CHAPTER 1
The Foundations: Logic and Proofs

SECTION 1.1 Propositional Logic

2. Propositions must have clearly defined truth values, so a proposition must be a declarative sentence with no free variables.

a) This is not a proposition; it's a command.

b) This is not a proposition; it's a question.

c) This is a proposition that is false, as anyone who has been to Maine knows.

d) This is not a proposition; its truth value depends on the value of x.

e) This is a proposition that is false.

f) This is not a proposition; its truth value depends on the value of n.

4. a) I did not buy a lottery ticket this week.

b) Either I bought a lottery ticket this week or [in the inclusive sense] I won the million dollar jackpot on Friday.

c) If I bought a lottery ticket this week, then I won the million dollar jackpot on Friday.

d) I bought a lottery ticket this week and I won the million dollar jackpot on Friday.

e) I bought a lottery ticket this week if and only if I won the million dollar jackpot on Friday.

f) If I did not buy a lottery ticket this week, then I did not win the million dollar jackpot on Friday.

g) I did not buy a lottery ticket this week, and I did not win the million dollar jackpot on Friday.

h) Either I did not buy a lottery ticket this week, or else I did buy one and won the million dollar jackpot on Friday.

6. a) The election is not decided.

b) The election is decided, or the votes have been counted.

c) The election is not decided, and the votes have been counted.

d) If the votes have been counted, then the election is decided.

e) If the votes have not been counted, then the election is not decided.

f) If the election is not decided, then the votes have not been counted.

g) The election is decided if and only if the votes have been counted.

h) Either the votes have not been counted, or else the election is not decided and the votes have been counted. Note that we were able to incorporate the parentheses by using the words *either* and *else*.

8. a) If you have the flu, then you miss the final exam.

b) You do not miss the final exam if and only if you pass the course.

c) If you miss the final exam, then you do not pass the course.

d) You have the flu, or miss the final exam, or pass the course.

e) It is either the case that if you have the flu then you do not pass the course or the case that if you miss the final exam then you do not pass the course (or both, it is understood).

f) Either you have the flu and miss the final exam, or you do not miss the final exam and do pass the course.

10. **a)** $r \wedge \neg q$ **b)** $p \wedge q \wedge r$ **c)** $r \rightarrow p$ **d)** $p \wedge \neg q \wedge r$ **e)** $(p \wedge q) \rightarrow r$ **f)** $r \leftrightarrow (q \vee p)$

12. **a)** This is $\mathbf{T} \leftrightarrow \mathbf{T}$, which is true.
b) This is $\mathbf{T} \leftrightarrow \mathbf{F}$, which is false.
c) This is $\mathbf{F} \leftrightarrow \mathbf{F}$, which is true.
d) This is $\mathbf{F} \leftrightarrow \mathbf{T}$, which is false.

14. **a)** This is $\mathbf{F} \rightarrow \mathbf{F}$, which is true.
b) This is $\mathbf{F} \rightarrow \mathbf{F}$, which is true.
c) This is $\mathbf{T} \rightarrow \mathbf{F}$, which is false.
d) This is $\mathbf{T} \rightarrow \mathbf{T}$, which is true.

16. **a)** The employer making this request would be happy if the applicant knew both of these languages, so this is clearly an inclusive *or*.

b) The restaurant would probably charge extra if the diner wanted both of these items, so this is an exclusive *or*.

c) If a person happened to have both forms of identification, so much the better, so this is clearly an inclusive *or*.

d) This could be argued either way, but the inclusive interpretation seems more appropriate. This phrase means that faculty members who do not publish papers in research journals are likely to be fired from their jobs during the probationary period. On the other hand, it may happen that they will be fired even if they do publish (for example, if their teaching is poor).

18. **a)** The necessary condition is the conclusion: If you get promoted, then you wash the boss's car.

b) If the winds are from the south, then there will be a spring thaw.

c) The sufficient condition is the hypothesis: If you bought the computer less than a year ago, then the warranty is good.

d) If Willy cheats, then he gets caught.

e) The "only if" condition is the conclusion: If you access the website, then you must pay a subscription fee.

f) If you know the right people, then you will be elected.

g) If Carol is on a boat, then she gets seasick.

20. **a)** If I am to remember to send you the address, then you will have to send me an e-mail message. (This has been slightly reworded so that the tenses make more sense.)

b) If you were born in the United States, then you are a citizen of this country.

c) If you keep your textbook, then it will be a useful reference in your future courses. (The word "then" is understood in English, even if omitted.)

d) If their goaltender plays well, then the Red Wings will win the Stanley Cup.

e) If you get the job, then you had the best credentials.

f) If there is a storm, then the beach erodes.

g) If you log on to the server, then you have a valid password.

h) If you do not begin your climb too late, then you will reach the summit.

22. **a)** You will get an A in this course if and only if you learn how to solve discrete mathematics problems.

b) You will be informed if and only if you read the newspaper every day. (It sounds better in this order; it would be logically equivalent to state this as "You read the newspaper every day if and only if you will be informed.")

c) It rains if and only if it is a weekend day.

d) You can see the wizard if and only if he is not in.

24. **a)** Converse: If I stay home, then it will snow tonight. Contrapositive: If I do not stay at home, then it will not snow tonight. Inverse: If it does not snow tonight, then I will not stay home.

b) Converse: Whenever I go to the beach, it is a sunny summer day. Contrapositive: Whenever I do not go to the beach, it is not a sunny summer day. Inverse: Whenever it is not a sunny day, I do not go to the beach.

c) Converse: If I sleep until noon, then I stayed up late. Contrapositive: If I do not sleep until noon, then I did not stay up late. Inverse: If I don't stay up late, then I don't sleep until noon.

26. A truth table will need 2^n rows if there are n variables.

a) $2^2 = 4$ **b)** $2^3 = 8$ **c)** $2^6 = 64$ **d)** $2^5 = 32$

28. To construct the truth table for a compound proposition, we work from the inside out. In each case, we will show the intermediate steps. In part **(d)**, for example, we first construct the truth tables for $p \wedge q$ and for $p \vee q$ and combine them to get the truth table for $(p \wedge q) \to (p \vee q)$. For parts **(a)** and **(b)** we have the following table (column three for part **(a)**, column four for part **(b)**).

p	$\neg p$	$p \to \neg p$	$p \leftrightarrow \neg p$
T	F	F	F
F	T	T	F

For parts **(c)** and **(d)** we have the following table.

p	q	$p \vee q$	$p \wedge q$	$p \oplus (p \vee q)$	$(p \wedge q) \to (p \vee q)$
T	T	T	T	F	T
T	F	T	F	F	T
F	T	T	F	T	T
F	F	F	F	F	T

For part **(e)** we have the following table.

p	q	$\neg p$	$q \to \neg p$	$p \leftrightarrow q$	$(q \to \neg p) \leftrightarrow (p \leftrightarrow q)$
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	T	T	T

For part **(f)** we have the following table.

p	q	$\neg q$	$p \leftrightarrow q$	$p \leftrightarrow \neg q$	$(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$
T	T	F	T	F	T
T	F	T	F	T	T
F	T	F	F	T	T
F	F	T	T	F	T

30. For parts **(a)** and **(b)** we have the following table (column two for part **(a)**, column four for part **(b)**).

p	$p \oplus p$	$\neg p$	$p \oplus \neg p$
T	F	F	T
F	F	T	T

For parts **(c)** and **(d)** we have the following table (columns five and six).

p	q	$\neg p$	$\neg q$	$p \oplus \neg q$	$\neg p \oplus \neg q$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

For parts **(e)** and **(f)** we have the following table (columns five and six). This time we have omitted the column explicitly showing the negation of q. Note that the first is a tautology and the second is a contradiction (see definitions in Section 1.2).

p	q	$p \oplus q$	$p \oplus \neg q$	$(p \oplus q) \vee (p \oplus \neg q)$	$(p \oplus q) \wedge (p \oplus \neg q)$
T	T	F	T	T	F
T	F	T	F	T	F
F	T	T	F	T	F
F	F	F	T	T	F

32. For parts **(a)** and **(b)**, we have

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$(p \vee q) \wedge r$
T	T	T	T	T	T
T	T	F	T	T	F
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	T	T	T
F	T	F	T	T	F
F	F	T	F	T	F
F	F	F	F	F	F

For parts **(c)** and **(d)**, we have

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$	$(p \wedge q) \wedge r$
T	T	T	T	T	T
T	T	F	T	T	F
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	F	T	F
F	T	F	F	F	F
F	F	T	F	T	F
F	F	F	F	F	F

Finally, for parts **(e)** and **(f)** we have

p	q	r	$\neg r$	$p \vee q$	$(p \vee q) \wedge \neg r$	$p \wedge q$	$(p \wedge q) \vee \neg r$
T	T	T	F	T	F	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	F	F	F
T	F	F	T	T	T	F	T
F	T	T	F	T	F	F	F
F	T	F	T	T	T	F	T
F	F	T	F	F	F	F	F
F	F	F	T	F	F	F	T

34. This time the truth table needs $2^4 = 16$ rows.

p	q	r	s	$p \to q$	$(p \to q) \to r$	$((p \to q) \to r) \to s$
T	T	T	T	T	T	T
T	T	T	F	T	T	F
T	T	F	T	T	F	T
T	T	F	F	T	F	T
T	F	T	T	F	T	T
T	F	T	F	F	T	F
T	F	F	T	F	T	T
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	T	T	F	T	T	F
F	T	F	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	T	T
F	F	T	F	T	T	F
F	F	F	T	T	F	T
F	F	F	F	T	F	T

36. a) Since the condition is true, the statement is executed, so x is incremented and now has the value 2.
b) Since the condition is false, the statement is not executed, so x is not incremented and now still has the value 1.
c) Since the condition is true, the statement is executed, so x is incremented and now has the value 2.
d) Since the condition is false, the statement is not executed, so x is not incremented and now still has the value 1.
e) Since the condition is true when it is encountered (since $x = 1$), the statement is executed, so x is incremented and now has the value 2. (It is irrelevant that the condition is now false.)

38. a) $1\ 1000 \wedge (0\ 1011 \vee 1\ 1011) = 1\ 1000 \wedge 1\ 1011 = 1\ 1000$
b) $(0\ 1111 \wedge 1\ 0101) \vee 0\ 1000 = 0\ 0101 \vee 0\ 1000 = 0\ 1101$
c) $(0\ 1010 \oplus 1\ 1011) \oplus 0\ 1000 = 1\ 0001 \oplus 0\ 1000 = 1\ 1001$
d) $(1\ 1011 \vee 0\ 1010) \wedge (1\ 0001 \vee 1\ 1011) = 1\ 1011 \wedge 1\ 1011 = 1\ 1011$

40. The truth value of "Fred and John are happy" is $\min(0.8, 0.4) = 0.4$. The truth value of "Neither Fred nor John is happy" is $\min(0.2, 0.6) = 0.2$, since this statement means "Fred is not happy, and John is not happy," and we computed the truth values of the two propositions in this conjunction in Exercise 35.

42. This cannot be a proposition, because it cannot have a truth value. Indeed, if it were true, then it would be truly asserting that it is false, a contradiction; on the other hand if it were false, then its assertion that it is false must be false, so that it would be true—again a contradiction. Thus this string of letters, while appearing to be a proposition, is in fact meaningless.

44. No. This is a classical paradox. (We will use the male pronoun in what follows, assuming that we are talking about males shaving their beards here, and assuming that all men have facial hair. If we restrict ourselves to beards and allow female barbers, then the barber could be female with no contradiction.) If such a barber existed, who would shave the barber? If the barber shaved himself, then he would be violating the rule that he shaves only those people who do not shave themselves. On the other hand, if he does not shave himself, then the rule says that he must shave himself. Neither is possible, so there can be no such barber.

46. a) If the explorer (a woman, so that our pronouns will not get confused here—the cannibals will be male) encounters a truth-teller, then he will honestly answer "no" to her question. If she encounters a liar, then the honest answer to her question is "yes," so he will lie and answer "no." Thus everybody will answer "no" to the question, and the explorer will have no way to determine which type of cannibal she is speaking to.

b) There are several possible correct answers. One is the following question: "If I were to ask you if you always told the truth, would you say that you did?" Then if the cannibal is a truth teller, he will answer yes (truthfully), while if he is a liar, then, since in fact he would have said that he did tell the truth if questioned, he will now lie and answer no.

48. **a)** "But" means "and": $r \wedge \neg p$.

b) "Whenever" means "if": $(r \wedge p) \to q$.

c) Access being denied is the negation of q, so we have $\neg r \to \neg q$.

d) The hypothesis is a conjunction: $(\neg p \wedge r) \to q$.

50. We write these symbolically: $u \to \neg a$, $a \to s$, $\neg s \to \neg u$. Note that we can make all the conclusion true by making a false, s true, and u false. Therefore if the users cannot access the file system, they can save new files, and the system is not being upgraded, then all the conditional statements are true. Thus the system is consistent.

52. This system is consistent. We use L, Q, N, and B to stand for the basic propositions here, "The file system is locked," "New messages will be queued," "The system is functioning normally," and "New messages will be sent to the message buffer," respectively. Then the given specifications are $\neg L \to Q$, $\neg L \leftrightarrow N$, $\neg Q \to B$, $\neg L \to B$, and $\neg B$. If we want consistency, then we had better have B false in order that $\neg B$ be true. This requires that both L and Q be true, by the two conditional statements that have B as their consequence. The first conditional statement therefore is of the form $\mathrm{F} \to \mathrm{T}$, which is true. Finally, the biconditional $\neg L \leftrightarrow N$ can be satisfied by taking N to be false. Thus this set of specifications is consistent. Note that there is just this one satisfying truth assignment.

54. This is similar to Example 17, about universities in New Mexico. To search for hiking in West Virginia, we could enter WEST **AND** VIRGINIA **AND** HIKING. If we enter (VIRGINIA **AND** HIKING) **NOT** WEST, then we'll get websites about hiking in Virginia but not in West Virginia, except for sites that happen to use the word "west" in a different context (e.g., "Follow the stream west until you come to a clearing").

56. If A is a knight, then his statement that both of them are knights is true, and both will be telling the truth. But that is impossible, because B is asserting otherwise (that A is a knave). If A is a knave, then B's assertion is true, so he must be a knight, and A's assertion is false, as it should be. Thus we conclude that A is a knave and B is a knight.

58. We can draw no conclusions. A knight will declare himself to be a knight, telling the truth. A knave will lie and assert that he is a knight. Since everyone will say "I am a knight," we can determine nothing.

60. **a)** We look at the three possibilities of who the innocent men might be. If Smith and Jones are innocent (and therefore telling the truth), then we get an immediate contradiction, since Smith said that Jones was a friend of Cooper, but Jones said that he did not even know Cooper. If Jones and Williams are the innocent truth-tellers, then we again get a contradiction, since Jones says that he did not know Cooper and was out of town, but Williams says he saw Jones with Cooper (presumably in town, and presumably if we was with him, then he knew him). Therefore it must be the case that Smith and Williams are telling the truth. Their statements do not contradict each other. Based on Williams' statement, we know that Jones is lying, since he said that he did not know Cooper when in fact he was with him. Therefore Jones is the murderer.

b) This is just like part **(a)**, except that we are not told ahead of time that one of the men is guilty. Can none of them be guilty? If so, then they are all telling the truth, but this is impossible, because as we just saw, some of the statements are contradictory. Can more than one of them be guilty? If, for example, they are all guilty, then their statements give us no information. So that is certainly possible.

62. This information is enough to determine the entire system. Let each letter stand for the statement that the person whose name begins with that letter is chatting. Then the given information can be expressed symbolically as follows: $\neg K \to H$, $R \to \neg V$, $\neg R \to V$, $A \to R$, $V \to K$, $K \to V$, $H \to A$, $H \to K$. Note that we were able to convert all of these statements into conditional statements. In what follows we will sometimes make use of the contrapositives of these conditional statements as well. First suppose that H is true. Then it follows that A and K are true, whence it follows that R and V are true. But R implies that V is false, so we get a contradiction. Therefore H must be false. From this it follows that K is true; whence V is true, and therefore R is false, as is A. We can now check that this assignment leads to a true value for each conditional statement. So we conclude that Kevin and Vijay are chatting but Heather, Randy, and Abby are not.

64. Note that Diana's statement is merely that she didn't do it.

a) John did it. There are four cases to consider. If Alice is the sole truth-teller, then Carlos did it; but this means that John is telling the truth, a contradiction. If John is the sole truth-teller, then Diana must be lying, so she did it, but then Carlos is telling the truth, a contradiction. If Carlos is the sole truth-teller, then Diana did it, but that makes John truthful, again a contradiction. So the only possibility is that Diana is the sole truth-teller. This means that John is lying when he denied it, so he did it. Note that in this case both Alice and Carlos are indeed lying.

b) Again there are four cases to consider. Since Carlos and Diana are making contradictory statements, the liar must be one of them (we could have used this approach in part **(a)** as well). Therefore Alice is telling the truth, so Carlos did it. Note that John and Diana are telling the truth as well here, and it is Carlos who is lying.

SECTION 1.2 Propositional Equivalences

2. There are two cases. If p is true, then $\neg(\neg p)$ is the negation of a false proposition, hence true. Similarly, if p is false, then $\neg(\neg p)$ is also false. Therefore the two propositions are logically equivalent.

4. a) We construct the relevant truth table and note that the fifth and seventh columns are identical.

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

b) Again we construct the relevant truth table and note that the fifth and seventh columns are identical.

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	F	T	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

6. We see that the fourth and seventh columns are identical.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

8. We need to negate each part and swap "and" with "or."

a) Kwame will not take a job in industry and will not go to graduate school.

b) Yoshiko does not know Java or does not know calculus.

c) James is not young, or he is not strong.

d) Rita will not move to Oregon and will not move to Washington.

10. We construct a truth table for each conditional statement and note that the relevant column contains only T's. For part **(a)** we have the following table.

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \to q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

For part **(b)** we have the following table. We omit the columns showing $p \to q$ and $q \to r$ so that the table will fit on the page.

p	q	r	$(p \to q) \to (q \to r)$	$q \to r$	$[(p \to q) \to (q \to r)] \to (p \to r)$
T	T	T	T	T	T
T	T	F	F	T	T
T	F	T	T	T	F
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	F	T	F
F	F	T	T	T	F
F	F	F	T	T	T

For part **(c)** we have the following table.

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

For part **(d)** we have the following table. We have omitted some of the intermediate steps to make the table fit.

p	q	r	$(p \vee q) \wedge (p \to r) \wedge (p \to r)$	$[(p \vee q) \wedge (p \to r) \wedge (p \to r)] \to r$
T	T	T	T	T
T	T	F	F	T
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

12. We argue directly by showing that if the hypothesis is true, then so is the conclusion. An alternative approach, which we show only for part **(a)**, is to use the equivalences listed in the section and work symbolically.

a) Assume the hypothesis is true. Then p is false. Since $p \vee q$ is true, we conclude that q must be true. Here is a more "algebraic" solution: $[\neg p \wedge (p \vee q)] \to q \equiv \neg[\neg p \wedge (p \vee q)] \vee q \equiv \neg\neg p \vee \neg(p \vee q)] \vee q \equiv p \vee \neg(p \vee q) \vee q \equiv (p \vee q) \vee \neg(p \vee q) \equiv \mathbf{T}$. The reasons for these logical equivalences are, respectively, Table 7, line 1; De Morgan's law; double negation; commutative and associative laws; negation law.

b) We want to show that if the entire hypothesis is true, then the conclusion $p \to r$ is true. To do this, we need only show that if p is true, then r is true. Suppose p is true. Then by the first part of the hypothesis, we conclude that q is true. It now follows from the second part of the hypothesis that r is true, as desired.

c) Assume the hypothesis is true. Then p is true, and since the second part of the hypothesis is true, we conclude that q is also true, as desired.

d) Assume the hypothesis is true. Since the first part of the hypothesis is true, we know that either p or q is true. If p is true, then the second part of the hypothesis tells us that r is true; similarly, if q is true, then the third part of the hypothesis tells us that r is true. Thus in either case we conclude that r is true.

14. This is not a tautology. It is saying that knowing that the hypothesis of an conditional statement is false allows us to conclude that the conclusion is also false, and we know that this is not valid reasoning. To show that it is not a tautology, we need to find truth assignments for p and q that make the entire proposition false. Since this is possible only if the conclusion if false, we want to let q be true; and since we want the hypothesis to be true, we must also let p be false. It is easy to check that if, indeed, p is false and q is true, then the conditional statement is false. Therefore it is not a tautology.

16. The first of these propositions is true if and only if p and q have the same truth value. The second is true if and only if either p and q are both true, or p and q are both false. Clearly these two conditions are saying the same thing.

18. It is easy to see from the definitions of conditional statement and negation that each of these propositions is false in the case in which p is true and q is false, and true in the other three cases. Therefore the two propositions are logically equivalent.

20. It is easy to see from the definitions of the logical operations involved here that each of these propositions is true in the cases in which p and q have the same truth value, and false in the cases in which p and q have opposite truth values. Therefore the two propositions are logically equivalent.

22. Suppose that $(p \to q) \wedge (p \to r)$ is true. We want to show that $p \to (q \wedge r)$ is true, which means that we want to show that $q \wedge r$ is true whenever p is true. If p is true, since we know that both $p \to q$ and $p \to r$ are true from our assumption, we can conclude that q is true and that r is true. Therefore $q \wedge r$ is true, as desired. Conversely, suppose that $p \to (q \wedge r)$ is true. We need to show that $p \to q$ is true and that $p \to r$ is true, which means that if p is true, then so are q and r. But this follows from $p \to (q \wedge r)$.

24. We determine exactly which rows of the truth table will have T as their entries. Now $(p \to q) \vee (p \to r)$ will be true when either of the conditional statements is true. The conditional statement will be true if p is false, or if q in one case or r in the other case is true, i.e., when $q \vee r$ is true, which is precisely when $p \to (q \vee r)$ is true. Since the two propositions are true in exactly the same situations, they are logically equivalent.

26. Applying the third and first equivalences in Table 7, we have $\neg p \to (q \to r) \equiv p \vee (q \to r) \equiv p \vee \neg q \vee r$. Applying the first equivalence in Table 7 to $q \to (p \vee r)$ shows that $\neg q \vee p \vee r$ is equivalent to it. But these are equivalent by the commutative and associative laws.

28. We know that $p \leftrightarrow q$ is true precisely when p and q have the same truth value. But this happens precisely when $\neg p$ and $\neg q$ have the same truth value, that is, $\neg p \leftrightarrow \neg q$.

30. The conclusion $q \vee r$ will be true in every case except when q and r are both false. But if q and r are both false, then one of $p \vee q$ or $\neg p \vee r$ is false, because one of p or $\neg p$ is false. Thus in this case the hypothesis $(p \vee q) \wedge (\neg p \vee r)$ is false. An conditional statement in which the conclusion is true or the hypothesis is false is true, and that completes the argument.

32. We just need to find an assignment of truth values that makes one of these propositions true and the other false. We can let p be true and the other two variables be false. Then the first statement will be $\mathbf{F} \to \mathbf{F}$, which is true, but the second will be $\mathbf{F} \wedge \mathbf{T}$, which is false.

34. We apply the rules stated in the preamble.
a) $p \wedge \neg q$ **b)** $p \vee (q \wedge (r \vee \mathbf{F}))$ **c)** $(p \vee \neg q) \wedge (q \vee \mathbf{T})$

36. If s has any occurrences of $\wedge$, $\vee$, $\mathbf{T}$, or $\mathbf{F}$, then the process of forming the dual will change it. Therefore $s^* = s$ if and only if s is simply one propositional variable (like p). A more difficult question is to determine when s^* will be logically equivalent to s. For example, $p \vee \mathbf{F}$ is logically equivalent to its dual $p \wedge \mathbf{T}$, because both are logically equivalent to p.

38. The table is in fact displayed so as to exhibit the duality. The two identity laws are duals of each other, the two domination laws are duals of each other, etc. The only law not listed with another, the double negation law, is its own dual, since there are no occurrences of $\wedge$, $\vee$, $\mathbf{T}$, or $\mathbf{F}$ to replace.

40. Following the hint, we easily see that the answer is $p \wedge q \wedge \neg r$.

42. The statement of the problem is really the solution. Each line of the truth table corresponds to exactly one combination of truth values for the n atomic propositions involved. We can write down a conjunction that is true precisely in this case, namely the conjunction of all the atomic propositions that are true and the negations of all the atomic propositions that are false. If we do this for *each* line of the truth table for which the value of the compound proposition is to be true, and take the disjunction of the resulting propositions, then we have the desired proposition in its disjunctive normal form.

44. Given a compound proposition p, we can, by Exercise 43, write down a proposition q that is logically equivalent to p and uses only $\neg$, $\wedge$, and $\vee$. Now by De Morgan's law we can get rid of all the $\vee$'s by replacing each occurrence of $p_1 \vee p_2 \vee \cdots \vee p_n$ with $\neg(\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$.

46. We write down the truth table corresponding to the definition.

p	q	$p \mid q$
T	T	F
T	F	T
F	T	T
F	F	T

48. We write down the truth table corresponding to the definition.

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

50. **a)** From the definition (or as seen in the truth table constructed in Exercise 48), $p \downarrow p$ is false when p is true and true when p is false, exactly as $\neg p$ is; thus the two are logically equivalent.

b) The proposition $(p \downarrow q) \downarrow (p \downarrow q)$ is equivalent, by part **(a)**, to $\neg(p \downarrow q)$, which from the definition (or truth table or Exercise 49) is clearly equivalent to $p \vee q$.

c) By Exercise 45, every compound proposition is logically equivalent to one that uses only $\neg$ and $\vee$. But by parts **(a)** and **(b)** of the present exercise, we can get rid of all the negations and disjunctions by using *NOR*'s. Thus every compound proposition can be converted into a logically equivalent compound proposition involving only *NOR*'s.

52. This exercise is similar to Exercise 50. First we can see from the truth tables that $(p \mid p) \equiv (\neg p)$ and that $((p \mid p) \mid (q \mid q)) \equiv (p \vee q)$. Then we argue exactly as in part **(c)** of Exercise 50: by Exercise 45, every compound proposition is logically equivalent to one that uses only $\neg$ and $\vee$. But by our observations at the beginning of the present exercise, we can get rid of all the negations and disjunctions by using *NAND*'s. Thus every compound proposition can be converted into a logically equivalent compound proposition involving only *NAND*'s.

54. To show that these are *not* logically equivalent, we need only find one assignment of truth values to p, q, and r for which the truth values of $p \mid (q \mid r)$ and $(p \mid q) \mid r$ differ. One such assignment is T for p and F for q and r. Then computing from the truth tables (or definitions), we see that $p \mid (q \mid r)$ is false and $(p \mid q) \mid r$ is true.

56. To say that p and q are logically equivalent is to say that the truth tables for p and q are identical; similarly, to say that q and r are logically equivalent is to say that the truth tables for q and r are identical. Clearly if the truth tables for p and q are identical, and the truth tables for q and r are identical, then the truth tables for p and r are identical (this is a fundamental axiom of the notion of equality). Therefore p and r are logically equivalent. (We are assuming—and there is no loss of generality in doing so—that the same atomic variables appear in all three propositions.)

58. If we want the first two of these to be true, then p and q must have the same truth value. If q is true, then the third and fourth expressions will be true, and if r is false, the last expression will be true. So all five of these disjunctions will be true if we set p and q to be true, and r to be false.

60. In each case we hunt for truth assignments that make all the disjunctions true.

a) Since p occurs in four of the five disjunctions, we can make p true, and then make q false (and make r and s anything we please). Thus this proposition is satisfiable.

b) This is satisfiable by, for example, setting p to be false (that takes care of the first, second, and fourth disjunctions), s to be false (for the third and sixth disjunctions), q to be true (for the fifth disjunction), and r to be anything.

c) It is not hard to find a satisfying truth assignment, such as p, q, and s true, and r false.

SECTION 1.3 Predicates and Quantifiers

2. a) This is true, since there is an a in *orange*. **b)** This is false, since there is no a in *lemon*.
c) This is false, since there is no a in *true*. **d)** This is true, since there is an a in *false*.

4. a) Here x is still equal to 0, since the condition is false.
b) Here x is still equal to 1, since the condition is false.
c) This time x is equal to 1 at the end, since the condition is true, so the statement $x := 1$ is executed.

6. The answers given here are not unique, but care must be taken not to confuse nonequivalent sentences. Parts **(c)** and **(f)** are equivalent; and parts **(d)** and **(e)** are equivalent. But these two pairs are not equivalent to each other.

a) Some student in the school has visited North Dakota. (Alternatively, there exists a student in the school who has visited North Dakota.)

b) Every student in the school has visited North Dakota. (Alternatively, all students in the school have visited North Dakota.)

c) This is the negation of part **(a)**: No student in the school has visited North Dakota. (Alternatively, there does not exist a student in the school who has visited North Dakota.)

d) Some student in the school has not visited North Dakota. (Alternatively, there exists a student in the school who has not visited North Dakota.)

e) This is the negation of part **(b)**: It is not true that every student in the school has visited North Dakota. (Alternatively, not all students in the school have visited North Dakota.)

f) All students in the school have not visited North Dakota. (This is technically the correct answer, although common English usage takes this sentence to mean—incorrectly—the answer to part **(e)**. To be perfectly clear, one could say that every student in this school has failed to visit North Dakota, or simply that no student has visited North Dakota.)

8. Note that part **(b)** and part **(c)** are not the sorts of things one would normally say.

a) If an animal is a rabbit, then that animal hops. (Alternatively, every rabbit hops.)

b) Every animal is a rabbit and hops.

c) There exists an animal such that if it is a rabbit, then it hops. (Note that this is trivially true, satisfied, for example, by lions, so it is not the sort of thing one would say.)

d) There exists an animal that is a rabbit and hops. (Alternatively, some rabbits hop. Alternatively, some hopping animals are rabbits.)

10. a) We assume that this means that one student has all three animals: $\exists x(C(x) \wedge D(x) \wedge F(x))$.
b) $\forall x(C(x) \vee D(x) \vee F(x))$ **c)** $\exists x(C(x) \wedge F(x) \wedge \neg D(x))$
d) This is the negation of part **(a)**: $\neg\exists x(C(x) \wedge D(x) \wedge F(x))$.
e) Here the owners of these pets can be different: $(\exists x\, C(x)) \wedge (\exists x\, D(x)) \wedge (\exists x\, F(x))$. There is no harm in using the same dummy variable, but this could also be written, for example, as $(\exists x\, C(x)) \wedge (\exists y\, D(y)) \wedge (\exists z\, F(z))$.

12. a) Since $0+1 > 2 \cdot 0$, we know that $Q(0)$ is true.
b) Since $(-1)+1 > 2 \cdot (-1)$, we know that $Q(-1)$ is true.
c) Since $1+1 \not> 2 \cdot 1$, we know that $Q(1)$ is false.
d) From part **(a)** we know that there is at least one x that makes $Q(x)$ true, so $\exists x\, Q(x)$ is true.
e) From part **(c)** we know that there is at least one x that makes $Q(x)$ false, so $\forall x\, Q(x)$ is false.
f) From part **(c)** we know that there is at least one x that makes $Q(x)$ false, so $\exists x\, \neg Q(x)$ is true.
g) From part **(a)** we know that there is at least one x that makes $Q(x)$ true, so $\forall x\, \neg Q(x)$ is false.

14. a) Since $(-1)^3 = -1$, this is true.

b) Since $(\frac{1}{2})^4 < (\frac{1}{2})^2$, this is true.

c) Since $(-x)^2 = ((-1)x)^2 = (-1)^2x^2 = x^2$, we know that $\forall x((-x)^2 = x^2)$ is true.

d) Twice a positive number is larger than the number, but this inequality is not true for negative numbers or 0. Therefore $\forall x(2x > x)$ is false.

16. a) true ($x = \sqrt{2}$) **b)** false ($\sqrt{-1}$ is not a real number)

c) true (the left-hand side is always at least 2) **d)** false (not true for $x = 1$ or $x = 0$)

18. Existential quantifiers are like disjunctions, and universal quantifiers are like conjunctions. See Examples 11 and 16.

a) We want to assert that $P(x)$ is true for some x in the domain, so either $P(-2)$ is true or $P(-1)$ is true or $P(0)$ is true or $P(1)$ is true or $P(2)$ is true. Thus the answer is $P(-2) \vee P(-1) \vee P(0) \vee P(1) \vee P(2)$. The other parts of this exercise are similar. Note that by De Morgan's laws, the expression in part **(c)** is logically equivalent to the expression in part **(f)**, and the expression in part **(d)** is logically equivalent to the expression in part **(e)**.

b) $P(-2) \wedge P(-1) \wedge P(0) \wedge P(1) \wedge P(2)$

c) $\neg P(-2) \vee \neg P(-1) \vee \neg P(0) \vee \neg P(1) \vee \neg P(2)$

d) $\neg P(-2) \wedge \neg P(-1) \wedge \neg P(0) \wedge \neg P(1) \wedge \neg P(2)$

e) This is just the negation of part **(a)**: $\neg(P(-2) \vee P(-1) \vee P(0) \vee P(1) \vee P(2))$

f) This is just the negation of part **(b)**: $\neg(P(-2) \wedge P(-1) \wedge P(0) \wedge P(1) \wedge P(2))$

20. Existential quantifiers are like disjunctions, and universal quantifiers are like conjunctions. See Examples 11 and 16.

a) We want to assert that $P(x)$ is true for some x in the domain, so either $P(-5)$ is true or $P(-3)$ is true or $P(-1)$ is true or $P(1)$ is true or $P(3)$ is true or $P(5)$ is true. Thus the answer is $P(-5) \vee P(-3) \vee P(-1) \vee P(1) \vee P(3) \vee P(5)$.

b) $P(-5) \wedge P(-3) \wedge P(-1) \wedge P(1) \wedge P(3) \wedge P(5)$

c) The formal translation is as follows: $((-5 \neq 1) \to P(-5)) \wedge ((-3 \neq 1) \to P(-3)) \wedge ((-1 \neq 1) \to P(-1)) \wedge ((1 \neq 1) \to P(1)) \wedge ((3 \neq 1) \to P(3)) \wedge ((5 \neq 1) \to P(5))$. However, since the hypothesis $x \neq 1$ is false when x is 1 and true when x is anything other than 1, we have more simply $P(-5) \wedge P(-3) \wedge P(-1) \wedge P(3) \wedge P(5)$.

d) The formal translation is as follows: $((-5 \geq 0) \wedge P(-5)) \vee ((-3 \geq 0) \wedge P(-3)) \vee ((-1 \geq 0) \wedge P(-1)) \vee ((1 \geq 0) \wedge P(1)) \vee ((3 \geq 0) \wedge P(3)) \vee ((5 \geq 0) \wedge P(5))$. Since only three of the x's in the domain meet the condition, the answer is equivalent to $P(1) \vee P(3) \vee P(5)$.

e) For the second part we again restrict the domain: $(\neg P(-5) \vee \neg P(-3) \vee \neg P(-1) \vee \neg P(1) \vee \neg P(3) \vee \neg P(5)) \wedge (P(-1) \wedge P(-3) \wedge P(-5))$. This is equivalent to $(\neg P(1) \vee \neg P(3) \vee \neg P(5)) \wedge (P(-1) \wedge P(-3) \wedge P(-5))$.

22. Many answer are possible in each case.

a) A domain consisting of a few adults in certain parts of India would make this true. If the domain were all residents of the United States, then this is certainly false.

b) If the domain is all residents of the United States, then this is true. If the domain is the set of pupils in a first grade class, it is false.

c) If the domain consists of all the United States Presidents whose last name is Bush, then the statement is true. If the domain consists of all United States Presidents, then the statement is false.

d) If the domain were all residents of the United States, then this is certainly true. If the domain consists of all babies born in the last five minutes, one would expect the statement to be false (it's not even clear that these babies "know" their mothers yet).

24. In order to do the translation the second way, we let $C(x)$ be the propositional function "x is in your class." Note that for the second way, we always want to use conditional statements with universal quantifiers and conjunctions with existential quantifiers.

a) Let $P(x)$ be "x has a cellular phone." Then we have $\forall x\, P(x)$ the first way, or $\forall x(C(x) \rightarrow P(x))$ the second way.

b) Let $F(x)$ be "x has seen a foreign movie." Then we have $\exists x\, F(x)$ the first way, or $\exists x(C(x) \wedge F(x))$ the second way.

c) Let $S(x)$ be "x can swim." Then we have $\exists x\, \neg S(x)$ the first way, or $\exists x(C(x) \wedge \neg S(x))$ the second way.

d) Let $Q(x)$ be "x can solve quadratic equations." Then we have $\forall x\, Q(x)$ the first way, or $\forall x(C(x) \rightarrow Q(x))$ the second way.

e) Let $R(x)$ be "x wants to be rich." Then we have $\exists x\, \neg R(x)$ the first way, or $\exists x(C(x) \wedge \neg R(x))$ the second way.

26. In all of these, we will let $Y(x)$ be the propositional function that x is in your school or class, as appropriate.

a) If we let $U(x)$ be "x has visited Uzbekistan," then we have $\exists x\, U(x)$ if the domain is just your schoolmates, or $\exists x(Y(x) \wedge U(x))$ if the domain is all people. If we let $V(x, y)$ mean that person x has visited country y, then we can rewrite this last one as $\exists x(Y(x) \wedge V(x, \text{Uzbekistan}))$.

b) If we let $C(x)$ and $P(x)$ be the propositional functions asserting that x has studied calculus and C++, respectively, then we have $\forall x(C(x) \wedge P(x))$ if the domain is just your schoolmates, or $\forall x(Y(x) \rightarrow (C(x) \wedge P(x)))$ if the domain is all people. If we let $S(x, y)$ mean that person x has studied subject y, then we can rewrite this last one as $\forall x(Y(x) \rightarrow (S(x, \text{calculus}) \wedge S(x, \text{C++})))$.

c) If we let $B(x)$ and $M(x)$ be the propositional functions asserting that x owns a bicycle and a motorcycle, respectively, then we have $\forall x(\neg(B(x) \wedge M(x)))$ if the domain is just your schoolmates, or $\forall x(Y(x) \rightarrow \neg(B(x) \wedge M(x)))$ if the domain is all people. Note that "no one" became "for all ... not." If we let $O(x, y)$ mean that person x owns item y, then we can rewrite this last one as $\forall x(Y(x) \rightarrow \neg(O(x, \text{bicycle}) \wedge O(x, \text{motorcycle})))$.

d) If we let $H(x)$ be "x is happy," then we have $\exists x\, \neg H(x)$ if the domain is just your schoolmates, or $\exists x(Y(x) \wedge \neg H(x))$ if the domain is all people. If we let $E(x, y)$ mean that person x is in mental state y, then we can rewrite this last one as $\exists x(Y(x) \wedge \neg E(x, \text{happy}))$.

e) If we let $T(x)$ be "x was born in the twentieth century," then we have $\forall x\, T(x)$ if the domain is just your schoolmates, or $\forall x(Y(x) \rightarrow T(x))$ if the domain is all people. If we let $B(x, y)$ mean that person x was born in the y^{th} century, then we can rewrite this last one as $\forall x(Y(x) \rightarrow B(x, 20))$.

28. Let $R(x)$ be "x is in the correct place"; let $E(x)$ be "x is in excellent condition"; let $T(x)$ be "x is a [or your] tool"; and let the domain of discourse be all things.

a) There exists something not in the correct place: $\exists x\, \neg R(x)$.

b) If something is a tool, then it is in the correct place place and in excellent condition: $\forall x\, (T(x) \rightarrow (R(x) \wedge E(x)))$.

c) $\forall x\, (R(x) \wedge E(x))$

d) This is saying that everything fails to satisfy the condition: $\forall x\, \neg(R(x) \wedge E(x))$.

e) There exists a tool with this property: $\exists x\, (T(x) \wedge \neg R(x) \wedge E(x))$.

30. **a)** $P(1,3) \vee P(2,3) \vee P(3,3)$ **b)** $P(1,1) \wedge P(1,2) \wedge P(1,3)$
c) $\neg P(2,1) \vee \neg P(2,2) \vee \neg P(2,3)$ **d)** $\neg P(1,2) \wedge \neg P(2,2) \wedge \neg P(3,2)$

32. In each case we need to specify some propositional functions (predicates) and identify the domain of discourse.

a) Let $F(x)$ be "x has fleas," and let the domain of discourse be dogs. Our original statement is $\forall x\, F(x)$. Its negation is $\exists x\, \neg F(x)$. In English this reads "There is a dog that does not have fleas."

b) Let $H(x)$ be "x can add," where the domain of discourse is horses. Then our original statement is $\exists x\, H(x)$. Its negation is $\forall x\, \neg H(x)$. In English this is rendered most simply as "No horse can add."

c) Let $C(x)$ be "x can climb," and let the domain of discourse be koalas. Our original statement is $\forall x\, C(x)$. Its negation is $\exists x\, \neg C(x)$. In English this reads "There is a koala that cannot climb."

d) Let $F(x)$ be "x can speak French," and let the domain of discourse be monkeys. Our original statement is $\neg\exists x\, F(x)$ or $\forall x\, \neg F(x)$. Its negation is $\exists x\, F(x)$. In English this reads "There is a monkey that can speak French."

e) Let $S(x)$ be "x can swim" and let $C(x)$ be "x can catch fish," where the domain of discourse is pigs. Then our original statement is $\exists x\, (S(x) \wedge C(x))$. Its negation is $\forall x\, \neg(S(x) \wedge C(x))$, which could also be written $\forall x\, (\neg S(x) \vee \neg C(x))$ by De Morgan's law. In English this is "No pig can both swim and catch fish," or "Every pig either is unable to swim or is unable to catch fish."

34. **a)** Let $S(x)$ be "x obeys the speed limit," where the domain of discourse is drivers. The original statement is $\exists x\, \neg S(x)$, the negation is $\forall x\, S(x)$, "All drivers obey the speed limit."

b) Let $S(x)$ be "x is serious," where the domain of discourse is Swedish movies. The original statement is $\forall x\, S(x)$, the negation is $\exists x\, \neg S(x)$, "Some Swedish movies are not serious."

c) Let $S(x)$ be "x can keep a secret," where the domain of discourse is people. The original statement is $\neg\exists x\, S(x)$, the negation is $\exists x\, S(x)$, "Some people can keep a secret."

d) Let $A(x)$ be "x has a good attitude," where the domain of discourse is people in this class. The original statement is $\exists x\, \neg A(x)$, the negation is $\forall x\, A(x)$, "Everyone in this class has a good attitude."

36. **a)** Since $1^2 = 1$, this statement is false; $x = 1$ is a counterexample. So is $x = 0$ (these are the only two counterexamples).

b) There are two counterexamples: $x = \sqrt{2}$ and $x = -\sqrt{2}$.

c) There is one counterexample: $x = 0$.

38. **a)** Some system is open. **b)** Every system is either malfunctioning or in a diagnostic state.

c) Some system is open, or some system is in a diagnostic state. **d)** Some system is unavailable.

e) No system is working. (We could also say "Every system is not working," as long as we understood that this is different from "Not every system is working.")

40. There are many ways to write these, depending on what we use for predicates.

a) Let $F(x)$ be "There is less than x megabytes free on the hard disk," with the domain of discourse being positive numbers, and let $W(x)$ be "User x is sent a warning message." Then we have $F(30) \rightarrow \forall x\, W(x)$.

b) Let $O(x)$ be "Directory x can be opened," let $C(x)$ be "File x can be closed," and let E be the proposition "System errors have been detected." Then we have $E \rightarrow ((\forall x\, \neg O(x)) \wedge (\forall x\, \neg C(x)))$.

c) Let B be the proposition "The file system can be backed up," and let $L(x)$ be "User x is currently logged on." Then we have $(\exists x\, L(x)) \rightarrow \neg B$.

d) Let $D(x)$ be "Product x can be delivered," and let $M(x)$ be "There are at least x megabytes of memory available" and $S(x)$ be "The connection speed is at least x kilobits per second," where the domain of discourse for the last two propositional functions are positive numbers. Then we have $(M(8) \wedge S(56)) \rightarrow D(\text{video on demand})$.

42. There are many ways to write these, depending on what we use for predicates.

a) Let $A(x)$ be "User x has access to an electronic mailbox." Then we have $\forall x\, A(x)$.

b) Let $A(x, y)$ be "Group member x can access resource y," and let $S(x, y)$ be "System x is in state y." Then we have $S(\text{file system, locked}) \rightarrow \forall x\, A(x, \text{system mailbox})$.

c) Let $S(x, y)$ be "System x is in state y." Recalling that "only if" indicates a necessary condition, we have $S(\text{firewall}, \text{diagnostic}) \to S(\text{proxy server}, \text{diagnostic})$.

d) Let $T(x)$ be "The throughput is at least x kbps," where the domain of discourse is positive numbers, let $M(x, y)$ be "Resource x is in mode y," and let $S(x, y)$ be "Router x is in state y." Then we have $(T(100) \land \neg T(500) \land \neg M(\text{proxy server}, \text{diagnostic})) \to \exists x\, S(x, \text{normal})$.

44. We want propositional functions P and Q that are sometimes, but not always, true (so that the second biconditional is $\mathbf{F} \leftrightarrow \mathbf{F}$ and hence true), but such that there is an x making one true and the other false. For example, we can take $P(x)$ to mean that x is an even number (a multiple of 2) and $Q(x)$ to mean that x is a multiple of 3. Then an example like $x = 4$ or $x = 9$ shows that $\forall x(P(x) \leftrightarrow Q(x))$ is false.

46. **a)** There are two cases. If A is true, then $(\forall x P(x)) \lor A$ is true, and since $P(x) \lor A$ is true for all x, $\forall x(P(x) \lor A)$ is also true. Thus both sides of the logical equivalence are true (hence equivalent). Now suppose that A is false. If $P(x)$ is true for all x, then the left-hand side is true. Furthermore, the right-hand side is also true (since $P(x) \lor A$ is true for all x). On the other hand, if $P(x)$ is false for some x, then both sides are false. Therefore again the two sides are logically equivalent.

b) There are two cases. If A is true, then $(\exists x P(x)) \lor A$ is true, and since $P(x) \lor A$ is true for some (really all) x, $\exists x(P(x) \lor A)$ is also true. Thus both sides of the logical equivalence are true (hence equivalent). Now suppose that A is false. If $P(x)$ is true for at least one x, then the left-hand side is true. Furthermore, the right-hand side is also true (since $P(x) \lor A$ is true for that x). On the other hand, if $P(x)$ is false for all x, then both sides are false. Therefore again the two sides are logically equivalent.

48. **a)** There are two cases. If A is false, then both sides of the equivalence are true, because a conditional statement with a false hypothesis is true. If A is true, then $A \to P(x)$ is equivalent to $P(x)$ for each x, so the left-hand side is equivalent to $\forall x\, P(x)$, which is equivalent to the right-hand side.

b) There are two cases. If A is false, then both sides of the equivalence are true, because a conditional statement with a false hypothesis is true (and we are assuming that the domain is nonempty). If A is true, then $A \to P(x)$ is equivalent to $P(x)$ for each x, so the left-hand side is equivalent to $\exists x\, P(x)$, which is equivalent to the right-hand side.

50. It is enough to find a counterexample. It is intuitively clear that the first proposition is asserting much more than the second. It is saying that one of the two predicates, P or Q, is universally true; whereas the second proposition is simply saying that for every x either $P(x)$ or $Q(x)$ holds, but which it is may well depend on x. As a simple counterexample, let $P(x)$ be the statement that x is odd, and let $Q(x)$ be the statement that x is even. Let the domain of discourse be the positive integers. The second proposition is true, since every positive integer is either odd or even. But the first proposition is false, since it is neither the case that all positive integers are odd nor the case that all of them are even.

52. **a)** This is false, since there are many values of x that make $x > 1$ true.

b) This is false, since there are two values of x that make $x^2 = 1$ true.

c) This is true, since by algebra we see that the unique solution to the equation is $x = 3$.

d) This is false, since there are no values of x that make $x = x + 1$ true.

54. There are only three cases in which $\exists x! P(x)$ is true, so we form the disjunction of these three cases. The answer is thus $(P(1) \land \neg P(2) \land \neg P(3)) \lor (\neg P(1) \land P(2) \land \neg P(3)) \lor (\neg P(1) \land \neg P(2) \land P(3))$.

56. A Prolog query returns a yes/no answer if there are no variables in the query, and it returns the values that make the query true if there are.

a) None of the facts was that Kevin was enrolled in EE 222. So the response is `no`.

b) One of the facts was that Kiko was enrolled in Math 273. So the response is `yes`.

c) Prolog returns the names of the courses for which Grossman is the instructor, namely just `cs301`.

d) Prolog returns the names of the instructor for CS 301, namely `grossman`.

e) Prolog returns the names of the instructors teaching any course that Kevin is enrolled in, namely `chan`, since Chan is the instructor in Math 273, the only course Kevin is enrolled in.

58. Following the idea and syntax of Example 28, we have the following rule:
`grandfather(X,Y) :- father(X,Z), father(Z,Y); father(X,Z), mother(Z,Y).`
Note that we used the comma to mean "and" and the semicolon to mean "or." For `X` to be the grandfather of `Y`, `X` must be either `Y`'s father's father or `Y`'s mother's father.

60. **a)** $\forall x(P(x) \rightarrow Q(x))$ **b)** $\exists x(R(x) \wedge \neg Q(x))$ **c)** $\exists x(R(x) \wedge \neg P(x))$
d) Yes. The unsatisfactory excuse guaranteed by part **(b)** cannot be a clear explanation by part **(a)**.

62. **a)** $\forall x(P(x) \rightarrow \neg S(x))$ **b)** $\forall x(R(x) \rightarrow S(x))$ **c)** $\forall x(Q(x) \rightarrow P(x))$ **d)** $\forall x(Q(x) \rightarrow \neg R(x))$
e) Yes. If x is one of my poultry, then he is a duck (by part **(c)**), hence not willing to waltz (part **(a)**). Since officers are always willing to waltz (part **(b)**), x is not an officer.

SECTION 1.4 Nested Quantifiers

2. **a)** There exists a real number x such that for every real number y, $xy = y$. This is asserting the existence of a multiplicative identity for the real numbers, and the statement is true, since we can take $x = 1$.

b) For every real number x and real number y, if x is nonnegative and y is negative, then the difference $x - y$ is positive. Or, more simply, a nonnegative number minus a negative number is positive (which is true).

c) For every real number x and real number y, there exists a real number z such that $x = y + z$. This is a true statement, since we can take $z = x - y$ in each case.

4. **a)** Some student in your class has taken some computer science course.

b) There is a student in your class who has taken every computer science course.

c) Every student in your class has taken at least one computer science course.

d) There is a computer science course that every student in your class has taken.

e) Every computer science course has been taken by at least one student in your class.

f) Every student in your class has taken every computer science course.

6. **a)** Randy Goldberg is enrolled in CS 252.

b) Someone is enrolled in Math 695.

c) Carol Sitea is enrolled in some course.

d) Some student is enrolled simultaneously in Math 222 and CS 252.

e) There exist two distinct people, the second of whom is enrolled in every course that the first is enrolled in.

f) There exist two distinct people enrolled in exactly the same courses.

8. **a)** $\exists x \exists y Q(x,y)$

b) This is the negation of part **(a)**, and so could be written either $\neg\exists x \exists y Q(x,y)$ or $\forall x \forall y \neg Q(x,y)$.

c) We assume from the wording that the statement means that the same person appeared on both shows: $\exists x(Q(x, \text{Jeopardy}) \wedge Q(x, \text{Wheel of Fortune}))$

d) $\forall y \exists x Q(x,y)$ **e)** $\exists x_1 \exists x_2 (Q(x_1, \text{Jeopardy}) \wedge Q(x_2, \text{Jeopardy}) \wedge x_1 \neq x_2)$

10. **a)** $\forall x F(x, \text{Fred})$ **b)** $\forall y F(\text{Evelyn}, y)$ **c)** $\forall x \exists y F(x,y)$ **d)** $\neg\exists x \forall y F(x,y)$ **e)** $\forall y \exists x F(x,y)$
f) $\neg\exists x(F(x, \text{Fred}) \wedge F(x, \text{Jerry}))$
g) $\exists y_1 \exists y_2 (F(\text{Nancy}, y_1) \wedge F(\text{Nancy}, y_2) \wedge y_1 \neq y_2 \wedge \forall y(F(\text{Nancy}, y) \rightarrow (y = y_1 \vee y = y_2)))$
h) $\exists y(\forall x F(x,y) \wedge \forall z(\forall x F(x,z) \rightarrow z = y))$ **i)** $\neg\exists x F(x,x)$
j) $\exists x \exists y(x \neq y \wedge F(x,y) \wedge \forall z((F(x,z) \wedge z \neq x) \rightarrow z = y))$ (We do not assume that this sentence is asserting that this person can or cannot fool her/himself.)

12. The answers to this exercise are not unique; there are many ways of expressing the same propositions symbolically. Note that $C(x,y)$ and $C(y,x)$ say the same thing.
a) $\neg I(\text{Jerry})$ **b)** $\neg C(\text{Rachel}, \text{Chelsea})$ **c)** $\neg C(\text{Jan}, \text{Sharon})$ **d)** $\neg\exists x\, C(x, \text{Bob})$
e) $\forall x(x \neq \text{Joseph} \leftrightarrow C(x, \text{Sanjay}))$ **f)** $\exists x\, \neg I(x)$ **g)** $\neg\forall x\, I(x)$ (same as **(f)**)
h) $\exists x \forall y(x = y \leftrightarrow I(y))$ **i)** $\exists x \forall y(x \neq y \leftrightarrow I(y))$ **j)** $\forall x(I(x) \rightarrow \exists y(x \neq y \wedge C(x,y)))$
k) $\exists x(I(x) \wedge \forall y(x \neq y \rightarrow \neg C(x,y)))$ **l)** $\exists x \exists y(x \neq y \wedge \neg C(x,y))$ **m)** $\exists x \forall y\, C(x,y)$
n) $\exists x \exists y(x \neq y \wedge \forall z\, \neg(C(x,z) \wedge C(y,z)))$ **o)** $\exists x \exists y(x \neq y \wedge \forall z(C(x,z) \vee C(y,z)))$

14. The answers to this exercise are not unique; there are many ways of expressing the same propositions symbolically. Our domain of discourse for persons here consists of people in this class. We need to make up a predicate in each case.
a) Let $S(x,y)$ mean that person x can speak language y. Then our statement is $\exists x\, S(x, \text{Hindi})$.
b) Let $P(x,y)$ mean that person x plays sport y. Then our statement is $\forall x \exists y\, P(x,y)$.
c) Let $V(x,y)$ mean that person x has visited state y. Then our statement is $\exists x(V(x, \text{Alaska}) \wedge \neg V(x, \text{Hawaii}))$.
d) Let $L(x,y)$ mean that person x has learned programming language y. Then our statement is $\forall x \exists y\, L(x,y)$.
e) Let $T(x,y)$ mean that person x has taken course y, and let $O(y,z)$ mean that course y is offered by department z. Then our statement is $\exists x \exists z \forall y(O(y,z) \rightarrow T(x,y))$.
f) Let $G(x,y)$ mean that persons x and y grew up in the same town. Then our statement is $\exists x \exists y(x \neq y \wedge G(x,y) \wedge \forall z(G(x,z) \rightarrow (x = y \vee x = z)))$.
g) Let $C(x,y,z)$ mean that persons x and y have chatted with each other in chat group z. Then our statement is $\forall x \exists y \exists z(x \neq y \wedge C(x,y,z))$.

16. We let $P(s,c,m)$ be the statement that student s has class standing c and is majoring in m. The variable s ranges over students in the class, the variable c ranges over the four class standings, and the variable m ranges over all possible majors.
a) The proposition is $\exists s \exists m P(s, \text{junior}, m)$. It is true from the given information.
b) The proposition is $\forall s \exists c P(s, c, \text{computer science})$. This is false, since there are some mathematics majors.
c) The proposition is $\exists s \exists c \exists m\big(P(s,c,m) \wedge (c \neq \text{junior}) \wedge (m \neq \text{mathematics})\big)$. This is true, since there is a sophomore majoring in computer science.
d) The proposition is $\forall s\big(\exists c P(s, c, \text{computer science}) \vee \exists m P(s, \text{sophomore}, m)\big)$. This is false, since there is a freshman mathematics major.
e) The proposition is $\exists m \forall c \exists s P(s,c,m)$. This is false. It cannot be that m is mathematics, since there is no senior mathematics major, and it cannot be that m is computer science, since there is no freshman computer science major. Nor, of course, can m be any other major.

18. **a)** $\forall f\,(H(f) \rightarrow \exists c\, A(c))$, where $A(x)$ means that console x is accessible, and $H(x)$ means that fault condition x is happening
b) $(\forall u \exists m\,(A(m) \wedge S(u,m))) \rightarrow \forall u\, R(u)$, where $A(x)$ means that the archive contains message x, $S(x,y)$ means that user x sent message y, and $R(x)$ means that the e-mail address of user x can be retrieved

c) $(\forall b \exists m\, D(m,b)) \leftrightarrow \exists p \neg C(p)$, where $D(x,y)$ means that mechanism x can detect breach y, and $C(x)$ means that process x has been compromised

d) $\forall x \forall y\, (x \neq y \rightarrow \exists p \exists q\, (p \neq q \land C(p,x,y) \land C(q,x,y)))$, where $C(p,x,y)$ means that path p connects endpoint x to endpoint y

e) $\forall x\, ((\forall u\, K(x,u)) \leftrightarrow x = \text{SysAdm})$, where $K(x,y)$ means that person x knows the password of user y

20. **a)** $\forall x \forall y((x < 0) \land (y < 0) \rightarrow (xy > 0))$ **b)** $\forall x \forall y((x > 0) \land (y > 0) \rightarrow ((x+y)/2 > 0))$

c) What does "necessarily" mean in this context? The best explanation is to assert that a certain universal conditional statement is not true. So we have $\neg \forall x \forall y((x < 0) \land (y < 0) \rightarrow (x - y < 0))$. Note that we do not want to put the negation symbol inside (it is not true that the difference of two negative integers is never negative), nor do we want to negate just the conclusion (it is not true that the sum is always nonnegative). We could rewrite our solution by passing the negation inside, obtaining $\exists x \exists y((x < 0) \land (y < 0) \land (x - y \geq 0))$.

d) $\forall x \forall y\, (|x+y| \leq |x| + |y|)$

22. $\exists x \forall a \forall b \forall c\, ((x > 0) \land x \neq a^2 + b^2 + c^2)$, where the domain of discourse consists of all integers

24. **a)** There exists an additive identity for the real numbers—a number that when added to every number does not change its value.

b) A nonnegative number minus a negative number is positive.

c) The difference of two nonpositive numbers is not necessarily nonpositive.

d) The product of two numbers is nonzero if and only if both factors are nonzero.

26. **a)** This is false, since $1 + 1 \neq 1 - 1$. **b)** This is true, since $2 + 0 = 2 - 0$.

c) This is false, since there are many values of y for which $1 + y \neq 1 - y$.

d) This is false, since the equation $x + 2 = x - 2$ has no solution.

e) This is true, since we can take $x = y = 0$. **f)** This is true, since we can take $y = 0$ for each x.

g) This is true, since we can take $y = 0$. **h)** This is false, since part **(d)** was false.

i) This is certainly false.

28. **a)** true (let $y = x^2$) **b)** false (no such y exists if x is negative) **c)** true (let $x = 0$)

d) false (the commutative law for addition always holds) **e)** true (let $y = 1/x$)

f) false (the reciprocal of y depends on y—there is not one x that works for all y) **g)** true (let $y = 1 - x$)

h) false (this system of equations is inconsistent)

i) false (this system has only one solution; if $x = 0$, for example, then no y satisfies $y = 2 \land -y = 1$)

j) true (let $z = (x+y)/2$)

30. We need to use the transformations shown in Table 2 of Section 1.3, replacing $\neg\forall$ by $\exists\neg$, and replacing $\neg\exists$ by $\forall\neg$. In other words, we push all the negation symbols inside the quantifiers, changing the sense of the quantifiers as we do so, because of the equivalences in Table 2 of Section 1.3. In addition, we need to use De Morgan's laws (Section 1.2) to change the negation of a conjunction to the disjunction of the negations and to change the negation of a disjunction to the conjunction of the negations. We also use the fact that $\neg\neg p \equiv p$.

a) $\forall y \forall x\, \neg P(x,y)$ **b)** $\exists x \forall y\, \neg P(x,y)$ **c)** $\forall y(\neg Q(y) \lor \exists x\, R(x,y))$

d) $\forall y(\forall x\, \neg R(x,y) \land \exists x\, \neg S(x,y))$ **e)** $\forall y(\exists x \forall z\, \neg T(x,y,z) \land \forall x \exists z\, \neg U(x,y,z))$

32. As we push the negation symbol toward the inside, each quantifier it passes must change its type. For logical connectives we either use De Morgan's laws or recall that $\neg(p \rightarrow q) \equiv p \land \neg q$ (Table 7 in Section 1.2) and that $\neg(p \leftrightarrow q) \equiv \neg p \leftrightarrow q$ (Exercise 21 in Section 1.2).

a)
$$\begin{aligned}\neg\exists z\forall y\forall x\, T(x,y,z) &\equiv \forall z\neg\forall y\forall x\, T(x,y,z)\\ &\equiv \forall z\exists y\neg\forall x\, T(x,y,z)\\ &\equiv \forall z\exists y\exists x\,\neg T(x,y,z)\end{aligned}$$

b)
$$\begin{aligned}\neg(\exists x\exists y\, P(x,y)\wedge\forall x\forall y\, Q(x,y)) &\equiv \neg\exists x\exists y\, P(x,y)\vee\neg\forall x\forall y\, Q(x,y)\\ &\equiv \forall x\neg\exists y\, P(x,y)\vee\exists x\neg\forall y\, Q(x,y)\\ &\equiv \forall x\forall y\,\neg P(x,y)\vee\exists x\exists y\,\neg\, Q(x,y)\end{aligned}$$

c)
$$\begin{aligned}\neg\exists x\exists y(Q(x,y)\leftrightarrow Q(y,x)) &\equiv \forall x\neg\exists y(Q(x,y)\leftrightarrow Q(y,x))\\ &\equiv \forall x\forall y\neg(Q(x,y)\leftrightarrow Q(y,x))\\ &\equiv \forall x\forall y(\neg Q(x,y)\leftrightarrow Q(y,x))\end{aligned}$$

d)
$$\begin{aligned}\neg\forall y\exists x\exists z\,(T(x,y,z)\vee Q(x,y)) &\equiv \exists y\neg\exists x\exists z\,(T(x,y,z)\vee Q(x,y))\\ &\equiv \exists y\forall x\neg\exists z\,(T(x,y,z)\vee Q(x,y))\\ &\equiv \exists y\forall x\forall z\,\neg(T(x,y,z)\vee Q(x,y))\\ &\equiv \exists y\forall x\forall z\,(\neg T(x,y,z)\wedge\neg Q(x,y))\end{aligned}$$

34. The logical expression is asserting that the domain consists of at most two members. (It is saying that whenever you have two unequal objects, any object has to be one of those two. Note that this is vacuously true for domains with one element.) Therefore any domain having one or two members will make it true (such as the female members of the United States Supreme Court in 2005), and any domain with more than two members will make it false (such as all members of the United States Supreme Court in 2005).

36. In each case we need to specify some predicates and identify the domain of discourse.

a) Let $L(x,y)$ mean that person x has lost y dollars playing the lottery. The original statement is then $\neg\exists x\exists y(y>1000\wedge L(x,y))$. Its negation of course is $\exists x\exists y(y>1000\wedge L(x,y))$; someone has lost more than \$1000 playing the lottery.

b) Let $C(x,y)$ mean that person x has chatted with person y. The given statement is $\exists x\exists y(y\neq x\wedge\forall z(z\neq x\rightarrow(z=y\leftrightarrow C(x,z))))$. The negation is therefore $\forall x\forall y(y\neq x\rightarrow\exists z(z\neq x\wedge\neg(z=y\leftrightarrow C(x,z))))$. In English, everybody in this class has either chatted with no one else or has chatted with two or more others.

c) Let $E(x,y)$ mean that person x has sent e-mail to person y. The given statement is $\neg\exists x\exists y\exists z(y\neq z\wedge x\neq y\wedge x\neq z\wedge\forall w(w\neq x\rightarrow(E(x,w)\leftrightarrow(w=y\vee w=z))))$. The negation is obviously $\exists x\exists y\exists z(y\neq z\wedge x\neq y\wedge x\neq z\wedge\forall w(w\neq x\rightarrow(E(x,w)\leftrightarrow(w=y\vee w=z))))$. In English, some student in this class has sent e-mail to exactly two other students in this class.

d) Let $S(x,y)$ mean that student x has solved exercise y. The statement is $\exists x\forall y\, S(x,y)$. The negation is $\forall x\exists y\,\neg S(x,y)$. In English, for every student in this class, there is some exercise that he or she has not solved. (One could also interpret the given statement as asserting that for every exercise, there exists a student—perhaps a different one for each exercise—who has solved it. In that case the order of the quantifiers would be reversed. Word order in English sometimes makes for a little ambiguity.)

e) Let $S(x,y)$ mean that student x has solved exercise y, and let $B(y,z)$ mean that exercise y is in section z of the book. The statement is $\neg\exists x\forall z\exists y(B(y,z)\wedge S(x,y))$. The negation is of course $\exists x\forall z\exists y(B(y,z)\wedge S(x,y))$. In English, some student has solved at least one exercise in every section of this book.

38. a) In English, the negation is "Some student in this class does not like mathematics." With the obvious propositional function, this is $\exists x\neg L(x)$.

b) In English, the negation is "Every student in this class has seen a computer." With the obvious propositional function, this is $\forall x S(x)$.

c) In English, the negation is "For every student in this class, there is a mathematics course that this student has not taken." With the obvious propositional function, this is $\forall x \exists c \neg T(x,c)$.

d) As in Exercise 15f, let $P(z,y)$ be "Room z is in building y," and let $Q(x,z)$ be "Student x has been in room z." Then the original statement is $\exists x \forall y \exists z \big(P(z,y) \wedge Q(x,z)\big)$. To form the negation, we change all the quantifiers and put the negation on the inside, then apply De Morgan's law. The negation is therefore $\forall x \exists y \forall z \big(\neg P(z,y) \vee \neg Q(x,z)\big)$, which is also equivalent to $\forall x \exists y \forall z \big(P(z,y) \to \neg Q(x,z)\big)$. In English, this could be read, "For every student there is a building such that for every room in that building, the student has not been in that room."

40. **a)** There are many counterexamples. If $x = 2$, then there is no y among the integers such that $2 = 1/y$, since the only solution of this equation is $y = 1/2$. Even if we were working in the domain of real numbers, $x = 0$ would provide a counterexample, since $0 = 1/y$ for no real number y.

b) We can rewrite $y^2 - x < 100$ as $y^2 < 100 + x$. Since squares can never be negative, no such y exists if x is, say, -200. This x provides a counterexample.

c) This is not true, since sixth powers are both squares and cubes. Trivial counterexamples would include $x = y = 0$ and $x = y = 1$, but we can also take something like $x = 27$ and $y = 9$, since $27^2 = 3^6 = 9^3$.

42. The distributive law is just the statement that $x(y+z) = xy+xz$ for all real numbers. Therefore the expression we want is $\forall x \forall y \forall z\, (x(y+z) = xy + xz)$, where the quantifiers are assumed to range over (i.e., the domain of discourse is) the real numbers.

44. We want to say that for each triple of coefficients (the a, b, and c in the expression $ax^2 + bx + c$, where we insist that $a \neq 0$ so that this actually is quadratic), there are at most two values of x making that expression equal to 0. The domain here is all real numbers. We write $\forall a \forall b \forall c (a \neq 0 \to \forall x_1 \forall x_2 \forall x_3 (ax_1^2 + bx_1 + c = 0 \wedge ax_2^2 + bx_2 + c = 0 \wedge ax_3^2 + bx_3 + c = 0) \to (x_1 = x_2 \vee x_1 = x_3 \vee x_2 = x_3))$.

46. This statement says that there is a number that is less than or equal to all squares.

a) This is false, since no matter how small a positive number x we might choose, if we let $y = \sqrt{x/2}$, then $x = 2y^2$, and it will not be true that $x \le y^2$.

b) This is true, since we can take $x = -1$, for example.

c) This is true, since we can take $x = -1$, for example.

48. We need to show that each of these propositions implies the other. Suppose that $\forall x P(x) \vee \forall x Q(x)$ is true. We want to show that $\forall x \forall y (P(x) \vee Q(y))$ is true. By our hypothesis, one of two things must be true. Either P is universally true, or Q is universally true. In the first case, $\forall x \forall y (P(x) \vee Q(y))$ is true, since the first expression in the disjunction is true, no matter what x and y are; and in the second case, $\forall x \forall y (P(x) \vee Q(y))$ is also true, since now the second expression in the disjunction is true, no matter what x and y are. Next we need to prove the converse. So suppose that $\forall x \forall y (P(x) \vee Q(y))$ is true. We want to show that $\forall x P(x) \vee \forall x Q(x)$ is true. If $\forall x P(x)$ is true, then we are done. Otherwise, $P(x_0)$ must be false for some x_0 in the domain of discourse. For this x_0, then, the hypothesis tells us that $P(x_0) \vee Q(y)$ is true, no matter what y is. Since $P(x_0)$ is false, it must be the case that $Q(y)$ is true for each y. In other words, $\forall y Q(y)$ is true, or, to change the name of the meaningless quantified variable, $\forall x Q(x)$ is true. This certainly implies that $\forall x P(x) \vee \forall x Q(x)$ is true, as desired.

50. **a)** By Exercises 45 and 46b in Section 1.3, we can simply bring the existential quantifier outside: $\exists x (P(x) \vee Q(x) \vee A)$.

b) By Exercise 48 of the current section, the expression inside the parentheses is logically equivalent to $\forall x \forall y (P(x) \vee Q(y))$. Applying the negation operation, we obtain $\exists x \exists y \neg (P(x) \vee Q(y))$.

c) First we rewrite this using Table 7 in Section 1.2 as $\exists x Q(x) \vee \neg\exists x P(x)$, which is equivalent to $\exists x Q(x) \vee \forall x \neg P(x)$. To combine the existential and universal statements we use Exercise 49b of the current section, obtaining $\forall x \exists y(\neg P(x) \vee Q(y))$, which is in prenex normal form.

52. We simply want to say that there exists an x such that $P(x)$ holds, and that every y such that $P(y)$ holds must be this same x. Thus we write $\exists x\big(P(x) \wedge \forall y(P(y) \rightarrow y = x)\big)$. Even more compactly, we can write $\exists x \forall y(P(y) \leftrightarrow y = x)$.

SECTION 1.5 Rules of Inference

2. This is modus tollens. The first statement is $p \rightarrow q$, where p is "George does not have eight legs" and q is "George is not an insect." The second statement is $\neg q$. The third is $\neg p$. Modus tollens is valid. We can therefore conclude that the conclusion of the argument (third statement) is true, given that the hypotheses (the first two statements) are true.

4. **a)** We have taken the conjunction of two propositions and asserted one of them. This is, according to Table 1, simplification.

b) We have taken the disjunction of two propositions and the negation of one of them, and asserted the other. This is, according to Table 1, disjunctive syllogism. See Table 1 for the other parts of this exercise as well.

c) modus ponens **d)** addition **e)** hypothetical syllogism

6. Let r be the proposition "It rains," let f be the proposition "It is foggy," let s be the proposition "The sailing race will be held," let l be the proposition "The life saving demonstration will go on," and let t be the proposition "The trophy will be awarded." We are given premises $(\neg r \vee \neg f) \rightarrow (s \wedge l)$, $s \rightarrow t$, and $\neg t$. We want to conclude r. We set up the proof in two columns, with reasons, as in Example 6. Note that it is valid to replace subexpressions by other expressions logically equivalent to them.

Step	**Reason**
1. $\neg t$	Hypothesis
2. $s \rightarrow t$	Hypothesis
3. $\neg s$	Modus tollens using (1) and (2)
4. $(\neg r \vee \neg f) \rightarrow (s \wedge l)$	Hypothesis
5. $(\neg(s \wedge l)) \rightarrow \neg(\neg r \vee \neg f)$	Contrapositive of (4)
6. $(\neg s \vee \neg l) \rightarrow (r \wedge f)$	De Morgan's law and double negative
7. $\neg s \vee \neg l$	Addition, using (3)
8. $r \wedge f$	Modus ponens using (6) and (7)
9. r	Simplification using (8)

8. First we use universal instantiation to conclude from "For all x, if x is a man, then x is not an island" the special case of interest, "If Manhattan is a man, then Manhattan is not an island." Then we form the contrapositive (using also double negative): "If Manhattan is an island, then Manhattan is not a man." Finally we use modus ponens to conclude that Manhattan is not a man. Alternatively, we could apply modus tollens.

10. **a)** If we use modus tollens starting from the back, then we conclude that I am not sore. Another application of modus tollens then tells us that I did not play hockey.

b) We really can't conclude anything specific here.

c) By universal instantiation, we conclude from the first conditional statement by modus ponens that dragonflies have six legs, and we conclude by modus tollens that spiders are not insects. We could say using existential generalization that, for example, there exists a non-six-legged creature that eats a six-legged creature, and that there exists a non-insect that eats an insect.

d) We can apply universal instantiation to the conditional statement and conclude that if Homer (respectively, Maggie) is a student, then he (she) has an Internet account. Now modus tollens tells us that Homer is not a student. There are no conclusions to be drawn about Maggie.

e) The first conditional statement is that if x is healthy to eat, then x does not taste good. Universal instantiation and modus ponens therefore tell us that tofu does not taste good. The third sentence says that if you eat x, then x tastes good. Therefore the fourth hypothesis already follows (by modus tollens) from the first three. No conclusions can be drawn about cheeseburgers from these statements.

f) By disjunctive syllogism, the first two hypotheses allow us to conclude that I am hallucinating. Therefore by modus ponens we know that I see elephants running down the road.

12. Applying Exercise 11, we want to show that the conclusion r follows from the five premises $(p \wedge t) \to (r \vee s)$, $q \to (u \wedge t)$, $u \to p$, $\neg s$, and q. From q and $q \to (u \wedge t)$ we get $u \wedge t$ by modus ponens. From there we get both u and t by simplification (and the commutative law). From u and $u \to p$ we get p by modus ponens. From p and t we get $p \wedge t$ by conjunction. From that and $(p \wedge t) \to (r \vee s)$ we get $r \vee s$ by modus ponens. From that and $\neg s$ we finally get r by disjunctive syllogism.

14. In each case we set up the proof in two columns, with reasons, as in Example 6.

a) Let $c(x)$ be "x is in this class," let $r(x)$ be "x owns a red convertible," and let $t(x)$ be "x has gotten a speeding ticket." We are given premises $c(\text{Linda})$, $r(\text{Linda})$, $\forall x(r(x) \to t(x))$, and we want to conclude $\exists x(c(x) \wedge t(x))$.

Step	**Reason**
1. $\forall x(r(x) \to t(x))$	Hypothesis
2. $r(\text{Linda}) \to t(\text{Linda})$	Universal instantiation using (1)
3. $r(\text{Linda})$	Hypothesis
4. $t(\text{Linda})$	Modus ponens using (2) and (3)
5. $c(\text{Linda})$	Hypothesis
6. $c(\text{Linda}) \wedge t(\text{Linda})$	Conjunction using (4) and (5)
7. $\exists x(c(x) \wedge t(x))$	Existential generalization using (6)

b) Let $r(x)$ be "r is one of the five roommates listed," let $d(x)$ be "x has taken a course in discrete mathematics," and let $a(x)$ be "x can take a course in algorithms." We are given premises $\forall x(r(x) \to d(x))$ and $\forall x(d(x) \to a(x))$, and we want to conclude $\forall x(r(x) \to a(x))$. In what follows y represents an arbitrary person.

Step	**Reason**
1. $\forall x(r(x) \to d(x))$	Hypothesis
2. $r(y) \to d(y)$	Universal instantiation using (1)
3. $\forall x(d(x) \to a(x))$	Hypothesis
4. $d(y) \to a(y)$	Universal instantiation using (3)
5. $r(y) \to a(y)$	Hypothetical syllogism using (2) and (4)
6. $\forall x(r(x) \to a(x))$	Universal generalization using (5)

c) Let $s(x)$ be "x is a movie produced by Sayles," let $c(x)$ be "x is a movie about coal miners," and let

$w(x)$ be "movie x is wonderful." We are given premises $\forall x(s(x) \to w(x))$ and $\exists x(s(x) \land c(x))$, and we want to conclude $\exists x(c(x) \land w(x))$. In our proof, y represents an unspecified particular movie.

Step	**Reason**
1. $\exists x(s(x) \land c(x))$	Hypothesis
2. $s(y) \land c(y)$	Existential instantiation using (1)
3. $s(y)$	Simplification using (2)
4. $\forall x(s(x) \to w(x))$	Hypothesis
5. $s(y) \to w(y)$	Universal instantiation using (4)
6. $w(y)$	Modus ponens using (3) and (5)
7. $c(y)$	Simplification using (2)
8. $w(y) \land c(y)$	Conjunction using (6) and (7)
9. $\exists x(c(x) \land w(x))$	Existential generalization using (8)

d) Let $c(x)$ be "x is in this class," let $f(x)$ be "x has been to France," and let $l(x)$ be "x has visited the Louvre." We are given premises $\exists x(c(x) \land f(x))$, $\forall x(f(x) \to l(x))$, and we want to conclude $\exists x(c(x) \land l(x))$. In our proof, y represents an unspecified particular person.

Step	**Reason**
1. $\exists x(c(x) \land f(x))$	Hypothesis
2. $c(y) \land f(y)$	Existential instantiation using (1)
3. $f(y)$	Simplification using (2)
4. $c(y)$	Simplification using (2)
5. $\forall x(f(x) \to l(x))$	Hypothesis
6. $f(y) \to l(y)$	Universal instantiation using (5)
7. $l(y)$	Modus ponens using (3) and (6)
8. $c(y) \land l(y)$	Conjunction using (4) and (7)
9. $\exists x(c(x) \land l(x))$	Existential generalization using (8)

16. **a)** This is correct, using universal instantiation and modus tollens.
b) This is not correct. After applying universal instantiation, it contains the fallacy of denying the hypothesis.
c) After applying universal instantiation, it contains the fallacy of affirming the conclusion.
d) This is correct, using universal instantiation and modus ponens.

18. We know that *some* s exists that makes $S(s, \text{Max})$ true, but we cannot conclude that Max is one such s. Therefore this first step is invalid.

20. **a)** This is invalid. It is the fallacy of affirming the conclusion. Letting $a = -2$ provides a counterexample.
b) This is valid; it is modus ponens.

22. We will give an argument establishing the conclusion. We want to show that all hummingbirds are small. Let Tweety be an arbitrary hummingbird. We must show that Tweety is small. The first premise implies that if Tweety is a hummingbird, then Tweety is richly colored. Therefore by (universal) modus ponens we can conclude that Tweety is richly colored. The third premise implies that if Tweety does not live on honey, then Tweety is not richly colored. Therefore by (universal) modus tollens we can now conclude that Tweety does live on honey. Finally, the second premise implies that if Tweety is a large bird, then Tweety does not live on honey. Therefore again by (universal) modus tollens we can now conclude that Tweety is not a large bird, i.e., that Tweety is small, as desired. Notice that we invoke universal generalization as the last step.

24. Steps 3 and 5 are incorrect; simplification applies to conjunctions, not disjunctions.

26. We want to show that the conditional statement $P(a) \to R(a)$ is true for all a in the domain; the desired conclusion then follows by universal generalization. Thus we want to show that if $P(a)$ is true for a particular a, then $R(a)$ is also true. For such an a, by universal modus ponens from the first premise we have $Q(a)$, and then by universal modus ponens from the second premise we have $R(a)$, as desired.

28. We want to show that the conditional statement $\neg R(a) \to P(a)$ is true for all a in the domain; the desired conclusion then follows by universal generalization. Thus we want to show that if $\neg R(a)$ is true for a particular a, then $P(a)$ is also true. For such an a, universal modus tollens applied to the second premise gives us $\neg(\neg P(a) \wedge Q(a))$. By rules from propositional logic, this gives us $P(a) \vee \neg Q(a)$. By universal generalization from the first premise, we have $P(a) \vee Q(a)$. Now by resolution we can conclude $P(a) \vee P(a)$, which is logically equivalent to $P(a)$, as desired.

30. Let a be "Allen is a good boy"; let h be "Hillary is a good girl"; let d be "David is happy." Then our assumptions are $\neg a \vee h$ and $a \vee d$. Using resolution gives us $h \vee d$, as desired.

32. We apply resolution to give the tautology $(p \vee \mathbf{F}) \wedge (\neg p \vee \mathbf{F}) \to (\mathbf{F} \vee \mathbf{F})$. The left-hand side is equivalent to $p \wedge \neg p$, since $p \vee \mathbf{F}$ is equivalent to p, and $\neg p \vee \mathbf{F}$ is equivalent to $\neg p$. The right-hand side is equivalent to $\mathbf{F}$. Since the conditional statement is true, and the conclusion is false, it follows that the hypothesis, $p \wedge \neg p$, is false, as desired.

34. Let us use the following letters to stand for the relevant propositions: d for "logic is difficult"; s for "many students like logic"; and e for "mathematics is easy." Then the assumptions are $d \vee \neg s$ and $e \to \neg d$. Note that the first of these is equivalent to $s \to d$, since both forms are false if and only if s is true and d is false. In addition, let us note that the second assumption is equivalent to its contrapositive, $d \to \neg e$. And finally, by combining these two conditional statements, we see that $s \to \neg e$ also follows from our assumptions.

a) Here we are asked whether we can conclude that $s \to \neg e$. As we noted above, the answer is yes, this conclusion is valid.

b) The question concerns $\neg e \to \neg s$. This is equivalent to its contrapositive, $s \to e$. That doesn't seem to follow from our assumptions, so let's find a case in which the assumptions hold but this conditional statement does not. This conditional statement fails in the case in which s is true and e is false. If we take d to be true as well, then both of our assumptions are true. Therefore this conclusion is not valid.

c) The issue is $\neg e \vee d$, which is equivalent to the conditional statement $e \to d$. This does *not* follow from our assumptions. If we take d to be false, e to be true, and s to be false, then this proposition is false but our assumptions are true.

d) The issue is $\neg d \vee \neg e$, which is equivalent to the conditional statement $d \to \neg e$. We noted above that this validly follows from our assumptions.

e) This sentence says $\neg s \to (\neg e \vee \neg d)$. The only case in which this is false is when s is false and both e and d are true. But in this case, our assumption $e \to \neg d$ is also violated. Therefore, in all cases in which the assumptions hold, this statement holds as well, so it *is* a valid conclusion.

SECTION 1.6 Introduction to Proofs

2. We must show that whenever we have two even integers, their sum is even. Suppose that a and b are two even integers. Then there exist integers s and t such that $a = 2s$ and $b = 2t$. Adding, we obtain $a+b = 2s+2t = 2(s+t)$. Since this represents $a+b$ as 2 times the integer $s+t$, we conclude that $a+b$ is even, as desired.

4. We must show that whenever we have an even integer, its negative is even. Suppose that a is an even integer. Then there exists an integer s such that $a = 2s$. Its additive inverse is $-2s$, which by rules of arithmetic and algebra (see Appendix 1) equals $2(-s)$. Since this is 2 times the integer $-s$, it is even, as desired.

6. An odd number is one of the form $2n+1$, where n is an integer. We are given two odd numbers, say $2a+1$ and $2b+1$. Their product is $(2a+1)(2b+1) = 4ab+2a+2b+1 = 2(2ab+a+b)+1$. This last expression shows that the product is odd, since it is of the form $2n+1$, with $n = 2ab+a+b$.

8. Let $n = m^2$. If $m = 0$, then $n+2 = 2$, which is not a perfect square, so we can assume that $m \geq 1$. The smallest perfect square greater than n is $(m+1)^2$, and we have $(m+1)^2 = m^2+2m+1 = n+2m+1 > n+2\cdot 1+1 > n+2$. Therefore $n+2$ cannot be a perfect square.

10. A rational number is a number that can be written in the form x/y where x and y are integers and $y \neq 0$. Suppose that we have two rational numbers, say a/b and c/d. Then their product is, by the usual rules for multiplication of fractions, $(ac)/(bd)$. Note that both the numerator and the denominator are integers, and that $bd \neq 0$ since b and d were both nonzero. Therefore the product is, by definition, a rational number.

12. This is true. Suppose that a/b is a nonzero rational number and that x is an irrational number. We must prove that the product xa/b is also irrational. We give a proof by contradiction. Suppose that xa/b were rational. Since $a/b \neq 0$, we know that $a \neq 0$, so b/a is also a rational number. Let us multiply this rational number b/a by the assumed rational number xa/b. By Exercise 26, the product is rational. But the product is $(b/a)(xa/b) = x$, which is irrational by hypothesis. This is a contradiction, so in fact xa/b must be irrational, as desired.

14. If x is rational and not zero, then by definition we can write $x = p/q$, where p and q are nonzero integers. Since $1/x$ is then q/p and $p \neq 0$, we can conclude that $1/x$ is rational.

16. We give a proof by contraposition. If it is not true than m is even or n is even, then m and n are both odd. By Exercise 6, this tells us that mn is odd, and our proof is complete.

18. **a)** We must prove the contrapositive: If n is odd, then $3n+2$ is odd. Assume that n is odd. Then we can write $n = 2k+1$ for some integer k. Then $3n+2 = 3(2k+1)+2 = 6k+5 = 2(3k+2)+1$. Thus $3n+2$ is two times some integer plus 1, so it is odd.

b) Suppose that $3n+2$ is even and that n is odd. Since $3n+2$ is even, so is $3n$. If we add subtract an odd number from an even number, we get an odd number, so $3n-n = 2n$ is odd. But this is obviously not true. Therefore our supposition was wrong, and the proof by contradiction is complete.

20. We need to prove the proposition "If 1 is a positive integer, then $1^2 \geq 1$." The conclusion is the true statement $1 \geq 1$. Therefore the conditional statement is true. This is an example of a trivial proof, since we merely showed that the conclusion was true.

22. We give a proof by contradiction. Suppose that we don't get a pair of blue socks or a pair of black socks. Then we drew at most one of each color. This accounts for only two socks. But we are drawing three socks. Therefore our supposition that we did not get a pair of blue socks or a pair of black socks is incorrect, and our proof is complete.

24. We give a proof by contradiction. If there were at most two days falling in the same month, then we could have at most $2 \cdot 12 = 24$ days, since there are 12 months. Since we have chosen 25 days, at least three of them must fall in the same month.

26. We need to prove two things, since this is an "if and only if" statement. First let us prove directly that if n is even then $7n + 4$ is even. Since n is even, it can be written as $2k$ for some integer k. Then $7n + 4 = 14k + 4 = 2(7k + 2)$. This is 2 times an integer, so it is even, as desired. Next we give a proof by contraposition that if $7n+4$ is even then n is even. So suppose that n is not even, i.e., that n is odd. Then n can be written as $2k+1$ for some integer k. Thus $7n+4 = 14k+11 = 2(7k+5)+1$. This is 1 more than 2 times an integer, so it is odd. That completes the proof by contraposition.

28. There are two things to prove. For the "if" part, there are two cases. If $m = n$, then of course $m^2 = n^2$; if $m = -n$, then $m^2 = (-n)^2 = (-1)^2 n^2 = n^2$. For the "only if" part, we suppose that $m^2 = n^2$. Putting everything on the left and factoring, we have $(m+n)(m-n) = 0$. Now the only way that a product of two numbers can be zero is if one of them is zero. Therefore we conclude that either $m + n = 0$ (in which case $m = -n$), or else $m - n = 0$ (in which case $m = n$), and our proof is complete.

30. We write these in symbols: $a < b$, $(a+b)/2 > a$, and $(a+b)/2 < b$. The latter two are equivalent to $a + b > 2a$ and $a + b < 2b$, respectively, and these are in turn equivalent to $b > a$ and $a < b$, respectively. It is now clear that all three statements are equivalent.

32. We give direct proofs that (*i*) implies (*ii*), that (*ii*) implies (*iii*), and that (*iii*) implies (*i*). That will suffice. For the first, suppose that $x = p/q$ where p and q are integers with $q \neq 0$. Then $x/2 = p/(2q)$, and this is rational, since p and $2q$ are integers with $2q \neq 0$. For the second, suppose that $x/2 = p/q$ where p and q are integers with $q \neq 0$. Then $x = (2p)/q$, so $3x - 1 = (6p)/q - 1 = (6p - q)/q$ and this is rational, since $6p - q$ and q are integers with $q \neq 0$. For the last, suppose that $3x - 1 = p/q$ where p and q are integers with $q \neq 0$. Then $x = (p/q + 1)/3 = (p+q)/(3q)$, and this is rational, since $p + q$ and $3q$ are integers with $3q \neq 0$.

34. No. This line of reasoning shows that *if* $\sqrt{2x^2 - 1} = x$, then we must have $x = 1$ or $x = -1$. These are therefore the only possible solutions, but we have no guarantee that they *are* solutions, since not all of our steps were reversible (in particular, squaring both sides). Therefore we *must* substitute these values back into the original equation to determine whether they do indeed satisfy it.

36. The only conditional statements not shown directly are $p_1 \leftrightarrow p_2$, $p_2 \leftrightarrow p_4$, and $p_3 \leftrightarrow p_4$. But these each follow with one or more intermediate steps: $p_1 \leftrightarrow p_2$, since $p_1 \leftrightarrow p_3$ and $p_3 \leftrightarrow p_2$; $p_2 \leftrightarrow p_4$, since $p_2 \leftrightarrow p_1$ (just established) and $p_1 \leftrightarrow p_4$; and $p_3 \leftrightarrow p_4$, since $p_3 \leftrightarrow p_1$ and $p_1 \leftrightarrow p_4$.

38. We must find a number that cannot be written as the sum of the squares of three integers. We claim that 7 is such a number (in fact, it is the smallest such number). The only squares that can be used to contribute to the sum are 0, 1, and 4. We cannot use two 4's, because their sum exceeds 7. Therefore we can use at most one 4, which means that we must get 3 using just 0's and 1's. Clearly three 1's are required for this, bringing the total number of squares used to four. Thus 7 cannot be written as the sum of three squares.

40. Suppose that we look at the ten groups of integers in three consecutive locations around the circle (first-second-third, second-third-fourth, ..., eighth-ninth-tenth, ninth-tenth-first, and tenth-first-second). Since each number from 1 to 10 gets used three times in these groups, the sum of the sums of the ten groups must equal three times the sum of the numbers from 1 to 10, namely $3 \cdot 55 = 165$. Therefore the average sum is $165/10 = 16.5$. By Exercise 39, at least one of the sums must be greater than or equal to 16.5, and since the sums are whole numbers, this means that at least one of the sums must be greater than or equal to 17.

42. We show that each of these is equivalent to the statement (v) n is odd, say $n = 2k+1$. Example 1 showed that (v) implies (i), and Example 8 showed that (i) implies (v). For $(v) \rightarrow (ii)$ we see that $1 - n = 1 - (2k + 1) = 2(-k)$ is even. Conversely, if n were even, say $n = 2m$, then we would have $1 - n = 1 - 2m = 2(-m) + 1$, so $1 - n$ would be odd, and this completes the proof by contraposition that $(ii) \rightarrow (v)$. For $(v) \rightarrow (iii)$, we see that $n^3 = (2k+1)^3 = 8k^3+12k^2+6k+1 = 2(4k^3+6k^2+3k)+1$ is odd. Conversely, if n were even, say $n = 2m$, then we would have $n^3 = 2(4m^3)$, so n^3 would be even, and this completes the proof by contraposition that $(iii) \rightarrow (v)$. Finally, for $(v) \rightarrow (iv)$, we see that $n^2 + 1 = (2k + 1)^2 + 1 = 4k^2 + 4k + 2 = 2(2k^2 + 2k + 1)$ is even. Conversely, if n were even, say $n = 2m$, then we would have $n^2 + 1 = 2(2m^2) + 1$, so $n^2 + 1$ would be odd, and this completes the proof by contraposition that $(iv) \rightarrow (v)$.

SECTION 1.7 Proof Methods and Strategy

2. The cubes that might go into the sum are 1, 8, 27, 64, 125, 216, 343, 512, and 729. We must show that no two of these sum to a number on this list. If we try the 45 combinations ($1+1$, $1+8$, ..., $1+729$, $8+8$, $8+27$, ... $8+729$, ..., $729+729$), we see that none of them works. Having exhausted the possibilities, we conclude that no cube less than 1000 is the sum of two cubes.

4. There are three main cases, depending on which of the three numbers is smallest. If a is smallest (or tied for smallest), then clearly $a \le \min(b, c)$, and so the left-hand side equals a. On the other hand, for the right-hand side we have $\min(a, c) = a$ as well. In the second case, b is smallest (or tied for smallest). The same reasoning shows us that the right-hand side equals b; and the left-hand side is $\min(a, b) = b$ as well. In the final case, in which c is smallest (or tied for smallest), the left-hand side is $\min(a, c) = c$, whereas the right-hand side is clearly also c. Since one of the three has to be smallest we have taken care of all the cases.

6. The number 1 has this property, since the only positive integer not exceeding 1 is 1 itself, and therefore the sum is 1. This is a constructive proof.

8. The only perfect squares that differ by 1 are 0 and 1. Therefore these two consecutive integers cannot both be perfect squares. This is a nonconstructive proof—we do not know which of them meets the requirement. (In fact, a computer algebra system will tell us that neither of them is a perfect square.)

10. Of these three numbers, at least two must have the same sign (both positive or both negative), since there are only two signs. (It is conceivable that some of them are zero, but we view zero as positive for the purposes of this problem.) The product of two with the same sign is nonnegative. This was a nonconstructive proof, since we have not identified which product is nonnegative. (In fact, a computer algebra system will tell us that all three are positive, so all three products are positive.)

12. An assertion like this one is implicitly universally quantified—it means that *for all* rational numbers a and b, a^b is rational. To disprove such a statement it suffices to provide one counterexample. Take $a = 2$ and $b = 1/2$. Then $a^b = 2^{1/2} = \sqrt{2}$, and we know from Example 10 in Section 1.6 that $\sqrt{2}$ is not rational.

14. We know from algebra that the following equations are equivalent: $ax + b = c$, $ax = c - b$. $x = (c - b)/a$. This shows, constructively, what the unique solution of the given equation is.

16. Given r, let a be the closest integer to r less than r, and let b be the closest integer to r greater than r. In the notation to be introduced in Section 2.3, $a = \lfloor r \rfloor$ and $b = \lceil r \rceil$. In fact, $b = a + 1$. Clearly the distance between r and any integer other than a or b is greater than 1 so cannot be less than $1/2$. Furthermore, since r is irrational, it cannot be exactly half-way between a and b, so exactly one of $r - a < 1/2$ and $b - r < 1/2$ holds.

18. Given x, let n be the greatest integer less than or equal to x, and let $\epsilon = x - n$. In the notation to be introduced in Section 2.3, $n = \lfloor x \rfloor$. Clearly $0 \le \epsilon < 1$, and ϵ is unique for this n. Any other choice of n would cause the required ϵ to be less than 0 or greater than or equal to 1, so n is unique as well.

20. We follow the hint. The square of every real number is nonnegative, so $(x - 1/x)^2 \ge 0$. Multiplying this out and simplifying, we obtain $x^2 - 2 + 1/x^2 \ge 0$, so $x^2 + 1/x^2 \ge 2$, as desired.

22. If $a = 5$ and $b = 8$, then the quadratic mean is $\sqrt{(5^2 + 8^2)/2} \approx 6.67$, and the arithmetic mean is $(5+8)/2 = 6.5$. If $a = 10$ and $b = 100$, then the quadratic mean is $\sqrt{(10^2 + 100^2)/2} \approx 71.06$, and the arithmetic mean is $(10 + 100)/2 = 55$. We conjecture that the quadratic mean of a and b is always greater than their arithmetic mean if a and b are distinct positive real numbers (clearly if $a = b$ then both means are this common value). So we want to verify the inequality $\sqrt{(a^2 + b^2)/2} > (a + b)/2$. Squaring both sides (this is legal because everything in sight is positive) and multiplying by 4 gives us the equivalent inequality $2a^2 + 2b^2 > a^2 + 2ab + b^2$, which is in turn equivalent to $(a - b)^2 > 0$ after putting everything on the left-hand side and factoring. This is clearly always true, and our proof is complete.

24. If we were to end up with nine 0's, then in the step before this we must have had either nine 0's or nine 1's, since each adjacent pair of bits must have been equal and therefore all the bits must have been the same. Thus if we are to start with something other than nine 0's and yet end up with nine 0's, we must have had nine 1's at some point. But in the step before that each adjacent pair of bits must have been different; in other words, they must have alternated 0, 1, 0, 1, and so on. This is impossible with an odd number of bits. This contradiction shows that we can never get nine 0's.

26. Clearly only the last two digits of n contribute to the last two digits of n^2. So we can compute 0^2, 1^2, 2^2, 3^2, ..., 99^2, and record the last two digits, omitting repetitions. We obtain 00, 01, 04, 09, 16, 25, 36, 49, 64, 81, 21, 44, 69, 96, 56, 89, 24, 61, 41, 84, 29, 76. From that point on, the list repeats in reverse order (as we take the squares from 25^2 to 49^2, and then it all repeats again as we take the squares from 50^2 to 99^2). The reason for these last two statements are that $(50 - n)^2 = 2500 - 100n + n^2$, so $(50 - n)^2$ and n^2 have the same two final digits, and $(50 + n)^2 = 2500 + 100n + n^2$, so $(50 + n)^2$ and n^2 have the same two final digits. Thus our list (which contains 22 numbers) is complete.

28. If $|y| \ge 2$, then $2x^2 + 5y^2 \ge 2x^2 + 20 \ge 20$, so the only possible values of y to try are 0 and ± 1. In the former case we would be looking for solutions to $2x^2 = 14$ and in the latter case to $2x^2 = 9$. Clearly there are no integer solutions to these equations, so there are no solutions to the original equation.

30. Following the hint, we let $x = m^2 - n^2$, $y = 2mn$, and $z = m^2 + n^2$. Then $x^2 + y^2 = (m^2 - n^2)^2 + (2mn)^2 = m^4 - 2m^2n^2 + n^4 + 4m^2n^2 = m^4 + 2m^2n^2 + n^4 = (m^2 + n^2)^2 = z^2$. Thus we have found infinitely many solutions, since m and n can be arbitrarily large.

32. One proof that $\sqrt[3]{2}$ is irrational is similar to the proof that $\sqrt{2}$ is irrational, given in Example 10 in Section 1.6. It is a proof by contradiction. Suppose that $2^{1/3}$ (or $\sqrt[3]{2}$, which is the same thing) is the rational number p/q, where p and q are positive integers with no common factors (the fraction is in lowest terms). Cubing, we see that $2 = p^3/q^3$, or, equivalently, $p^3 = 2q^3$. Thus p^3 is even. Since the product of odd numbers is odd, this means that p is even, so we can write $p = 2s$. Substituting into the equation $p^3 = 2q^3$, we obtain $8s^3 = 2q^3$, which simplifies to $4s^3 = q^3$.

Now we play the same game with q. Since q^3 is even, q must be even. We have now concluded that p and q are both even, that is, that 2 is a common divisor of p and q. This contradicts the choice of p/q to be in lowest terms. Therefore our original assumption—that $\sqrt[3]{2}$ is rational—is in error, so we have proved that $\sqrt[3]{2}$ is irrational.

34. The average of two different numbers is certainly always between the two numbers. Furthermore, the average a of rational number x and irrational number y must be irrational, because the equation $a = (x + y)/2$ leads to $y = 2a - x$, which would be rational if a were rational.

36. The solution is not unique, but here is one way to measure out four gallons. Fill the 5-gallon jug from the 8-gallon jug, leaving the contents $(3, 5, 0)$, where we are using the ordered triple to record the amount of water in the 8-gallon jug, the 5-gallon jug, and the 3-gallon jug, respectively. Next fill the 3-gallon jug from the 5-gallon jug, leaving $(3, 2, 3)$. Pour the contents of the 3-gallon jug back into the 8-gallon jug, leaving $(6, 2, 0)$. Empty the 5-gallon jug's contents into the 3-gallon jug, leaving $(6, 0, 2)$, and then fill the 5-gallon jug from the 8-gallon jug, producing $(1, 5, 2)$. Finally, top off the 3-gallon jug from the 5-gallon jug, and we'll have $(1, 4, 3)$, with four gallons in the 5-gallon jug.

38. **a)** $16 \to 8 \to 4 \to 2 \to 1$
b) $11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$
c) $35 \to 106 \to 53 \to 160 \to 80 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$
d) $113 \to 340 \to 170 \to 85 \to 256 \to 128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$

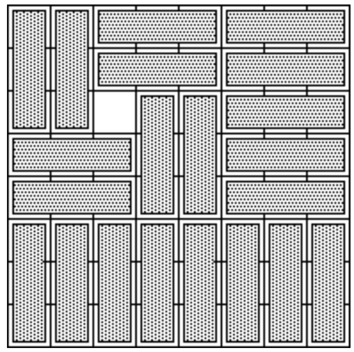
40. This is easily done, by laying the dominoes horizontally, three in the first and last rows and four in each of the other six rows.

42. Without loss of generality, we number the squares from 1 to 25, starting in the top row and proceeding left to right in each row; and we assume that squares 5 (upper right corner), 21 (lower left corner), and 25 (lower right corner) are the missing ones. We argue that there is no way to cover the remaining squares with dominoes.

By symmetry we can assume that there is a domino placed in 1-2 (using the obvious notation). If square 3 is covered by 3-8, then the following dominoes are forced in turn: 4-9, 10-15, 19-20, 23-24, 17-22, and 13-18, and now no domino can cover square 14. Therefore we must use 3-4 along with 1-2. If we use all of 17-22, 18-23, and 19-24, then we are again quickly forced into a sequence of placements that lead to a contradiction. Therefore without loss of generality, we can assume that we use 22-23, which then forces 19-24, 15-20, 9-10, 13-14, 7-8, 6-11, and 12-17, and we are stuck once again. This completes the proof by contradiction that no placement is possible.

44. The barriers shown in the diagram split the board into one continuous closed path of 64 squares, each adjacent to the next (for example, start at the upper left corner, go all the way to the right, then all the way down, then all the way to the left, and then weave your way back up to the starting point). Because each square in the path is adjacent to its neighbors, the colors alternate. Therefore, if we remove one black square and one white square, this closed path decomposes into two paths, each of which starts in one color and ends in the other color (and therefore has even length). Clearly each such path can be covered by dominoes by starting at one end. This completes the proof.

46. If we study Figure 7, we see that by rotating or reflecting the board, we can make any square we wish nonwhite, with the exception of the squares with coordinates $(3,3)$, $(3,6)$, $(6,3)$, and $(6,6)$. Therefore the same argument as was used in Example 22 shows that we cannot tile the board using straight triominoes if any one of those other 60 squares is removed. The following drawing (rotated as necessary) shows that we can tile the board using straight triominoes if one of those four squares is removed.



48. We will use a coloring of the 10×10 board with four colors as the basis for a proof by contradiction showing that no such tiling exists. Assume that 25 straight tetrominoes can cover the board. Some will be placed horizontally and some vertically. Because there is an odd number of tiles, the number placed horizontally and the number placed vertically cannot both be odd, so assume without loss of generality that an even number of tiles are placed horizontally. Color the squares in order using the colors red, blue, green, yellow in that order repeatedly, starting in the upper left corner and proceeding row by row, from left to right in each row. Then it is clear that every horizontally placed tile covers one square of each color and each vertically placed tile covers either zero or two squares of each color. It follows that in this tiling an even number of squares of each color are covered. But this contradicts the fact that there are 25 squares of each color. Therefore no such coloring exists.

SUPPLEMENTARY EXERCISES FOR CHAPTER 1

2. The truth table is as follows.

p	q	r	$p \vee q$	$p \wedge \neg r$	$(p \vee q) \to (p \wedge \neg r)$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	F	F	T
F	F	F	F	F	T

4. a) The converse is "If I drive to work today, then it will rain." The contrapositive is "If I do not drive to work today, then it will not rain." The inverse is "If it does not rain today, then I will not drive to work."

b) The converse is "If $x \geq 0$ then $|x| = x$." The contrapositive is "If $x < 0$ then $|x| \neq x$." The inverse is "If $|x| \neq x$, then $x < 0$."

c) The converse is "If n^2 is greater than 9, then n is greater than 3." The contrapositive is "If n^2 is not greater than 9, then n is not greater than 3." The inverse is "If n is not greater than 3, then n^2 is not greater than 9."

6. The inverse of $p \to q$ is $\neg p \to \neg q$. Therefore the inverse of the inverse is $\neg\neg p \to \neg\neg q$, which is equivalent to $p \to q$ (the original proposition). The converse of $p \to q$ is $q \to p$. Therefore the inverse of the converse is $\neg q \to \neg p$, which is the contrapositive of the original proposition. The inverse of the contrapositive is $q \to p$, which is the same as the converse of the original statement.

8. Let t be "Sergei takes the job offer"; let b be "Sergei gets a signing bonus"; and let h be "Sergei will receive a higher salary." The given statements are $t \to b$, $t \to h$, $b \to \neg h$, and t. By modus ponens we can conclude b and h from the first two conditional statements, and therefore we can conclude $\neg h$ from the third conditional statement. We now have the contradiction $h \wedge \neg h$, so these statements are inconsistent.

10. Since both knights and knaves claim that they are knights (the former truthfully and the latter deceivingly), we know that A is a knave. But since A's statement must be false, and the first part of the conjunction is true, the second part must be false, so we know that B must be a knave as well. If C were a knight, then B's statement would be true, and knaves must lie, so C must also be a knave. Thus all three are knaves.

12. If S is a proposition, then it is either true or false. If S is false, then the statement "If S is true, then unicorns live" is vacuously true; but this statement *is* S, so we would have a contradiction. Therefore S is true, so the statement "If S is true, then unicorns live" is true and has a true hypothesis. Hence it has a true conclusion (modus ponens), and so unicorns live. But we know that unicorns do not live. It follows that S cannot be a proposition.

14. **a)** The answer is $\exists x P(x)$ if we do not read any significance into the use of the plural, and $\exists x \exists y (P(x) \wedge P(y) \wedge x \neq y)$ if we do.
b) $\neg \forall x P(x)$, or, equivalently, $\exists x \neg P(x)$ **c)** $\forall y Q(y)$
d) $\forall x P(x)$ (the class has nothing to do with it) **e)** $\exists y \neg Q(y)$

16. The given statement tells us that there are exactly two elements in the domain. Therefore the statement will be true as long as we choose the domain to be anything with size 2, such as the United States presidents named Bush.

18. We want to say that for every y, there do not exist four different people each of whom is the grandmother of y. Thus we have $\forall x \neg \exists a \exists b \exists c \exists d (a \neq b \wedge a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \wedge c \neq d \wedge G(a,y) \wedge G(b,y) \wedge G(c,y) \wedge G(d,y))$.

20. **a)** Since there is no real number whose square is -1, it is true that there exist exactly 0 values of x such that $x^2 = -1$.
b) This is true, because 0 is the one and only value of x such that $|x| = 0$.
c) This is true, because $\sqrt{2}$ and $-\sqrt{2}$ are the only values of x such that $x^2 = 2$.
d) This is false, because there are more than three values of x such that $x = |x|$, namely all positive real numbers.

22. Let us assume the hypothesis. This means that there is some x_0 such that $P(x_0, y)$ holds for all y. Then it is certainly true that for all y there exists an x such that $P(x,y)$ is true, since in each case we can take $x = x_0$. Note that the converse is not always a tautology, since the x in $\forall y \exists x P(x,y)$ can depend on y.

24. No. Here is an example. Let $P(x,y)$ be $x > y$, where we are talking about integers. Then for every y there does exist an x such that $x > y$; we could take $x = y + 1$, for example. However, there does not exist an x such that for *every* y, $x > y$; in other words, there is no superlarge integer (if for no other reason than that no integer can be larger than itself).

26. **a)** It will snow today, but I will not go skiing tomorrow.
b) Some person in this class does not understand mathematical induction.
c) All students in this class like discrete mathematics.
d) There is some mathematics class in which all the students stay awake during lectures.

28. Let $W(r)$ means that room r is painted white. Let $I(r,b)$ mean that room r is in building b. Let $L(b,u)$ mean that building b is on the campus of United States university u. Then the statement is that there is some university u and some building on the campus of u such that every room in b is painted white. In symbols this is $\exists u \exists b(L(b,u) \wedge \forall r(I(r,b) \to W(r)))$.

30. To say that there are exactly two elements that make the statement true is to say that two elements exist that make the statement true, and that every element that makes the statement true is one of these two elements. More compactly, we can phrase the last part by saying that an element makes the statement true if and only if it is one of these two elements. In symbols this is $\exists x \exists y(x \neq y \wedge \forall z(P(z) \leftrightarrow (z = x \vee z = y)))$. In English we might express the rule as follows. The hypotheses are that $P(x)$ and $P(y)$ are both true, that $x \neq y$, and that every z that satisfies $P(z)$ must be either x or y. The conclusion is that there are exactly two elements that make P true.

32. We give a proof by contraposition. If x is rational, then $x = p/q$ for some integers p and q with $q \neq 0$. Then $x^3 = p^3/q^3$, and we have expressed x^3 as the quotient of two integers, the second of which is not zero. This by definition means that x^3 is rational, and that completes the proof of the contrapositive of the original statement.

34. Let m be the square root of n, rounded down if it is not a whole number. (In the notation to be introduced in Section 2.3, we are letting $m = \lfloor \sqrt{n} \rfloor$.) We can see that this is the unique solution in a couple of ways. First, clearly the different choices of m correspond to a partition of $\mathbf{N}$, namely into $\{0\}$, $\{1,2,3\}$, $\{4,5,6,7,8\}$, $\{9,10,11,12,13,14,15\}$, ... So every n is in exactly one of these sets. Alternatively, take the square root of the given inequalities to give $m \le \sqrt{n} < m+1$. That m is then the floor of $\sqrt{n}$ (and that m is unique) follows from statement (1a) of Table 1 in Section 2.3.

36. A constructive proof seems indicated. We can look for examples by hand or with a computer program. The smallest ones to be found are $50 = 5^2 + 5^2 = 1^2 + 7^2$ and $65 = 4^2 + 7^2 = 1^2 + 8^2$.

38. We claim that the number 7 is not the sum of at most two squares and a cube. The first two positive squares are 1 and 4, and the first positive cube is 1, and these are the only numbers that could be used in forming the sum. Clearly no sum of three or fewer of these is 7. This counterexample disproves the statement.

40. We give a proof by contradiction. If $\sqrt{2} + \sqrt{3}$ were rational, then so would be its square, which is $5 + 2\sqrt{6}$. Subtracting 5 and dividing by 2 then shows that $\sqrt{6}$ is rational, but this contradicts the theorem we are told to assume.