What Is i^i

(i to the power of i)

$$e^{i\pi} + 1 = 0$$

$$e^{i\pi} = -1$$

$$\sqrt{e^{i\pi}} = \sqrt{-1}$$

$$\left(e^{i\pi}\right)^{1/2} = i$$

$$e^{i\pi/2} = i$$

$$\left(e^{i\pi/2}\right)^{i} = i^i$$

$$e^{i^2\pi/2} = i^i$$

$$e^{-\pi/2} = i^i$$

$$i^i = e^{-\pi/2} \sim 0.20788 \sim \frac{1}{5}$$

2.38. Consider a discrete-time LTI system with impulse response $h[n]$ given by

$$h[n] = \alpha^n u[n]$$

(*a*) Is this system causal?

(*b*) Is this system BIBO stable?

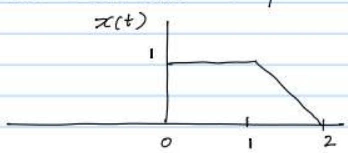
(*a*) Since $h[n] = 0$ for $n < 0$, the system is causal.

(*b*) Using Eq. (1.91) (Prob. 1.19), we have

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} \left|\alpha^k u[n]\right| = \sum_{k=0}^{\infty} |\alpha|^k = \frac{1}{1-|\alpha|} \qquad |\alpha| < 1$$

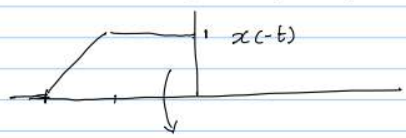
Therefore, the system is BIBO stable if $|\alpha| < 1$ and unstable if $|\alpha| \geq 1$.

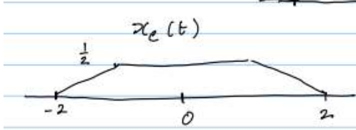
Example: Find the even and odd parts of

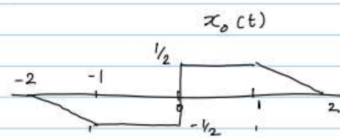


$$x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right] \checkmark$$

$$x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right] \checkmark$$







1.6. Find the even and odd components of $x(t) = e^{jt}$.

Let $x_e(t)$ and $x_o(t)$ be the even and odd components of e^{jt}, respectively.

$$e^{jt} = x_e(t) + x_o(t)$$

From Eqs. (1.5) and (1.6) and using Euler's formula, we obtain

$$x_e(t) = \frac{1}{2}(e^{jt} + e^{-jt}) = \cos t$$

$$x_o(t) = \frac{1}{2}(e^{jt} - e^{-jt}) = j\sin t$$

Example 2. *What is the energy from $t = 0$ to $t = 10$ in $x(t) = e^{j\omega t}$? Integrating gives*

$$E = \int_0^{10} |x(t)|^2 \, \mathrm{d}t = \int_0^{10} |e^{j\omega t}|^2 \, \mathrm{d}t = \int_0^{10} 1 \, \mathrm{d}t = 10.$$

1.3. Given the continuous-time signal specified by

$$x(t) = \begin{cases} 1 - |t| & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

determine the resultant discrete-time sequence obtained by uniform sampling of $x(t)$ with a sampling interval of (*a*) 0.25 s, (*b*) 0.5 s, and (*c*) 1.0 s.

It is easier to take the graphical approach for this problem. The signal $x(t)$ is plotted in Fig. 1-21(*a*). Figs. 1-21(*b*) to (*d*) give plots of the resultant sampled sequences obtained for the three specified sampling intervals.

(*a*) $T_s = 0.25$ s. From Fig. 1-21(*b*) we obtain

$$x[n] = \{\ldots, 0, 0.25, 0.5, 0.75, \underset{\uparrow}{1}, 0.75, 0.5, 0.25, 0, \ldots\}$$

(*b*) $T_s = 0.5$ s. From Fig. 1-21(*c*) we obtain

$$x[n] = \{\ldots, 0, 0.5, \underset{\uparrow}{1}, 0.5, 0, \ldots\}$$

(*c*) $T_s = 1$ s. From Fig. 1-21(*d*) we obtain

$$x[n] = \{\ldots, 0, \underset{\uparrow}{1}, 0, \ldots\} = \delta[n]$$

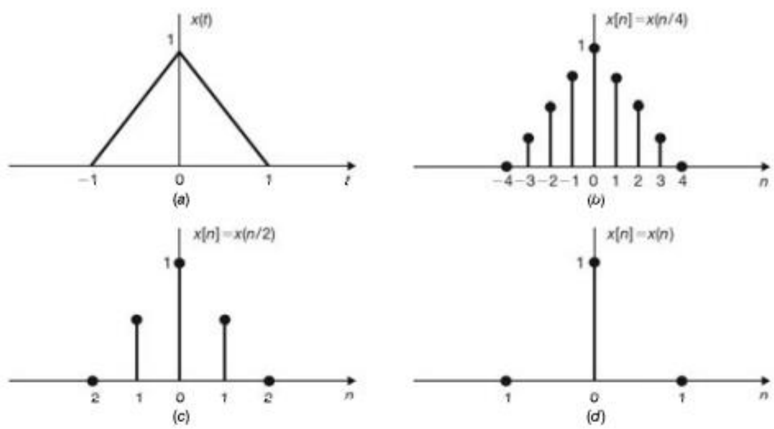
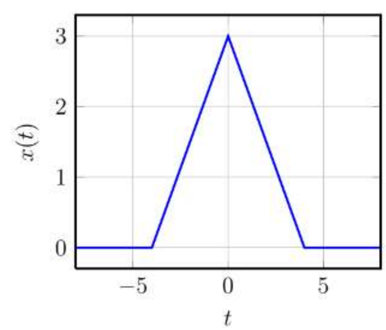


Fig. 1-21

Any signal with finite energy (i.e., $E_\infty < \infty$) has power $P_\infty = 0$ and is sometimes called an "energy-type" signal. Any signal with $0 < P_\infty < \infty$ has $E_\infty = \infty$ and is sometimes called a "power-type" signal.

$$x(t) = \begin{cases} 3\left(1 - \frac{t}{4}\right) & \text{if } 0 < x \le 4 \\ 3\left(1 + \frac{t}{4}\right) & \text{if } -4 < x \le 0 \\ 0 & \text{otherwise} \end{cases}$$



Exa *r the above signal, what is the total energy? Integrating gives*

$$\begin{aligned} E_\infty &= \int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t = \int_{-4}^{4} |x(t)|^2 \, \mathrm{d}t \\ &= 2\int_0^4 3^2 \left(1 - \frac{t}{4}\right)^2 \mathrm{d}t = 18 \int_0^4 \left(1 - \frac{t}{4}\right)^2 \mathrm{d}t \\ &= 18 \int_0^4 \left(1 - \frac{t}{2} + \frac{t^2}{16}\right)^2 \mathrm{d}t = 18\left(4 - \frac{16}{4} + \frac{4^3}{48}\right) = 24. \end{aligned}$$

Example 1

Find the fundamental frequency of the following continuous signal

$$x(t) = cos\left(\frac{10\pi}{3}t\right) + sin\left(\frac{5\pi}{4}t\right)$$

The frequencies and periods of the two terms are, respectively,

$$w_1 = \frac{10\pi}{3}, f_1 = \frac{5}{3}, T_1 = \frac{3}{5}$$

$$\text{and } w_2 = \frac{5\pi}{4}, f_2 = \frac{5}{8}, T_2 = \frac{8}{5}$$

The fundamental frequency f_0 is the GCD of $f_1 = 5/3$ **and** $f_2 = 5/8$

$$f_0 = GCD\left(\frac{5}{3}, \frac{5}{8}\right) = GCD\left(\frac{40}{24}, \frac{15}{24}\right) = \frac{5}{24}$$

Alternatively, the period of the fundamental T_0 is the LCM of $T_1 = \frac{3}{5}$ **and** $T_1 = \frac{8}{5}$

$$T_0 = LCM\left(\frac{3}{5}, \frac{8}{5}\right) = \frac{24}{5}$$

Now we get $w_0 = 2\pi f_0 = \frac{2\pi}{T_0} = \frac{5\pi}{12}$ and the signal can be written as

$$x(t) = cos\left(8\frac{5\pi}{12}t\right) + sin\left(3\frac{5\pi}{12}t\right) = cos(8w_0t) + sin(3w_0t)$$

i.e., the two terms are the 3rd and 8th harmonic of the fundamental frequency w_0, respectively.

1.36. The discrete-time system shown in Fig. 1-36 is known as the *unit delay* element. Determine whether the system is (*a*) memoryless, (*b*) causal, (*c*) linear, (*d*) time-invariant, or (*e*) stable.

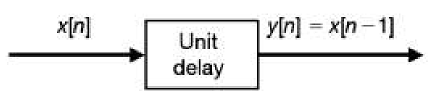


Fig. 1-36 Unit delay element

(*a*) The system input-output relation is given by

$$y[n] = \mathbf{T}\{x[n]\} = x[n-1] \tag{1.111}$$

Since the output value at n depends on the input values at $n - 1$, the system is not memoryless.

(*b*) Since the output does not depend on the future input values, the system is causal.

(*c*) Let $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$. Then

$$y[n] = \mathbf{T}\{\alpha_1 x_1[n] + \alpha_2 x_2[n]\} = \alpha_1 x_1[n-1] + \alpha_2 x_2[n-1]$$
$$= \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

Thus, the superposition property (1.68) is satisfied and the system is linear.

(*d*) Let $y_1[n]$ be the response to $x_1[n] = x[n - n_0]$. Then

$$y_1[n] = \mathbf{T}\{x_1[n]\} = x_1[n-1] = x[n-1-n_0]$$

and
$$y[n-n_0] = x[n-n_0-1] = x[n-1-n_0] = y_1[n]$$

Hence, the system is time-invariant.

(*e*) Since

$$|y[n]| = |x[n-1]| \le k \qquad \text{if } |x[n]| \le k \text{ for all } n$$

the system is BIBO stable.

1.16. Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

(a) $x(t) = \cos\left(t + \dfrac{\pi}{4}\right)$ (b) $x(t) = \sin\dfrac{2\pi}{3}t$

(c) $x(t) = \cos\dfrac{\pi}{3}t + \sin\dfrac{\pi}{4}t$ (d) $x(t) = \cos t + \sin\sqrt{2}t$

(e) $x(t) = \sin^2 t$ (f) $x(t) = e^{j[(\pi/2)t-1]}$

(g) $x[n] = e^{j(\pi/4)n}$ (h) $x[n] = \cos\dfrac{1}{4}n$

(i) $x[n] = \cos\dfrac{\pi}{3}n + \sin\dfrac{\pi}{4}n$ (j) $x[n] = \cos^2\dfrac{\pi}{8}n$

(a) $x(t) = \cos\left(t + \dfrac{\pi}{4}\right) = \cos\left(\omega_0 t + \dfrac{\pi}{4}\right) \to \omega_0 = 1$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 2\pi$.

(b) $x(t) = \sin\dfrac{2\pi}{3}t \to \omega_0 = \dfrac{2\pi}{3}$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 3$.

(c) $x(t) = \cos\dfrac{\pi}{3}t + \sin\dfrac{\pi}{4}t = x_1(t) + x_2(t)$

where $x_1(t) = \cos(\pi/3)t = \cos\omega_1 t$ is periodic with $T_1 = 2\pi/\omega_1 = 6$ and $x_2(t) = \sin(\pi/4)t = \sin\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = 8$. Since $T_1/T_2 = \frac{6}{8} = \frac{3}{4}$ is a rational number, $x(t)$ is periodic with fundamental period $T_0 = 4T_1 = 3T_2 = 24$.

(d) $x(t) = \cos t + \sin\sqrt{2}t = x_1(t) + x_2(t)$

where $x_1(t) = \cos t = \cos\omega_1 t$ is periodic with $T_1 = 2\pi/\omega_1 = 2\pi$ and $x_2(t) = \sin\sqrt{2}t = \sin\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = \sqrt{2}\pi$. Since $T_1/T_2 = \sqrt{2}$ is an irrational number, $x(t)$ is nonperiodic.

(e) Using the trigonometric identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, we can write

$$x(t) = \sin^2 t = \frac{1}{2} - \frac{1}{2}\cos 2t = x_1(t) + x_2(t)$$

where $x_1(t) = \frac{1}{2}$ is a dc signal with an arbitrary period and $x_2(t) = -\frac{1}{2}\cos 2t = -\frac{1}{2}\cos\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = \pi$. Thus, $x(t)$ is periodic with fundamental period $T_0 = \pi$.

(f) $x(t) = e^{j[(\pi/2)t-1]} = e^{-j}e^{j(\pi/2)t} = e^{-j}e^{j\omega_0 t} \to \omega_0 = \frac{\pi}{2}$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 4$.

(g) $x[n] = e^{j(\pi/4)n} = e^{j\Omega_0 n} \to \Omega_0 = \frac{\pi}{4}$

Since $\Omega_0/2\pi = \frac{1}{8}$ is a rational number, $x[n]$ is periodic, and by Eq. (1.55) the fundamental period is $N_0 = 8$.

(h) $x[n] = \cos\frac{1}{4}n = \cos\Omega_0 n \to \Omega_0 = \frac{1}{4}$

Since $\Omega_0/2\pi = 1/8\pi$ is not a rational number, $x[n]$ is nonperiodic.

(i) $x[n] = \cos\frac{\pi}{3}n + \sin\frac{\pi}{4}n = x_1[n] + x_2[n]$

where

$$x_1[n] = \cos\frac{\pi}{3}n = \cos\Omega_1 n \to \Omega_1 = \frac{\pi}{3}$$

$$x_2[n] = \sin\frac{\pi}{4}n = \cos\Omega_2 n \to \Omega_2 = \frac{\pi}{4}$$

Since $\Omega_1/2\pi = \frac{1}{6}$ (= rational number), $x_1[n]$ is periodic with fundamental period $N_1 = 6$, and since $\Omega_2/2\pi = \frac{1}{8}$ (= rational number), $x_2[n]$ is periodic with fundamental period $N_2 = 8$. Thus, from the result of Prob. 1.15, $x[n]$ is periodic and its fundamental period is given by the least common multiple of 6 and 8, that is, $N_0 = 24$.

(j) Using the trigonometric identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, we can write

$$x[n] = \cos^2\frac{\pi}{8}n = \frac{1}{2} + \frac{1}{2}\cos\frac{\pi}{4}n = x_1[n] + x_2[n]$$

where $x_1[n] = \frac{1}{2} = \frac{1}{2}(1)^n$ is periodic with fundamental period $N_1 = 1$ and $x_2[n] = \frac{1}{2}\cos(\pi/4)n = \frac{1}{2}\cos\Omega_2 n \to \Omega_2 = \pi/4$. Since $\Omega_2/2\pi = \frac{1}{8}$ (= rational number), $x_2[n]$ is periodic with fundamental period $N_2 = 8$. Thus, $x[n]$ is periodic with fundamental period $N_0 = 8$ (the least common multiple of N_1 and N_2).