Presentation Topics

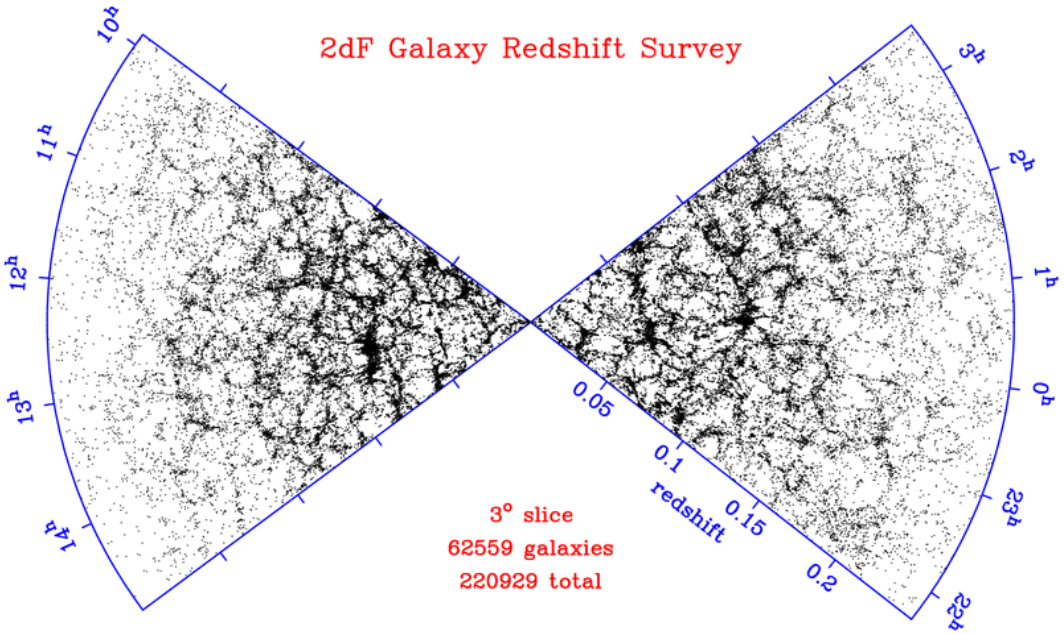
- May 1
  - Max
  - Ryan
  - TBA
- May 6
  - Kendra
  - Adrian
  - Andrew
- May 8
  - Rahul
  - Marina
  - Donald

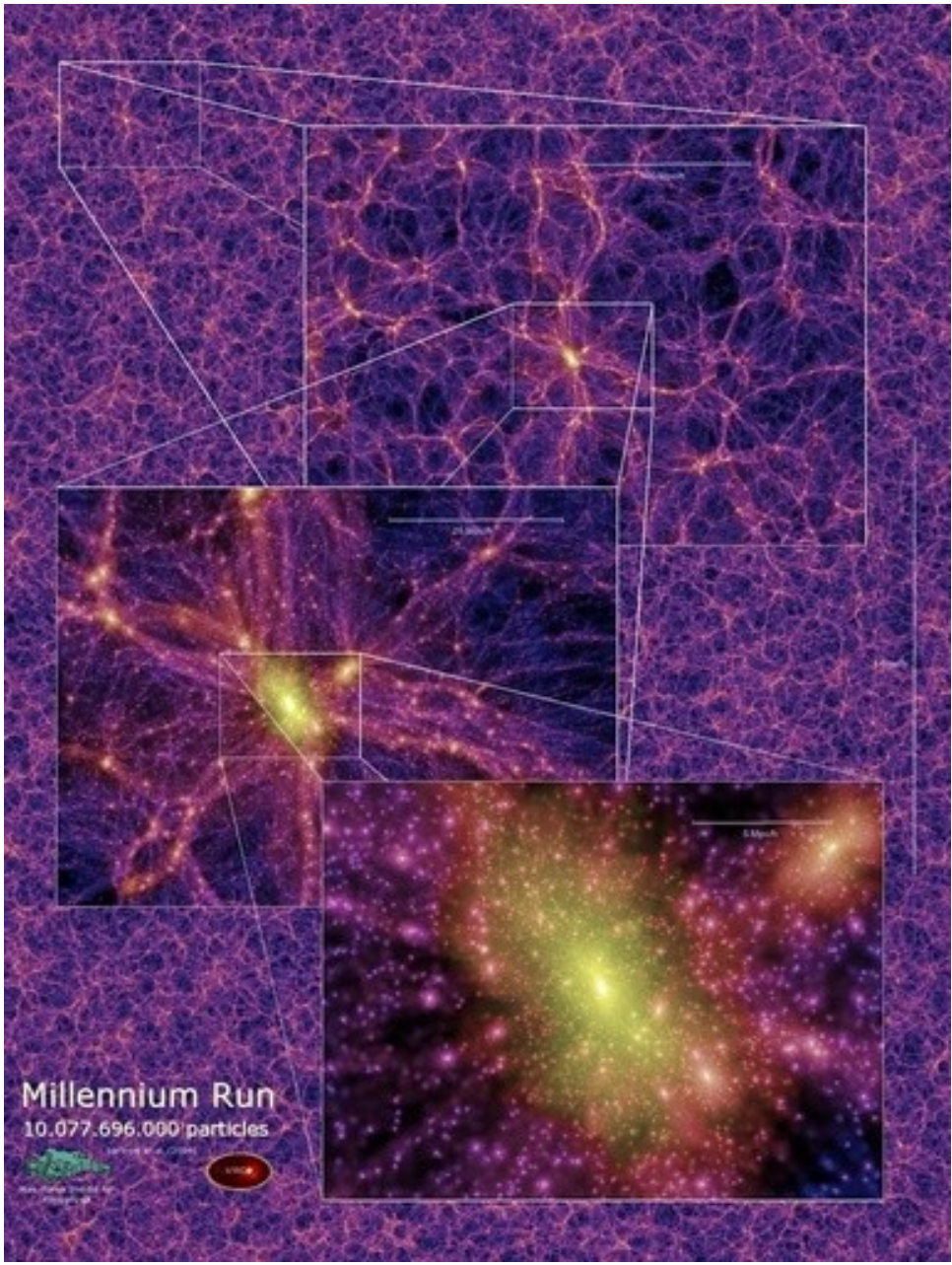
Hydrodynamics

- When we discussed PDEs, we focused so far on scalar PDEs
- Often we wish to study systems of PDEs.
- Here we'll look at the equations of hydrodynamics
  - Nonlinear system of hyperbolic PDEs
  - We'll see many of our ideas extend to this case
  - These same ideas can be applied to other hyperbolic systems

Computation on the Cosmological Scale

- Small inhomogeneities in the early Universe seed structure
- More than 10 billion particles
- Self-gravity dominates the evolution



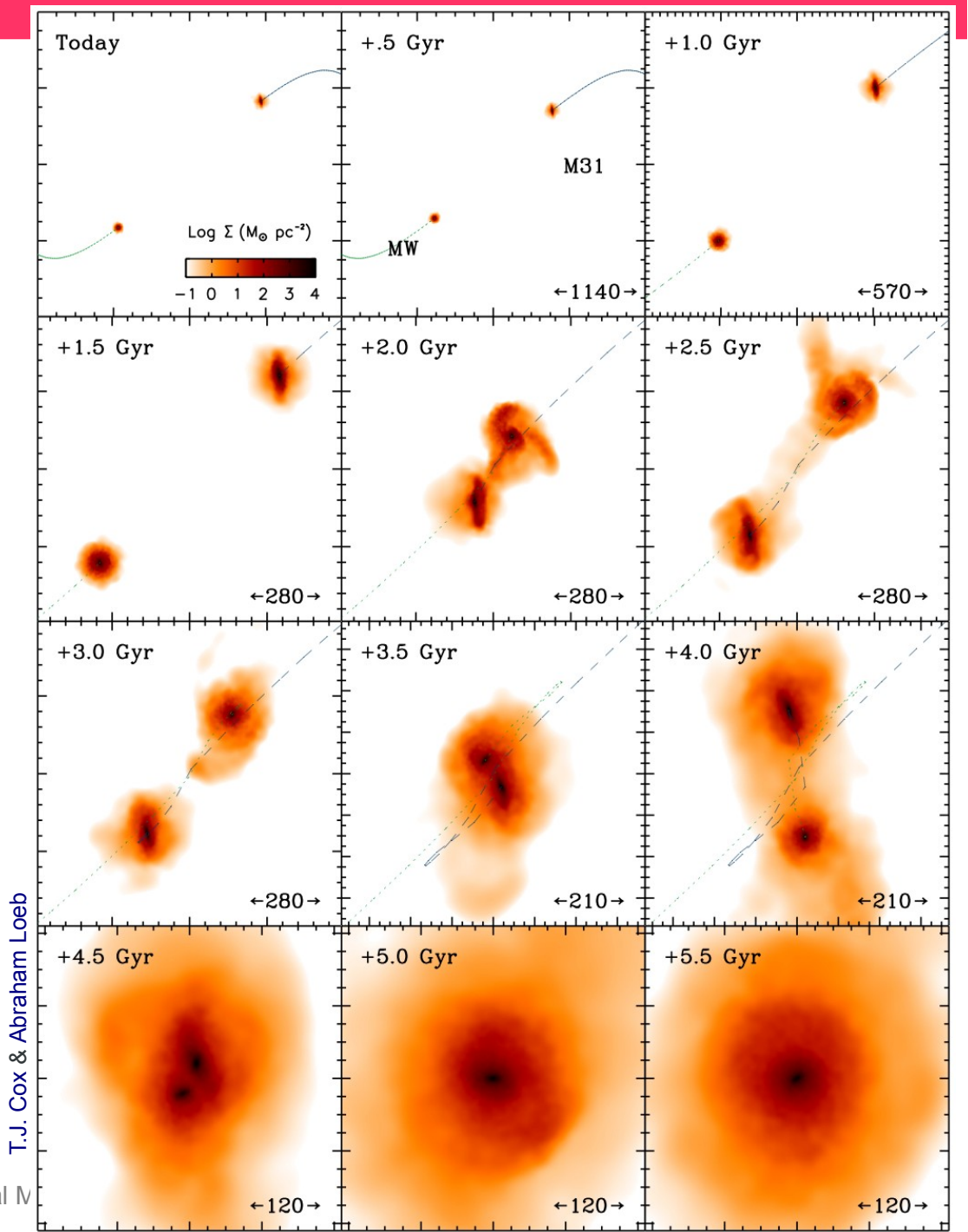


Simulating the growth of structure and the formation of galaxies. (Springel et al. 2005)

Computation on the Galactic Scale

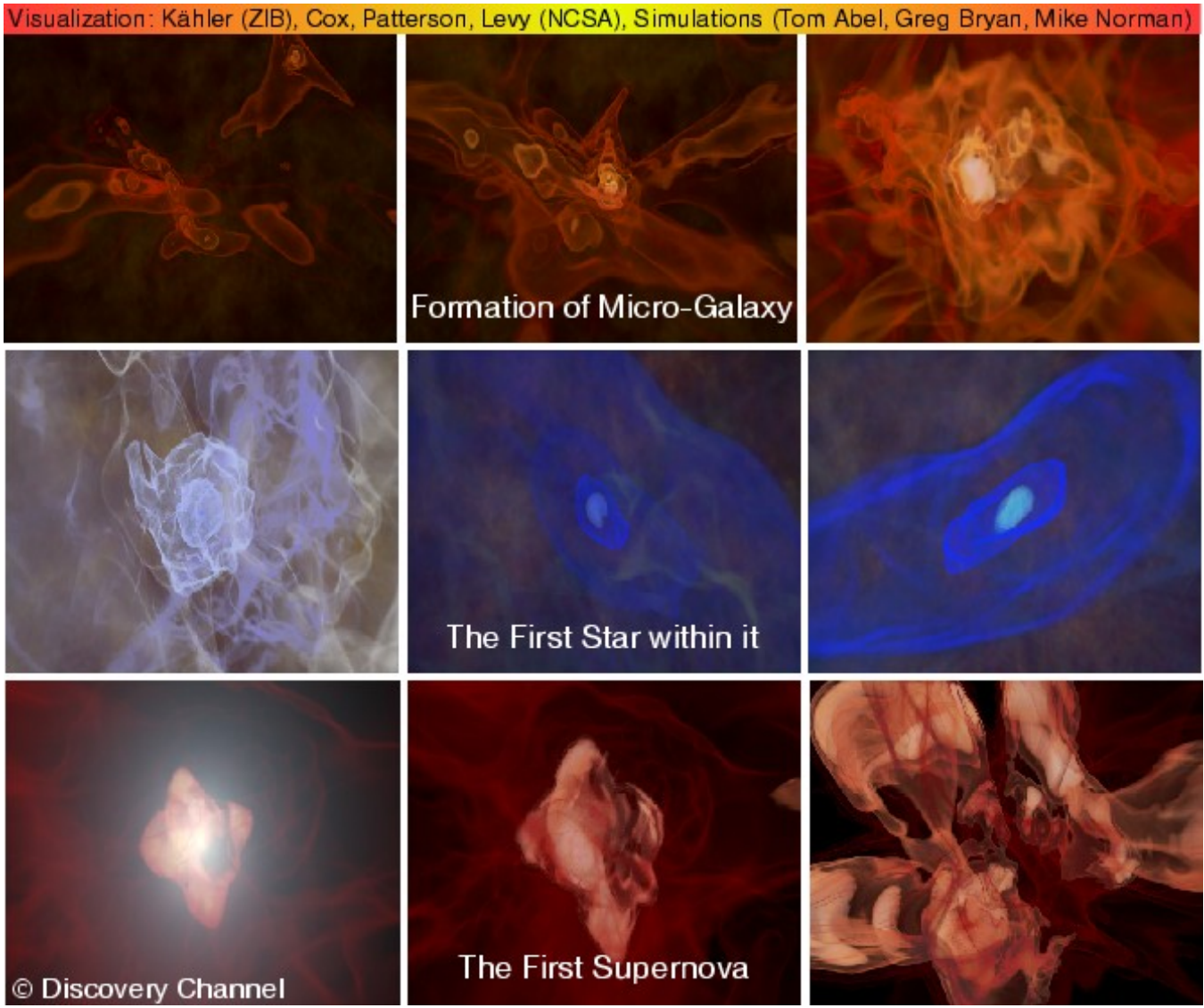
The merger of the Milky Way and Andromeda

What happens to the Sun?



Computation on the Galactic Scale

An exceptionally detailed simulation of the formation of the first stars in the universe.



Computation on the Stellar Scale

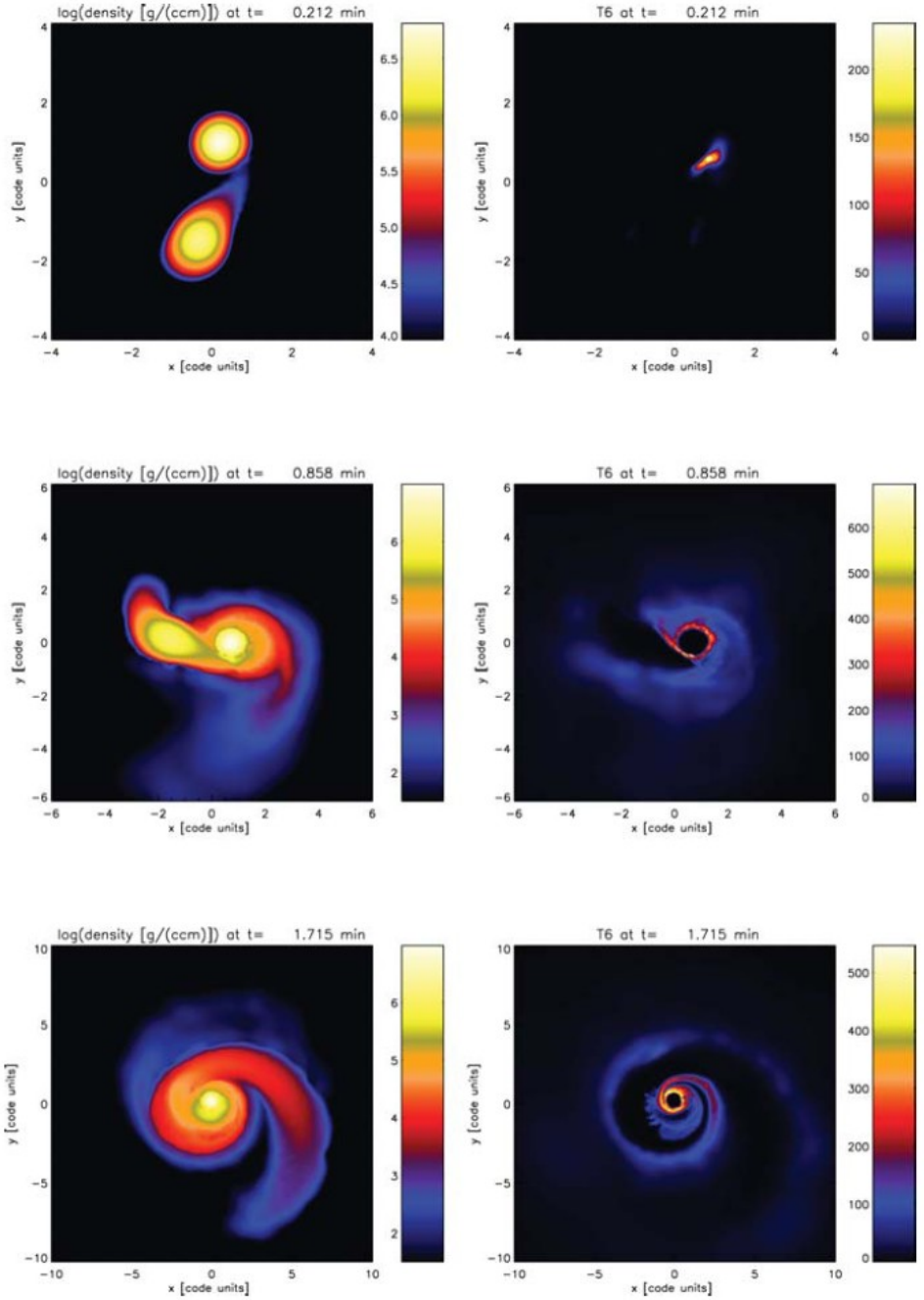
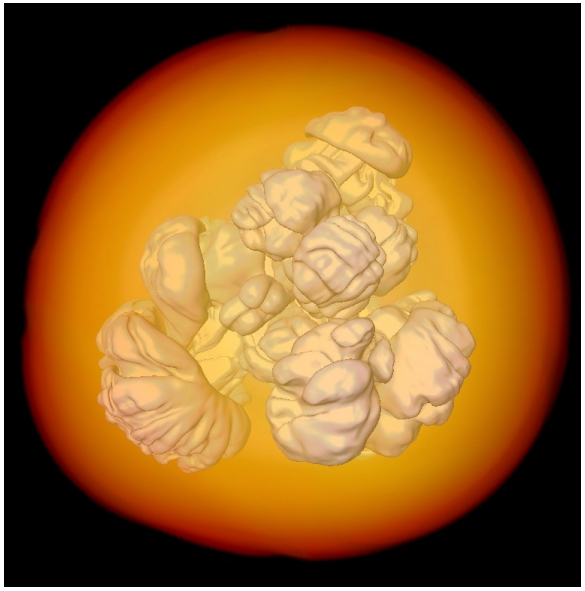


Figure 2. Dynamical evolution of the coalescence of a 0.6 $M_\odot$ + 0.9 $M_\odot$ CO white dwarf binary. The panels in the left-hand column show the density in the orbital plane, the panels in the right-hand column the temperature in units of 10^6 K. Lengths are in code units (= 10^9 cm).

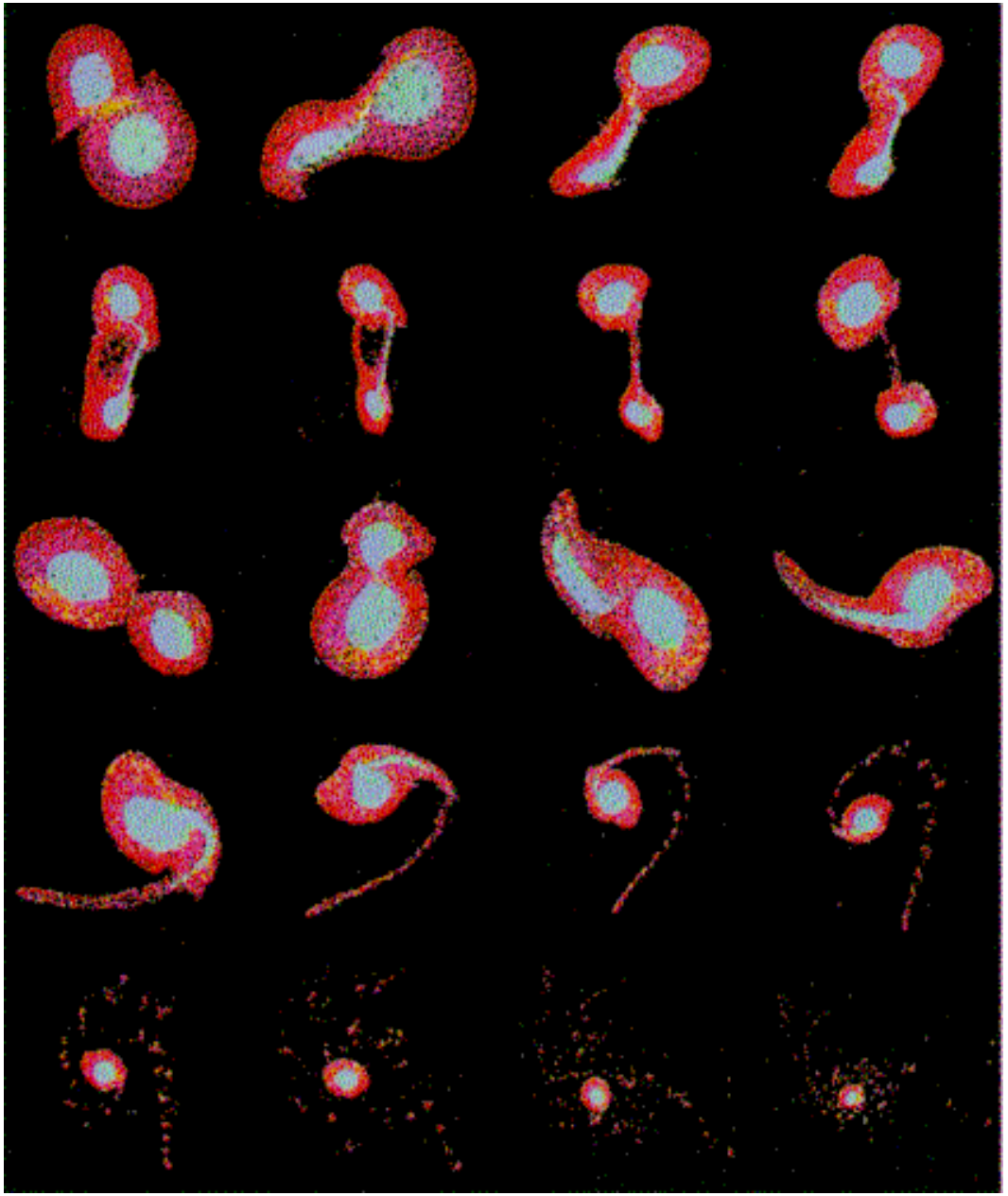
Is it a single white dwarf or merging white dwarfs?—model both and see which looks more like nature.



(Roepke and Hillebrandt 2005)

Computation on the Planetary Scale

Giant Impact hypothesis for the formation of the Moon (Alastair Cameron)



Computing A Star

- Suppose you wanted to write down an equation of motion for every atom/ion/electron in a star
  - How much information do we need to store?
    - Number of particles in a carbon white dwarf:

$$N \sim \frac{M_\odot}{12 m_p} = \frac{2 \times 10^{33}\ \mathrm{g}}{12 \cdot 1.67 \times 10^{-24}\ \mathrm{g}} \sim 10^{56}$$

That's just the nuclei

    - For each particle, we store position and velocity, and composition (at a minimum)
  - Each number you store on a computer requires 8 bytes of memory
    - Memory = (3 + 3 + 1) * 10^{56} * 8 bytes ~ 5 x 10^{48} GB
    - A typical computer has ~ 1 GB of memory, so we would need 5 x 10^{48} computers!

The Fluid Approximation

(ref: Shu, Ch. 1)

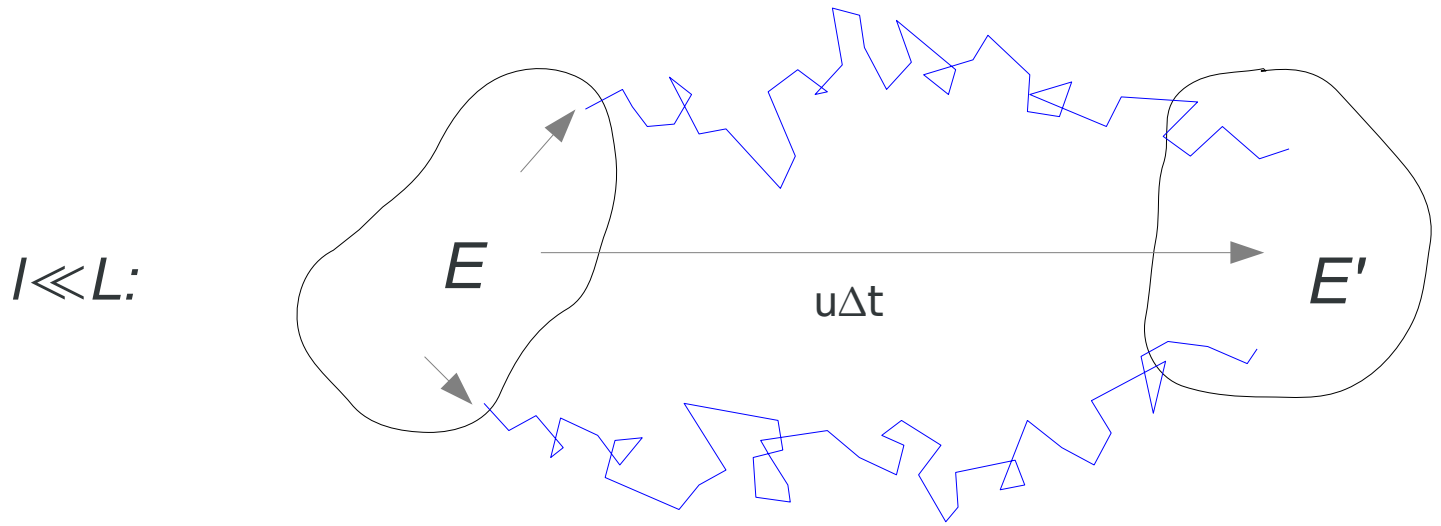
- Think of water coming out of a faucet and how it flows
  - We don't think about individual atoms, but rather some average (bulk) properties of the fluid
- Fluid approximation: average over the random atomic motions and focus on scales that are large compared to individual atoms but small compared to the star as a whole

(Roger McLassus; Wikipedia)

The Mean Free Path

(ref: Shu, Ch. 1)

- Main quantity to consider: mean free path
  - Distance a particle travels between interactions (collisions, ...)
  - Compare to the size of the system

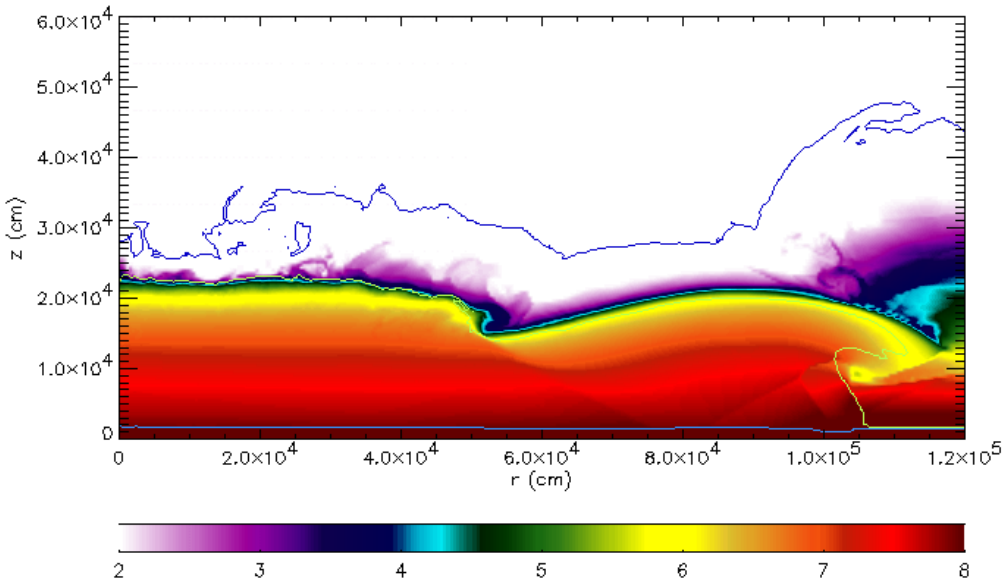


- On the scale we are interested in (cm on up), the random motions average out, and we can treat the star as a fluid.

The Fluid Approximation

ISS image of Cyclone Catarina

(NASA)



thermonuclear runaway on a neutron star (Zingale et al. 2001)

- Now we just need to worry about the bulk properties of the fluid—its density, velocity, pressure, etc.
  - These are quantities that we have some experience with in our everyday life (think weather forecast)

Approximations In Astrophysics

- Fluid
  - We'll focus on the fluid approximation
  - Mean free path is much smaller than the system we are modeling
  - Bulk fluid quantities provide a good description of the physics
- N-body
  - Popular with dark matter simulations
  - Models only the $1/r^2$ gravitational interaction between masses
  - No continuous description of the system

CFD

- ~~Colorful~~ Computational Fluid Dynamics
  - We model our system of interest as a fluid
  - Evolution dictated by conservation laws:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0 \qquad \text{Conservation of mass}$$

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u u + p)}{\partial x} = 0 \qquad \text{Conservation of momentum}$$

$$\frac{\partial (\rho E)}{\partial t} + \frac{\partial (\rho u E + u p)}{\partial x} = 0 \qquad \text{Conservation of energy}$$

Conservative System

- This can be written in conservation law form:

$$U_t + [F(U)]_x = 0$$

  - With

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} \qquad F(U) = \begin{pmatrix} \rho u \\ \rho u u + p \\ \rho u E + u p \end{pmatrix}$$

- To close this system, we need an equation of state
- Simplest: gamma-law

$$p = \rho e(\gamma - 1)$$

$$\rho e = \rho E - \frac{1}{2}\rho u^2$$

Conservative System

- To make it look more like advection, we write this in quasi-linear form.
  - Express flux vector in terms of $u_1 = \rho, u_2 = \rho\, u, u_3 = \rho\, E$
  - Compute the Jacobian:

$$A(U) = \frac{\partial F}{\partial U} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2}u^2(3-\gamma) & u(3-\gamma) & \gamma - 1 \\ \frac{1}{2}(\gamma-2)u^3 - \frac{uc^2}{\gamma-1} & \frac{3-2\gamma}{2}u^2 + \frac{c^2}{\gamma-1} & u\gamma \end{pmatrix}$$

    - Here sound speed is: $c = \sqrt{\gamma p/\rho}$
  - System becomes:

$$U_t + A(U)U_x = 0$$

  - This matrix is complicated and hard to work with

Primitive Variable Formulation

- We can instead cast things in terms of the primitive variables: density, velocity, and pressure
- Our system becomes (blackboard):

$$q_t + A(q) q_x = 0$$

$$q = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix} \qquad A(q) = \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u \end{pmatrix}$$

  - Notice that the Jacobian for this formulation is much simpler

Eigensystem

- Recall that a system is hyperbolic if the eigenvalues are real and finite.
- For our system, the eigenvalues are:

$$\lambda^{(-)} = u - c, \lambda^{(\circ)} = u, \lambda^{(+)} = u + c$$

  - These are the speeds at which information propagates in our system
  - Three distinct wave speeds for 3 equations
  - We'd get the same eigenvalues from the Jacobian of the conserved system
- There is a rich mathematical description of the theory of hyperbolic systems of conservation laws. The book by LeVeque is an excellent introduction.

Eigensystem

- We can also find the eigenvectors:

$$A\, r^{(\nu)} = \lambda^{(\nu)} r^{(\nu)} \; ; \qquad l^{(\nu)}\, A = \lambda^{(\nu)} l^{(\nu)}$$

$$r^{(-)} = \begin{pmatrix} 1 \\ -c/\rho \\ c^2 \end{pmatrix} \qquad r^{(\circ)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad r^{(+)} = \begin{pmatrix} 1 \\ c/\rho \\ c^2 \end{pmatrix}$$

$$l^{(-)} = \begin{pmatrix} 0 & -\frac{\rho}{2c} & \frac{1}{2c^2} \end{pmatrix}$$

$$l^{(\circ)} = \begin{pmatrix} 1 & 0 & -\frac{1}{c^2} \end{pmatrix}$$

$$l^{(+)} = \begin{pmatrix} 0 & \frac{\rho}{2c} & \frac{1}{2c^2} \end{pmatrix}$$

  - These are normalized such that $l^{(i)} \cdot r^{(j)} = \delta_{ij}$

Characteristic Variables

- A final form of the system is in terms of the characteristic variables
  - Construct matrices of the left and right eigenvectors

$$R = (r^{(-)}|r^{(\circ)}|r^{(+)})$$

$$L = \begin{pmatrix} l^{(-)} \\ - \; \overline{l}^{(\circ)} \; - \\ - \; \overline{l}^{(+)} \; - \end{pmatrix}$$

    - Satisfy: $LR = RL = I$
  - Define $dw = L\, dq$
  - Our system can be written as:

$$w_t + \Lambda w_x = 0$$

Characteristic Variables

- Here, w are the characteristic variables
- The three equations are decoupled:

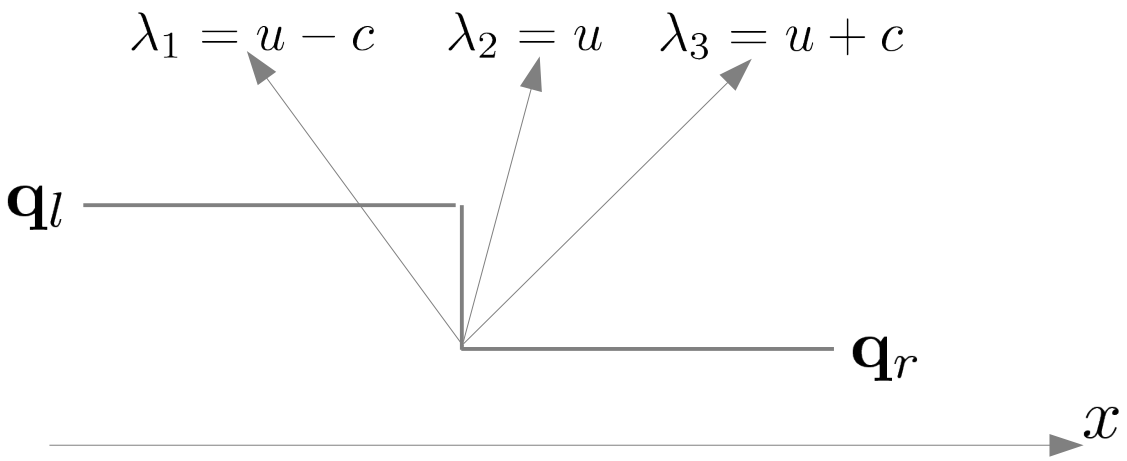
$$\Lambda = LAR = \begin{pmatrix} \lambda^{(-)} & & \\ & \lambda^{(\circ)} & \\ & & \lambda^{(+)} \end{pmatrix}$$

  - If our system were linear, we'd be done:
    - Transform into characteristic variables
    - Each characteristic variable advects at a given wave speed, without interacting with one-another
    - Solve and then transform back to primitive form (or conserved form)
  - We're non-linear: the wave-speeds and eigenvectors change with the solution

Jumps Across Waves

- The characteristic system is telling us something interesting already.
  - Consider an initial discontinuity in the primitive variables
  - Each wave will carry a jump in their associate characteristic quantity away from the discontinuity at their speed
  - The corresponding jump in the primitive variable is just $dq = L^{-1} dw = R\, dw$

Jumps Across Waves



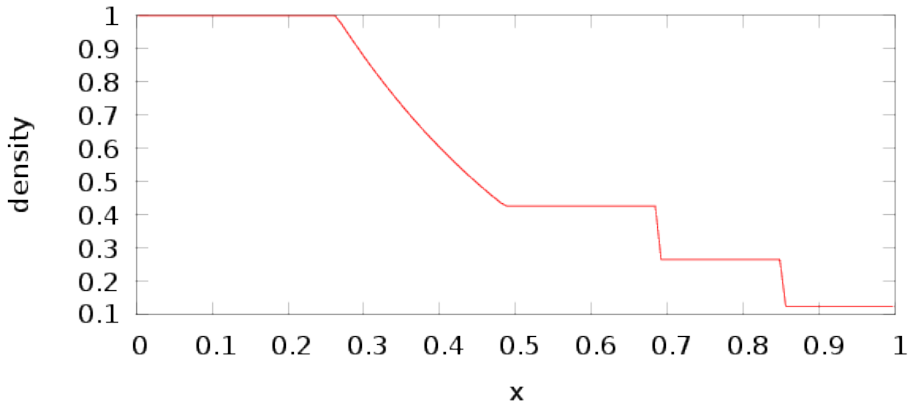
$$r^{(-)} = \begin{pmatrix} 1 \\ -c/\rho \\ c^2 \end{pmatrix} \qquad r^{(\circ)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad r^{(+)} = \begin{pmatrix} 1 \\ c/\rho \\ c^2 \end{pmatrix}$$

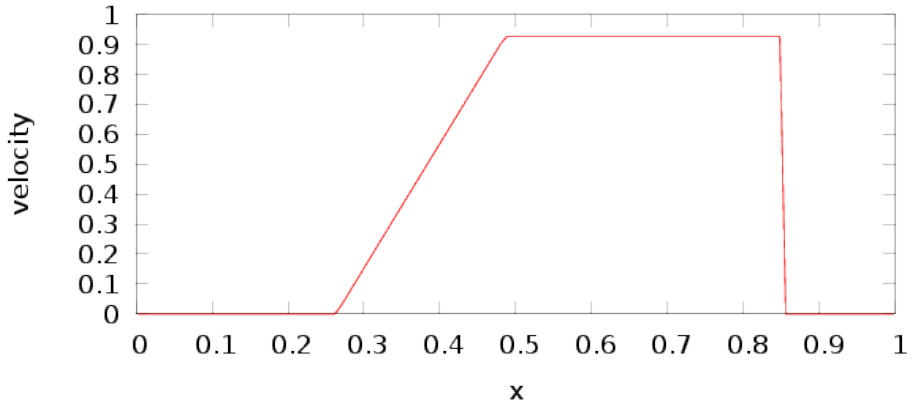
All primitive quantities jump across the u-c wave

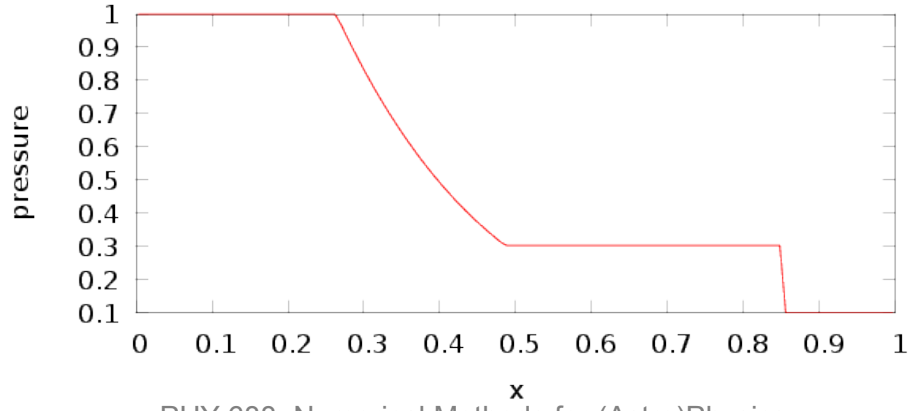
Only density jumps across the middle (u) wave

All primitive varaibles jump across the u+c wave

Jumps Across Waves







Solution Methodology

- This system has similarities to the advection equation we already studied.
  - We can take a similar approach
- We will use the finite-volume discretization:

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U_t = -\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial}{\partial x} F(U) dx$$

$$\frac{\partial}{\partial t} U_i = -\frac{1}{\Delta x}\left\{[F(U)]_{i+1/2} - [F(U)]_{i-1/2}\right\}$$

  - Second-order in time requires time-centering the righthand side

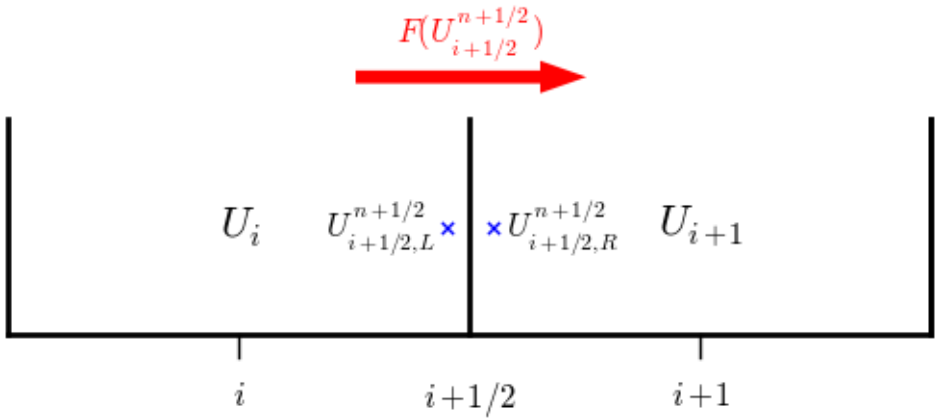
$$U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x}(F_{i-1/2}^{n+1/2} - F_{i+1/2}^{n+1/2})$$

Solution Methodology

- As we saw in our homework (Burger's eq.), we need to solve this in conservative form to get the shock speed correct
- Basic idea: evaluate fluxes at interfaces by predicting the fluid state there, then solve Riemann problem:

$$F_{i+1/2}^{n+1/2} = F(U_{i+1/2}^{n+1/2})$$

$$U_{i+1/2}^{n+1/2} = \mathcal{R}(U_{i+1/2,L}^{n+1/2}, U_{i+1/2,R}^{n+1/2})$$



Interface State Prediction

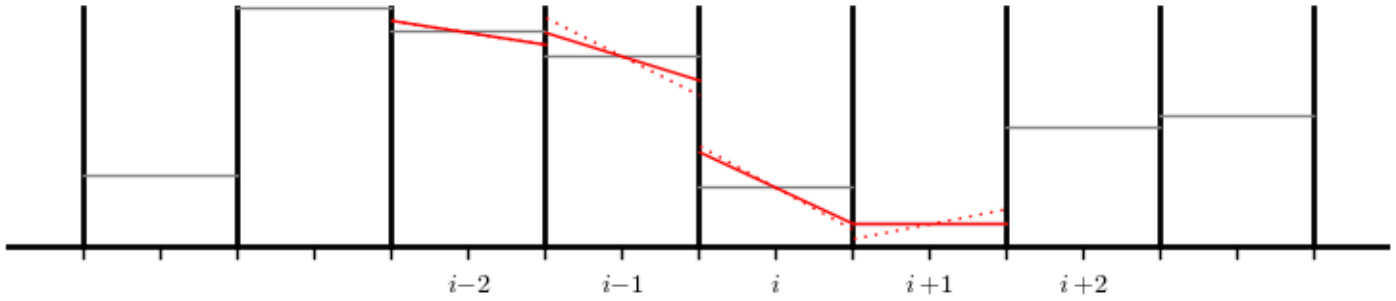
- The simplest method to construct the interface states is to simply use the cell averages

$$U_{i+1/2,L} = U_i\,; \qquad U_{i+1/2,R} = U_{i+1}$$

  - This is first order accurate in space and time
  - This method is called Godunov's method
  - Doesn't consider how U is changing in space or over the timestep

Piecewise Linear Reconstruction

- We can get second-order accuracy by reconstructing the cell-average data to be piecewise linear



  - It is easier to work with the primitive variables
  - We reconstruct each quantity and limit the slopes (just like we did with advection)
  - Predict edge-centered, time-centered state via Taylor expansion (blackboard...)

Piecewise Linear Reconstruction

- We have:

$$q^{n+1/2}_{i+1/2,L} = q^n_i + \frac{1}{2}\left[1 - \frac{\Delta t}{\Delta x} A\right] \overline{\Delta q}$$

  - This is the amount of the primitive variable that reaches the interface over the timestep
  - Recall that the jumps in q are carried by 3 waves
  - We only want to include a jump if that wave was moving toward the interface—we need to do a characteristic decomposition

$$q^{n+1/2}_{i+1/2,L} = q^n_i + \frac{1}{2}\left[RL - \frac{\Delta t}{\Delta x} R\Lambda L\right] \overline{\Delta q}$$

(blackboard derivation...)

$$= q^n_i + \frac{1}{2}\sum_\nu \left[1 - \frac{\Delta t}{\Delta x}\lambda^{(\nu)}\right] (l^{(\nu)} \cdot \overline{\Delta q}) r^{(\nu)}$$

Piecewise Linear Reconstruction

- Notice the quantity: $\Delta t\, \lambda \,/\, \Delta x$ —this is the CFL number for each wave
- To consider only the waves moving toward the interface, we alter the sum slightly:

$$q^{n+1/2}_{i+1/2,L} = q^n_i + \frac{1}{2} \sum_{\nu;\lambda^{(\nu)} \geq 0} \left[1 - \frac{\Delta t}{\Delta x} \lambda^{(\nu)} \right] (l^{(\nu)} \cdot \overline{\Delta q}) r^{(\nu)}$$

  - The right state at this interface is similarly constructed:

$$q^{n+1/2}_{i+1/2,R} = q^n_{i+1} - \frac{1}{2} \sum_{\nu;\lambda^{(\nu)} \leq 0} \left[1 + \frac{\Delta t}{\Delta x} \lambda^{(\nu)}_{i+1} \right] (l^{(\nu)}_{i+1} \cdot \overline{\Delta q}_{i+1}) r^{(\nu)}_{i+1}$$

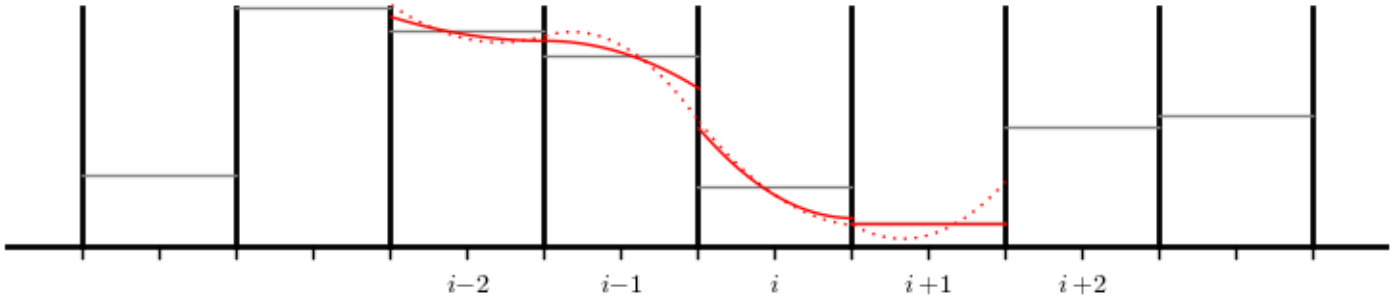
Piecewise Linear Reconstruction

- The decomposition of the jumps into the sum over the left and right eigenvectors is sometimes called a characteristic projection
  - Note that we explictly see here that each wave carries a jump proportional to the right eigenvector
  - $(l \cdot \Delta q)$ is the projection of the primitive variable jump into the characteristic variables
- Many sources introduce a reference state chosen so as to minimize the effect of the characteristic decomposition—this is because we are linearizing a nonlinear system

$$q^{n+1/2}_{i+1/2,L} = \tilde{q}_+ - \sum_{\nu;\lambda^{(\nu)}\geq 0} l_i^{(\nu)} \cdot \left\{ \tilde{q}_+ - \left[q_i^n + \frac{1}{2}\left(1 - \frac{\Delta t}{\Delta x}\lambda_i^{(\nu)}\right)\overline{\Delta q}_i \right] \right\} r_i^{(\nu)}$$

Piecewise Parabolic

- The most popular method in astrophysics is PPM (piecewise parabolic method)
  - Reconstruct the data in each cell as a parabola



  - The parabolic profiles are limited to prevent generation of new extrema (see Colella & Woodward 1984 for a description of the original method)
  - General form:

$$q_i(x) = q_{-,i} + \xi(x)(\Delta q_i + q_{6,i}(1 - \xi(x))) \qquad \xi(x) = \frac{x - x_{i-1/2}}{\Delta x}$$

Piecewise Parabolic

- The reconstruction polynomial recovers the cell-average:

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x)dx = q_i$$

- To use the parabolic interpolant, we start with our interface state for piecewise linear:

$$q_{i+1/2,L}^{n+1/2} = \tilde{q}_+ - \sum_{\nu;\lambda^{(\nu)}\geq 0} l_i^{(\nu)} \cdot \left\{ \tilde{q}_+ - \left[q_i^n + \frac{1}{2}\left(1 - \frac{\Delta t}{\Delta x}\lambda_i^{(\nu)}\right)\overline{\Delta q}_i \right] \right\} r_i^{(\nu)}$$

  - Recognize that for a linear profile:

$$q_i^n + \frac{1}{2}\left(1 - \frac{\Delta t}{\Delta x}\lambda_i^{(\nu)}\right)\overline{\Delta q}_i \approx \frac{1}{\lambda\Delta t}\int_{x_{i+1/2}-\lambda\Delta t}^{x_{i+1/2}} q(x)dx$$

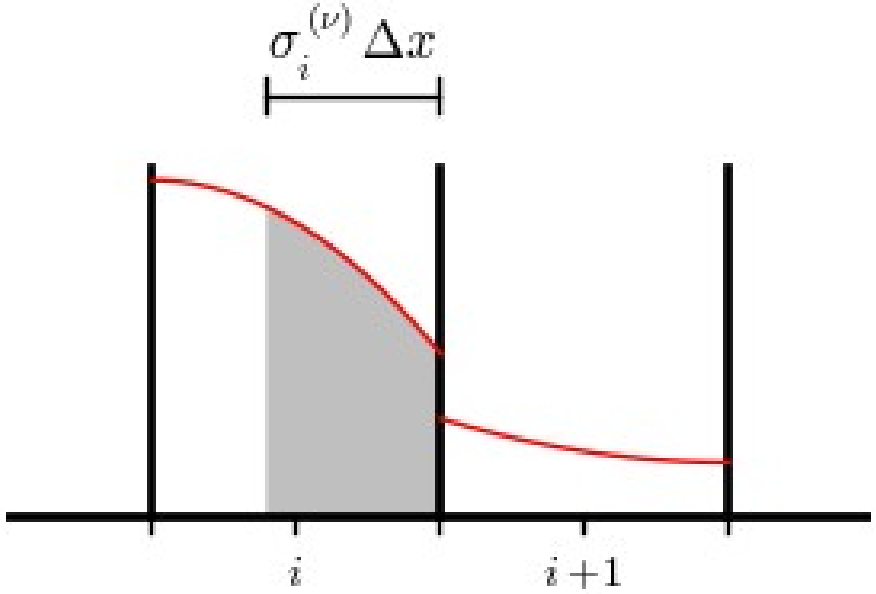
Piecewise Parabolic

- This suggests we can insert the average under our parabolic profile to construct the interface state:

$$\mathcal{I}_+^{(\nu)}(q_i) = \frac{1}{\sigma^{(\nu)}\Delta x}\int_{x_{i+1/2}-\sigma^{(\nu)}\Delta x}^{x_{i+1/2}} q(x)dx$$

  - Where

$$\sigma^{(\nu)} = |\lambda^{(\nu)}|\Delta t/\Delta x$$



Piecewise Parabolic

- The PPM interface states are then:

$$q^{n+1/2}_{i+1/2,L} = \tilde{q}_+ - \sum_{\nu;\lambda^{(\nu)}\geq 0} l_i^{(\nu)} \cdot \left(\tilde{q}_+ - \mathcal{I}_+^{(\nu)}(q_i)\right) r_i^{(\nu)}$$

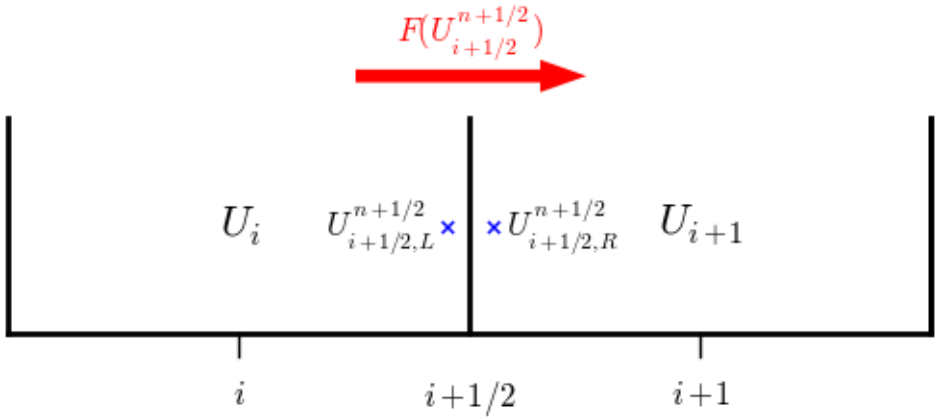
$$q^{n+1/2}_{i-1/2,R} = \tilde{q}_- - \sum_{\nu;\lambda_\nu\leq 0} l_i^{(\nu)} \cdot \left(\tilde{q}_- - \mathcal{I}_-^{(\nu)}(q_i)\right) r_i^{(\nu)}$$

  - Where

$$\mathcal{I}_-^{(\nu)}(q) = \frac{1}{\sigma^{(\nu)}\Delta x}\int_{x_{i-1/2}}^{x_{i-1/2}+\sigma^{(\nu)}\Delta x} q(x)dx$$

Riemann Problem

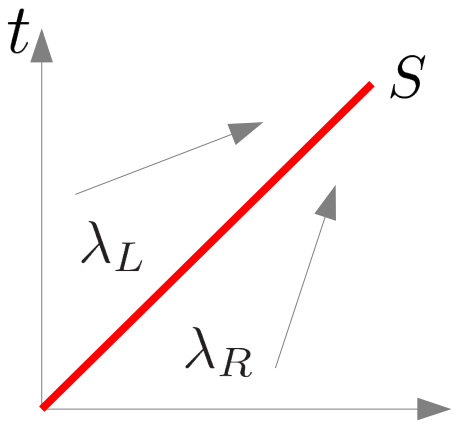
- No matter the method used to predict the interface states, we now have left and right states at each interface



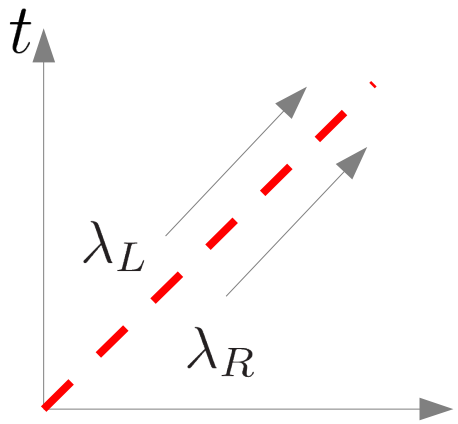
- Unlike the linear advection or Burger's equation, we rarely solve the Riemann problem for the Euler equations exactly.
  - We need to consider what is carried by each wave
  - Different types of waves are present depending on the behavior of the characteristics

Riemann Problem

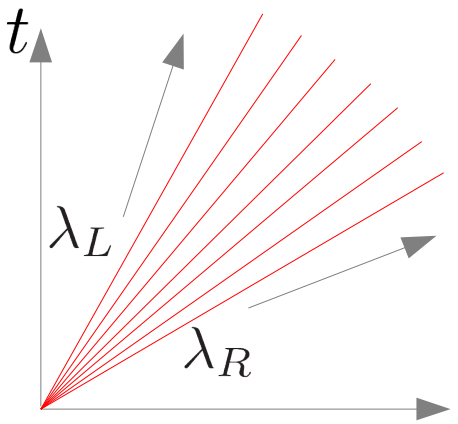
- Wave type is determined by whether the characteristics converge, are parallel, or diverge



Shock wave



Contact Discontinuity



Rarefaction

  - We already saw shocks and rarefactions with Burger's equation
  - A contact discontinuity is where the solution jumps, but there is no compression or expansion

Riemann Problem

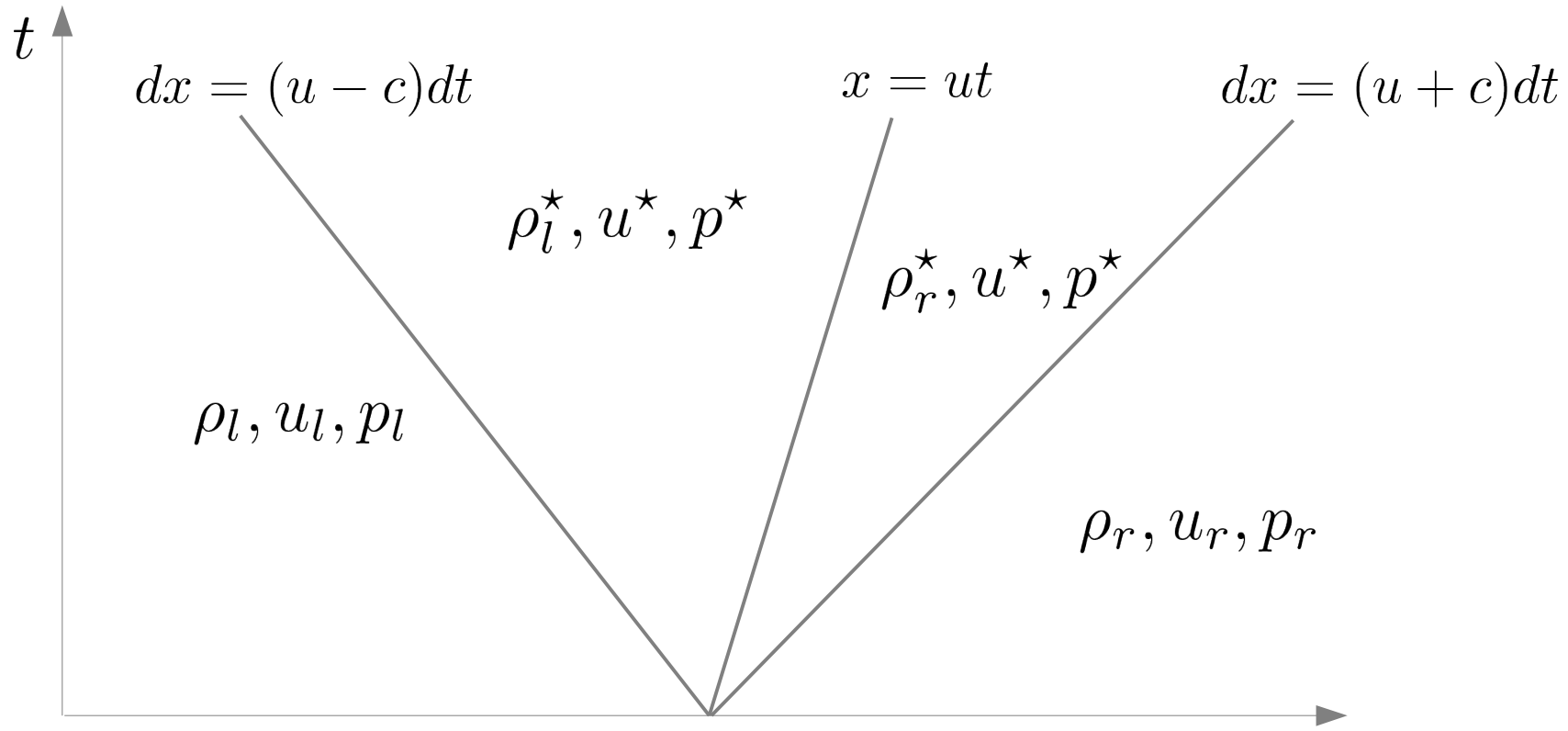
- Across the middle wave ($\lambda = u$), only the density jumps

$$r^{(-)} = \begin{pmatrix} 1 \\ -c/\rho \\ c^2 \end{pmatrix} \qquad r^{(\circ)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad r^{(+)} = \begin{pmatrix} 1 \\ c/\rho \\ c^2 \end{pmatrix}$$

  - So the velocity is the same on either side—no convergence or divergence.
  - The middle wave is always a contact discontinuity
- The left and right waves can be either a shock or rarefaction

Riemann Problem

- The Riemann problem for the Euler equations looks like:



  - Solving the Riemann problem means finding the 4 quantities: $\rho_l^\star, \rho_r^\star, u^\star, p^\star$

Riemann Problem

- If the flow is compressing, then we have a shock
  - Rankine-Hugoniot jump conditions provide the solution
- If we are not compressing, then we are a rarefaction
  - It can be shown that entropy is constant across the left and right waves (consider the system of ρ, u, s and look at the right eigenvectors)
  - We can integrate the characteristic variables along these waves and determine what must be constant
  - We find:

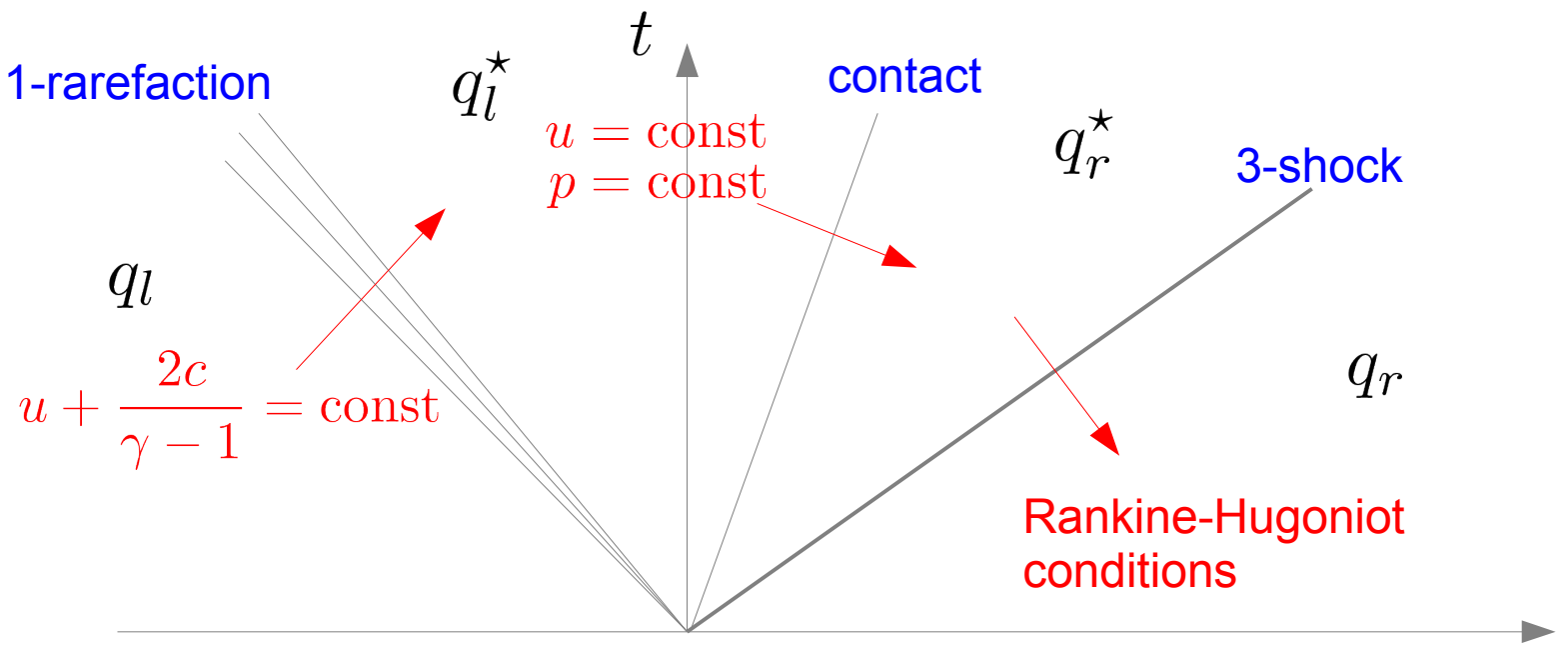
$$u + 2c/(\gamma - 1) = \text{const} \qquad \text{Across left rarefaction}$$

$$u - 2c/(\gamma - 1) = \text{const} \qquad \text{Across right rarefaction}$$

(see E.g. LeVeque Eq. 14.48)

Wave Structure

- The characteristic structure becomes:

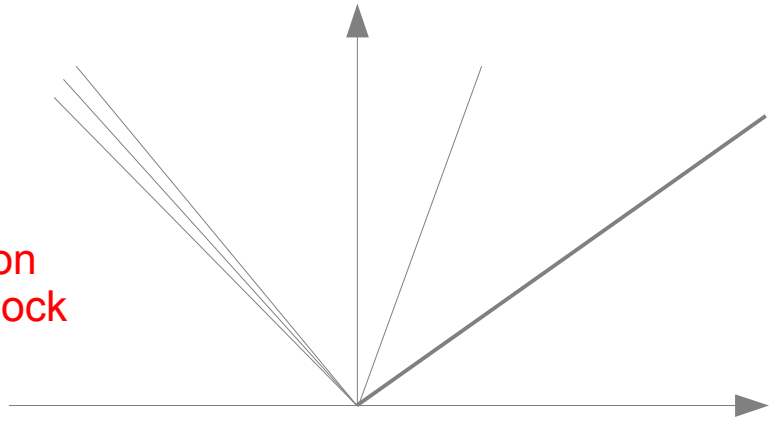


Using the correct relation across each wave allows us to link the 4-states and solve the Riemann problem.

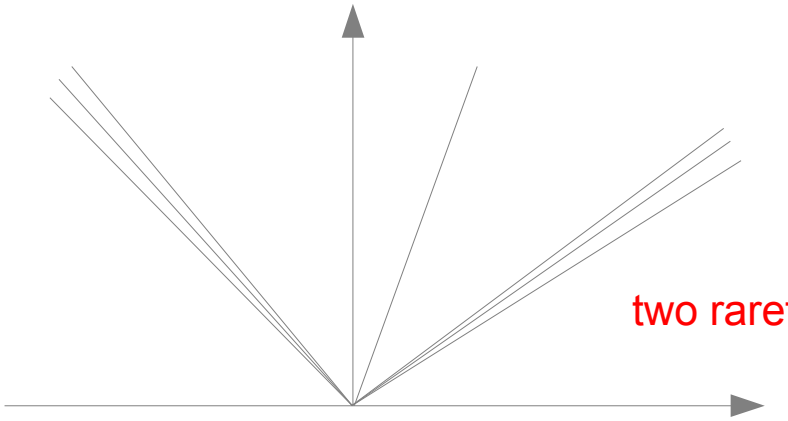
Wave Structure

- There are several different wave configurations

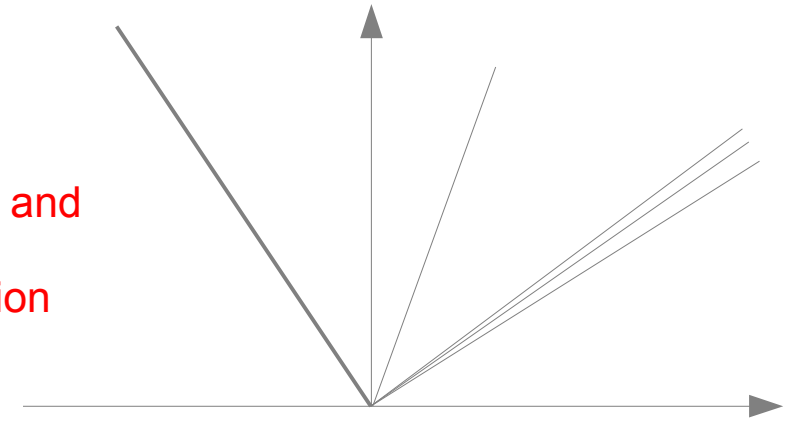
1-rarefaction and 3-shock



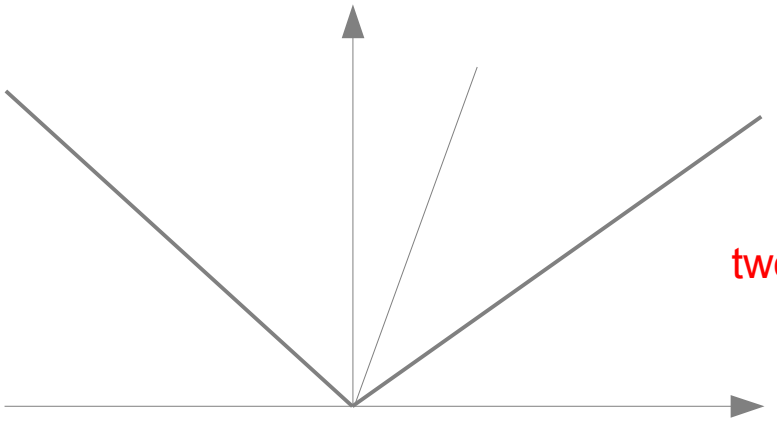
two rarefactions



1-shock and 3-rarefaction

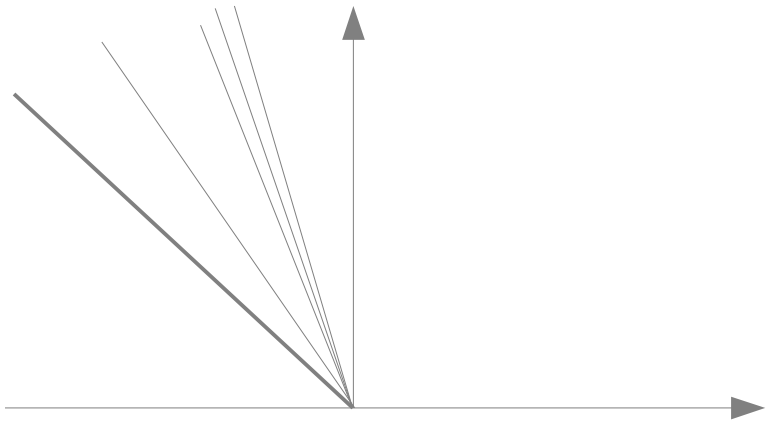


two shocks

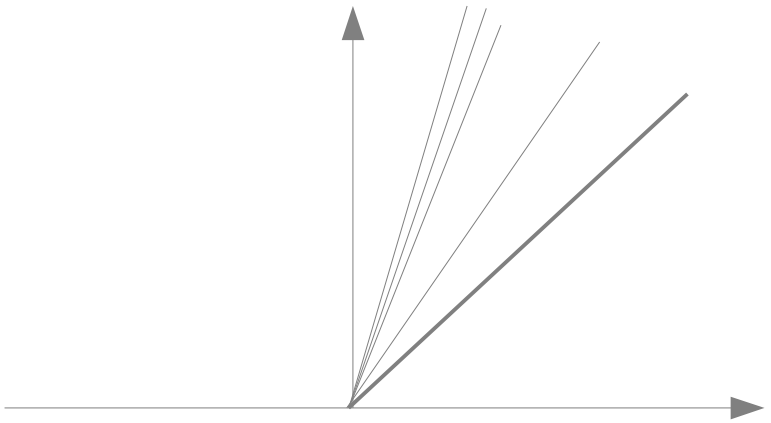


Wave Structure

- And the can span the initial interface or all be on one side



supersonic flow to the left



supersonic flow to the right

Riemann Solution

- It can be shown that the solution to the Euler equation Riemann problem appears as:

$$u^\star \equiv \phi_l(p^\star) = u_l + \begin{cases} \frac{2c_l}{\sqrt{2\gamma(\gamma-1)}} \frac{1-p^\star/p_l}{\sqrt{1+\frac{\gamma+1}{\gamma-1}\frac{p^\star}{p_l}}} & p^\star > p_l \\ \frac{2c_l}{\gamma-1}\left[1-(p^\star/p_l)^{(\gamma-1)/(2\gamma)}\right] & p^\star \leq p_l \end{cases}$$

(LeVeque Eq. 14.51, 14.52)

$$u^\star \equiv \phi_r(p^\star) = u_r - \begin{cases} \frac{2c_r}{\sqrt{2\gamma(\gamma-1)}} \frac{1-p^\star/p_r}{\sqrt{1+\frac{\gamma+1}{\gamma-1}\frac{p^\star}{p_r}}} & p^\star > p_r \\ \frac{2c_r}{\gamma-1}\left[1-(p^\star/p_r)^{(\gamma-1)/(2\gamma)}\right] & p^\star \leq p_r \end{cases}$$

(LeVeque Eq. 14.54, 14.55)

  - The constancy of the pressure and velocity across the contact gives:

$$\phi_l(p^\star) = \phi_r(p^\star)$$

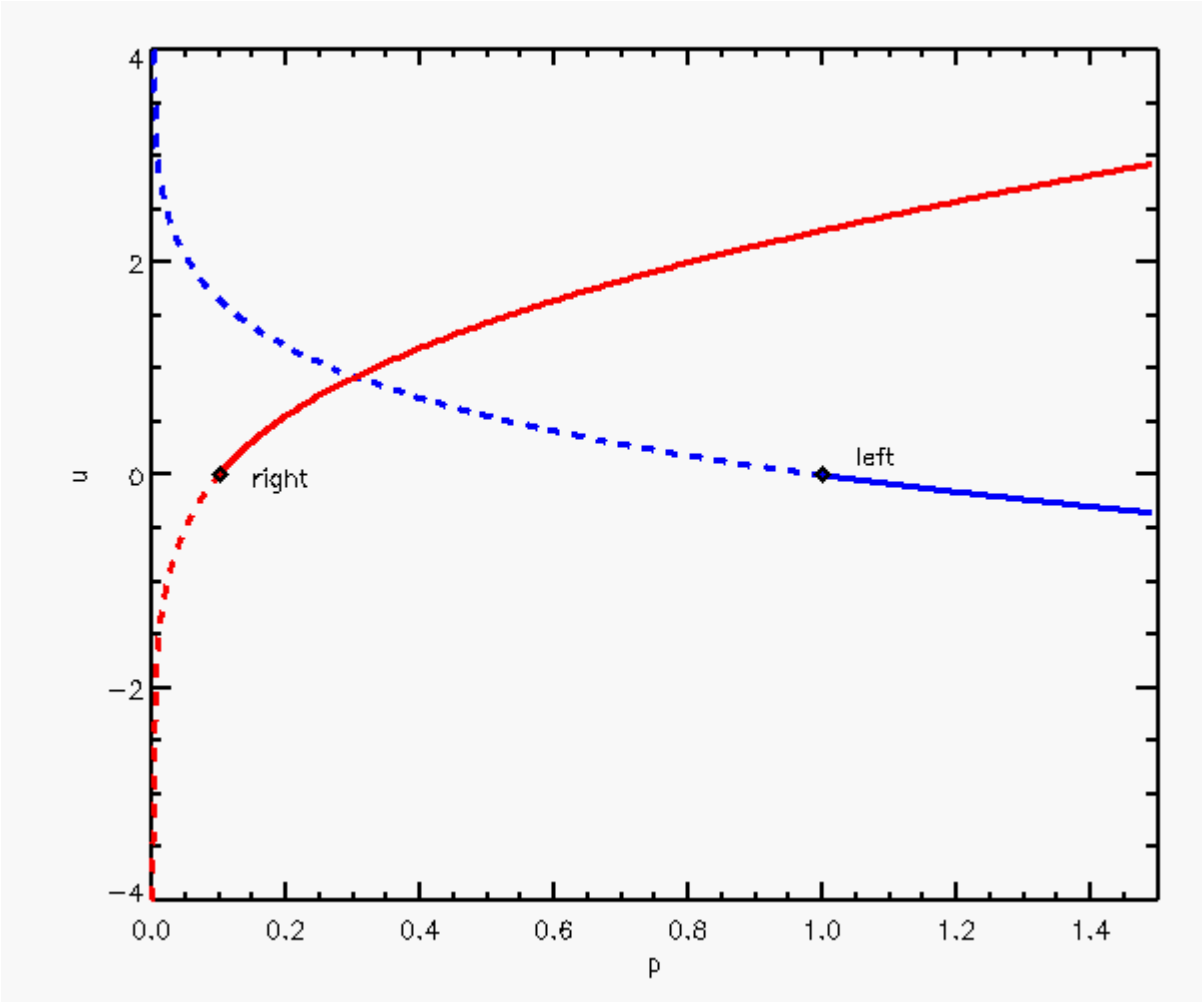
Riemann Solution

- We can look at the solution to the Riemann problem graphically
- Sod problem:

$$\rho_l = 1, u_l = 0, p_l = 1$$

$$\rho_l = 1/8, u_l = 0, p_l = 1/10$$

  - Plot shows the curves that each state can reach through a shock (solid) or rarefaction (dashed)
  - Solution is the intersection
  - Here we must have a left rarefaction and a right shock



Riemann Solution

- We only need to know the Riemann solution on our interface
- Once we get the state on the interface, we can construct the fluxes through the interface:

$$F^{n+1/2}_{i+1/2} = \begin{pmatrix} \rho^{n+1/2}_{i+1/2} u^{n+1/2}_{i+1/2} \\ \rho^{n+1/2}_{i+1/2} (u^{n+1/2}_{i+1/2})^2 + p^{n+1/2}_{i+1/2} \\ u^{n+1/2}_{i+1/2} p^{n+1/2}_{i+1/2} / (\gamma - 1) + \frac{1}{2} \rho^{n+1/2}_{i+1/2} (u^{n+1/2}_{i+1/2})^3 + u^{n+1/2}_{i+1/2} p^{n+1/2}_{i+1/2} \end{pmatrix}$$