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An Introduction to Chromatic Polynomials*

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ABSTRACT

This expository paper is a general introduction to the theory of chromatic polynomials. Chromatic polynomials are defined, their salient properties are derived, and some practical methods for computing them are given. A brief mention is made of the connection between the theory of chromatic polynomials and map coloring problems. The paper concludes with some unsolved problems relating to chromatic polynomials and to applications of the theory to practical problems in operations research.

1. INTRODUCTION

There are many interesting problems which arise when one considers the ways of coloring the nodes of a graph subject to certain restrictions. The object of this paper is to give a brief account of the fundamentals of this branch of graph theory.

A *coloring* of a graph is the result of giving to each node of the graph one of a specified set of colors. In more mathematical terms it is a mapping of the nodes into (or onto) a specified finite set C (the set of colors). We shall leave for the moment the question of whether the mapping is to be into or onto.

By a *proper coloring* of a graph will be meant a coloring which satisfies the restriction that adjacent nodes are not given (i.e., mapped onto) the same color (element) of C. A coloring for which this is not true will be called an *improper coloring.*

These are the definitions; but it so happens that we shall nearly always be concerned with proper colorings only, and it will therefore be convenient to drop the term "proper" and agree that by "colorings" of a graph we mean "proper colorings" unless the contrary is stated.

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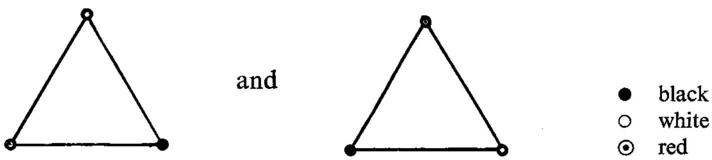
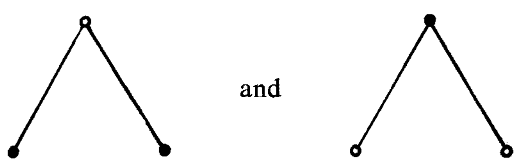
Of particular interest is the function, associated with a given graph G, which expresses the number of different ways of coloring G as a function of the number of specified colors. The number of colors is, for some obscure reason, usually denoted by λ, and the function will be written as $M_G(\lambda)$. Before we can define it we must decide, among other things, whether the mappings are to be into or onto the color set. It turns out to be more convenient, algebraically, to work with "into" mappings, and this is what we shall do. Thus $M_G(\lambda)$ is the number of ways of coloring the Graph G with λ colors, with no stipulation that all the λ colors are in fact used. If we wish to make this stipulation (as we shall later on) we can describe the colorings as being in "exactly λ colors."

Two other questions must be settled before the meaning of $M_G(\lambda)$ is clear. First, are we regarding the nodes as fixed, or can we permute them? For example, do we regard the colorings



of the triangle as being the same or different? Insofar as they differ only by a cyclic permutation of the nodes (which leaves the graph unchanged) it might be felt that they should be regarded as equivalent, but we shall not take this view. Instead we shall regard our graphs as if their nodes were points fixed in space, so that, for example, a triangle whose apex (as we look at it) is colored white will be differently colored from one whose apex has some other colors.

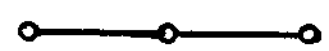
Second, we shall agree that the actual colors used (or rather the distinctions between them) are important. There are two ways of looking at this. One can think of the allocation of colors to the nodes merely as a convenient means of partitioning the nodes into a number of disjoint sets. Thus the colorings of the triangle given above have the effect of dividing the set of nodes into three subsets, each having one node. If this were the only purpose that the coloring had to serve, then permuting the colors would give an equivalent coloring. There would, for example, be no point in making a distinction between the colorings



since they give the same partition of the nodes. Colorings counted in this way will be called "colorings with color indifference," but we shall not usually count them in this way. Instead we shall take account of different colors, and hence regard the two colorings of the graph just given as being distinct.

2. Some Illustrations

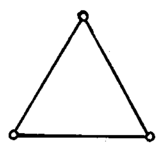
By way of illustration let us calculate the function $M_G(\lambda)$ for some simple graphs. Take first the graph



as the graph G. We can color the centre node in any of the λ colors. When this has been done, this color is no longer available for coloring the outer nodes, by the condition for a proper coloring. Hence the outer nodes can be colored independently each in $\lambda - 1$ ways. Thus

$$M_G(\lambda) = \lambda(\lambda - 1)^2.$$

Next let us take the triangle



as G. There are λ ways of coloring, say, the top node. There are then $\lambda - 1$ ways of coloring an adjacent node, and $\lambda - 2$ ways of coloring the remaining node, since no two nodes may be given the same color. Thus

$$M_G(\lambda) = \lambda(\lambda - 1)(\lambda - 2).$$

This can clearly be generalized. Suppose G is the complete graph on n nodes. We choose a node and color it; this is possible in λ ways. Picking another node we have $\lambda - 1$ colors with which it can be colored, since it is adjacent to the first node. Pick another node; it is adjacent to both nodes already colored, and can therefore be colored in $\lambda - 2$ ways. We continue in this way; the last node can be given any of the remaining $\lambda - (n - 1)$ colors. Hence

$$M_G(\lambda) = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1).$$

We shall use the notation $\lambda^{(n)}$ for this factorial expression.

Finally let G be the empty graph on n nodes, i.e., the graph having no edges. Its n isolated nodes can be colored independently, each in λ ways. Hence for this graph

$$M_G(\lambda) = \lambda^n.$$

It will be seen that, in each of the above examples, $M_G(\lambda)$ is a polynomial in λ. This is always so, as we shall shortly prove. The function $M_G(\lambda)$ is called the "chromatic polynomial" of the graph G.

3. A Fundamental Theorem

Let us consider a particular graph G, such as that in figure 1, and concentrate on a particular pair of non-adjacent nodes, for example those marked A and B in the figure. Now the colorings of G in λ colors are of two types:

(i) those in which A and B are given different colors, and
(ii) those in which A and B are given the same color.

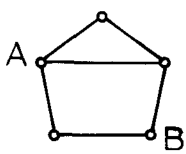


Figure 1

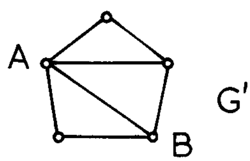


Figure 2

A coloring of G of type (i) will be a coloring of the graph G' obtained from G by adding the edge AB (see Fig. 2), since the addition of this edge does not infringe the requirements for a proper coloring. Conversely, to any coloring of G' corresponds a type (i) coloring of G.

Further, a coloring of G of type (ii) will be a coloring of the graph G'' obtained from G by identifying the nodes A and B (see Fig. 3). (Note that we can replace multiple edges, if any arise, by single edges, since a multiple edge represents exactly the same restriction on the colors as does a single edge joining the same nodes). Conversely, any coloring of G'' corresponds to a type (ii) coloring of G.

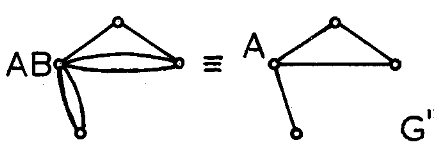
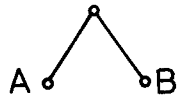


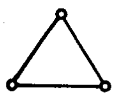
Figure 3

From these two results we derive

THEOREM 1. $M_G(\lambda) = M_{G'}(\lambda) + M_{G''}(\lambda)$
—*a theorem of fundamental importance.*

By way of example, if we take G to be



then G' and G'' are  and 

respectively. We have already seen that

$$M_G(\lambda) = \lambda(\lambda - 1)^2$$

and

$$M_{G'}(\lambda) = \lambda(\lambda - 1)(\lambda - 2),$$

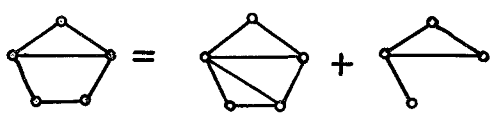
while it is easy to see that

$$M_{G''}(\lambda) = \lambda(\lambda - 1).$$

Thus the theorem is verified in this particular case.

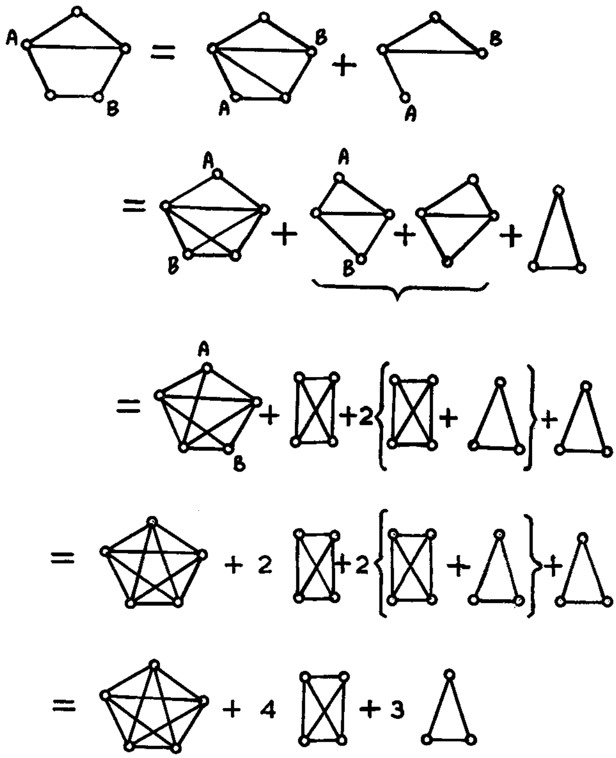
By means of Theorem 1 the chromatic polynomial of a graph can be expressed in terms of the chromatic polynomials of a graph with an extra edge, and another with one fewer nodes. The theorem can then be applied again to these graphs, and so on, the process terminating (as it must do) when none of these graphs has a pair of non-adjacent nodes. The chromatic polynomial of the given graph will then have been expressed as the sum of the chromatic polynomials of complete graphs; and these, as we have seen, are known.

To do this in practice it is convenient to adopt a convention whereby the actual picture of a graph serves to denote its chromatic polynomial (with λ understood).[1] Thus instead of writing $M_G(\lambda) = M_{G'}(\lambda) + M_{G''}(\lambda)$, and having to explain what G, G', and G'' stand for, we can simply write



Applying Theorem 1 repeatedly to this graph, and indicating by A and B the nodes being considered at each stage, we have

[1] This useful notational device was introduced by Zykov [9].



Hence

$$M_G(\lambda) = \lambda^{(5)} + 4\lambda^{(4)} + 3\lambda^{(3)}.$$

It can be seen that the chromatic polynomial of any graph can be reduced to the sum of a number of factorials, and hence is indeed a polynomial. When a chromatic polynomial is expressed in this way we shall say that it is in "factorial form." If the factorial form is known the polynomial itself is readily found. (Tables of Stirling's numbers which, in effect, give the $\lambda^{(n)}$ as polynomials in λ, come in handy here.) For the above graph we have

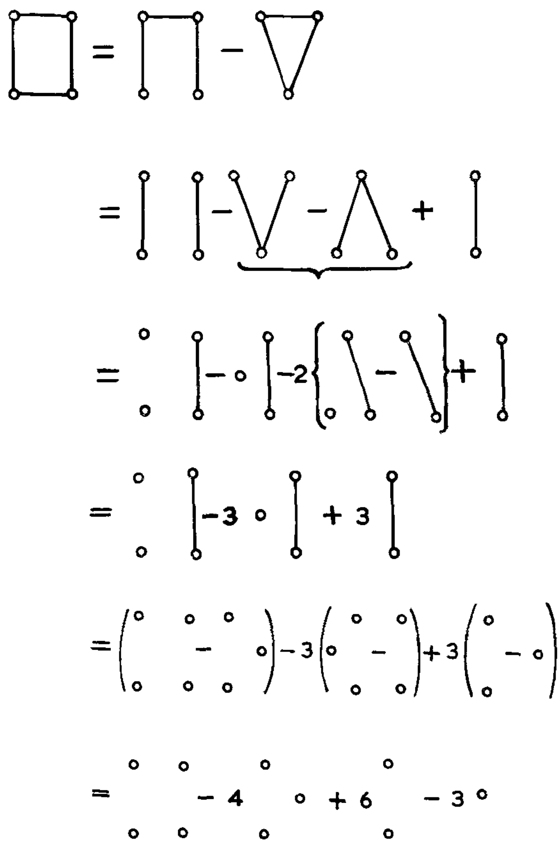
$$M_G(\lambda) = \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda.$$

Theorem 1 can also be used round the other way, id the form

$$M_{G'}(\lambda) = M_G(\lambda) - M_{G''}(\lambda).$$

Here the process is that of removing edges, and we end up with our chromatic polynomial expressed in terms of the chromatic polynomials of empty graphs.

Thus



so that

$$M_G(\lambda) = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda.$$

We shall call this process of expressing chromatic polynomials in terms of chromatic polynomials of complete or empty graphs "chromatic reduction."

4. Some Short Cuts

Various short cuts are available whereby the calculation of chromatic polynomials by this sort of method can be facilitated. Some of these are contained in the following theorems.

THEOREM 2. *If a graph G has connected components $G_1, G_2, \ldots, G_k$, then*

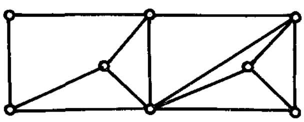
$$M_G(\lambda) = M_{G_1}(\lambda) \cdot M_{G_2}(\lambda) \cdots M_{G_k}(\lambda).$$

PROOF. Since the components are disjoint, the coloring of each is quite independent of the coloring of the others. Hence the number of ways of coloring the whole graph is simply the product of the numbers of colorings of the separate components.

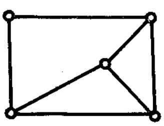
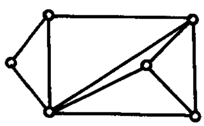
THEOREM 3. *If two graphs X and Y "overlap" in a complete graph on k nodes then the chromatic polynomial of the graph formed by X and Y together is*

$$\frac{M_X(\lambda) \cdot M_Y(\lambda)}{\lambda^{(k)}}.$$

(By "overlapping" is meant the following: The node sets of X and Y are not disjoint, but have k nodes in common, and every pair of these k nodes is joined both in X and in Y. As an example take the graph



for which

$X =$  and $Y =$ 

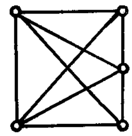
having the triangle ($k = 3$) in common.)

PROOF. The number of ways of coloring the common part is $\lambda^{(k)}$. If we fix the colors of these k nodes there will be $M_X(\lambda)/\lambda^{(k)}$ ways of coloring the remaining nodes of X, and $M_Y(\lambda)/\lambda^{(k)}$ ways of coloring the remaining nodes of Y. Hence the total number of colorings is

$$\lambda^{(k)} \frac{M_X(\lambda)}{\lambda^{(k)}} \cdot \frac{M_Y(\lambda)}{\lambda^{(k)}} = \frac{M_X(\lambda) \cdot M_Y(\lambda)}{\lambda^{(k)}}.$$

THEOREM 4. *The chromatic polynomial of the product of two graphs X and Y is $M_X(\lambda) \odot M_Y(\lambda)$ where $\odot$ denotes a type of multiplication in which factorials are treated as powers (further explained below).*

(By the product of two (disjoint) graphs X and Y is meant the graph obtained by adding to X and Y in all possible ways, edges joining a node of X to a node of Y. Thus the product of

 and  is  .)

PROOF. Let us apply chromatic reduction to X and Y separately. If they have m and n nodes, respectively, we shall end up with

$$M_X(\lambda) = M_{K_m}(\lambda) + a_1 M_{K_{m-1}}(\lambda) + a_2 M_{K_{m-2}}(\lambda) + \cdots \tag{1}$$

when $M_X(\lambda)$ has been expressed in terms of the chromatic polynomials of the complete graphs K_m, K_{m-1}, K_{m-2},... on m, $m-1$, $m-2$,... nodes. Similarly

$$M_Y(\lambda) = M_{K_n}(\lambda) + b_1 M_{K_{n-1}}(\lambda) + b_2 M_{K_{n-2}}(\lambda) + \cdots. \tag{2}$$

If we now perform the chromatic reduction of the product of X and Y, then the X and Y portions of the product graph will be reduced in exactly the above way. Moreover, at every stage, every node of each graph obtained in the reduction of X will be joined to every node of each graph in the reduction of Y. Hence we shall finish by expressing the chromatic polynomial of the product in terms of all possible products of a complete graph from (1) and a complete graph from (2). But the product of a complete graph on (say) p nodes and one on q nodes is itself a complete graph, on $p + q$ nodes. Hence to a term $\lambda^{(p)}$ in the factorial form of $M_X(\lambda)$, and a term $\lambda^{(q)}$ in the factorial form of $M_Y(\lambda)$, there will be a term in $\lambda^{(p+q)}$ in the factorial form of the chromatic polynomial of the product graph.

It follows that this chromatic polynomial can be found in its factorial form by taking the factorial forms of $M_X(\lambda)$ and $M_Y(\lambda)$ and multiplying them as if the factorials were powers. This is the process that we denoted symbolically by $M_X(\lambda) \odot M_Y(\lambda)$ in the statement of the theorem.

By way of example, if

$X =$  and $Y =$ 

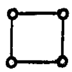
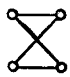
then $M_X(\lambda) = \lambda^{(2)}$ and $M_Y(\lambda) = \lambda^{(3)} + \lambda^{(2)}$.

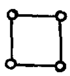
Hence

$$M_X(\lambda) \odot M_Y(\lambda) = \lambda^{(2)} \odot (\lambda^{(3)} + \lambda^{(2)})$$
$$= \lambda^{(5)} + \lambda^{(4)}.$$

5. Illustrations

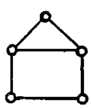
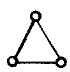
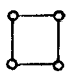
To illustrate the use of these theorems let us calculate a few chromatic polynomials, beginning with two that have already been found.

First,  =  is the product of  and , and the chromatic polynomial of  is $\lambda^2 = \lambda^{(2)} + \lambda^{(1)}$. Therefore, by Theorem 4,



$$= (\lambda^{(2)} + \lambda^{(1)}) \odot (\lambda^{(2)} + \lambda^{(1)})$$
$$= \lambda^{(4)} + 2\lambda^{(3)} + \lambda^{(2)}$$
$$= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda.$$

Second, by Theorem 3, with $k = 2$,

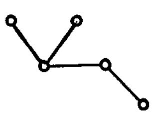
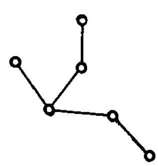
 =  $\cdot$  $\Big/ \lambda^{(2)}$

$$= \frac{\lambda(\lambda - 1)(\lambda - 2)(\lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda)}{\lambda(\lambda - 1)}$$
$$= \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda.$$

Finally we have two general results:

Theorem 5. *The chromatic polynomial of any tree having n nodes is* $\lambda(\lambda - 1)^{n-1}$.

Proof. We can build up any tree by starting with a single edge and adding edges one by one, each added edge having one node in common with the tree so far constructed. For example

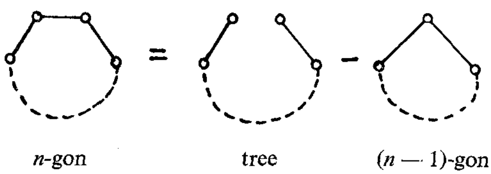
 and  give 

Hence by Theorem 3 with $k = 1$, the chromatic polynomial of the new tree (having the added edge) is obtained from that of the old tree by multiplying by the chromatic polynomial of the edge, viz., $\lambda(\lambda - 1)$, and dividing by $\lambda^{(1)}$, i.e., λ. Hence the addition of each edge multiplies the chromatic polynomial by $\lambda - 1$. Since we started with a single edge (chromatic polynomial $\lambda(\lambda - 1)$) the theorem follows.

THEOREM 6. *The chromatic polynomial of an n-gon is*

$$(\lambda - 1)^n + (-1)^n(\lambda - 1).$$

PROOF. Let $P_n(\lambda)$ denote the chromatic polynomial of an n-gon. Then by Theorem 1 we have



by deleting an edge. Hence we have

$$P_n(\lambda) = \lambda(\lambda - 1)^{n-1} - P_{n-1}(\lambda)$$

by Theorem 5. This can be written

$$P_n(\lambda) - (\lambda - 1)^n = (\lambda - 1)^{n-1} - P_{n-1}(\lambda),$$

from which it follows that $(-1)^n\{P_n(\lambda) - (\lambda - 1)^n\}$ is a constant, which can be found by putting $n = 3$. For

$$\begin{aligned} -\{P_3(\lambda) - (\lambda - 1)^3\} &= -\{\lambda(\lambda - 1)(\lambda - 2) - (\lambda - 1)^3\} \\ &= \lambda - 1. \end{aligned}$$

Thus

$$P_n(\lambda) - (\lambda - 1)^n = (-1)^n(\lambda - 1),$$

i.e.,

$$P_n(\lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1).$$

6. PROPERTIES OF CHROMATIC POLYNOMIALS

We shall now list and prove some properties of the chromatic polynomial $M_G(\lambda)$ of a graph G. We let n denote the number of nodes of G.

THEOREM 7. *The degree of $M_G(\lambda)$ is n.*

PROOF: By the nature of the reduction process there is, at each stage, exactly one graph having n nodes. This is therefore true at the final stage, and from this the result follows. From the fact that there is exactly one such graph we also have

THEOREM 8. *The coefficient of λ^n in $M_G(\lambda)$ is 1.*

THEOREM 9. *$M_G(\lambda)$ has no constant term.*

PROOF: If not, then $M(0) \neq 0$, whereas clearly the number of ways of coloring a graph in no colors must be zero!

THEOREM 10. *The terms in $M_G(\lambda)$ alternate in sign.*

PROOF. (It is worth remarking here that many properties of chromatic polynomials can be found by the use of the very elegant theory of Möbius functions developed recently by Gian-Carlo Rota [5]. From the standpoint of this theory the theorem we are now considering is almost trivially true; but in the absence of this background we shall work from first principles. The proof is by two-way mathematical induction.)

It is easily verified that the theorem is true for all graphs having 1, 2, 3 nodes. Let n be any integer such that the theorem is true for all graphs on n nodes or less. Consider graphs on $n + 1$ nodes.

The empty graph on $n + 1$ nodes certainly satisfies the theorem. Let k be any integer such that the theorem is true for all graphs on $n + 1$ nodes and k or fewer edges. Consider any graph G' with $n + 1$ nodes and $k + 1$ edges. By Theorem 1 we have

$$M_{G'}(\lambda) = M_G(\lambda) - M_{G''}(\lambda),$$

where G has $n + 1$ nodes and k edges, and G'' has n nodes. Since the theorem is therefore true for G and G'' we can write

$$M_G(\lambda) = \lambda^{n+1} - a_1\lambda^n + a_2\lambda^{n-1} - a_3\lambda^{n-2} + \cdots$$

and

$$M_{G''}(\lambda) = \lambda^n - b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots$$

with every a_i, b_i positive. Hence

$$M_{G'}(\lambda) = \lambda^{n+1} - (a_1 + 1)\,\lambda^n + (a_2 + b_1)\,\lambda^{n+1} - \cdots \tag{3}$$

in which the coefficients alternate in sign.

By mathematical induction on k the theorem is true for all graphs on $n+1$ nodes; and by mathematical induction on n it is true for all graphs.

By observing that, in this proof, every deletion of an edge added 1 to the absolute value of the coefficient of the second term of the chromatic polynomial we derive

THEOREM 11. *The absolute value of the second coefficient of $M_G(\lambda)$ is the number of edges in G.*

THEOREM 12. *For a connected graph, $M_G(\lambda) \leqslant \lambda(\lambda-1)^{n-1}$ (λ a positive integer).*

PROOF: Consider any spanning tree T of G. Its nodes can be colored in $\lambda(\lambda-1)^{n-1}$ ways. Now every coloring of G is a coloring of T, but in general some colorings of T will not be colorings of G. Hence

$$M_G(\lambda) \leqslant \lambda(\lambda-1)^{n-1}.$$

We have already seen that equality holds if G is a tree. The converse also holds, and we can state

THEOREM 13. *A necessary and sufficient condition for a graph G on n nodes to be a tree is that $M_G(\lambda) = \lambda(\lambda-1)^{n-1}$.*

PROOF: The necessary part has already been proved. Let G be a graph whose chromatic polynomial is $\lambda(\lambda-1)^{n-1}$. Then G is connected, for otherwise its chromatic polynomial would be the product of those of its components (Theorem 2), each of which has a factor λ (Theorem 9). Hence $M_G(\lambda)$ would have a factor λ^2 at least, which it has not.

The coefficient of λ^{n-1} in $M_G(\lambda)$ is $-(n-1)$; hence G has $n-1$ edges. Since G is connected, has n nodes and $n-1$ edges it must be a tree.

The method of considering a spanning tree of a connected graph also gives us the following theorem:

THEOREM 14. *If G is connected then the absolute value of the coefficient of λ^r in $M_G(\lambda)$ is not less than $\binom{n-1}{r-1}$.*

PROOF. Let T be a spanning tree of G. From equation (3) it follows that the addition of an edge to a graph cannot decrease the absolute value of any coefficient. But G can be obtained by adding suitable edges to T. Hence, since

$$M_T(\lambda) = \lambda(\lambda-1)^{n-1} = \sum_{r=1}^{n} (-1)^{n-r-1} \binom{n-1}{r-1} \lambda^r$$

it follows that the absolute value of the coefficient of λ^r in $M_G(\lambda)$ is not less than $\binom{n-1}{r-1}$. In particular, the coefficient of λ for a connected graph cannot be zero (its absolute value must be $\geqslant 1$). From this result, and Theorem 2 we deduced the following corollary:

COROLLARY. *The smallest number r such that λ^r has a nonzero coefficient in $M_G(\lambda)$ is the number of components of G.*

7. Interpretation of the Coefficients

It is possible to give an interpretation of the coefficients in the chromatic polynomial, both in its usual form and in its factorial form. We shall take the factorial form first.

Let $P_G(r)$ denote the number of ways of coloring a graph G with *exactly* r colors, with color indifference (as explained in Section 1). This is then the number of ways of partitioning the set of nodes into r disjoint subsets such that no two nodes of the same subset are joined. To take account of of the differences between the colors we must allot a color to each subset, and this is possible in $r!$ ways. Thus the number of colorings of G in *exactly* r colors, but recognizing the different colors is $r!P_G(r)$.

Let us now reconstruct $M_G(\lambda)$. This is the number of ways of coloring G in λ colors, but not necessarily using all λ of them. Consider those colorings in which exactly r colors are used. Their number is $\binom{\lambda}{r}r!P_G(r)$, since there are $\binom{\lambda}{r}$ ways of choosing which r colors are to be used. Summing over all r, we have

$$M_G(\lambda) = \sum_{r=1}^{\lambda} \binom{\lambda}{r} r!P_G(r)$$

$$= \sum_{r=1}^{\lambda} \lambda^{(r)} P_G(r).$$

The right-hand side is now the chromatic polynomial in its factorial form, and we therefore have

THEOREM 15. *The coefficient of $\lambda^{(r)}$ in the factorial form of $M_G(\lambda)$ is the number of ways of coloring G in* exactly *r colors with color indifference.*

The interpretation of the coefficients in the usual form of the chromatic polynomial is rather less obvious, and requires the use of the principle of inclusion and exclusion, well known in combinatorial analysis (see, for example, Riordan [6]). We shall attempt to find the chromatic polynomial of a graph G by starting with the total number of colorings, both proper and improper, and subtracting the improper colorings.

The total number of colorings in λ colors, including improper colorings, is clearly λ^n, where n is the number of nodes. Let us consider such a coloring of G, and let us delete any edge of G which joins nodes of different colors. We shall get a subgraph of G having the property that adjacent nodes are always colored *alike* (what one might call a "highly improper" coloring!). There is a single color associated with each connected component of this subgraph, and thus if the subgraph has p components there are λ^p of these highly improper colorings. We observe that to every coloring (proper or improper) of G there is a highly improper coloring of some subgraph of G. For proper colorings of G this subgraph is the empty graph. Let $N(p, r)$ denote the number of subgraphs of G which have p components and r edges.

We first subtract from the total λ_n the number of highly improper colorings of those subgraphs having just one edge. If we subtract $\sum_p N(p, 1)\,\lambda^p$ we shall have subtracted these, but much more besides. For, if AB and CD are edges of G, then the contribution from the subgraph consisting of the edge AB alone will include colorings in which C and D are given the same colors, and this will be a highly improper coloring of the subgraph consisting of AB and CD. Moreover it will have been subtracted twice, once for AB and once for CD. Similarly colorings for subgraphs of 3, 4 and more edges will also have been subtracted an appropriate number of times.

To redress the balance we can add the term $\sum_p N(p, 2)\,\lambda^p$. This will compensate for the double subtraction of the colorings of the two-edge subgraphs, but will now necessitate a compensation for the three-edge subgraphs, and so on. We obtain, for the number of *proper* colorings of G,

$$\lambda^n - \sum_p N(p, 1)\,\lambda^p + \sum_p N(p, 2)\,\lambda^p - \sum_p N(p, 3)\,\lambda^p + \cdots.$$

Since $N(n, 0)$ is clearly $= 1$ (the empty graph), we can write this expression as

$$\begin{aligned} M_G(\lambda) &= \sum_{r=0}^{k} \sum_{p=1}^{n} (-1)^r\, N(p, r)\,\lambda^p \\ &= \sum_{p=1}^{p} \left\{ \sum_{r=0}^{k} (-1)^r\, N(p, r) \right\} \lambda^p. \end{aligned} \tag{4}$$

From this we obtain

THEOREM 16. *The coefficient of λ^p in $M_G(\lambda)$ is $\sum_{r=0}^{k}(-1)^r N(p, r)$,* *where $N(p,r)$ is the number of subgraphs of G with p components and r edges, and k is the number of edges in G.*

Many results concerning chromatic polynomials, including our Theorems 7, 8, 9, 11, follow readily from the above form for the coefficients.

To calculate a chromatic polynomial by means of Theorem 16 would entail examining all the 2^k subgraphs of G (spanning subgraphs to be quite unambiguous—they have all n nodes but only some subset of the edges of G). It was shown by Whitney [7], however, that one need only consider a comparatively small number of these subgraphs. Let us list all the circuits in the graph G, and from each circuit remove one edge. To do this systematically, we can imagine the edges to have been numbered in some arbitrary fashion; then from each circuit we remove the edge with highest number. We obtain a set of what are called "broken circuits."

THEOREM 17. *Equation* (4) *still holds if, in finding $N(p,r)$ we consider only those subgraphs of G which do not contain any broken circuits.*

PROOF: The full proof of this is contained in the paper [7] by Whitney, and we shall not give it here, but merely indicate the lines on which it proceeds. Briefly, the method is to show that those subgraphs which contain a broken circuit can be paired off so that the contributions to $M_G(\lambda)$ from the subgraphs of a pair will cancel.

If a subgraph H of G contains a broken circuit B, but does not contain the edge (b say) that was deleted to form B, then there is also a subgraph H^* which is H augmented by the edge b. Now H and H^* have the same number of components, since the two nodes of H^* joined by b were already joined in H by the broken circuit. Therefore since H^* has one more edge than H, the contributions of these two subgraphs to $M_G(\lambda)$ (in equation 4) will cancel.

In general there may be several possible companions for a given subgraph (if it contains more than one broken circuit) and the part of the proof that we have omitted is that which ensures that *all* the subgraphs which contain some broken circuit can be paired so that their contributions vanish. It then follows that only the other subgraphs need be considered.

8. MAP COLORING PROBLEMS

Problems concerning the coloring of the regions of a map are special cases of graph coloring problems. For we can take a point in the interior of each region, call these points the nodes of a graph G, and join two of these nodes if and only if the regions to which they belong have a boundary in common. The condition for a map coloring, viz., that two regions with a common boundary must be given different colors, becomes the condition that two adjacent nodes must be colored differently, that is, the condition for a proper coloring of the resultant graph G. If the map is drawn on a plane, then G will be a planar graph.

Map coloring problems are therefore equivalent to problems in the coloring of planar graphs. In particular, the famous four-color conjecture, that any planar map can be colored in four colors, becomes

$$\text{“}M_G(4) \neq 0, \qquad \text{if } G \text{ is planar”}$$

since $M_G(4) = 0$ means that G cannot be colored in four colors.

For a discussion of various results concerning chromatic polynomials of maps, and other applications to specific map coloring problems see Birkhoff [2–4] and Whitney [7, 8].

9. Unsolved Problems

There are many unsolved problems in connection with graph colorings; we shall mention only a few. First and foremost is the question "What makes a polynomial chromatic?" We have derived various necessary conditions for a polynomial to be the chromatic polynomial of some graph (Theorems 7, 8, 9, 10, 12) but none of them is sufficient. For example, the polynomial

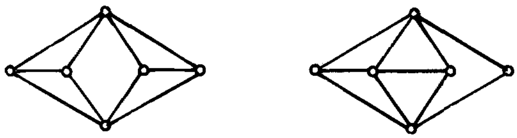
$$\lambda^4 - 3\lambda^3 + 3\lambda^2$$

satisfies these conditions but is not the chromatic polynomial of any graph. The problem of characterizing chromatic polynomials is unsolved. Another unsolved problem in a similar vein is that of determining what numbers can be roots of some chromatic polynomial.

A property that is very noticeable when one has calculated a few chromatic polynomials is that the coefficients first increase in absolute magnitude, and then decrease; two successive coefficients may be equal, but it seems that one never finds a coefficient flanked by larger coefficients, and it is natural to conjecture that the coefficients always behave in this way. It is fairly easy to show that the coefficients are bounded in absolute magnitude by the corresponding coefficients in the chromatic polynomial of the complete graph on the same number of nodes (the proof of this will be left as an exercise for the reader); and certainly these upper bounds first increase and then decrease. But whether this is true for all chromatic polynomials is, as far as I know, still an open question.

Again, this increase and decrease in the coefficients suggest that for large values of n the coefficients in the chromatic polynomials of "most" graphs on n nodes might approximate to some well-known unimodal statistical distribution. Several unsolved problems can be formulated along these lines.

It is clear that distinct graphs may have the same chromatic polynomial. For example, all trees with n nodes have the same chromatic polynomial. Less trivially, the following distinct graphs have the same chromatic polynomial:



This prompts the question "What is a necessary and sufficient condition for two graphs to have the same chromatic polynomial?" This question also is unsolved.

10. Two Applications

This introduction may well have left the reader with the impression that chromatic polynomials are not of any particular practical importance. To show that this is not necessarily the case, we give two possible applications:

Application 1. *Allocation of channels to television stations.*

Assume that there are k possible channels (frequencies) available for use by the n television stations in a certain country. As is well known, stations that are near to each other cannot use the same channel without causing interference. Thus, given any two stations, it may or may not be the case that they can use the same channel. The problem is to allocate a channel to each station in such a way that any two stations which need to have different channels get different channels.

Let us construct a graph G whose nodes represent the stations. We join two nodes by an edge if and only if the corresponding stations cannot use the same channel. Then any allocation of channels is, effectively, a coloring of G in k colors, and if it is proper then the condition about nearby stations being given different channels is satisfied. Thus the problem reduces to that of coloring a graph, and the chromatic polynomial will give the number of ways of allocating the k channels.

The chromatic number $\chi(G)$ of a graph G is the smallest number of colors with which G can be properly colored; that is, it is the smallest integer λ for which $M_G(\lambda) \neq 0$. In the above example it is the minimum number of channels that will suffice to satisfy the allocation conditions.

APPLICATION 2. *Construction of timetables.*

Let us formulate the problem of constructing a timetable, say the weekly schedule of lectures for courses at a university.

We have a list of lectures to be given in the various subjects. Certain of these lectures must not be given at the same time as certain others, however, since there are students who wish to attend both. Therefore we have also a list of pairs of lectures that must not clash. We now construct a graph L in the following way. For each lecture that has to be given there is a node of L; and two nodes of L are joined by an edge if and only if the lectures they represent may not be given at the same time.

There will be a set of periods during the week, probably seven or eight per day, when lectures may be given. We shall take these as constituting our set of colors. If we can construct a proper coloring of L with these "colors" we shall have a timetable; for lectures corresponding to nodes allocated a certain color will be given during the period represented by that color; and lectures that may not clash will not be scheduled for the same period, since the nodes that represent them, being joined, will not have been given the same color. If it should happen that the chromatic number of L is greater than the total number of available periods, then no timetable is possible with the given restrictions.

For an actual timetable the graph L would have a large number of nodes, and it would be impracticable to compute its chromatic polynomial. (The same remark applies to Application 1.) However, what is needed is not the number of ways of coloring L but, first of all, the value of $\chi(L)$ (in case the construction of a timetable should turn out to be impossible), and one (at least) actual coloring. This is a practical possibility; in fact the problem can be stated as a special kind of linear programming problem.

We define a "maximal independent set" of nodes of a graph G as a set of nodes no two of which are joined by an edge, and which is maximal for this property. Let the maximal independent sets of G be $M_1, M_2, ..., M_s$. The union of these sets is clearly the set of nodes of G. In general two of these sets will have non-empty intersection.

To find a coloring of G in k colors it is sufficient to select k of these maximal independent sets of nodes whose union is the set of all nodes of G. For then we can allocate one color to the nodes of each set. Nodes belonging to more than one set may be given the color associated with any of the sets to which they belong. Further, the coloring will be proper, since nodes of the same set are not joined.

Let $x_i = 1$ if M_i is chosen in the selection of maximal independent sets, and $x_i = 0$ if it is not. Let a_{ij} be 1 if node i belongs to M_j, and 0 if it does not. Then for a coloring of G we must have

$$\sum a_{ij}x_j > 0 \tag{5}$$

since node i must belong to at least one of the sets, and

$$\sum_i x_i = \text{number of sets} \tag{6}$$

should be a minimum, for the coloring in the fewest colors.

The minimizing of (6) subject to the constraints (5) is a problem in integer programming. It is of a rather special kind in that all coefficients are 0's or 1's. Insofar as methods are known [1] for the solution of such problems, and are well adapted to computation by electronic means, the handling of problems such as the channel allocation problem and the construction of timetables should be feasible, even when the graphs in question are quite large.

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