



BLM2502

Theory of

Computation

Spring 2016

BLM2502 Theory of Computation

» Course Outline

» Week Content

- » 1 Introduction to Course
- » 2 Computability Theory, Complexity Theory, Automata Theory, Set Theory, Relations, Proofs, Pigeonhole Principle
- » 3 Regular Expressions
- » 4 Finite Automata
- » 5 Deterministic and Nondeterministic Finite Automata
- » 6 Epsilon Transition, Equivalence of Automata
- » 7 Pumping Theorem
- » 8 April 10 - 14 week is the first midterm week
- » 9 Context Free Grammars, Parse Tree, Ambiguity
- » 10 Pumping Theorem, Normal Forms
- » 11 Pushdown Automata
- » 12 **Turing Machines, Recognition and Computation, Church-Turing Hypothesis**
- » 13 Turing Machines, Recognition and Computation, Church-Turing Hypothesis
- » 14 May 22 – 27 week is the second midterm week
- » 15 Review
- » 16 Final Exam date will be announced



Turing Machines

The Language Hierarchy

$a^n b^n c^n$?

ww ?

Context-Free Languages

$a^n b^n$

ww^R

Regular Languages

a^*

$a^* b^*$

The diagram consists of three concentric ellipses. The outermost ellipse is labeled 'Languages accepted by Turing Machines'. Inside it is an ellipse labeled 'Context-Free Languages'. Inside that is the innermost ellipse labeled 'Regular Languages'. Each ellipse contains examples of languages belonging to that class.

Languages accepted by
Turing Machines

$a^n b^n c^n$

ww

Context-Free Languages

$a^n b^n$

ww^R

Regular Languages

a^*

$a^* b^*$

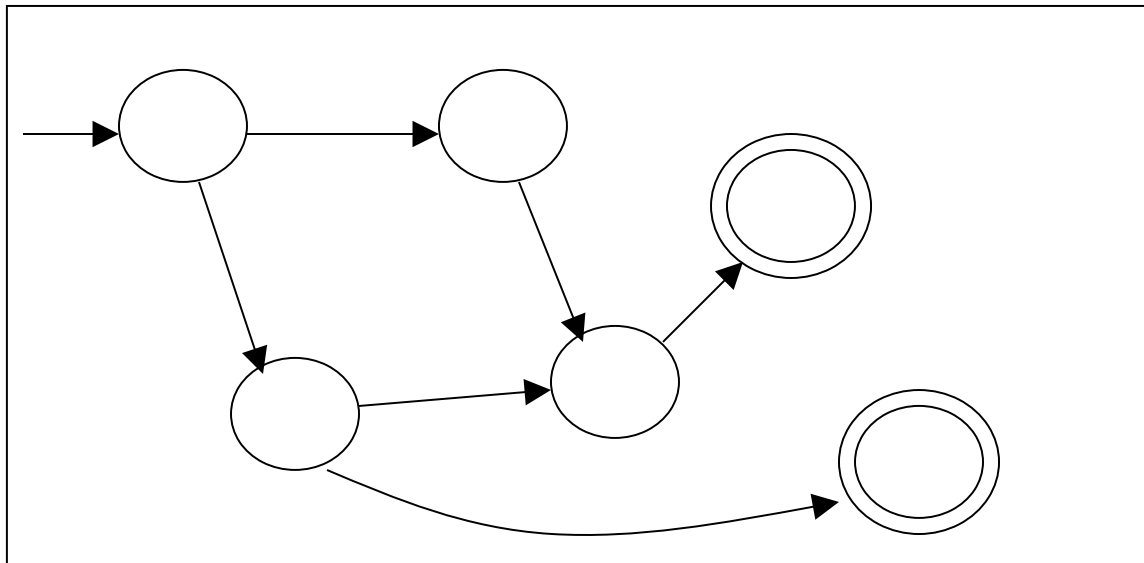
A Turing Machine

Tape



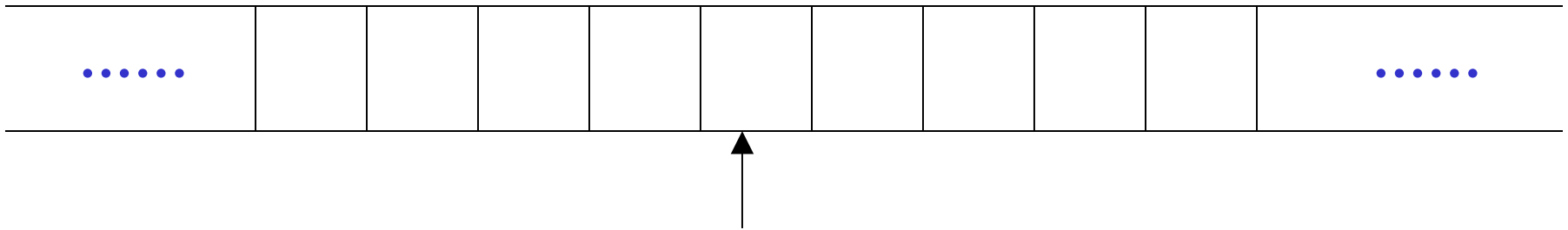
Read-Write head

Control Unit



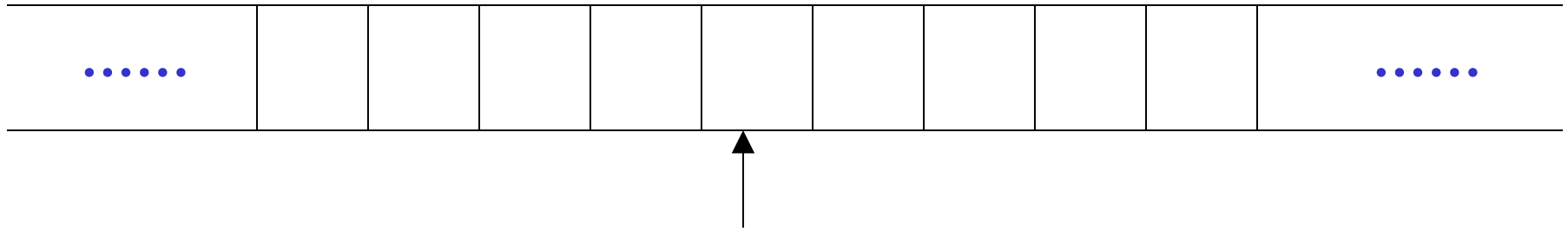
The Tape

No boundaries -- infinite length



Read-Write head

The head moves Left or Right



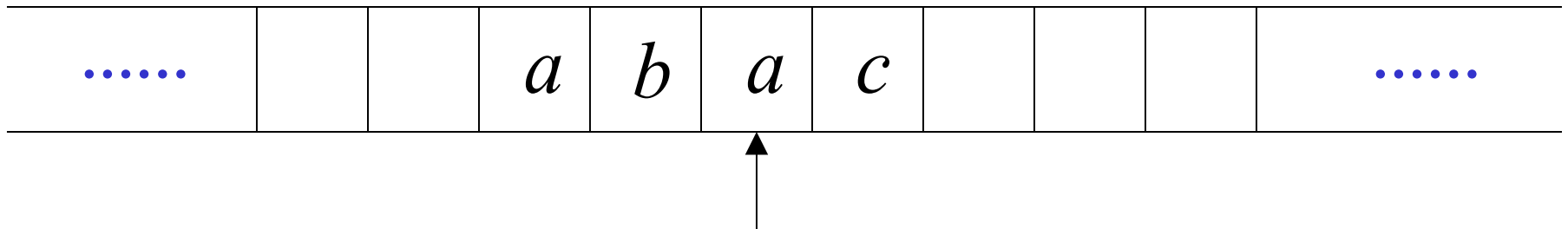
Read-Write head

The head at each transition (time step):

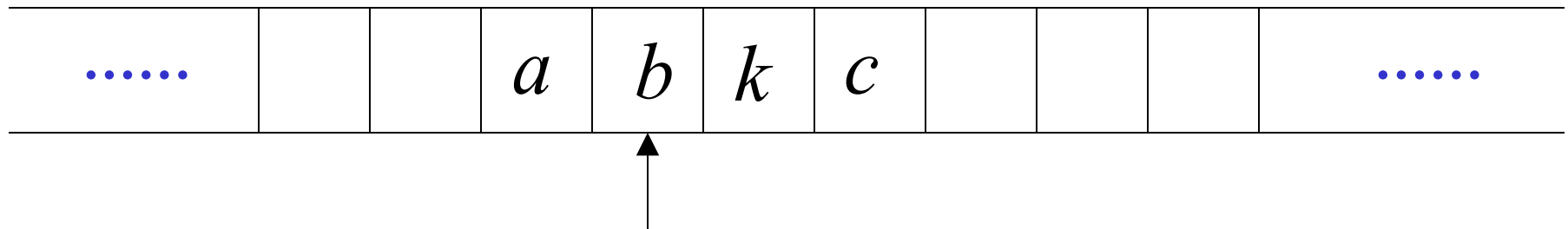
1. Reads a symbol
2. Writes a symbol
3. Moves Left or Right

Example:

Time 0

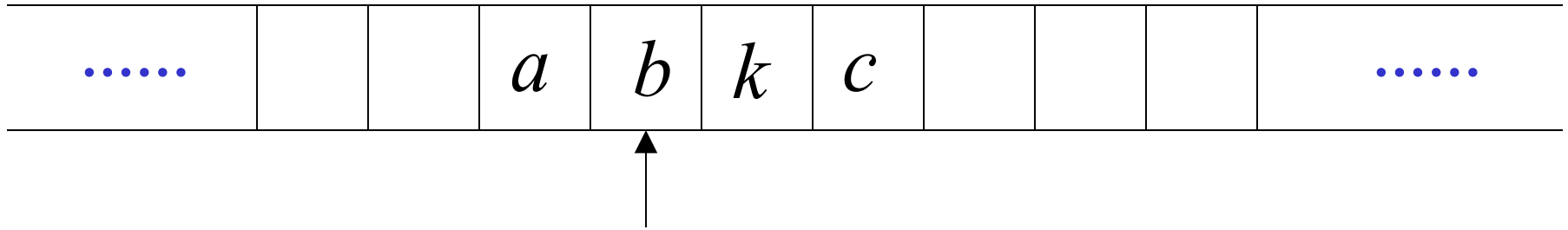


Time 1

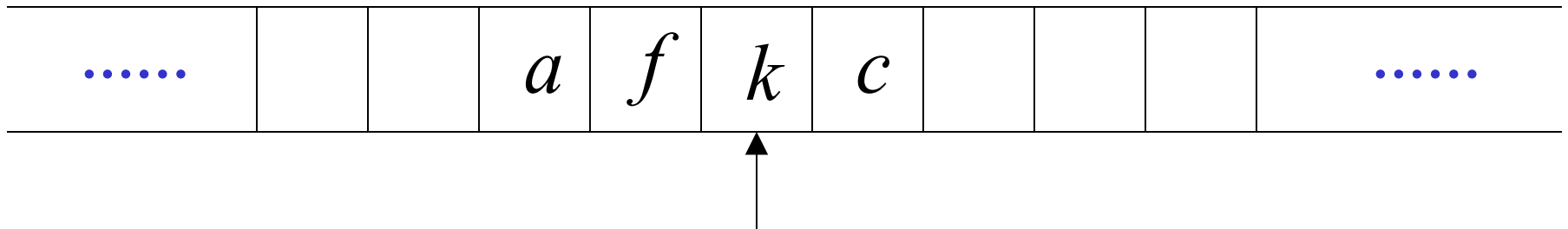


1. Reads a
2. Writes k
3. Moves Left

Time 1

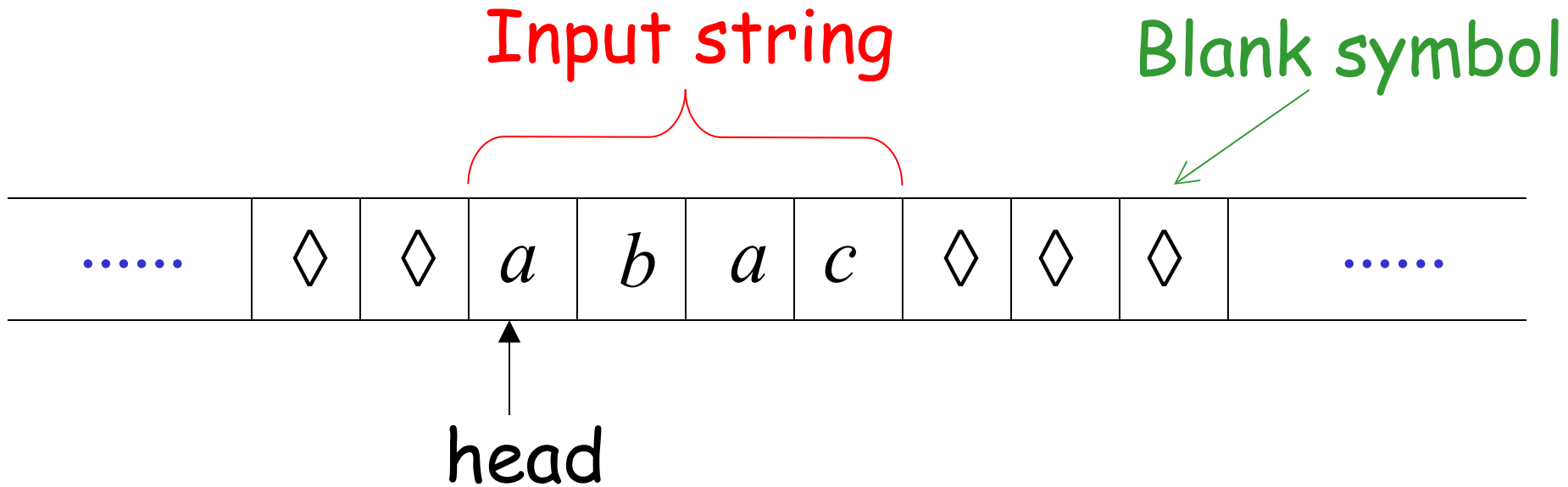


Time 2



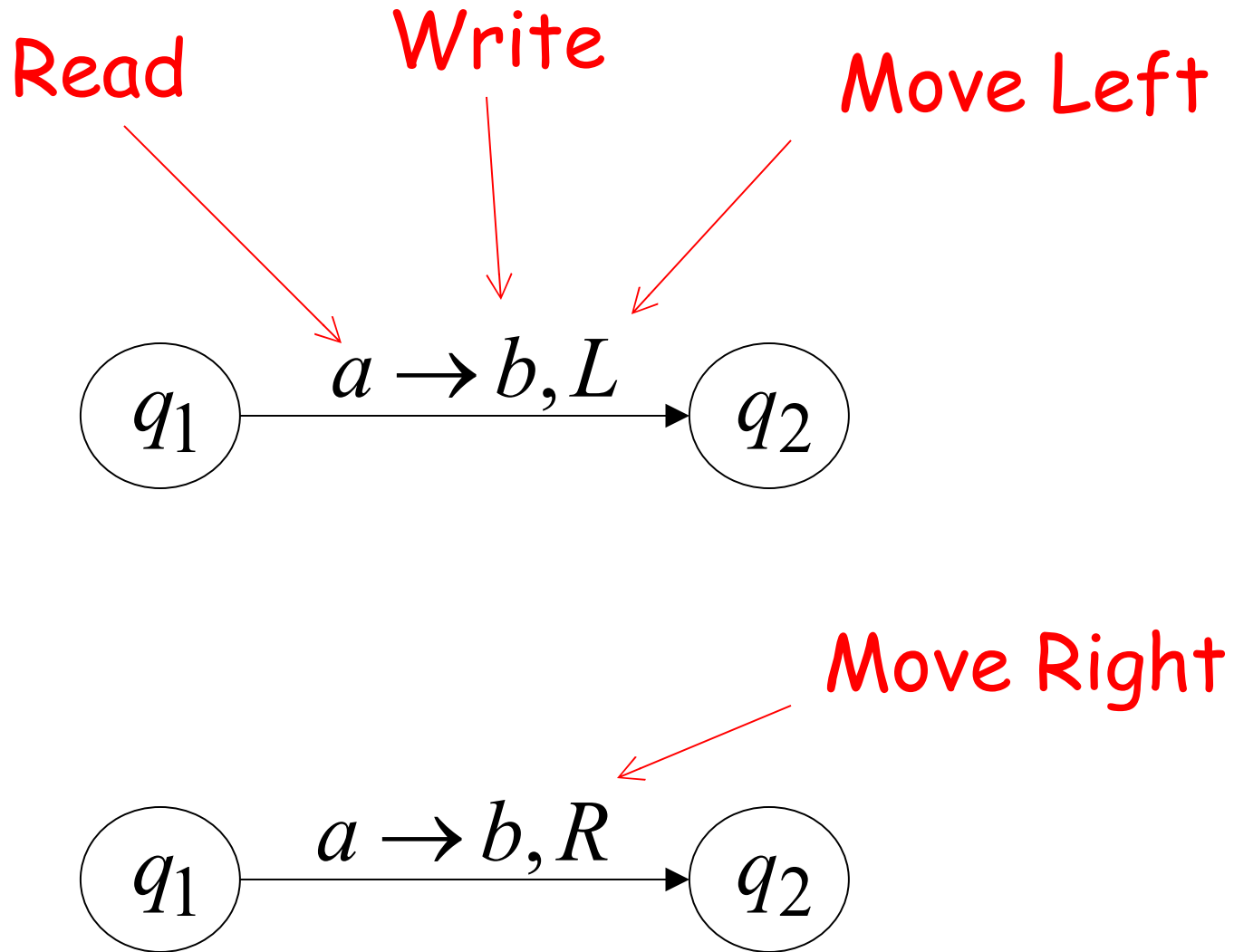
1. Reads b
2. Writes f
3. Moves Right

The Input String



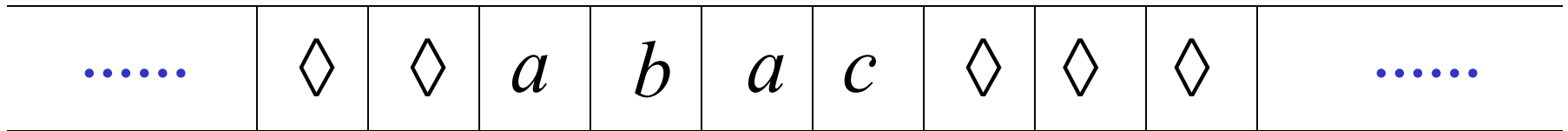
Head starts at the leftmost position
of the input string

States & Transitions



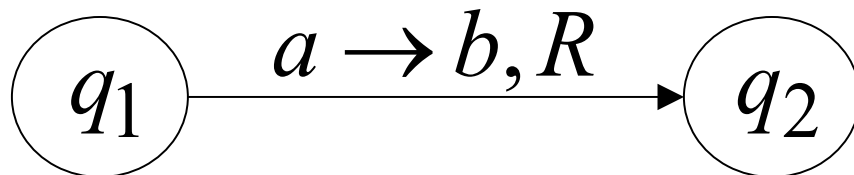
Example:

Time 1

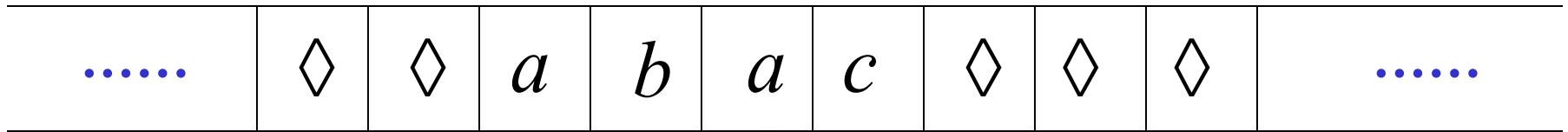


q_1

current state

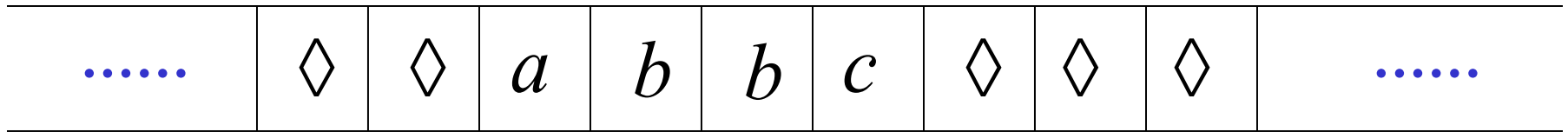


Time 1

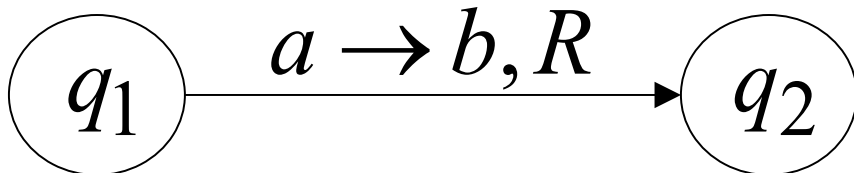


q_1

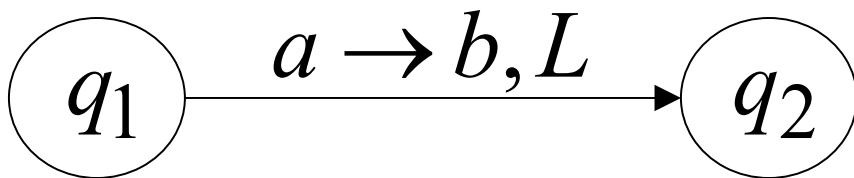
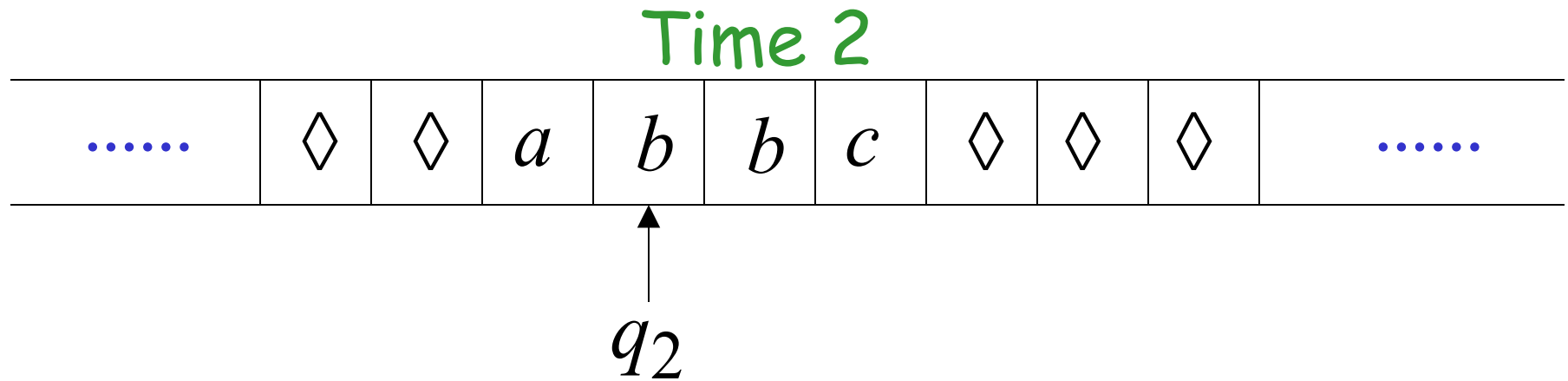
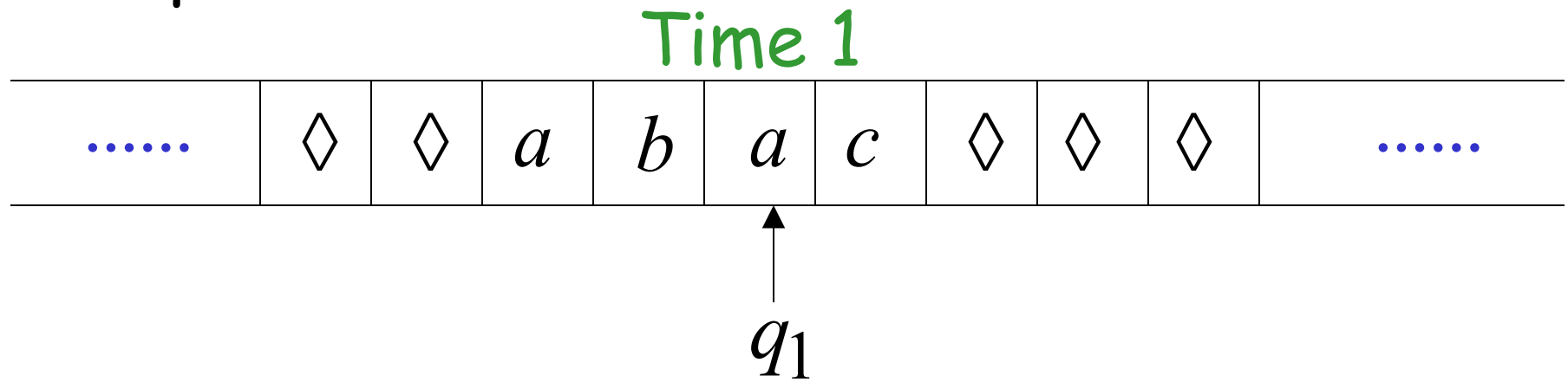
Time 2



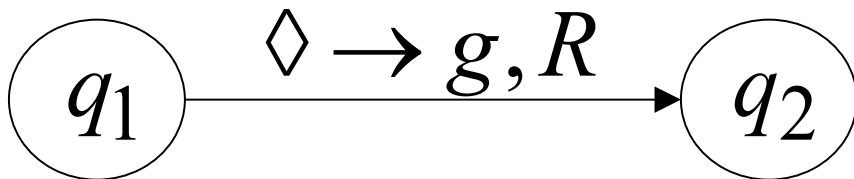
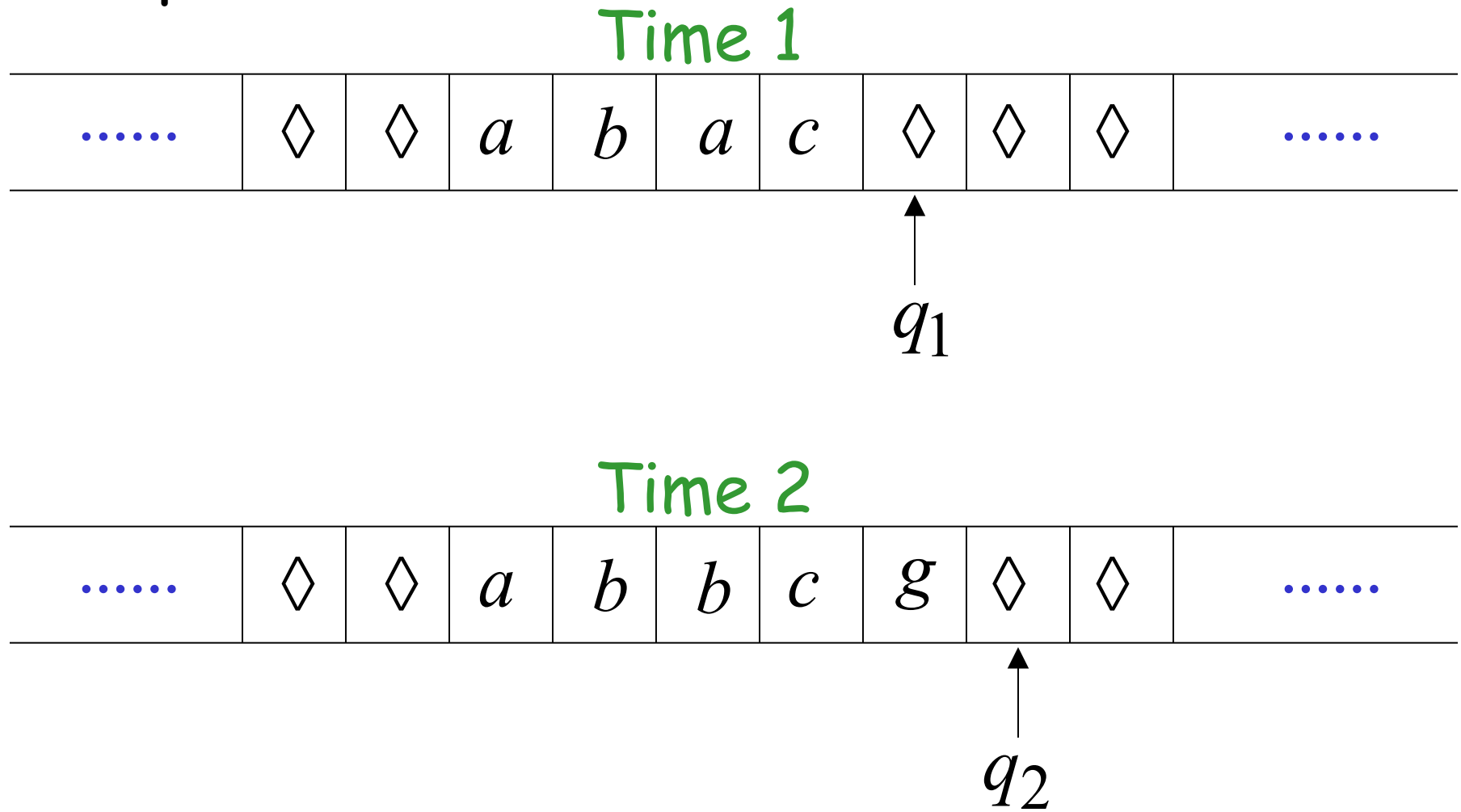
q_2



Example:



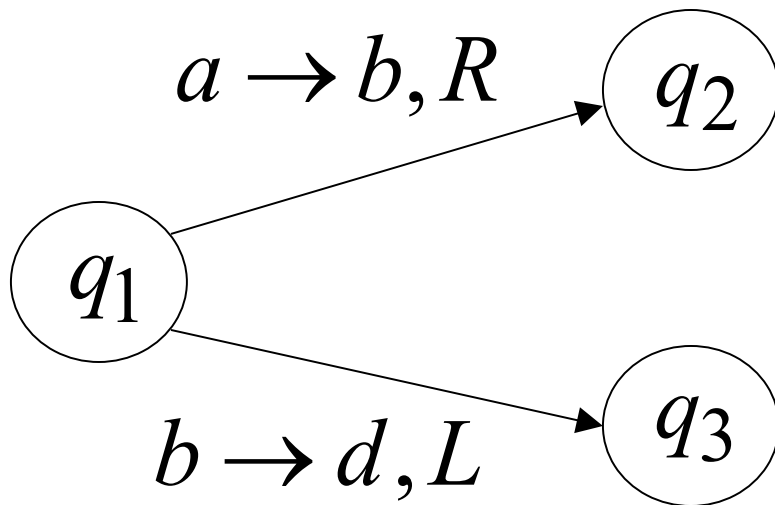
Example:



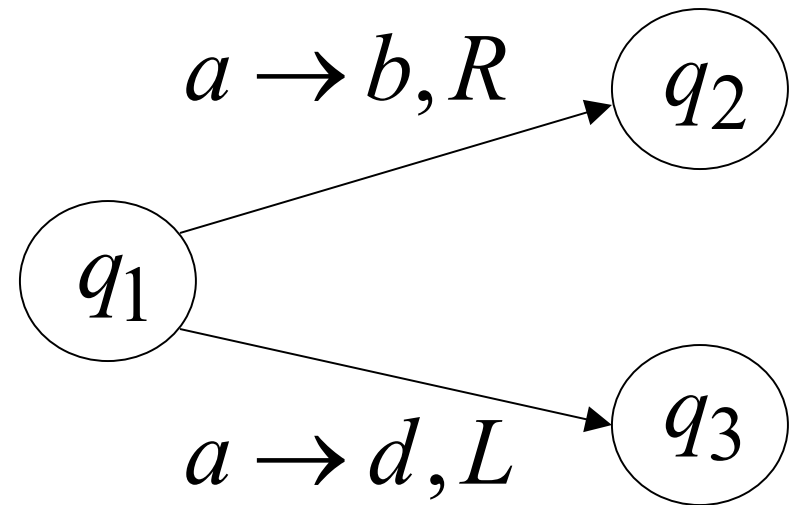
Determinism

Turing Machines are deterministic

Allowed



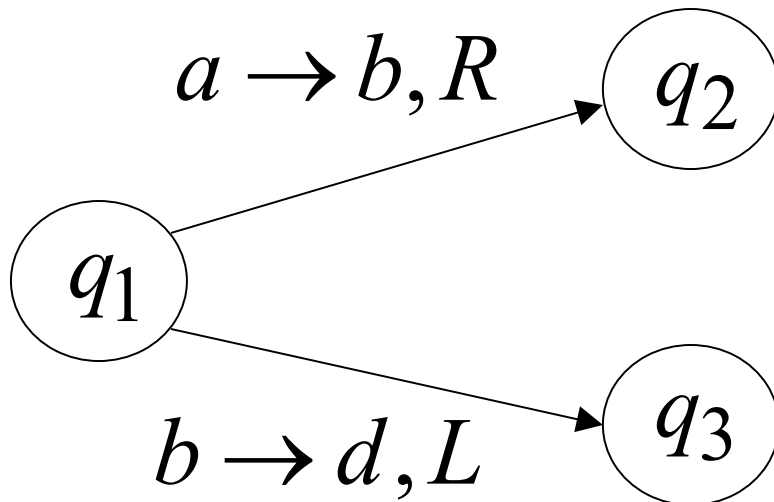
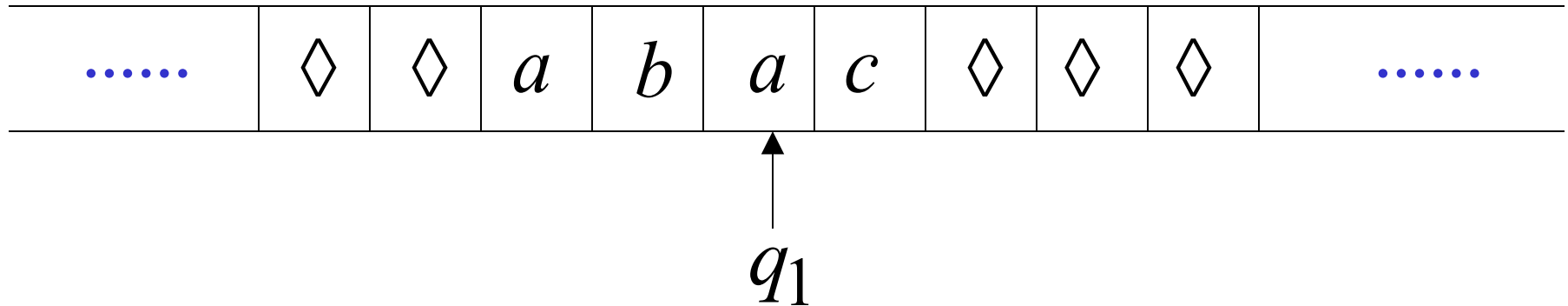
Not Allowed



No epsilon transitions allowed

Partial Transition Function

Example:



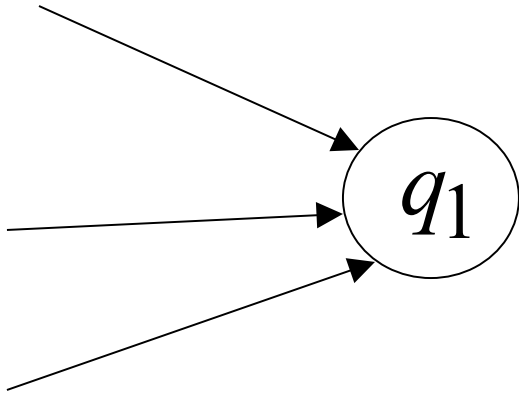
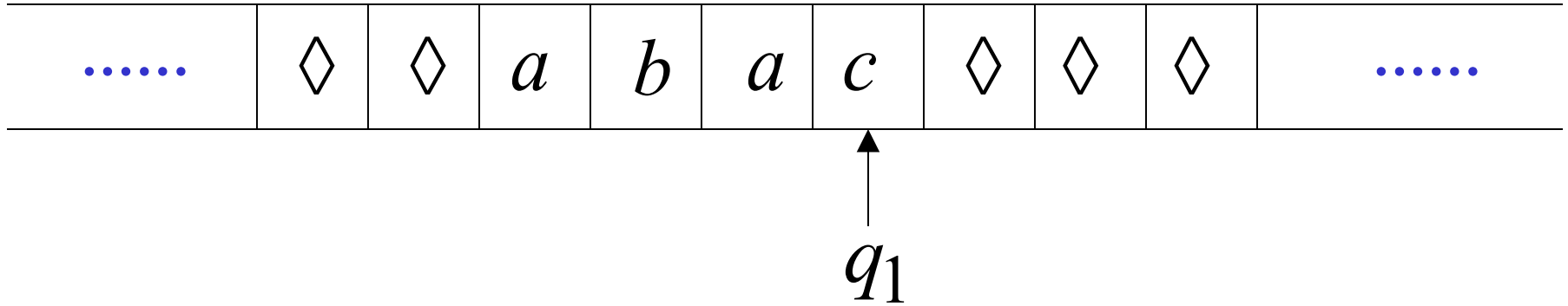
Allowed:

No transition
for input symbol c

Halting

The machine **halts** in a state if there is no transition to follow

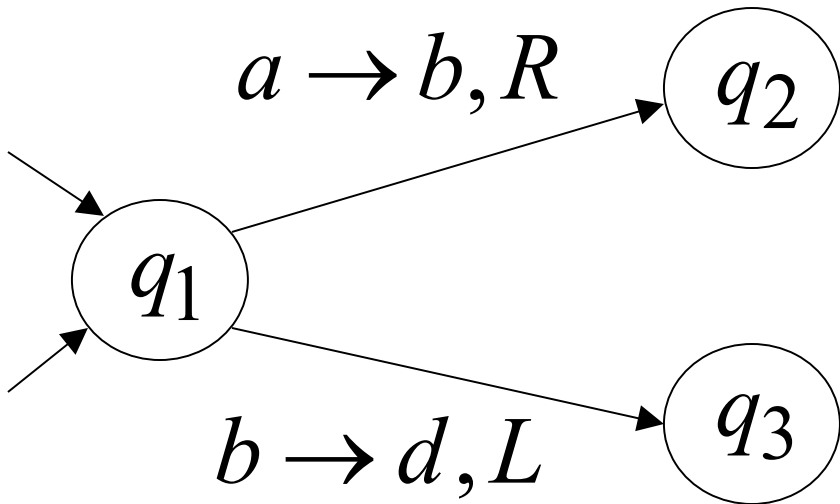
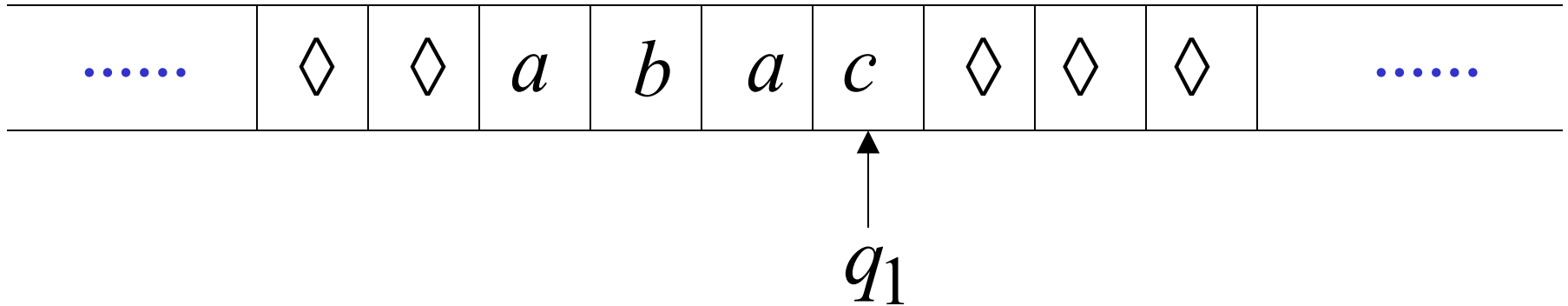
Halting Example 1:



No transition from q_1

HALT!!!

Halting Example 2:



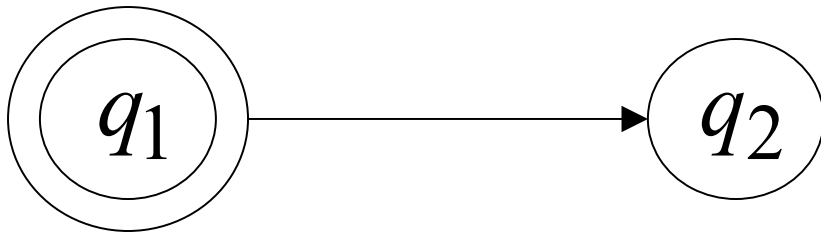
No possible transition
from q_1 and symbol c

HALT!!!

Accepting States



Allowed



Not Allowed

- Accepting states have no outgoing transitions
- The machine halts and accepts

Acceptance

Accept Input
string



If machine halts
in an accept state

Reject Input
string



If machine halts
in a non-accept state

or

If machine enters
an *infinite loop*

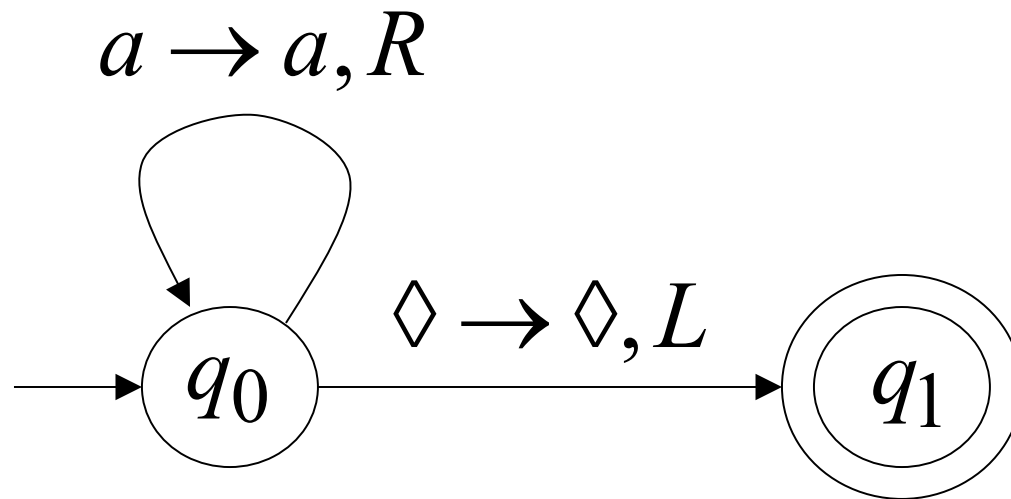
Observation:

In order to accept an input string,
it is not necessary to scan all the
symbols in the string

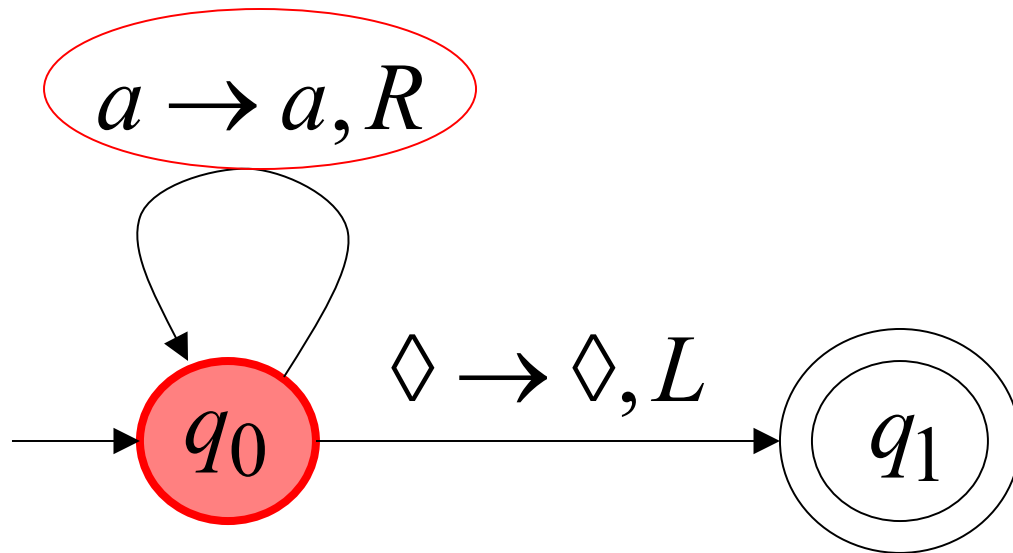
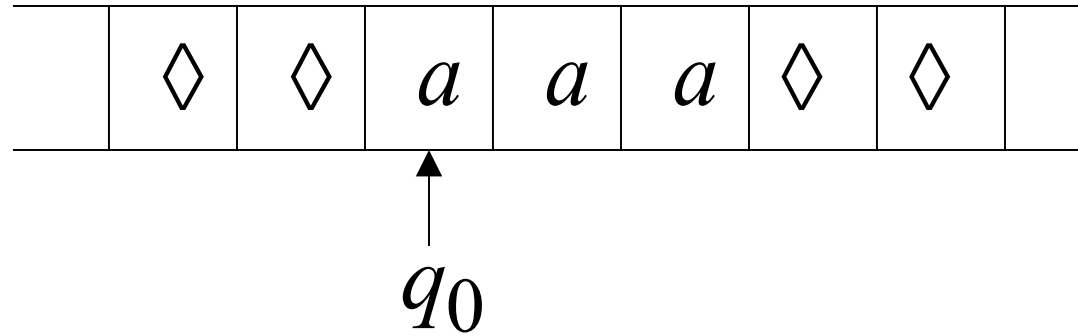
Turing Machine Example

Input alphabet $\Sigma = \{a, b\}$

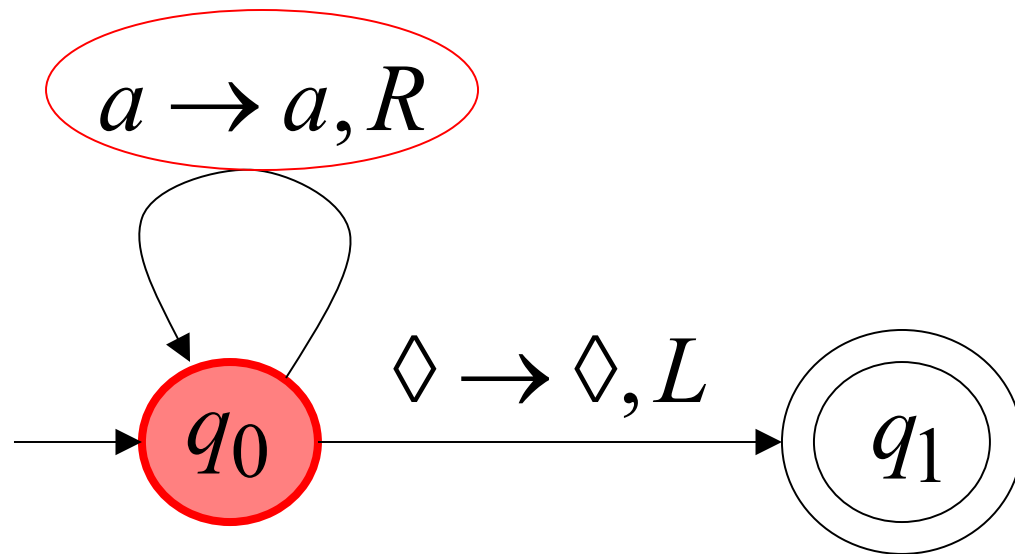
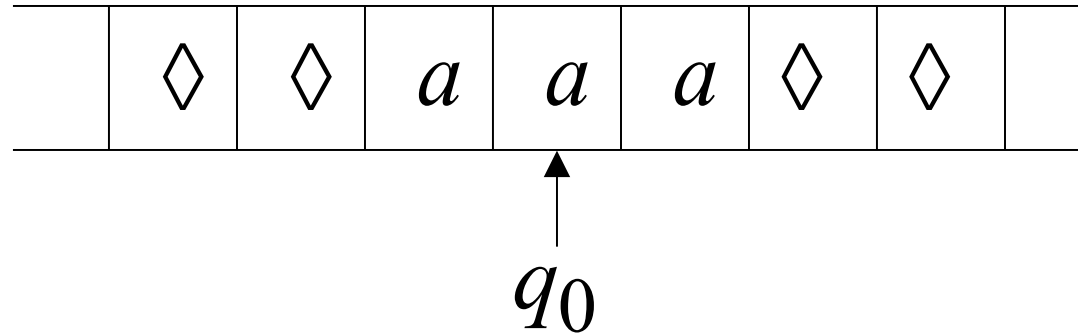
Accepts the language: a^*



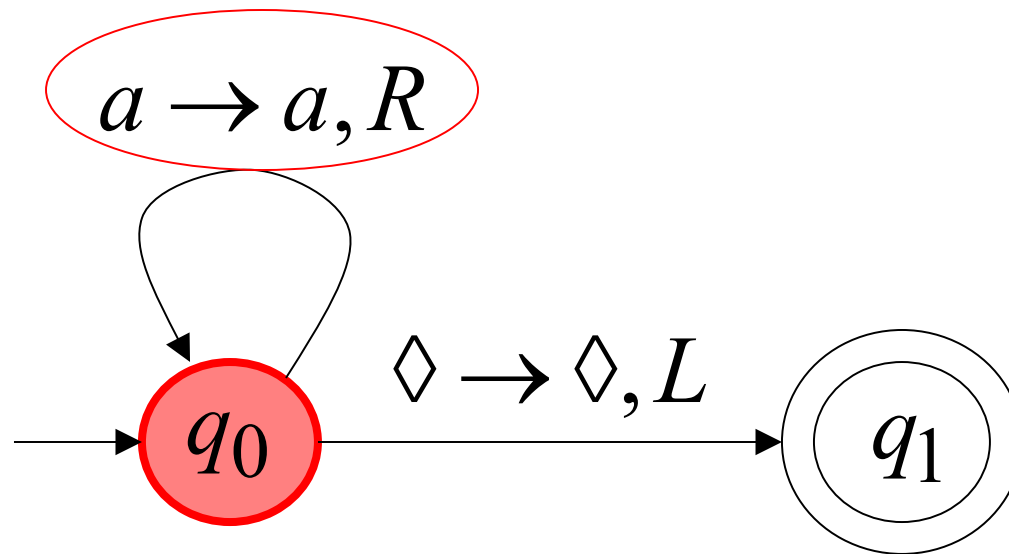
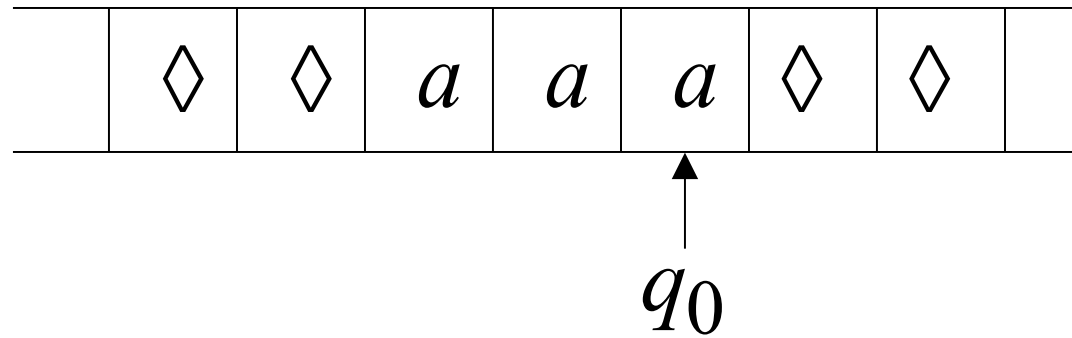
Time 0



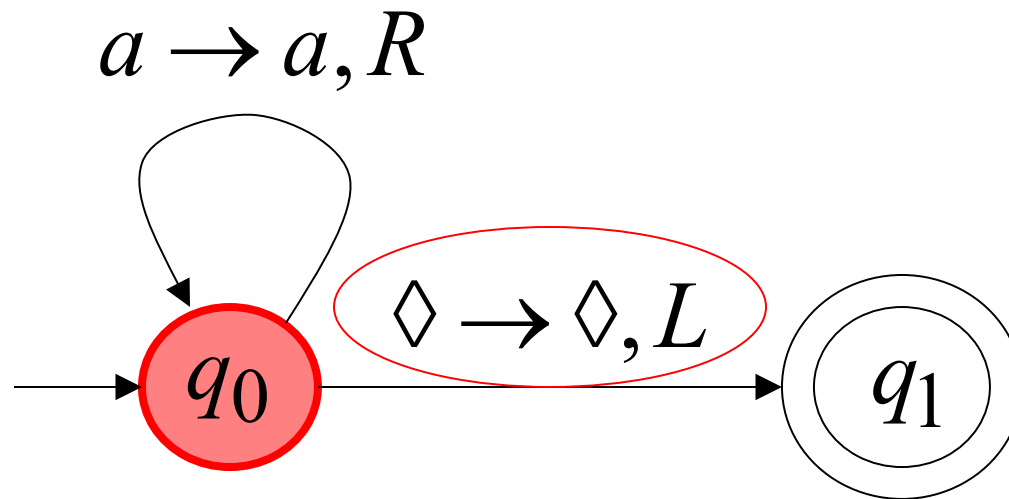
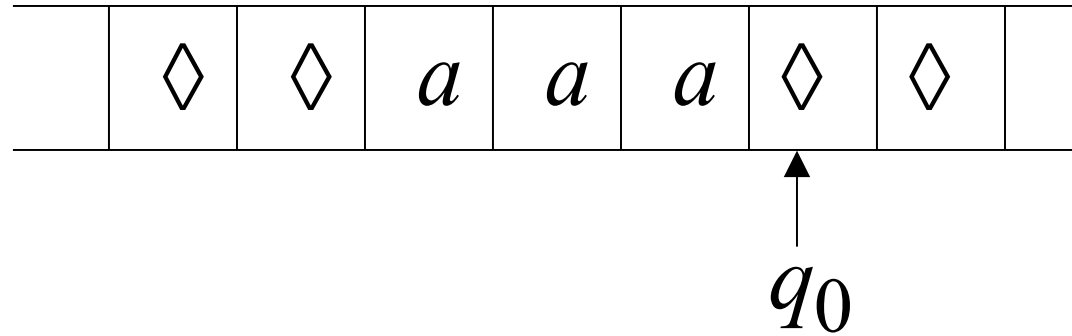
Time 1



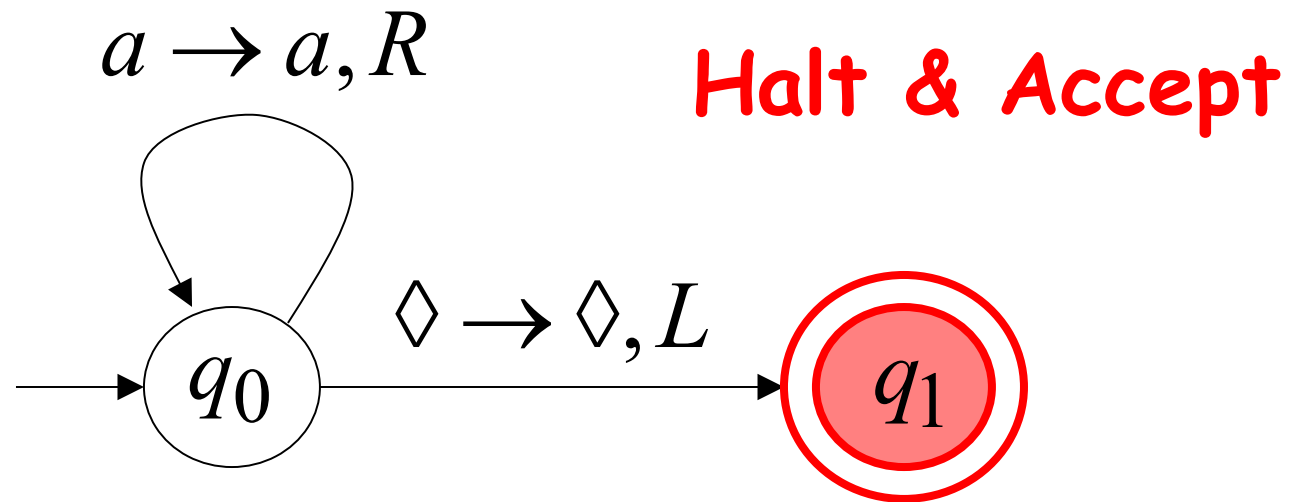
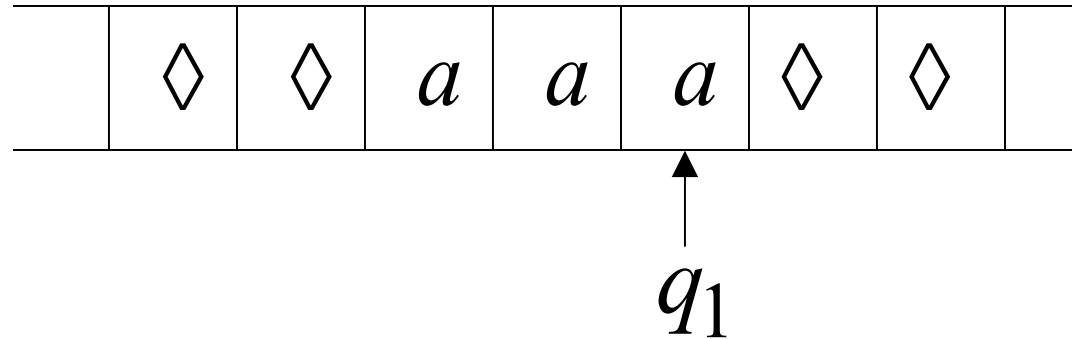
Time 2



Time 3

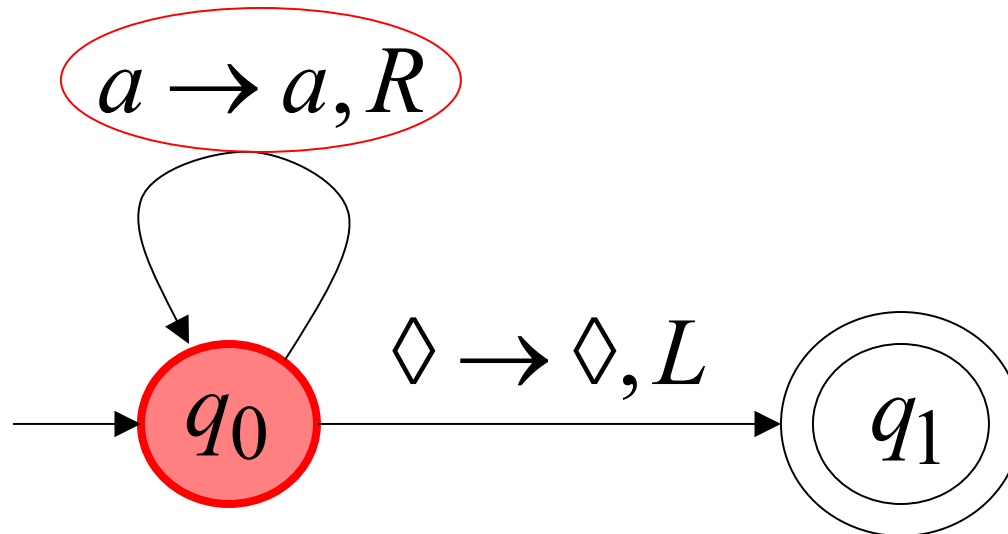
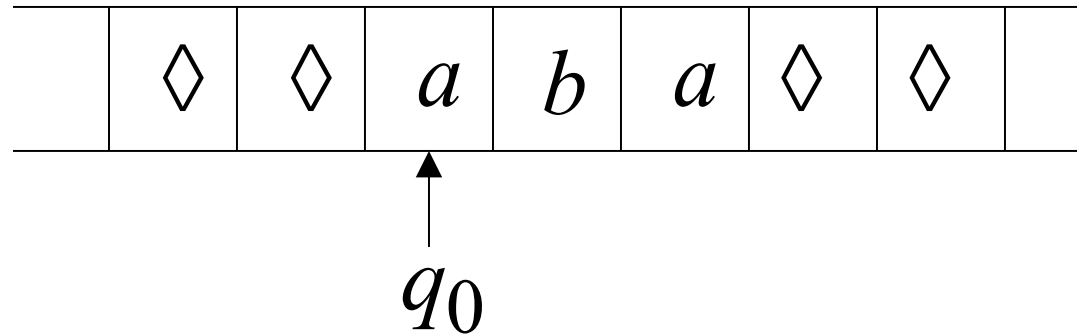


Time 4

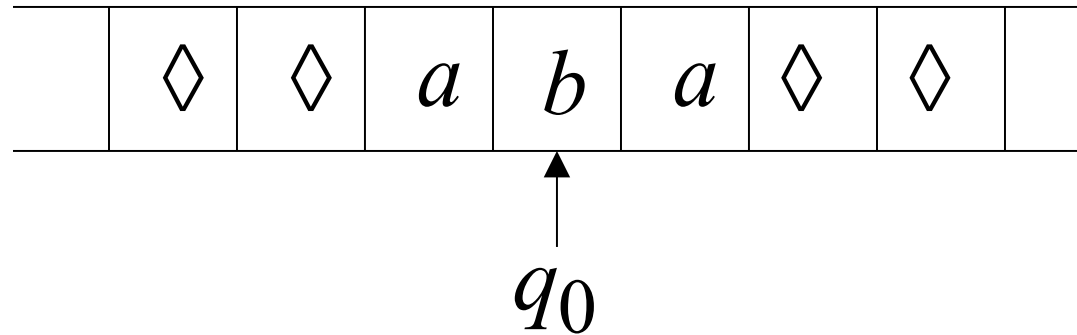


Rejection Example

Time 0

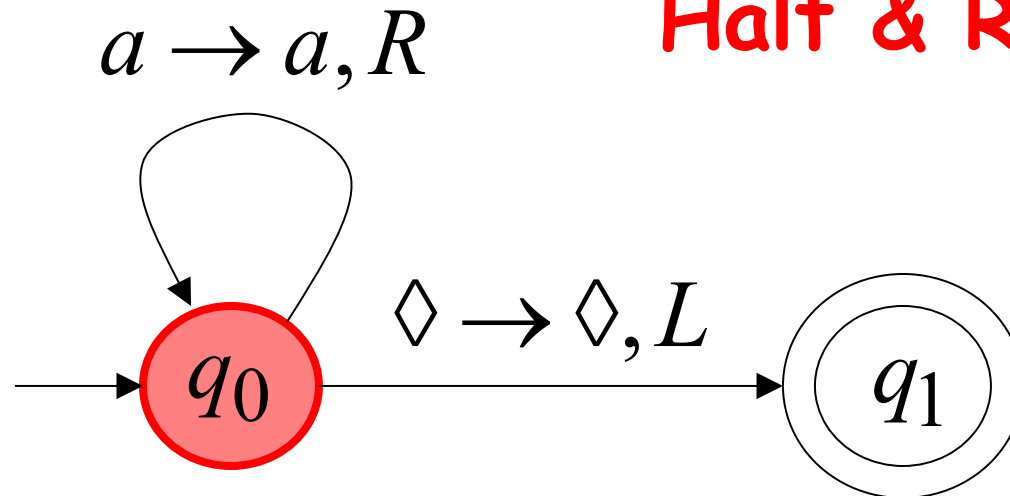


Time 1



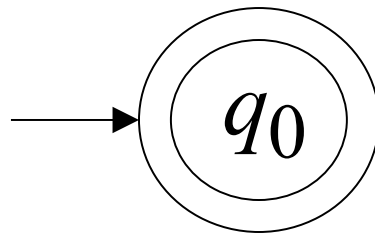
No possible Transition

Halt & Reject

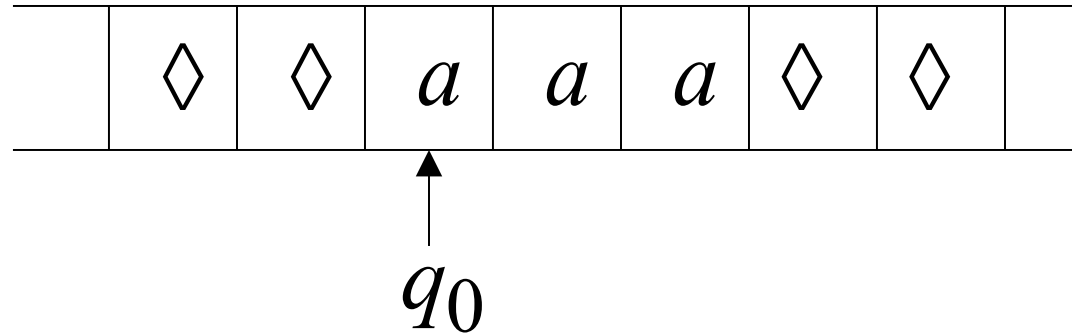


A simpler machine for same language
but for input alphabet $\Sigma = \{a\}$

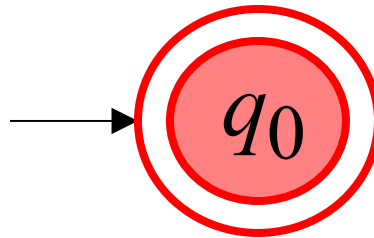
Accepts the language: a^*



Time 0



Halt & Accept



Not necessary to scan input

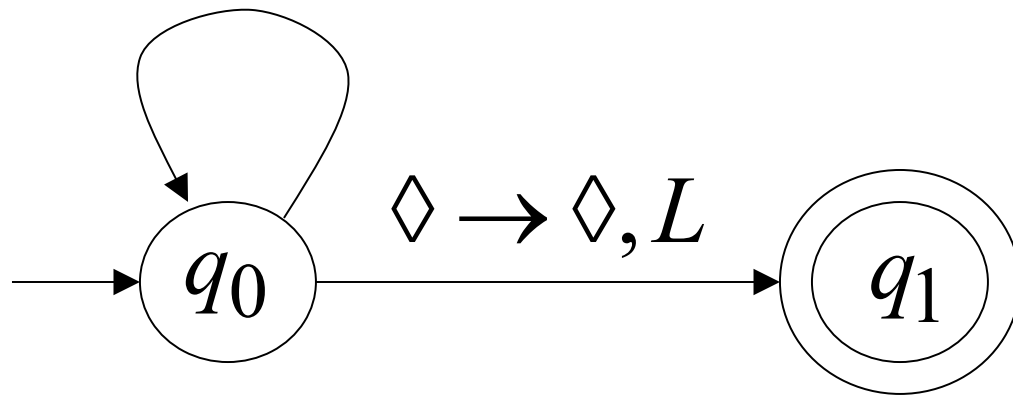
Infinite Loop Example

A Turing machine

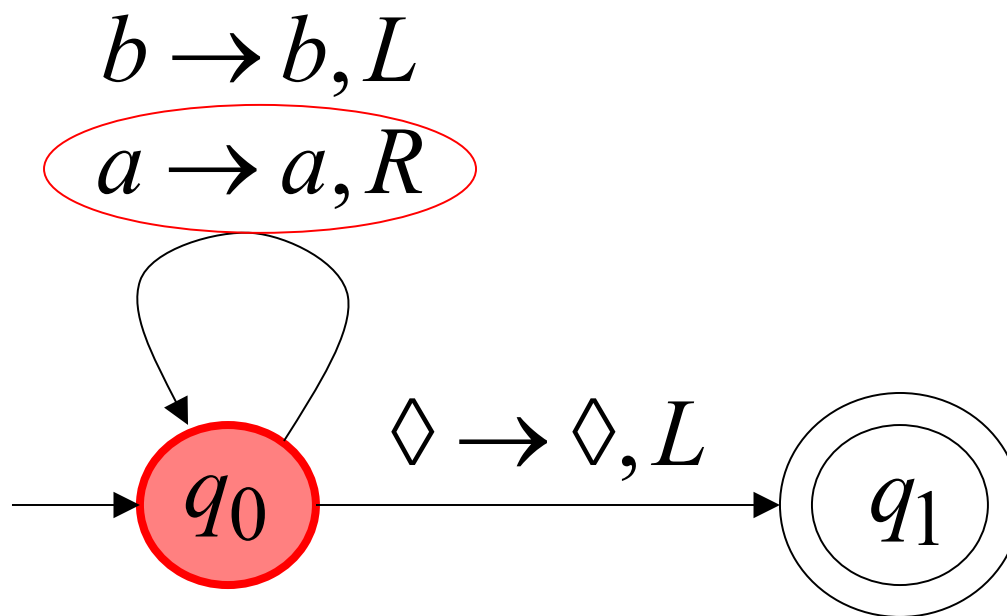
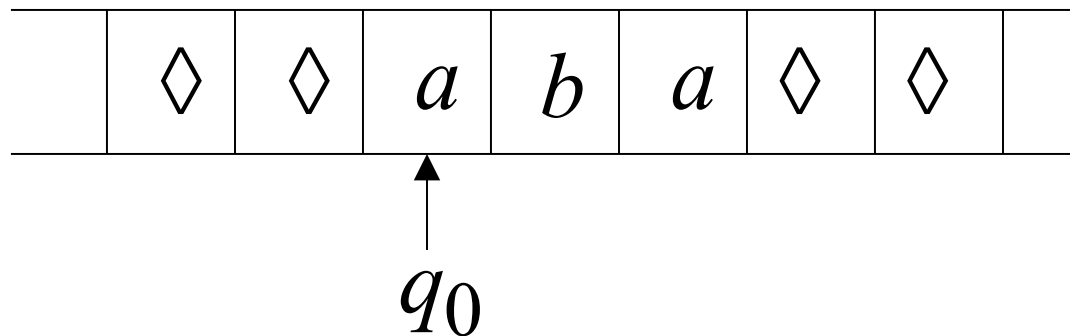
for language $a^* + b(a + b)^*$

$b \rightarrow b, L$

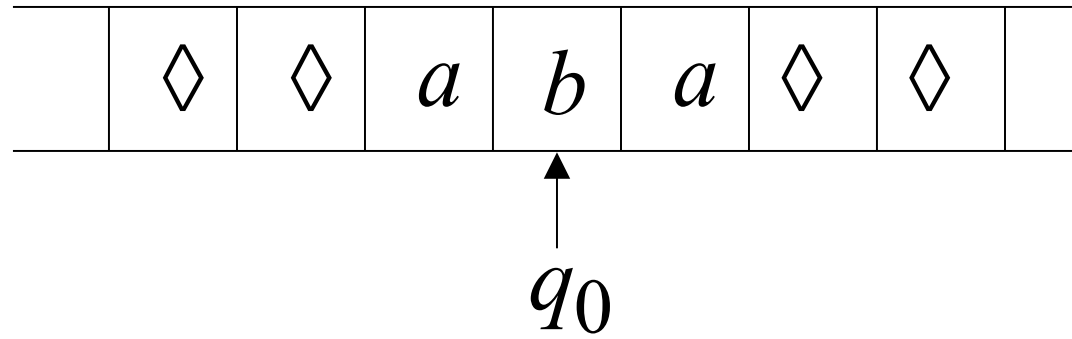
$a \rightarrow a, R$



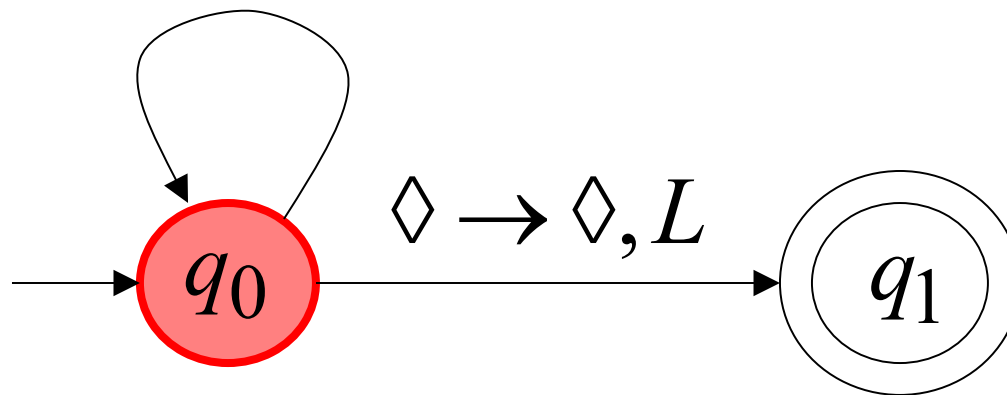
Time 0



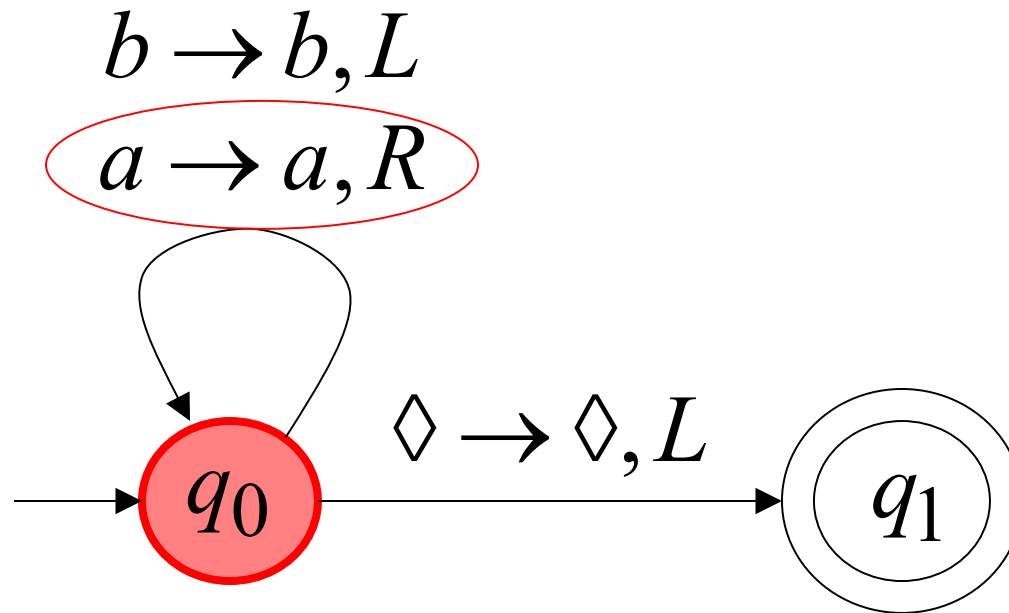
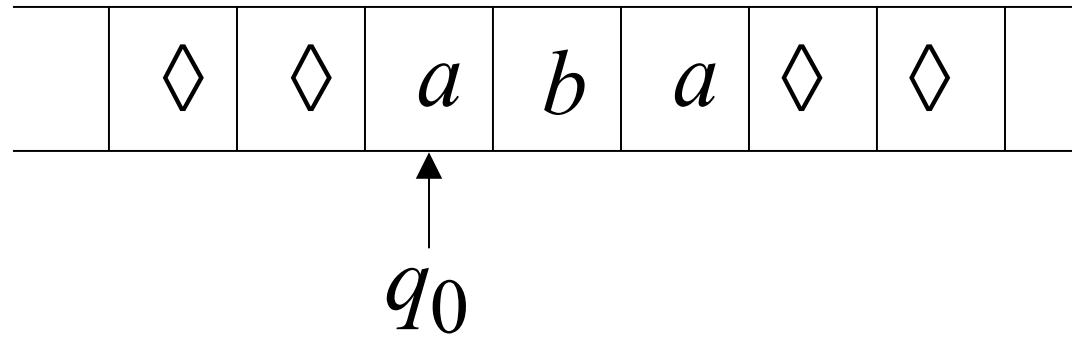
Time 1



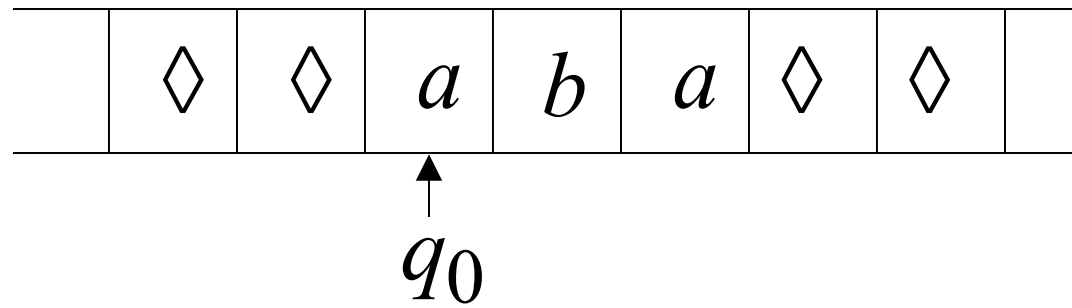
$b \rightarrow b, L$
 $a \rightarrow a, R$



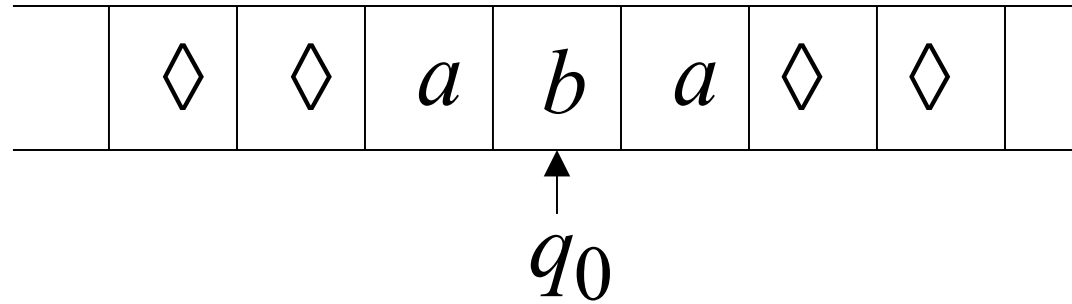
Time 2



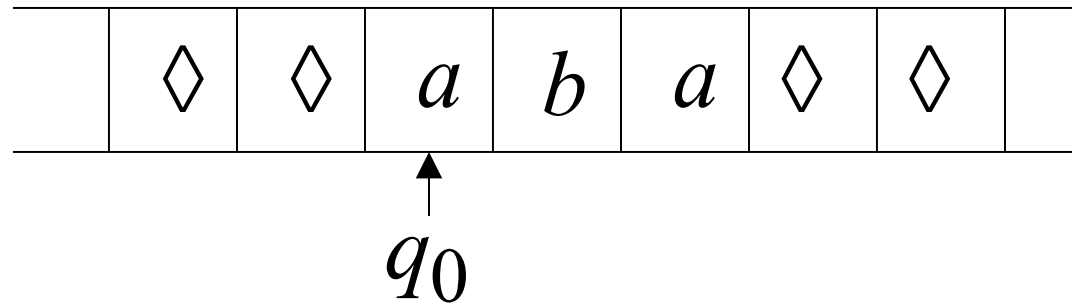
Time 2



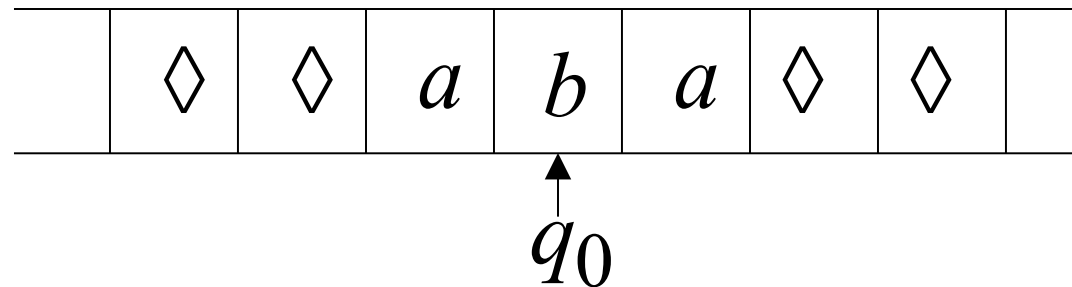
Time 3



Time 4



Time 5



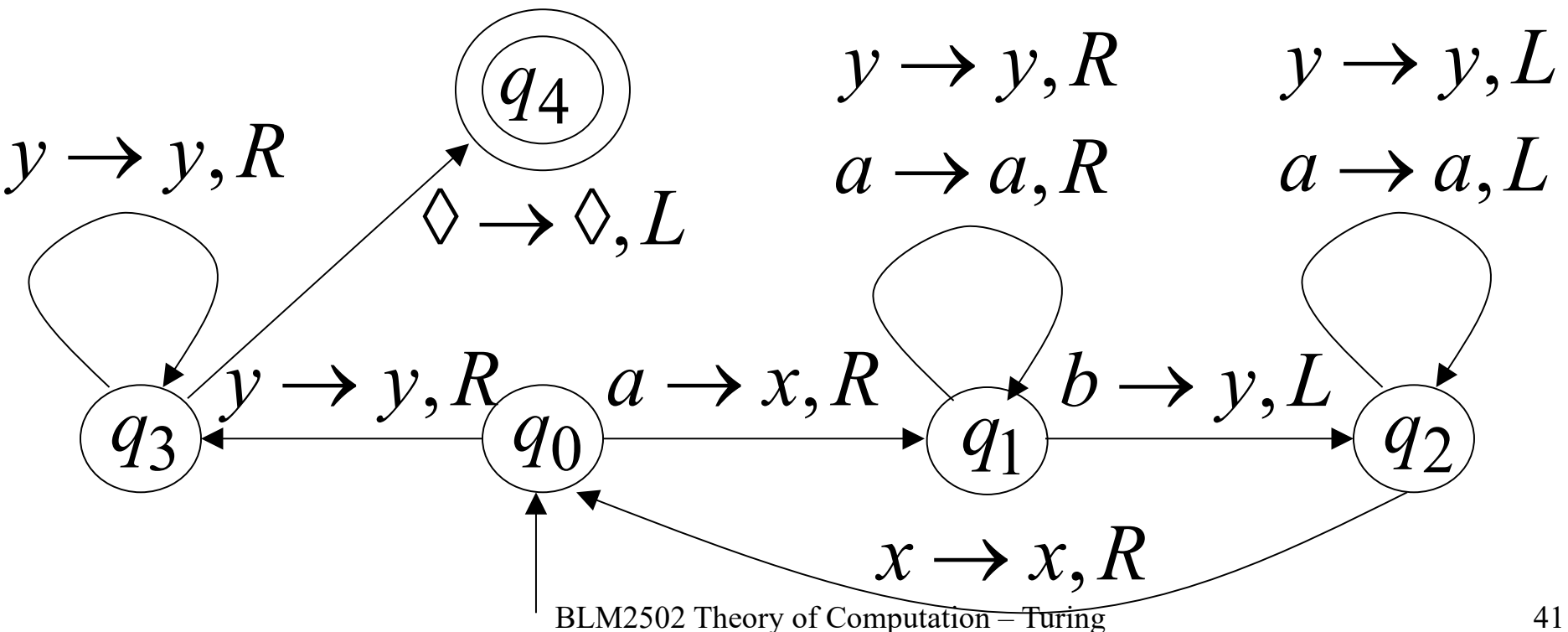
Infinite loop

Because of the **infinite loop**:

- The accepting state cannot be reached
- The machine never halts
- The input string is **rejected**

Another Turing Machine Example

Turing machine for the language $\{a^n b^n\}$
 $n \geq 1$



Basic Idea:

Match **a**'s with **b**'s:

Repeat:

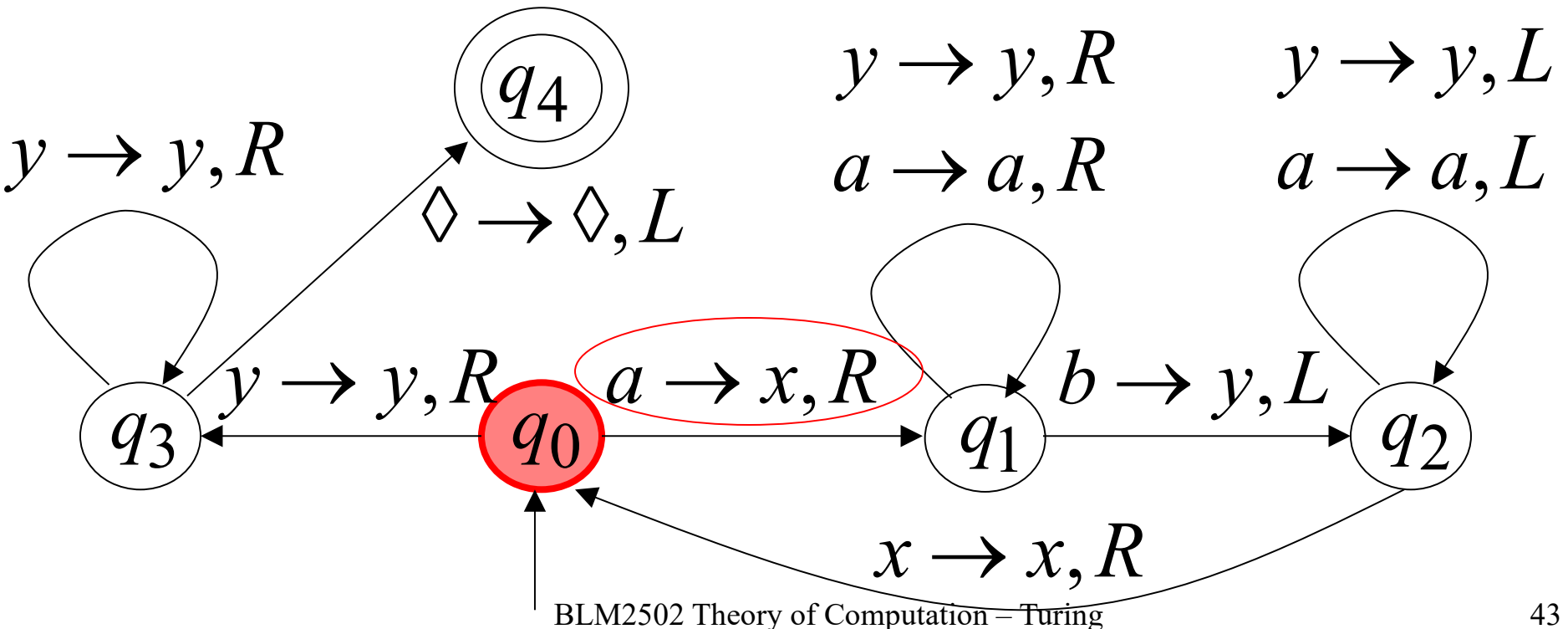
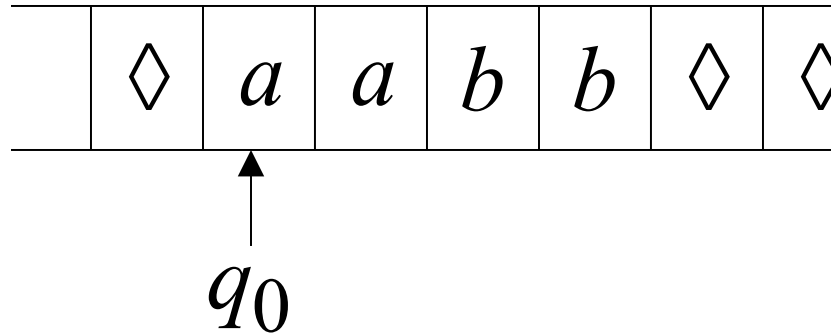
replace leftmost **a** with **x**

find leftmost **b** and replace it with **y**

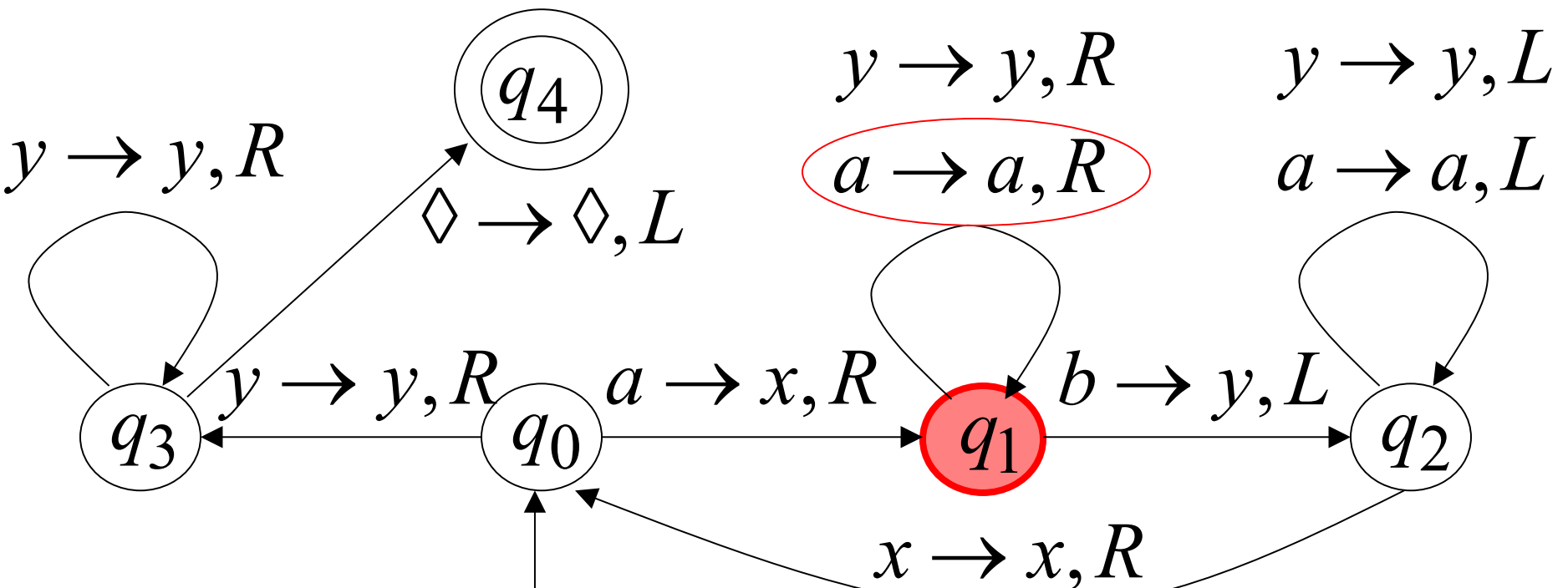
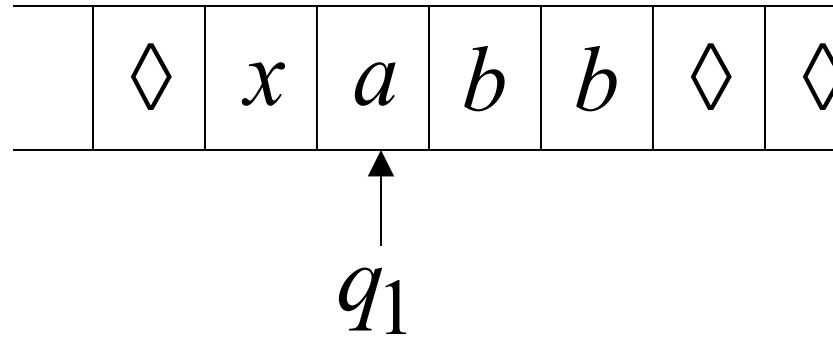
Until there are no more **a**'s or **b**'s

If there is a remaining **a** or **b** reject

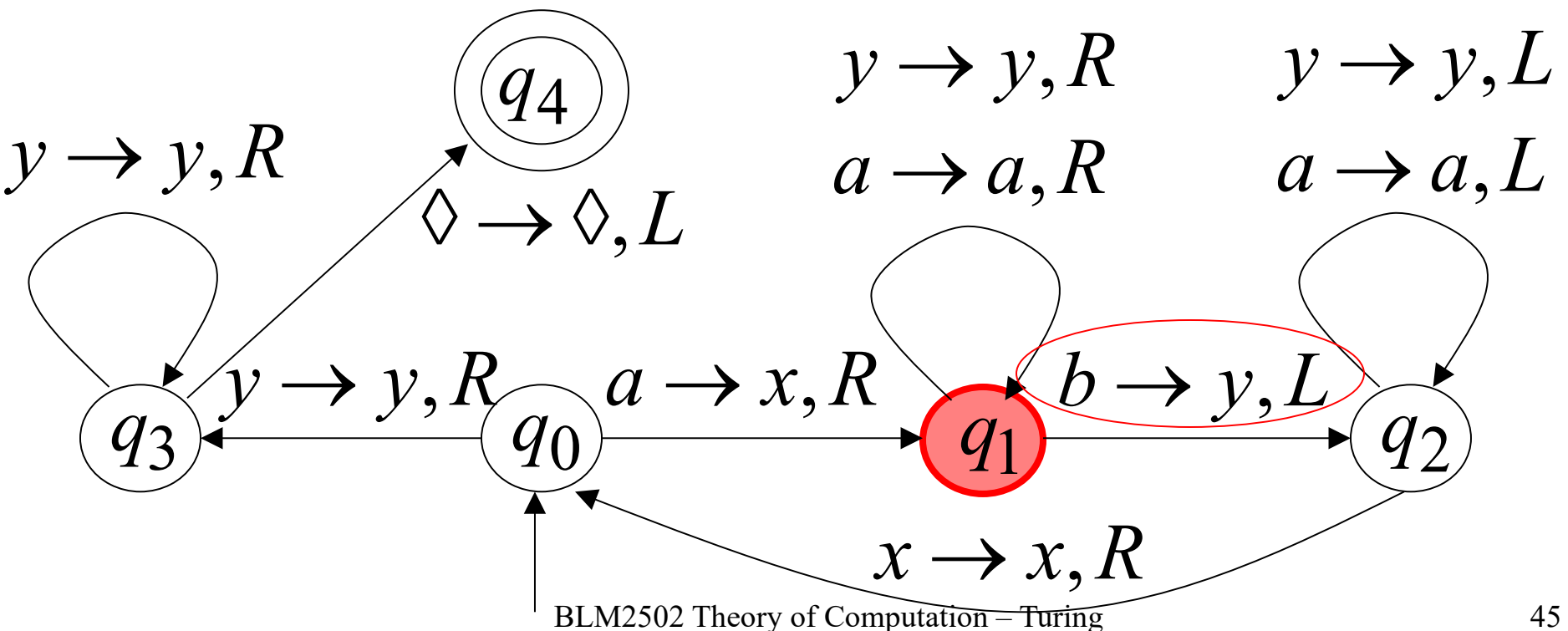
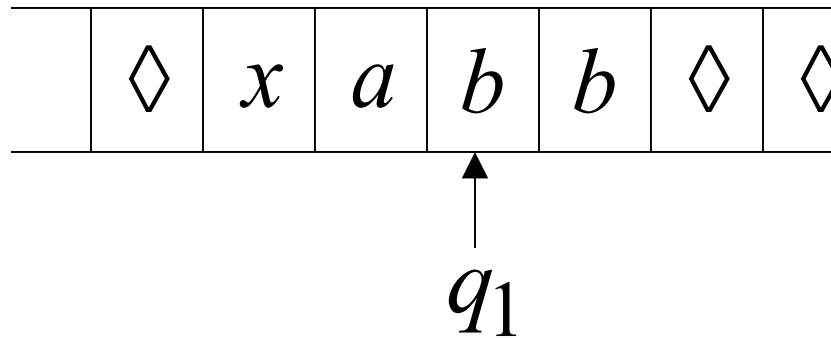
Time 0



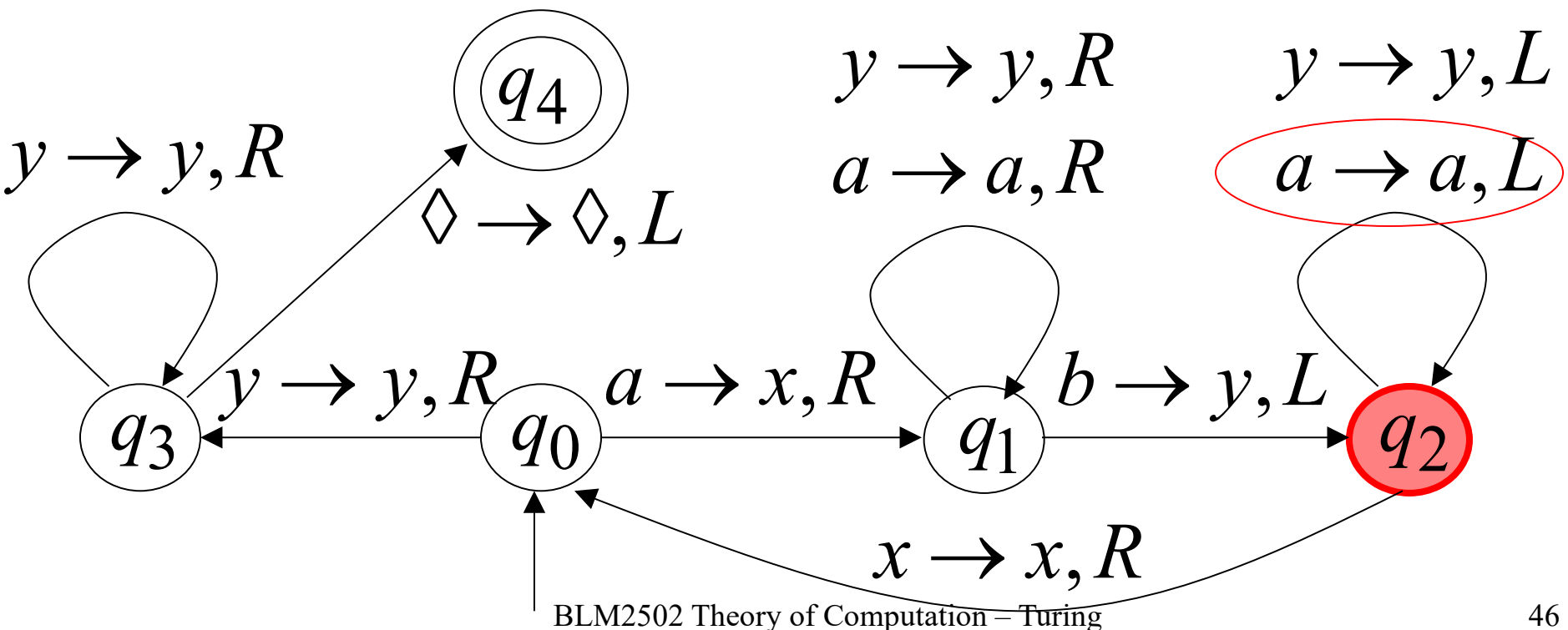
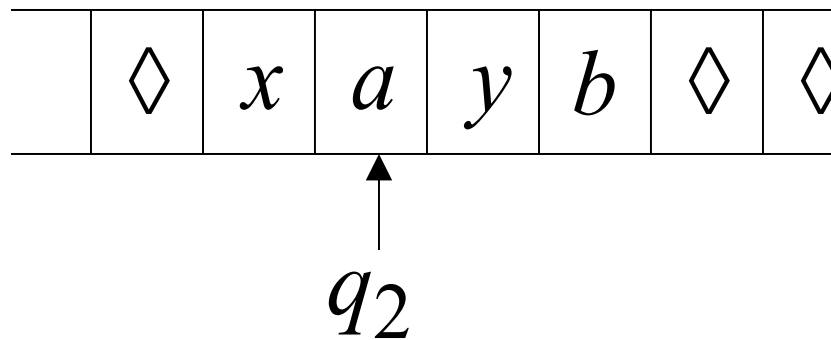
Time 1



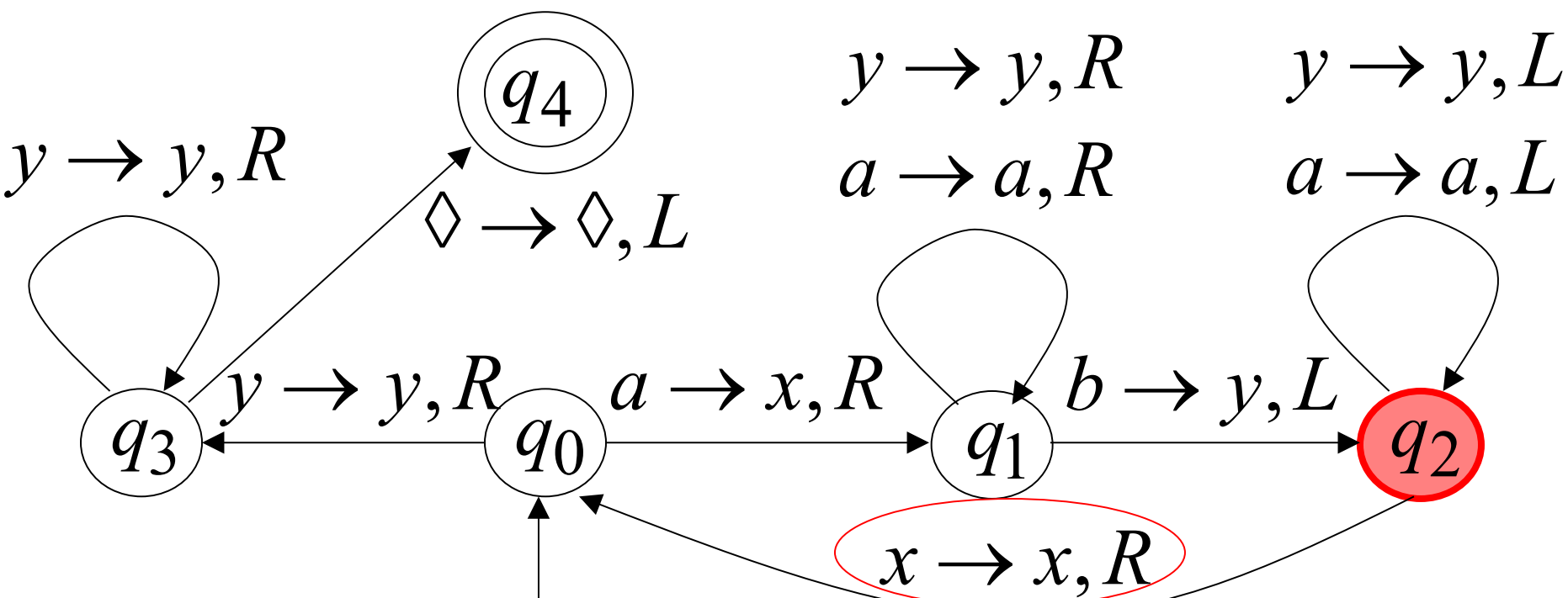
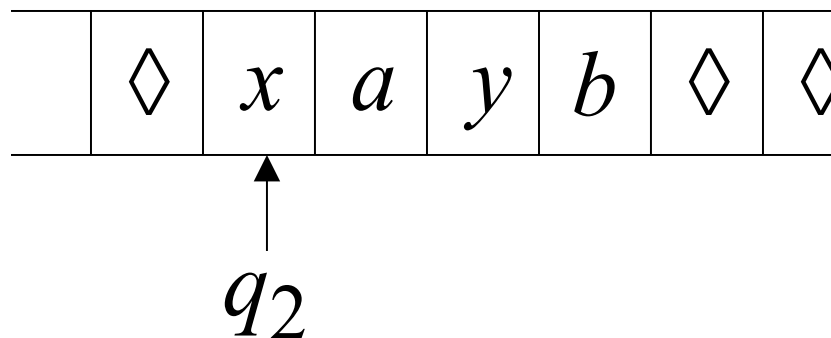
Time 2



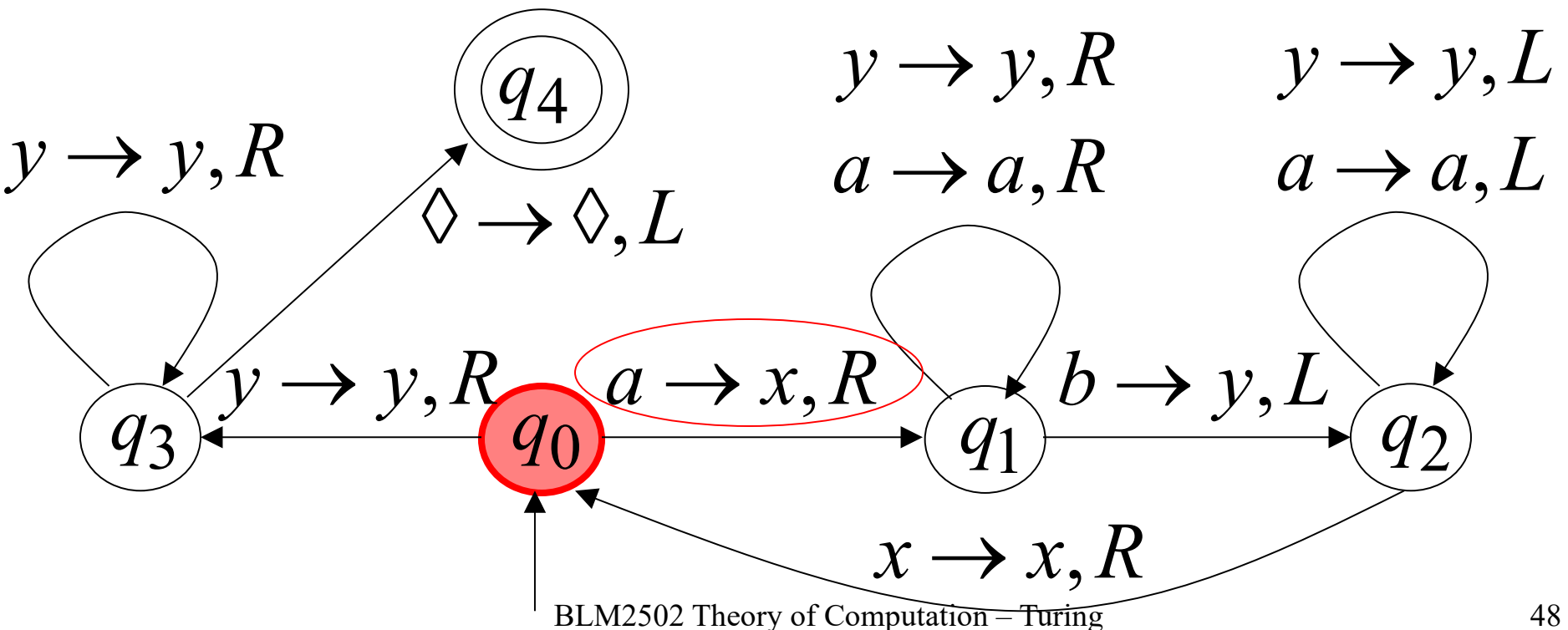
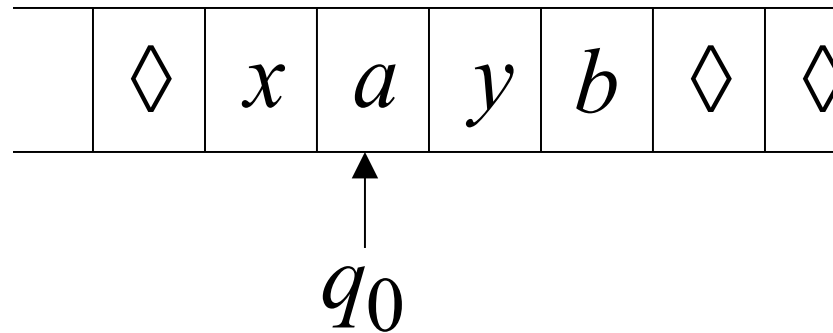
Time 3



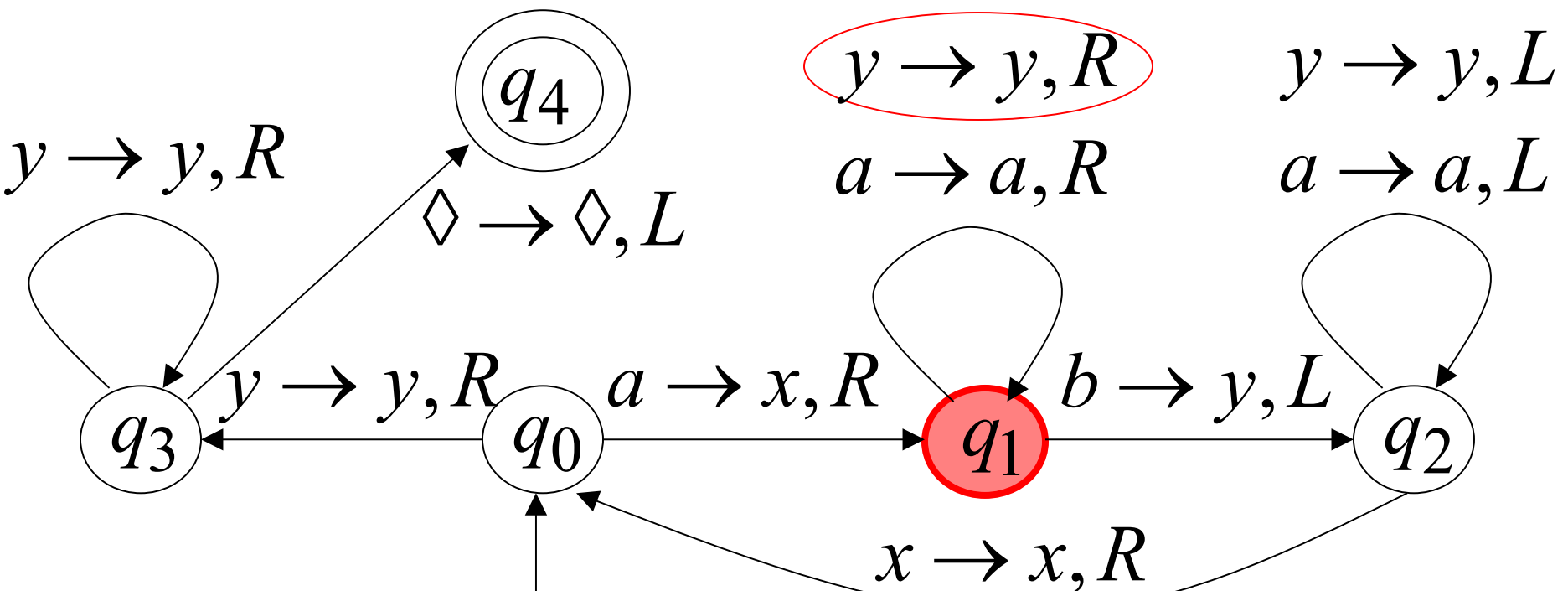
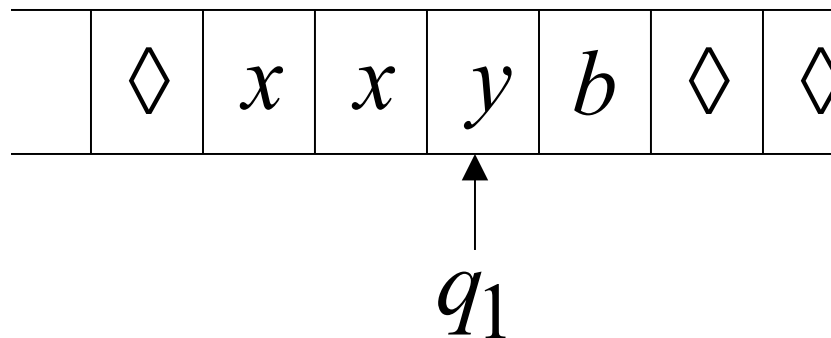
Time 4



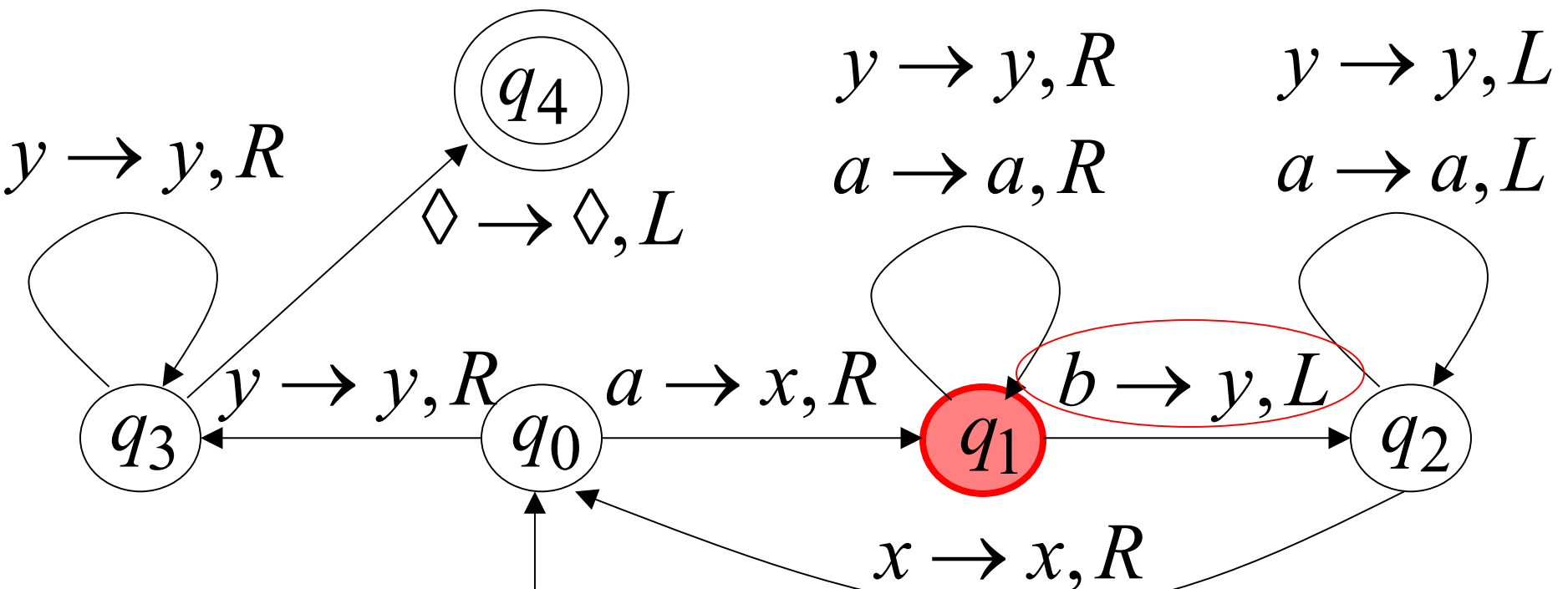
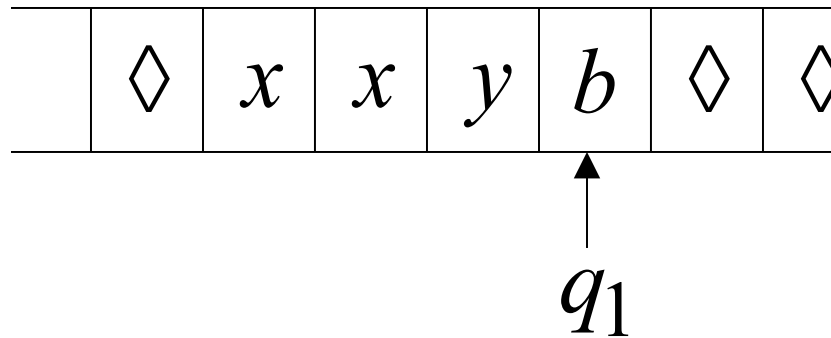
Time 5



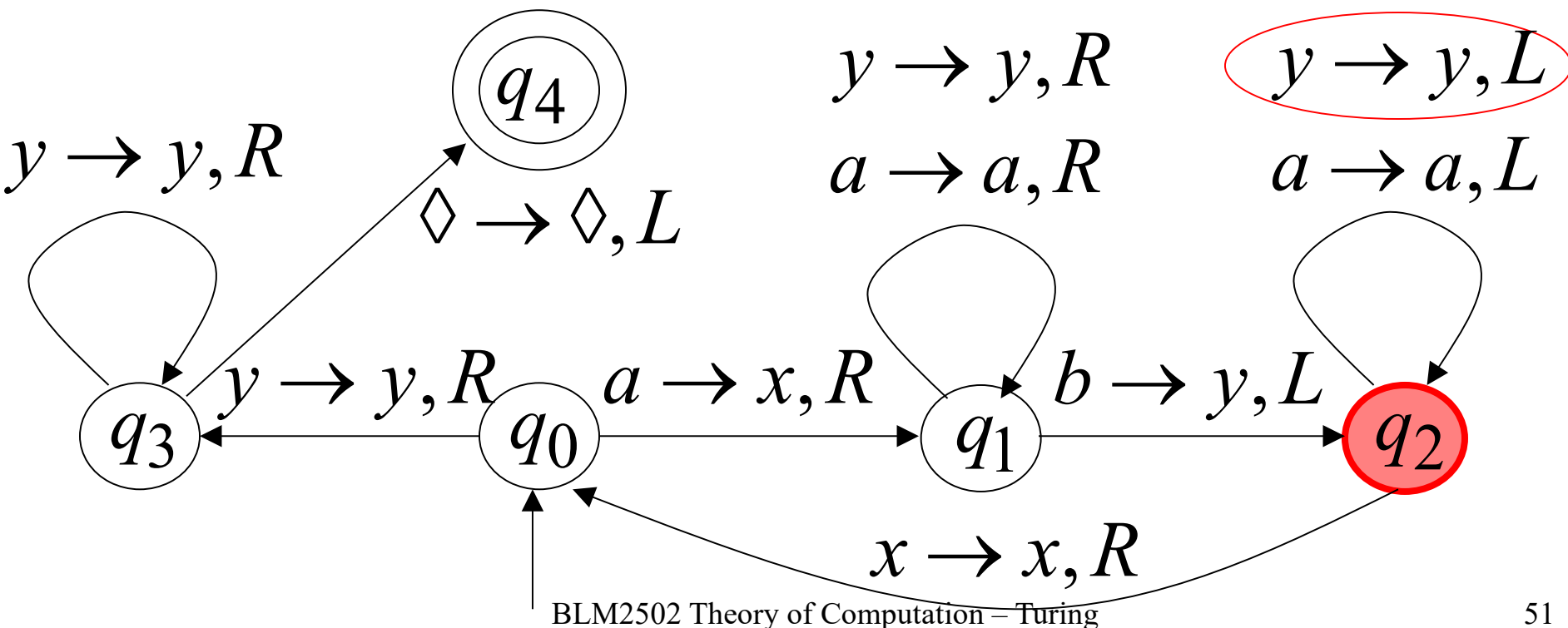
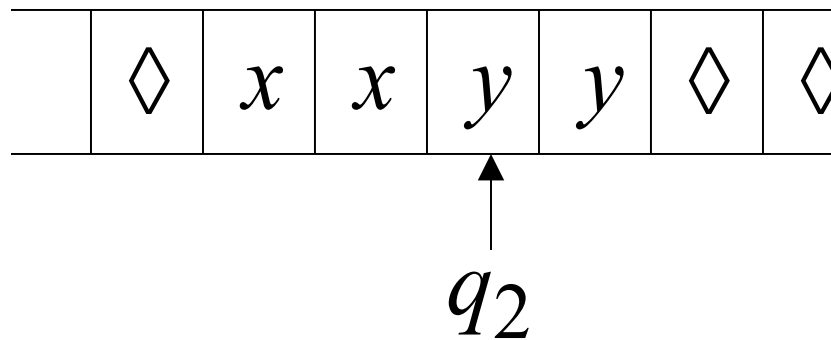
Time 6



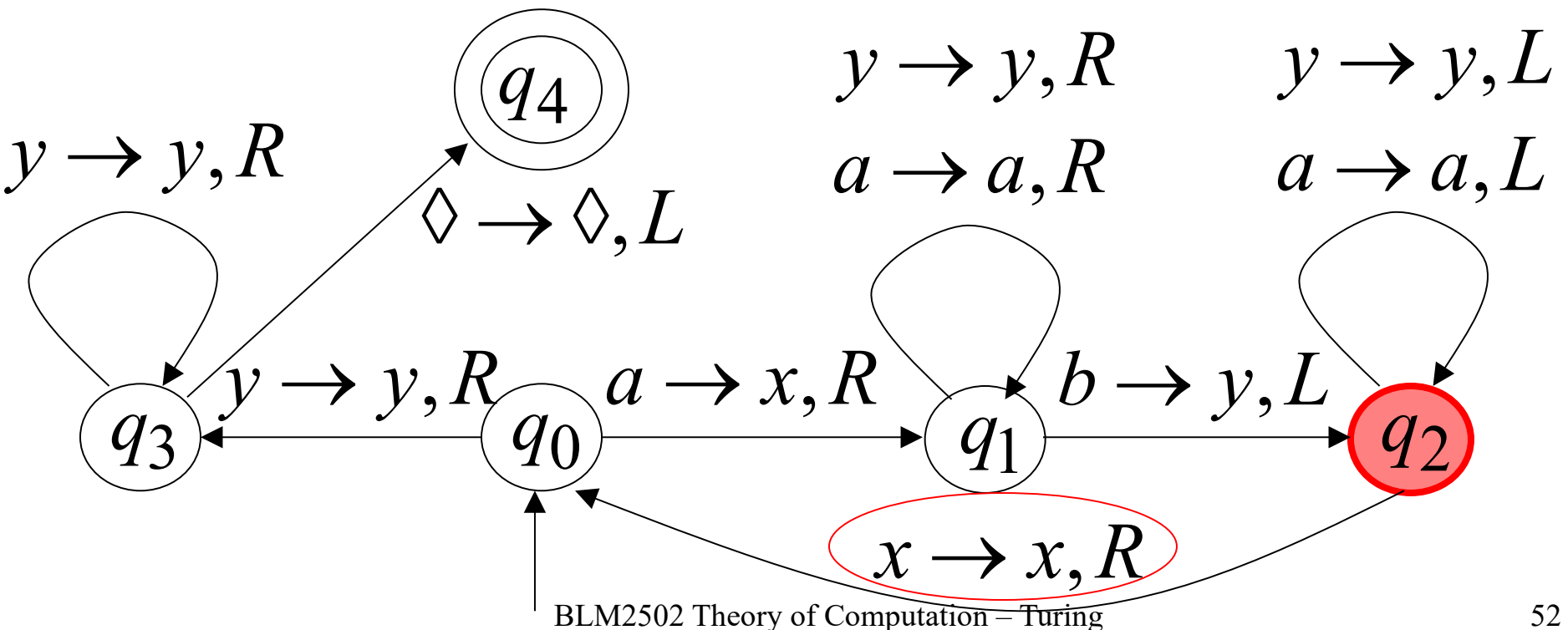
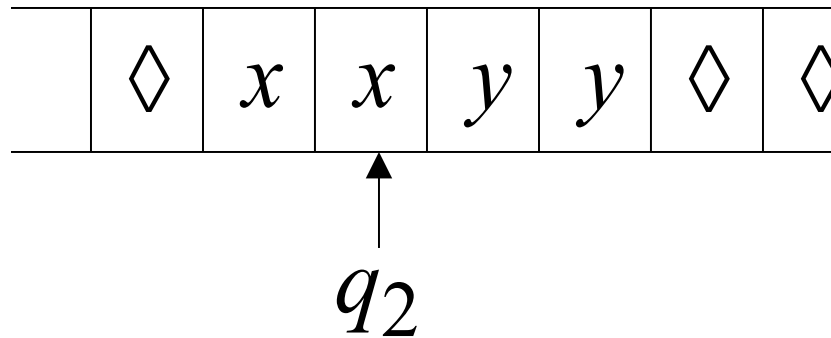
Time 7



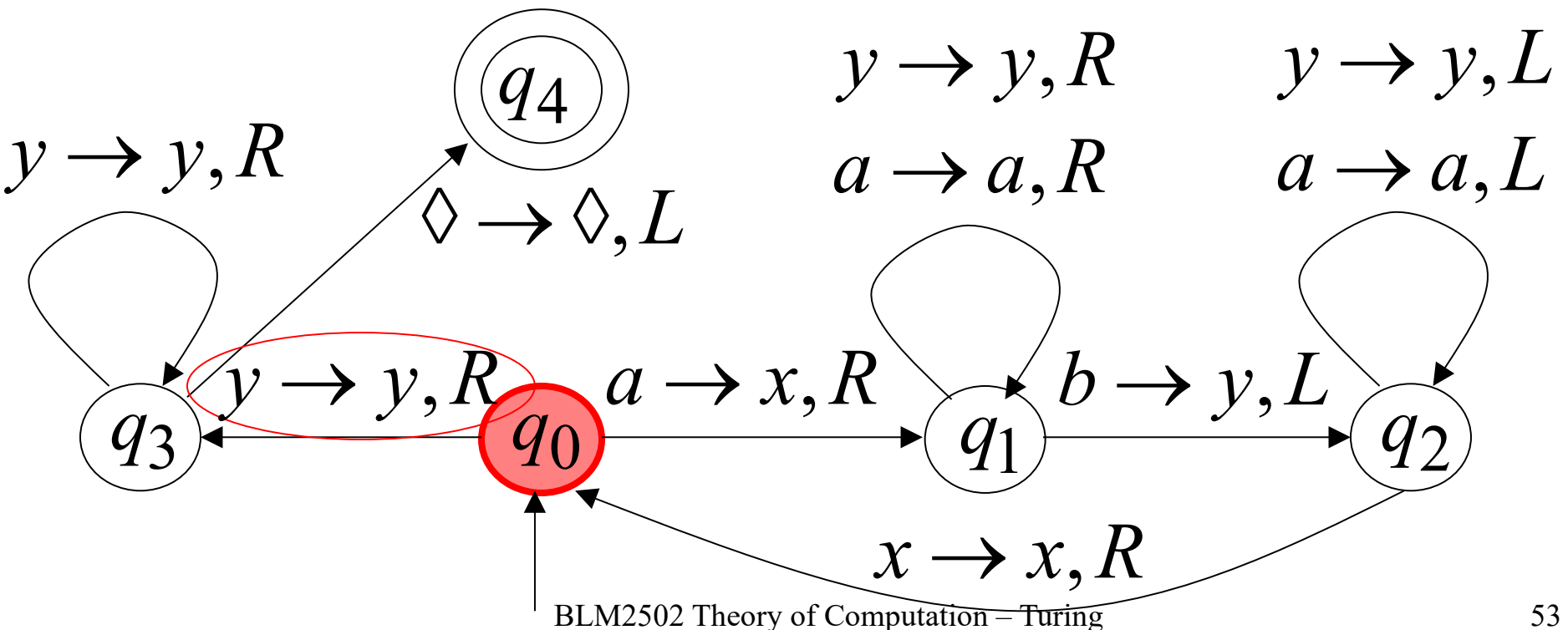
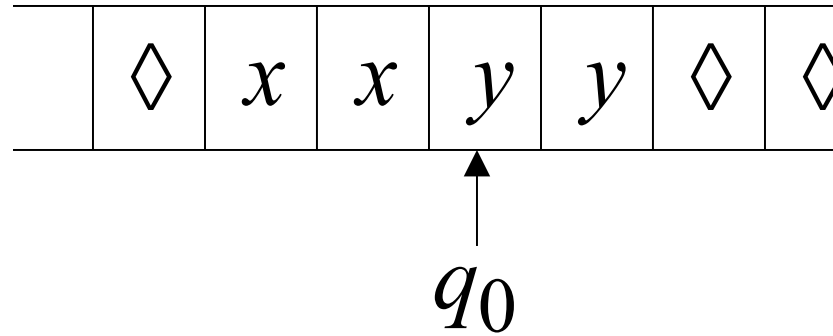
Time 8



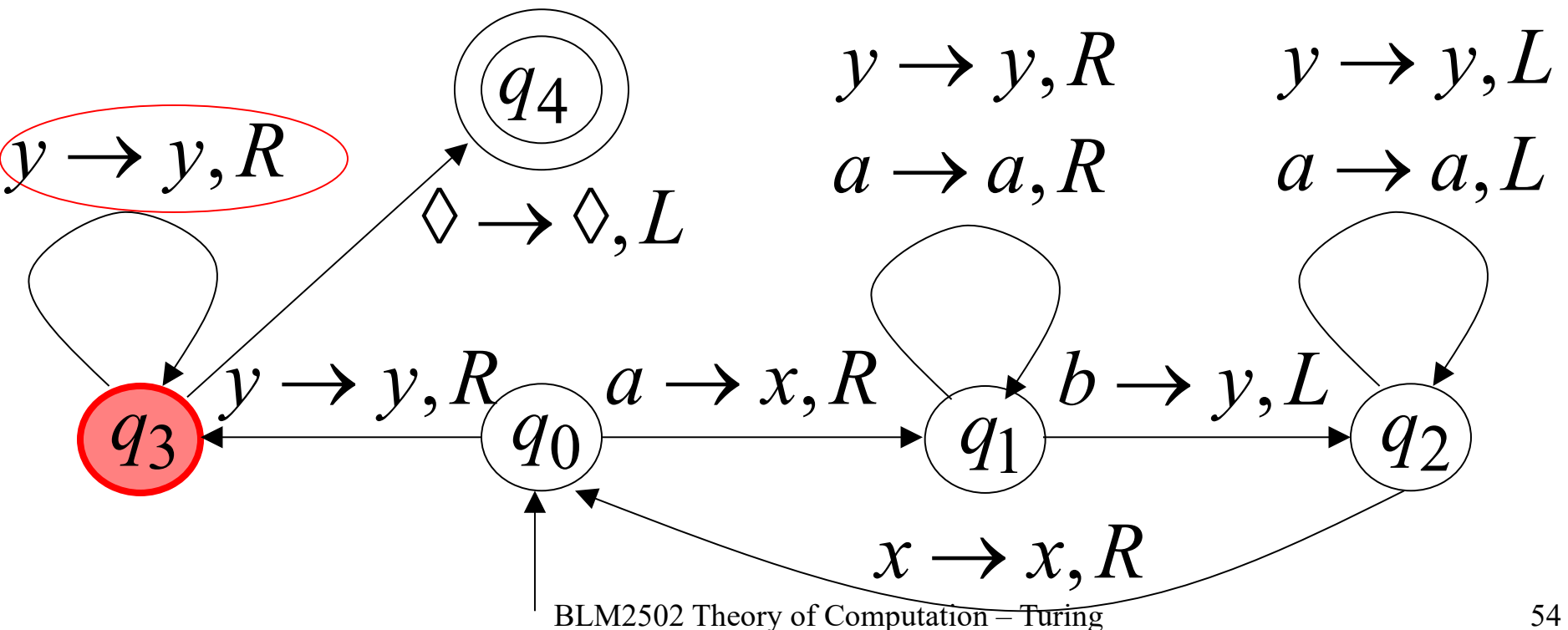
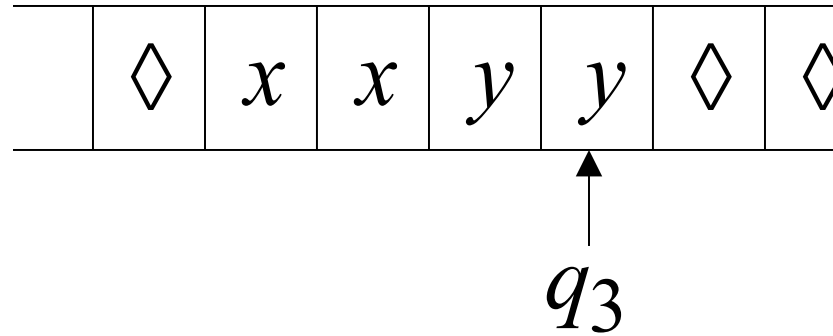
Time 9



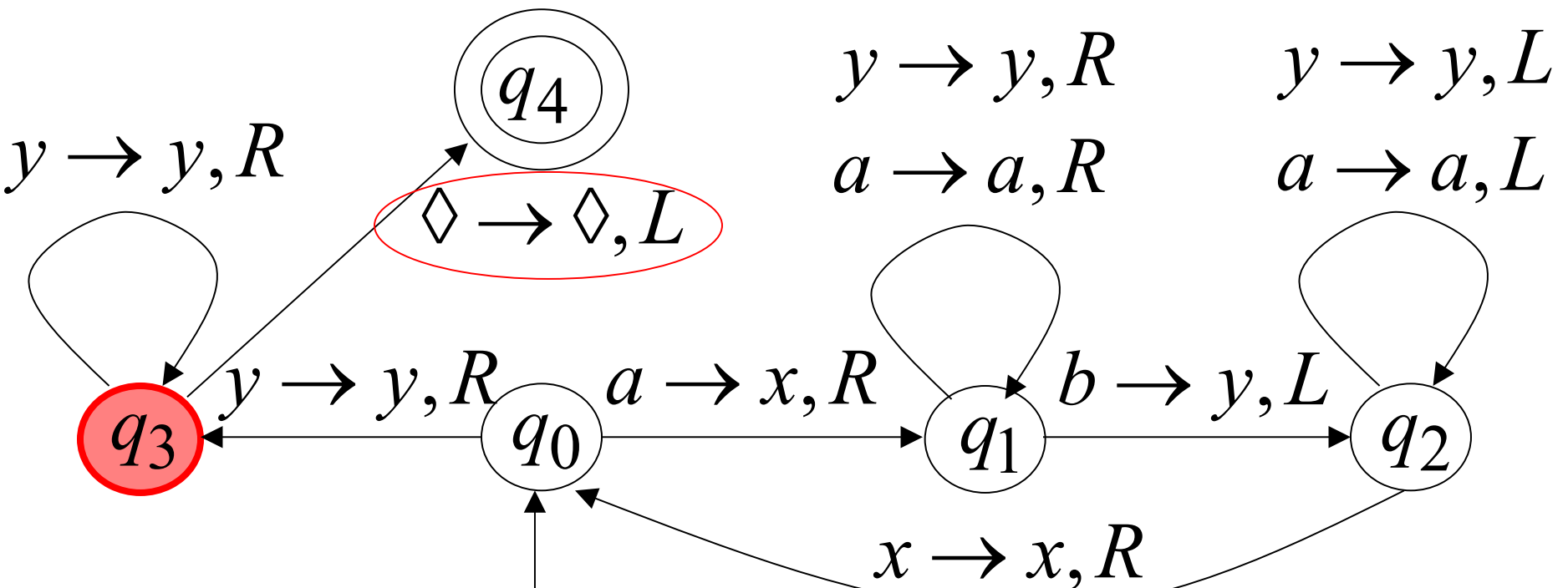
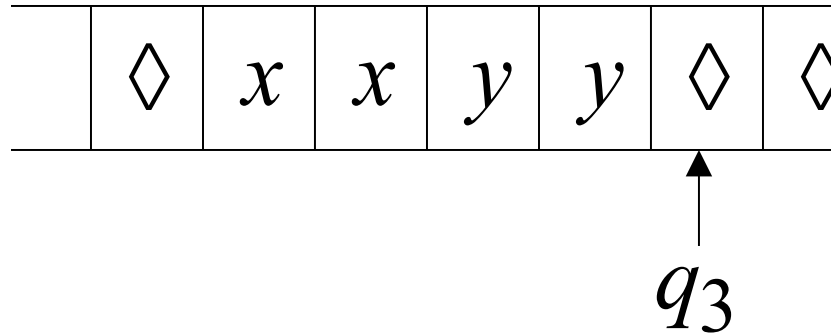
Time 10



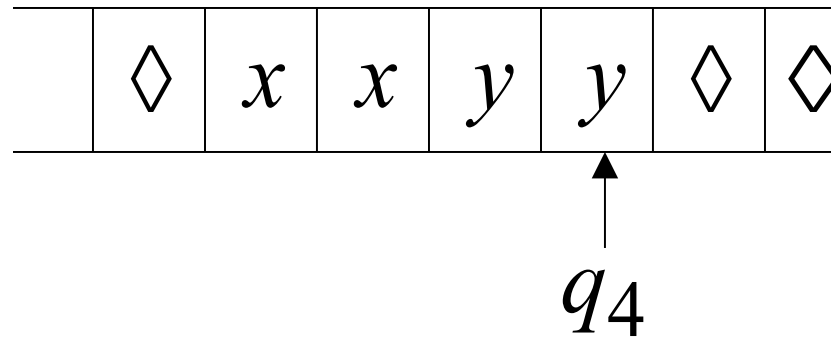
Time 11



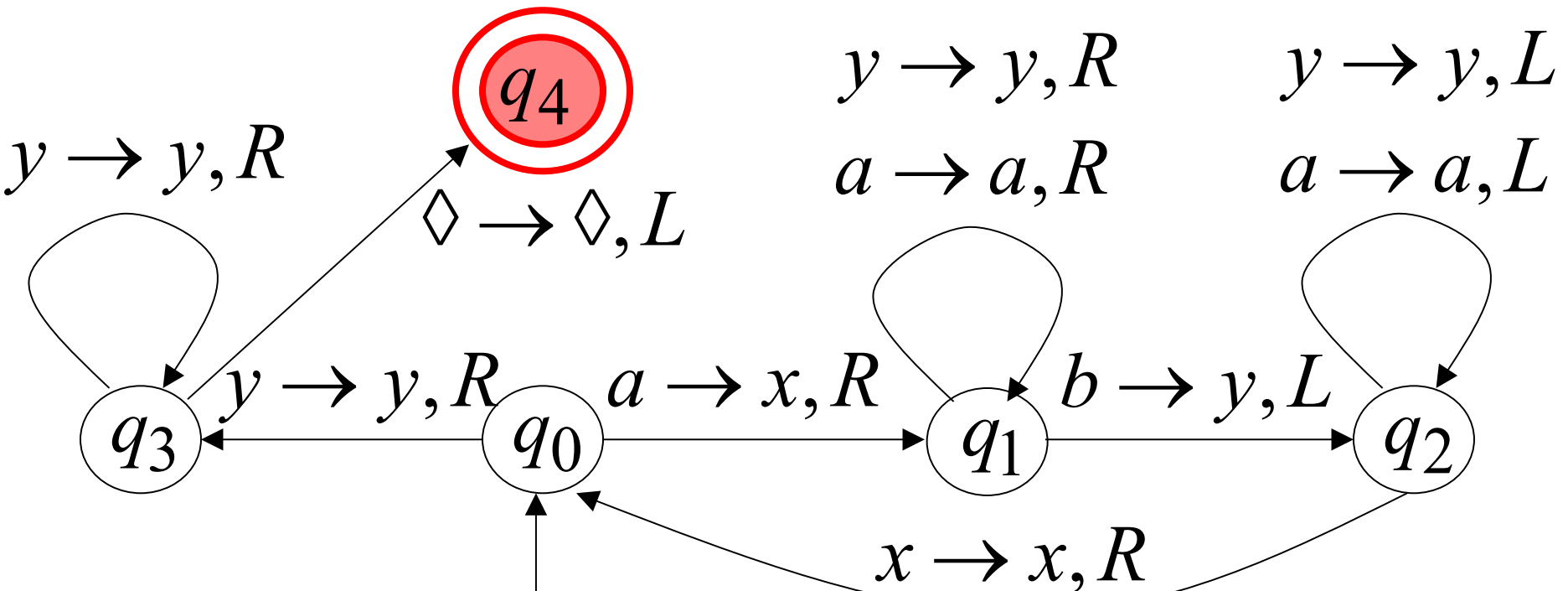
Time 12



Time 13



Halt & Accept



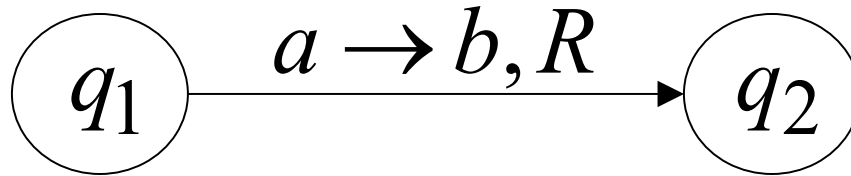
Observation:

If we modify the
machine for the language $\{a^n b^n\}$

we can easily construct
a machine for the language $\{a^n b^n c^n\}$

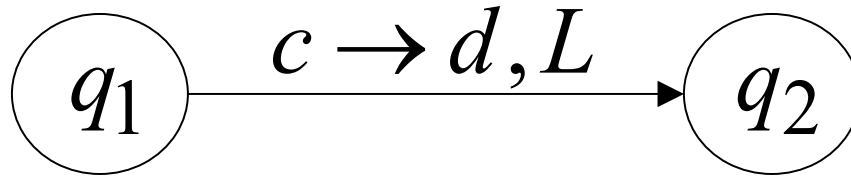
Formal Definitions for Turing Machines

Transition Function



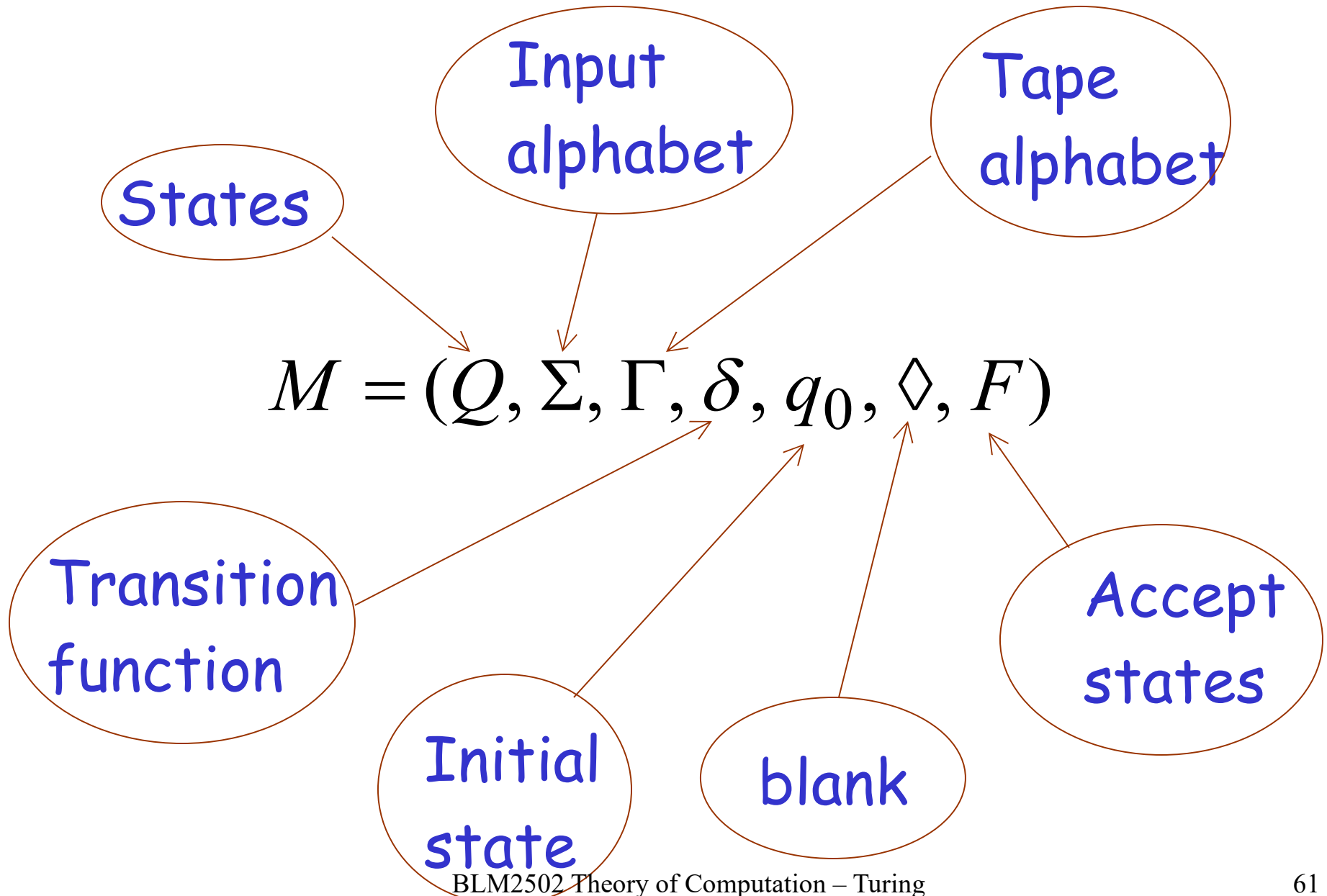
$$\delta(q_1, a) = (q_2, b, R)$$

Transition Function

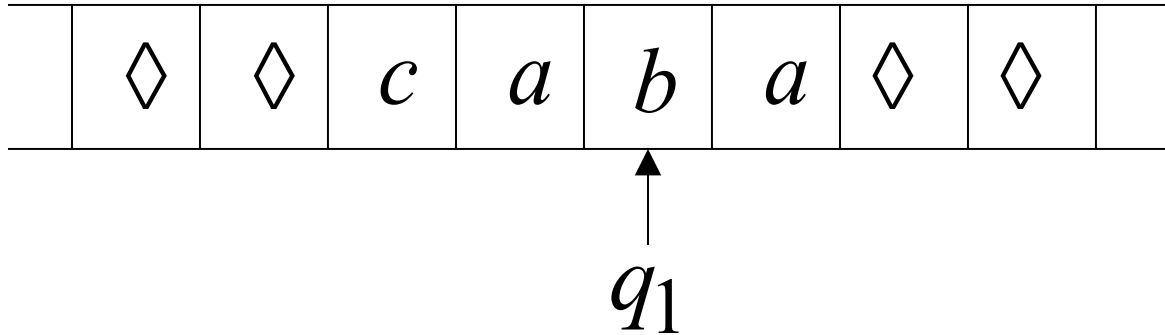


$$\delta(q_1, c) = (q_2, d, L)$$

Turing Machine:

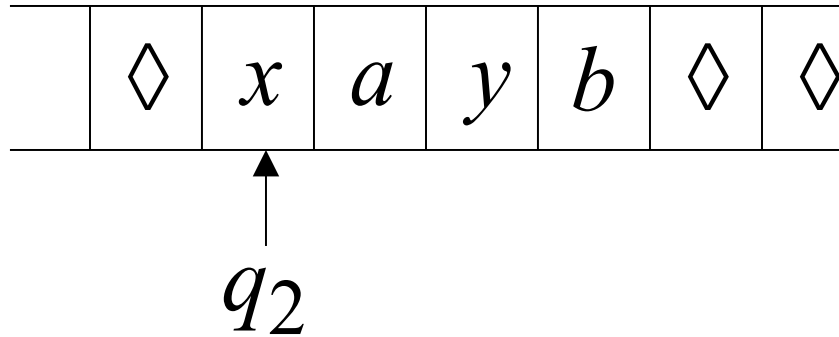


Configuration

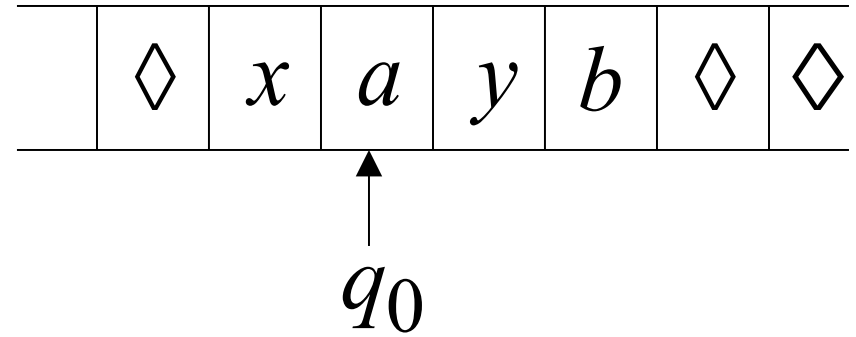


Instantaneous description: $ca\ q_1\ ba$

Time 4



Time 5

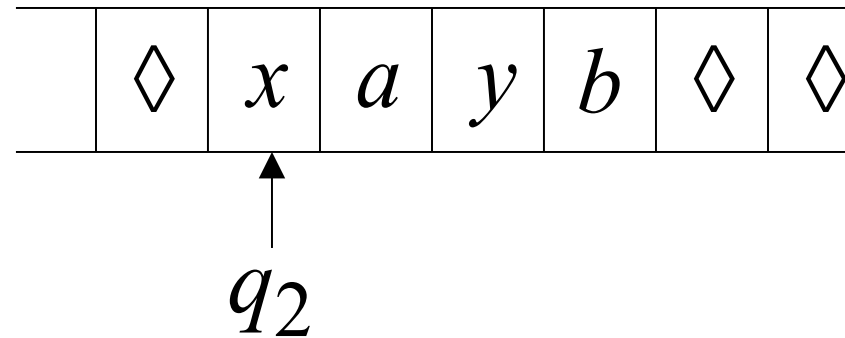


A Move:

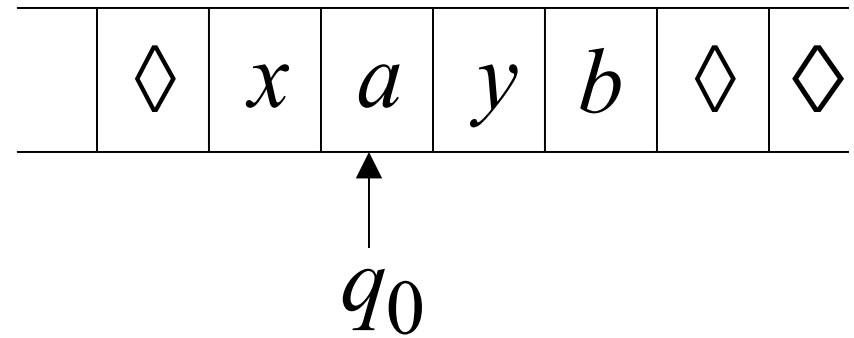
$$q_2 xayb \succ x q_0 ayb$$

(yields in one move)

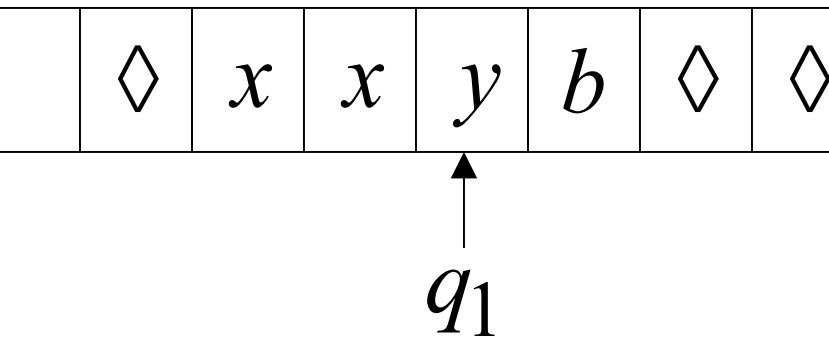
Time 4



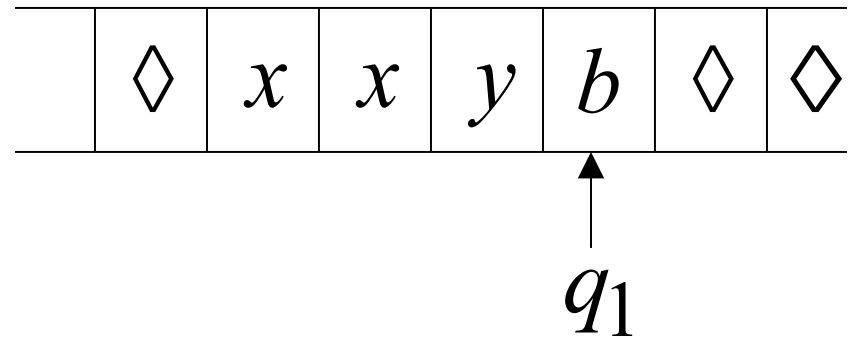
Time 5



Time 6



Time 7



A computation

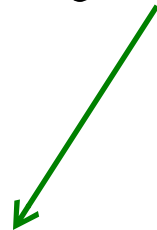
$q_2 \ x a y b \succ x \ q_0 \ a y b \succ x x \ q_1 \ y b \succ x x y \ q_1 \ b$

$$q_2 xayb \succ x q_0 ayb \succ xx q_1 yb \succ xxy q_1 b$$

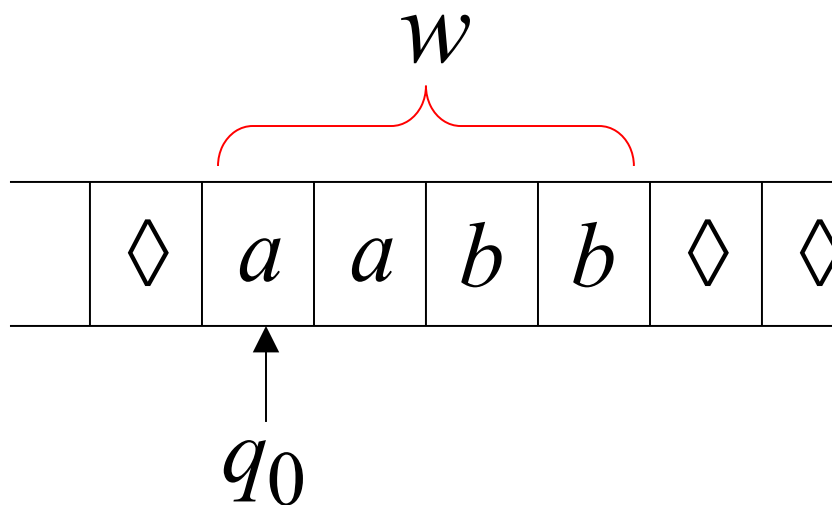
Equivalent notation:

$$q_2 xayb \overset{*}{\succ} xxy q_1 b$$

Initial configuration: $q_0 w$



Input string



The Accepted Language

For any Turing Machine M

$$L(M) = \{w : q_0 w \xrightarrow{*} x_1 q_f x_2\}$$



Initial state



Accept state

If a language L is accepted
by a Turing machine M
then we say that L is:

- Turing Recognizable

Other names used:

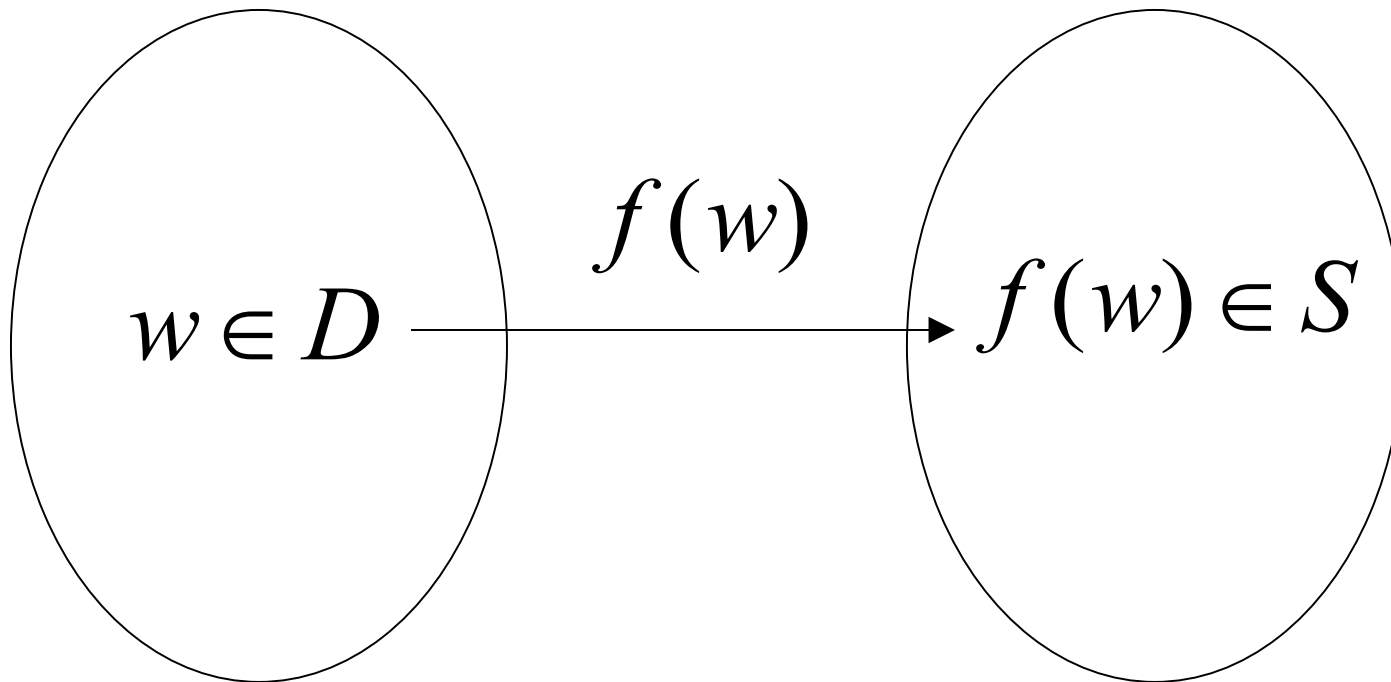
- Turing Acceptable
- Recursively Enumerable

Computing Functions with Turing Machines

A function $f(w)$ has:

Domain: D

Result Region: S



A function may have many parameters:

Example: Addition function

$$f(x, y) = x + y$$

Integer Domain

Decimal: 5

Binary: 101

Unary: 11111

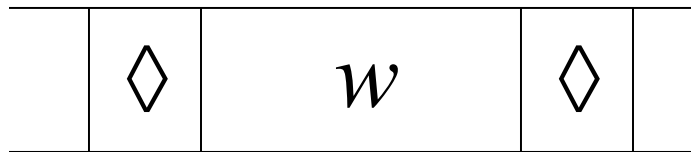
We prefer **unary** representation:

easier to manipulate with Turing machines

Definition:

A function f is computable if
there is a Turing Machine M such that:

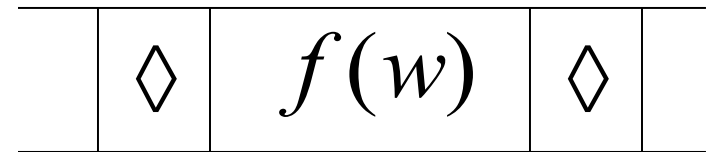
Initial configuration



q_0

initial state

Final configuration



q_f

accept state

For all $w \in D$ Domain

In other words:

A function f is computable if
there is a Turing Machine M such that:

$$q_0 w \stackrel{*}{\succ} q_f f(w)$$

Initial
Configuration

Final
Configuration

For all $w \in D$ Domain

Example

The function $f(x, y) = x + y$ is computable

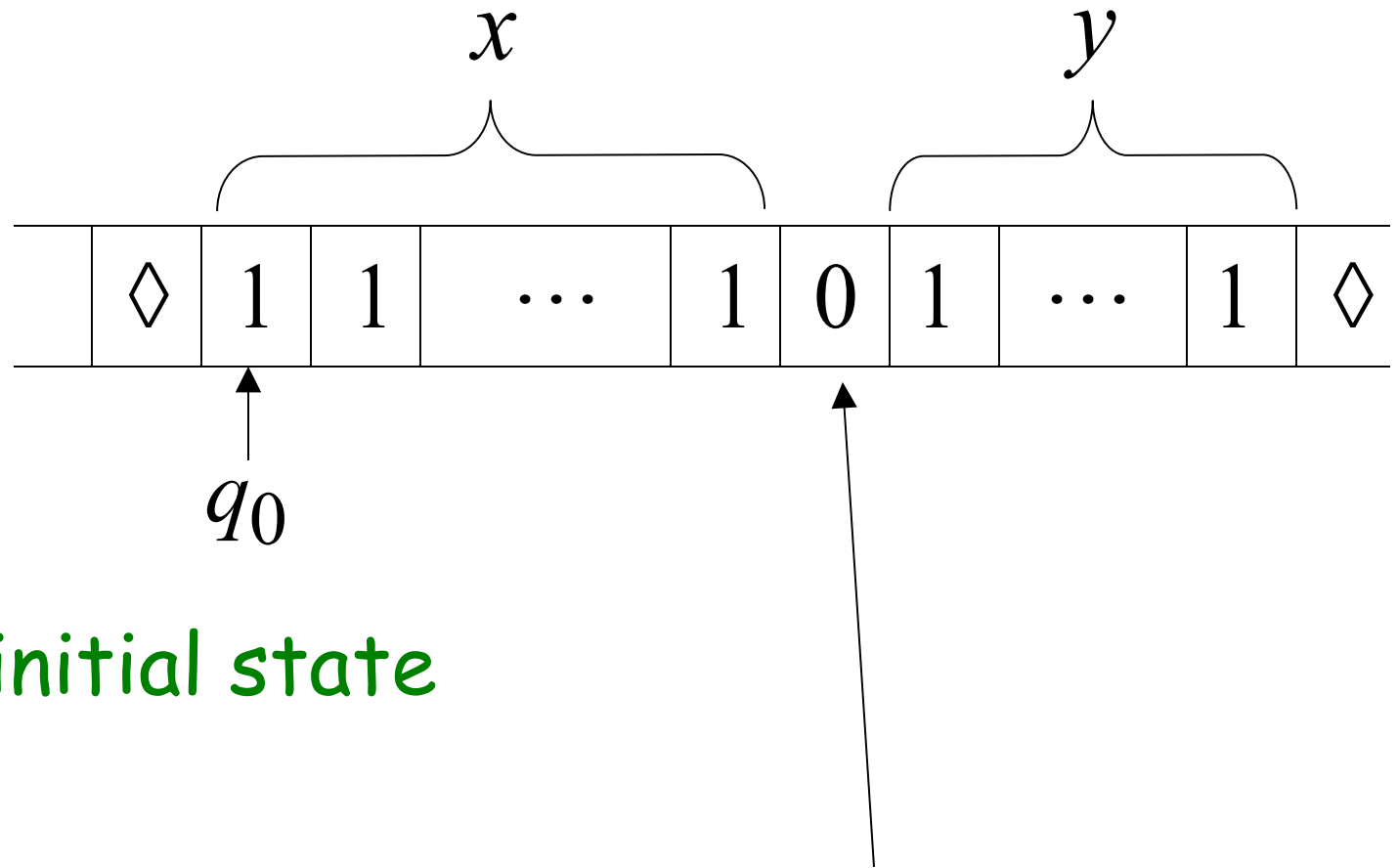
x, y are integers

Turing Machine:

Input string: $x0y$ unary

Output string: $xy0$ unary

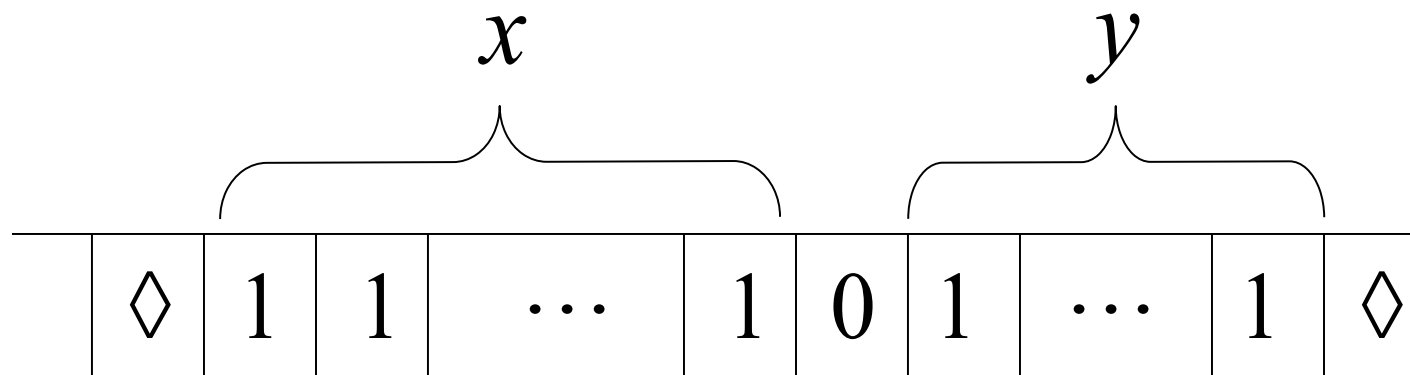
Start



initial state

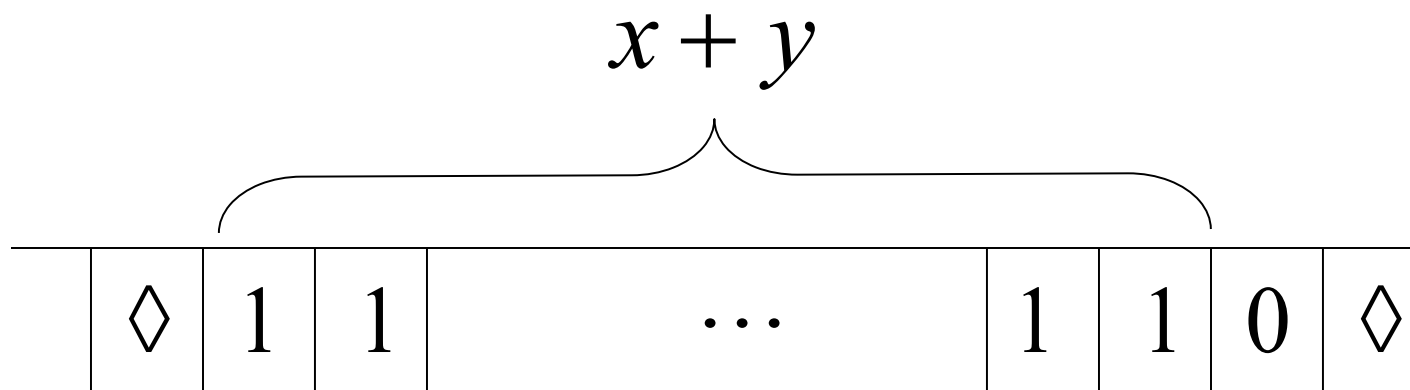
The 0 is the delimiter that separates the two numbers

Start



q_0 initial state

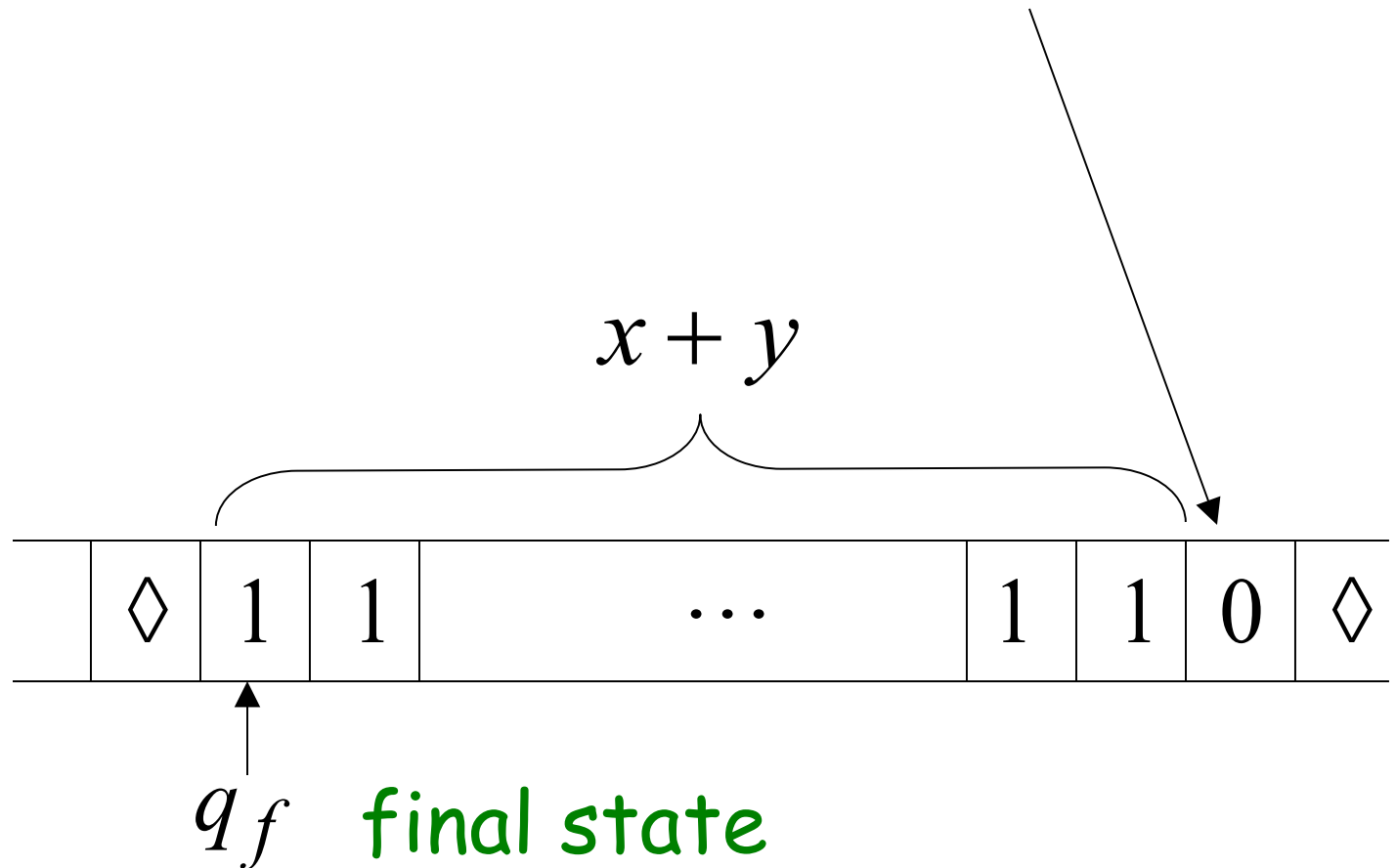
Finish



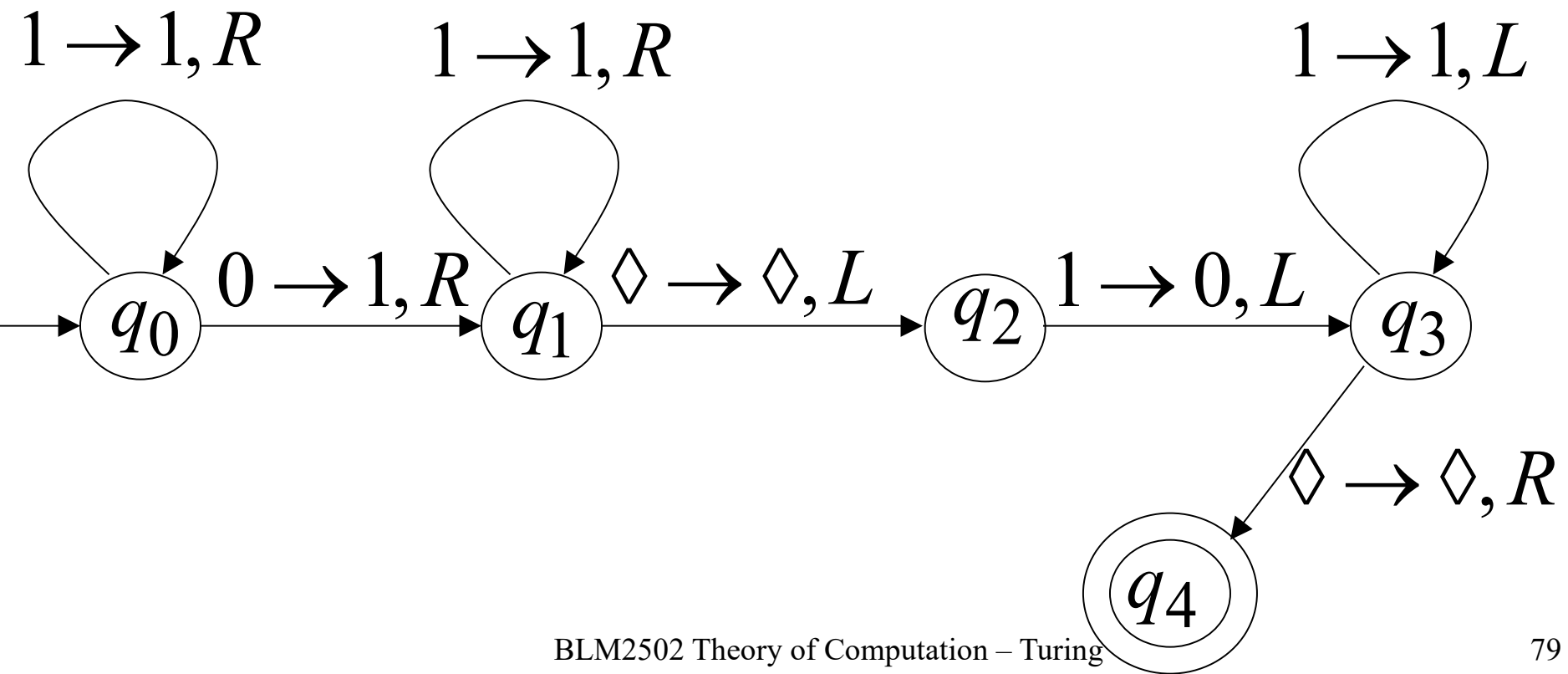
q_f final state

The 0 here helps when we use the result for other operations

Finish



Turing machine for function $f(x, y) = x + y$

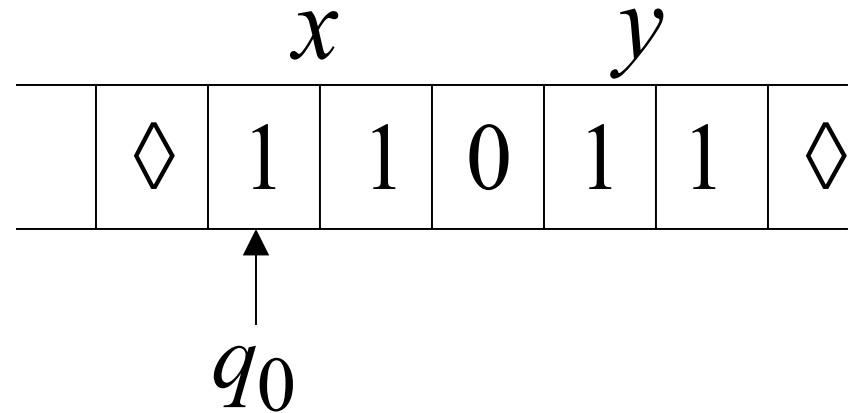


Execution Example:

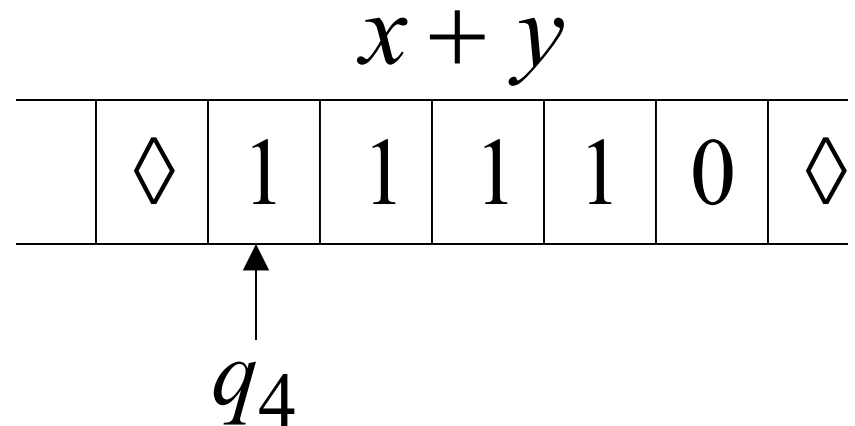
$$x = 11 \quad (=2)$$

$$y = 11 \quad (=2)$$

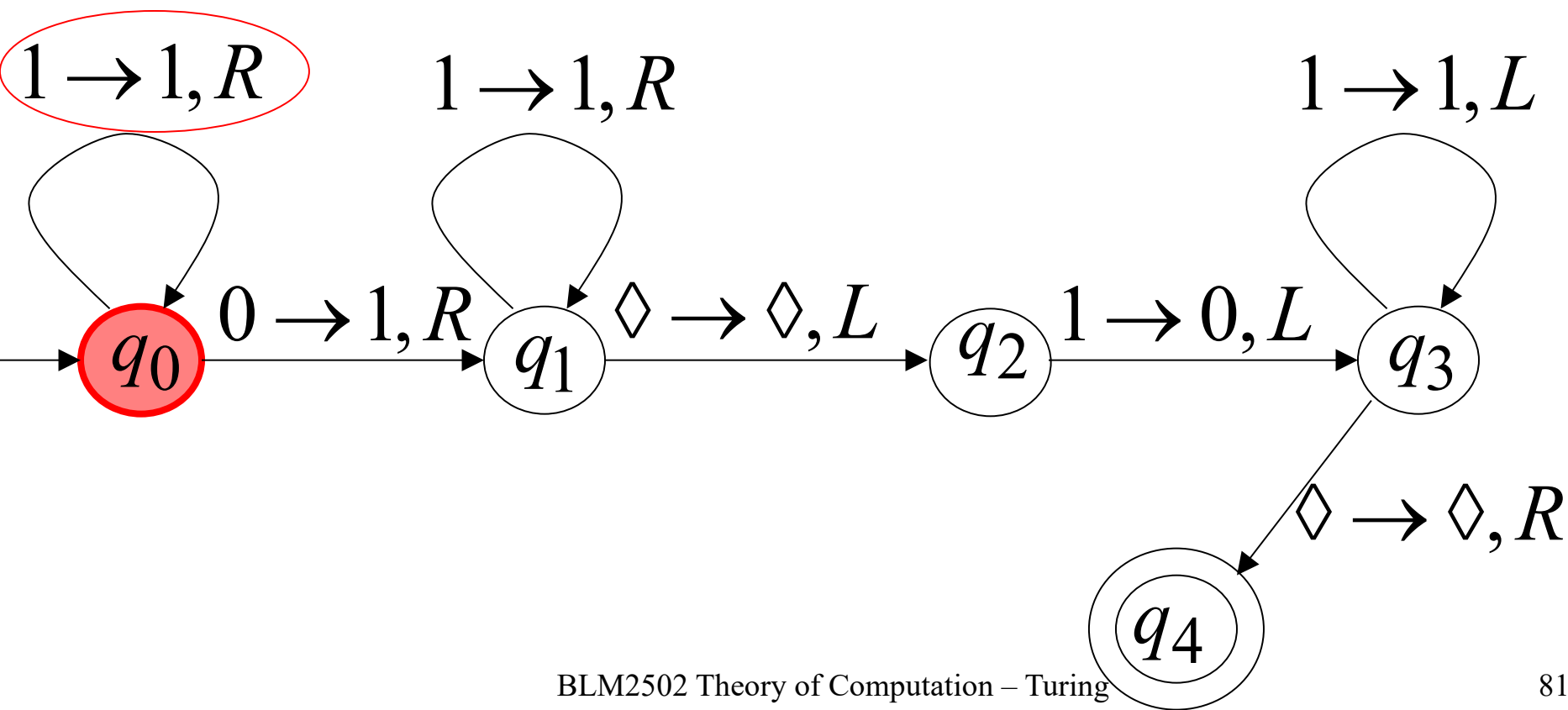
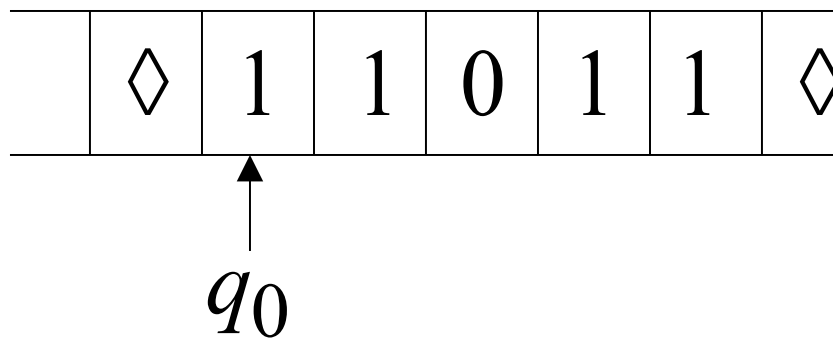
Time 0



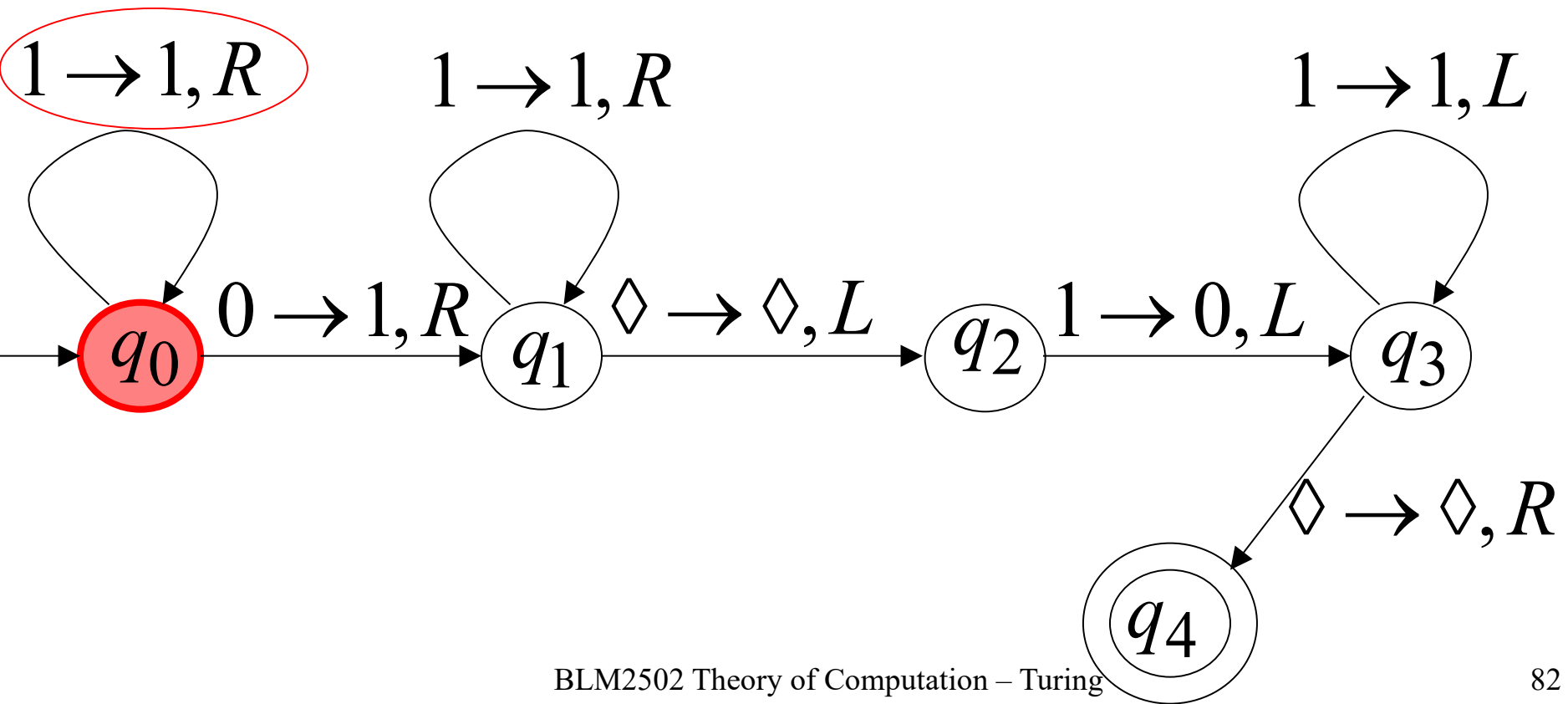
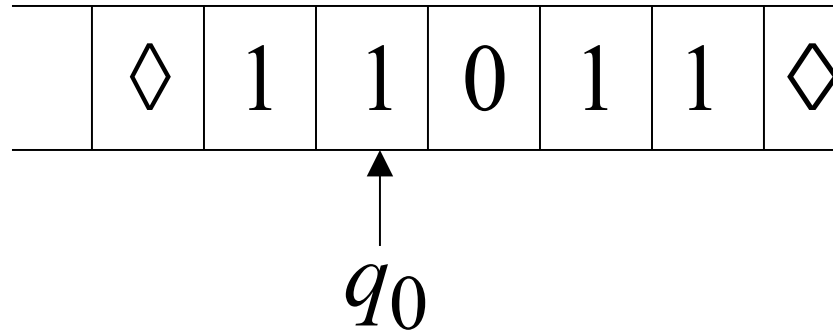
Final Result



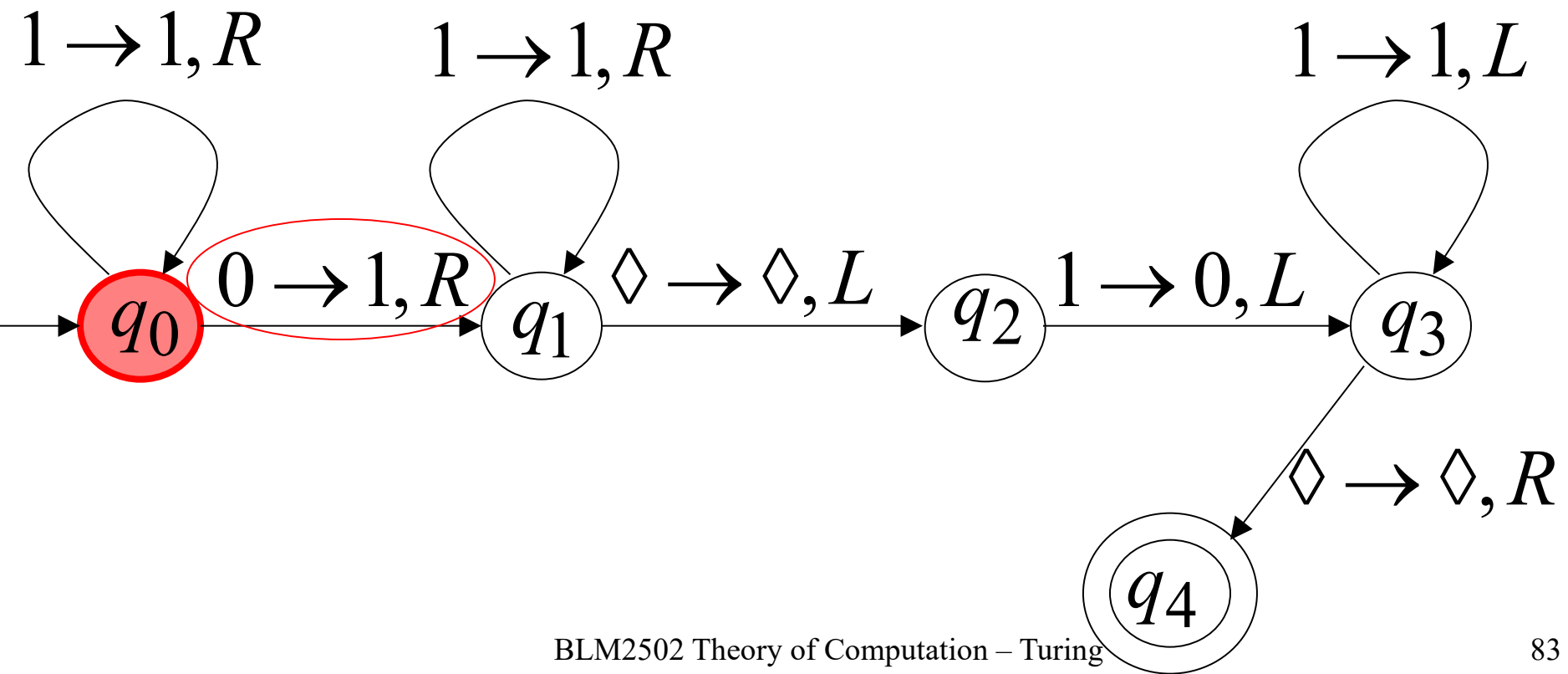
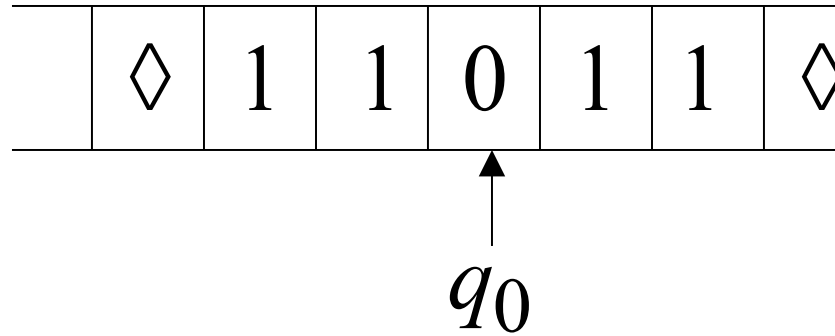
Time 0



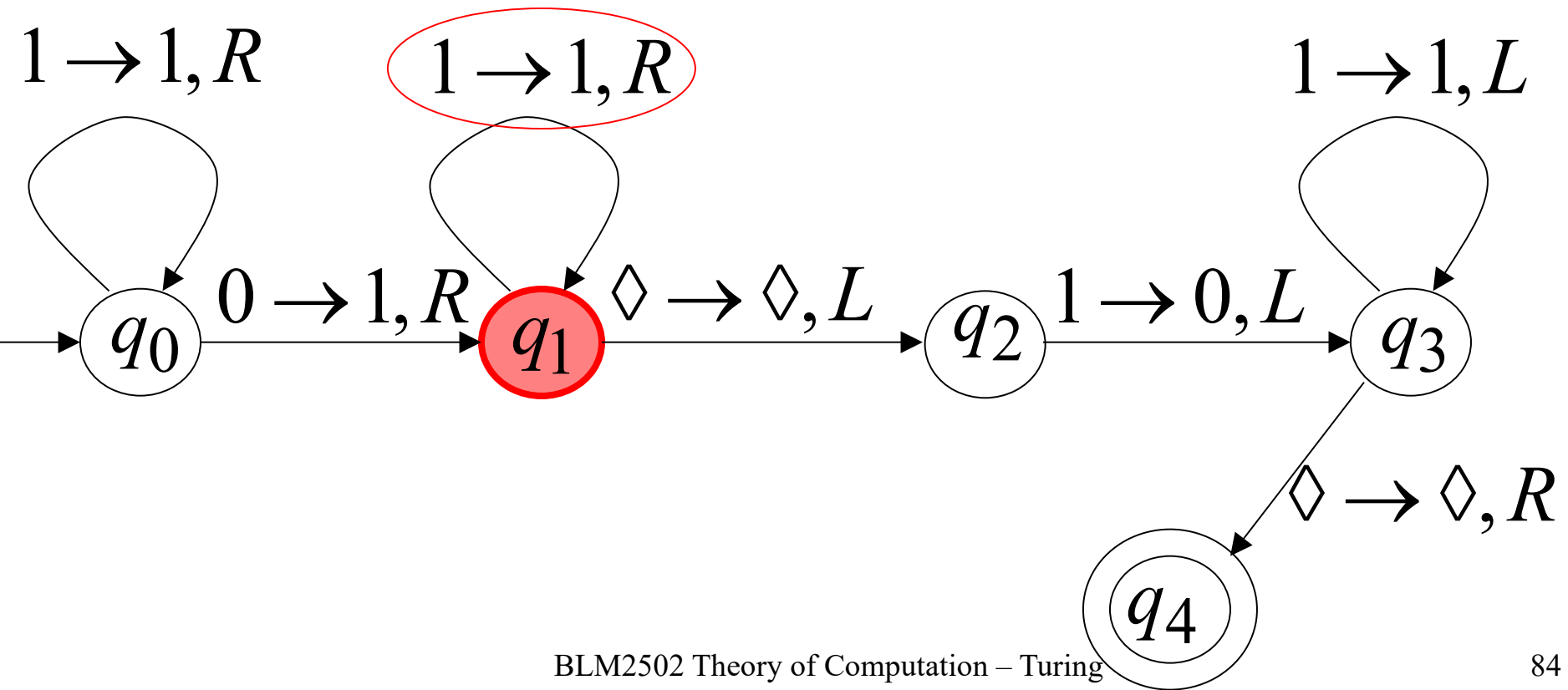
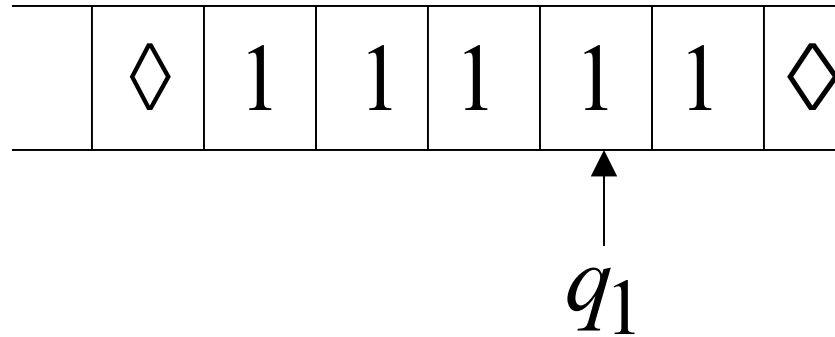
Time 1



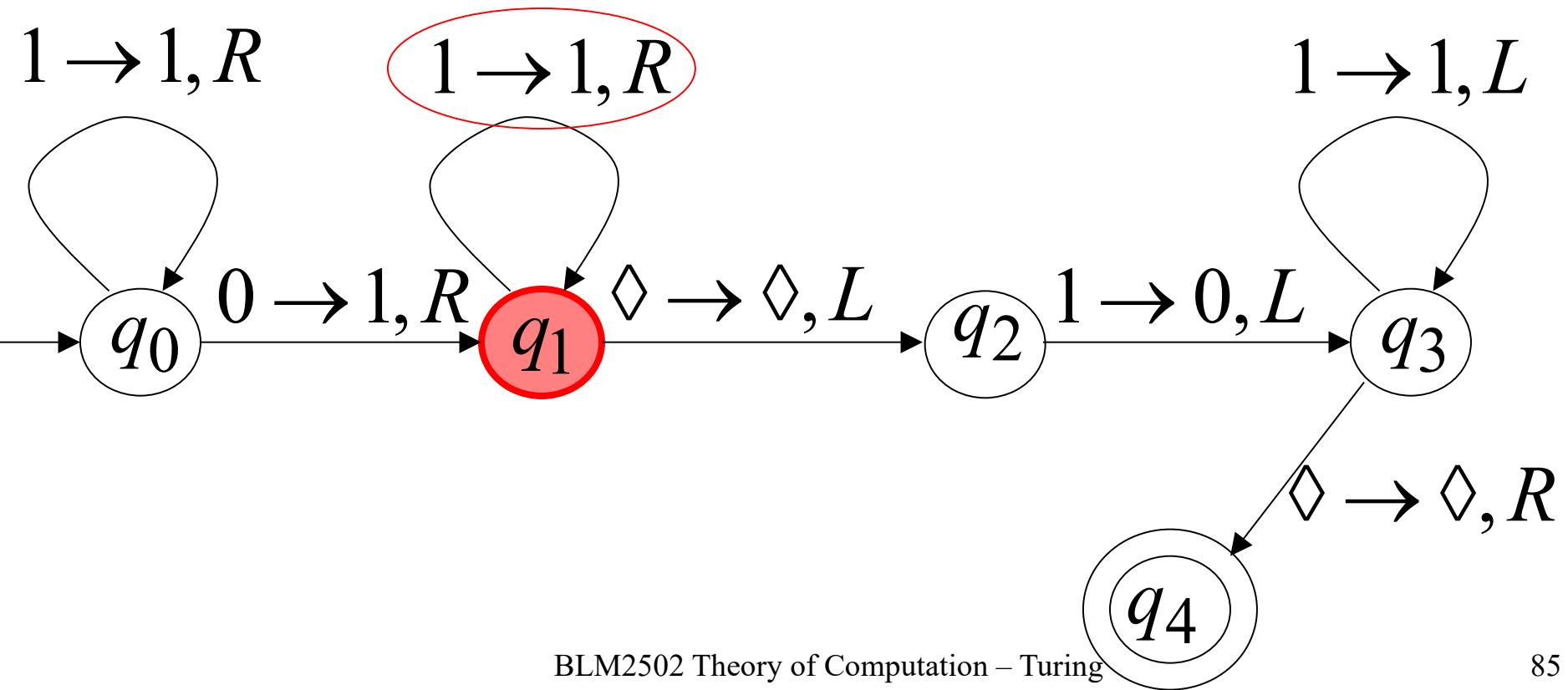
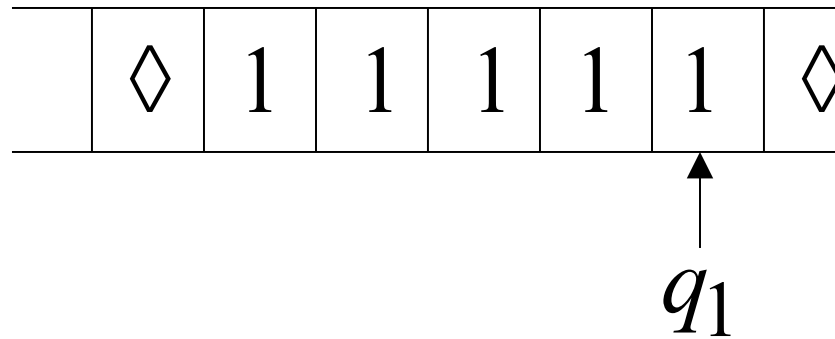
Time 2



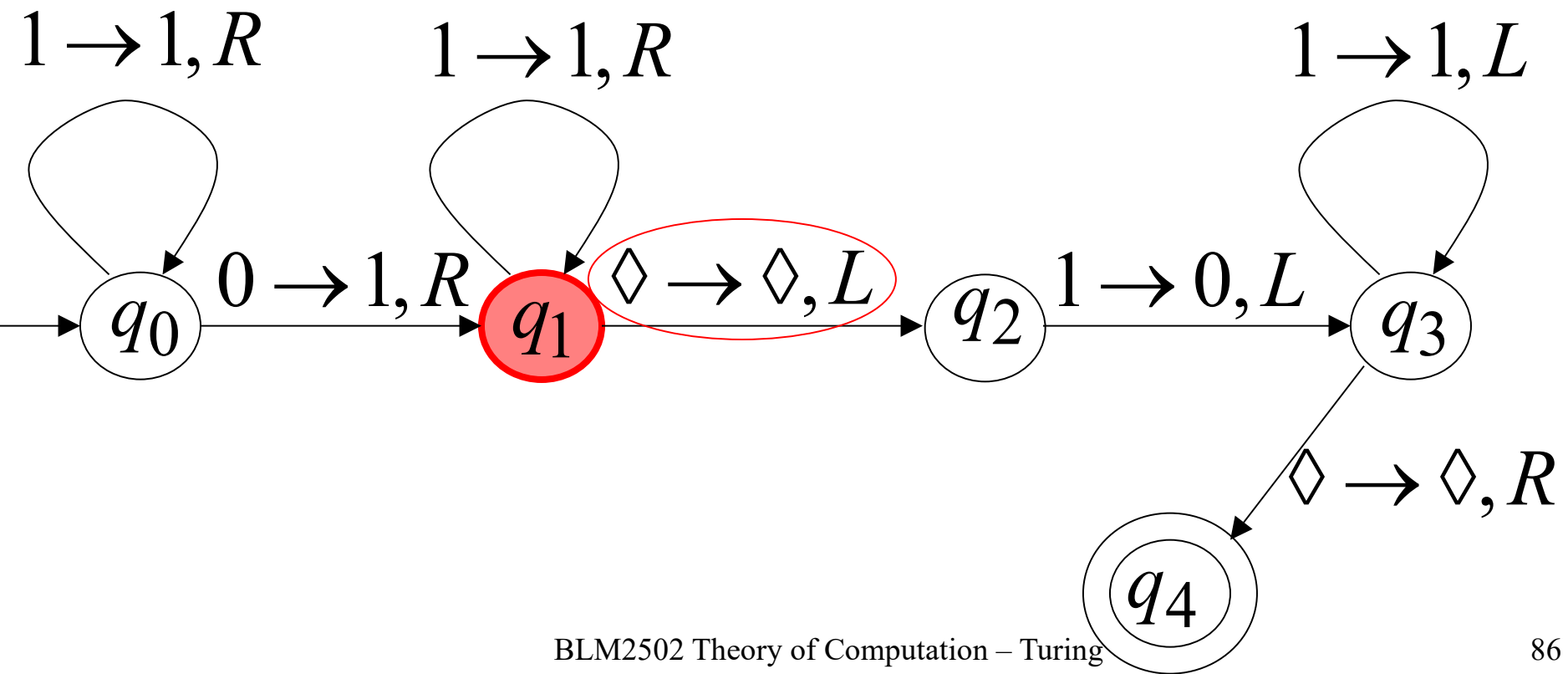
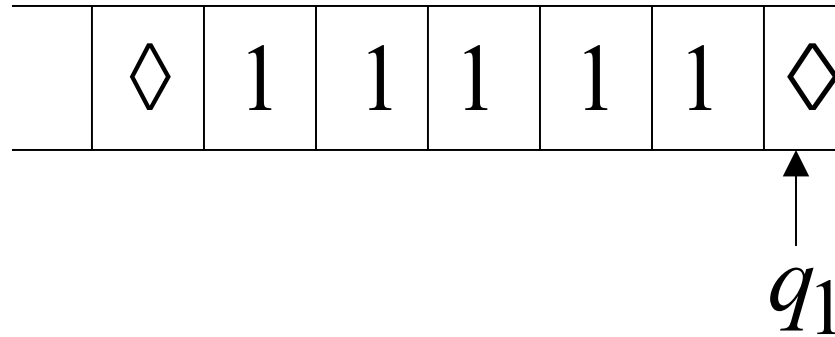
Time 3



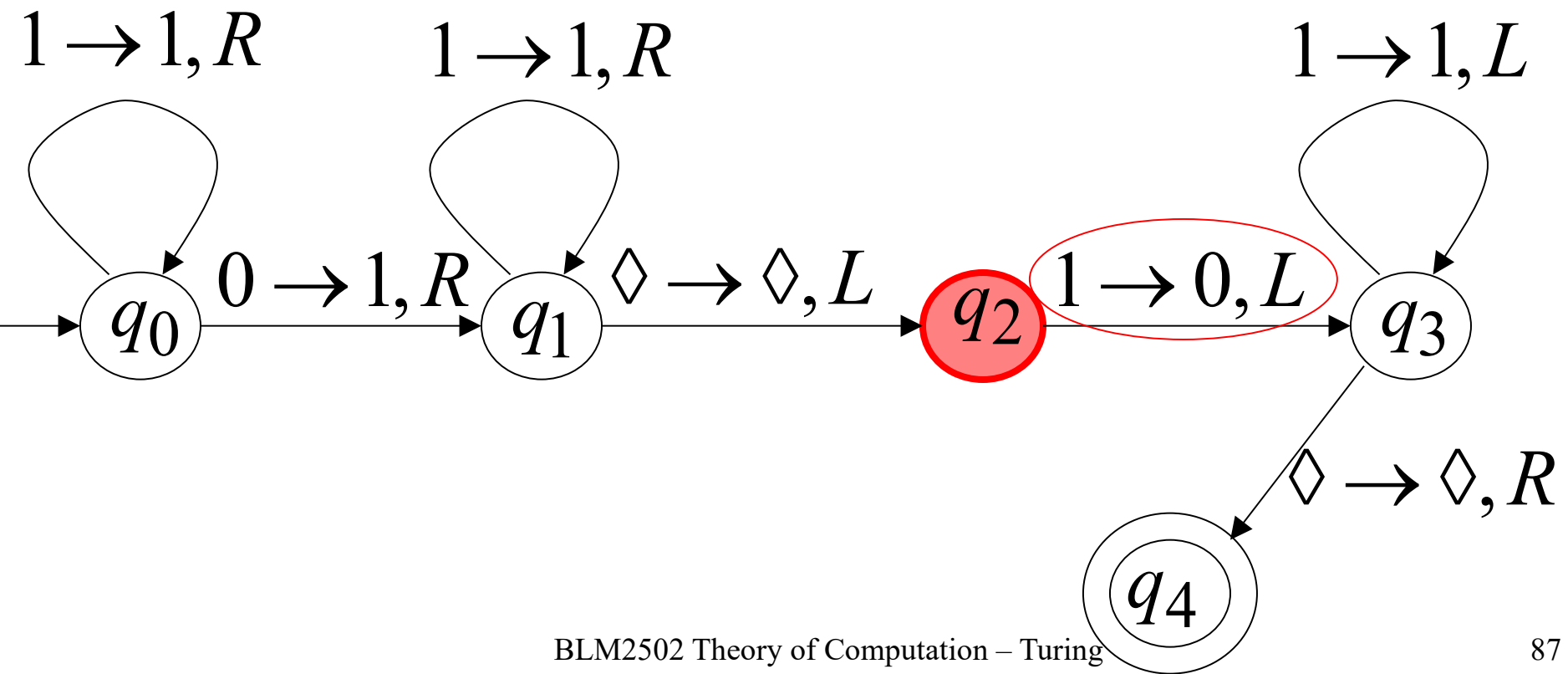
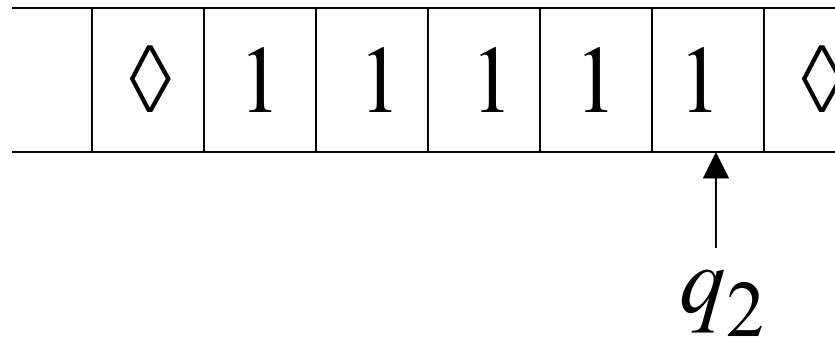
Time 4



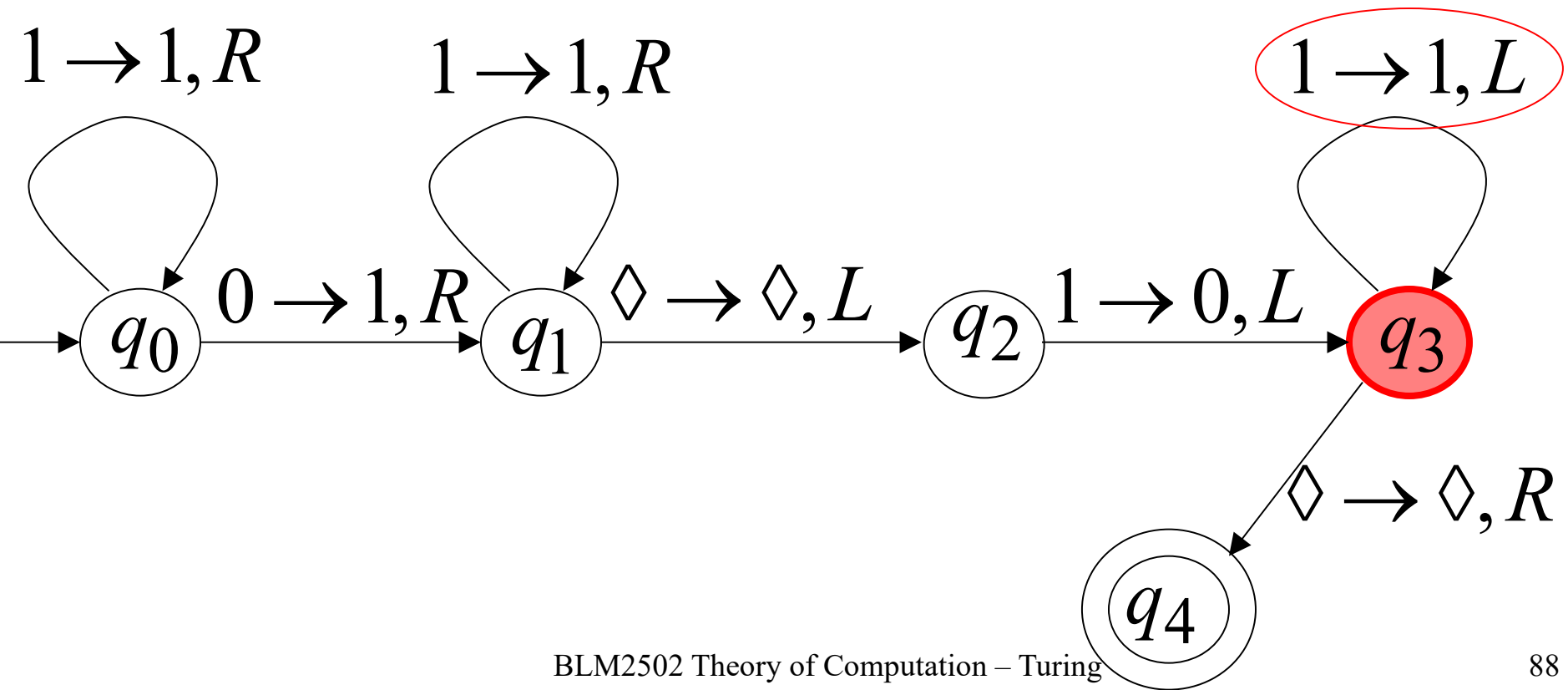
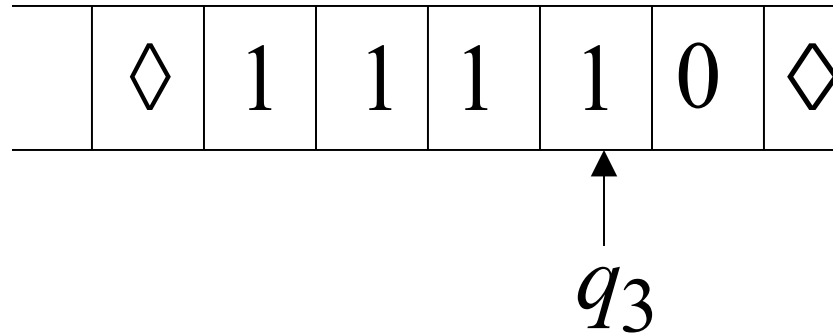
Time 5



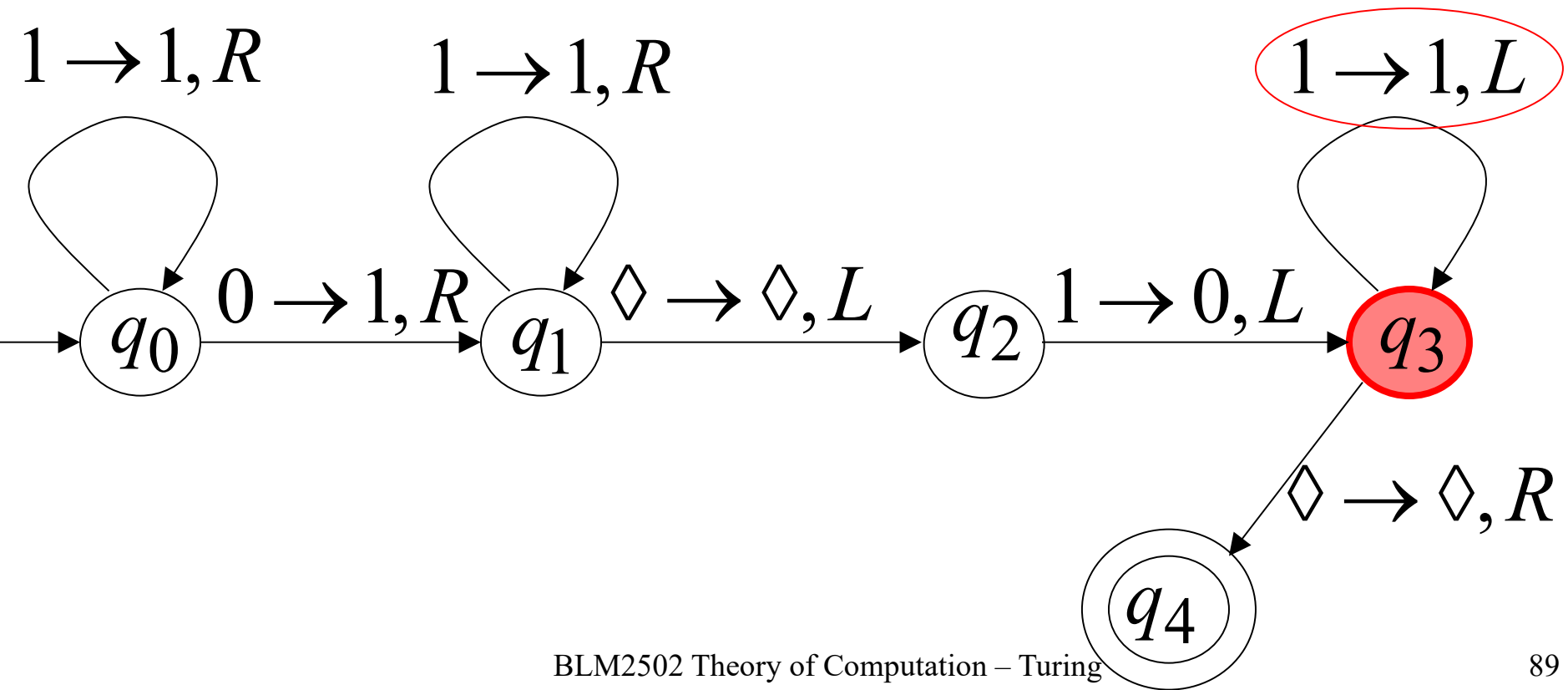
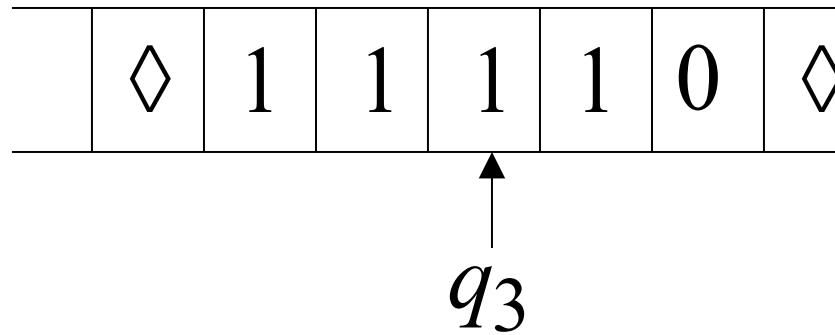
Time 6



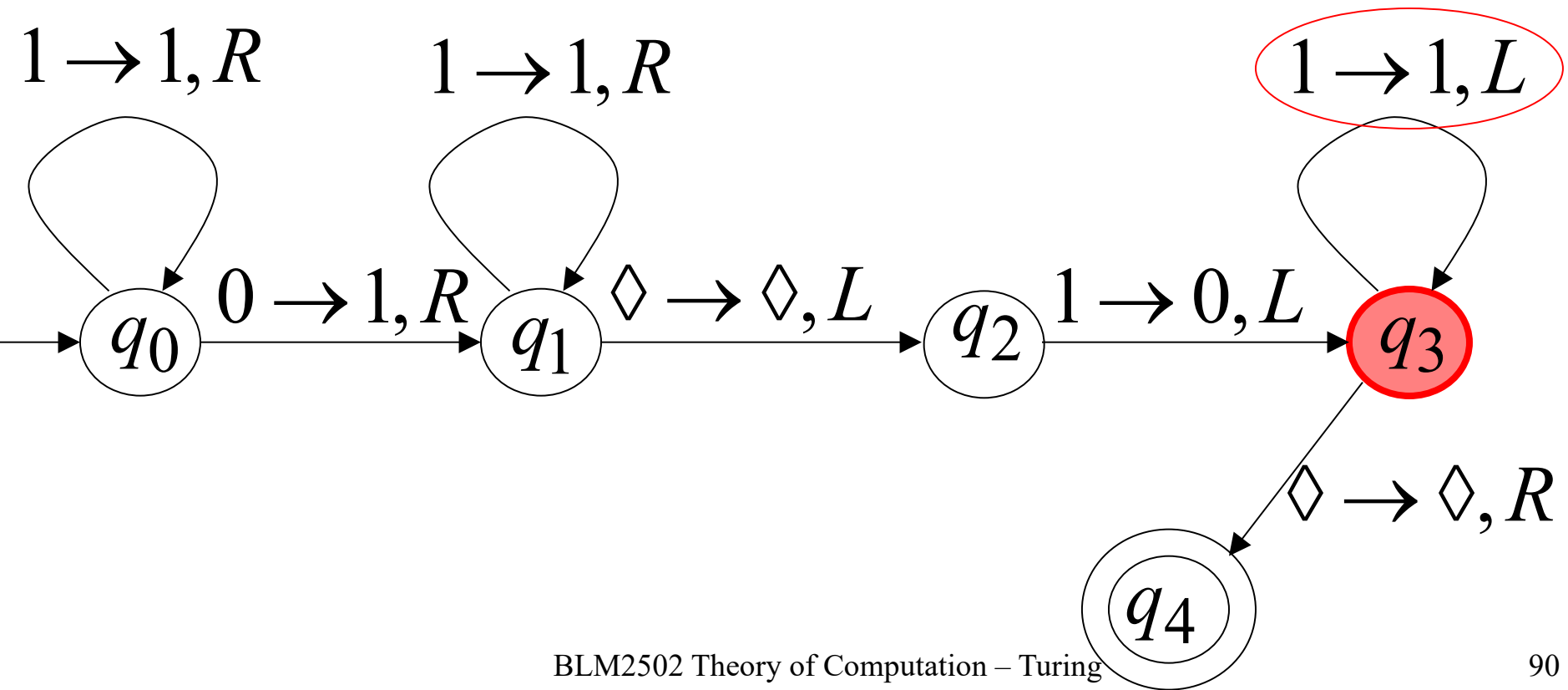
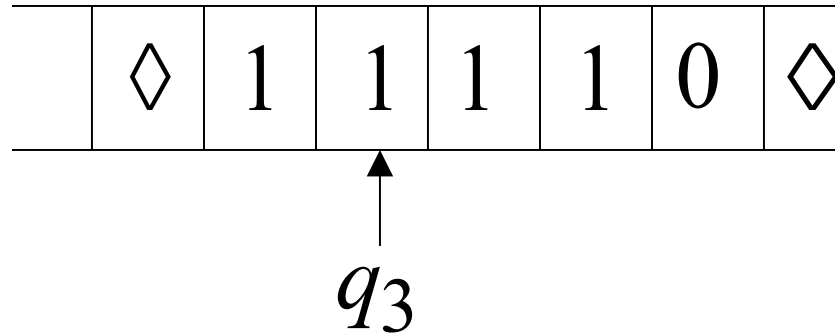
Time 7



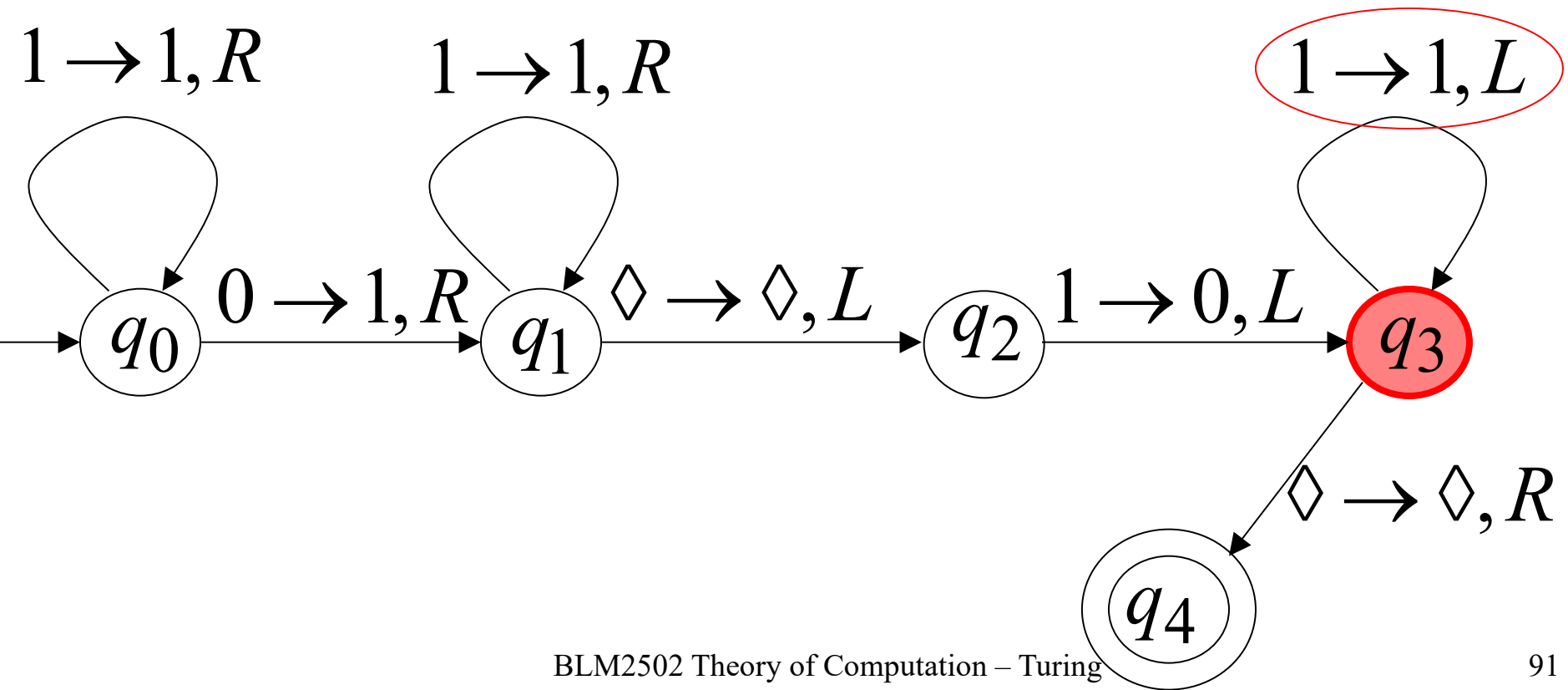
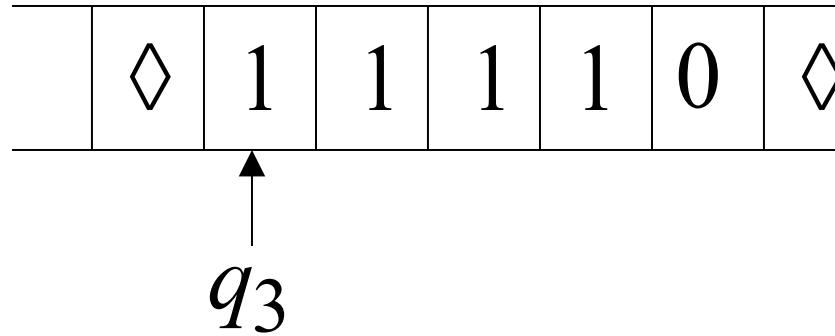
Time 8



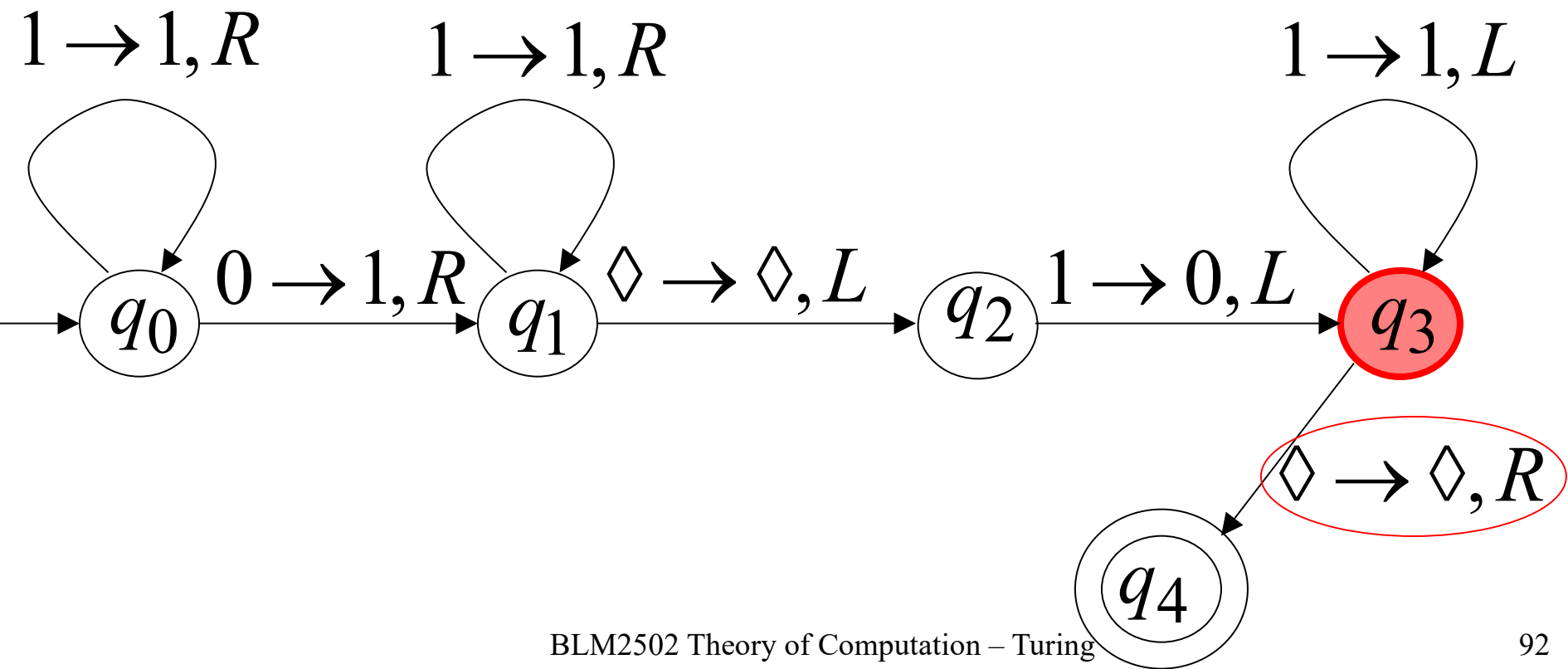
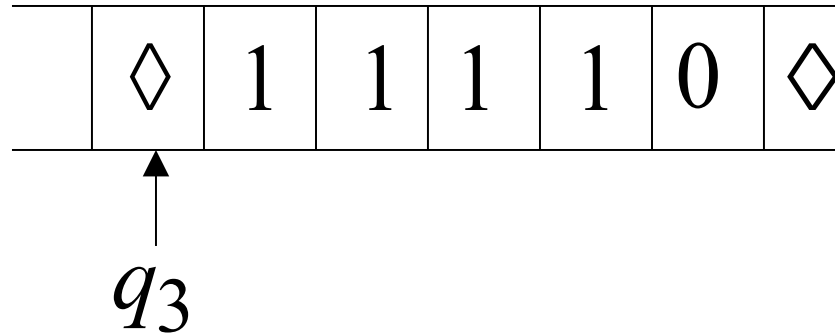
Time 9



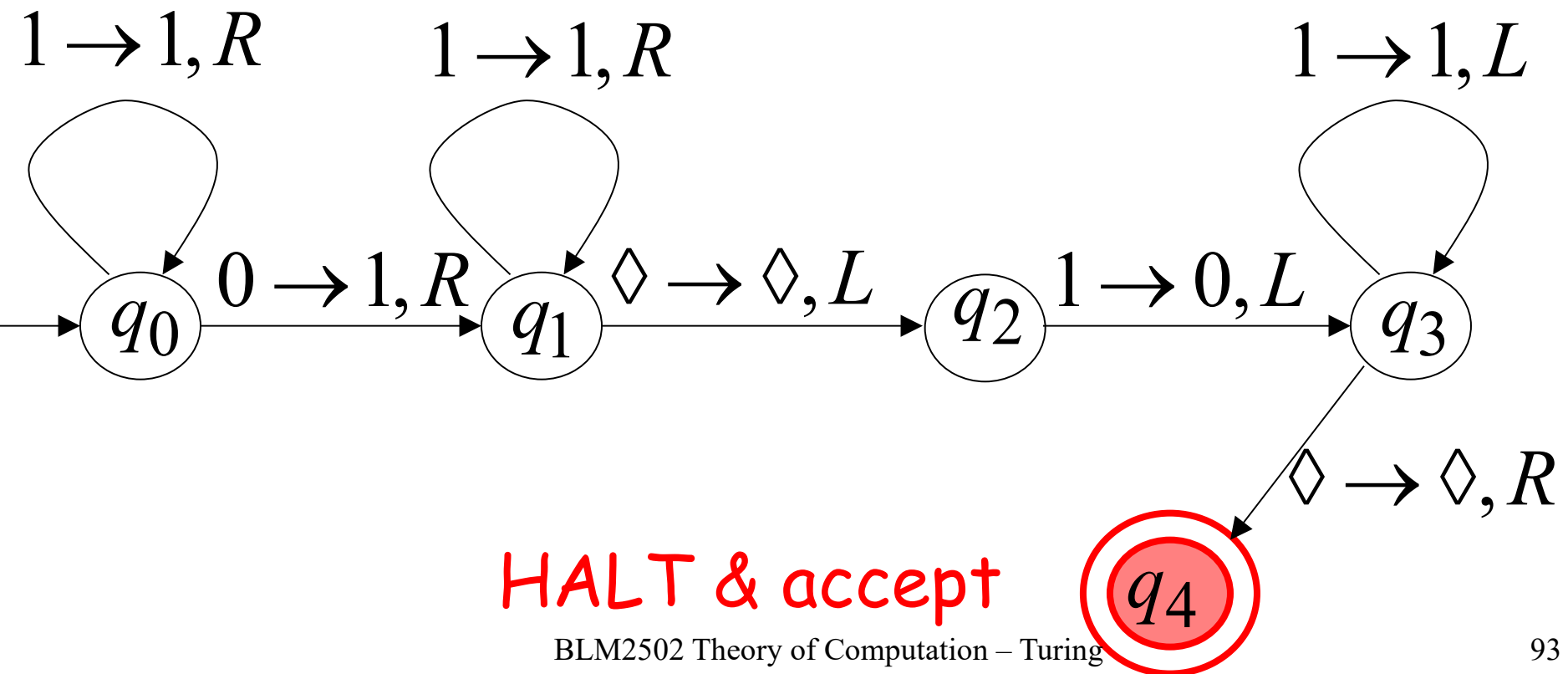
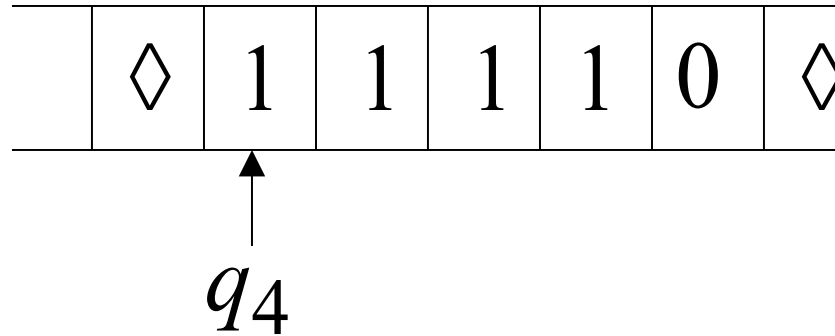
Time 10



Time 11



Time 12



Another Example

The function $f(x) = 2x$ is computable

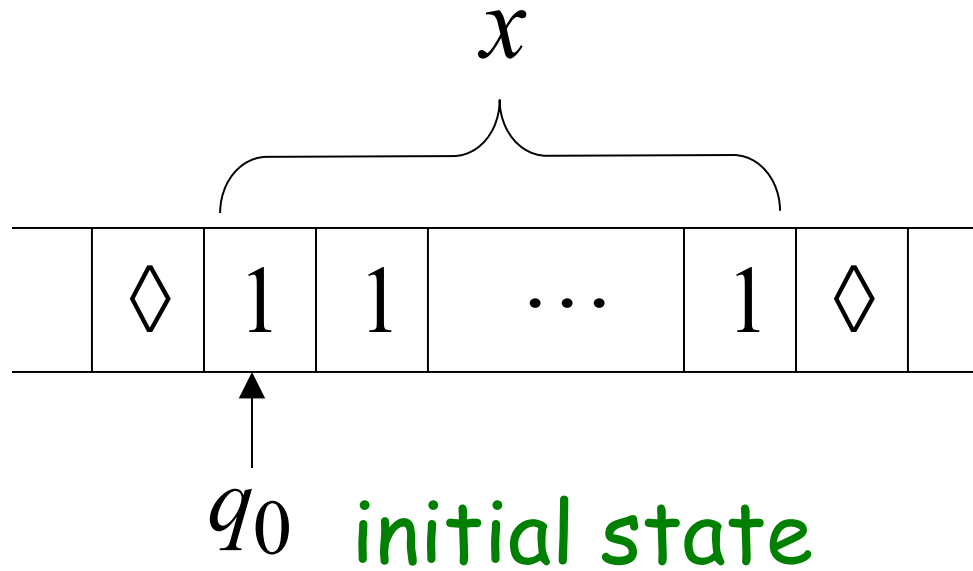
x is integer

Turing Machine:

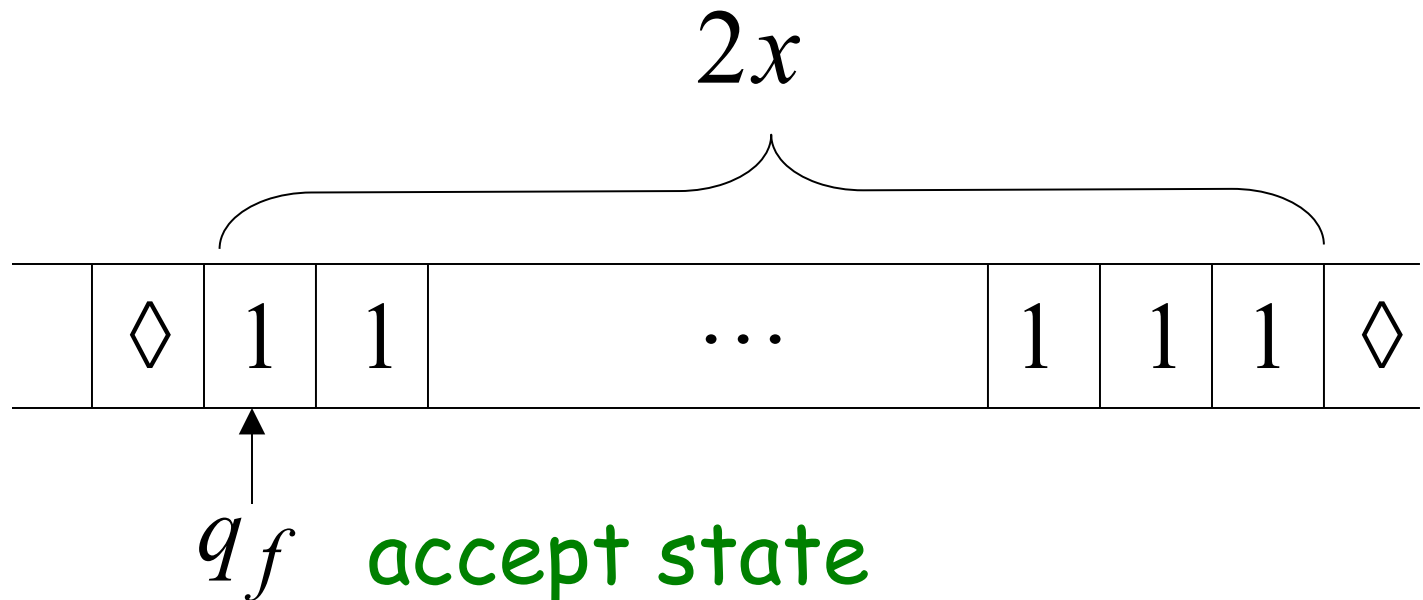
Input string: x unary

Output string: xx unary

Start



Finish

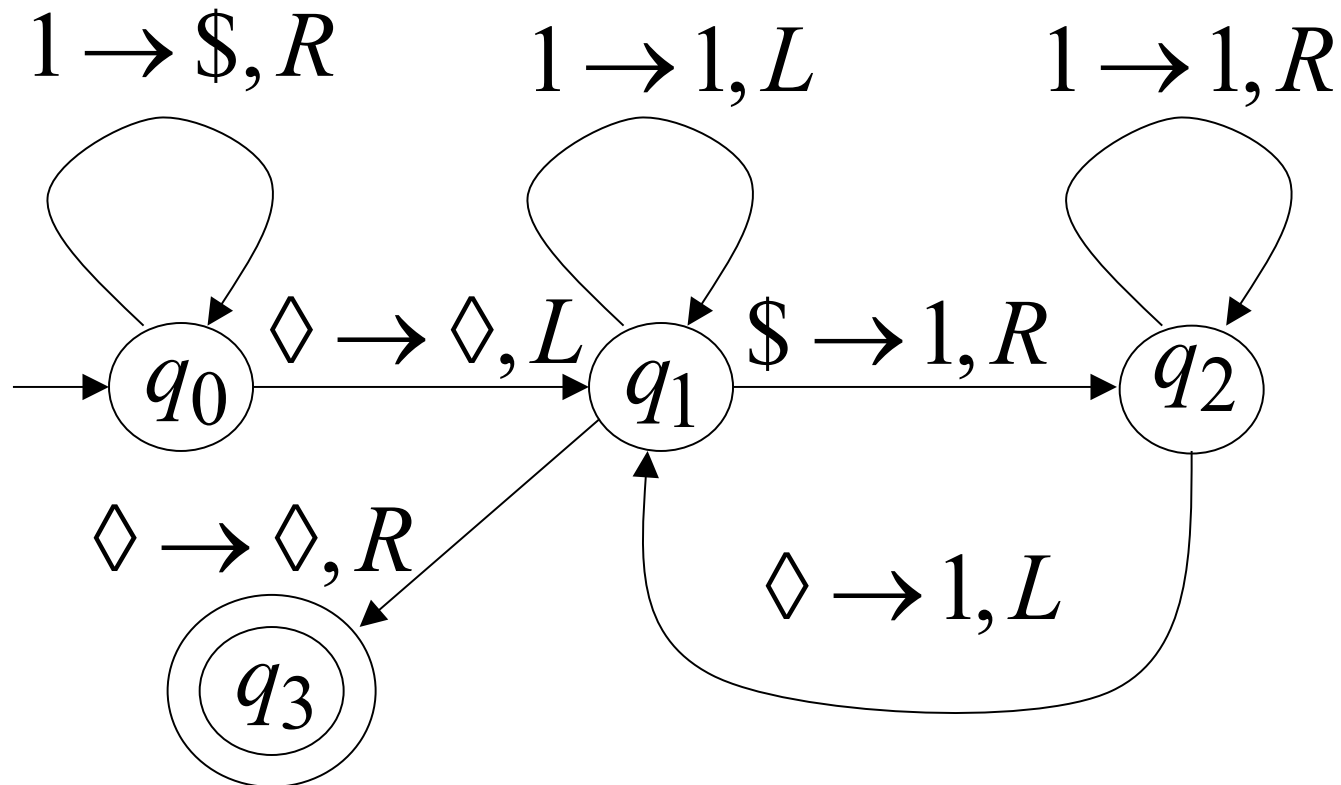


Turing Machine Pseudocode for $f(x) = 2x$

- Replace every 1 with \$
- Repeat:
 - Find rightmost \$, replace it with 1
 - Go to right end, insert 1

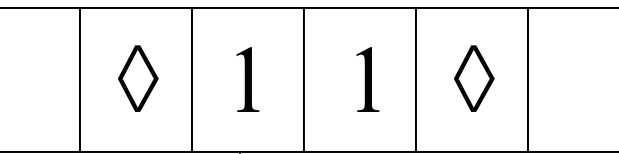
Until no more \$ remain

Turing Machine for $f(x) = 2x$



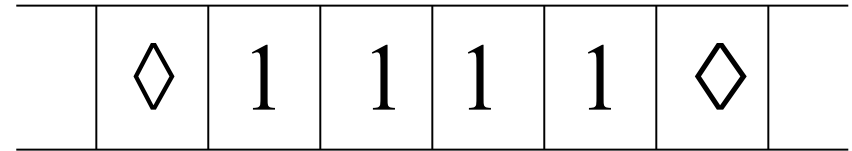
Example

Start

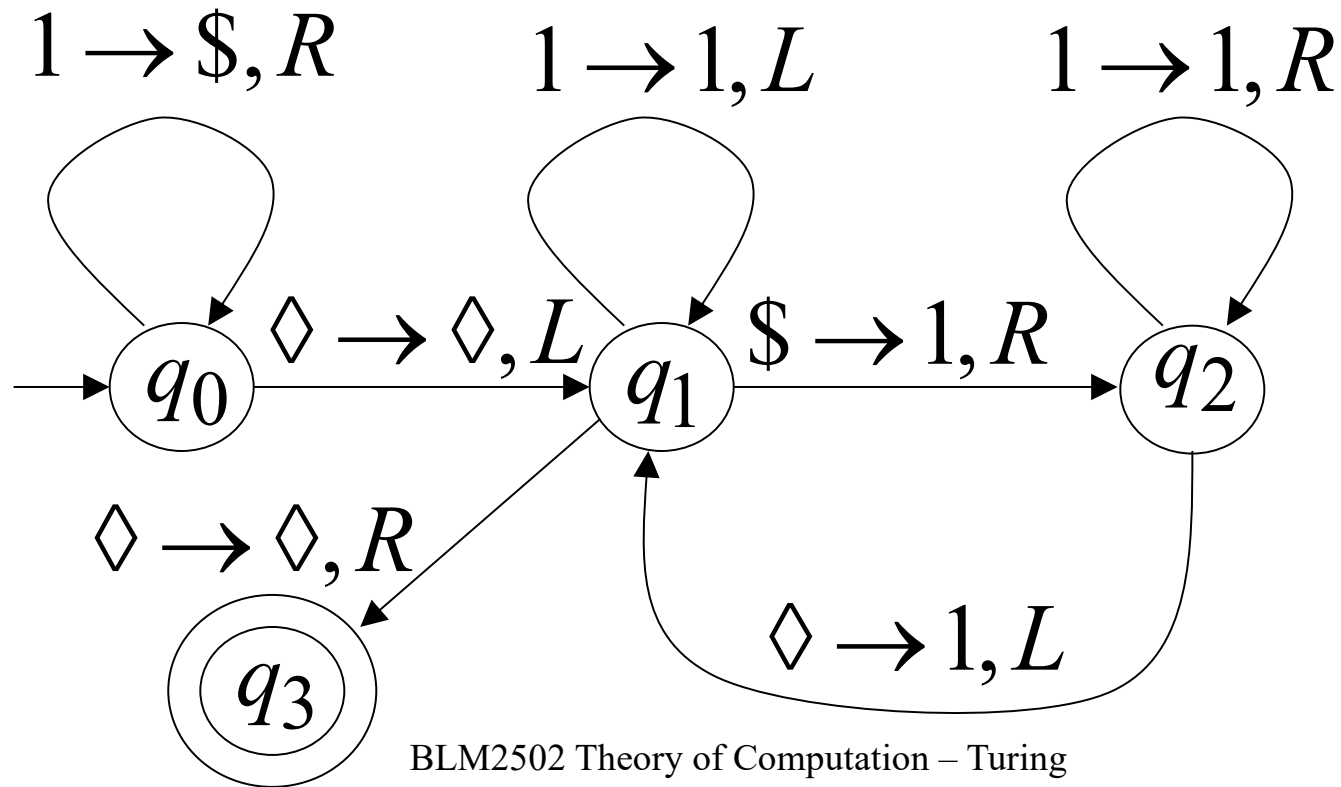


q_0

Finish



q_3



Another Example

The function $f(x, y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$ is computable

Input: $x0y$

Output: 1 or 0

Turing Machine Pseudocode:

- Repeat

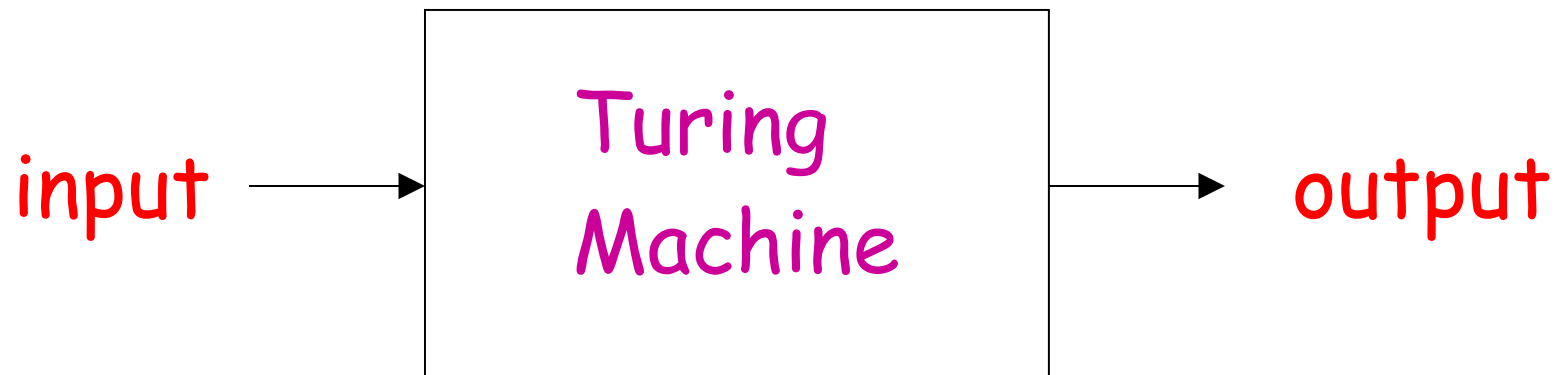
Match a 1 from x with a 1 from y

Until all of x or y is matched

- If a 1 from x is not matched
erase tape, write 1 $(x > y)$
else
erase tape, write 0 $(x \leq y)$

Combining Turing Machines

Block Diagram



Example:

$$f(x, y) = \begin{cases} x + y & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$

