Solution of Initial Value Problems

- The Laplace transform is named for the French mathematician Laplace, who studied this transform in 1782.
- The techniques described here were developed primarily by Oliver Heaviside (1850-1925), an English electrical engineer.
- We see how the Laplace transform can be used to solve initial value problems for linear differential equations with constant coefficients.
- The Laplace transform is useful in solving these differential equations because the transform of f' is related in a simple way to the transform of f

Theorem

- Suppose that f is a function for which the following hold:
  - (1) f is continuous and f' is piecewise continuous on [0, b] for all $b > 0$.
  - (2) $|f(t)| \le Ke^{at}$ when $t \ge M$, for constants a, K, M, with K, $M > 0$.
- Then the Laplace Transform of f' exists for $s > a$, with

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

- **Proof** (outline): For f and f' continuous on [0, b], we have

$$\begin{aligned}\lim_{b\to\infty}\int_0^b e^{-st} f'(t)dt &= \lim_{b\to\infty}\left[e^{-st} f(t)\Big|_0^b - \int_0^b (-s)e^{-st} f(t)dt \right] \\ &= \lim_{b\to\infty}\left[e^{-sb} f(b) - f(0) + s\int_0^b e^{-st} f(t)dt \right], (e^{-\infty} \approx 0,\ e^{\infty} \approx \infty)\end{aligned}$$

- Similarly for f' piecewise continuous on [0, b]

The Laplace Transform of f'

- Thus if f and f' satisfy the hypotheses of Theorem , then

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

- Now suppose f' and f'' satisfy the conditions specified for f and f' of Theorem. We then obtain

$$\begin{aligned} L\{f''(t)\} &= sL\{f'(t)\} - f'(0) \\ &= s\left[sL\{f(t)\} - f(0)\right] - f'(0) \\ &= s^2 L\{f(t)\} - sf(0) - f'(0) \end{aligned}$$

- Similarly, we can derive an expression for $L\{f^{(n)}\}$, provided f and its derivatives satisfy suitable conditions. This result is given in Corollary:

Corollary

- Suppose that f is a function for which the following hold:

  (1) $f, f', f'', \ldots, f^{(n-1)}$ are continuous, and $f^{(n)}$ piecewise continuous, on $[0, b]$ for all $b > 0$.

  (2) $|f(t)| \le Ke^{at}$, $|f'(t)| \le Ke^{at}$, ..., $|f^{(n-1)}(t)| \le Ke^{at}$ for $t \ge M$, for constants a, K, M, with $K, M > 0$.

  Then the Laplace Transform of $f^{(n)}$ exists for $s > a$, with

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

Example 1

- Consider the initial value problem

$$y'' + 5y' + 6y = 0, \quad y(0) = 2, \; y'(0) = 3$$

We now solve this problem using Laplace Transforms

- Assume that our IVP has a solution ϕ and that $\phi'(t)$ and $\phi''(t)$ satisfy the conditions of Corollary. Then

$$L\{y'' + 5y' + 6y\} = L\{y''\} + 5L\{y'\} + 6L\{y\} = L\{0\} = 0$$

and hence

$$\left[s^2 L\{y\} - sy(0) - y'(0)\right] + 5\left[sL\{y\} - y(0)\right] + 6L\{y\} = 0$$

Example 1

- Letting *Y*(*s*) = *L*{*y*}, we have

$$\left(s^2+5s+6\right)Y(s)-\left(s+5\right)y(0)-y'(0)=0$$

- Substituting in the initial conditions, we obtain

$$\left(s^2+5s+6\right)Y(s)-2\left(s+5\right)-3=0$$

- Thus

$$L\{y\}=Y(s)=\frac{2s+13}{\left(s+3\right)\left(s+2\right)}$$

Example 1: Partial Fractions

- Using partial fraction decomposition, *Y*(*s*) can be rewritten:

$$\frac{2s+13}{(s+3)(s+2)} = \frac{A}{(s+3)} + \frac{B}{(s+2)}$$

$$2s+13 = A(s+2) + B(s+3)$$

$$2s+13 = (A+B)s + (2A+3B)$$

$$A+B=2, \;\; 2A+3B=13$$

$$A=-7, \; B=9$$

- Thus

$$L\{y\} = Y(s) = -\frac{7}{(s+3)} + \frac{9}{(s+2)}$$

Example 1: Solution

- Recall

$$L\left\{e^{at}\right\} = F(s) = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \frac{1}{s-a}, \quad s > a$$

- Thus

$$Y(s) = -\frac{7}{(s+3)} + \frac{9}{(s+2)} = -7L\{e^{-3t}\} + 9L\{e^{-2t}\}, \; s > -2,$$

- Recalling $Y(s) = L\{y\}$, we have

$$L\{y\} = L\{-7e^{-3t} + 9e^{-2t}\}$$

and hence

$$y(t) = -7e^{-3t} + 9e^{-2t}$$

General Laplace Transform Method

- Consider the constant coefficient equation

$$ay'' + by' + cy = f(t)$$

- Assume that this equation has a solution $y = \phi(t)$, and that $\phi'(t)$ and $\phi''(t)$ satisfy the conditions of Corollary. Then

$$L\{ay'' + by' + cy\} = aL\{y''\} + bL\{y'\} + cL\{y\} = L\{f(t)\}$$

- If we let $Y(s) = L\{y\}$ and $F(s) = L\{f\}$, then

$$a\left[s^2 L\{y\} - sy(0) - y'(0)\right] + b\left[sL\{y\} - y(0)\right] + cL\{y\} = F(s)$$

$$\left(as^2 + bs + c\right)Y(s) - \left(as + b\right)y(0) - ay'(0) = F(s)$$

$$Y(s) = \frac{\left(as + b\right)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

Algebraic Problem

- Thus the differential equation has been transformed into the the algebraic equation

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

  for which we seek $y = \phi(t)$ such that $L\{\phi(t)\} = Y(s)$.

- Note that we do not need to solve the homogeneous and nonhomogeneous equations separately, nor do we have a separate step for using the initial conditions to determine the values of the coefficients in the general solution.

Characteristic Polynomial

- Using the Laplace transform, our initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \; y'(0) = y_0'$$

becomes

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

- The polynomial in the denominator is the **characteristic polynomial** associated with the differential equation.
- The partial fraction expansion of $Y(s)$ used to determine ϕ requires us to find the roots of the characteristic equation.
- For higher order equations, this may be difficult, especially if the roots are irrational or complex.

Example 2

- Find the inverse Laplace Transform of the given function.

$$Y(s) = \frac{2}{s}$$

- To find *y*(*t*) such that *y*(*t*) = L^{-1}\{*Y*(*s*)\}, we first rewrite *Y*(*s*):

$$Y(s) = \frac{2}{s} = 2\left(\frac{1}{s}\right)$$

- Using Tables,

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{2}{s}\right\} = 2L^{-1}\left\{\frac{1}{s}\right\} = 2(1) = 2$$

- Thus

$$y(t) = 2$$

Example 3

- Find the inverse Laplace Transform of the given function.

$$Y(s) = \frac{3}{s-5}$$

- To find $y(t)$ such that $y(t) = L^{-1}\{Y(s)\}$, we first rewrite $Y(s)$:

$$Y(s) = \frac{3}{s-5} = 3\left(\frac{1}{s-5}\right)$$

- Using Table ,

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{3}{s-5}\right\} = 3L^{-1}\left\{\frac{1}{s-5}\right\} = 3e^{5t}$$

- Thus

$$y(t) = 3e^{5t}$$

Example 4

- Find the inverse Laplace Transform of the given function.

$$Y(s) = \frac{6}{s^4}$$

- To find $y(t)$ such that $y(t) = L^{-1}\{Y(s)\}$, we first rewrite $Y(s)$:

$$Y(s) = \frac{6}{s^4} = \frac{3!}{s^4}$$

- Using Table,

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{3!}{s^4}\right\} = t^3$$

- Thus

$$y(t) = t^3$$

Example 5

- Find the inverse Laplace Transform of the given function.

$$Y(s) = \frac{8}{s^3}$$

- To find $y(t)$ such that $y(t) = L^{-1}\{Y(s)\}$, we first rewrite $Y(s)$:

$$Y(s) = \frac{8}{s^3} = \left(\frac{8}{2!}\right)\left(\frac{2!}{s^3}\right) = 4\left(\frac{2!}{s^3}\right)$$

- Using Table,

$$L^{-1}\{Y(s)\} = L^{-1}\left\{4\left(\frac{2!}{s^3}\right)\right\} = 4L^{-1}\left\{\frac{2!}{s^3}\right\} = 4t^2$$

- Thus

$$y(t) = 4t^2$$

Example 6

- Find the inverse Laplace Transform of the given function.

$$Y(s) = \frac{4s+1}{s^2+9}$$

- To find $y(t)$ such that $y(t) = L^{-1}\{Y(s)\}$, we first rewrite $Y(s)$:

$$Y(s) = \frac{4s+1}{s^2+9} = 4\left[\frac{s}{s^2+9}\right] + \frac{1}{3}\left[\frac{3}{s^2+9}\right]$$

- Using Table,

$$L^{-1}\{Y(s)\} = 4L^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{1}{3}L^{-1}\left\{\frac{3}{s^2+9}\right\} = 4\cos 3t + \frac{1}{3}\sin 3t$$

- Thus

$$y(t) = 4\cos 3t + \frac{1}{3}\sin 3t$$

Example 7

- Find the inverse Laplace Transform of the given function.

$$Y(s) = \frac{4s+1}{s^2-9}$$

- To find $y(t)$ such that $y(t) = L^{-1}\{Y(s)\}$, we first rewrite $Y(s)$:

$$Y(s) = \frac{4s+1}{s^2-9} = 4\left[\frac{s}{s^2-9}\right] + \frac{1}{3}\left[\frac{3}{s^2-9}\right]$$

- Using Table,

$$L^{-1}\{Y(s)\} = 4L^{-1}\left\{\frac{s}{s^2-9}\right\} + \frac{1}{3}L^{-1}\left\{\frac{3}{s^2-9}\right\} = 4\cosh 3t + \frac{1}{3}\sinh 3t$$

- Thus

$$y(t) = 4\cosh 3t + \frac{1}{3}\sinh 3t$$

Example 8

- Find the inverse Laplace Transform of the given function.

$$Y(s) = -\frac{10}{(s+1)^3}$$

- To find $y(t)$ such that $y(t) = L^{-1}\{Y(s)\}$, we first rewrite $Y(s)$:

$$Y(s) = -\frac{10}{(s+1)^3} = -\frac{10}{2!}\left[\frac{2!}{(s+1)^3}\right] = -5\left[\frac{2!}{(s+1)^3}\right]$$

- Using Table,

$$L^{-1}\{Y(s)\} = -5L^{-1}\left\{\frac{2!}{(s+1)^3}\right\} = -5t^2e^{-t}$$

- Thus

$$y(t) = -5t^2e^{-t}$$

Example 9

- For the function $Y(s)$ below, we find $y(t) = L^{-1}\{Y(s)\}$ by using a partial fraction expansion, as follows.

$$Y(s) = \frac{3s+1}{s^2+s-12} = \frac{3s+1}{(s+4)(s-3)} = \frac{A}{s+4} + \frac{B}{s-3}$$

$$3s+1 = A(s-3) + B(s+4)$$

$$3s+1 = (A+B)s + (4B-3A)$$

$$A+B = 3, \quad 4B-3A = 1$$

$$A = 11/7, \quad B = 10/7$$

$$Y(s) = \frac{11}{7}\left[\frac{1}{s+4}\right] + \frac{10}{7}\left[\frac{1}{s-3}\right] \Rightarrow y(t) = \frac{11}{7}e^{-4t} + \frac{10}{7}e^{3t}$$

Example 10

- For the function $Y(s)$ below, we find $y(t) = L^{-1}\{Y(s)\}$ by **completing the square in the denominator and rearranging the numerator**, as follows.

$$Y(s) = \frac{4s-10}{s^2-6s+10} = \frac{4s-10}{\left(s^2-6s+9\right)+1} = \frac{4s-12+2}{(s-3)^2+1}$$

$$= \frac{4(s-3)+2}{(s-3)^2+1} = 4\left[\frac{s-3}{(s-3)^2+1}\right] + 2\left[\frac{1}{(s-3)^2+1}\right]$$

- Using Table, we obtain

$$y(t) = 4e^{3t}\cos t + 2e^{3t}\sin t$$

Example 11: Initial Value Problem

- Consider the initial value problem

$$y'' - 8y' + 25y = 0, \quad y(0) = 0, \;\; y'(0) = 6$$

- Taking the Laplace transform of the differential equation, and assuming the conditions of Corollary are met, we have

$$\left[s^2 L\{y\} - sy(0) - y'(0)\right] - 8\left[sL\{y\} - y(0)\right] + 25L\{y\} = 0$$

- Letting $Y(s) = L\{y\}$, we have

$$\left(s^2 - 8s + 25\right)Y(s) - \left(s - 8\right)y(0) - y'(0) = 0$$

- Substituting in the initial conditions, we obtain

$$\left(s^2 - 8s + 25\right)Y(s) - 6 = 0$$

- Thus

$$L\{y\} = Y(s) = \frac{6}{s^2 - 8s + 25}$$

Example 11----control !!

- Completing the square, we obtain

$$Y(s) = \frac{6}{s^2 - 8s + 25} = \frac{6}{\left(s^2 - 8s + 16\right) + 9}$$

- Thus

$$Y(s) = 2\left[\frac{3}{\left(s-4\right)^2 + 9}\right]$$

- Using Table 6.2.1, we have

$$L^{-1}\{Y(s)\} = 2\,e^{4t}\sin 3t$$

- Therefore our solution to the initial value problem is

$$y(t) = 2\,e^{4t}\sin 3t$$

Example 12: Nonhomogeneous Problem

- Consider the initial value problem

$$y'' + y = \sin 2t, \quad y(0) = 2, \;\; y'(0) = 1$$

- Taking the Laplace transform of the differential equation, and assuming the conditions of Corollary are met, we have

$$\left[s^2 L\{y\} - sy(0) - y'(0)\right] + L\{y\} = 2/(s^2 + 4)$$

- Letting $Y(s) = L\{y\}$, we have

$$\left(s^2 + 1\right) Y(s) - sy(0) - y'(0) = 2/(s^2 + 4)$$

- Substituting in the initial conditions, we obtain

$$\left(s^2 + 1\right) Y(s) - 2s - 1 = 2/(s^2 + 4)$$

- Thus

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)}$$

Example 12: Solution (2 of 2)

- Using partial fractions,

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

- Then

$$2s^3 + s^2 + 8s + 6 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$
$$= (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D)$$

- Solving, we obtain $A = 2$, $B = 5/3$, $C = 0$, and $D = -2/3$. Thus

$$Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}$$

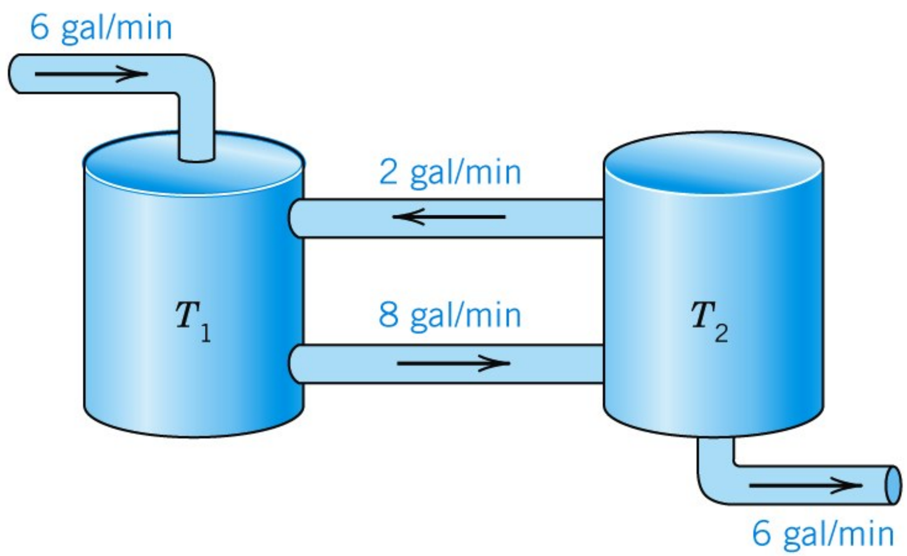
- Hence

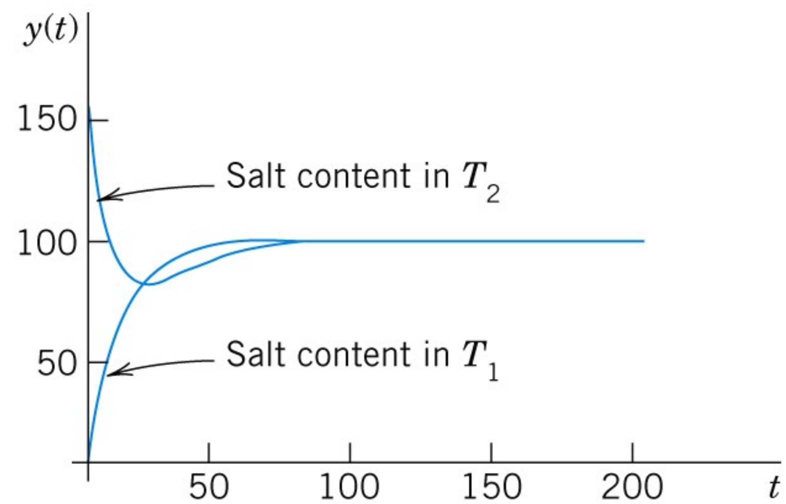
$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

Mixing Problem Involving Two Tanks

- Tank T_1 in Figure initially contains 100 gal of pure water. Tank T_2 initially contains 100 gal of water in which 150 lb of salt are dissolved. The inflow into T_1 is 2 gal/min
- from T_2 and 6 gal/min containing 6 lb of salt from the outside. The inflow into T_2 is 8 gal/min from T_1.
- The outflow from T_2 is $2 + 6 = 8$ gal/min, as shown in the figure. The mixtures are kept uniform by stirring. Find and plot the salt contents $y_1(t)$ and $y_2(t)$ in T_1 and T_2, respectively.

Mixing Problem Involving Two Tanks





Mixing Problem Involving Two Tanks

- The model is obtained in the form of two equations

Time rate of change = Inflow/min − Outflow/min for the two tanks. Thus,

$$y_1' = -\frac{8}{100} y_1 + \frac{2}{100} y_2 + 6.$$
$$y_2' = \frac{8}{100} y_1 - \frac{8}{100} y_2.$$

- The initial conditions are $y_1(0) = 0$, $y_2(0) = 150$. From this
- we see that the subsidiary system (2) is

$$(-0.08 - s)Y_1 + 0.02Y_2 = -\frac{6}{s}$$
$$0.08Y_1 + (-0.08 - s)Y_2 = -150$$

We solve this algebraically for Y_1 and Y_2 by elimination (or by Cramer's rule), and we write the solutions in terms of partial fractions,

$$Y_1 = \frac{9s + 0.48}{s(s+0.12)(s+0.04)} = \frac{100}{s} - \frac{62.5}{s+0.12} - \frac{37.5}{s+0.04}$$

$$Y_2 = \frac{150s^2 + 12s + 0.48}{s(s+0.12)(s+0.04)} = \frac{100}{s} + \frac{125}{s+0.12} - \frac{75}{s+0.04}$$

By taking the inverse transform we arrive at the solution

$$y_1 = 100 - 62.5e^{-0.12t} - 37.5e^{-0.04t}$$
$$y_2 = 100 + 125e^{-0.12t} - 75e^{-0.04t}.$$

- Check the plot of these functions. Can you give physical explanations for their main features?
- Why do they have the limit 100?
- Why is y_2 not monotone, whereas y_1 is?
- Why is y_1 from some time on suddenly larger than y_2?