Alice and Bob Chess Game

Instructions

Alice and Bob play a game of chess, choosing each round to play **aggressive**, **balanced**, or **defensive**. A win gives +1 point, a draw gives +0.5, and a loss gives 0 points. The payoff matrix is based on current points of Alice (n_A) and Bob (n_B). Let $M = 10^9 + 7$.

Question 1

If Alice and Bob choose to attack all the time:

(a) What is the probability that after T rounds, Alice wins T_1T_2 matches and Bob wins T_3T_4 matches, where $T = T_1T_2 + T_3T_4$ and T_1, T_2, T_3, T_4 are the last 4 digits of your Entry Number? (Replace any 0s in your Entry Number with 9).

(b) Define a random variable X_i as:

$$X_i = \begin{cases} 1 & \text{if Alice wins round } i \\ 0 & \text{if it's a draw} \\ -1 & \text{if Alice loses} \end{cases}$$

Compute $\mathbb{E}\left[\sum_{i=1}^{T} X_i\right]$ and $\mathrm{Var}\left(\sum_{i=1}^{T} X_i\right)$, where $T = T_3T_4$.

Question 2

Bob's form is influenced by the result of the previous round:

- If he won: plays defensively
- If it was a draw: plays balanced
- If he lost: plays aggressively

(a) What is the optimal strategy for Alice to maximize her points in the current round? Perform a Monte Carlo simulation to demonstrate the advantage.

(b) Is it optimal for Alice to always use the greedy strategy derived in (a)? If not, provide a scenario where a non-greedy strategy outperforms it. Validate using Monte Carlo simulation.

(c) Let τ be the number of rounds Alice takes to reach T wins. For the greedy strategy, estimate $\mathbb{E}[\tau]$ via Monte Carlo simulation where $T = T_3T_4$.

Question 3

Now Bob plays uniformly at random each round.

(a) What should Alice's strategy be to maximize expected points in a single round? Validate using Monte Carlo simulation.

(b) What should Alice's strategy be to maximize her expected points over T future rounds (where $T = T_3T_4$)? Also compute the expected number of points she will have at the end.