BLM2502 Theory of Computation

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» **Course Outline**

Week	Content
1.	Introduction to Course
2.	Computability Theory, Complexity Theory, Automata Theory, Set Theory, Relations, Proofs, Pigeonhole Principle
3.	Regular Expressions
4.	Finite Automata
5.	**Deterministic and Nondeterministic Finite Automata**
6.	**Epsilon Transition, Equivalence of Automata**
7.	Pumping Theorem
8.	Context Free Grammars
9.	Parse Tree, Ambiguity,
10.	Pumping Theorem
11.	Turing Machines, Recognition and Computation, Church-Turing Hypothesis
12.	Turing Machines, Recognition and Computation, Church-Turing Hypothesis
13.	Review

NFA
Non-Deterministic
Finite Automata

Formal Definition of NFA

$$M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$$

$Q:$ Set of states, i.e. $\{q_0, q_1, q_2\}$

$\Sigma:$ Input aplhabet, i.e. $\{a, b\}$ $\varepsilon \notin \Sigma$

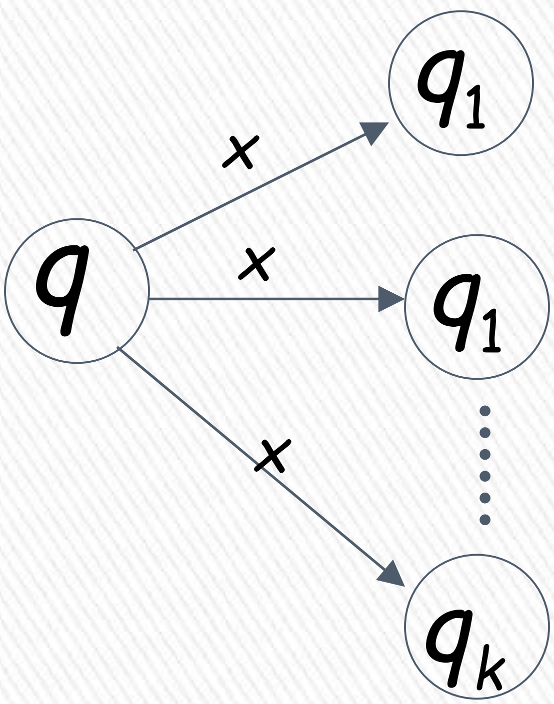
$\delta:$ Transition function Q x Σ -> 2^Q

$q_0:$ Initial state

$F:$ Accepting states

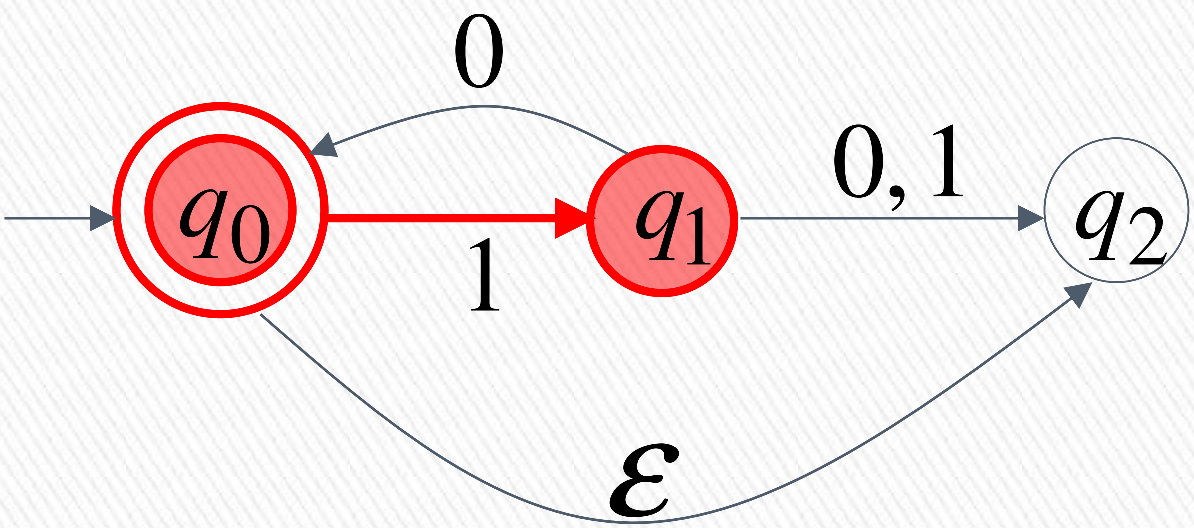
Transition Function δ

$$\delta(q, x) = \{q_1, q_2, \ldots, q_k\}$$

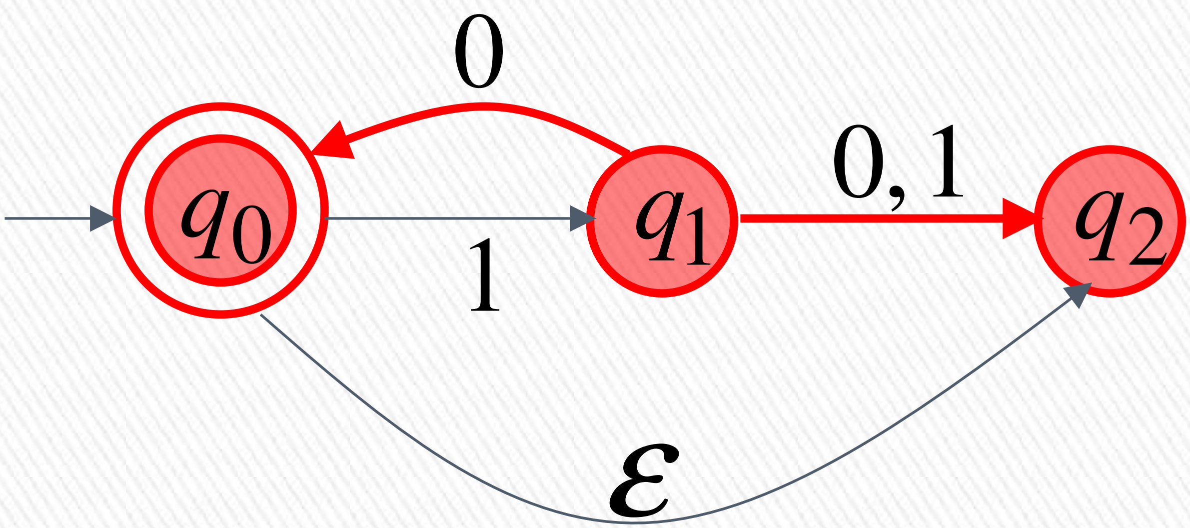


resulting states with following one transition with symbol x

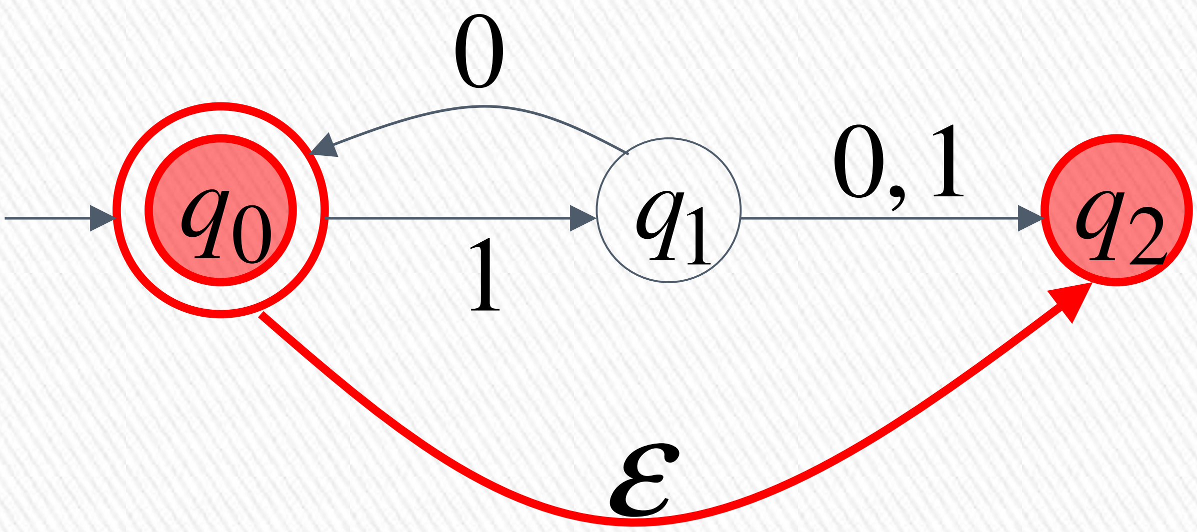
$$\delta(q_0, 1) = \{q_1\}$$



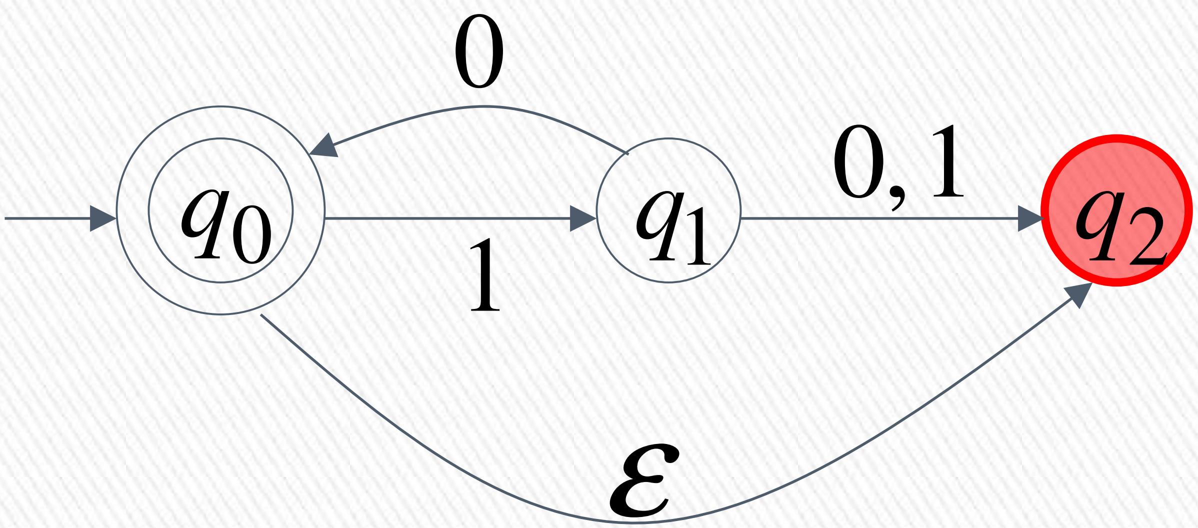
$$\delta(q_1, 0) = \{q_0, q_2\}$$



$$\delta(q_0, \varepsilon) = \{q_2\}$$

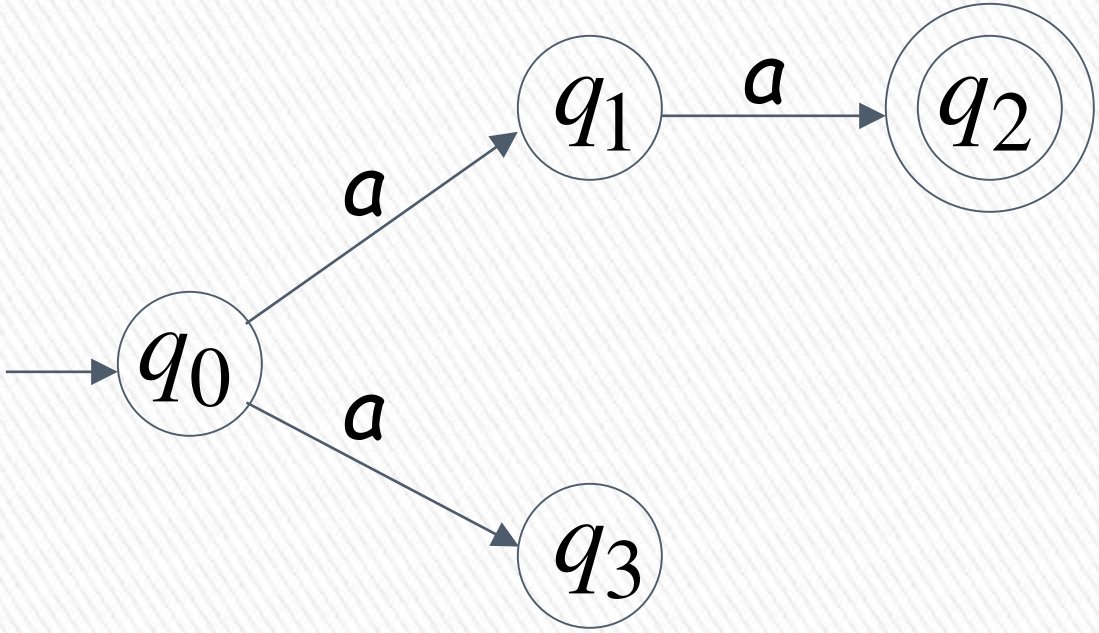


$$\delta(q_2, 1) = \varnothing$$

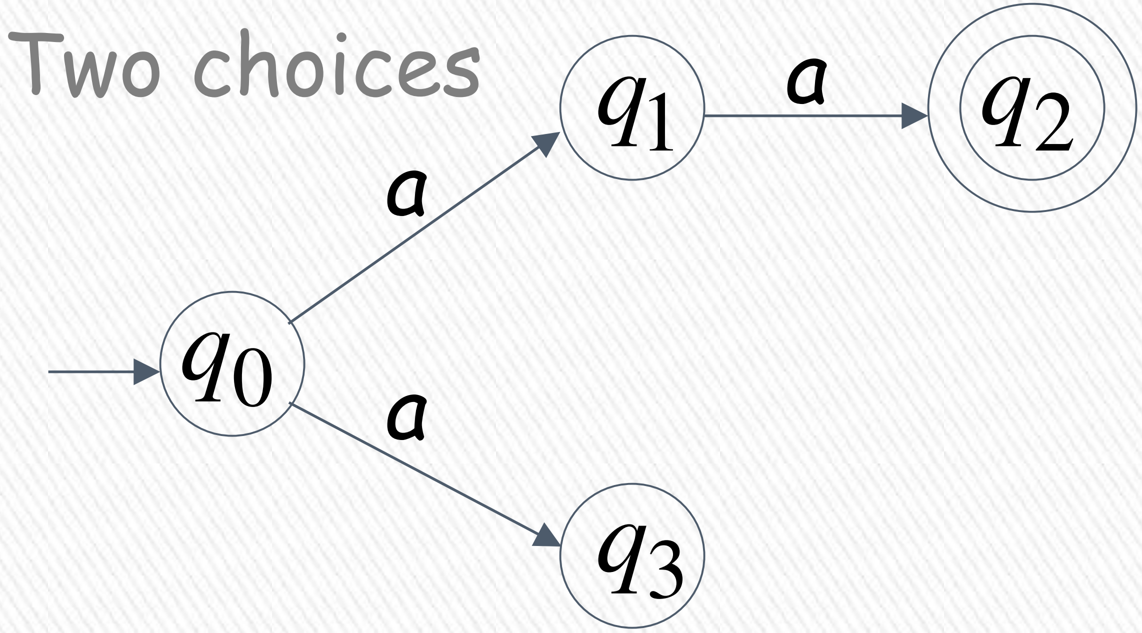


Nondeterministic Finite Automaton (NFA)

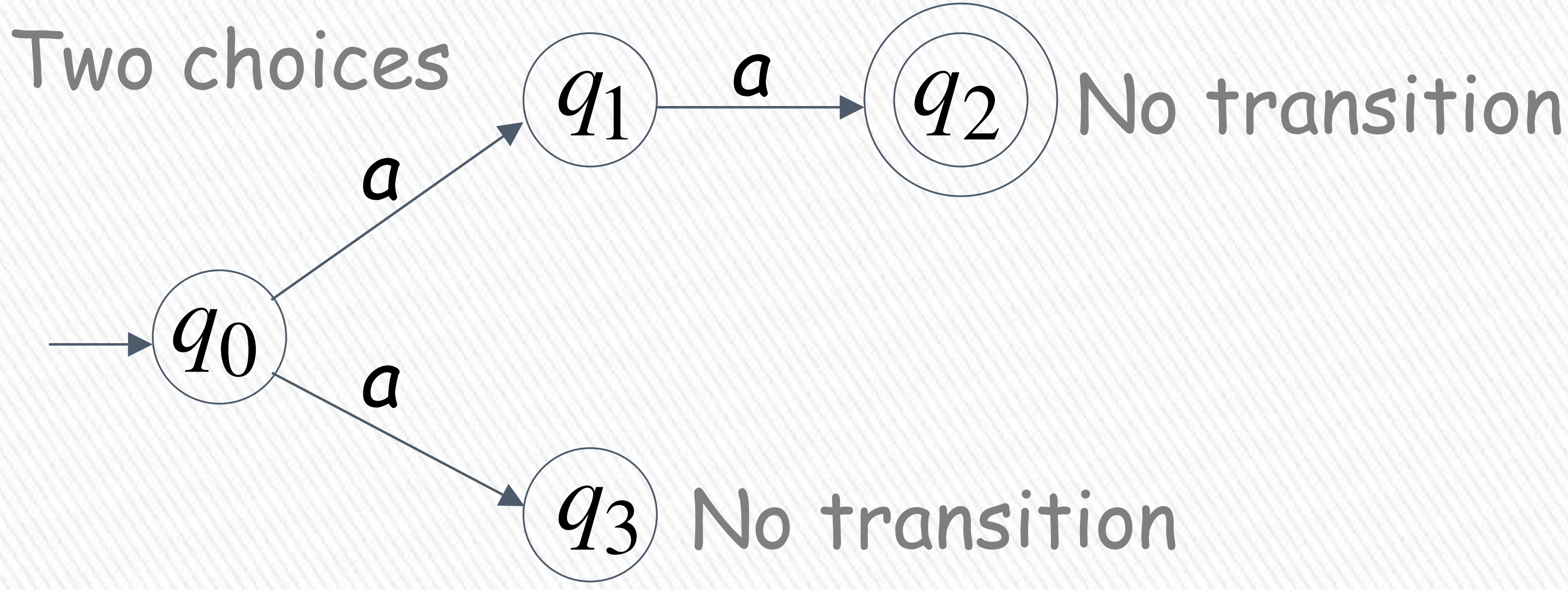
Alphabet = $\{a\}$



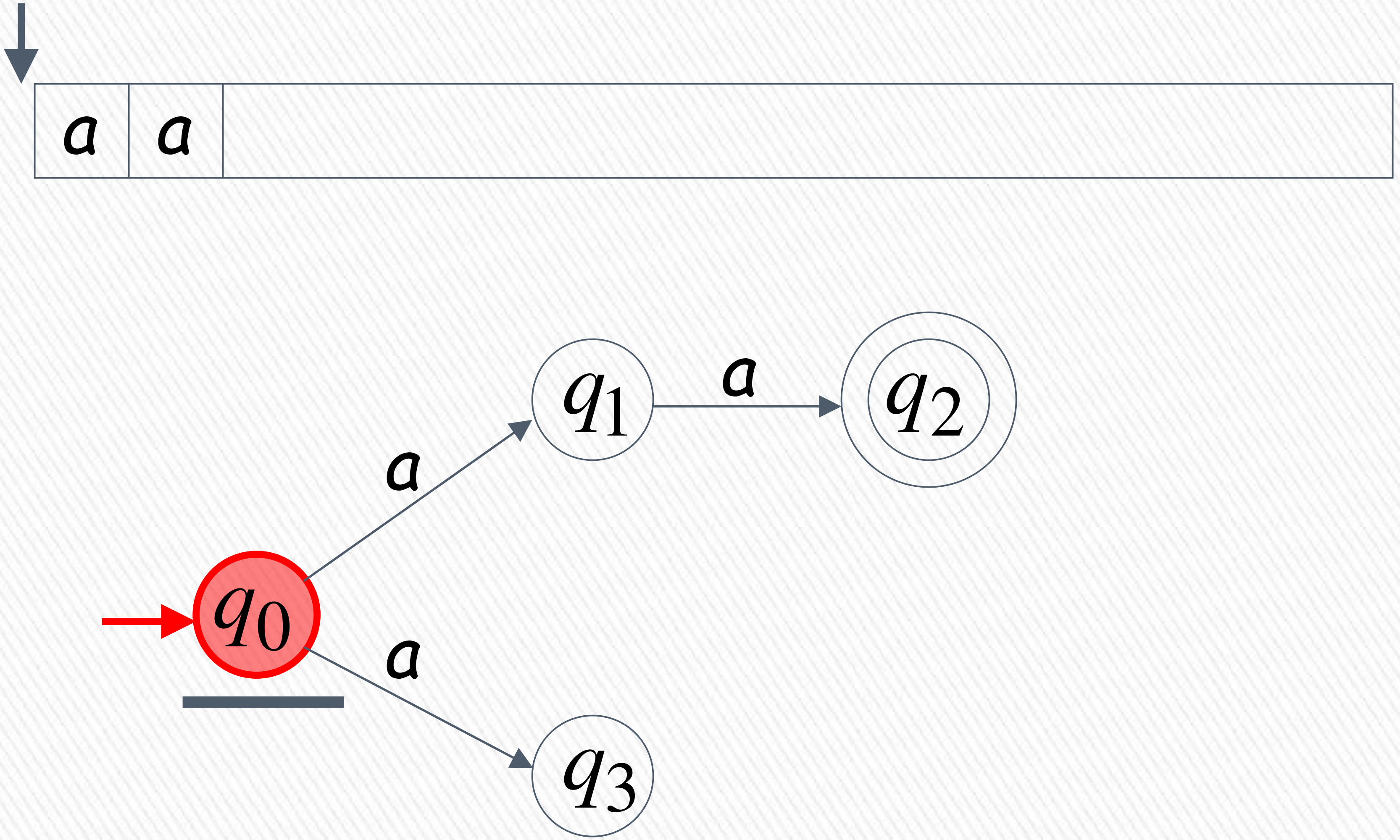
Alphabet = $\{a\}$



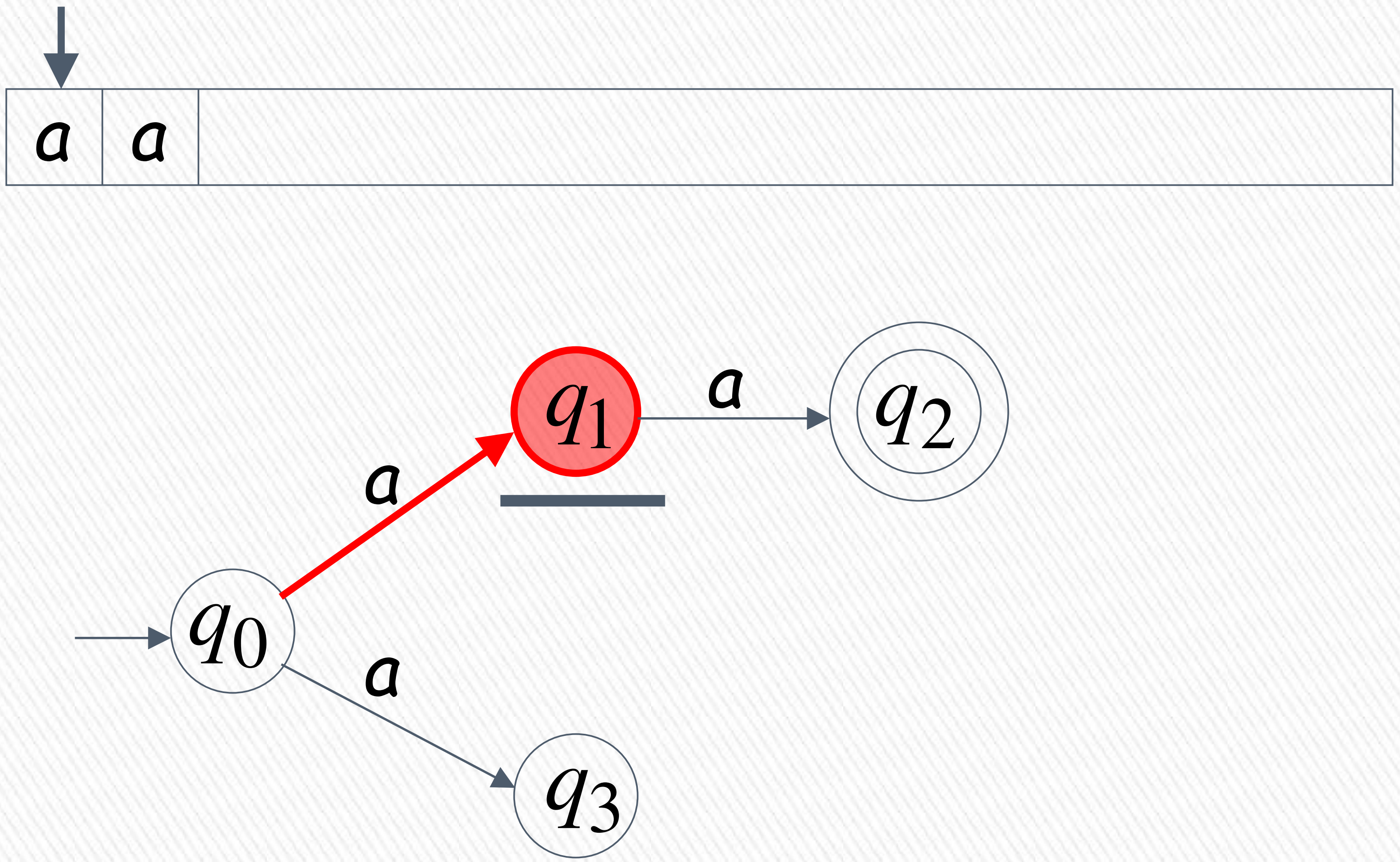
Alphabet = $\{a\}$



First Choice

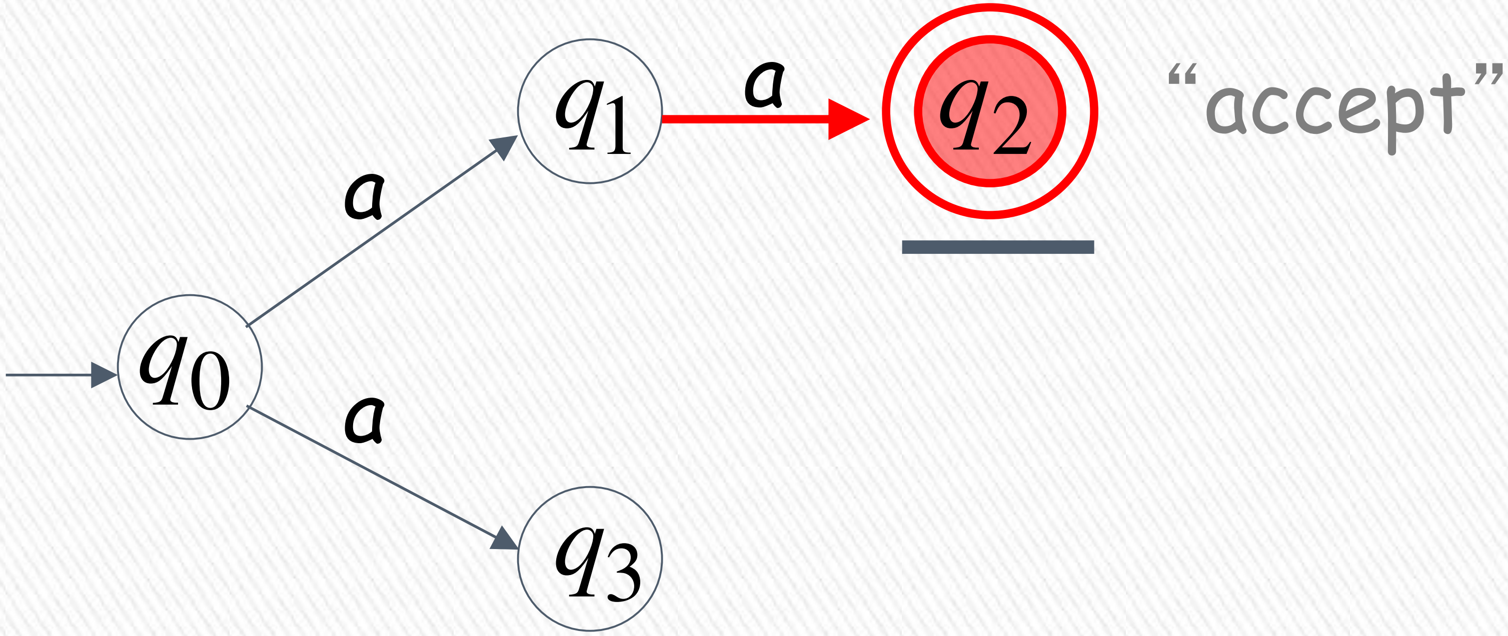


First Choice

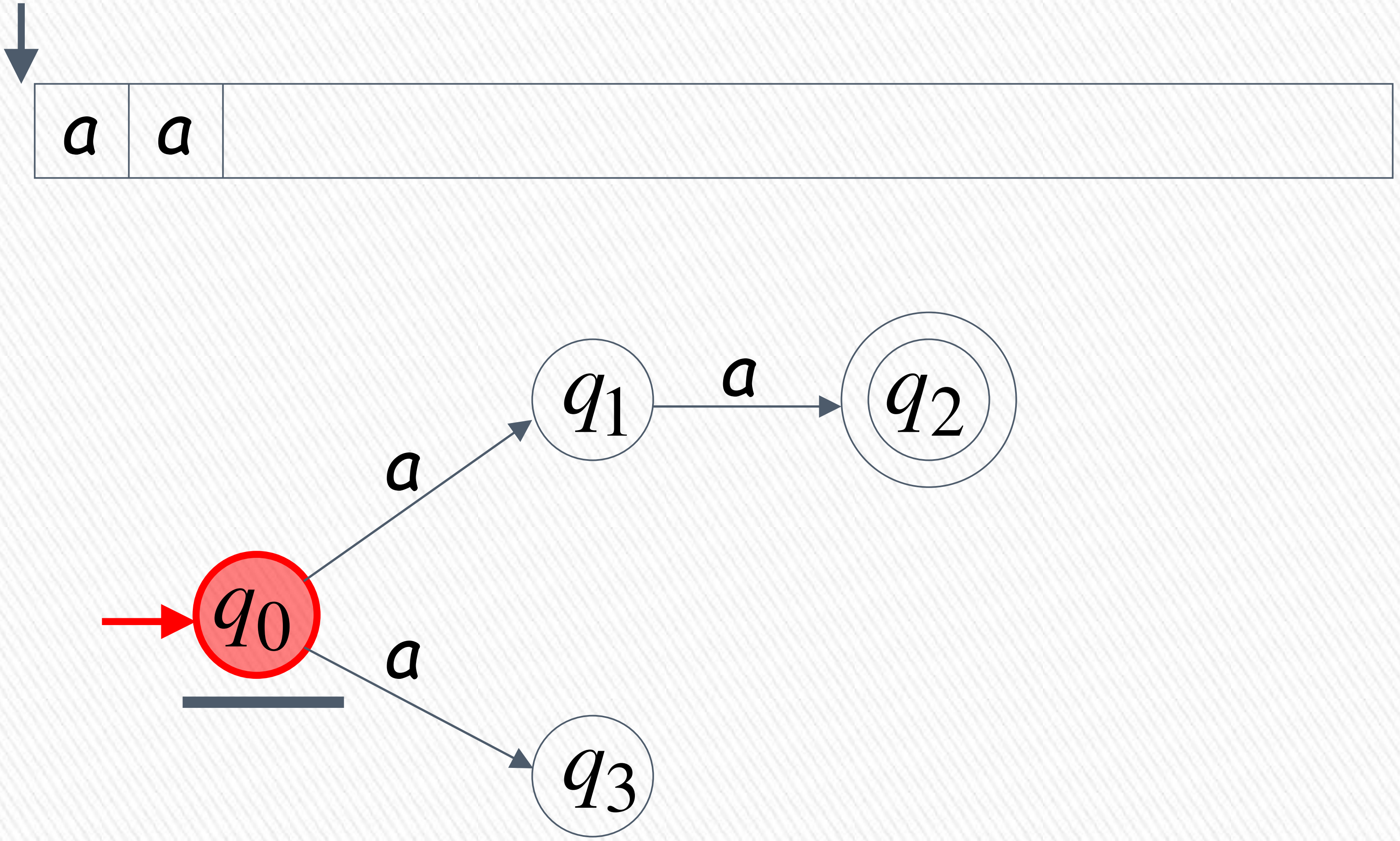


First Choice

All input is consumed

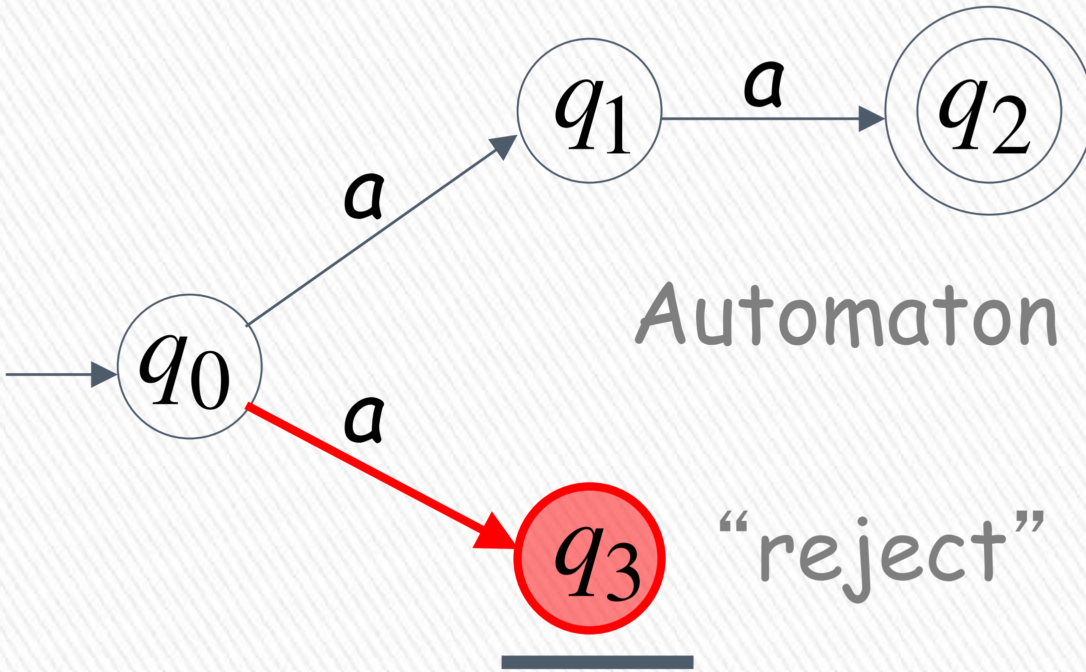


Second Choice



Second Choice

Input cannot be consumed



Automaton Halts

"reject"

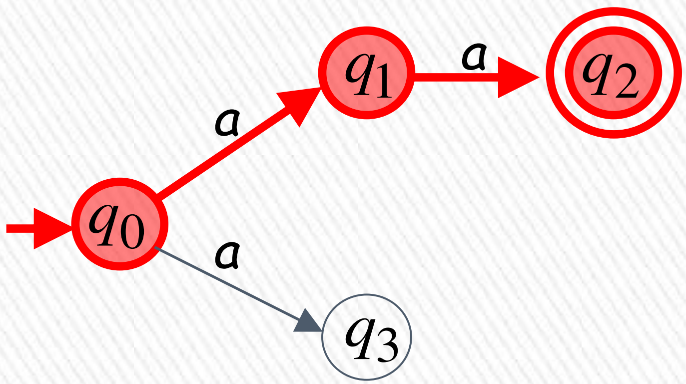
An NFA accepts a string:

if there exists a computation of the NFA that accepts the string

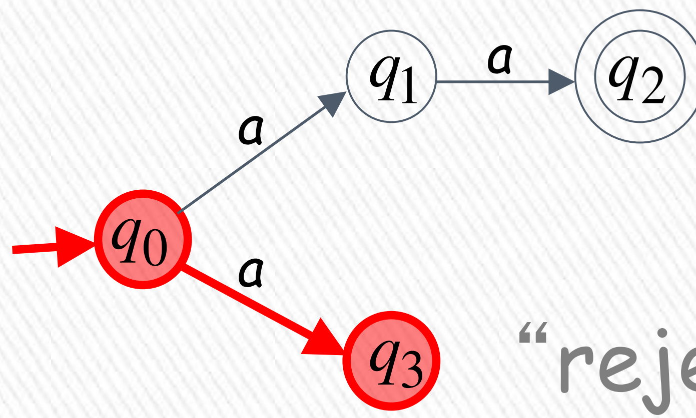
i.e., all the input string is processed and the automaton is in an accepting state

aa is accepted by the NFA:

"accept"



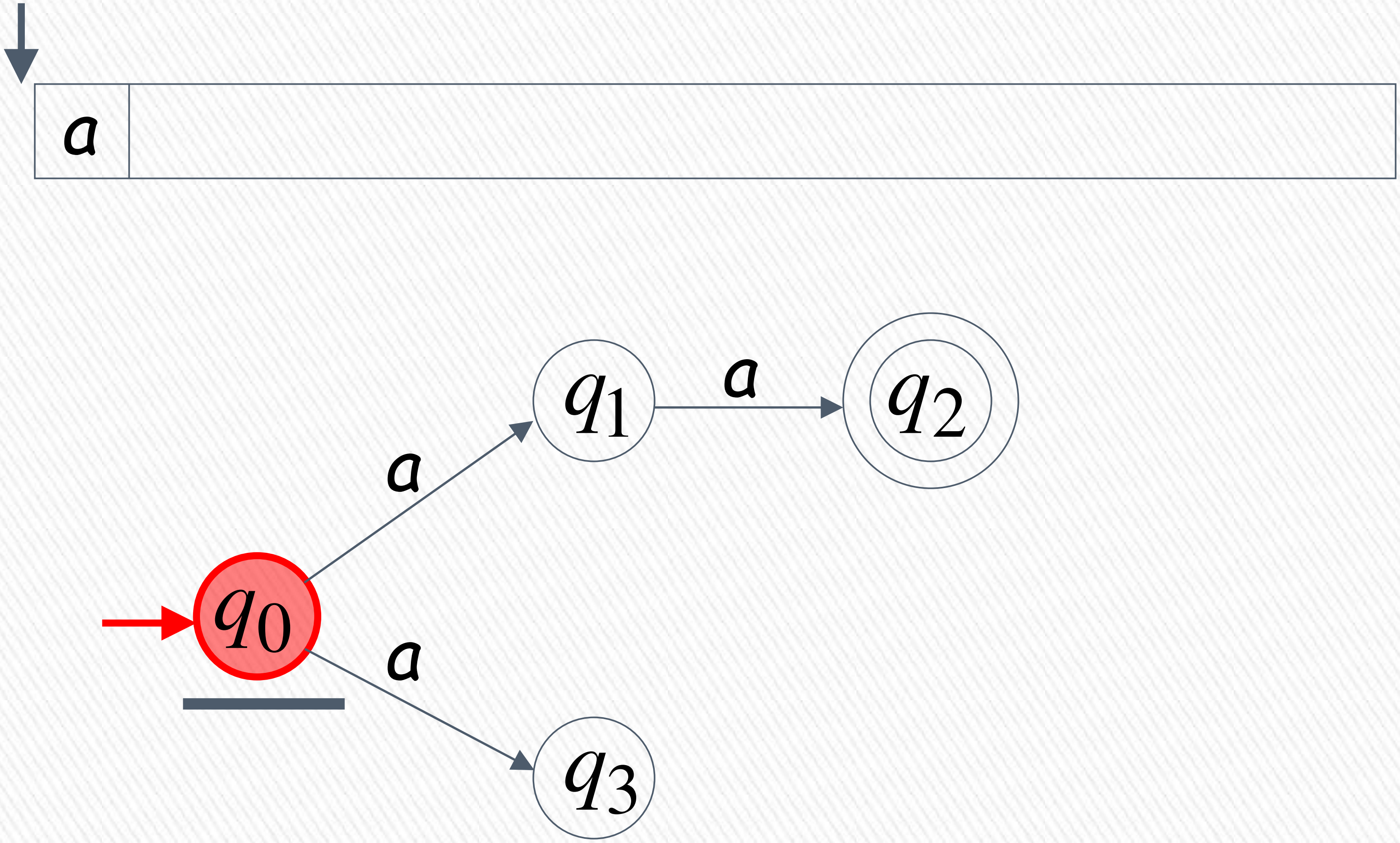
because this computation accepts aa



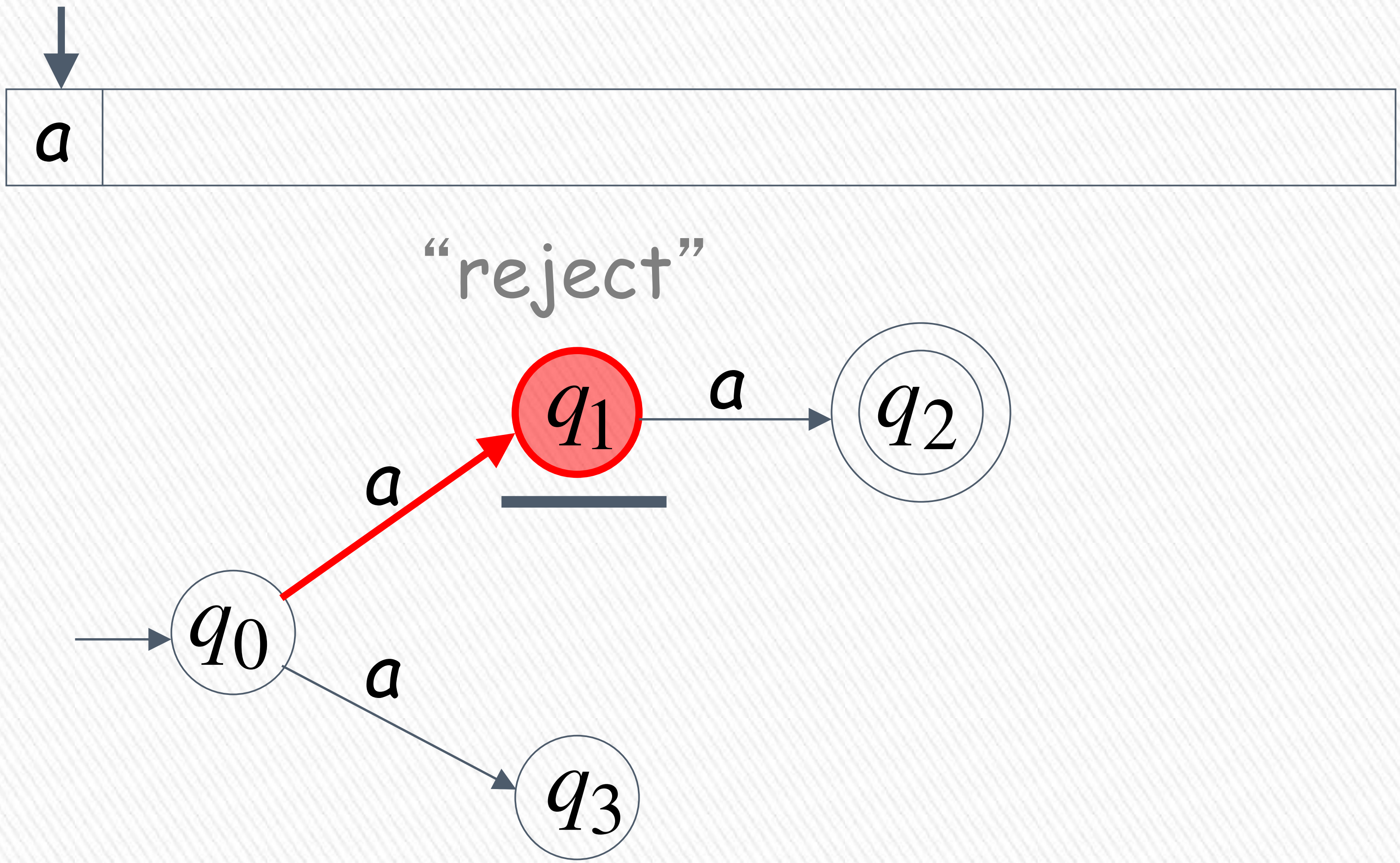
"reject"

this computation is ignored

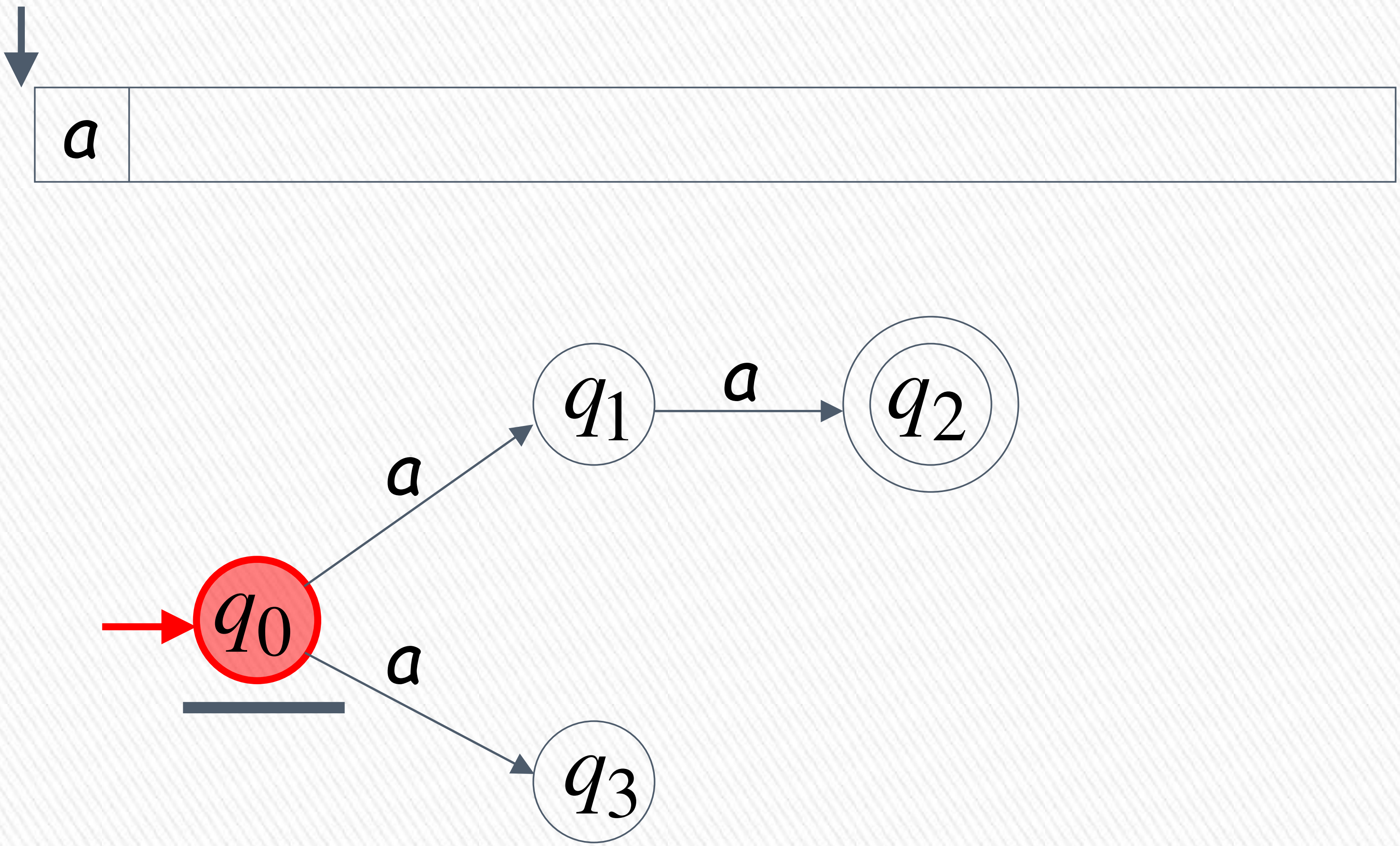
Rejection example



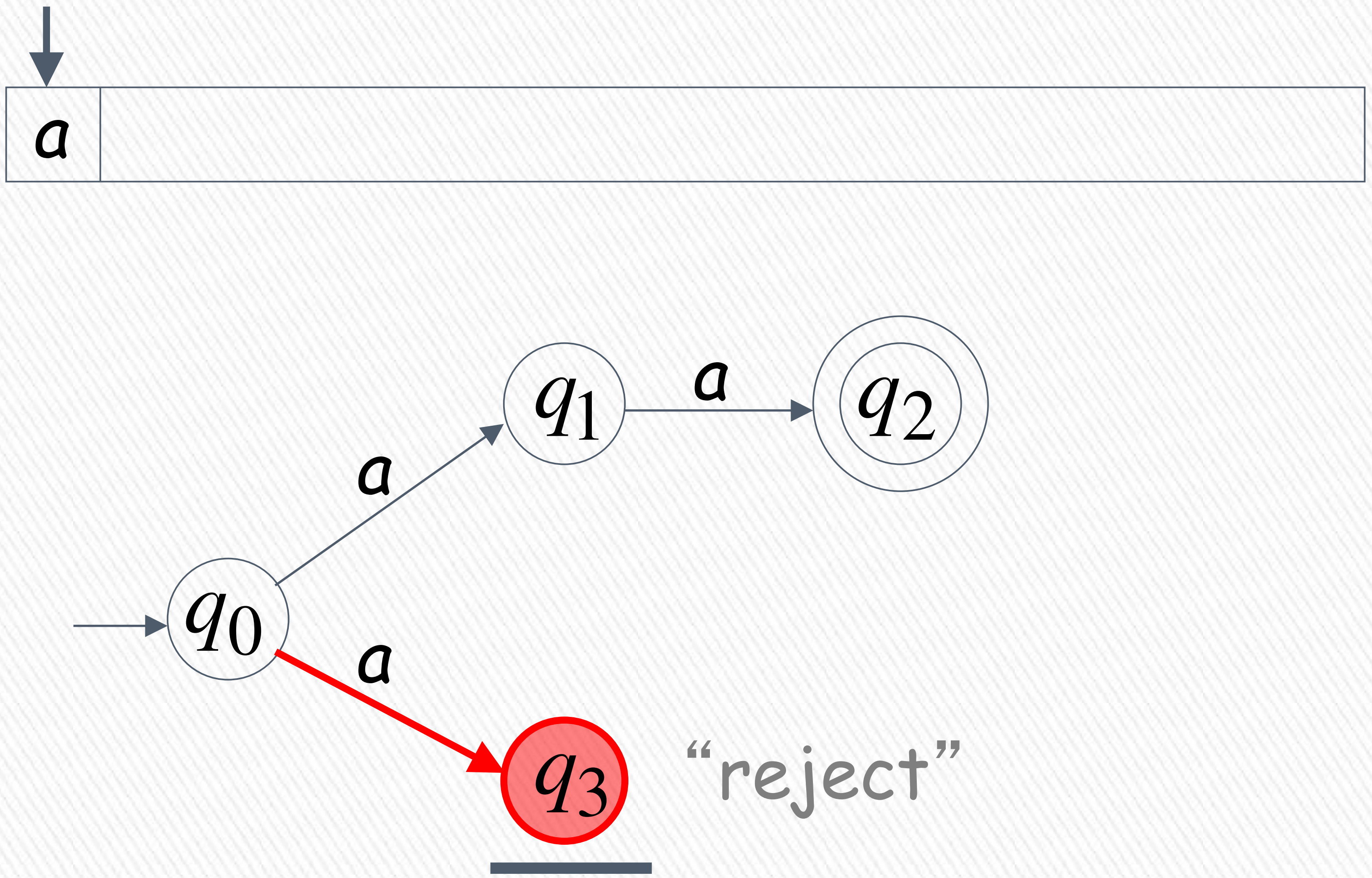
First Choice



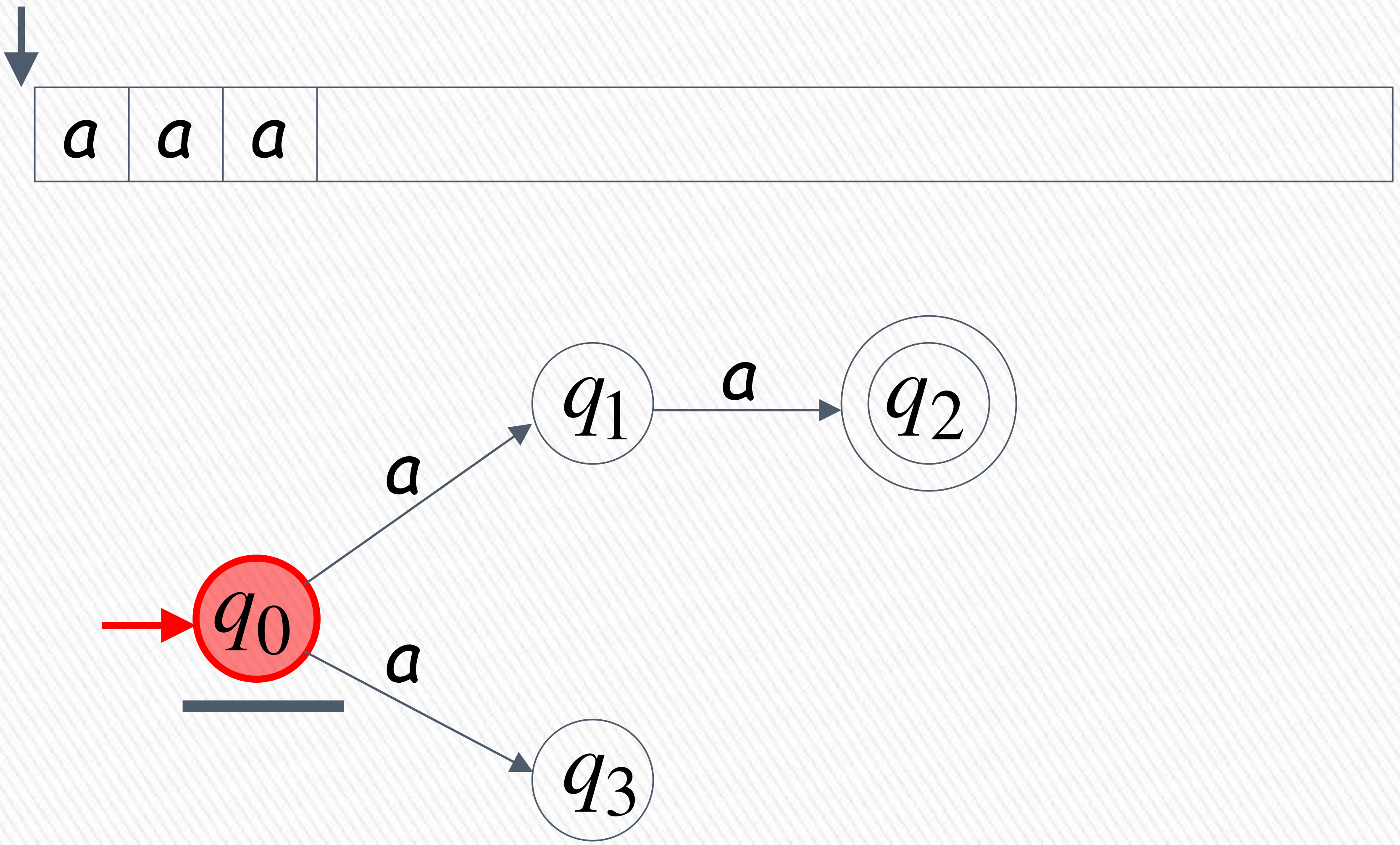
Second Choice



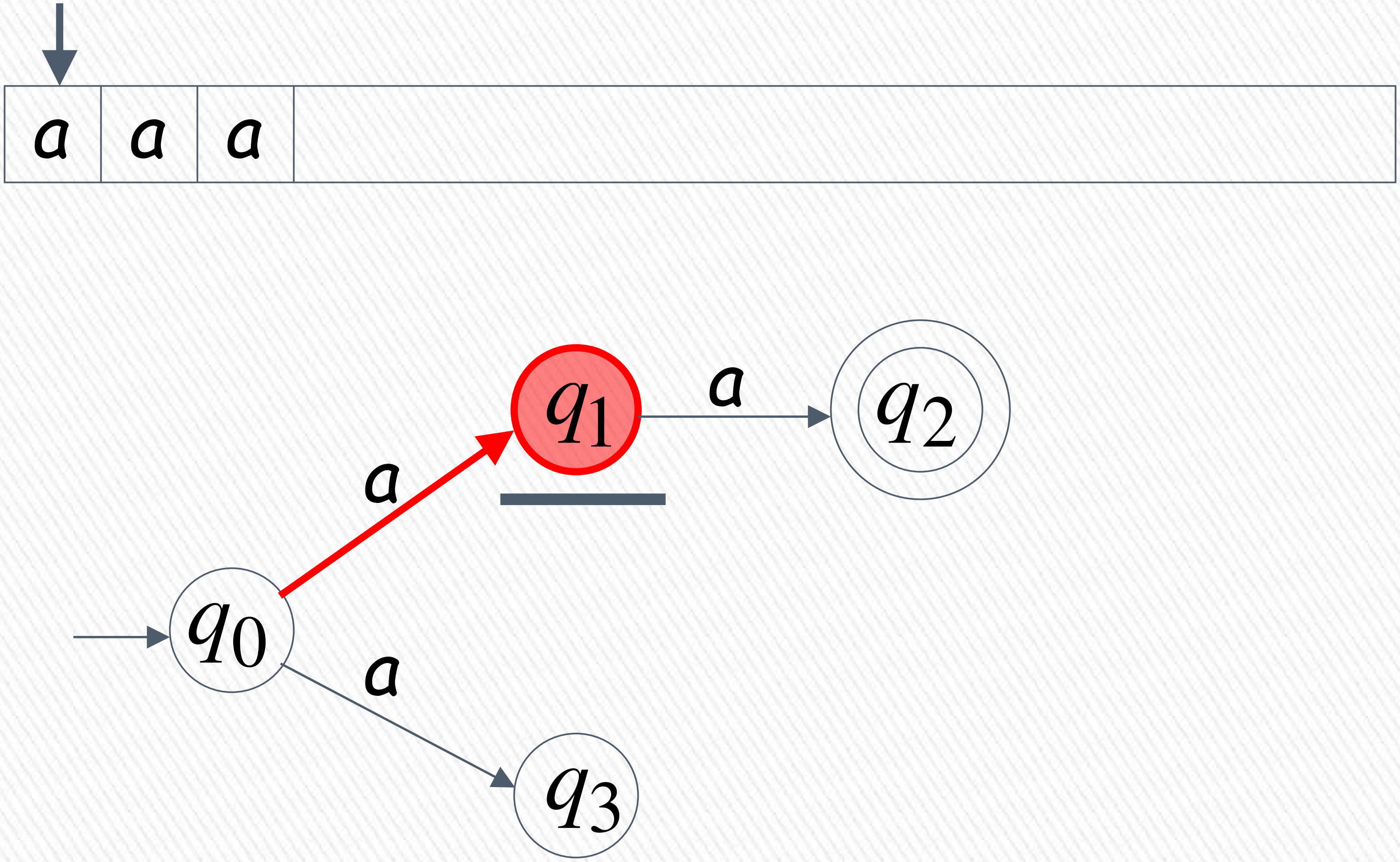
Second Choice



Another Rejection example

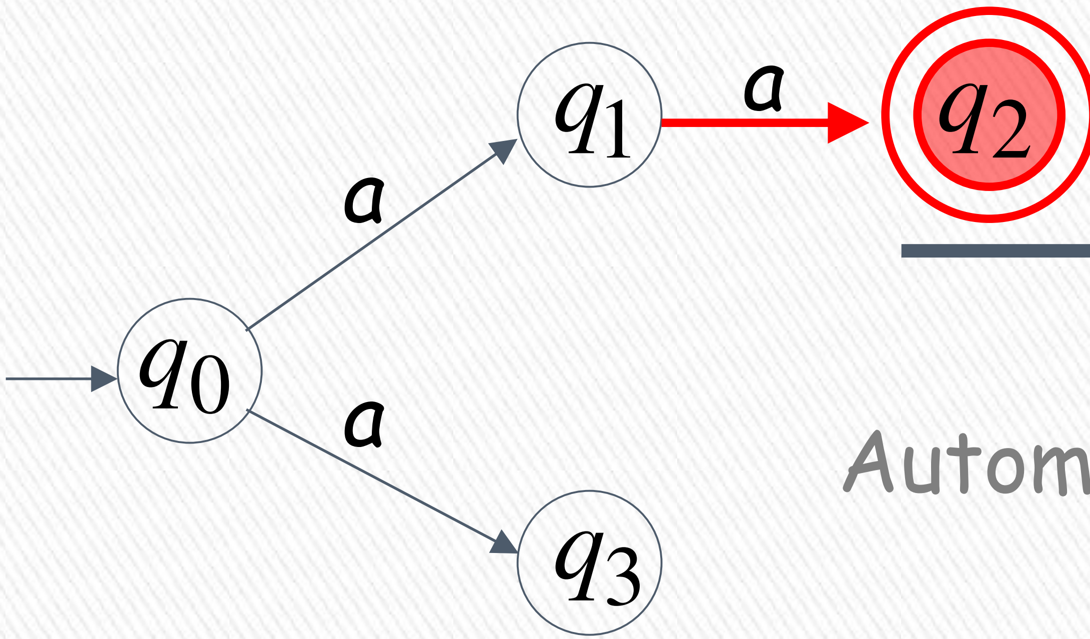


First Choice



First Choice

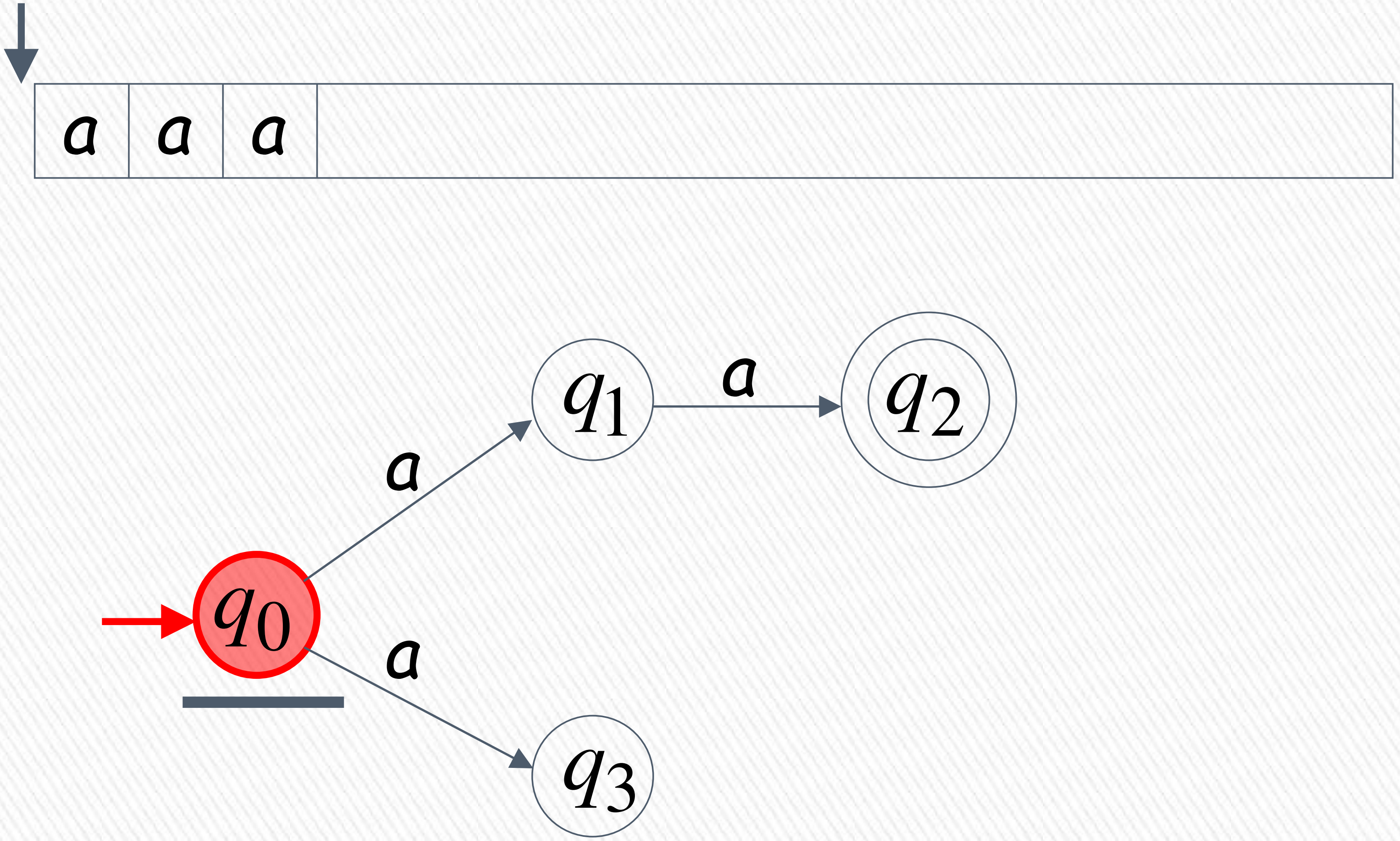
Input cannot be consumed



"reject"

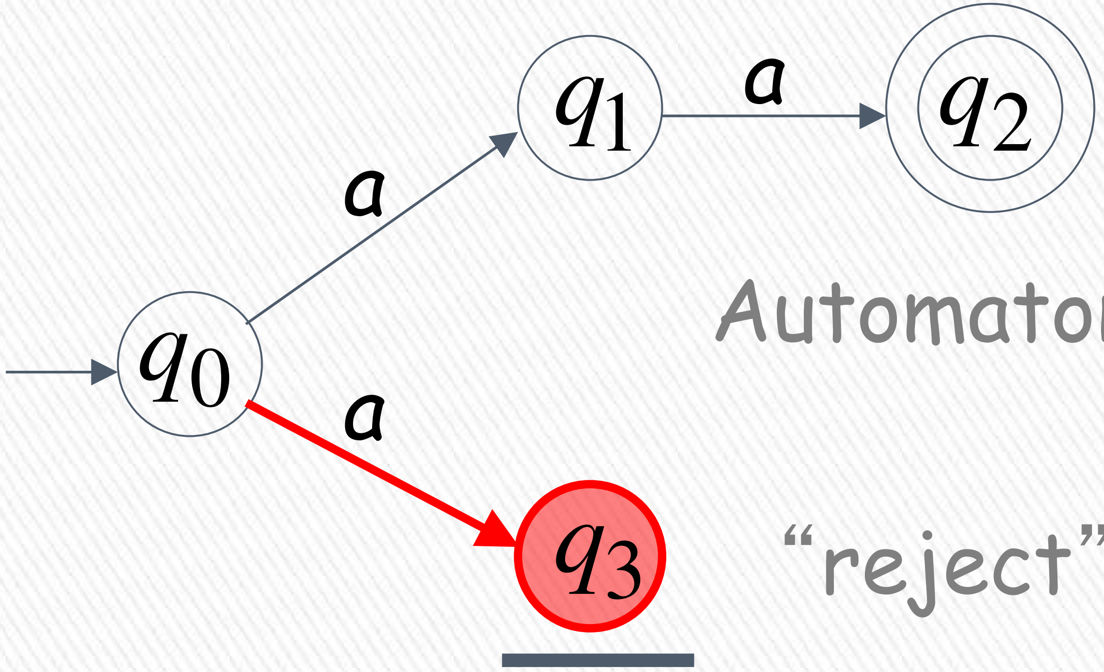
Automaton halts

Second Choice



Second Choice

Input cannot be consumed



Automaton halts

"reject"

An NFA rejects a string:

if there is no computation of the NFA that accepts the string.

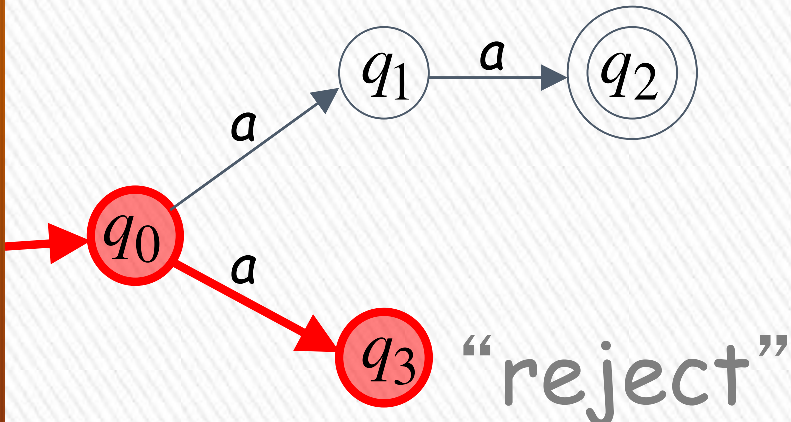
For each computation:

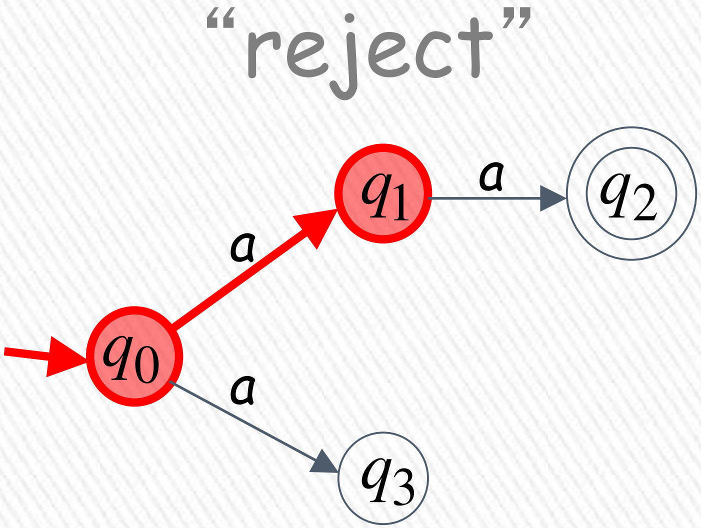
- All the input is consumed and the automaton is in a non final state

OR

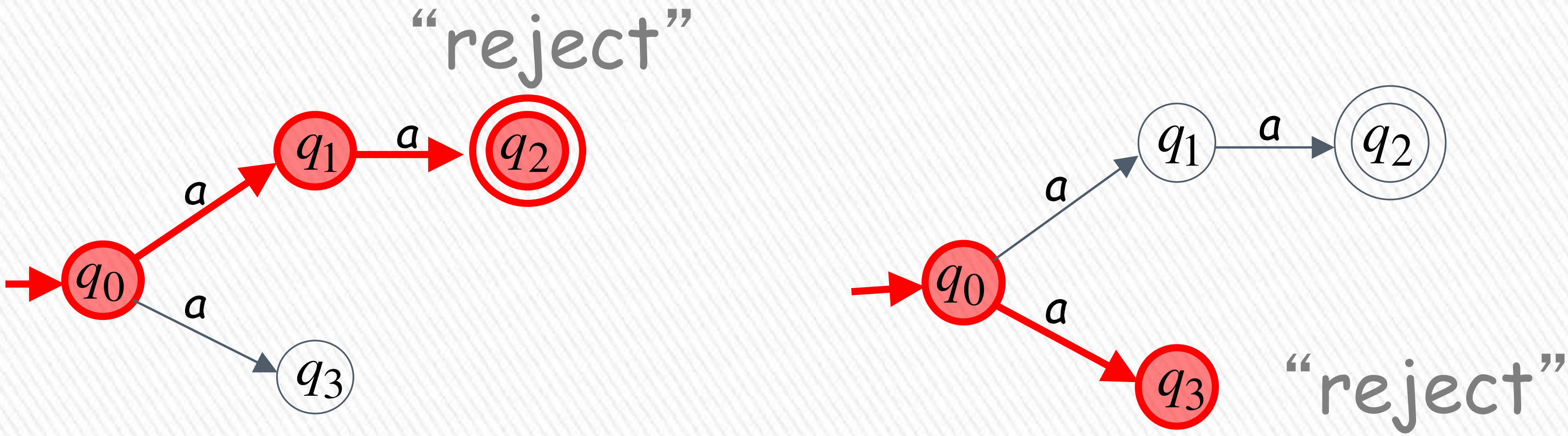
- The input cannot be consumed

a is rejected by the NFA:



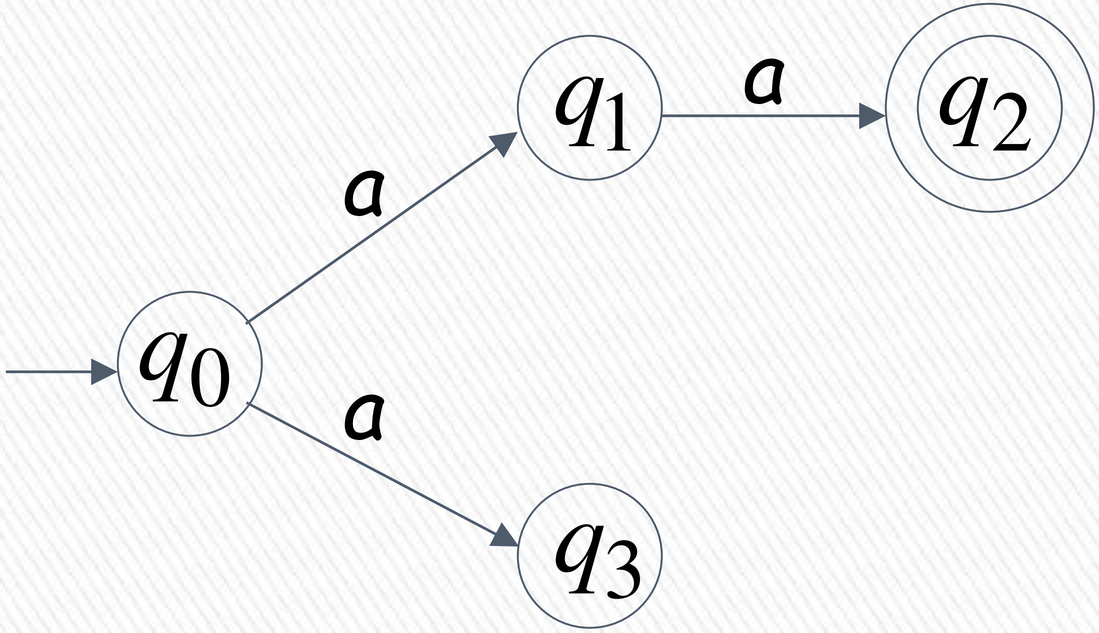


All possible computations lead to rejection aaa is rejected by the NFA:

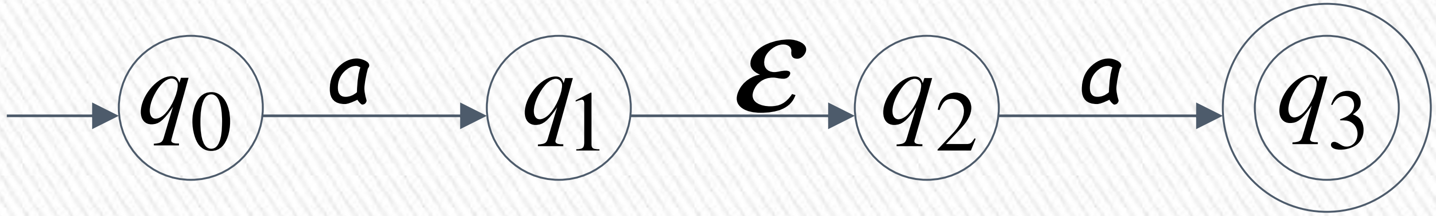


All possible computations lead to rejection

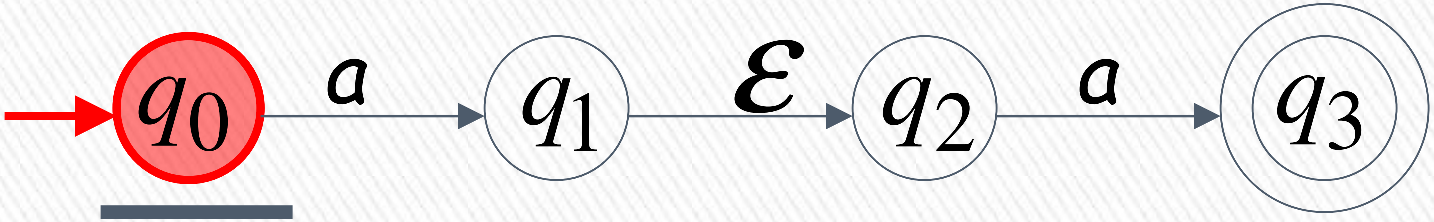
Language accepted: $L = \{aa\}$

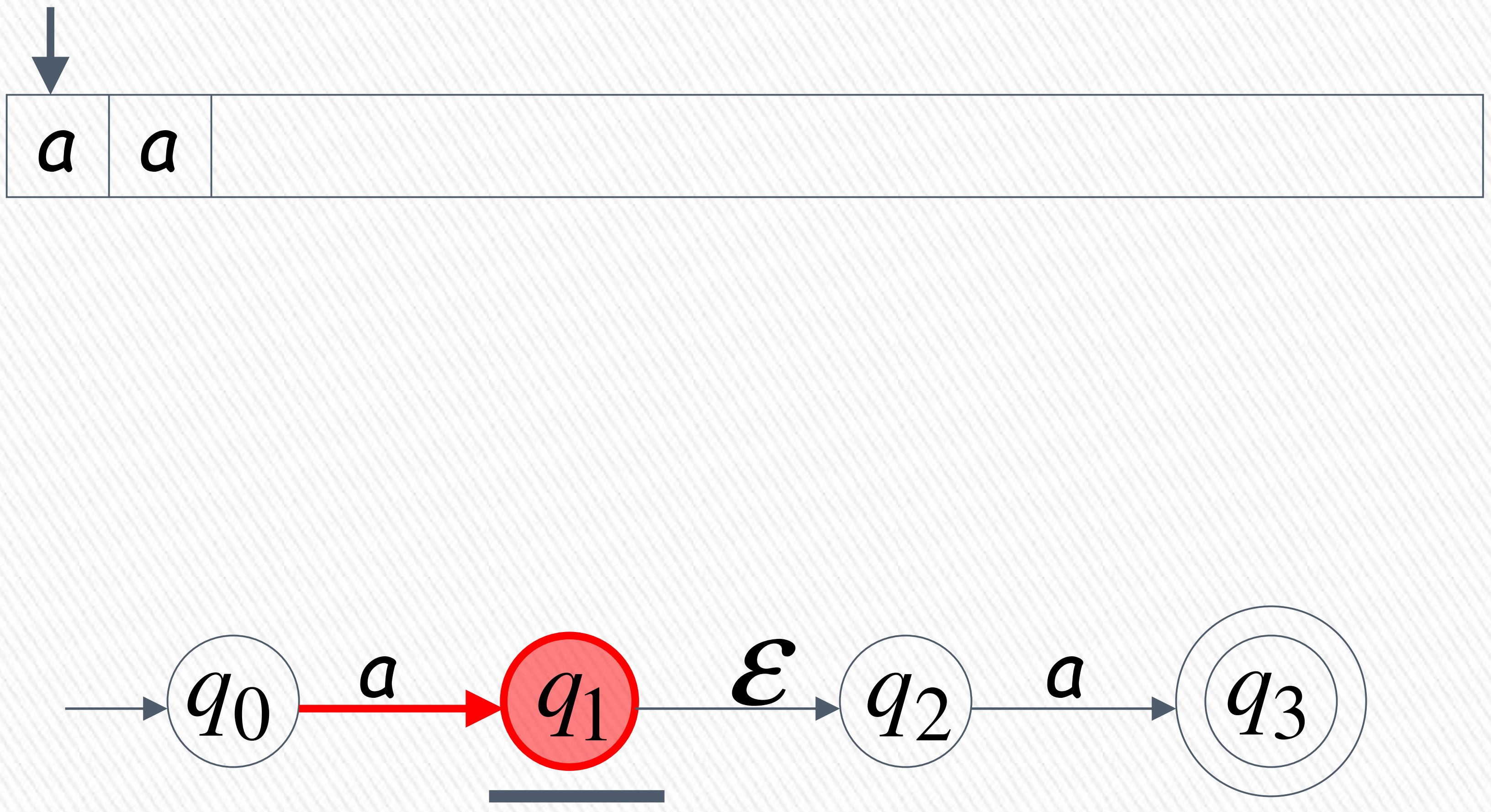


Epsilon Transition

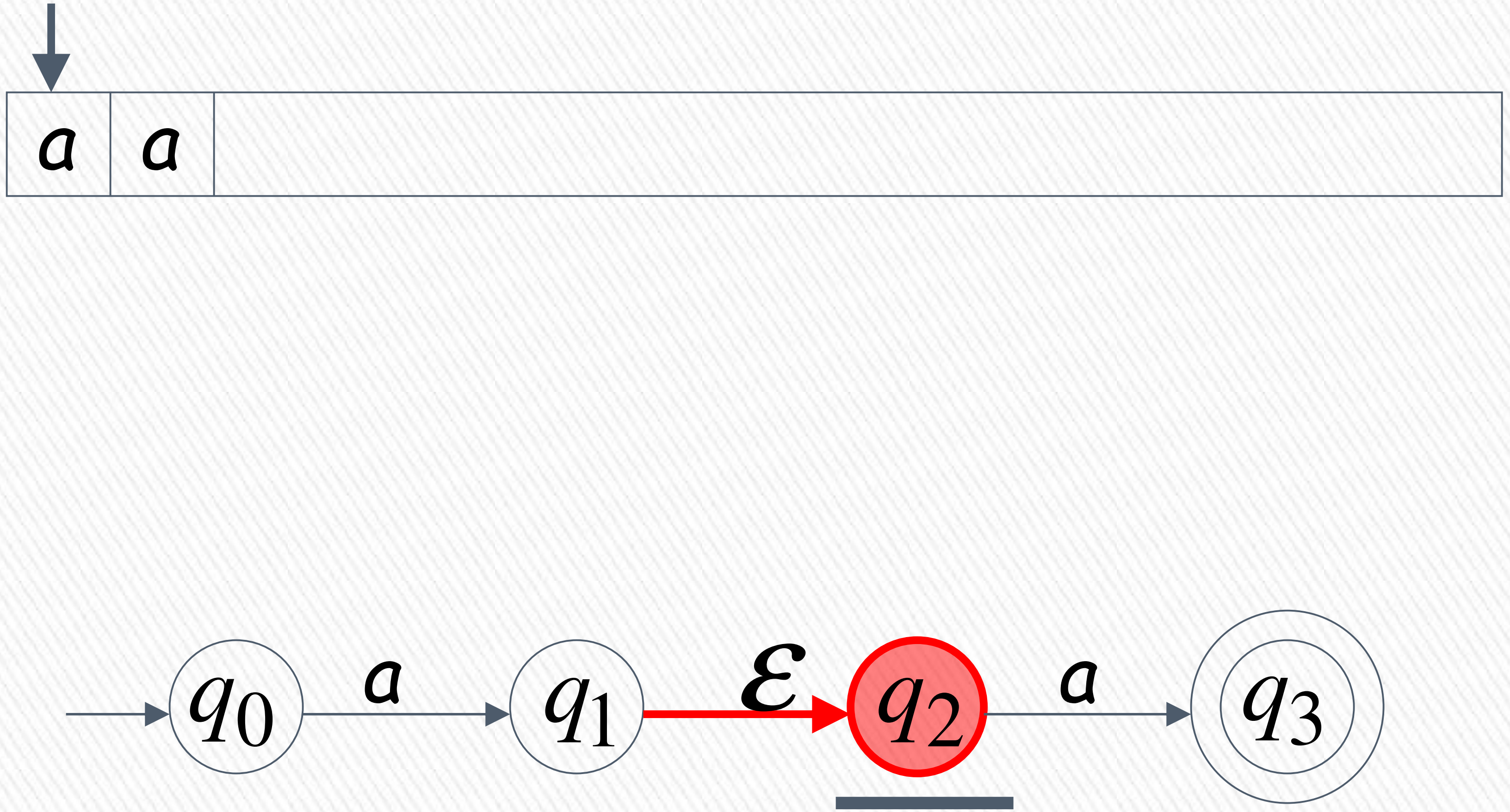



a
a

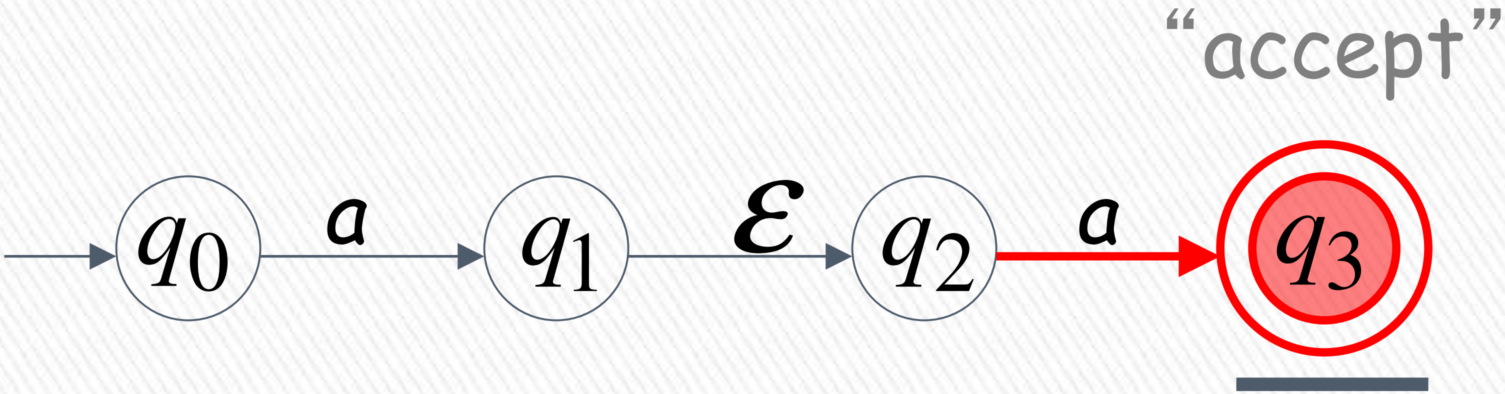

q_0
a
q_1
ε
q_2
a
q_3


a
a
a
ε
a
q_0
q_1
q_2
q_3

input tape head does not move

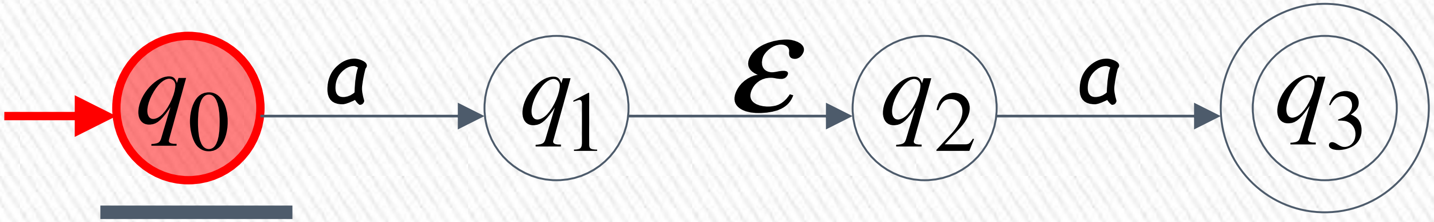


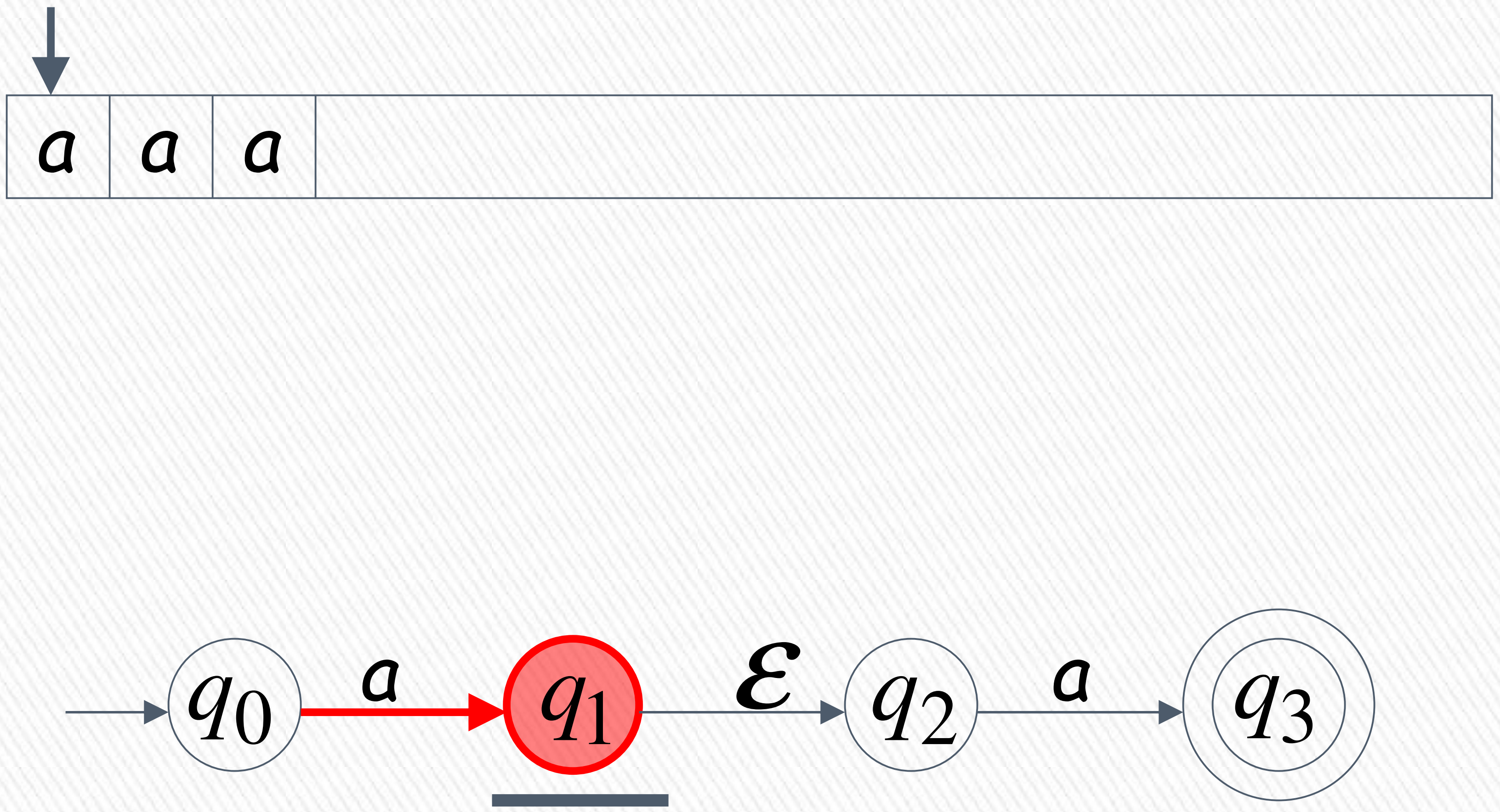
all input is consumed



String aa is accepted

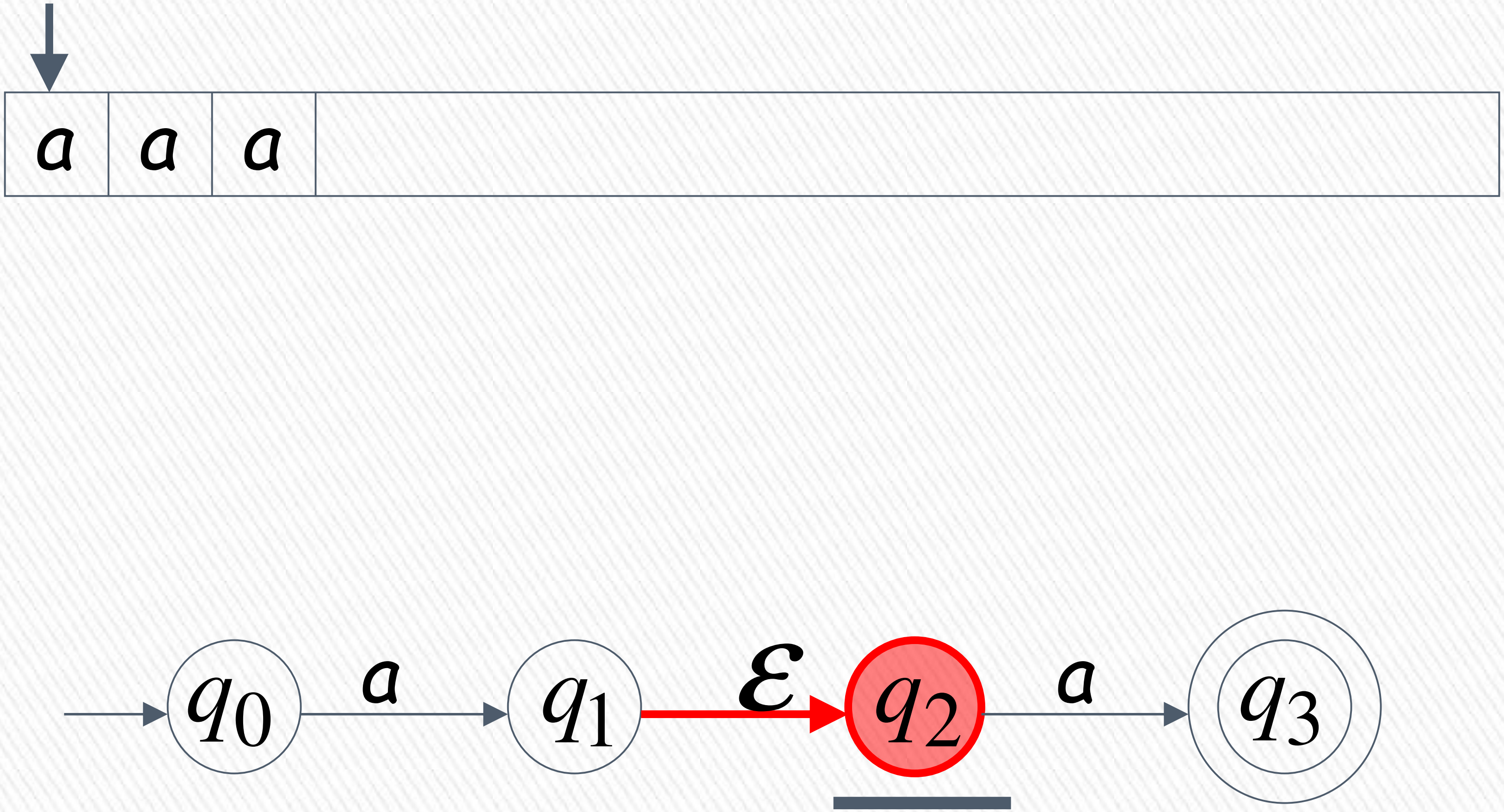
Rejection Example





a a a
q0 a q1 ε q2 a q3

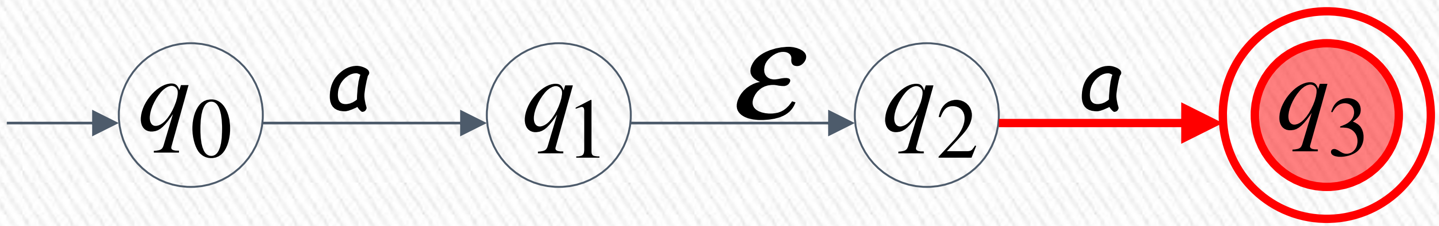
(read head doesn't move)



Input cannot be consumed

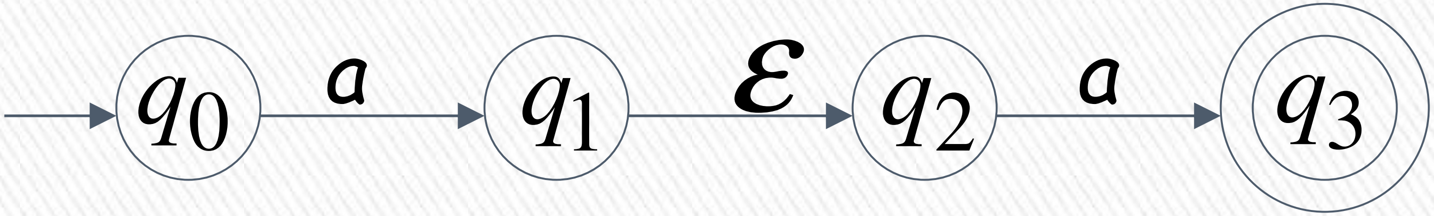
Automaton halts

“reject”

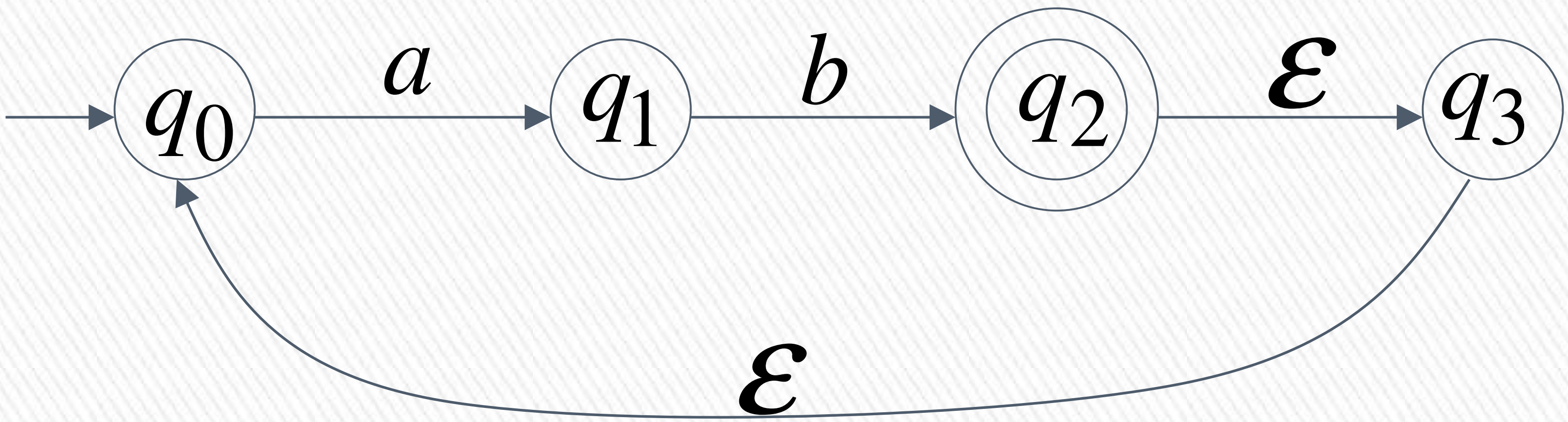


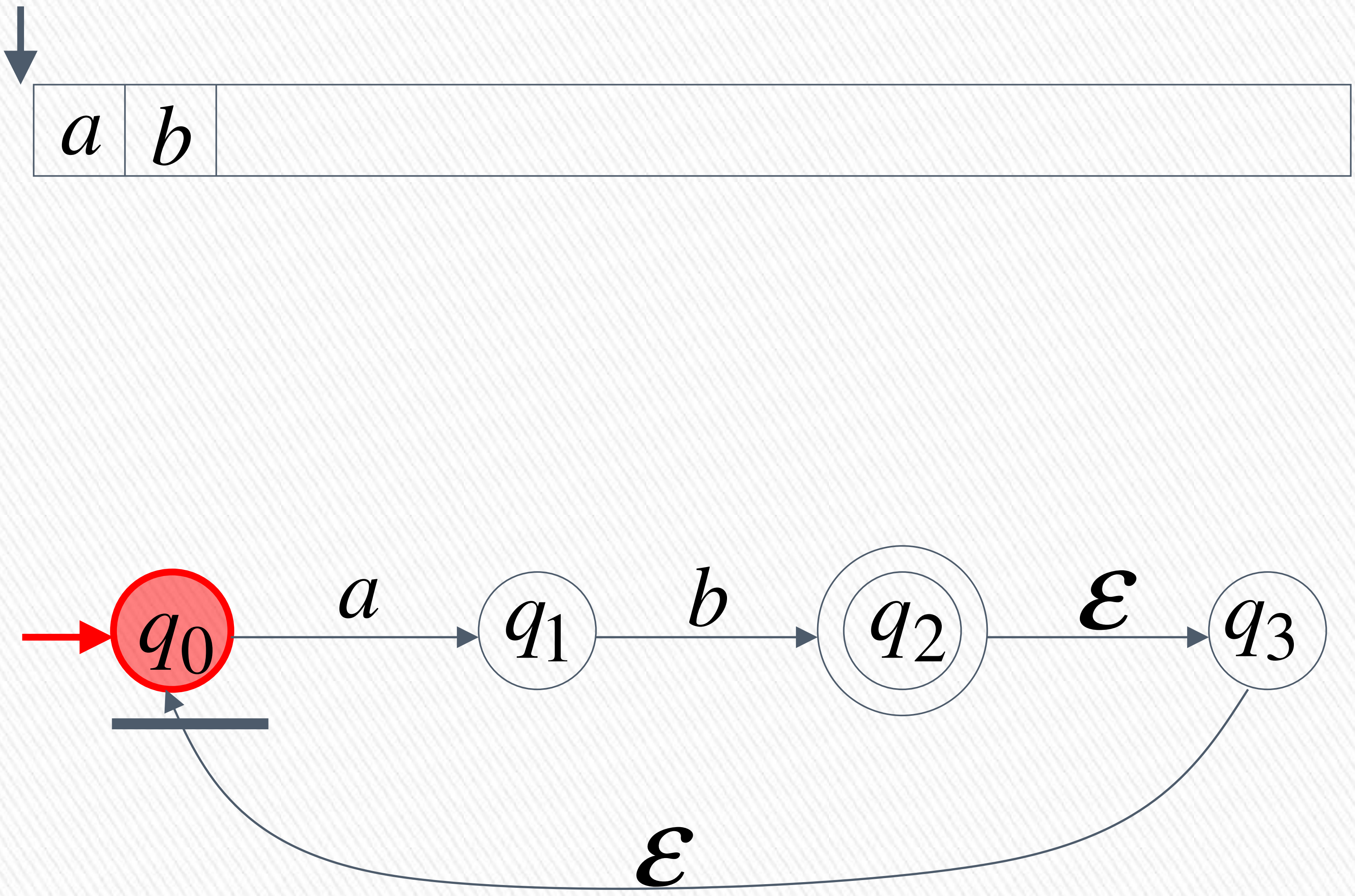
String **aaa** is rejected

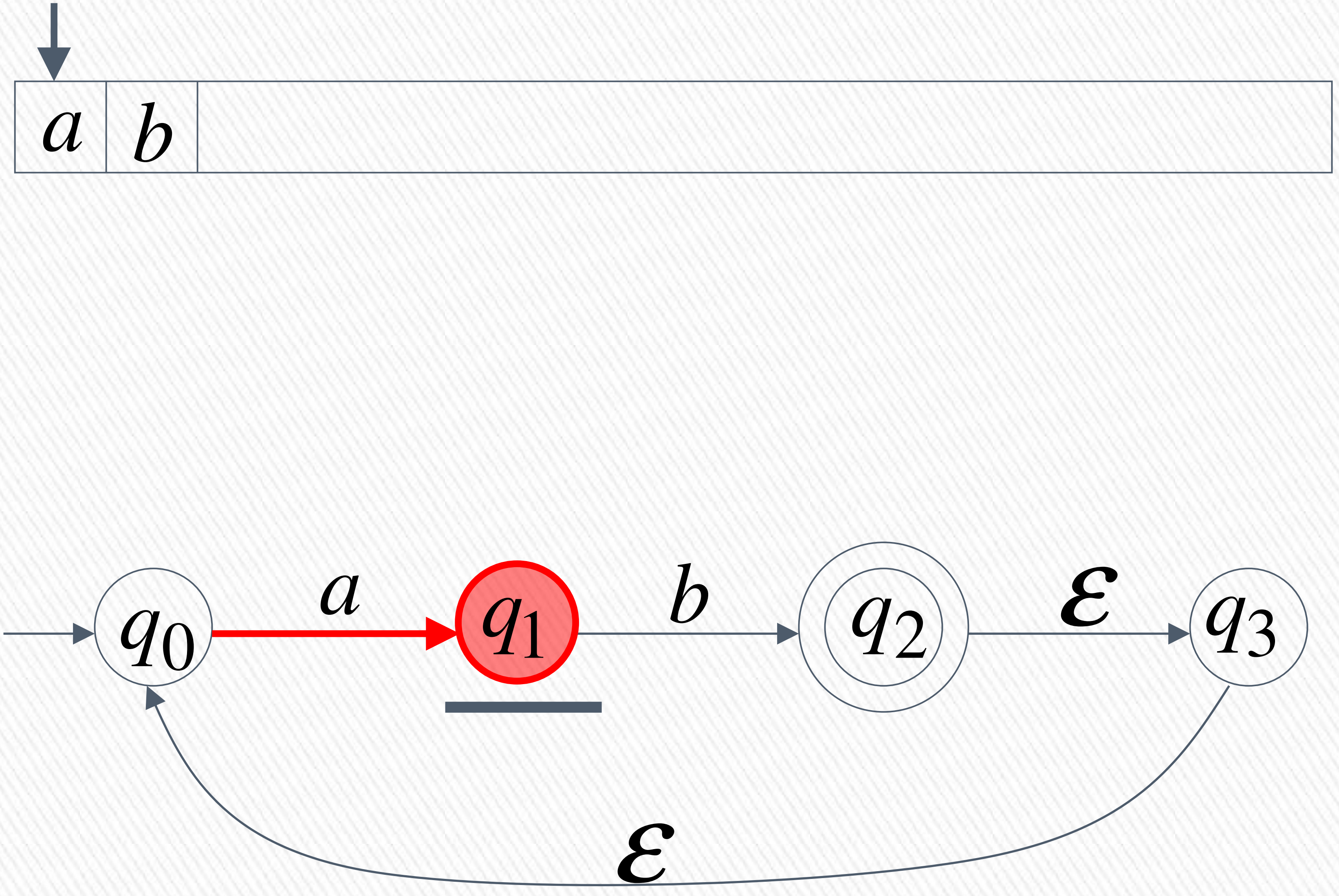
Language accepted: $L = \{aa\}$



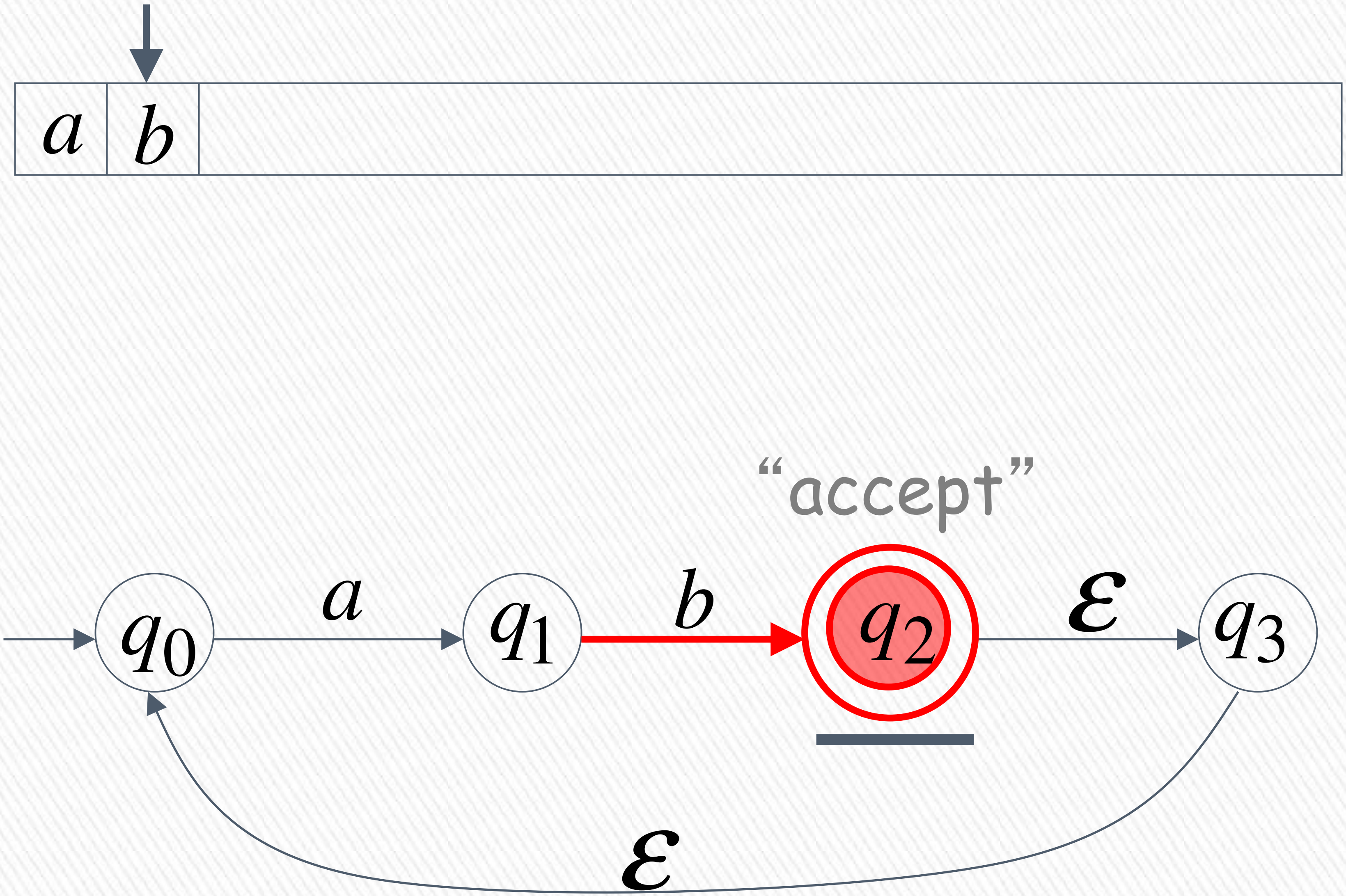
Another NFA Example




a
b
q_0
a
q_1
b
q_2
ε
q_3
ε

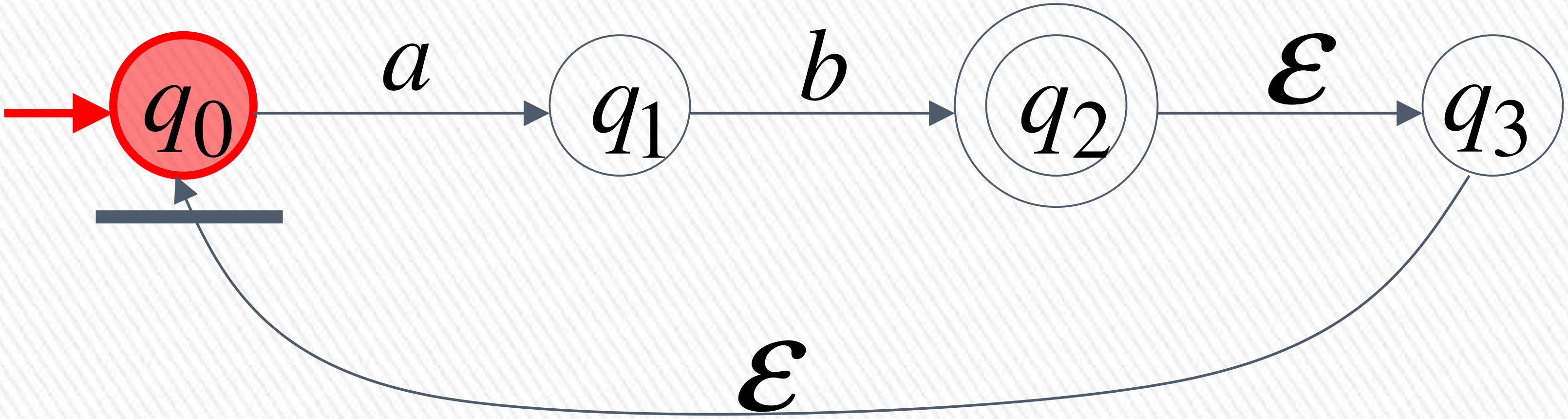


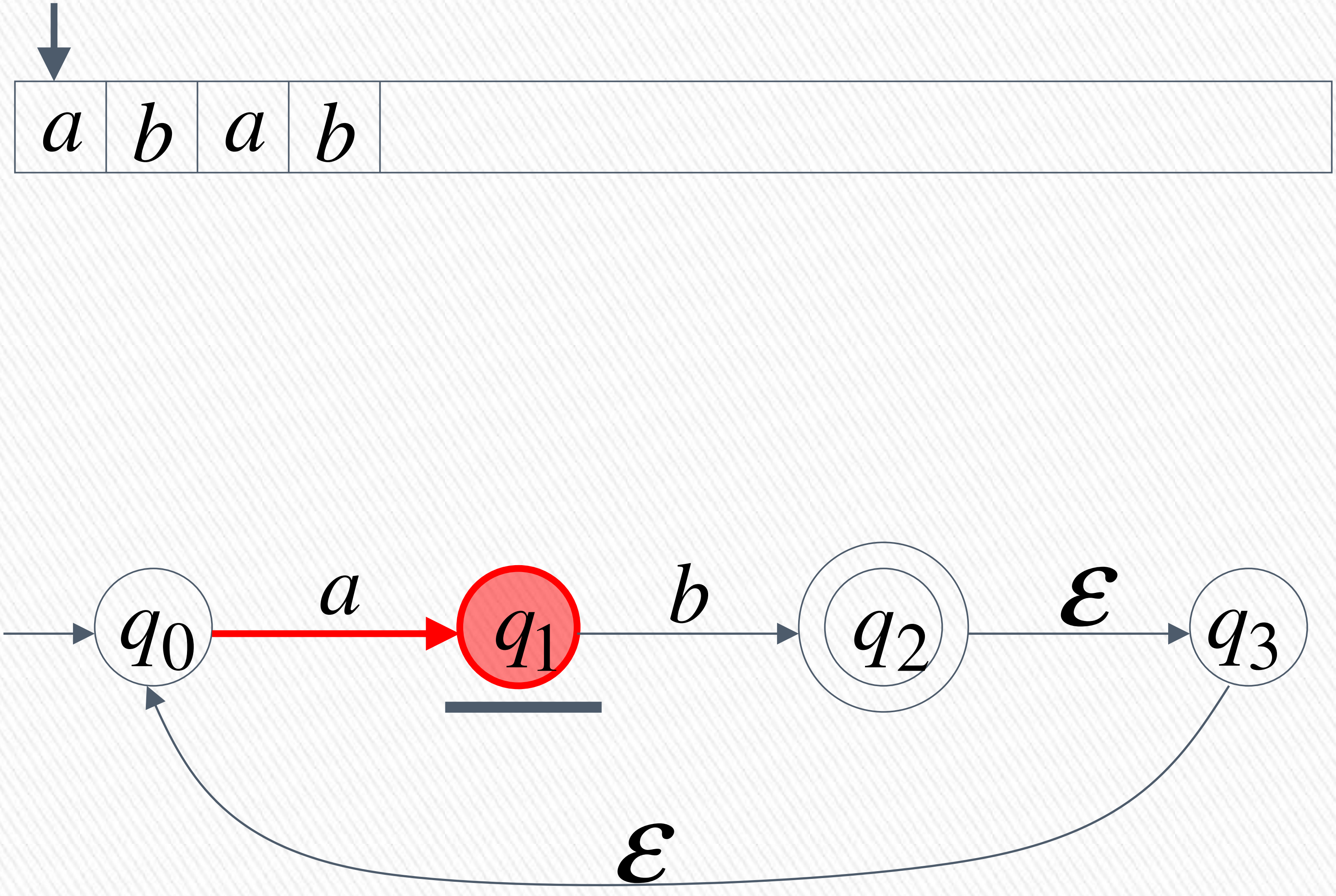
a
b
a
b
ε
ε
q_0
q_1
q_2
q_3



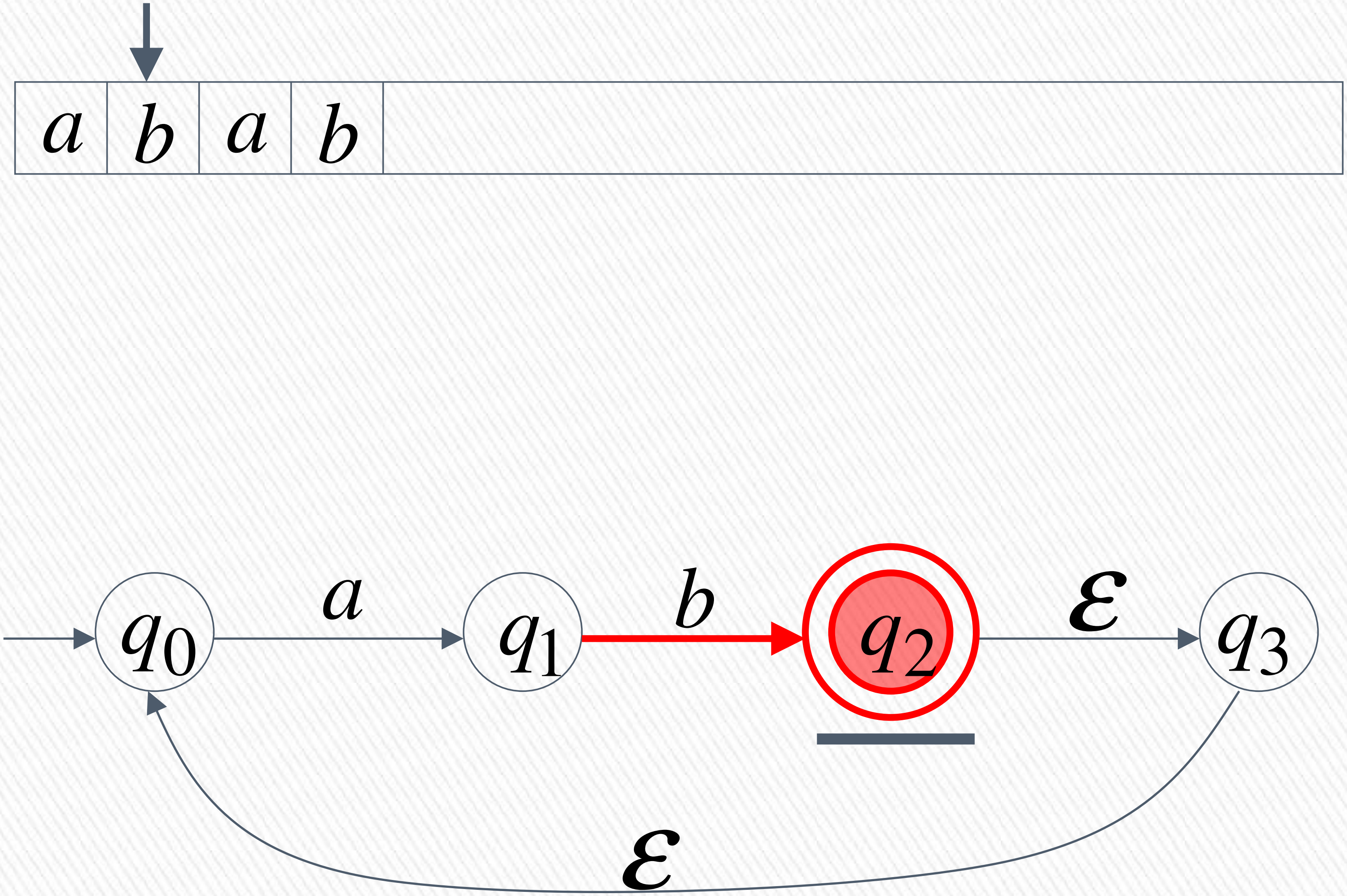
a
b
"accept"
q_0
a
q_1
b
q_2
ε
q_3
ε

Another String

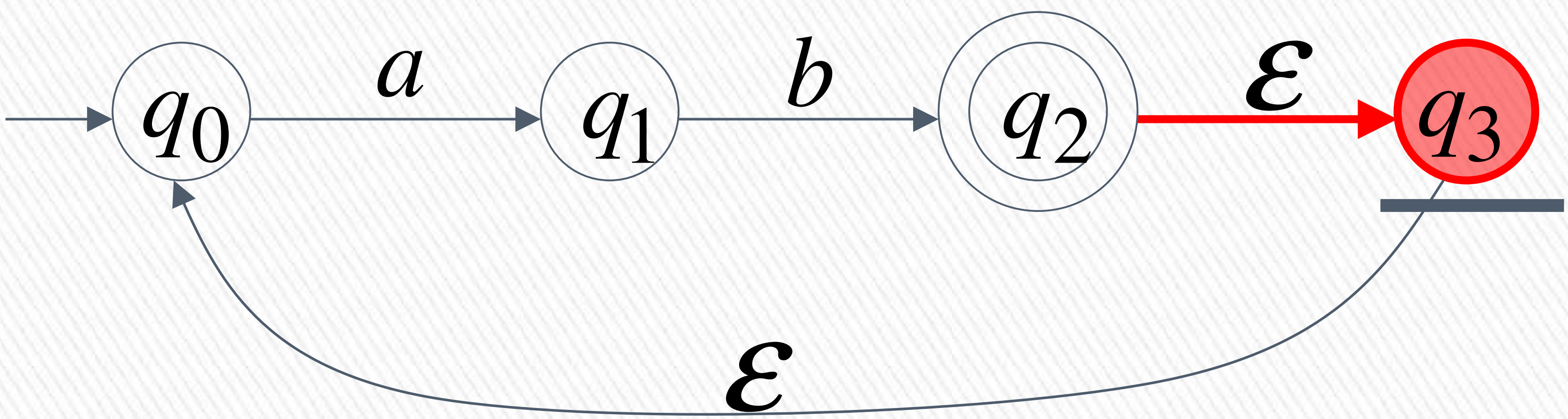




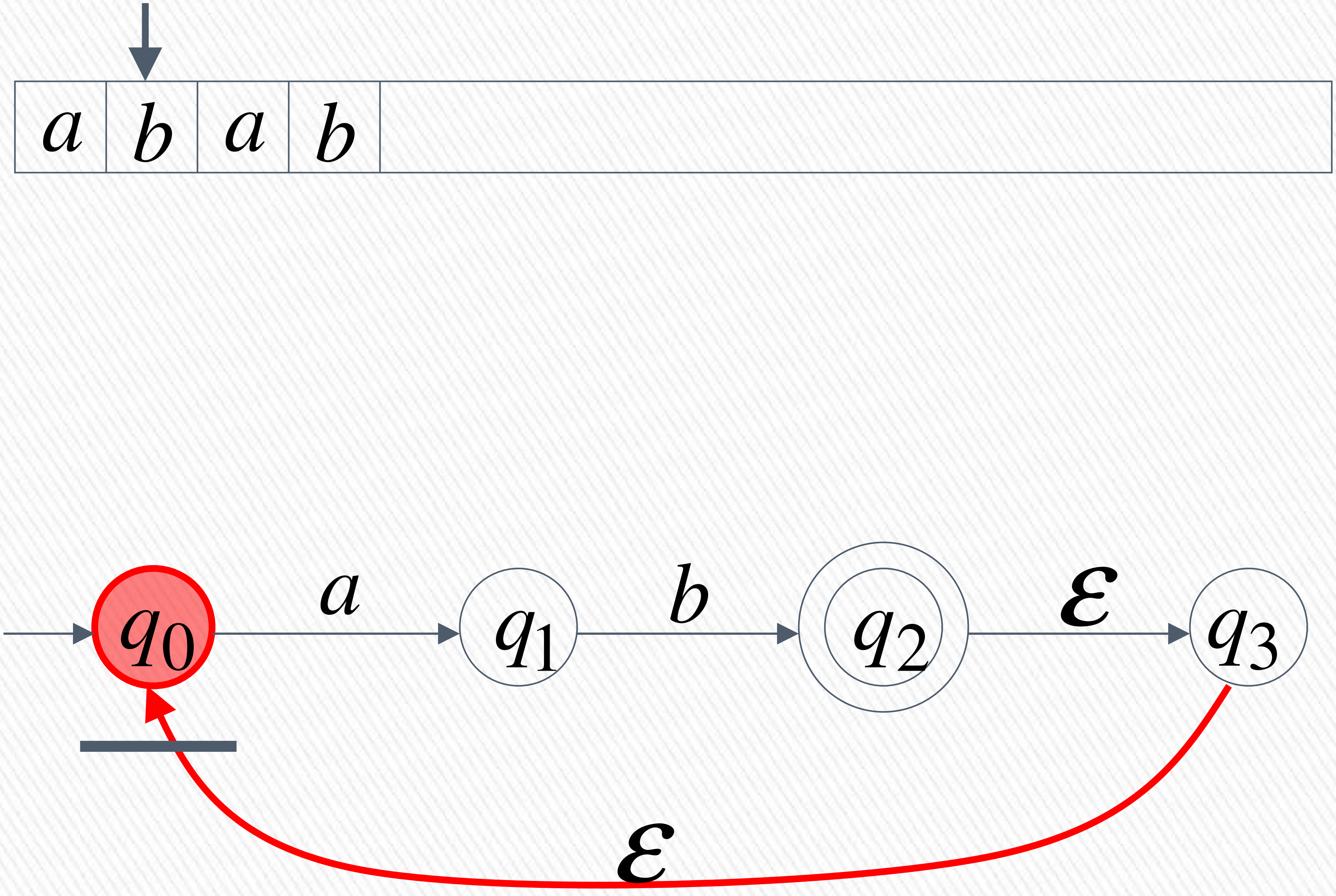
a
b
a
b
a
b
ε
ε
q_0
q_1
q_2
q_3


a b a b
a
b
ε
ε
q_0
q_1
q_2
q_3

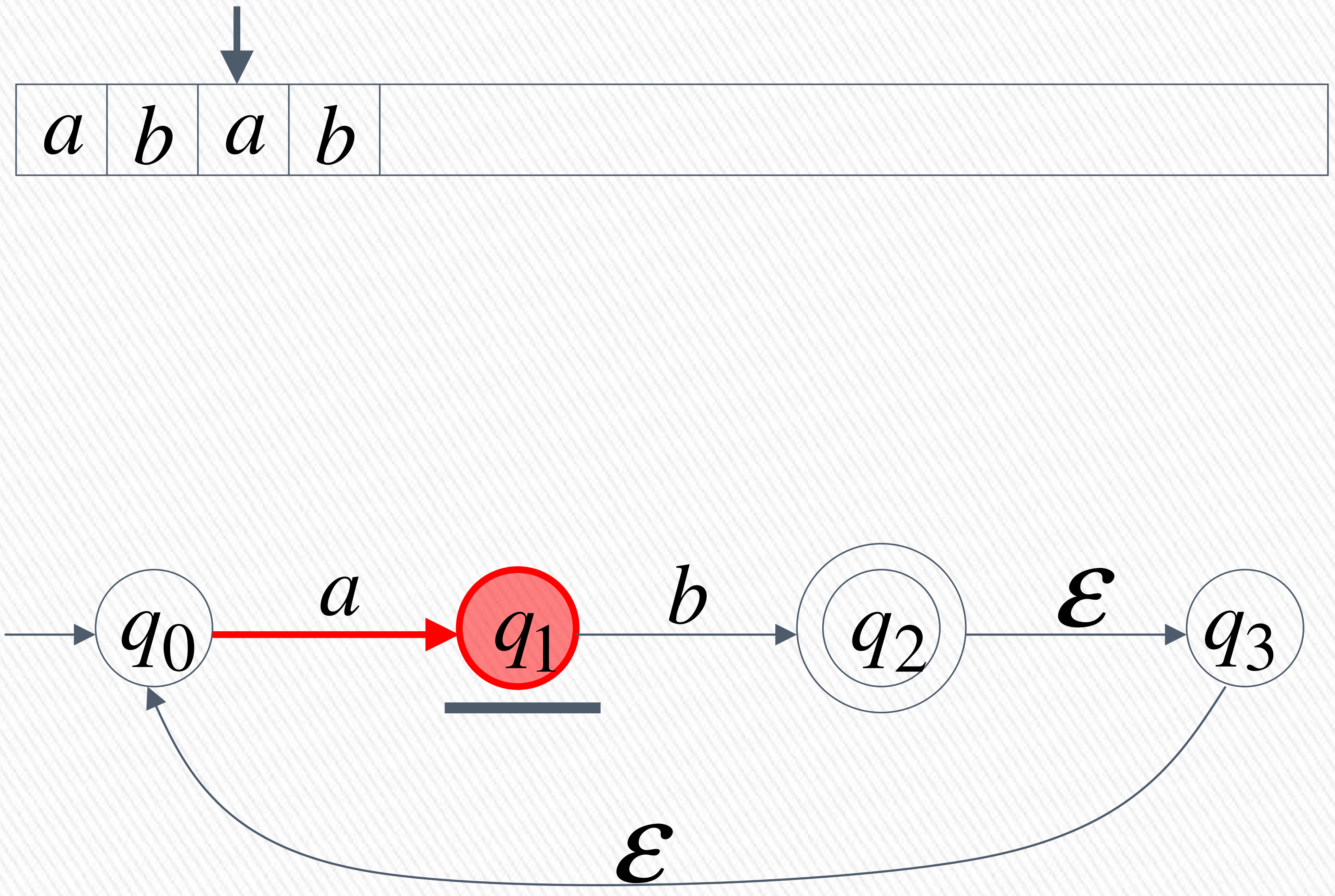
a
b
a
b



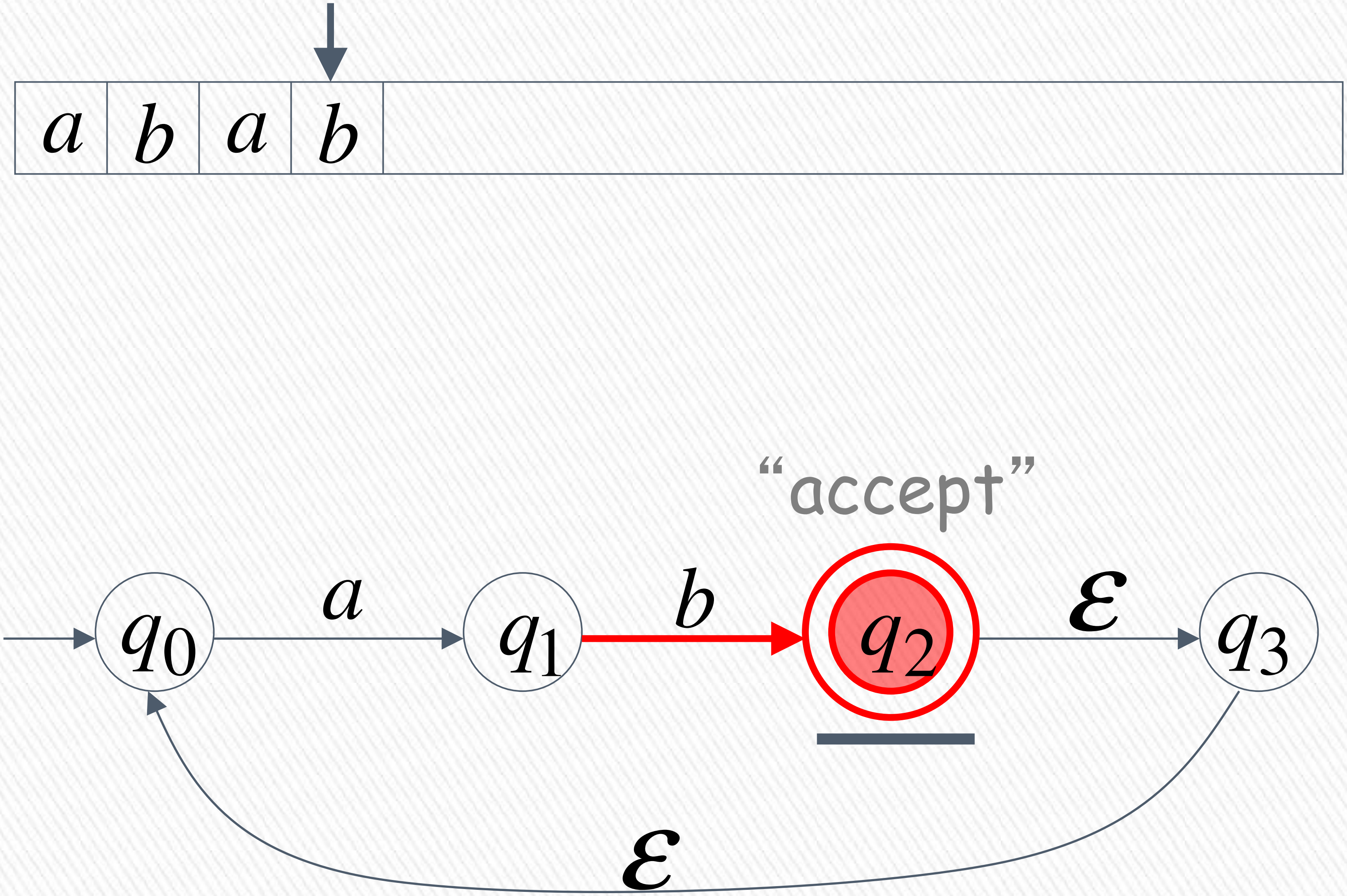
q_0
a
q_1
b
q_2
ε
q_3
ε



a b a b
a
b
ε
q0 q1 q2 q3
ε



a
b
a
b
a
b
ε
ε
q0
q1
q2
q3

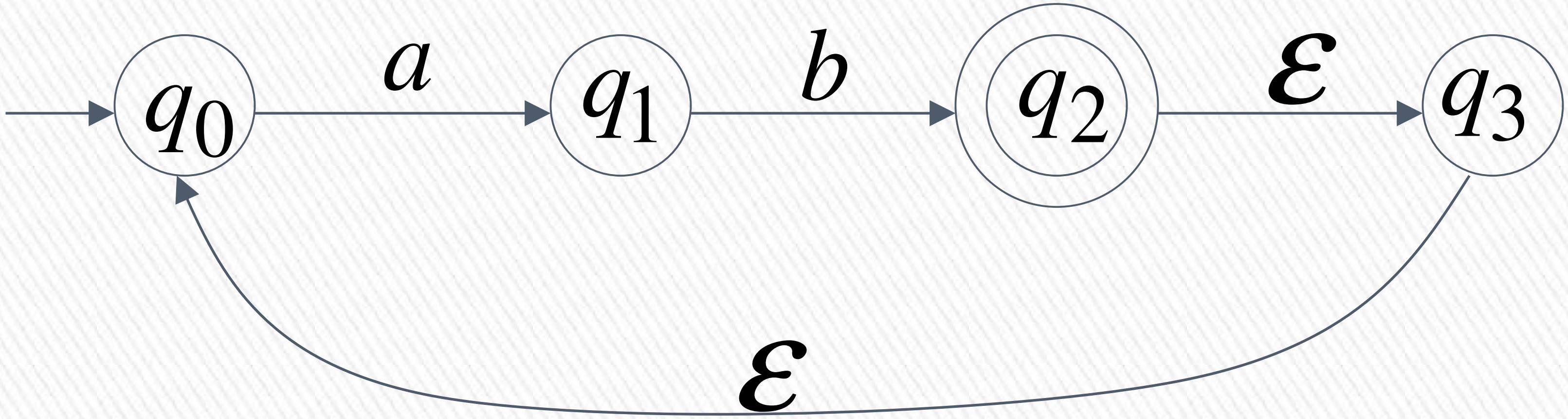


a
b
a
b
"accept"
q_0
q_1
q_2
q_3
a
b
ε
ε

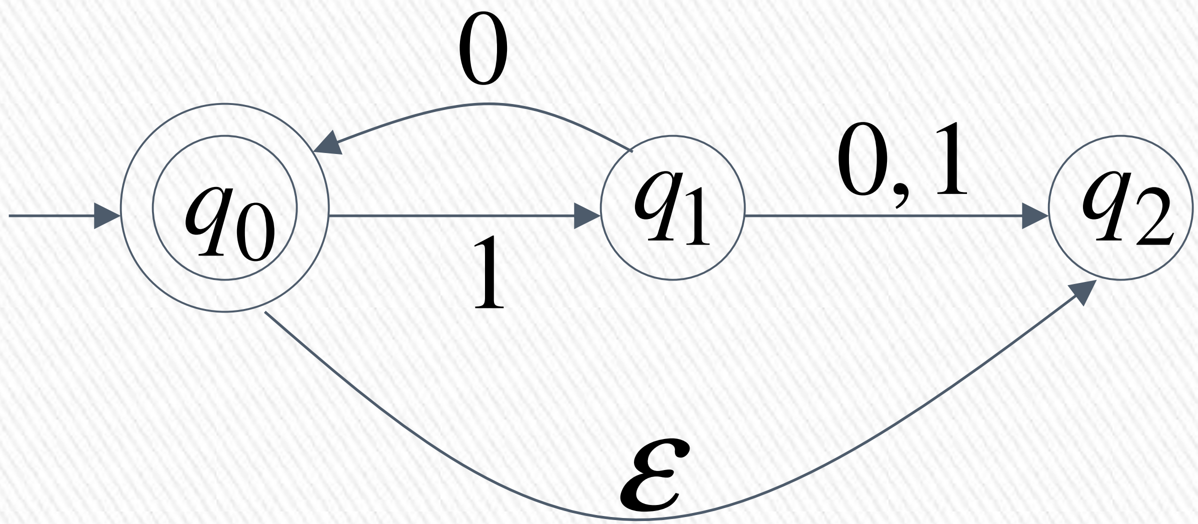
Language accepted

$$L = \{ab,\ abab,\ ababab,\ ...\}$$

$$= \{ab\}^+$$

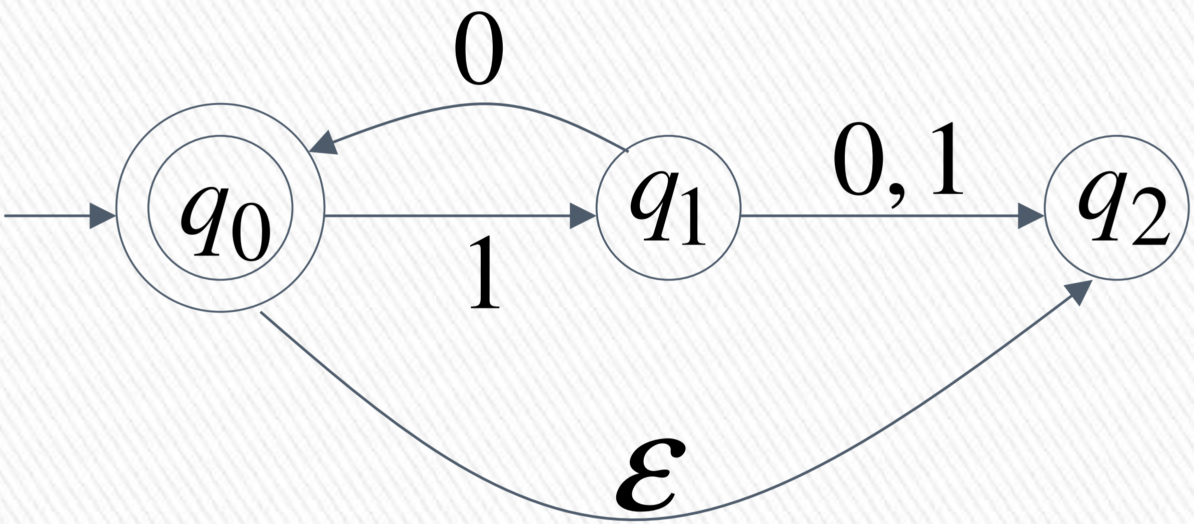


Another NFA Example



Language accepted

$$L(M) = \{\varepsilon,\ 10,\ 1010,\ 101010,\ \ldots\}$$
$$= \{10\}^*$$

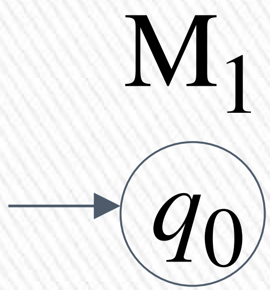


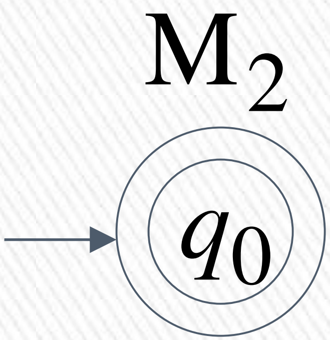
(redundant state)

Remarks:

- The ε symbol never appears on the input tape

- Simple automata:



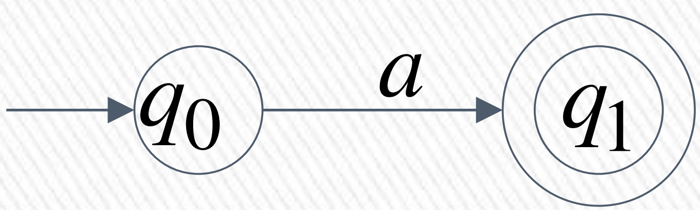


$$L(M_1) = \{\}$$

$$L(M_2) = \{\varepsilon\}$$

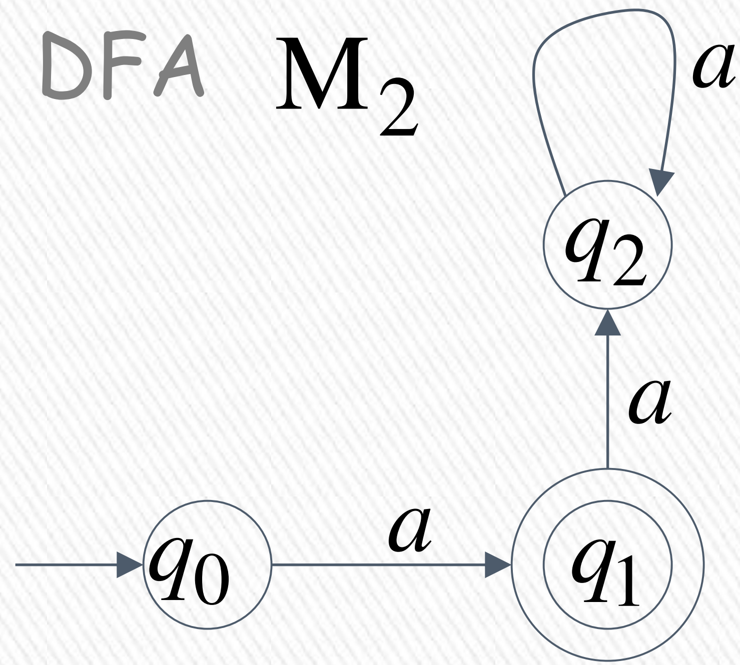
- NFAs are interesting because we can express languages easier than DFAs

NFA M_1



$$L(M_1) = \{a\}$$

DFA M_2

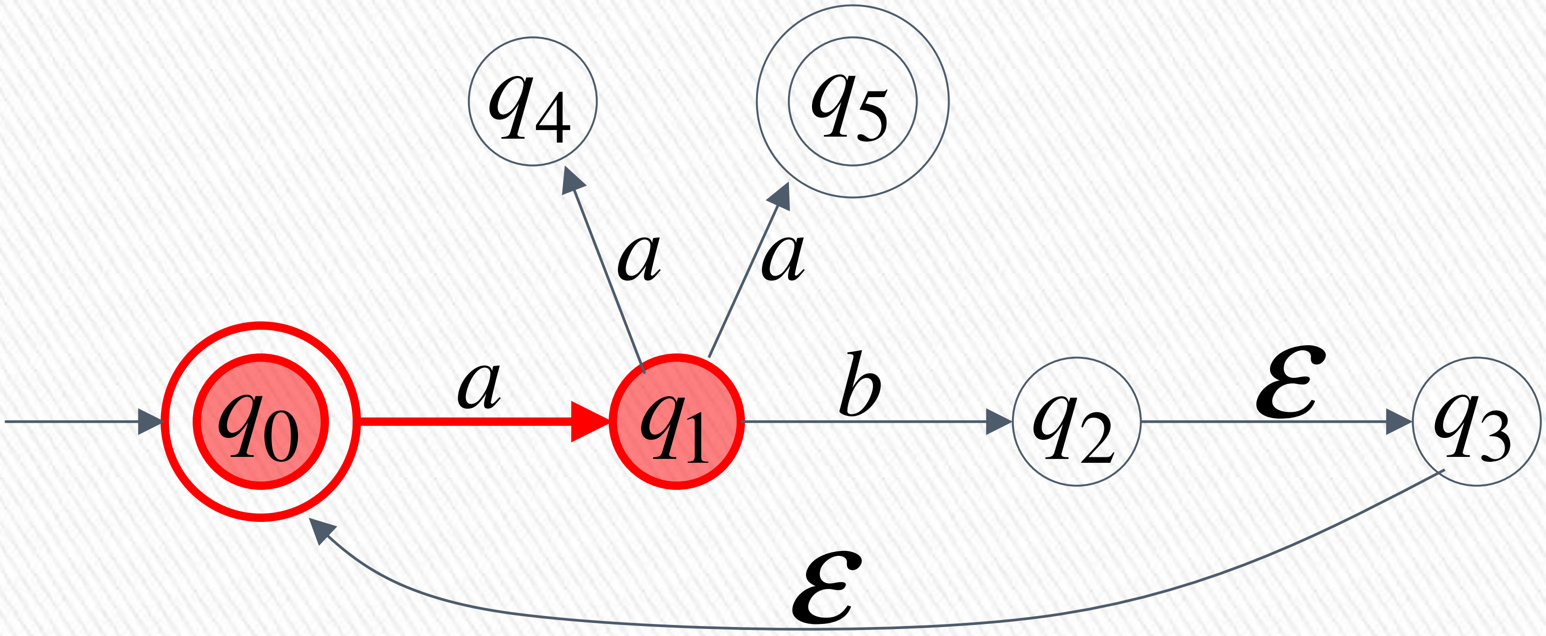


$$L(M_2) = \{a\}$$

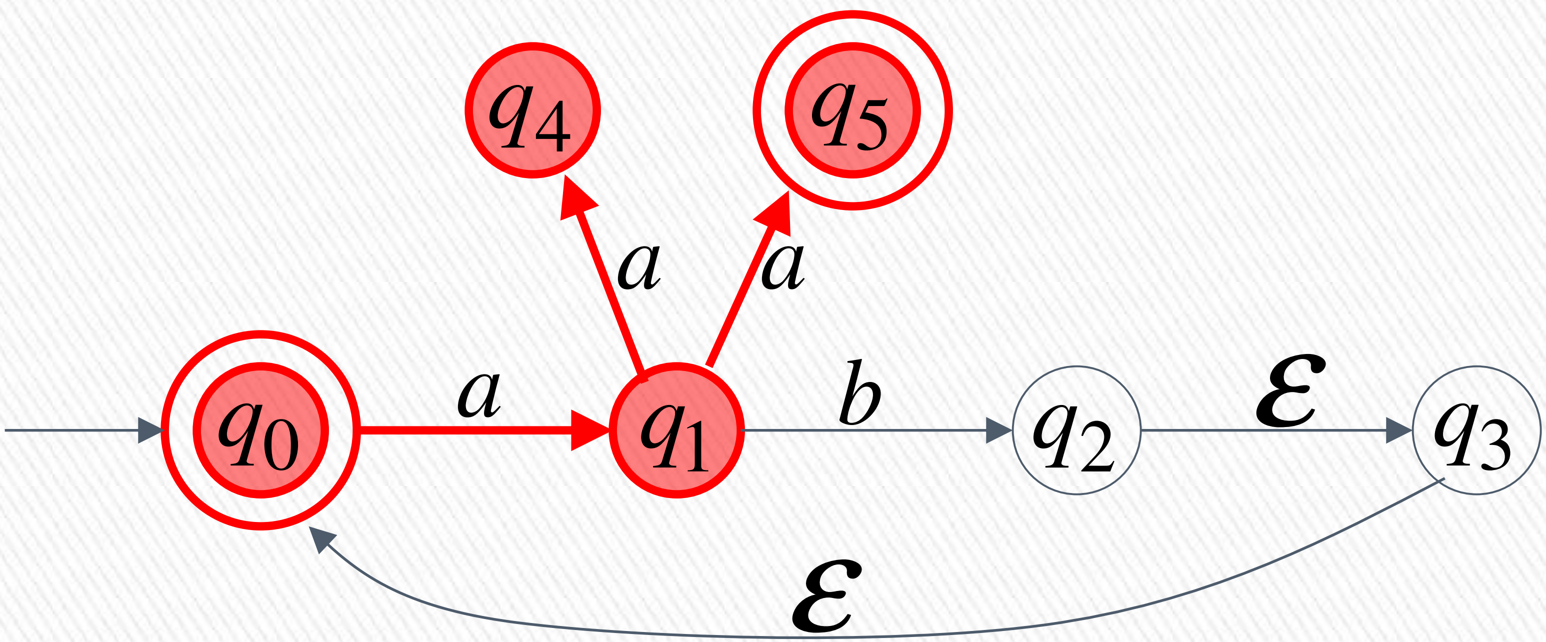
Extended Transition Function δ^*

Same with δ but applied on strings

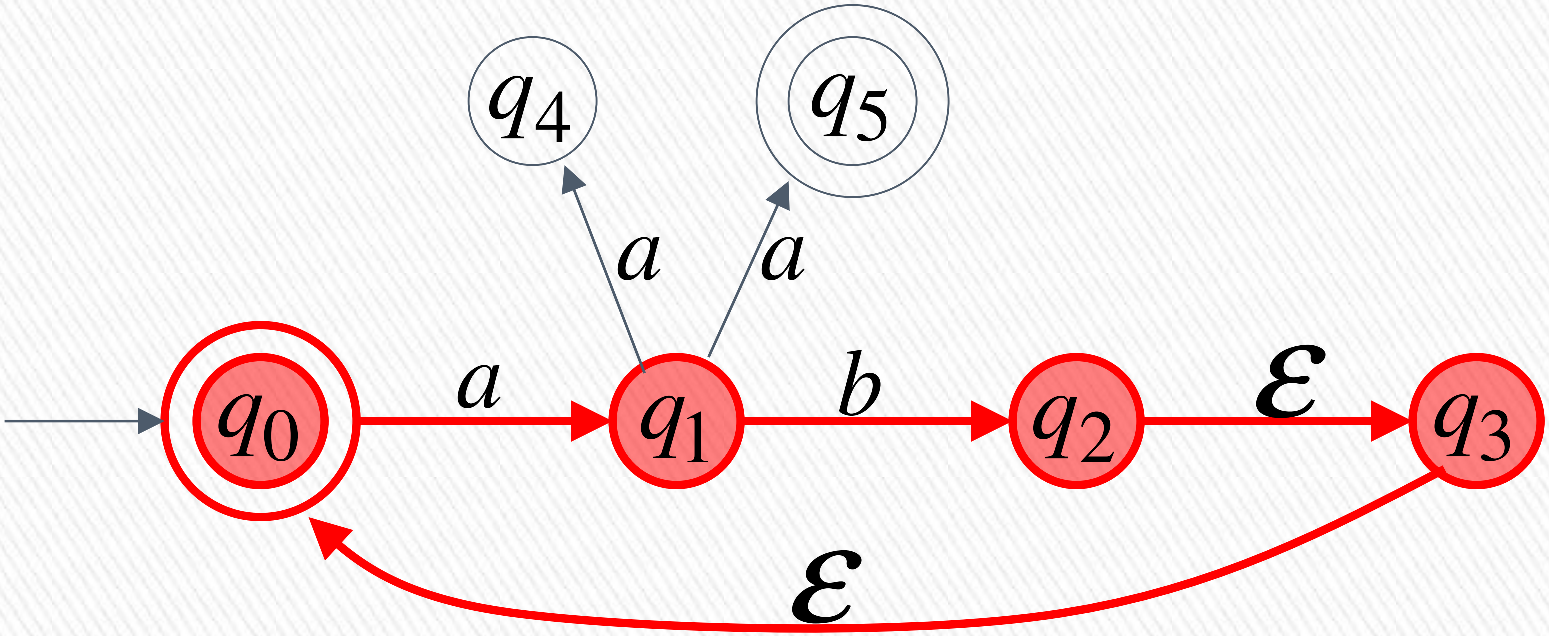
$$\delta^*(q_0, a) = \{q_1\}$$



$$\delta^*(q_0, aa) = \{q_4, q_5\}$$



$$\delta^*(q_0, ab) = \{q_2, q_3, q_0\}$$



In general

$q_j \in \delta^*(q_i, w)$: there is a walk from q_i to q_j with label w

$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$

The Language of an NFA

» The language accepted by M is:

$$L(M) = \{w_1, w_2, \ldots w_n\}$$

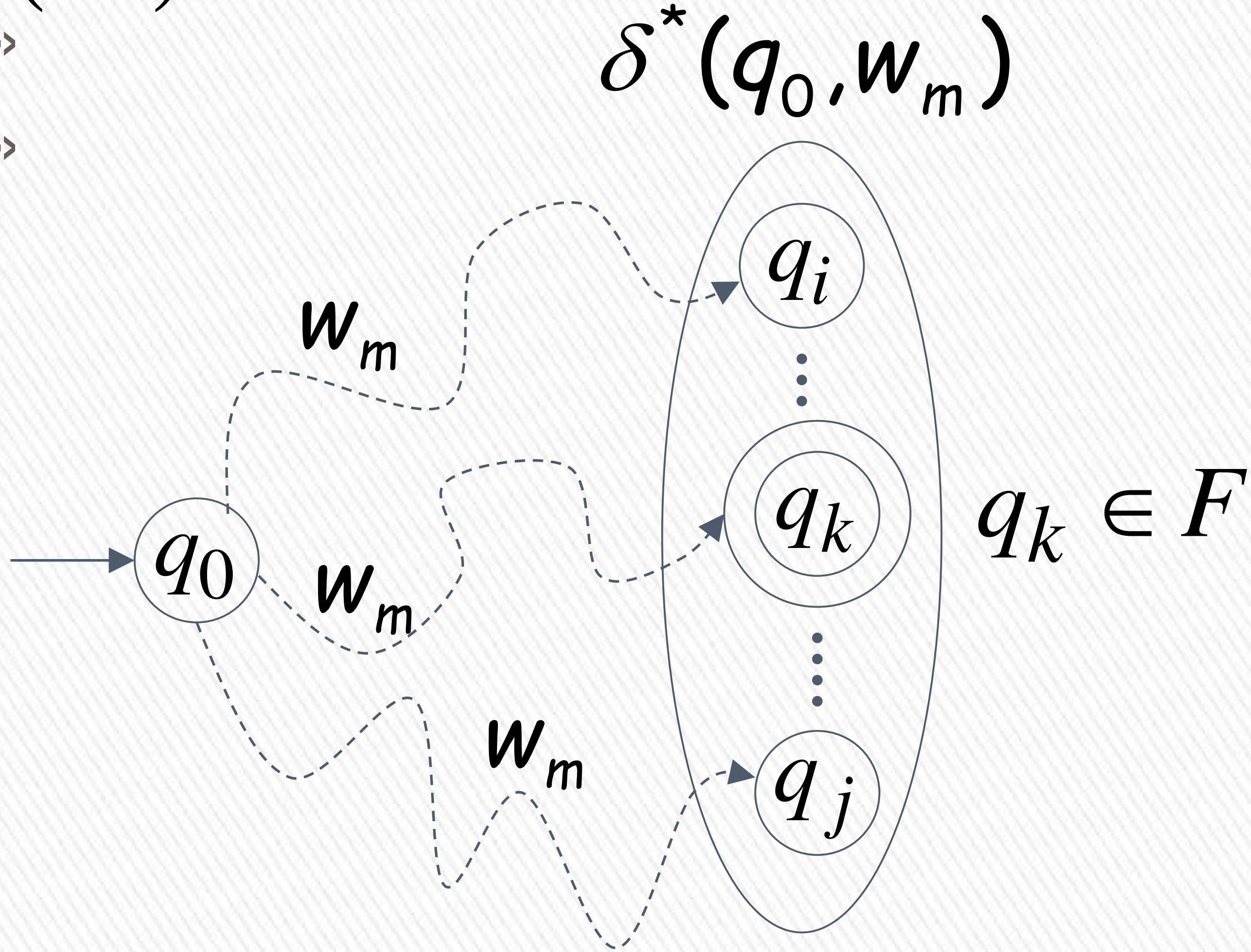
» where

$$\delta^*(q_0, w_m) = \{q_i, \ldots, q_k, \ldots, q_j\}$$

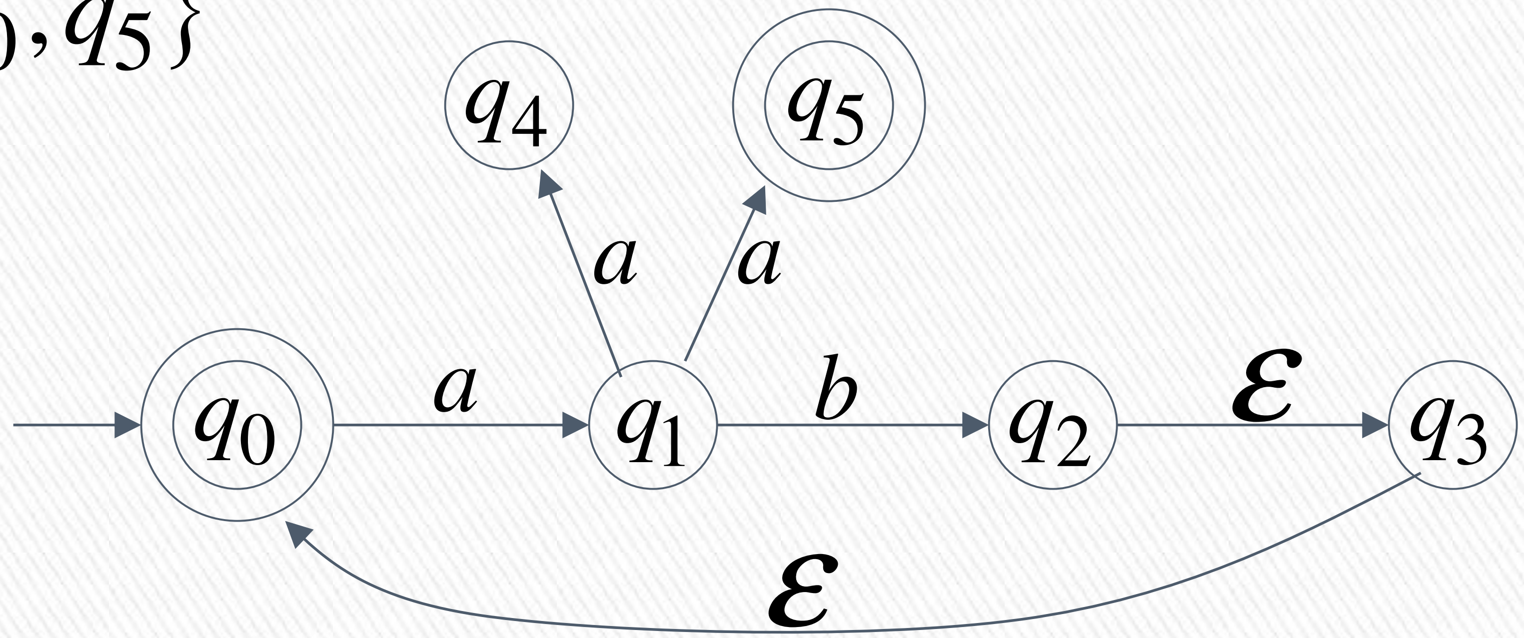
» and there is some $q_k \in F$

(accepting state)

$w_m \in L(M)$



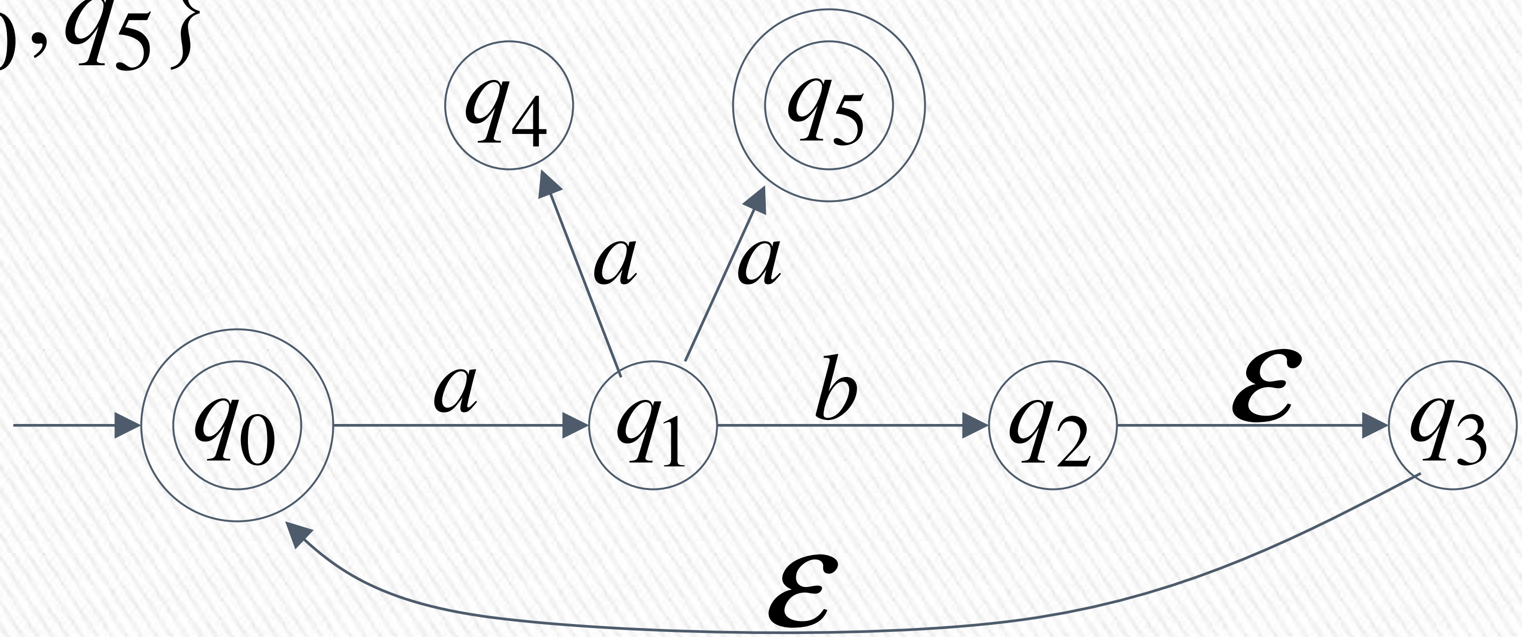
$$F = \{q_0, q_5\}$$



$$\delta^*(q_0, aa) = \{q_4, \underline{q_5}\} \Rightarrow aa \in L(M)$$

$\in F$

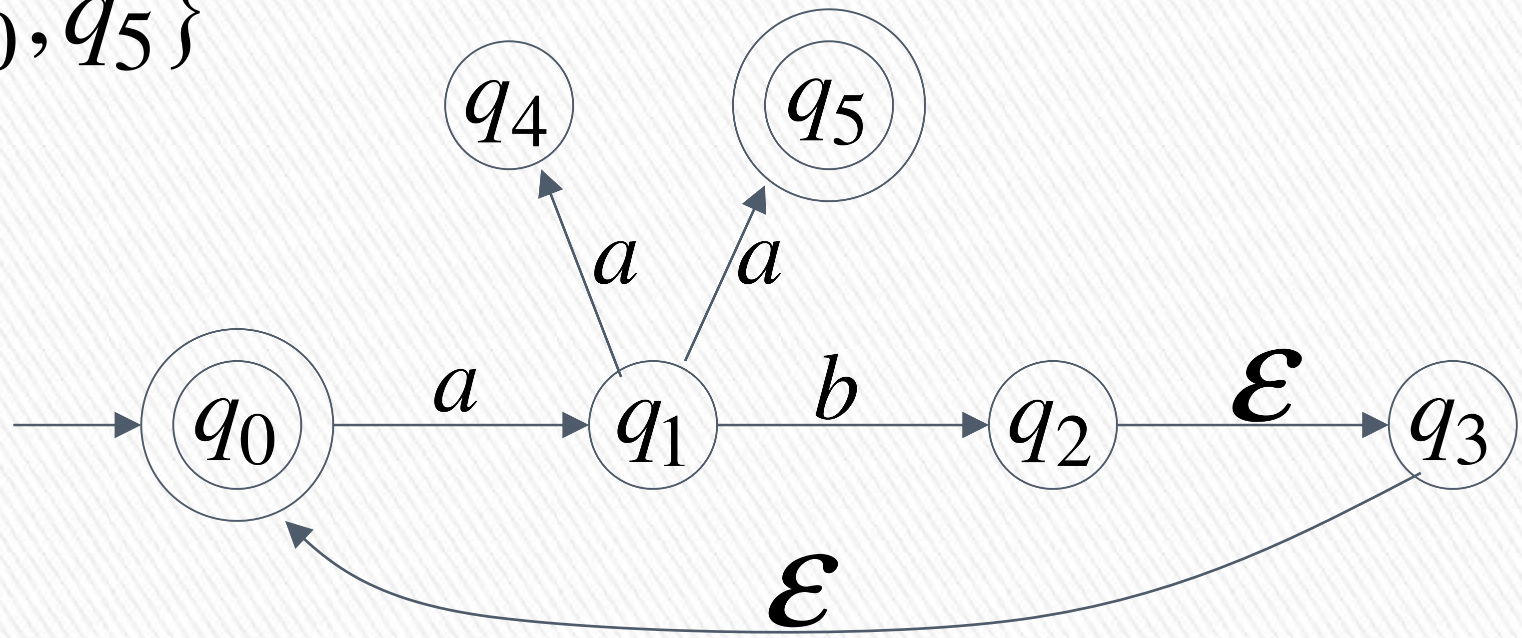
$$F = \{q_0, q_5\}$$



$$\delta^*(q_0, ab) = \{q_2, q_3, \underline{q_0}\} \Longrightarrow ab \in L(M)$$

$\in F$

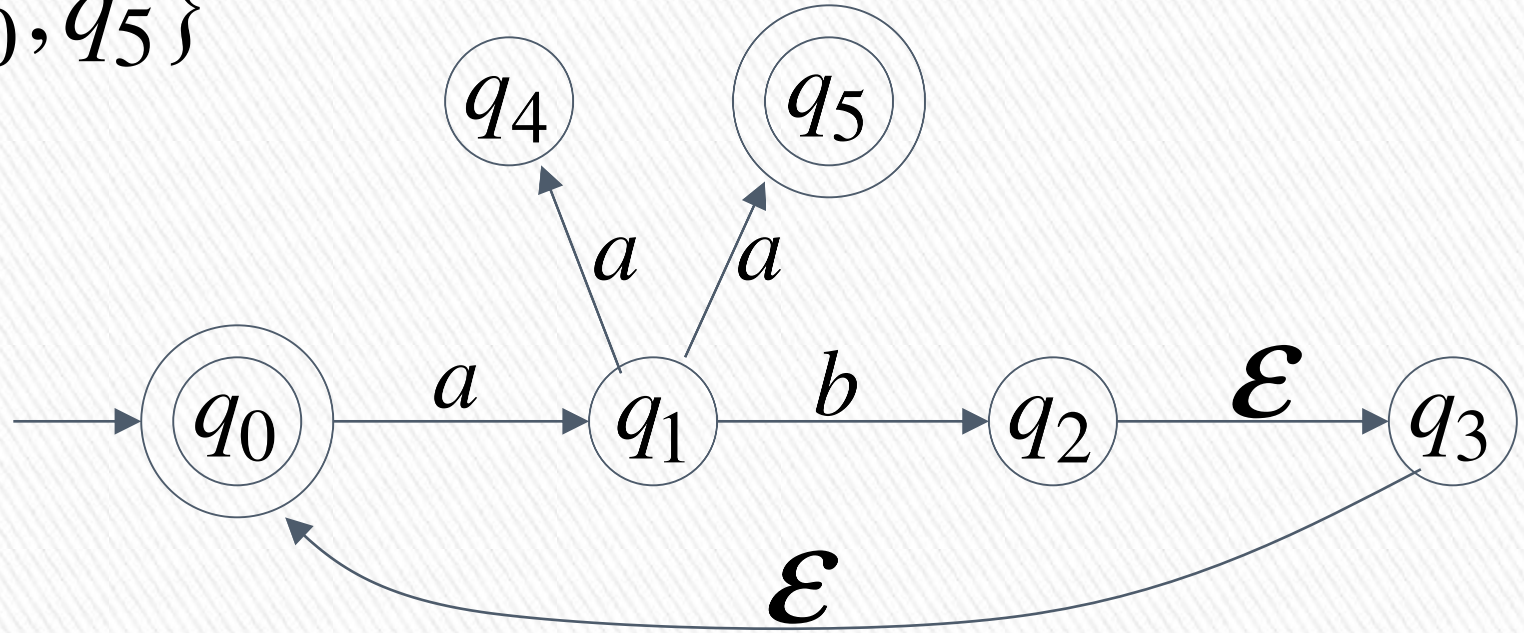
$$F = \{q_0, q_5\}$$



$$\delta^*(q_0, abaa) = \{q_4, \underline{q_5}\} \qquad abaa \in L(M)$$

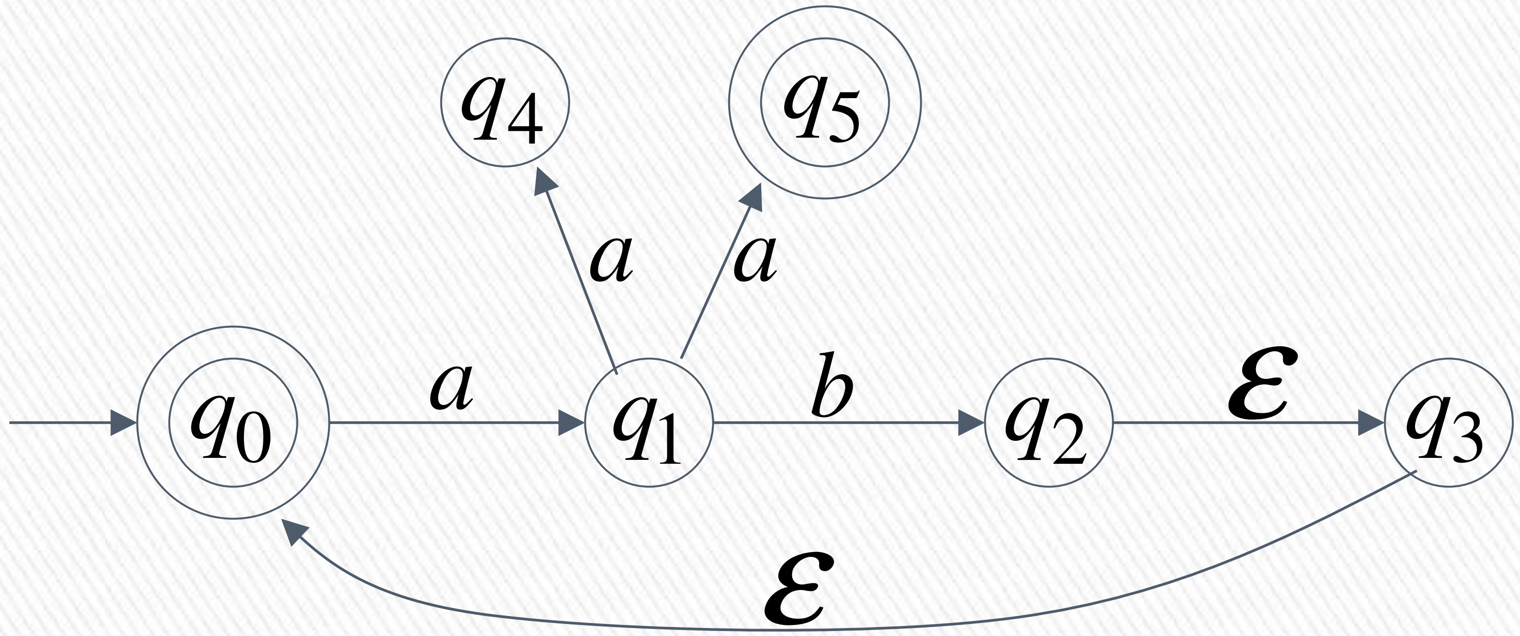
$$\in F$$

$$F = \{q_0, q_5\}$$



$$\delta^*(q_0, aba) = \{q_1\} \Longrightarrow aba \notin L(M)$$

$$\notin F$$



$$L(M) = \{ab\}^* \cup \{ab\}^* \{aa\}$$

Equivalence of Machines

- Definition:

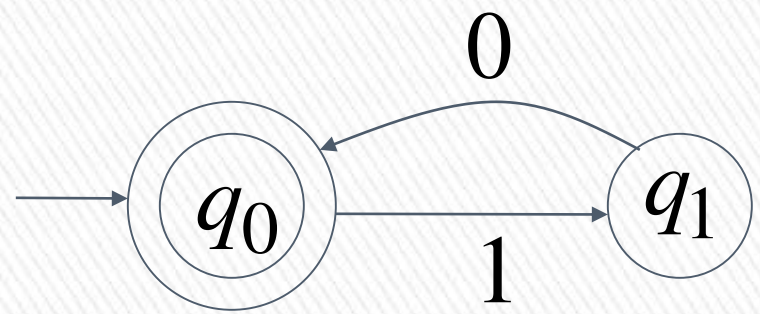
- Machine M_1 is equivalent to machine M_2

  if $L(M_1) = L(M_2)$

Example of equivalent machines

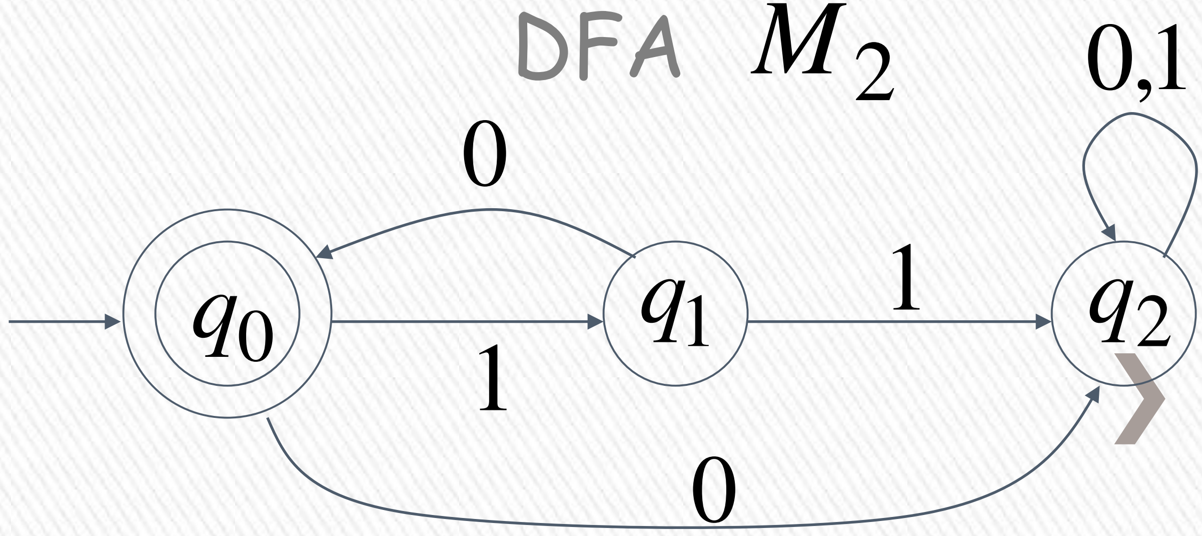
NFA M_1

$$L(M_1) = \{10\}^*$$



DFA M_2

$$L(M_2) = \{10\}^*$$



NFAs accept Regular Languages

Theorem:

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} = \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Languages Accepted
by DFAs

NFAs and DFAs have the same computation power, accept the same set of languages

Proof: we only need to show

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

AND

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Proof-Step 1

$$\left\{\begin{array}{l}\text{Languages}\\ \text{accepted}\\ \text{by NFAs}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$$

Every DFA is trivially an NFA

$\Downarrow$

Any language L accepted by a DFA is also accepted by an NFA

Proof-Step 2

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Any NFA can be converted to an equivalent DFA

$\Downarrow$

Any language L accepted by an NFA is also accepted by a DFA

Lemma:

If we convert NFA M to DFA M' then the two automata are equivalent:

$$L(M) = L(M')$$

Proof:

We only need to show: $L(M) \subseteq L(M')$

AND

$L(M) \supseteq L(M')$

First we show: $L(M) \subseteq L(M')$

We only need to prove:

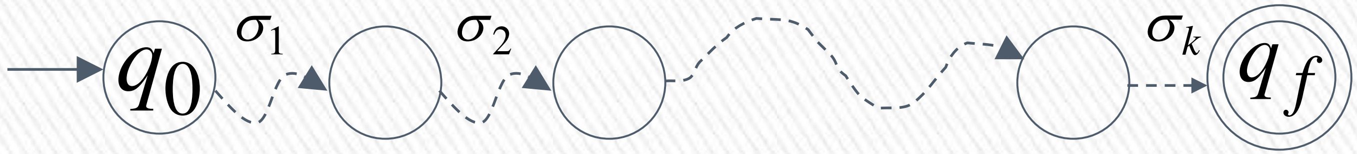
$w \in L(M)$ $\Rightarrow$ $w \in L(M')$

NFA

Consider $w \in L(M)$

symbols

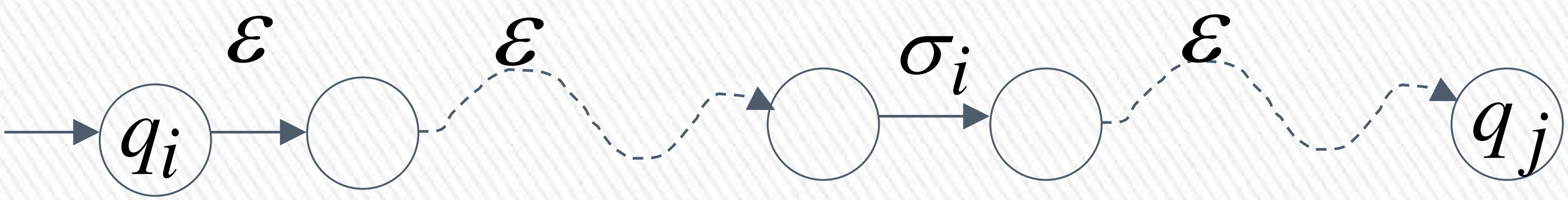
$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$



symbol

denotes a possible sub-path like

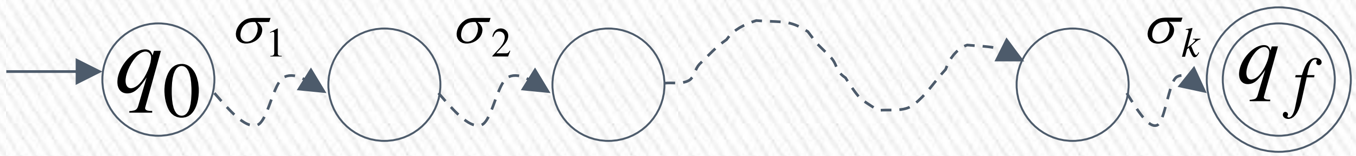
symbol



We will show that if $w \in L(M)$

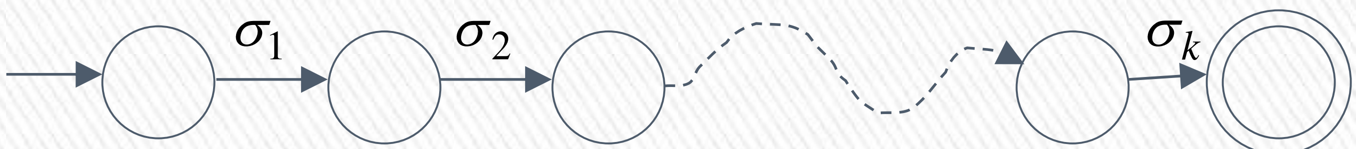
$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$

NFA $M:$



then

DFA $M':$



$\{q_0\}$

state label

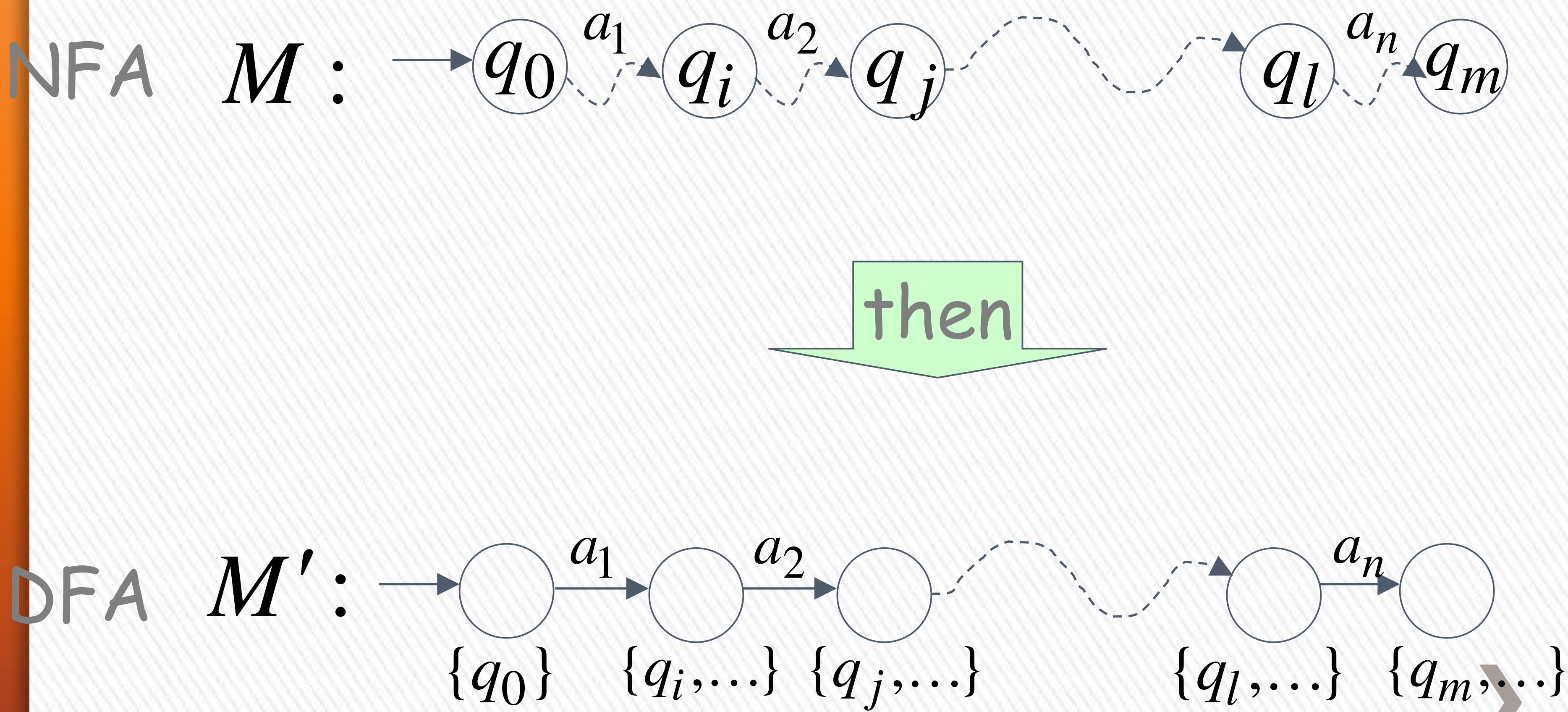
$$w \in L(M')$$

$\{q_f, \ldots\}$

state label

More generally, we will show that if in M:

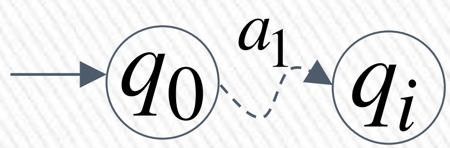
(arbitrary string) $v = a_1 a_2 \cdots a_n$



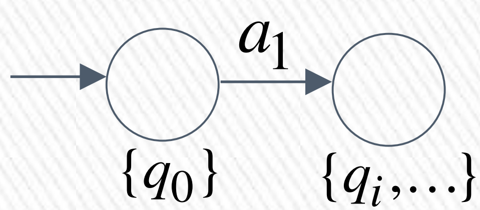
Proof by induction on $|v|$

Induction Basis: $|v| = 1$ $\qquad v = a_1$

NFA M:



DFA M':

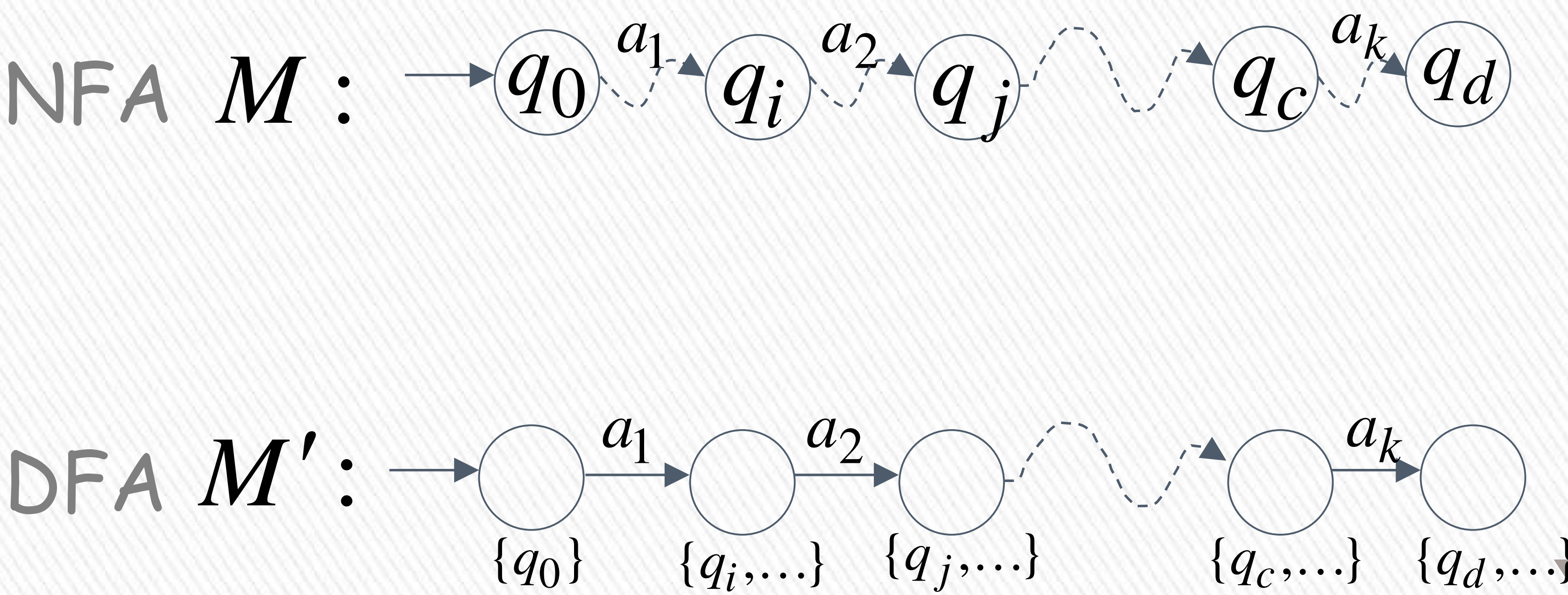


is true by construction of M'

Induction hypothesis: $1 \leq |v| \leq k$

$$v = a_1 a_2 \cdots a_k$$

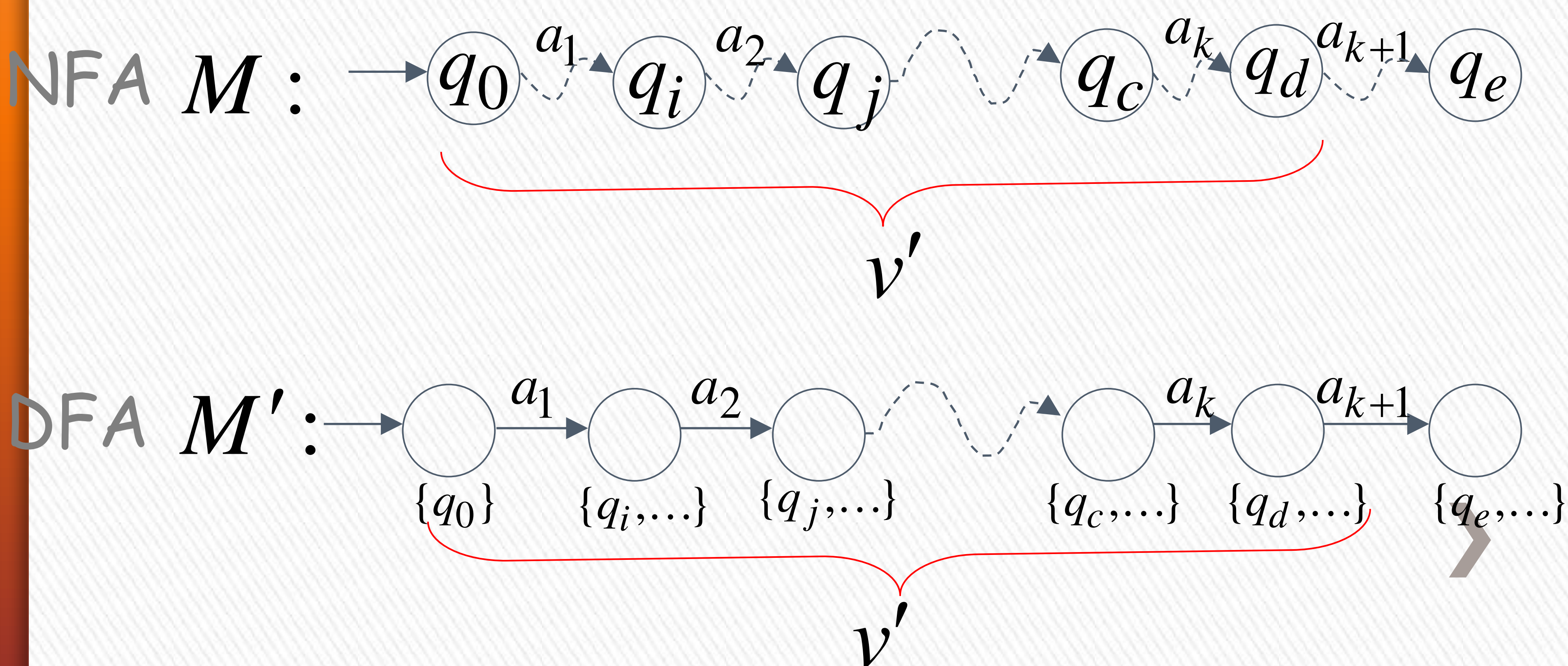
Suppose that the following hold



Induction Step: $|v| = k+1$

$$v = \underbrace{a_1 a_2 \cdots a_k}_{v'} a_{k+1} = v' a_{k+1}$$

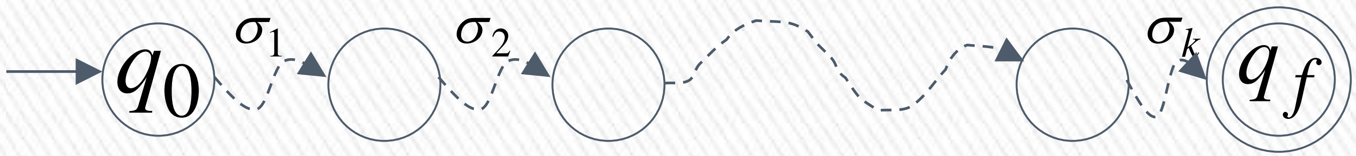
Then this is true by construction of M'



Therefore if $w \in L(M)$

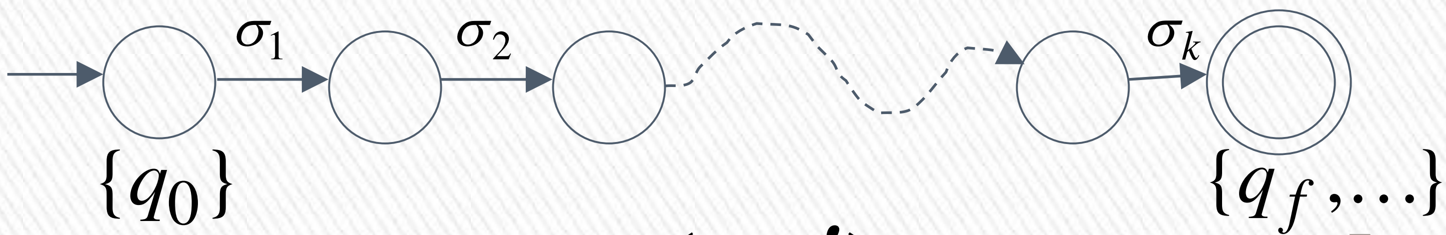
$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$

NFA $M:$



then

DFA $M':$



$$w \in L(M')$$

We have shown: $L(M) \subseteq L(M')$

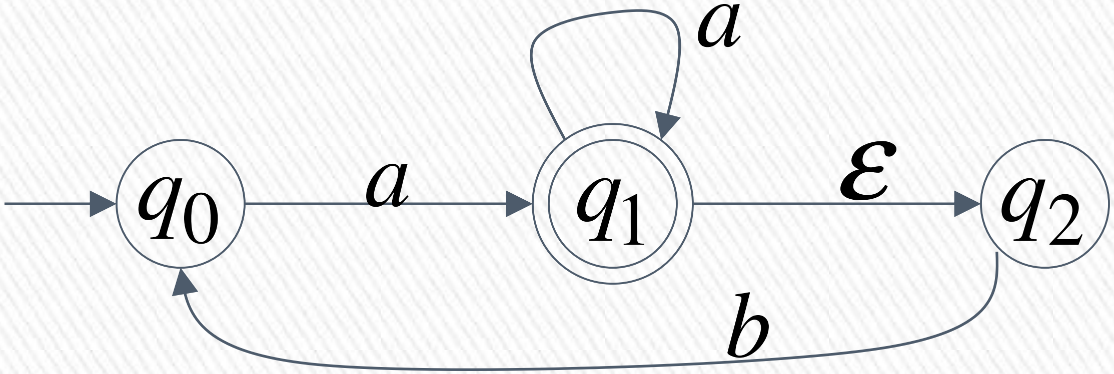
With a similar proof

we can show: $L(M) \supseteq L(M')$

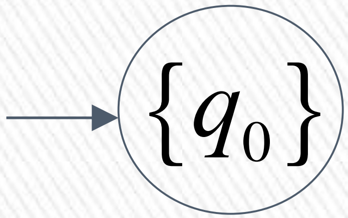
Therefore: $L(M) = L(M')$

END OF LEMMA PROOF

Conversion NFA to DFA

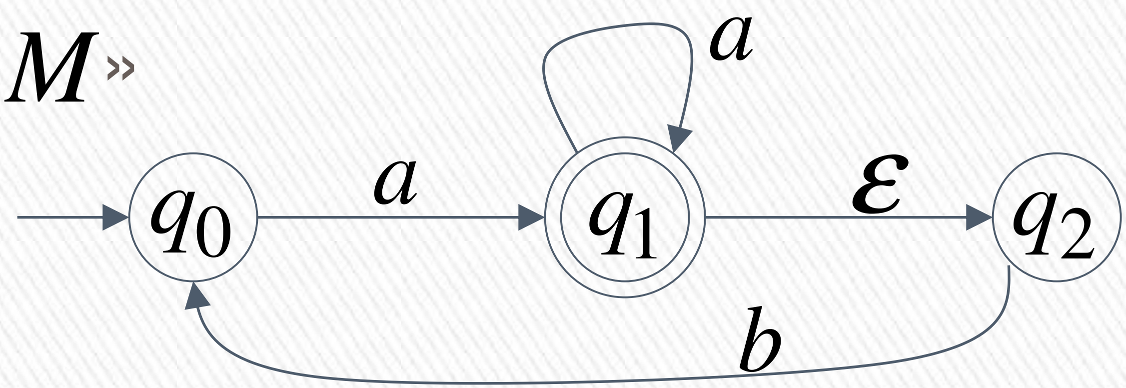


NFA M



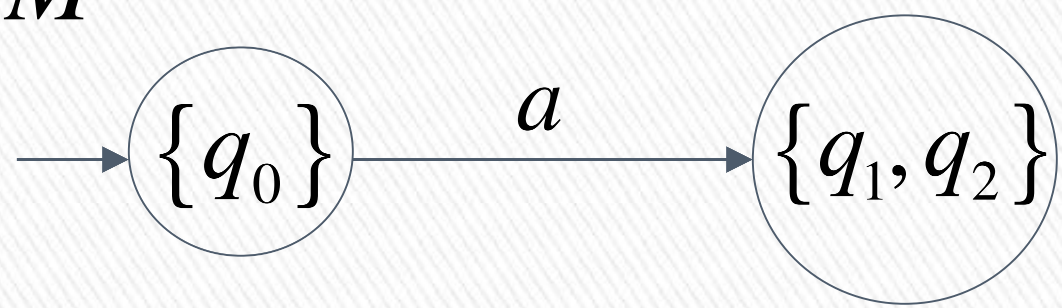
DFA M'

NFA M



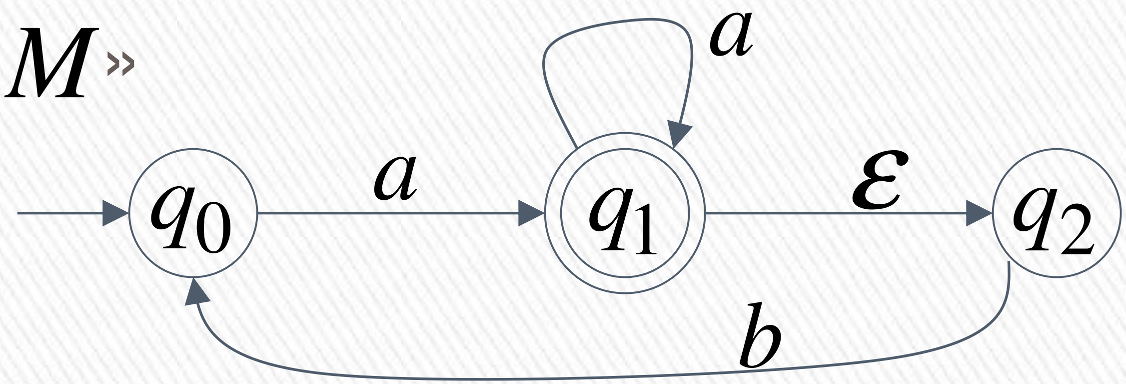
$$\delta^*(q_0, a) = \{q_1, q_2\}$$

DFA M'

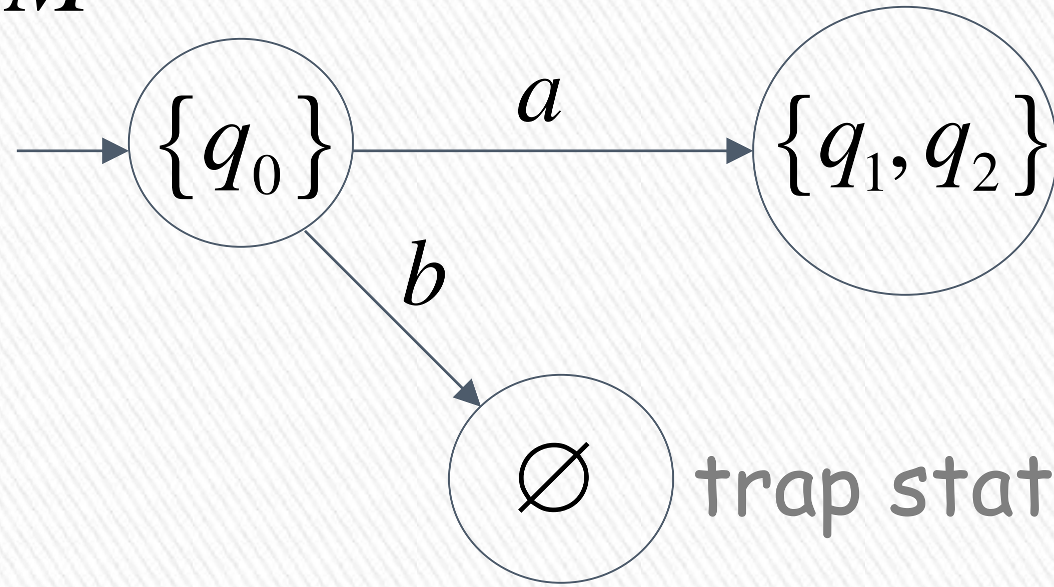


$\delta^*(q_0, b) = \varnothing$ empty set

NFA M



DFA M'

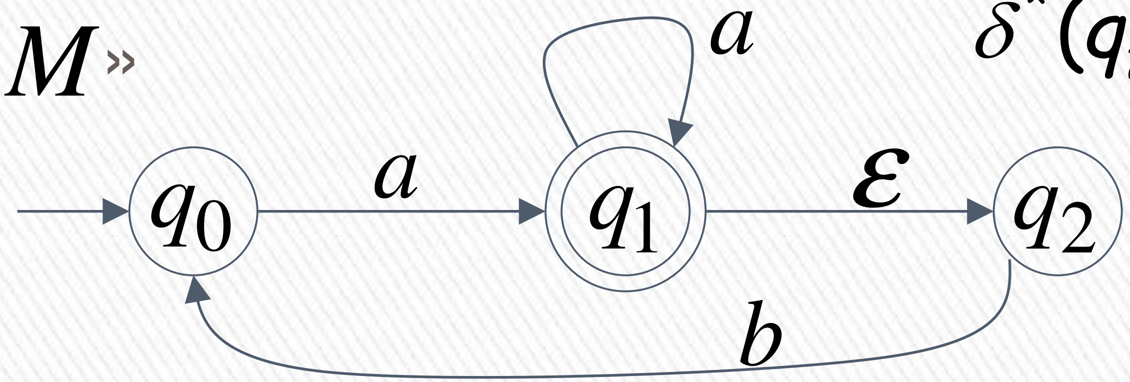


trap state

NFA M

$\delta^*(q_1,a)=\{q_1,q_2\}$

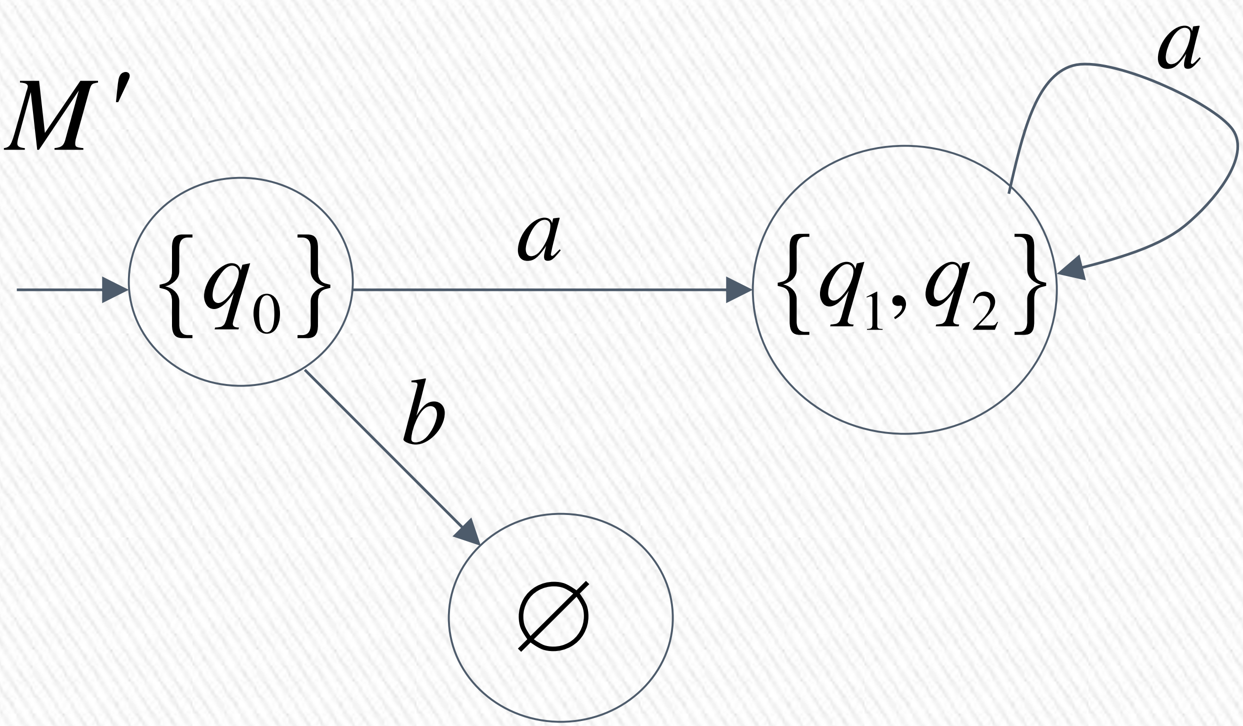
$\delta^*(q_2,a)=\varnothing$



union

$\{q_1,q_2\}$

DFA M'



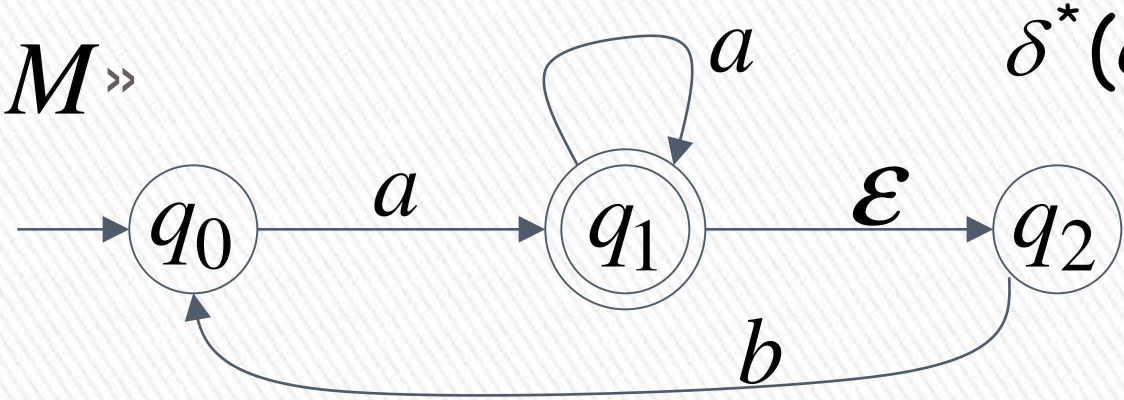
$$\delta^*(q_1, b) = \{q_0\}$$

$$\delta^*(q_2, b) = \{q_0\}$$

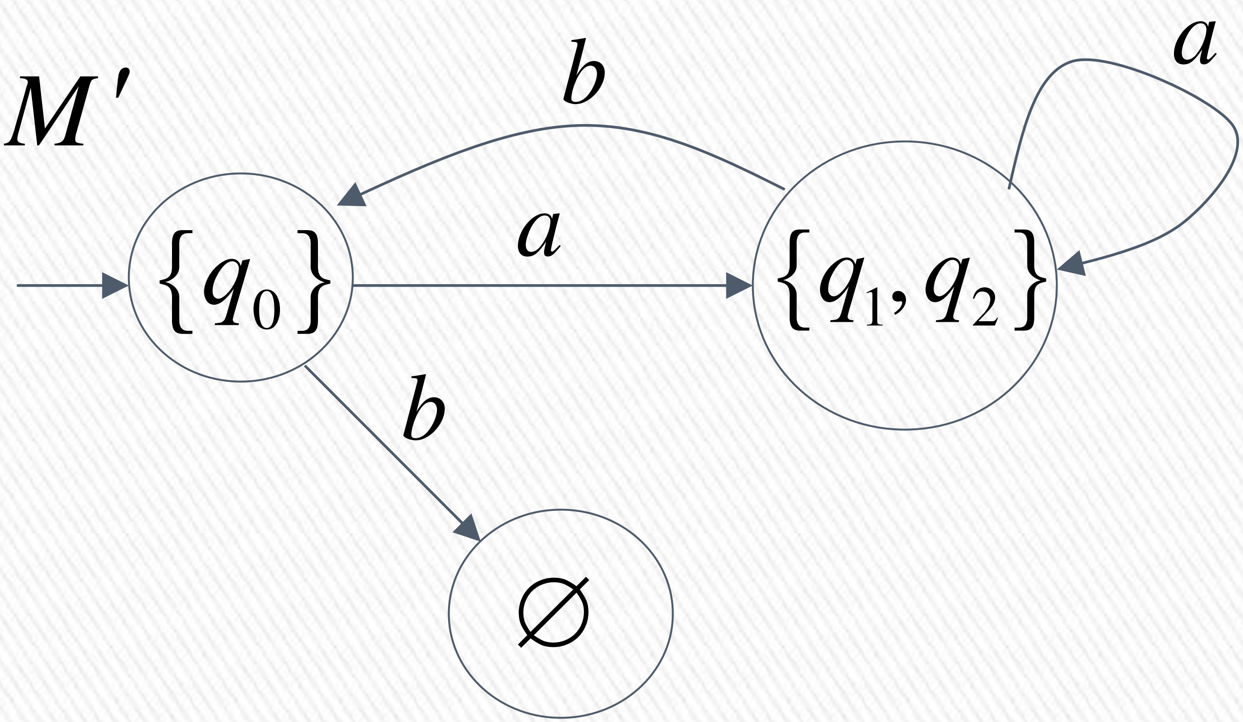
union

$\{q_0\}$

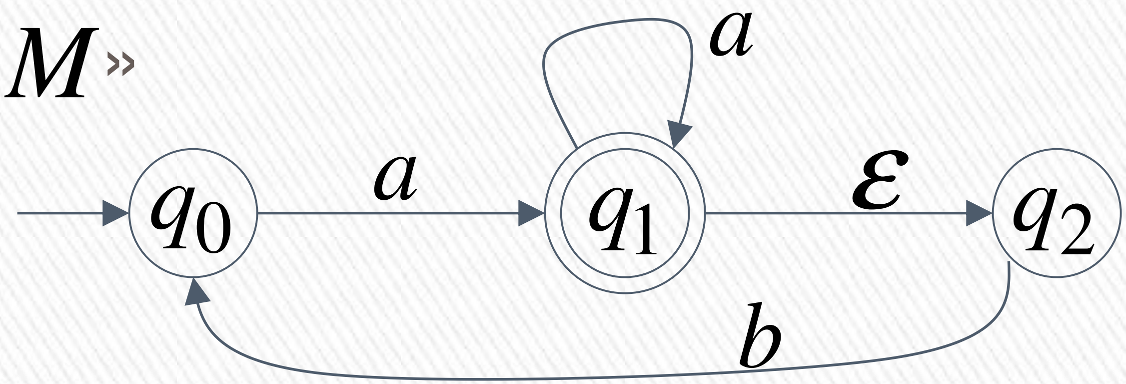
NFA M



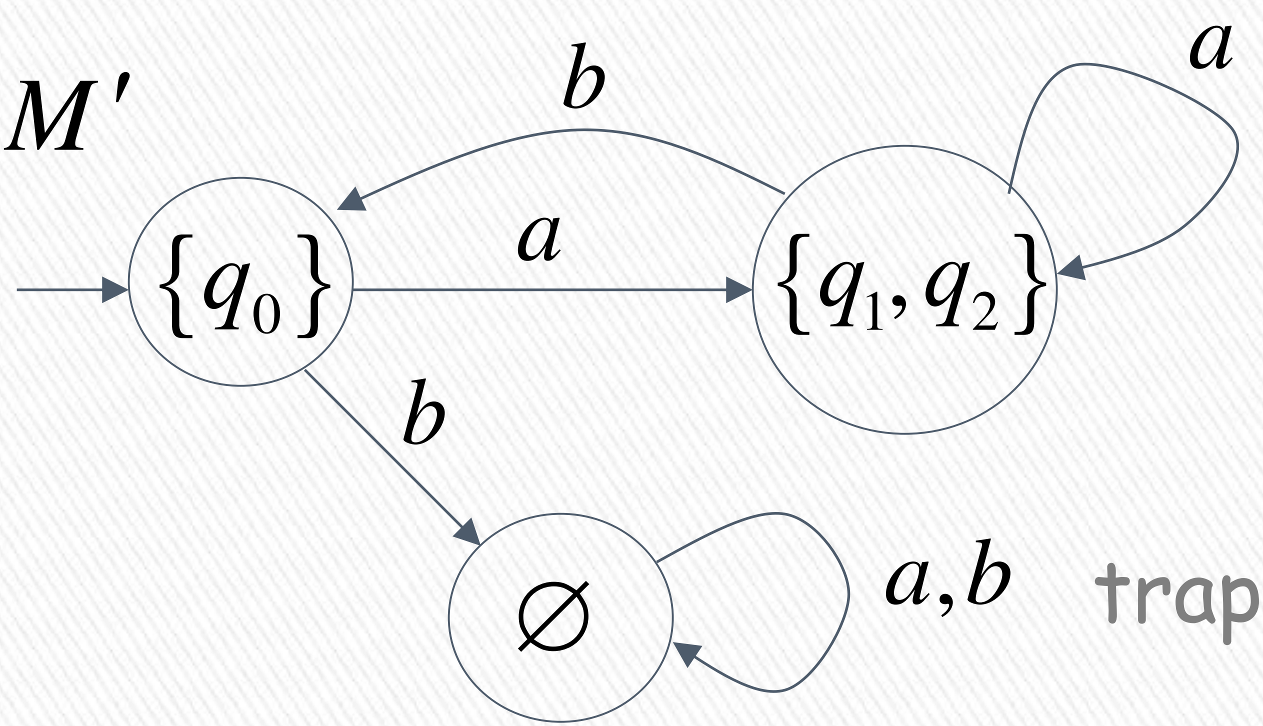
DFA M'



NFA M



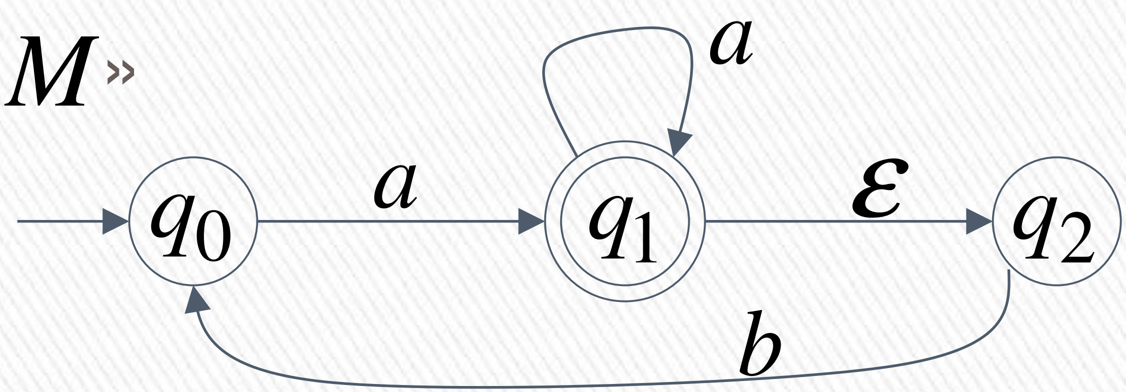
DFA M'



trap state

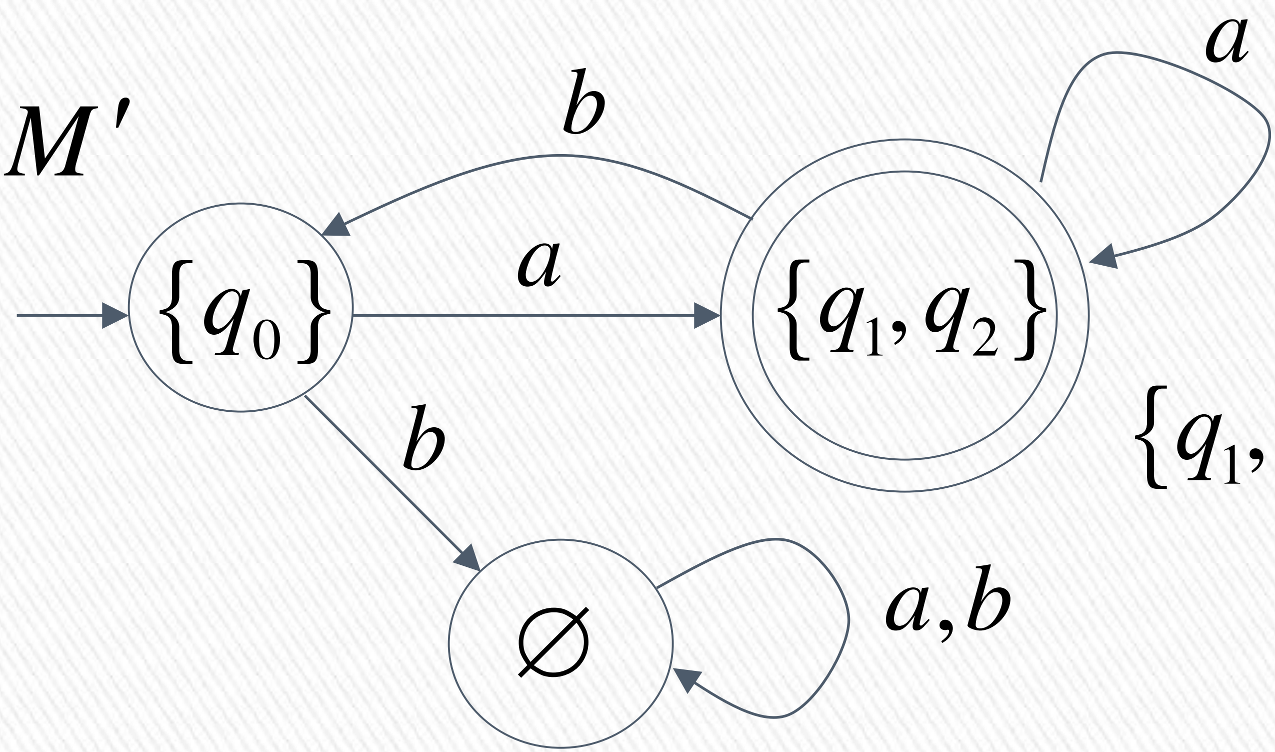
END OF CONSTRUCTION

NFA M



$q_1 \in F$

DFA M'



$\{q_1, q_2\} \in F'$

General Conversion Procedure

- Input: an NFA M
- Output: an equivalent DFA M'

  with $L(M) = L(M')$

- The NFA has states

$$q_0, q_1, q_2, \ldots$$

- The DFA has states from the power set

$$\emptyset, \{q_0\}, \{q_1\}, \{q_0, q_1\}, \{q_1, q_2, q_3\}, \ldots.$$

Conversion Procedure Steps

»

Step 1

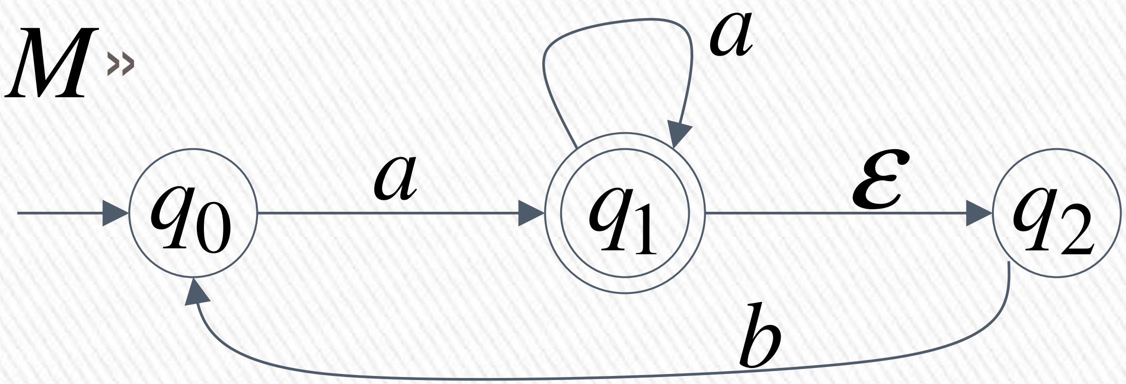
» Initial state of NFA: q_0

$$\Downarrow$$

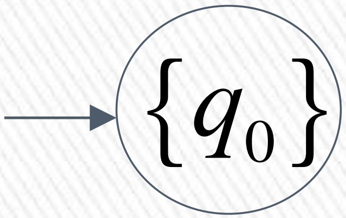
» Initial state of DFA: $\{q_0\}$

Example

NFA M



DFA M'



Step 2

For every DFA's state $\{q_i, q_j, \ldots, q_m\}$

compute in the NFA

$$\left.\begin{array}{l} \delta^*(q_i, a) \\ \cup\, \delta^*(q_j, a) \\ \ldots \\ \cup\, \delta^*(q_m, a) \end{array}\right\} = \{q'_k, q'_l, \ldots, q'_n\}$$

Union

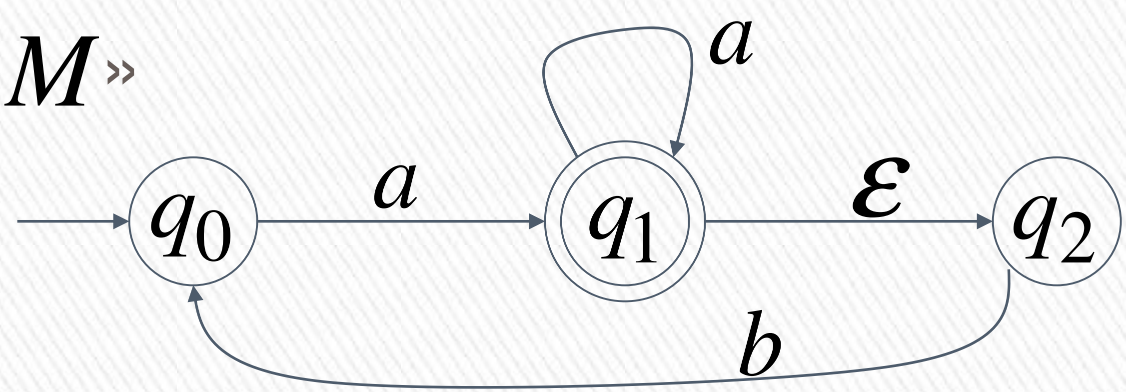
add transition to DFA

$$\delta(\{q_i, q_j, \ldots, q_m\},\ a) = \{q'_k, q'_l, \ldots, q'_n\}$$

Example

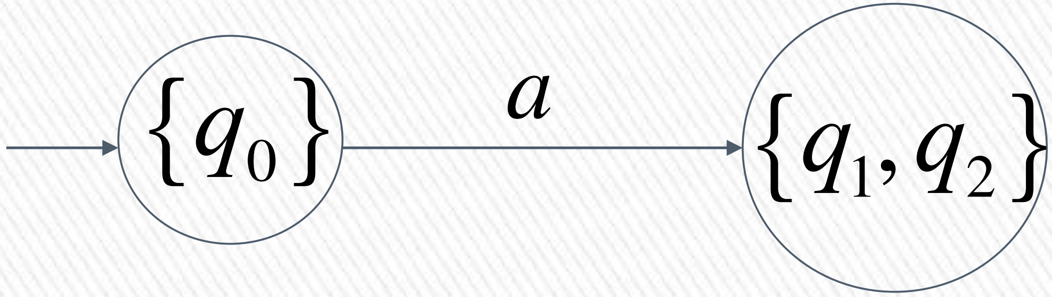
$$\delta^*(q_0, a) = \{q_1, q_2\}$$

NFA M



DFA M'

$$\delta(\{q_0\}, a) = \{q_1, q_2\}$$

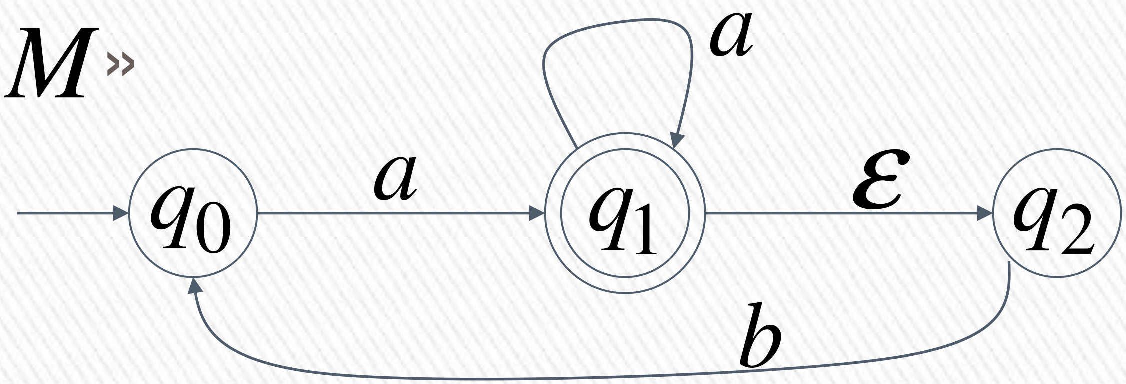


Step 3

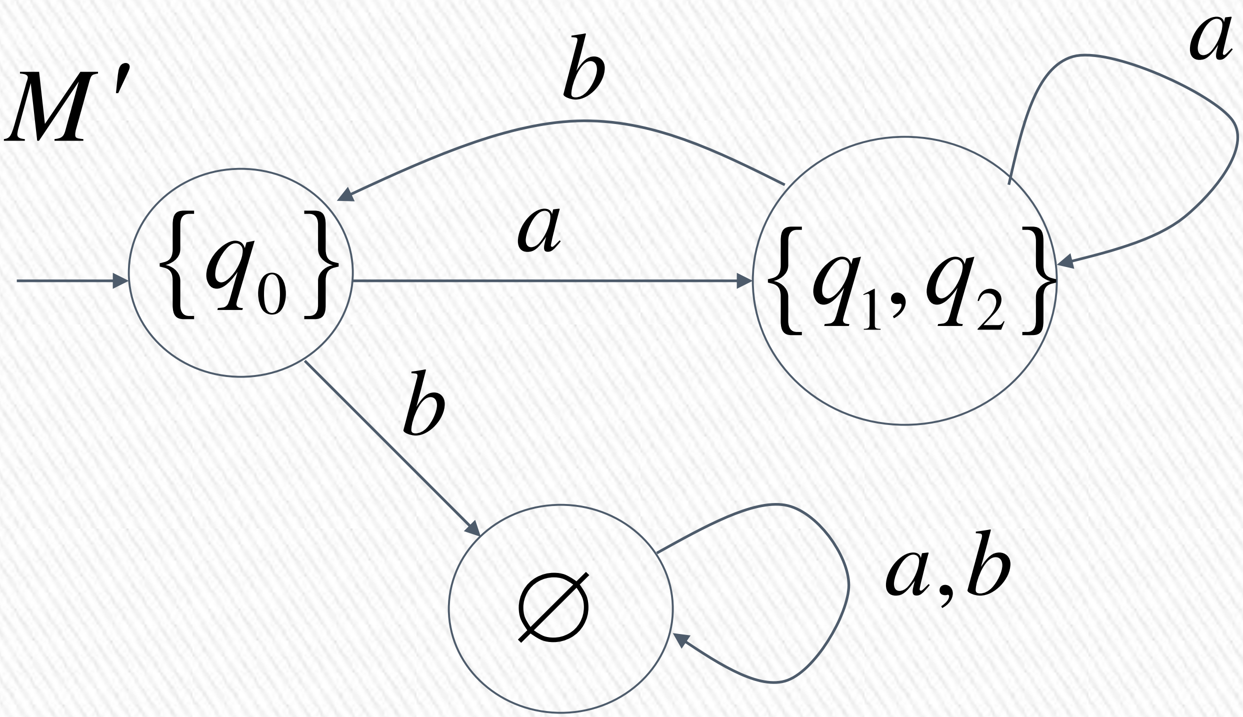
- Repeat Step 2 for every state in DFA and symbols in alphabet until no more states can be added in the DFA

Example

NFA M



DFA M'

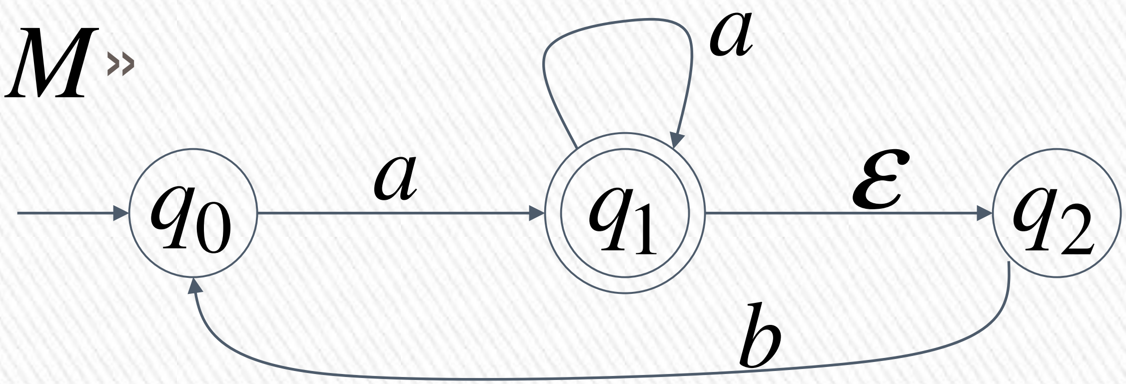


Step 4

- For any DFA state $\{q_i, q_j, \ldots, q_m\}$
- if some q_j is accepting state in NFA
- Then, $\{q_i, q_j, \ldots, q_m\}$ is accepting state in DFA

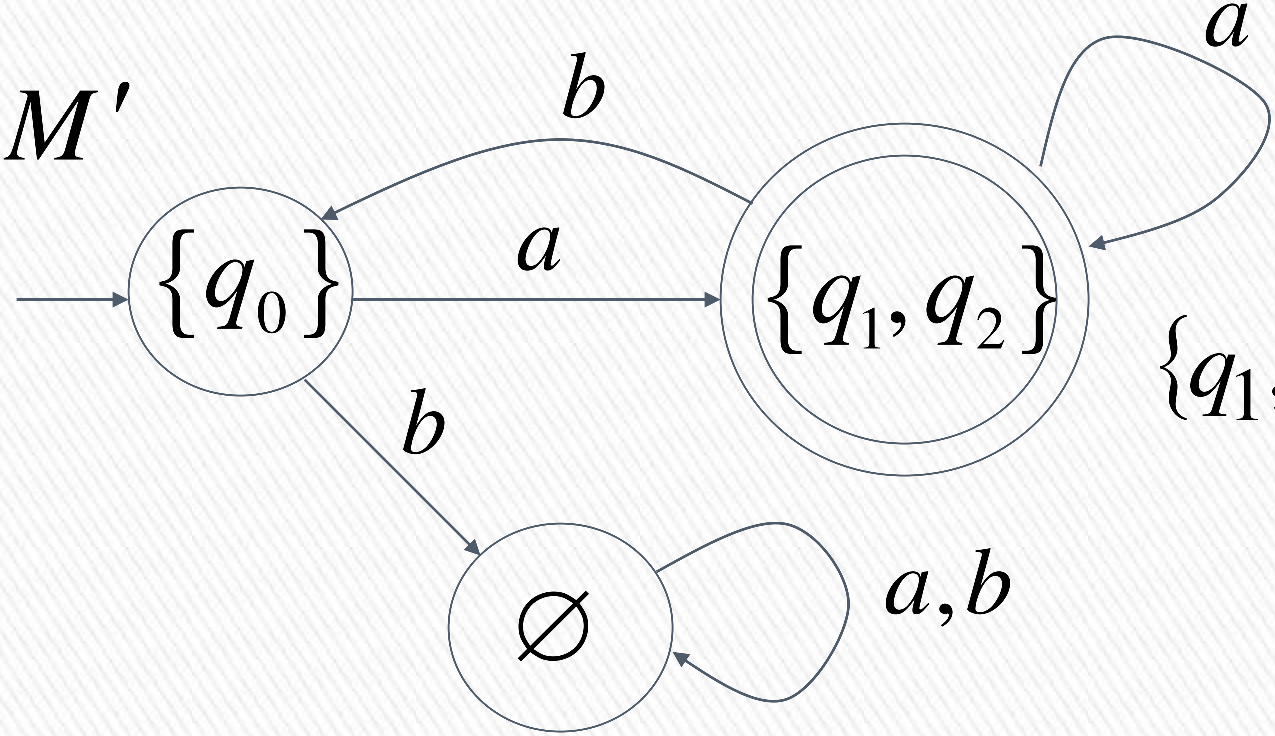
Example

NFA M



$q_1 \in F$

DFA M'



$\{q_1, q_2\} \in F'$