

CENG 222

Statistical Methods for Computer Engineering

Week 5

Chapter 4

Continuous Distributions: Gamma and Normal Distributions, Central Limit Theorem

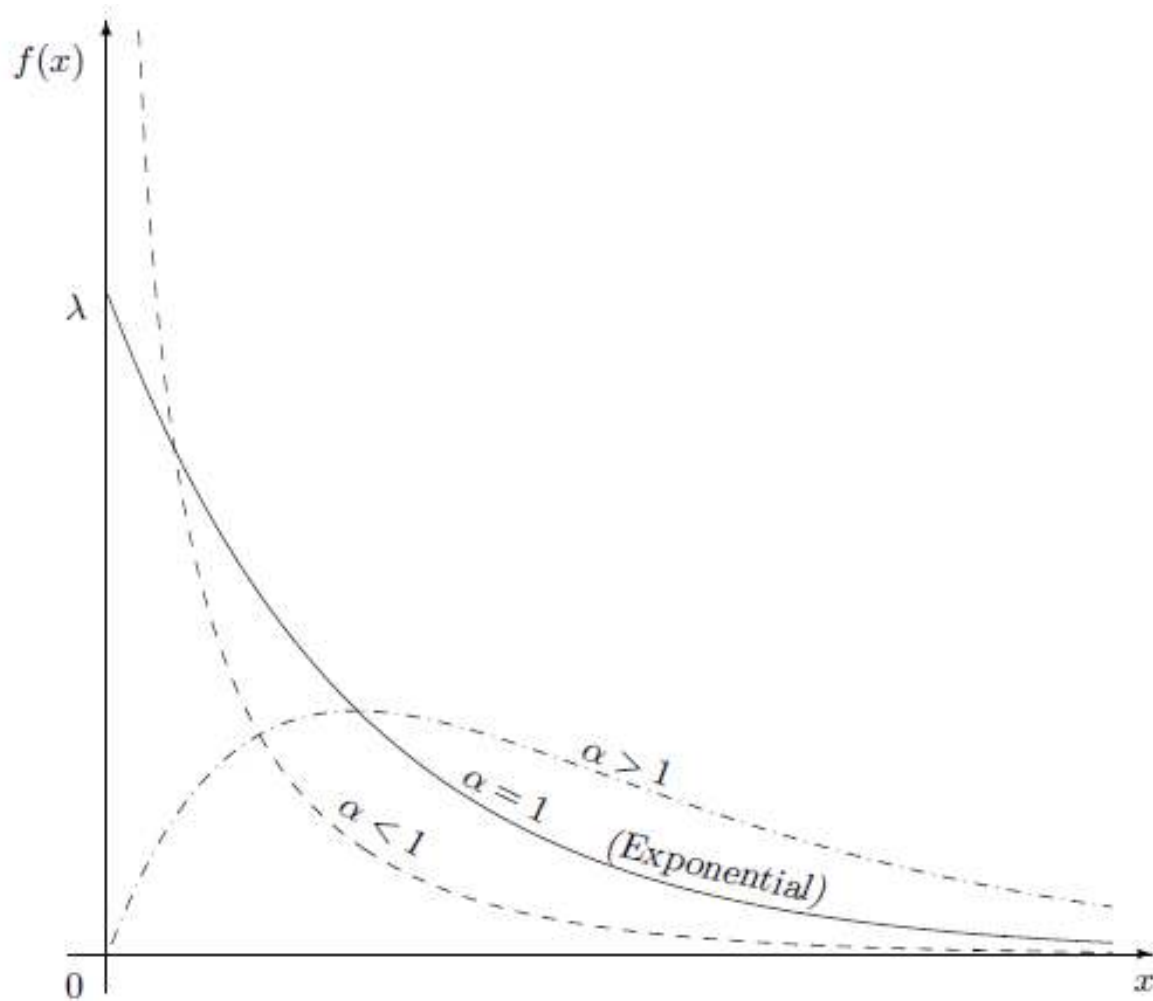
Gamma distribution

- X = the total time of observing α rare and independent events each with exponential waiting times (with parameter λ)
 - i.e., it is the sum of α exponential rvs
- Expectation and variance can be found using linearity of expectation.
 - $E(X) = \frac{\alpha}{\lambda}$, $Var(X) = \frac{\alpha}{\lambda^2}$

Gamma pdf

- $f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda}, \quad x > 0$
- $\Gamma(\alpha) = (\alpha - 1)!$

α does not need to be an integer

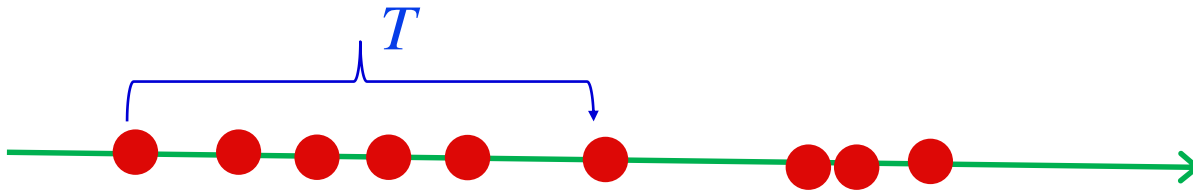


Gamma distribution

- Is widely used to model random variables other than waiting times (since α does not need to be an integer)
 - Amount of money spent
 - Amount of resources used (electricity, gas, etc.)

Gamma-Poisson formula

- Rare events



- $T = \text{time of the } \alpha\text{th rare event} = \text{Gamma } (\alpha, \lambda)$
 - The event $\{T > t\}$ means that fewer than α events occur in t time.
 - Let X be a Poisson rv with parameter λt
 - $\{T > t\} = \{X < \alpha\}$ hence $P(T > t) = P(X < \alpha)$
 - $\rightarrow P(T \leq t) = P(X \geq \alpha)$
 - \rightarrow we can use the Poisson table for computation of Gamma probabilities (Caution: T is continuous, X is discrete)

Example 4.9

- Lifetimes for computer chips have Gamma distribution with expectation $\mu=12$ years and standard deviation $\sigma=4$ years. What is the probability that such a chip has a lifetime between 8 and 10 years?
- Step 1: what are the parameters of this Gamma rv?
$$-\frac{\alpha}{\lambda} = 12, \frac{\alpha}{\lambda^2} = 16 \rightarrow \lambda = 12/16 = 0.75, \alpha = 12*0.75 = 9$$

Example 4.9 continued

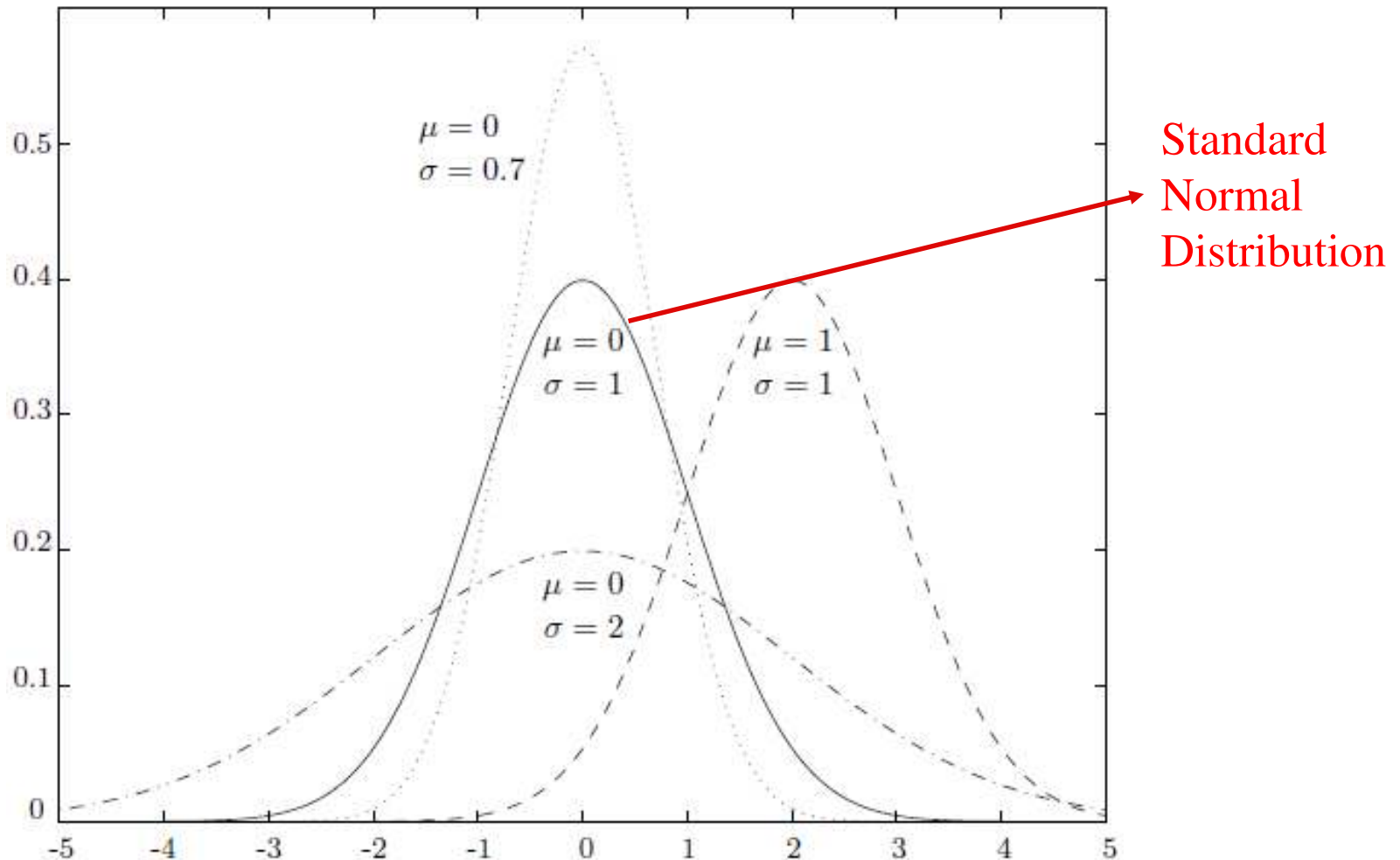
- Step 2: Compute the probability
 - $P(8 < T < 10) = F_T(10) - F_T(8)$
 - $F_T(10) = P(T \leq 10) = P(X_1 \geq 9)$ where $X_1 = \text{Poisson}(7.5)$
 - $P(X_1 \geq 9) = 1 - F_{X_1}(8) = 0.338$
 - $F_T(8) = P(T \leq 8) = P(X_2 \geq 9)$ where $X_2 = \text{Poisson}(6)$
 - $P(X_2 \geq 9) = 1 - F_{X_2}(8) = 0.153$
 - $P(8 < T < 10) = 0.338 - 0.153 = 0.185$

Normal (Gaussian) distribution

- A good model for physical variables like weight, height, temperature, etc.
- Sums and averages of arbitrarily distributed rvs are also normally distributed (Central Limit Theorem)
 - Thus, very popular for modelling errors
- Normal pdf:
 - $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left\{ \frac{-(x-\mu)^2}{2\sigma^2} \right\} , \quad -\infty < x < +\infty$

Normal distribution

- The mean and the std. dev. are also called *location* and *scale* parameters.



Standard Normal Distribution

- Any non-standard Normal rv X with $\text{Normal}(\mu, \sigma)$ can be standardized as follows:
 - $Z = \text{Normal}(0,1) = \frac{X-\mu}{\sigma}$
 - and vice versa: $X = \mu + \sigma Z$
 - \rightarrow we only need the Standard Normal Distribution table only
- Example 4.11 – computing non-standard probabilities using the standard normal table
- Example 4.12 – solving inverse problems

Central Limit Theorem

- Let X_1, \dots, X_n be random variables from any distribution with $\mu = \mathbf{E}(X_i)$ and $\sigma^2 = \mathbf{Var}(X_i)$ (n rvs from the same distribution)

As $n \rightarrow \infty$,

$$\frac{(X_1 + \dots + X_n) - n\mu}{\sigma\sqrt{n}} \rightarrow \text{Normal}(0,1)$$

$$\rightarrow \mathbf{P} \left(\frac{(X_1 + \dots + X_n) - n\mu}{\sigma\sqrt{n}} \leq x \right) \rightarrow F_{\text{Normal}(0,1)}(x)$$

Examples:

$\text{Binomial}(n, p) \approx \text{Normal}(\mu, \sigma)$ for large n

$\text{Gamma}(\alpha, \lambda) \approx \text{Normal}(\mu, \sigma)$ for large α

Central Limit Theorem

- Example 4.13
- Example 4.14

Normal Approximation to Binomial

- $\text{Binomial}(n, p) \approx \text{Normal}(\mu = np, \sigma = \sqrt{np(1-p)})$
- We need continuity correction
 - $P(X=x) = 0$ for a continuous variable X
 - If we want to find $f_B(b)$ for a Binomial variable B
 - $f_B(b) = P(B = b) = P(b - 0.5 < B < b + 0.5)$
 - We expand the interval for the discrete variable 0.5 units in each direction and use the Normal approximation to compute the probability of an interval, not the probability of a point.
- Example 4.15