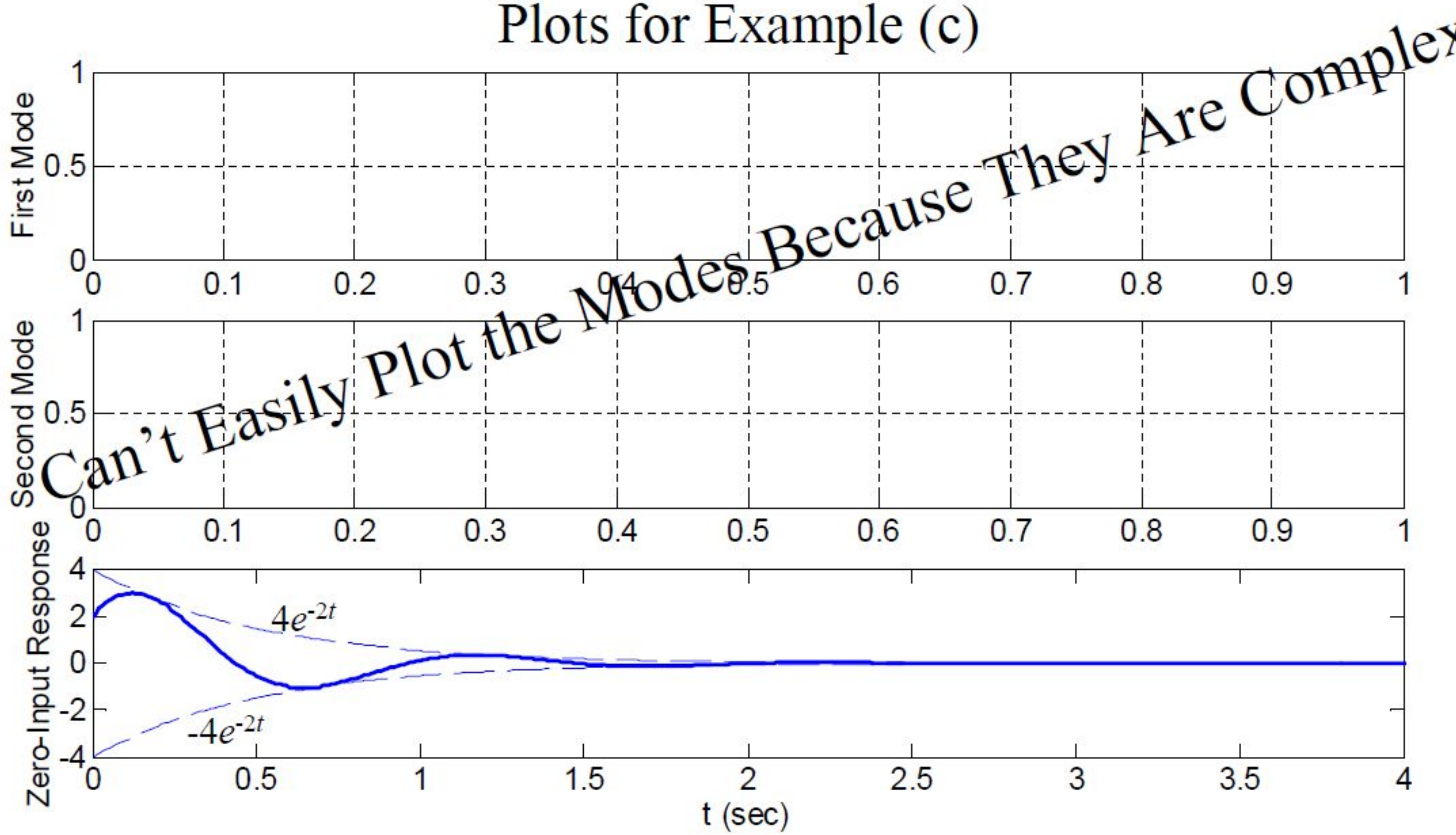
Set by the ICs

$$y_{zi}(t) = 4e^{-2t}\cos(6t + \pi/3)$$

Imag. part of root controls oscillation

Real part of root controls Decay

Plots for Example (c)



Because the characteristic roots are <u>complex</u>… have oscillations!
Because real part of root is negative… decays to zero !!!

https://www.youtube.com/watch?v=XggxeuFDaDU

Tacoma Bridge Collapse: The Wobbliest Bridge in the World? (1940)

Big Picture…

The structure of the D.E. determines the char. roots, which determine the "character" of the response:

- Decaying vs. Exploding (controlled by real part of root)
- Oscillating or Not (controlled by imag part of root)

The D.E. structure is determined by the physical system's structure and component values.