Exercise 6

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Statistical Parameter Estimation 2024

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Task 1 (Exercise 7.1 in the book)

Consider the following non-linear state space model:

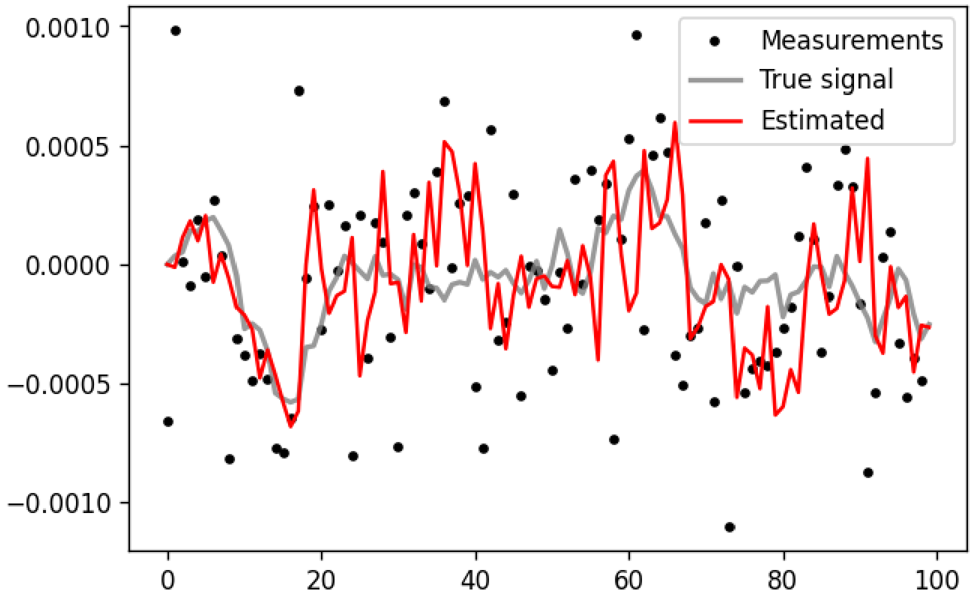
$$\begin{aligned} x_k &= x_{k-1} - 0.01\sin(x_{k-1}) + q_{k-1}, \\ y_k &= 0.5\ \sin(2x_k) + r_k \end{aligned},$$

where q_{k-1} has a variance of 0.01^2 and r_k has a variance of 0.02 (I think here might be a typo in the task description as the second variance would be so much larger in this form without the square. I assumed in my code that the second variance would be squared also). Derive the required derivatives for an EKF and implement the EKF for the model. Simulate trajectories from the model, compute the RMSE values, and plot the result.

Required derivatives

$$\begin{aligned} \mathbf{f}(\mathbf{x}_{k-1}) &= x_{k-1} - 0.01\sin(x_{k-1}) \\ \mathbf{F_x}(\mathbf{x}) &= 1 - 0.01\cos(x) \\ \mathbf{h}(\mathbf{x}_k) &= 0.5\sin(2x_k) \\ \mathbf{H_x}(\mathbf{x}) &= \cos(2x) \end{aligned}$$

EKF estimate



RMSE

RMSE of the estimate vs ground truth ≈ 0.000253.

Task 2. (Exercise 7.2 in the book)

The state of the target at time step k consists of the position (x_k, y_k) and the velocity $(\dot{x}_k, \dot{y}_k)$ The dynamics of the state vector $x_k = (x_k, y_k, \dot{x}_k, \dot{y}_k)^T$ are modeled with the discretized Wiener velocity model:

$$\begin{pmatrix} x_k \\ y_k \\ \dot{x}_k \\ \dot{y}_k \end{pmatrix} = \begin{pmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{k-1} \\ y_{k-1} \\ \dot{x}_{k-1} \\ \dot{y}_{k-1} \end{pmatrix} + q_{k-1},$$

where q_{k-1} is a zero mean Gaussian process noise with covariance

$$Q = \begin{pmatrix} q_1^c \Delta t^3/3 & 0 & q_1^c \Delta t^2/2 & 0 \\ 0 & q_2^c \Delta t^3/3 & 0 & q_2^c \Delta t^2/2 \\ q_1^c \Delta t^2/2 & 0 & q_1^c \Delta t & 0 \\ 0 & q_2^c \Delta t^2/2 & 0 & q_2^c \Delta t \end{pmatrix}.$$

In this scenario the diffusion coefficients are $q_1^c = q_2^c = 0.1$ and the sampling period is $\Delta t = 0.1$. The measurement model for sensor $i \in 1, 2$ is:

$$\theta_k^i = \tan^{-1}\left(\frac{y_k - s_y^i}{x_k - s_x^i}\right) + r_k^i,$$

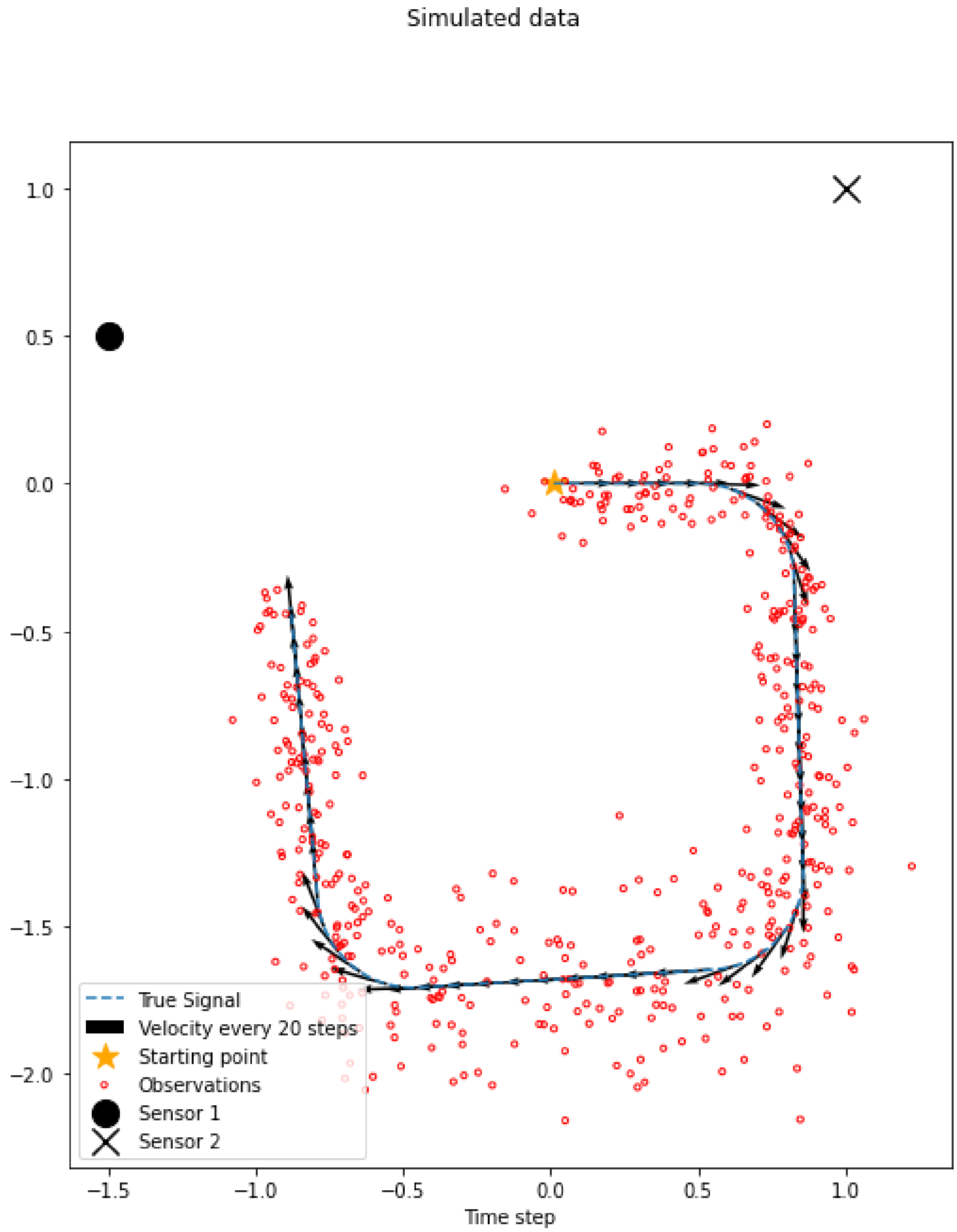
where (s_x^i, s_y^i) is the position of the sensor i, and $r_k^i \sim N(0, \sigma^2)$ is a Gaussian measurement noise with a standard deviation of $\sigma = 0.05$ radians. At each sampling time, which occurs 10 times per second (i.e., $\Delta t = 0.1$), both of the two sensors produce a measurement.

Task is to implement an EKF for the problem and compare the results graphically and in the RMSE sense.

Required derivatives for the EKF

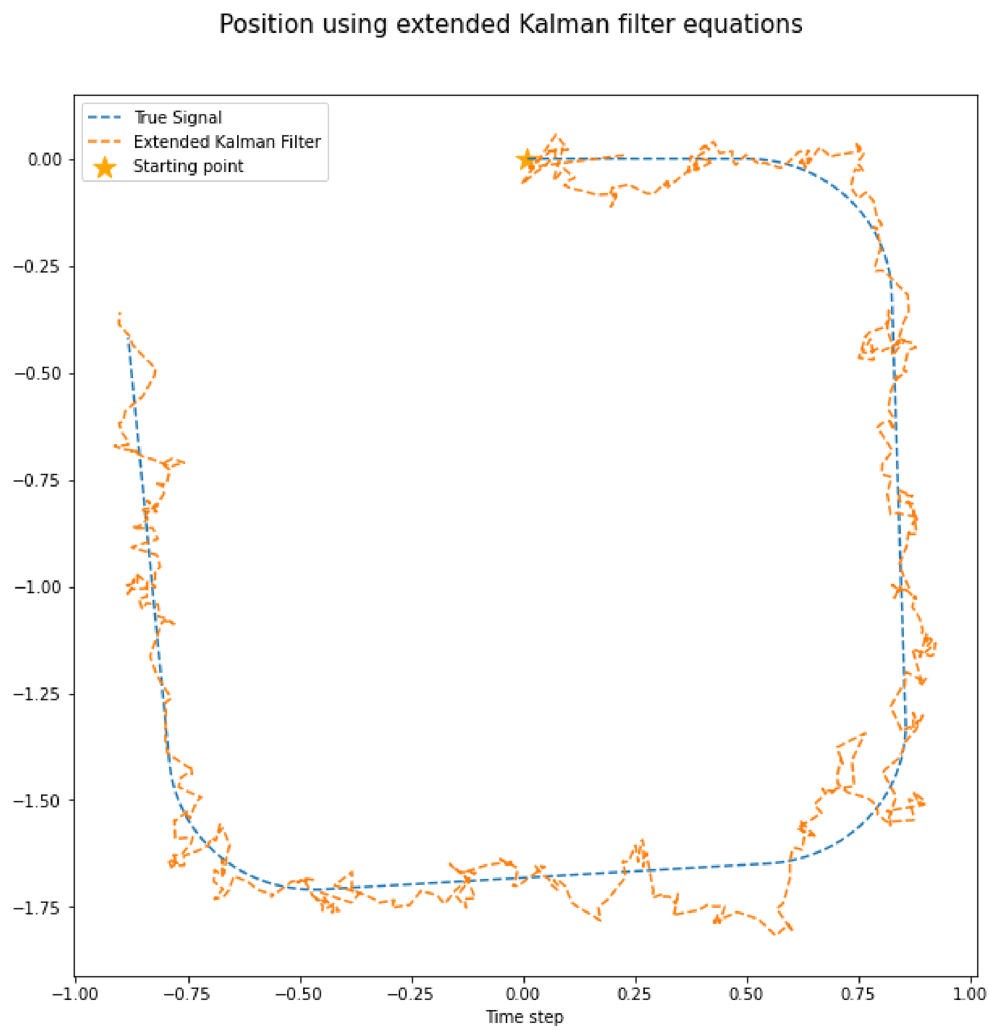
$$
\begin{aligned}
\mathbf{f}(\mathbf{x}_{k-1}) &= \mathbf{A}\mathbf{x_{k-1}} \\
\mathbf{F_x}(\mathbf{x}) &= \mathbf{A} \\
\mathbf{h}(\mathbf{x}_k) &= \tan^{-1}\left(\frac{y_k - s_y^i}{x_k - s_x^i}\right) \\
\mathbf{H_x}(\mathbf{x}) &= \frac{s_y^i - y_k}{x_k^2 - 2x_k s_x^i + (s_x^i)^2 + (s_y^i - y_k)^2} \\
\mathbf{H_y}(\mathbf{x}) &= \frac{s_x^i - x_k}{y_k^2 - 2y_k s_y^i + (s_x^i)^2 - 2s_x^i x_k + (s_y^i)^2 + x_k^2}
\end{aligned}
\qquad \text{where } \mathbf{A} = \begin{pmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Original trace of the trajectory



Baseline solution

EKF solution



RMSE

RMSE of the EKF estimate vs true signal ≈ 0.315