BLM2041 Signals and Systems

Week 9

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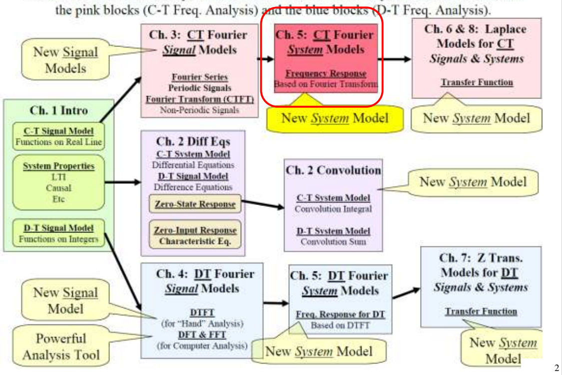
Where are we now?

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



Frequency-Domain Analysis of Systems

Our main interest in this chapter is:

How do we use the FT to analyze LTI systems?

We'll focus on the zero-state response here…

(The zero-input response can be found using the characteristic equation method or the more complete methods we'll study later)

We'll look first at CT systems using three steps:

8.1: Find out how sinusoids go through a C-T LTI

8.2: Because a periodic signal is a sum of sinusoids we use linearity to extend section 8.1 results to periodic signals.

8.2: Non-periodic signals also can be viewed as a sum (really an integral) of sinusoids so we can extend the result again!

Frequency-Domain Analysis of Systems

LTI: **Linear**, Time-Invariant

Q: How does a sinusoid go through an LTI System?

Consider: $x(t) = A\cos(\omega_o t + \theta) \rightarrow \boxed{h(t)} \rightarrow y(t) = ?$

To make this easier to answer (yes… this makes it easier!!) we use Euler's Formula:

$$x(t) = A\cos(\omega_0 t + \theta) = \underbrace{\tfrac{A}{2}e^{j(\omega_0 t+\theta)}} + \underbrace{\tfrac{A}{2}e^{-j(\omega_0 t+\theta)}}$$

The input is now viewed as the sum of two parts… By linearity of the system we can find the response to each part and then add them together.

So we now re-form our question…

Frequency-Domain Analysis of Systems

Q: How does a *complex* sinusoid go through an LTI System?

Consider: $x_1(t) = \tfrac{A}{2}e^{j(\omega_0 t+\theta)} \rightarrow \boxed{h(t)} \rightarrow y(t) = ?$

With convolution as a tool we can now easily answer this question:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

Plug in our input for $x(t-\tau)$

$$= \int_{-\infty}^{\infty} \tfrac{A}{2}e^{j[\omega_o(t-\tau)+\theta]}h(\tau)d\tau = \int_{-\infty}^{\infty} \tfrac{A}{2}e^{j[\omega_o t+\theta]}e^{-j\omega_o \tau}h(\tau)d\tau$$

Use rules for exponentials

$$= \underbrace{\tfrac{A}{2}e^{j[\omega_o t+\theta]}}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-j\omega_o \tau}d\tau}_{\triangleq H(\omega_o)}$$

Pull out part that does not depend on variable of integration… Note that it is just $x_1(t)$

Evaluates to some complex number that depends on $h(t)$ and ω_o

So… the output is just this complex sinusoidal input multiplied by some complex number!!!

Frequency-Domain Analysis of Systems

So… $\boxed{y(t) = H(\omega_o)\tfrac{A}{2}e^{j(\omega_o t+\theta)}}$

Complex-valued

Let's work this equation a bit more to get a more useful, but equivalent form…

Because it is complex we can write $H(\omega_o) = |H(\omega_o)|e^{j\angle H(\omega_o)}$

So using this gives: $y(t) = \left(|H(\omega_o)|e^{j\angle H(\omega_o)}\right)\tfrac{A}{2}e^{j(\omega_o t+\theta)}$

$$= \left(|H(\omega_o)|\tfrac{A}{2}\right)e^{j(\omega_o t+\theta+\angle H(\omega_o))}$$

$$\boxed{y(t) = \underbrace{|H(\omega_o)|\tfrac{A}{2}}_{\text{System changes the amplitude}}e^{j(\omega_o t+\theta+\underbrace{\angle H(\omega_o)}_{\text{System changes the phase}})}}$$

Frequency-Domain Analysis of Systems

Now… we can re-visit our first question…

Q: How does a sinusoid go through an LTI System?

Consider: $x(t) = A\cos(\omega_0 t + \theta) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = ?$

This is equivalent to:

$$x(t) = \tfrac{A}{2}e^{j(\omega_0 t+\theta)} + \tfrac{A}{2}e^{-j(\omega_0 t+\theta)} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = ?$$

And due to linearity and the previous result used twice we have:

$$y(t) = |H(\omega_o)|\tfrac{A}{2}e^{j(\omega_o t+\theta+\angle H(\omega_o))} + |H(-\omega_o)|\tfrac{A}{2}e^{j(-\omega_o t-\theta+\angle H(-\omega_o))}$$

Later we'll see that $|H(\omega_o)| = |H(-\omega_o)|$ $\quad \angle H(-\omega_o) = -\angle H(\omega_o)$

So we get: $y(t) = |H(\omega_o)|A\underbrace{\left[\tfrac{1}{2}e^{j(\omega_o t+\theta+\angle H(\omega_o))} + \tfrac{1}{2}e^{-j(\omega_o t+\theta+\angle H(\omega_o))}\right]}_{\cos(\omega_0 t+\theta+\angle H(\omega_0))}$

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Frequency-Domain Analysis of Systems

So… breaking a signal into sinusoidal parts makes our job EASY!!
(As long we know what the $H(\omega)$ function looks like)

So… How does a sinusoid go through an LTI System?

Consider: $x(t) = A\cos(\omega_0 t + \theta) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = A|H(\omega_o)|\cos(\omega_o t + \theta + \angle H(\omega_o))$

The only thing an LTI system does to a sinusoid is change its amplitude and its phase!!!!

But what about when we have more complicated input signals???

We've already seen that we have to do convolution to solve that case!!!

But… if we have a signal that is a sum of sinusoids then we could use this easy result because of linearity and superposition!!!

$$x(t) = A_1\cos(\omega_1 t + \theta_1) + A_2\cos(\omega_2 t + \theta_2) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = A_1|H(\omega_1)|\cos(\omega_1 t + \theta_1 + \angle H(\omega_1)) + A_2|H(\omega_2)|\cos(\omega_2 t + \theta_2 + \angle H(\omega_2))$$

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Frequency-Domain Analysis of Systems

In the previous slides we saw that it is easy to state how a complex sinusoid goes through a C-T LTI system :

$$x(t) = Ae^{j(\omega_0 t+\theta)} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = Ae^{j(\omega_0 t+\theta)}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-j\omega_0\tau}d\tau}_{= H(\omega_0)}$$

We now know that this is the FT of the system's impulse response, evaluated at $\omega = \omega_0$

$$\boxed{y(t) = AH(\omega_0)e^{j(\omega_0 t+\theta)}}$$

$$\boxed{y(t) = |H(\omega_0)|Ae^{j(\omega_0 t+\theta+\angle H(\omega_0))}}$$

Same frequency sinusoid comes out… the system just changes the input sinusoid's amplitude and phase

An LTI acts to change a complex sinusoid's amplitude and phase

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Frequency-Domain Analysis of Systems

We also saw how a *real* sinusoid goes through a C-T LTI System

$x(t) = A\cos(\omega_0 t + \theta) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = A|H(\omega_o)|\cos(\omega_o t + \theta + \angle H(\omega_o))$

The only thing an LTI system does to a real sinusoid is change its amplitude and its phase!!!!

Of course, you already knew that from circuits!!

So… The big result is:

$h(t)$ = impulse response $\xleftrightarrow{\text{FT}}$ $H(\omega)$ = frequency response

$A\cos(\omega_0 t + \theta) \longrightarrow \boxed{h(t) \; H(\omega)} \longrightarrow |H(\omega_0)|A\cos(\omega_0 t + \theta + \angle H(\omega_0))$

***$H(\omega)$* is called the "frequency response" of the system**

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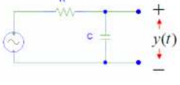
Frequency-Domain Analysis of Systems

Example: Connecting these general ideas to sinusoidal analysis of circuits.

$x(t) = A\cos(\omega t + \theta)$



To go from the circuit view to the system view… we need $H(\omega)$

$A\cos(\omega_0 t + \theta) \longrightarrow \boxed{h(t) \; H(\omega)} \longrightarrow y(t) = ?$

When you did sinusoidal analysis in Circuits you did this!!!

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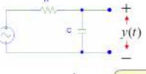
Frequency-Domain Analysis of Systems

Sinusoidal Analysis of Circuit gives the System's Frequency Response $H(\omega)$

$x(t) = A\cos(\omega t + \theta)$

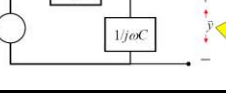


1. Convert capacitor into impedance: $Z_c(\omega) = \dfrac{1}{j\omega C}$ ← Small impedance at high ω Large impedance at low ω

2. Write input as phasor: $Ae^{j\theta} = \tilde{x}$ ← Phasor captures amplitude and phase of cosine… the only things the system can change!!

3. Now analyze the circuit as if it were a DC circuit with a complex voltage in (the phasor) and complex resistors (the impedances):

$Ae^{j\theta} = \tilde{x}$



Now find the output phasor as a function of the input phasor… Here this is easiest using voltage divider!

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Frequency-Domain Analysis of Systems

Voltage Divider: $\tilde{y} = \dfrac{Z_c(\omega)}{R + Z_c(\omega)}\tilde{x} = \underbrace{\left[\dfrac{1}{1 + j\omega RC}\right]}_{=H(\omega)}\tilde{x}$

Output Phasor: $\tilde{y} = H(\omega)\tilde{x} = |H(\omega)|e^{j\angle H(\omega)}\tilde{x}$

$= |H(\omega)|e^{j\angle H(\omega)}Ae^{j\theta}$

$= (|H(\omega)|A)e^{j(\theta + \angle H(\omega))}$

4. Convert the "phasor solution" into the "sinusoidal solution":

Remember that a phasor is a complex number that holds:

- sinusoid's amplitude in its magnitude
- sinusoid's phase in its angles

$$\tilde{y} = \overbrace{(|H(\omega)|A)}e^{\overbrace{j(\theta + \angle H(\omega))}} \quad \Rightarrow \quad y(t) = \overbrace{|H(\omega)|A}\cos(\omega t + \overbrace{\theta + \angle H(\omega)})$$

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Frequency-Domain Analysis of Systems

To see how different frequencies are affected by the RC circuit we plot

$|H(\omega)|$ & $\angle H(\omega)$

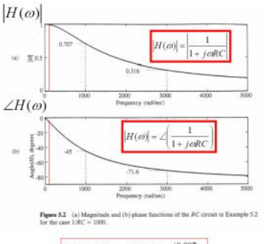


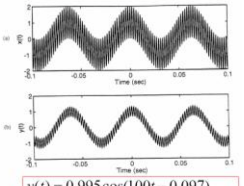
Figure 5.2 (a) Magnitude and (b) phase functions of the *RC* circuit in Example 5.2 for the case $1/RC = 1000$.

$H(100) = 0.995e^{-j0.097}$

$H(3000) = 0.316e^{-j1.249}$

Input has equal amounts at the 2 frequencies…

$x(t) = \cos(100t) + \cos(3000t)$



$y(t) = 0.995\cos(100t - 0.097)$
$\qquad + 0.316\cos(3000t - 1.249)$

Output has almost all of the low frequency component but much reduced high frequency component!

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Frequency-Domain Analysis of Systems

So what have we seen:

- We can find the frequency response function $H(\omega)$ by doing a simple sinusoidal analysis of the circuit
- The frequency response function tells how a circuit changes the input sinusoid's amplitude and phase
- The amount of change in each of these is different for different input frequencies… and a plot of $H(\omega)$ magnitude and phase shows this dependence
- RLC circuits can be used to allow certain frequency components to pass mostly unchanged while others are drastically reduced in amplitude
  - We can "filter out" undesired frequency components

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Response to Periodic Signals

periodic $x(t)$ → $h(t)$, $H(\omega)$ → $y(t) = ?$

Since $x(t)$ is periodic, write it as FS: $x(t) = \sum_{k=-\infty}^{\infty} c_k^x e^{jk\omega_0 t}$ — Indicates "for $x(t)$"

So, the input is a sum of terms

Linear System: So… Output = Sum of Individual Responses

But each individual response is to a complex sinusoid input ⇒ EASY!

$c_k^x e^{jk\omega_0 t}$ → $H(\omega)$ → $H(\omega_0 k)c_k^x e^{jk\omega_0 t}$ (complex: magnitude & phase)

Sum these to get input: $x(t) = \sum_{k=-\infty}^{\infty} c_k^x e^{jk\omega_0 t}$

Sum these to get output: $y(t) = \sum_{k=-\infty}^{\infty}[\underbrace{H(\omega_0 k)c_k^x}]e^{jk\omega_0 t}$

FS coefficient of $y(t)$

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Response to Periodic Signals

General Insights from this Analysis

1. periodic in, periodic out
2. The system's frequency response $H(\omega)$ works to modify the input FS coefficients to create the output FS coefficients:

$$\boxed{c_k^y = H(k\omega_0)c_k^x}$$

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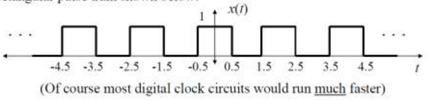
Response to Periodic Signals

Problem: suppose you have a circuit board that has a digital clock circuit on it. It makes the rectangular pulse train shown below:

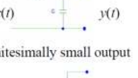


(Of course most digital clock circuits would run much faster)

Suppose you need to connect this clock signal to a circuit on another circuit board using a twisted pair of wires:

Q: What effect does the cable have on the clock signal at the 2nd board???

Pair of wires can be modeled as an RC circuit:



Assume: The circuit "driving" the cable has an infinitesimally small output impedance (that is good!):

Thevenin of driver: $x(t)$

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Response to Periodic Signals

Assume: The circuit being "driven" by the cable has infinite input impedance (that is good!) i.e. No loading of the RC circuit

So…

$y(t) \longrightarrow$ (goes to driven circuit having infinite input impedance)

Goal: Perform an analysis to enable you to recommend an acceptable value of cable RC time constant **(Analysis Drives Design!)**

Step 1: Analytically find FS of input and compute truncated FS sum:

From Ex. 3.4 we get:

Indicates "for $x(t)$"

$$c_k^x = \begin{cases} \dfrac{1}{k\pi}, & k = \pm 1, \pm 5, \pm 9, \ldots \\ -\dfrac{1}{k\pi}, & k = \pm 3, \pm 7, \pm 11, \ldots \\ 0, & k = \pm 2, \pm 4, \pm 6, \ldots \\ \dfrac{1}{2}, & k = 0 \end{cases}$$

$$x(t) \approx \sum_{k=-N}^{N} c_k^x e^{jk\omega_0 t}$$

Then plot vs. time t

Step 2: Find cable's frequency response as a function of RC: $H(\omega) = \dfrac{1}{1 + j\omega RC}$

Response to Periodic Signals

Step 3 (optional) (But it really helps you see what is going on!)

Look at frequency domain plots of Input and System (for various RC values)

"stem" plot of FS coefficients' Magnitude $|c_k^x|$

"continuous" plot of Magnitude of system's Frequency Resp. $|H(\omega)|$

Step 4 (optional) (This also really helps you see what is going on)

Compute output FS coefficients: $c_k^y = H(k\omega_0)c_k^x$

Look at the result → "stem" plot of $|c_k^y|$

Step 5: Compute truncated FS sum to see output signal

$$y(t) \approx \sum_{k=-N}^{N} c_k^y e^{jk\omega_0 t}$$

Plot vs. time t

See plots on next 3 pages for three RC time constant values:

RC = 0.01 s

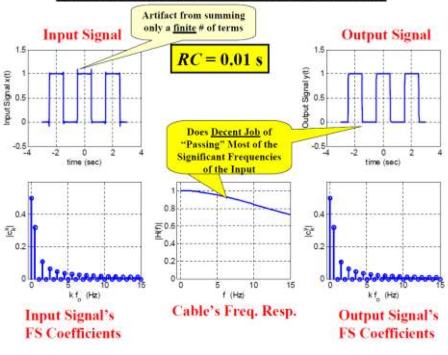
RC = 0.1 s

RC = 1 s

Note: Short RC time constant passes high frequencies better than long RC time constant

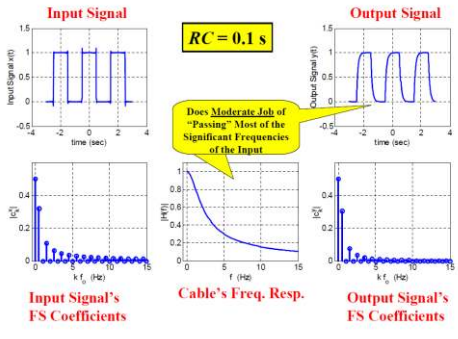
Response to Periodic Signals

RC Circuit Analysis w/ Square Wave Input



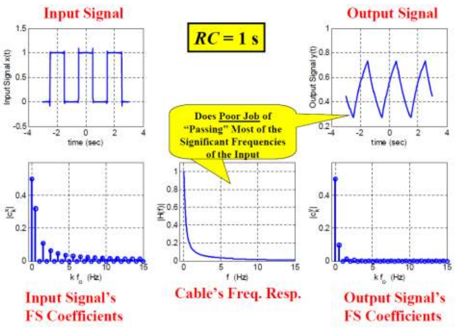
Response to Periodic Signals

RC Circuit Analysis w/ Square Wave Input



Response to Periodic Signals

RC Circuit Analysis w/ Square Wave Input



Response to Periodic Signals

Insight from Example:

- We used a simple model for the cable to make it easy to analyze
  - But… the method would be the same even if we had a more detailed model for the cable
- The input clock signal has nice sharp transitions due to its significant high frequency components
- Cables that significantly suppressed the input's high frequency components provided a low-quality clock signal to the 2nd board
- We made assumptions about the driver circuit and the driven circuit
  - The driver was assumed to have zero output resistance
    - If that were not true, its output impedance gets added to the resistor and that would further degrade the performance (in fact the driver's output impedance may be more than the cable resistance in which case it would be the dominant factor
  - The driven circuit was assumed to have infinite input impedance
    - If that were not true we would have to combine it in parallel with the capacitor's impedance… this would further degrade the performance
- Typically the RC value of a cable increases with length
  - So performance would decrease with length of cable

Response to Aperiodic Signals

Recall:

$$Ae^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(\omega)Ae^{j\omega t}$$

where

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

Thus: Frequency Response = FT{Impulse Response}

-Impulse Response $h(t)$ is a time-domain description of the system

-Frequency Response $H(\omega)$ is a frequency-domain description of the system

Recall that:

Because $h(t)$ and $H(\omega)$ form a FT pair, one completely defines the other.

$h(t)$ and convolution completely describe the zero-state response of an LTI to an input – i.e. $h(t)$ completely describes the system.

Thus: $H(\omega)$ must also completely describes the LTI system HOW????

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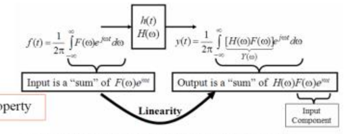
Response to Aperiodic Signals



Conv. Property

"Proof"

****

**** This says: FT of Output = [FT of Input]×[Freq Resp] ****

****

Step 1: Think of the input as a sum of complex sinusoids

-Each component $= F(\omega)e^{j\omega t}$

Step 2: We know how each component passes through an LTI

-This is the idea of frequency response $F(\omega)e^{j\omega t}$

- $H(\omega)F(\omega)e^{j\omega t}$ is the out. component that is due to the input component

Step 3: Exploit System Linearity (again – Step 2 was the first time)

-Total output is a sum of output components $\boxed{y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[H(\omega)F(\omega)]e^{j\omega t}d\omega}$

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Response to Aperiodic Signals

Input-Output Relationship Characterized Two Ways

1. Time-Domain: $y(t) = h(t)*f(t)$
2. Freq-Domain: $Y(\omega) = H(\omega)F(\omega)$

Given input $f(t)$ and impulse response $h(t)$, to analyze the system we could either:

1. Compute the convolution $h(t)*f(t)$

or…

2. Do the following:
   - (a) Compute $H(\omega)$ & compute $F(\omega)$
   - (b) Compute the product $Y(\omega) = H(\omega)F(\omega)$
   - (c) Compute the IFT: $y(t) = \mathfrak{F}^{-1}\{H(\omega)F(\omega)\}$

Method #2 (Freq-Domain Method) may not be necessarily easier, but it usually provides a lot more insight than Method #1!!!!

From the Freq-Domain view we can see how $H(\omega)$ boosts or cuts the amounts of the various frequency components

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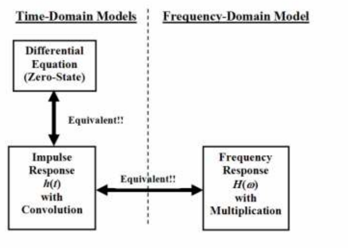
Response to Aperiodic Signals

Relationships between various modeling methods

Recall: we are trying to find ways to model… CT Linear Time-Invariant Systems in Zero-State



Since these are all equivalent…we can use any or all of them to solve a given problem!!

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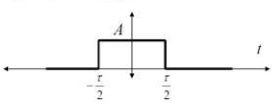
Response to Aperiodic Signals

Example

Scenario: You need to send a pulse signal into a computer's interface circuit to initiate an event (e.g. "next PTT slide")

Q: What kind of signal should you use?

Possibility: A rectangular pulse: $Ap_\tau(t)$



Q: Will this work?

It depends on the interface circuitry already in the computer!

Suppose the interface circuitry consists of an "AC Coupled" transistor amplifier as shown below

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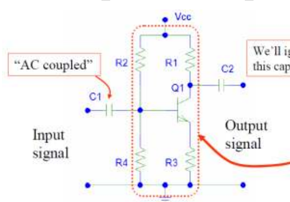
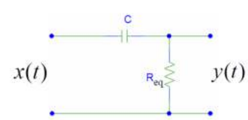
Response to Aperiodic Signals



We'll ignore the effects of this capacitor in our analysis

Model this as an equivalent Input Impedance… simplify here: R_{eq}

"Equivalent Circuit Model"



Now we need to find the System Model viewpoint!

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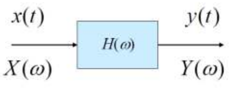
Response to Aperiodic Signals

"Equivalent System Model"



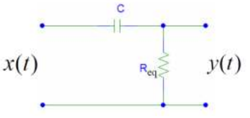
Actually… one LIKE it!

What is $H(\omega)$??

Use Sinusoidal Analysis to find it… we did that once already for this circuit…

Use Phasors, Impedances, and Voltage Divider:

$$\vec{V}_0 = \left[\frac{R_{eq}}{R_{eq} + \frac{1}{j\omega C}}\right]\vec{V}_i$$



$$\Rightarrow \boxed{H(\omega) = \frac{j\omega R_{eq}C}{1 + j\omega R_{eq}C}}$$

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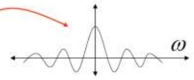
Response to Aperiodic Signals

Now…what does the input pulse look like in the frequency domain?

From FT table:

$$p_\tau(t) \leftrightarrow \underbrace{\tau\,\text{sinc}\left(\frac{\tau\omega}{2\pi}\right)}_{X(\omega)}$$



So the output FT looks like: $Y(\omega) = H(\omega)X(\omega) = \tau\,\text{sinc}\left(\frac{\tau\omega}{2\pi}\right)\underbrace{\left[\frac{j\omega R_{eq}C}{1 + j\omega R_{eq}C}\right]}$

Now how do we find $y(t)$? $y(t) = \mathfrak{F}^{-1}\{Y(\omega)\}$

So find IFT of this…

YUCK!!! HARD!!!

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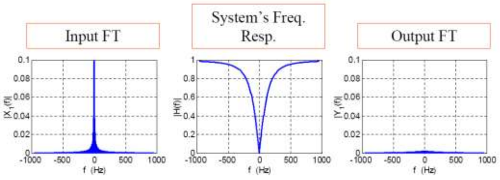
Response to Aperiodic Signals

Well…do we need to "go back to the time domain"? NO!

Just look at $Y(\omega)$ and see what it tells

Think Parseval's theorem

The plots below show that very little energy gets through the system



So this pulse signal is not usable here because very little of its energy gets through the interface circuitry!!!

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Response to Aperiodic Signals

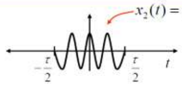
The problem lies in that $|H(\omega)|$ is small where $|X(\omega)|$ is big

(and vice versa)

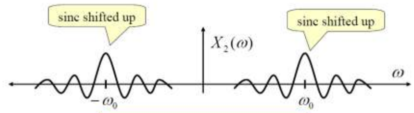
$\Rightarrow$ Pick an $X(\omega)$ that does not do that!!

Use a pulse that is "Modulated Up" to where $|H(\omega)|$ allows it to pass

$x_2(t) = p_\tau(t)\cos(\omega_0 t)$



"A modulated pulse"



See actual plots on next page

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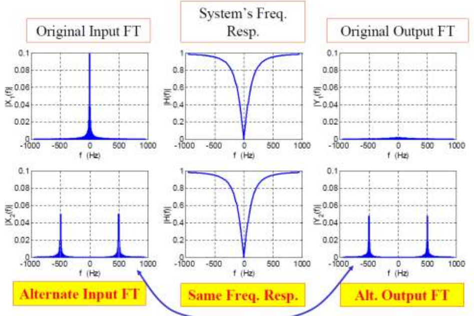
Response to Aperiodic Signals



Output FT is not changed much from Input FT: this is a viable pulse!!!

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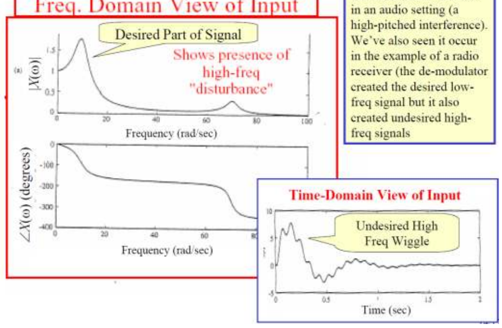
Response to Aperiodic Signals

Example: Attenuation of high frequency Disturbance



This scenario could occur in an audio setting (a high-pitched interference). We've also seen it occur in the example of a radio receiver (the de-modulator created the desired low-freq signal but it also created undesired high-freq signals

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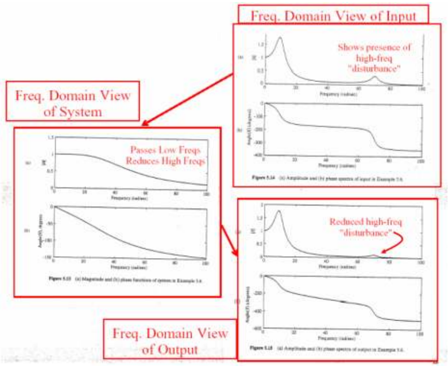
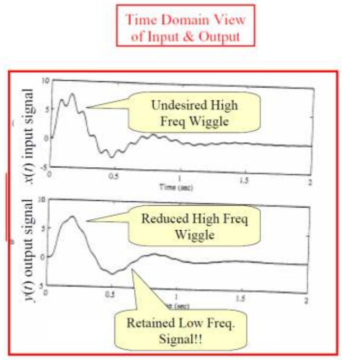
Response to Aperiodic Signals



Response to Aperiodic Signals



Response to Aperiodic Signals

Comments on This Example

- We can use the FT to "see" at what frequencies there are undesired signals
- Then we can specify a desired system frequency response $H(\omega)$ that will reduce (or "attenuate") the undesired signal while keeping the desired signal
  - Note that it would be virtually impossible to try to *directly* specify a desired system impulse response that will do this
- Once we have specified the desired $H(\omega)$ we could try to find a circuit (i.e., a physical system) that will implement it (either exactly or approximately)
  - This is the "design" or "system synthesis" problem
  - We haven't yet learned how to do this!! Tools we'll learn later will help!
  - However, if we have $H(\omega)$ specified as a mathematical function we could *possibly* compute the inverse FT to get the impulse response $h(t)$… then we could implement this "digitally" like we did earlier to simulate an RC circuit using D-T convolution.

Filters

Ideal Filters

Often we have a scenario where we have a "good" signal, $x_g(t)$, corrupted by a "bad" signal, $x_b(t)$, and we want to use an LTI system to remove (or filter out) the bad signal, leaving only the good signal.

$$x(t) = x_g(t) + x_b(t) \longrightarrow \boxed{h(t) \quad H(\omega)} \longrightarrow y(t) = x_g(t)$$

Desired Output

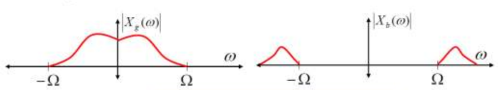
Called "a Filter"

How do we do this? What $H(\omega)$ do we want?

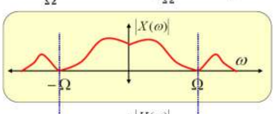
Note: You cannot design the circuit until you know which $H(\omega)$ the circuit must implement

Filters

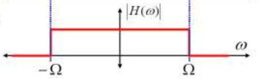
Case #1: $x_g(t)$ is a low-frequency signal $x_b(t)$ is a high-frequency signal



Spectrum of the Input Signal



In this case, we want a filter like this:



Mathematically: $|H(\omega)| = \begin{cases} 1, & -\Omega < \omega < \Omega \\ 0, & otherwise \end{cases}$

"Passband"

"Stopband"

Filters

Then:

$$X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow Y(\omega) = X(\omega)H(\omega)$$

$$\begin{aligned} |Y(\omega)| &= |H(\omega)||X_g(\omega)| + |H(\omega)||X_b(\omega)| \\ &= |X_g(\omega)| \qquad\qquad\quad = 0 \\ &= |X_g(\omega)| \qquad\qquad\quad \text{as desired} \end{aligned}$$

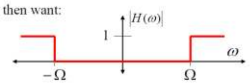
Such a filter is called a "low-pass filter"

Filters

Case #2: $X_g(\omega)$ is a high-frequency signal

$X_b(\omega)$ is a low-frequency signal

We then want:



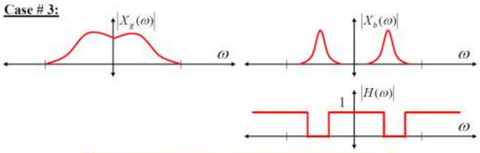
$$|H(\omega)| = \begin{cases} 0, & -\Omega < \omega < \Omega \\ 1, & otherwise \end{cases}$$

"Stopband"

"Passband"

This is called a "high-pass filter"

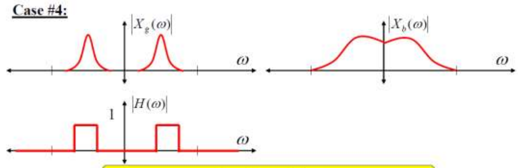
Case # 3:



"Bandstop Filter" or "Notch Filter"

Filters

Case #4:



Bandpass Filter

Note that in Cases #3 and #4 the filter can't remove the bad signal without causing some damage to the desired signal…

…this is not specific to bandpass and bandstop filters…

…it can also happen with low-pass and high-pass filters.

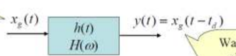
In practice this is almost always the case!!

Filters

What about the ***phase*** of the filter's $H(\omega)$?

Well…we could tolerate a small delay in the output so…

Put in the signal we want "passed"



Want to get out the signal we want "passed"… but we can accept a "small" delay

From the time-shift property of the FT then we need:

$$\boxed{Y(\omega) = X_g(\omega)e^{-j\omega t_d}}$$

Thus we should treat the exponential term here as $H(\omega)$, so we have:

$$|H(\omega)| = |e^{-j\omega t_d}| = 1$$

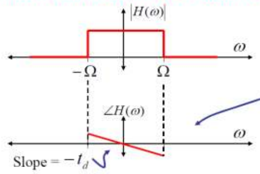
$$\angle H(\omega) = \angle e^{-j\omega t_d} = -\omega t_d$$

For ω in the "pass band" of the filter

Line of slope $-t_d$ "Linear Phase"

Filters

So… for an ideal low-pass filter (LPF) we have:



$$H(\omega) = \begin{cases} 1e^{-j\omega t_d}, & -\Omega < \omega < \Omega \\ 0, & otherwise \end{cases}$$

Phase is undefined here:

$0 = 0e^{j\theta}$

$\angle 0 = ?$

i.e. phase is undefined for frequencies outside the ideal passband

Summary of *Ideal* Filters

1. Magnitude Response:
   a. Constant in Passband
   b. Zero in Stopband
2. Phase Response
   a. Linear in Passband (negative slope = delay)
   b. Undefined in Stopband

Filters

Example of the effect of a nonlinear phase but an ideal magnitude

Here is the scenario: Imagine we have a a signal $x(t)$ given by

$$x(t) = 9 - 5\cos(2\pi t) - 3\cos(2\pi 2t) - \cos(2\pi 3t)$$

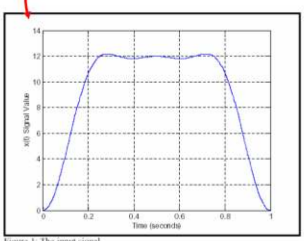
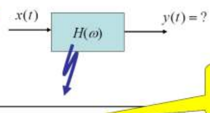


Figure 1: The input signal.

Filters

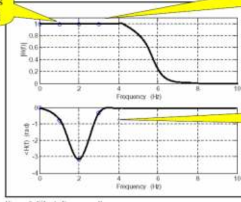
0 Hz 1 Hz 2 Hz 3 Hz

$$x(t) = 9 - 5\cos(2\pi t) - 3\cos(2\pi 2t) - \cos(2\pi 3t)$$



Circles Show the Frequencies in Input Signal

Filter does ***NOT*** change ***amplitudes*** of input components



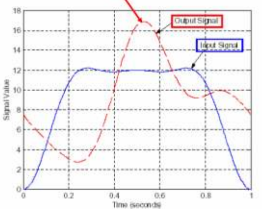
Filter has ***Non***-Linear Phase

Figure 2: Filter's Frequency Response

Filters

So, at the filter's output we have four sinusoids at the same frequencies and amplitudes as at the input…BUT, they are not aligned in time in the same way they were at the input

$$y(t) = 9 - 5\cos(2\pi t - \frac{\pi}{4}) - 3\cos(2\pi 2t - \pi) - \cos(2\pi 3t - \frac{\pi}{10})$$



Point of this Example

A filter with an ideal magnitude response but non-ideal phase response can degrade a signal as much as a filter with a non-ideal magnitude response!!!