# CENG 222
## Statistical Methods for Computer Engineering

## Week 13

Chapter 6 Stochastic Processes

Markov Processes and Markov Chains

# Definitions and Classification

- $X(t, \omega)$ denotes a stochastic process where $t \in T$ is time and $\omega \in S$ is an outcome
- At any fixed time $X_t(\omega)$ is a random variable.
- If we fix an outcome, $X_\omega(t)$ is a function of time and is called a *realization*, a *sample path*, or a *trajectory* of a process $X(t, \omega)$.
- If the set of times is discrete, the process is called a *discrete-time* process. Otherwise, it is called a *continuous-time* process.
- Similarly, if the outcomes are discrete, the process is called *discrete-state* process (and *continuous-state* otherwise)

# Example stochastic processes

- Temperature
- Stock value
- Number of jobs in a queue
- Number of internet connections
- Football score
- *Poisson process*
- *Binomial process*
- *Brownian motion*

# Markov Process

- A stochastic process $X(t)$ is a Markov process if for any $t_1 < \cdots < t_n < \mathrm{t}$

$$P(X(t) \in A \mid X(t_1) = x_1, \ldots, X(t_n) = x_n)$$
$$= P(X(t) \in A \mid X(t_n) = x_n)$$

which means

$$P(\text{future} \mid \text{past, present}) = P(\text{future} \mid \text{present})$$

# Markov Chain

- A Markov chain is a discrete-time, discrete-state Markov process
- $T = \{0,1,2,\dots\}$
- A Markov chain is a random sequence
- $\{X(0), X(1), X(2), \dots\}$
- Markov property implies that the value of $X(t+1)$ can be predicted by only looking at $X(t)$

# Transition probability

- $p_{ij}(t) = P(X(t+1) = j \mid X(t) = i)$

  is the probability of the Markov chain $X$ to make a transition from state $i$ to state $j$ at time $t$.

- $p_{ij}^{(h)}(t) = P(X(t+h) = j \mid X(t) = i)$

  is the $h$-step transition probability

# Homogeneity

- A Markov chain is *homogeneous* if all its transition probabilities are independent of $t$, i.e., the transition from state $i$ to state $j$ is the same at any time.
- Hence, all the one-step transition probabilities can be represented as an $n \times n$ matrix, if we have $n$ states.

# State distribution

- At each time step, we have a probability mass function that shows the likelihood of outcomes/states at that time point.
- $P_t$ is the probability mass function for $X(t)$
- $P_0$ is the initial distribution
- The distribution of a Markov chain is completely determined by $P_0$ and the transition probabilities $p_{ij}$

# Things we can compute from $P_0$ and $p_{ij}$

- $h$-step transition probabilities $p_{ij}^{(h)}$
- $P_h$, i.e. the state distribution at time $h$.
- The limit of $P_h$ as $h \to \infty$, i.e., the long-term forecast.

# One-step transition probabilities

$$\begin{array}{ll|c} & & \text{From} \\ & & \text{state:} \\ P \quad = & \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} \\ \hline \text{To state:} & \quad\; 1 \qquad 2 \qquad \cdots \qquad n & \end{array}$$

# *h*-step transition probabilities

- $P^{(h)} = P^h$
- The $h^{th}$ power of the one-step transition probability matrix gives the $h$-step transition probability matrix.

# The state distribution at time *h*

- $P_h = P_0 P^h$
- Caution: The state distributions $P_h$ and $P_0$ are row vectors, i.e., $1 \times n$ matrices; whereas the transition probability matrices $P$, $P^{(h)}$, and $P^h$ are $n \times n$ matrices.
- The transition probability matrices are always row normalized, i.e., sum of probabilities in a row is 1.
- State distributions are *pmf*s, i.e., the probabilities also add up to 1 in a state distribution.

# Steady-state distribution

- The state distribution at the limit is called the steady-state distribution
- $\pi_x = \lim_{h \to \infty} P_h(x)$
- In the limit, the state distribution does not change from time *t* to time *t*+1.
- Hence, it can be found by solving

$$\pi = \pi P$$

This equation has infinitely many solutions (scaled by a constant factor *c*), but a unique state distribution as the solution.

# Limit of $P^h$

- $\Pi = \lim_{h\to\infty} P^{(h)} = \begin{pmatrix} \pi_1 \pi_2 & \cdots & \pi_n \\ \vdots & \ddots & \vdots \\ \pi_1 \pi_2 & \cdots & \pi_n \end{pmatrix}$
- Each row of the matrix is the steady-state distribution.

# Existence of a Steady State

- Periodic Markov chains do not have steady state distributions
- Example:
- $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $P^{(h)} = P^h = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ for all odd } h \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ for all even } h \end{cases}$

# Regular Markov Chains

- A Markov chain is regular if for some step $h$, all the $h$-step transition probabilities between states are strictly greater than 0.
- Any regular Markov chain has a steady-state distribution.
- Example 6.15.
  - If the one-step transition matrix contains 0s, can the Markov chain be regular?
    - Yes, if its $h$-step step transition matrix contains values all $> 0$.