

Continuous Probabilities

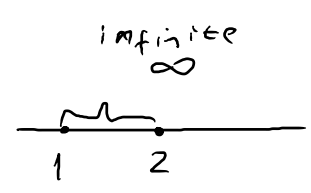
- Discrete r.v.

$\hookrightarrow R_X \rightarrow$ countable set

- Continuous r.v.

$R_X \rightarrow$ an interval in the real line

$R_X \in [1, 2]$



$$P_X(x) = 0 \rightarrow \frac{1}{\infty} = 0$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

PDF f(x)
 \hookrightarrow integration

$$E[X] \rightarrow P(1 < X < 1.2) = \frac{1.2-1}{2-1} = \frac{0.2}{1} = 0.2$$

$$E[X] = \sum_x x \cdot P(x)$$

\downarrow
PMF x

Discrete R.V. $R_{Jes} \approx$ Continuous R.V. R_{Jes}

Continuous r.v.'s X with CDF $F_X(x)$ is said to be continuous if

$F_X(x)$ is a continuous function for all $x \in \mathbb{R}$.

\rightarrow Also assume $F_X(x)$ is differentiable almost everywhere.

Ex: If we choose a real number uniformly at random in the interval $[a, b]$ and call it X .

$$F_X(x) = ?$$

$$F_X(x) = P(X \leq x)$$

if $x < a$

$$F_X(x) = 0$$

if $x > b$

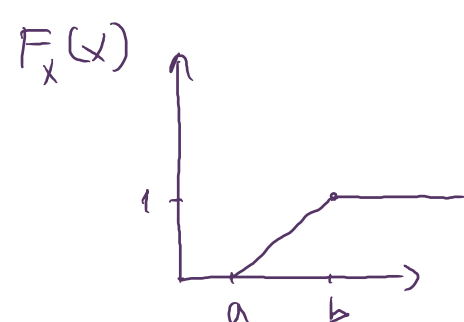
$$F_X(x) = 1$$

if $a \leq x \leq b$

$$F_X(x) = \frac{x-a}{b-a} = P(X \leq x) = P(X \in [a, x])$$

$$F_X(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } b < x \end{cases}$$

Graph CDF



Probability density function (PDF)

PDF of an absolutely continuous r.v. X is (PDF of X on x)

$$f_X(x) = \frac{d}{dx} F_X(x)$$

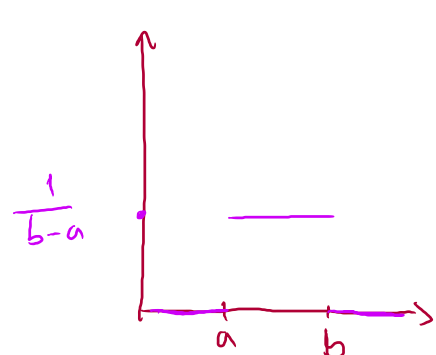
\leftarrow differentiation

Ex: continued

$$f_X(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{for } x > b \end{cases}$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} \frac{d}{dx} 0 = 0 & \text{for } x < a \\ \frac{d}{dx} \left(\frac{x-a}{b-a} \right) = \frac{1}{b-a} & \text{for } a \leq x \leq b \\ \frac{d}{dx} 1 = 0 & \text{for } x > b \end{cases}$$

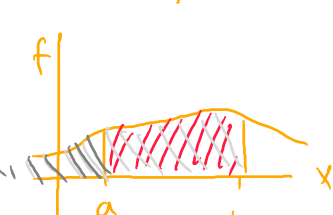
Graph of PDF



PDF and CDF

$$f_X(x) = \frac{d}{dx} F_X(x) \quad F_X(x) = \int_{-\infty}^x f_X(u) du$$

Prob of a range



$$P(a \leq X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u) du$$

$$\text{Total Prob is } 1 \rightarrow \int_{-\infty}^{\infty} f_X(u) du = 1$$

Prob of a set

$$Ex: P(X \in [0, 1] \cup [3, 4]) = \int_0^1 f_X(x) dx + \int_3^4 f_X(x) dx$$

$$P(X \in A) = \int_A f_X(x) dx$$

Range of C.R.V.

set of possible values of X

If X is continuous, R_X can be defined in terms of set of real numbers whose density is larger than 0

$$R_X = \{x | f_X(x) > 0\}$$

Ex: Lifetime in years of some electronic component is a continuous r.v. with $f_X(x) = \begin{cases} \frac{k}{x^3} & \text{for } x > 1 \\ 0 & \text{for } x < 1 \end{cases}$

$$f_X(x) = ? \rightarrow \text{Graph } a$$

$k = ?$

$$P\{\text{lifetime exceeding 5 years}\} = ? <$$

$$b) \int_{-\infty}^{\infty} f_X(x) dx = \int_1^{\infty} f_X(x) dx = \int_1^{\infty} \frac{k}{x^3} dx = \left[-\frac{k}{2x^2} \right]_1^{\infty} = \frac{k}{2} = 1 \quad \text{Total prob} = 1$$

$$a) F_X(x) = \int_{-\infty}^x f_X(u) du \rightarrow \int_1^x f_X(u) du = \int_1^x \frac{k}{u^3} du = \left[-\frac{k}{2u^2} \right]_1^x = -\frac{k}{2x^2} + \frac{k}{2} = 1 - \frac{1}{x^2}$$

$$F_X(x) = \begin{cases} 0 & \text{for } x \leq 1 \\ 1 - \frac{1}{x^2} & \text{for } x > 1 \end{cases}$$

$$c) \int_0^{\infty} f_X(x) dx = \left[-\frac{1}{x^2} \right]_1^{\infty} = \frac{1}{1} = 1$$

Expected value of C.R.V.

$$EX = E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) dx$$

$$EX = E[X] = \sum_{x_i} x_i \cdot P(x_i)$$

$$\text{Expected value of Function of C.R.V. } EX: f_X(x) = 2x^{-3} \quad x > 1$$

$$Y = g(X) \quad \int_{-\infty}^{\infty} x \cdot f_X(x) dx = EX = \int_1^{\infty} 2x^{-2} dx = \left[-\frac{2}{x} \right]_1^{\infty} = 2$$

$$EY = \int_{-\infty}^{\infty} g(y) \cdot f_Y(y) dy \quad EX: E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f_X(x) dx = 2 \cdot \lim_{x \rightarrow \infty} \left[\frac{1}{x} \right] = \infty$$

EXPECTATION IS LINEAR

$$E(ax + b) = a \cdot EX + b \quad \text{for all } a, b \in \mathbb{R}$$

$$E(X_1 + X_2 + \dots + X_n) = EX_1 + EX_2 + \dots + EX_n$$

\hookrightarrow Geom Negative binomial...

Variance of a C.R.V.

$$Var(X) = E((X - M_X)^2) = \int_{-\infty}^{\infty} (x - M_X)^2 \cdot f_X(x) dx \quad \leftarrow \text{form I}$$

$$b) E(X^2 - M_X^2) = \left(\int_{-\infty}^{\infty} x^2 \cdot f_X(x) dx \right) - M_X^2 = \int_{-\infty}^{\infty} x^2 \cdot f_X(x) dx - M_X^2 \quad \leftarrow \text{form II}$$

EX: continued

$$f_X(x) = 2x^{-3} \quad \text{for } x \geq 1 \quad Var(X) = ?$$

$$EX = 2 \quad EX^2 = \infty \quad Var(X) = EX^2 - M_X^2 = \infty - 2^2 = \infty$$

Properties of Variance

$$Var(ax + b) = a^2 Var(X)$$

$$\text{For independent } X \text{ and } Y \quad Var(X + Y) = Var(X) + Var(Y)$$

Functions of Continuous Random Variables

if X is a C.R.V. and $Y = g(X)$ then Y itself is a r.v.

- we can find the $F_Y(y), f_Y(y), \dots$

$$\textcircled{1} \text{ find } R_Y = \{g(x) | f_X(x) > 0\}$$

$$\textcircled{2} F_Y(y) = P(Y \leq y) = P(g(X) \leq y)$$

$$\textcircled{3} f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$Ex: \text{ Let } X \sim \text{uniform}(0, 1) \quad \text{Let } Y = e^X$$

$$a) F_Y(y) = ? \quad b) f_Y(y) = ? \quad c) EY$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 < x < 1 \\ 1 & x > 1 \end{cases} \quad a, b = 1$$

$$R_Y = \{g(x) | f_X(x) > 0\} = [g(0), g(1)] = [1, e]$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases} \quad \begin{cases} 0 & y < 1 \\ \ln y & 1 < y \leq e \\ 1 & y > e \end{cases}$$

$$R_X = [0, 1]$$



$$F_Y(y) = \begin{cases} 0 & y < 1 \\ 1 & e < y \\ \frac{\ln y}{e-1} & 1 < y \leq e \end{cases} \quad \begin{aligned} &\text{for } 1 \leq y \leq e \\ &F_Y(y) = P(e^X \leq y) \\ &= P(X \leq \ln y) \\ &= F_X(\ln y) = \text{from } f_X(x) \end{aligned}$$

$$F_Y(y) = \begin{cases} 0 & y < 1 \\ \ln y & 1 \leq y \leq e \\ 1 & y > e \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$EY = \int_{-\infty}^{\infty} y \cdot f_Y(y) dy$$

$$\begin{cases} \frac{1}{y} & 1 < y \leq e \\ 0 & y > e \end{cases}$$

$$\int_1^e \frac{1}{y} dy = \ln y \Big|_1^e = 1$$