

# Linear Advection Equation

- The linear advection equation provides a simple problem to explore methods for hyperbolic problems

$$a_t + ua_x = 0$$

- Here,  $u$  represents the speed at which information propagates
- First order, linear PDE
  - We'll see later that many hyperbolic systems can be written in a form that looks similar to advection, so what we learn here will apply later.

# Linear Advection Equation

- We need initial conditions

$$a(x, 0) = a_0(x)$$

- and boundary conditions

$$a(0, t) = a_l(t)$$

$$a(L, t) = a_r(t)$$

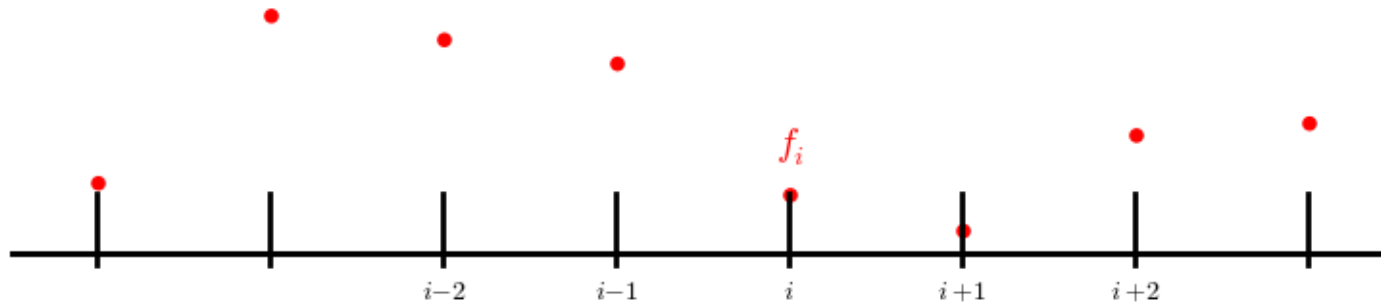
- We'll see in a moment that we only really need 1 boundary condition, since this is a first-order equation

# Linear Advection Equation

- Solution is trivial—any initial configuration simply shifts to the right (for  $u > 0$ )
  - e.g.  $a(x - ut)$  is a solution
  - This demonstrates that the solution is constant on lines  $x = ut$  — these are called the characteristics
- This makes the advection problem an ideal test case
  - Evolve in a periodic domain
  - Compare original profile with evolved profile after 1 period
  - Differences are your numerical error

# Finite-Difference Approximation

- We store the function value at each point in our grid:



- We will use the notation:  $a_i^n$ 
  - Superscripts = time discretization; subscripts = spatial discretization
  - We'll use 0-based indexing
- Simple discretization:

$$\frac{a_i^{n+1} - a_i^n}{\Delta t} = -u \frac{a_{i+1}^n - a_{i-1}^n}{2\Delta x}$$

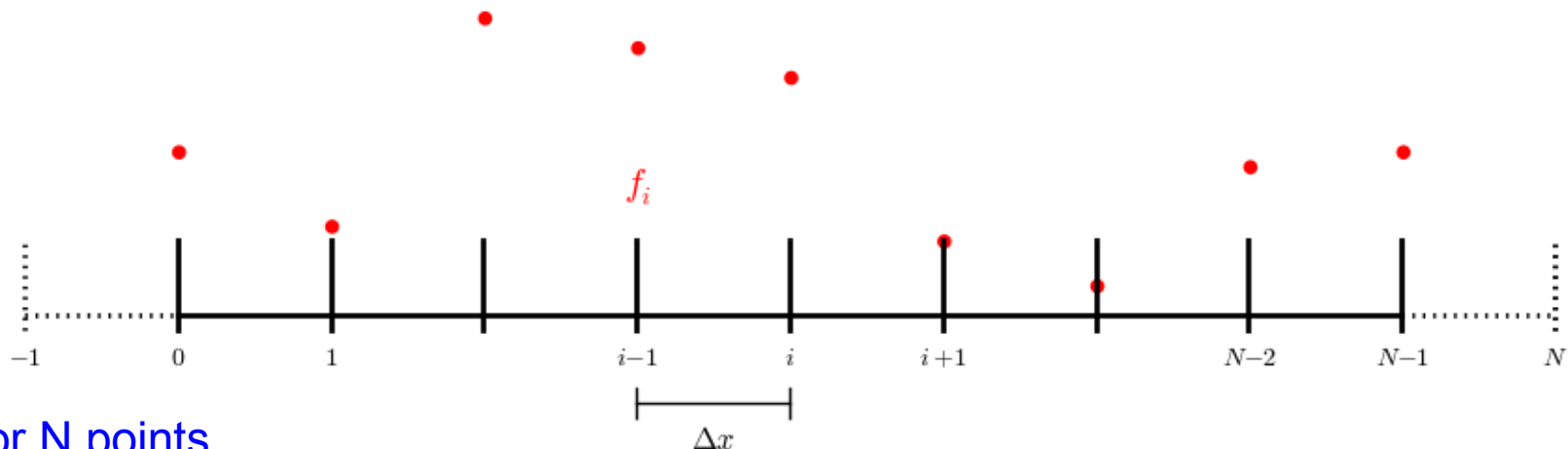
- Explicit
- 2<sup>nd</sup> order in space, 1<sup>st</sup> order in time (FTCS method)

# Boundary Conditions

- We want to be able to apply the same update equation to all the grid points:

$$a_i^{n+1} = a_i^n - \frac{C}{2}(a_{i+1}^n - a_{i-1}^n)$$

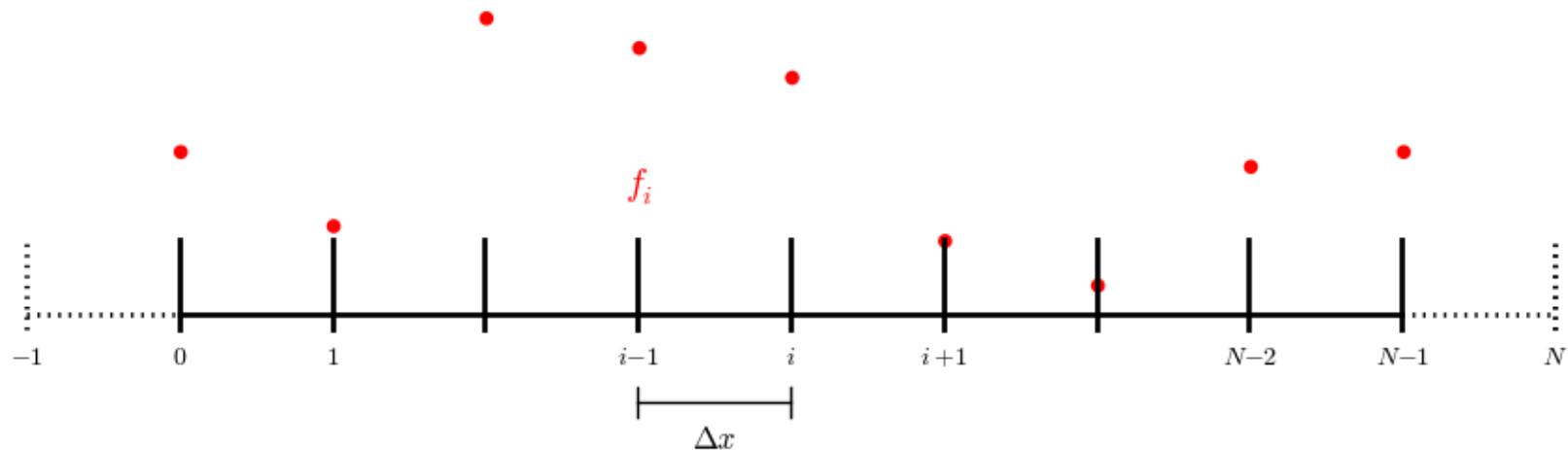
- Here,  $C = u\Delta t / \Delta x$  is the fraction of a zone we cross per timestep—this is called the **Courant-Friedrichs-Lewy number** (or **CFL number**)
- Notice that if we attempt to update zone  $i=0$  we “fall off” the grid
- Solution: ghost points (for finite-volume, we'll call them ghost cells)



Grid for N points

# Boundary Conditions

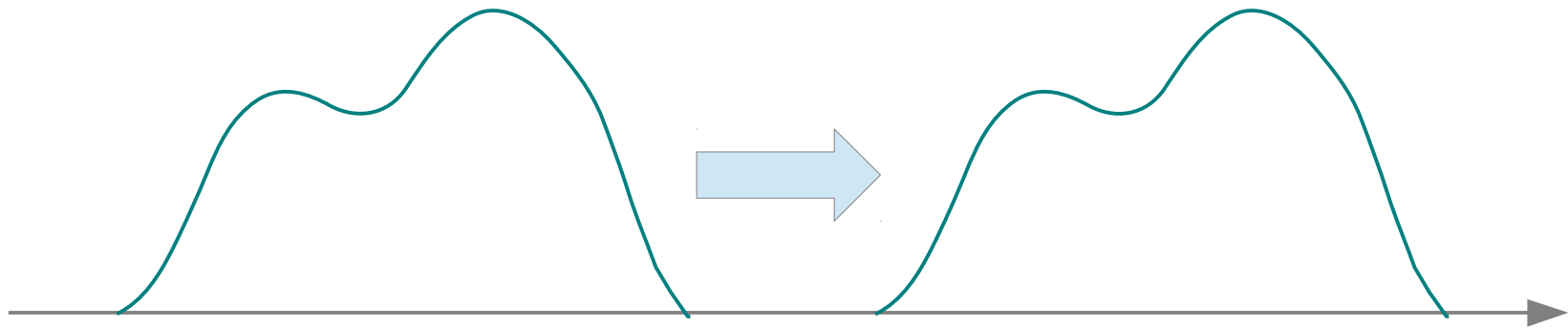
- Before each timestep, we will fill the ghost points with data that represents the boundary conditions



- Note that with this discretization, we have a point exactly on each boundary (we only really need to update one of them)
- Periodic BCs would mean:  $a_0^n = a_{N-1}^n$  ← General grid of N points
  - $a_N^n = a_1^n$ ;  $a_{-1}^n = a_{N-2}^n$
- Other common BCs are outflow (zero derivative at boundary)

# Implementation and Testing

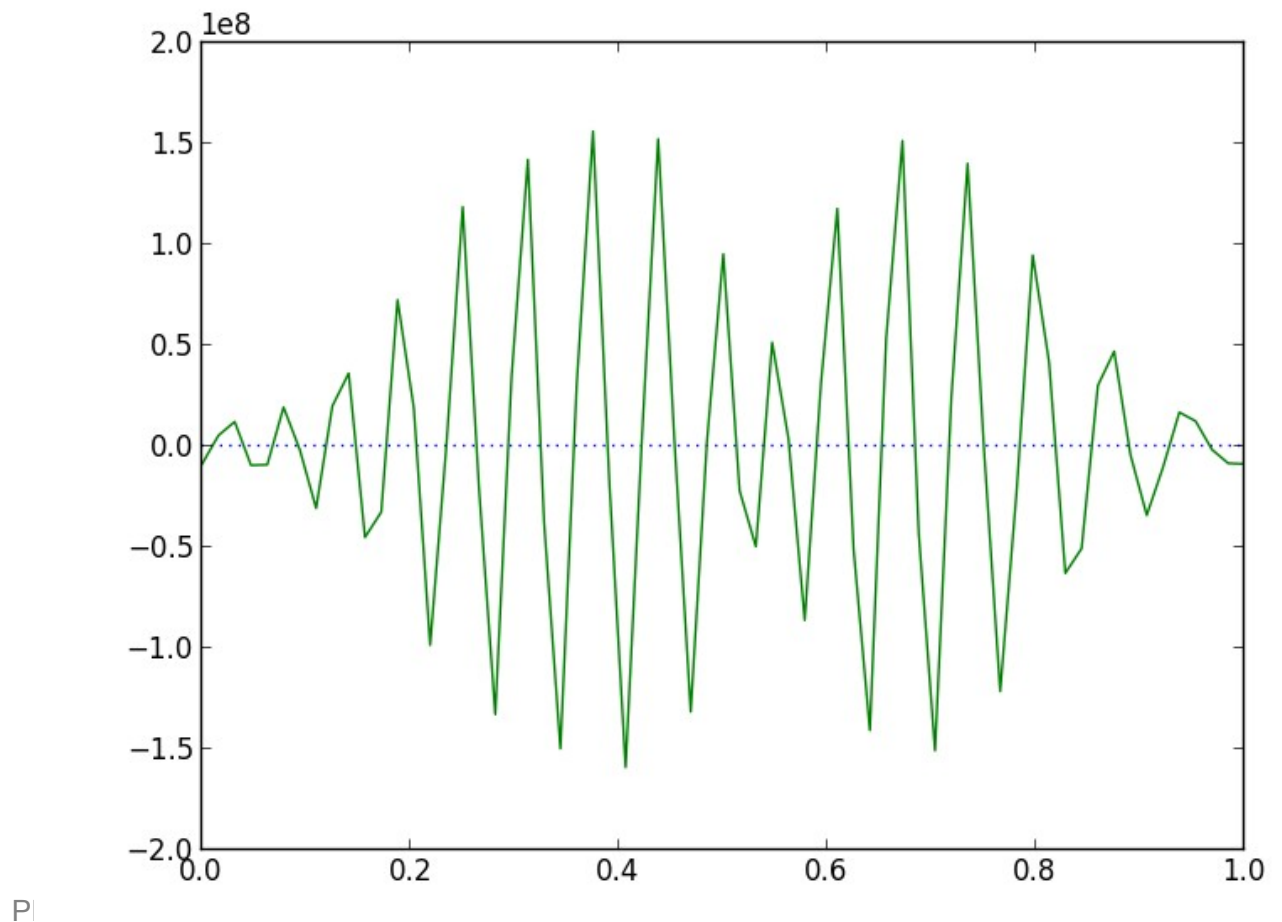
- Recall that the solution is to just propagate any initial shape to the right:



- We'll code this up with periodic BCs and compare after 1 period
  - On a domain  $[0,1]$ , one period is simply:  $1/u$
- We'll use a tophat initial profile
- Let's look at the code...

# FTCS Linear Advection

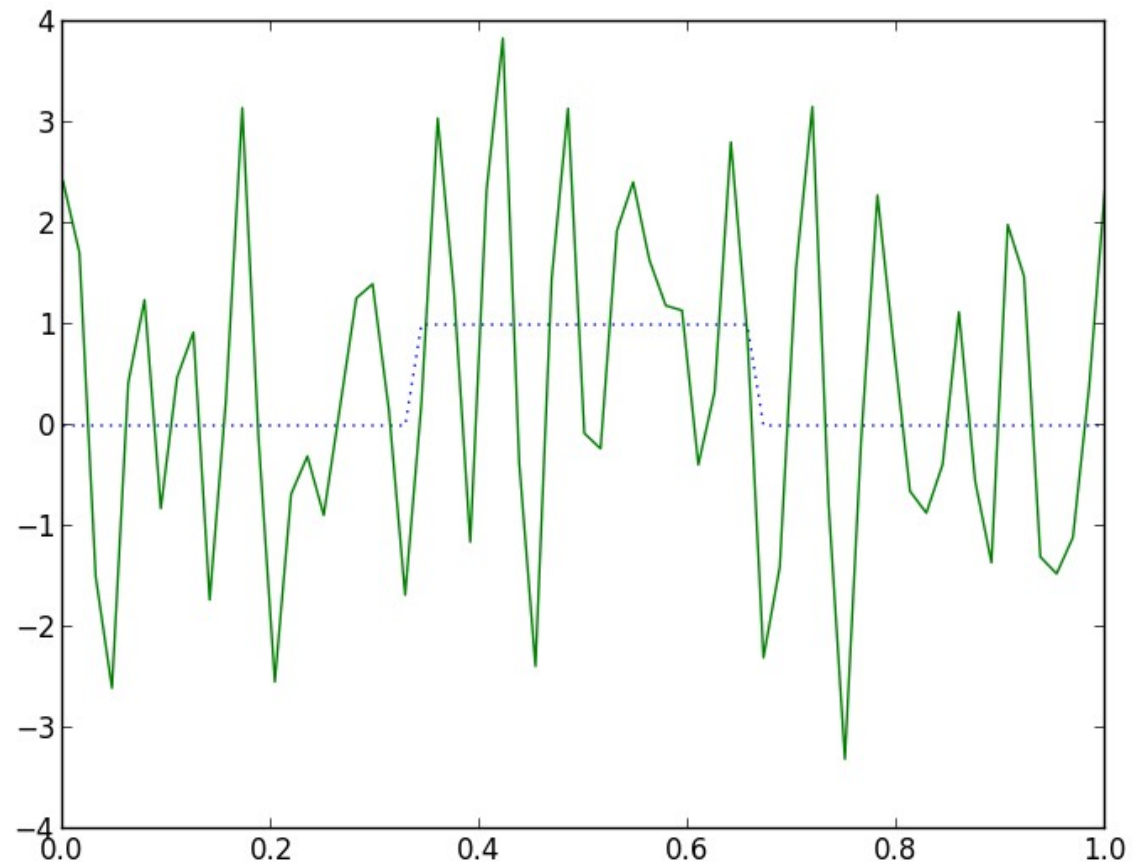
- Evolution with  $N=65$ ,  $C=0.9$ , after 1 period
  - Note the vertical scale!!!!
  - We see nothing that looks like a tophat
  - Any ideas?





# FTCS Linear Advection

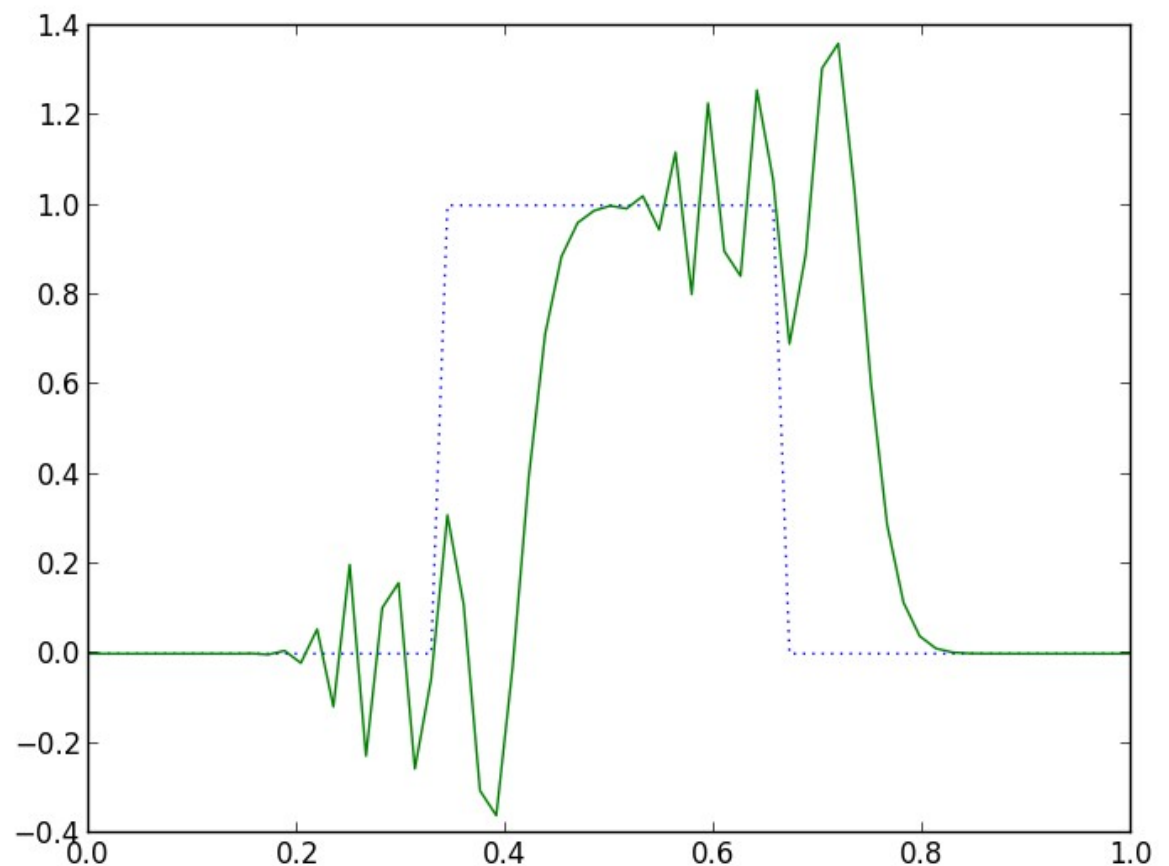
- Reducing the timestep is equivalent to reducing the CFL number
  - $CFL = 0.1$ , still 1 full period
  - The scale is much reduced, but it is still horrible
  - Let's look how these features develop



# FTCS Linear Advection

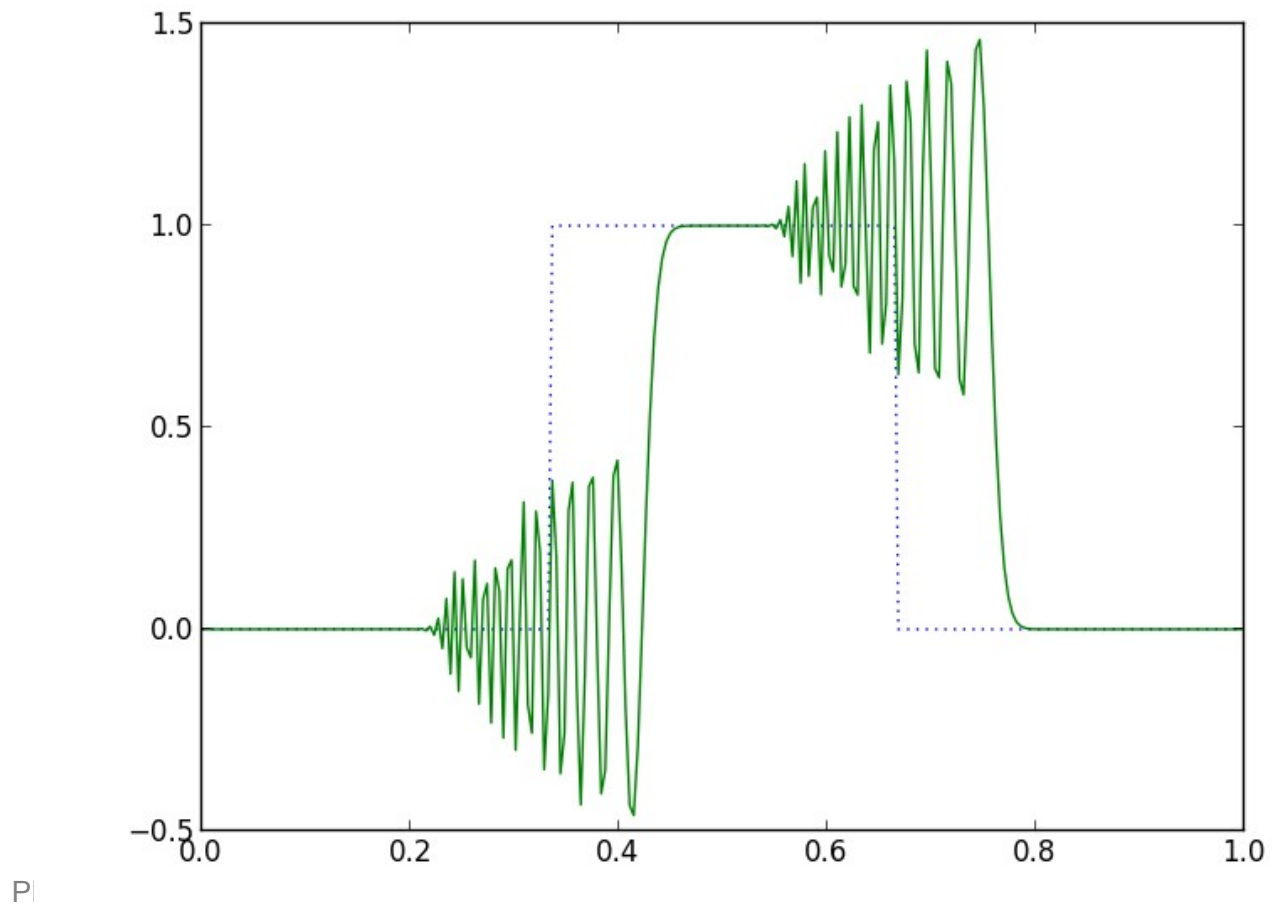
- Here we evolve for only 1/10th of a period

- CFL = 0.1
- Notice that the oscillations are developing right near the discontinuities
- So far we've looked at timestep, what about resolution?



# FTCS Linear Advection

- This is with  $N = 257$  points
  - There is something more fundamental than resolution or CFL number going on here...



# Homework #2 Comments

- Problem 2: this was to be analytic (similar to how we showed angular momentum conservation in class for E-C)
  - You can express the new energy in terms of the original energy + a term that looks like  $\Delta t^2$
  - Two approaches: small-angle approx. or assume  $\Delta t \ll 1$
  - Later gives:

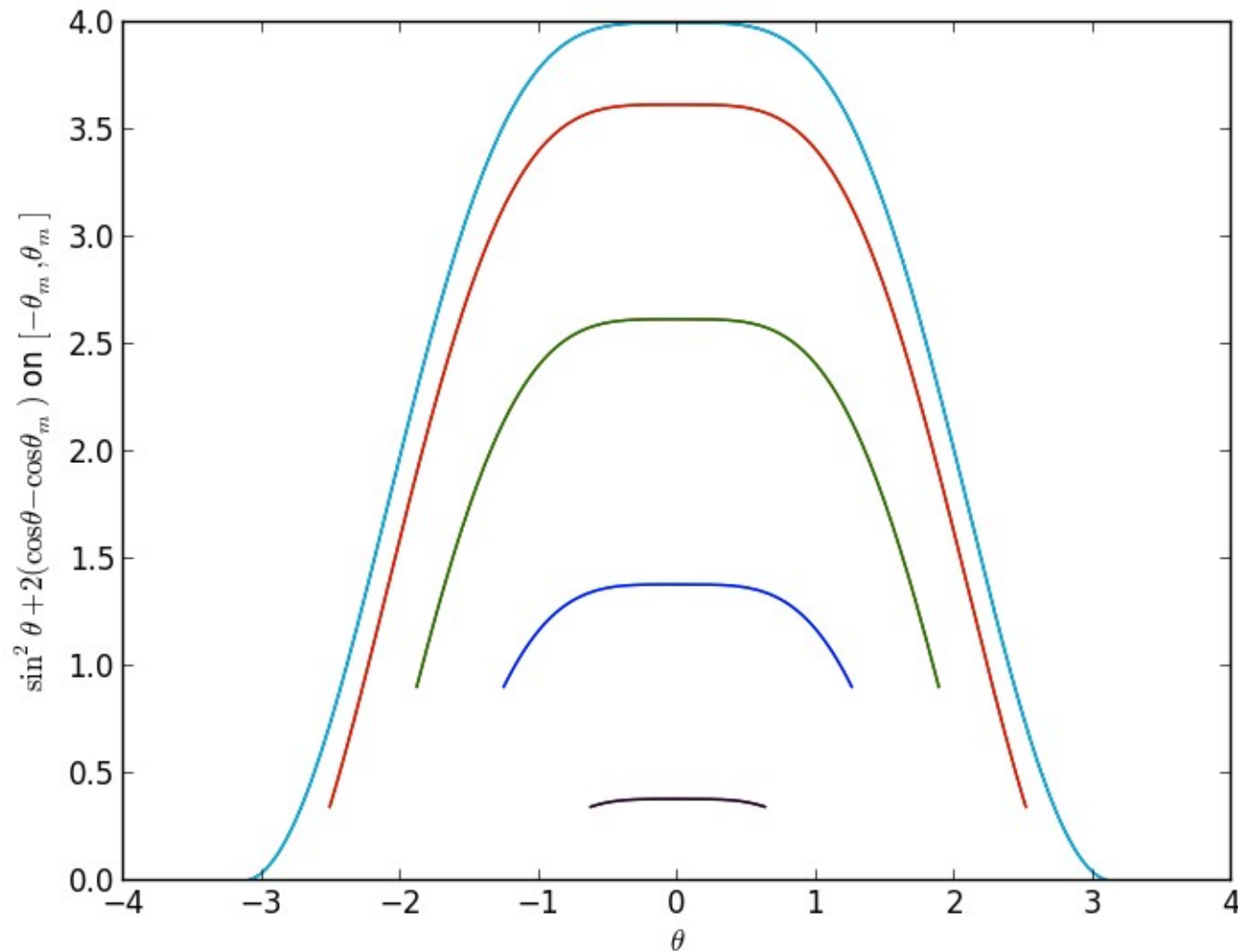
$$E^{n+1} = E^n + \frac{1}{2}mg(g \sin^2 \theta^n + L\omega^{n2} \cos \theta^n)\Delta t^2$$

- Energy conservation tells us that:  $L\omega^2 = 2g(\cos \theta - \cos \theta_m)$

$$\Delta E = \frac{1}{2}mg^2(\sin^2 \theta^n + 2(\cos \theta^n - \cos \theta_m))$$

We need to show that this is  $> 0$

# Homework #2 Comments



For the range of angles the pendulum takes on, it is  $> 0$ , therefore energy always increases

# Homework #2 Comments

- Problem 3: here you were to demonstrate convergence
  - Some of you did not pick small enough timesteps to be in the converging regime (i.e. your errors were  $O(1)$ !)
  - You should estimate the convergence rate—this means look to see what sort of powerlaw behavior it takes
  - Curiously, the convergence seems much better when you end exactly on a period boundary (this is what the plot in the solutions I passed out did). This suggests that perhaps some other metric for convergence is more robust.
- General comments: include the code! And hardcopies of plots with some sentences never hurts...

# Online Notes

- We'll follow a development of methods for advection that is common in astrophysics.
  - Neither text does things in the same way (although Garcia has some nice discussions)
  - More verbose (LaTeX-ed) notes of mine are linked online

# Stability

- The problem is that FTCS is not **stable**
- There are a lot of ways we can investigate the stability
  - It is instructive to just work through an update using pencil + paper to see how things grow
  - Growth of a single Fourier mode
  - Truncation analysis
  - Graphically: we can trace back the characteristics to see what information the solution depends on.



# Fourier Mode Analysis of FTCS

- Consider a solution consisting of a single Fourier mode:

$$a_i^n = A^n e^{ji\theta} \quad j = \sqrt{-1}$$

- Since we are linear, we don't have to worry about different modes “talking” to one another, so we can just deal with a single mode
- Stability will require that:

$$\left| \frac{A^{n+1}}{A^n} \right| \leq 1$$

- Putting this mode into our difference equation we find (blackboard...)

$$\left| \frac{A^{n+1}}{A^n} \right|^2 = 1 + C^2 \sin^2 \theta \geq 1$$

- Note there is no dependence on  $n$  – all modes grow the same
- No value of  $C$  makes things work

# Fourier Mode Analysis of FTCS

- FTCS is unconditionally unstable
- Note that although this method of stability analysis only works for linear equations, it can still guide our insight into nonlinear equations
- This methodology was developed by von Neumann during WWII at LANL and allegedly, it was originally classified...

# Truncation Analysis

- Observe that:

*Finite-difference methods solve linear advection equations approximately, but they solve modified linear advection equations exactly*

—Laney (p. 265)

- Let's figure out what physical equation this difference approximation better represents
- Substitute in

$$a_i^{n+1} = a_i^n + \dot{a}\Delta t + \frac{1}{2}\ddot{a}\Delta t^2 + \mathcal{O}(\Delta t^3)$$

$$a_{i\pm 1}^n = a_i^n \pm a'\Delta x + \frac{1}{2}a''\Delta x^2 + \mathcal{O}(\Delta x^3)$$

- And lot's of algebra...

# Truncation Analysis


- We find:

$$\underbrace{a_t + ua_x}_{\text{This is our original equation}} = \underbrace{-\frac{1}{2}\Delta t u^2 a_{xx}}_{\text{The looks like diffusion, but look at the sign!}}$$


- We are more accurately solving an advection/diffusion equation
- But the diffusion is negative!
  - This means that it acts to take smooth features and make them strongly peaked—this is unphysical!
- The presence of a numerical diffusion (or numerical viscosity) is quite common in difference schemes, but it should behave physically!

# Upwinding

- Let's go back to the original equation and try a different discretization
- Instead of a second-order (centered-difference) spatial difference, let's try first order.
  - We have two choices:

$$a_x = \frac{a_i^n - a_{i-1}^n}{\Delta x}$$


upwind

$$a_x = \frac{a_{i+1}^n - a_i^n}{\Delta x}$$


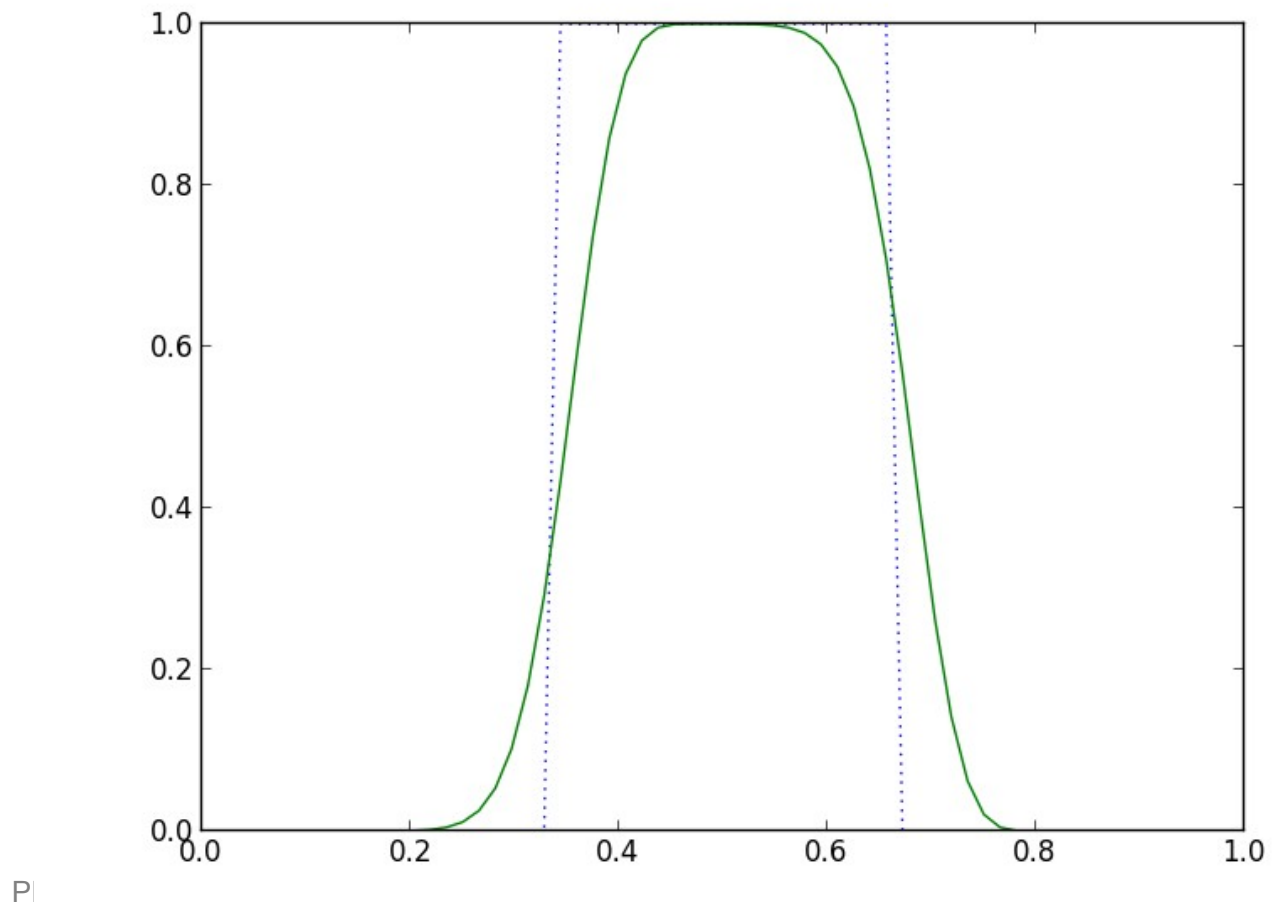
downwind

- Let's try the upwinded difference—that gives:

$$a_i^{n+1} = a_i^n - C(a_i^n - a_{i-1}^n)$$

# Upwinding

- Upwinding solution with  $N = 65$ ,  $C = 0.9$ , after 1 period
  - Much better
  - There still is some error, but it is not unstable
  - Numerical diffusion evident



# Upwinding

- Notice that if we do  $C = 1$ , we get an exact translation of the data from one zone to the next:

$$a_i^{n+1} = a_i^n - C(a_i^n - a_{i-1}^n)$$

- However, as we will see, for nonlinear eqs. and systems of linear advection eqs., we cannot in general do  $C = 1$ , because there is not a single speed over all the grid.
- Let's play with the code and explore resolution, number of periods, etc...

# Stability Analysis of Upwinded Eq.

- We can do the same single-mode Fourier analysis and we find:

$$\left| \frac{A^{n+1}}{A^n} \right|^2 = 1 - 2C(1 - C)(1 - \cos \theta)$$

- We know that  $1 - \cos \theta \geq 0$
- We need:  $2C(1 - C) \geq 0$
- **Blackboard...**
- The upwind method is stable for  $0 \leq C \leq 1$ 
  - This is the statement that information cannot propagate more than one cell per timestep
  - We'll see that this is a pretty standard timestep restriction for hyperbolic PDEs



# Stability Analysis of Upwinded Eq.

- Truncation analysis would show this method is equivalent to

$$a_t + ua_x = \frac{1}{2}u\Delta x(1 - C)a_{xx}$$

- Represents physical diffusion so long as  $1 - C > 0$
  - This also shows that we get the exact solution for  $C = 1$
- Note that if we used the downwind difference, our method would be unconditionally unstable
  - Direction of difference based on sign of velocity
- Physically, the choice of upwinding means that we make use of the information from the direction the wind is blowing
  - This term originated in the weather forecasting community

# Stability Analysis of Upwinded Eq.

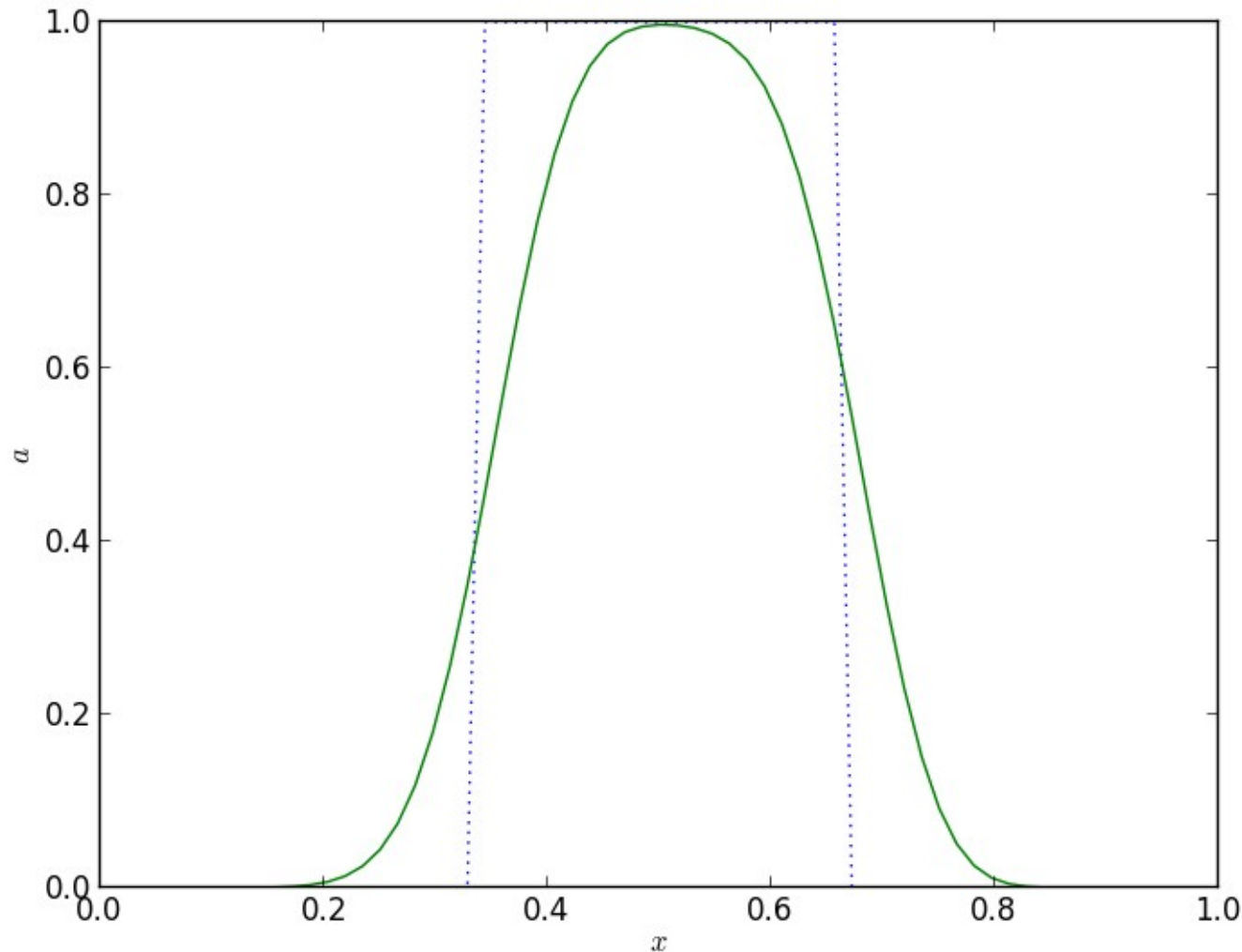
- Note that there is 0 diffusion for  $C = 1$ 
  - Look at the update when this is the case—solution is advected perfectly (blackboard)
  - However, when we look at systems or nonlinear equations, we will see that we cannot in general use  $C = 1$

# Phase Error

- Examining the behavior of a Fourier mode can also tell us if the method introduces a phase lag
  - Look at the ratio of imaginary to real parts of the amplification factor
  - Some methods have severe phase lag
  - We won't go into this here...

# Upwind Results

- Finite-difference (node-centered) grid, with  $N=65$ , 1 period:
  - Tophat initial conditions,  $C=0.8$

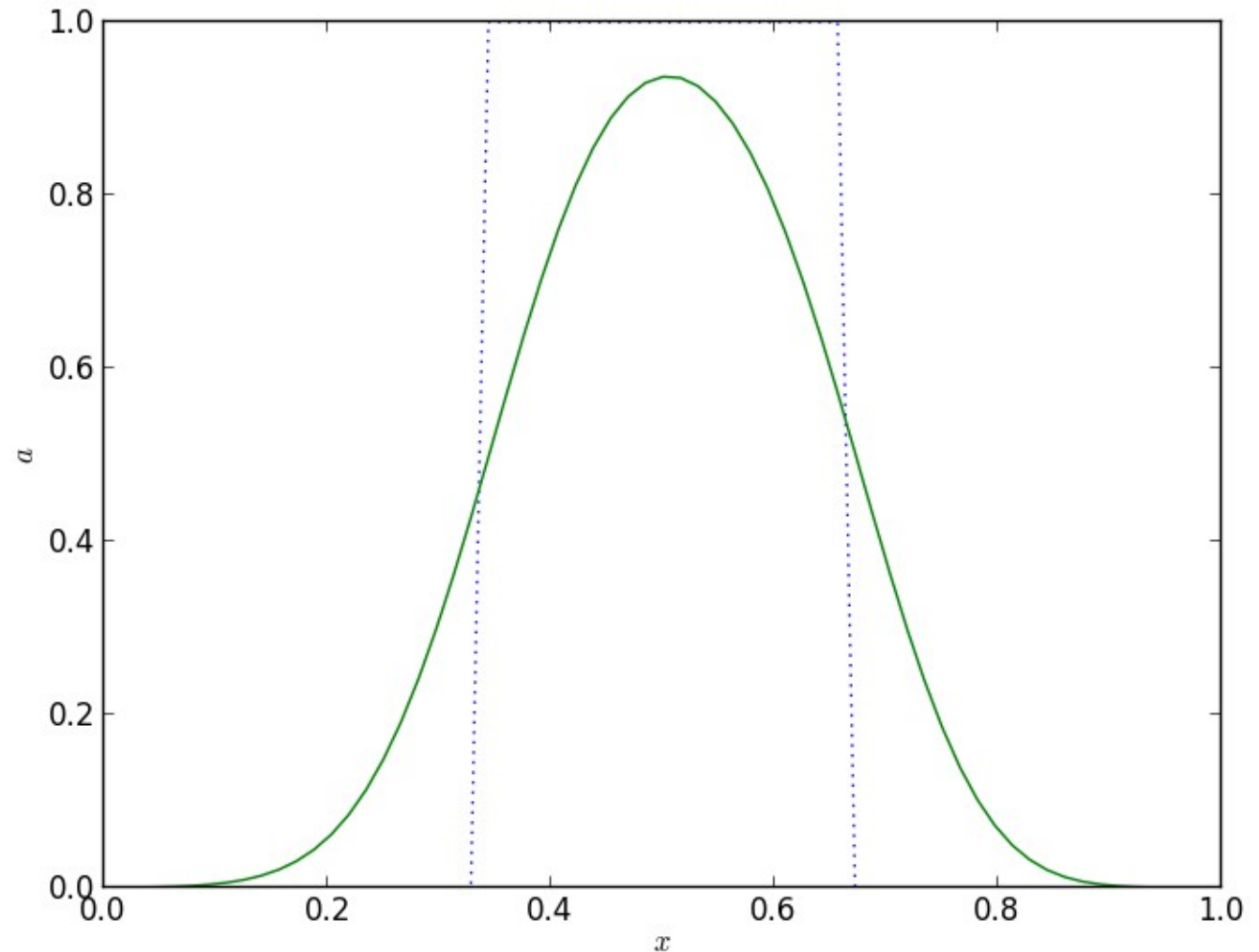


# Upwind Results

- Finite-difference (node-centered) grid, with  $N=65$ , 5 periods:
  - Tophat initial conditions,  $C=0.8$

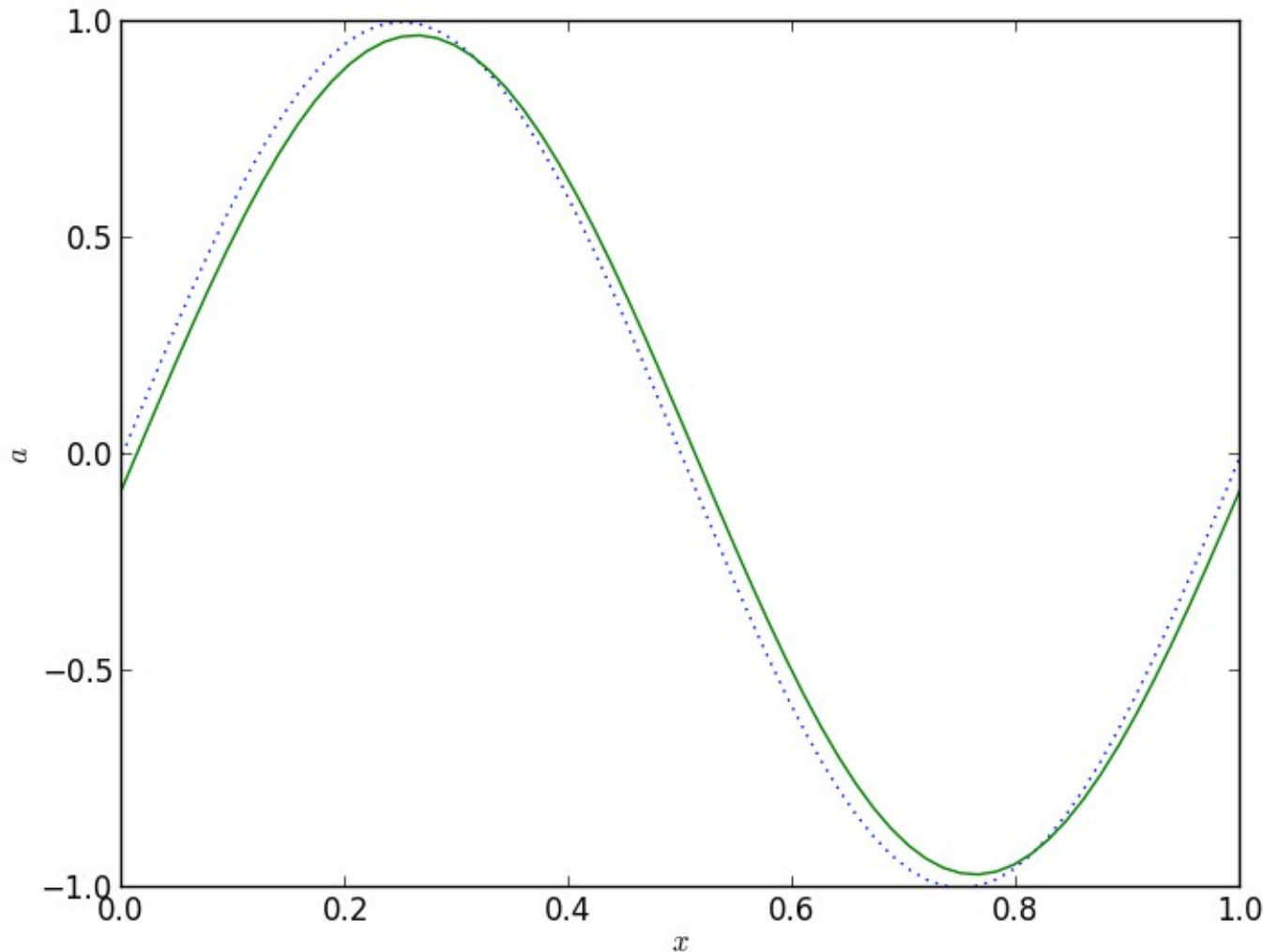
Notice that the numerical diffusion is strongly apparent here

What about smooth initial conditions?



# Upwind Results

- Finite-difference (node-centered) grid, with  $N=65$ , 1 period:
  - sine wave,  $C=0.8$



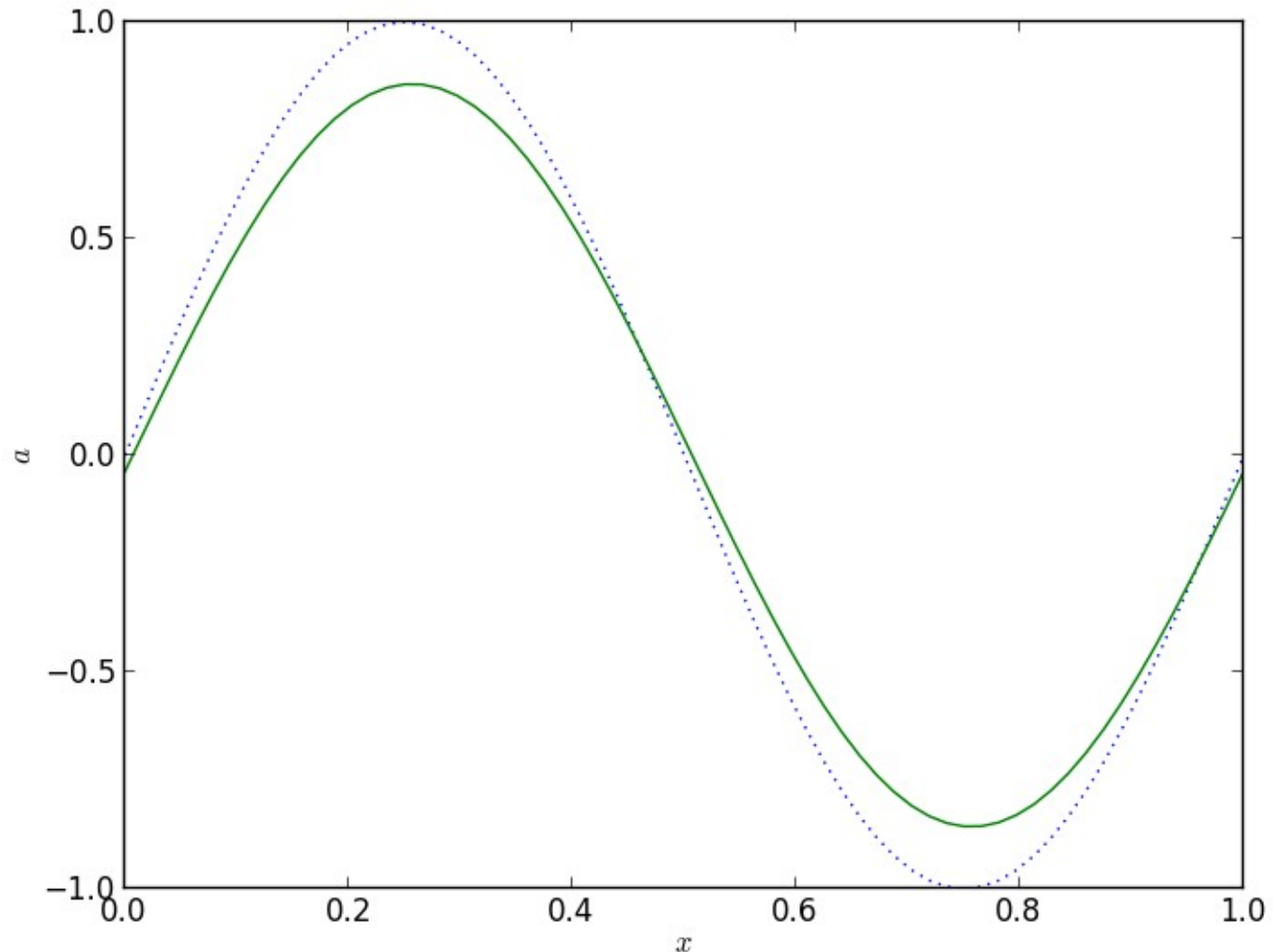
# Upwind Results

- Finite-difference (node-centered) grid, with  $N=65$ , 5 periods
  - sine wave,  $C=0.8$

Note that the sine wave stays in phase (that's a good thing)

Diffusion still apparent.

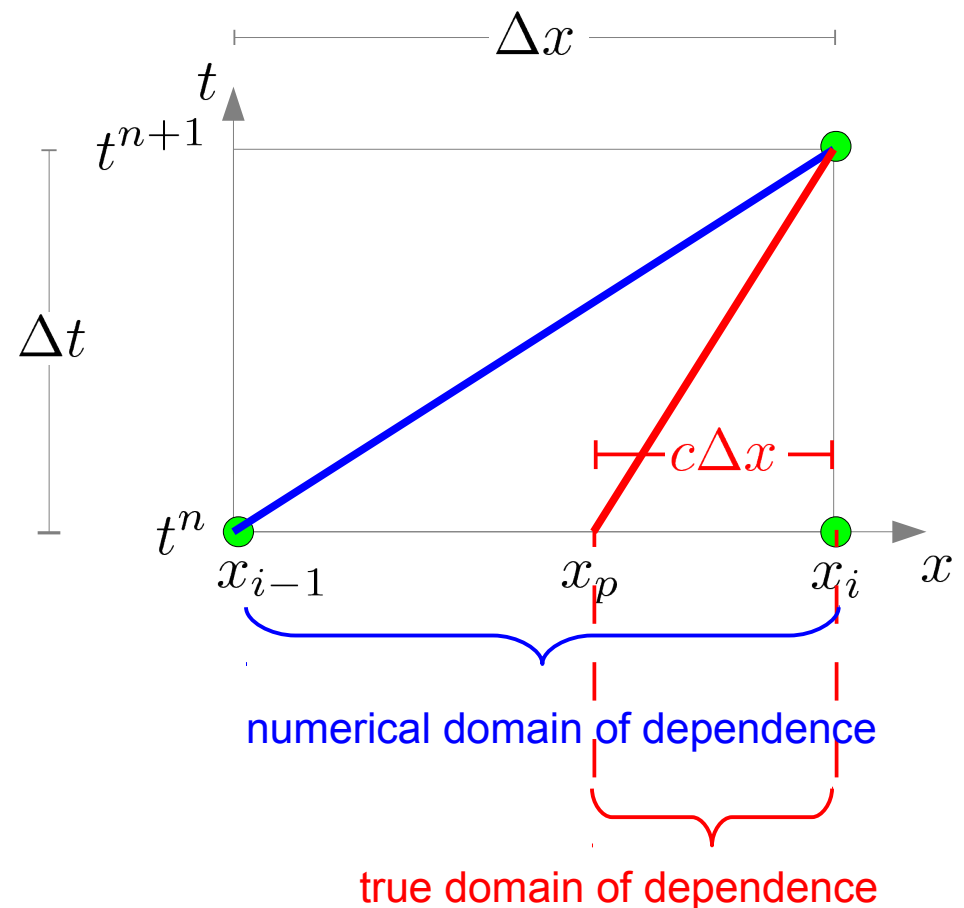
Just for fun, let's try a downwinded method...



# A Little More Insight on Stability

- A necessary (**but not sufficient**) condition for stability is that the **numerical domain of dependence** contain the **true domain of dependence** (see Lahey, Ch. 12 for a nice discussion)
  - True domain of dependence is found by tracing the characteristics backwards in time

Upwind method ►  
For  $u > 0$





# A Little More Insight on Stability

- The downwind method clearly violates this

- Ex from Lahey:

- Consider initially discontinuous data

$$a_i^0 = a(i\Delta x, 0) = \begin{cases} 1 & i < 0 \\ 0 & i \geq 0 \end{cases}$$

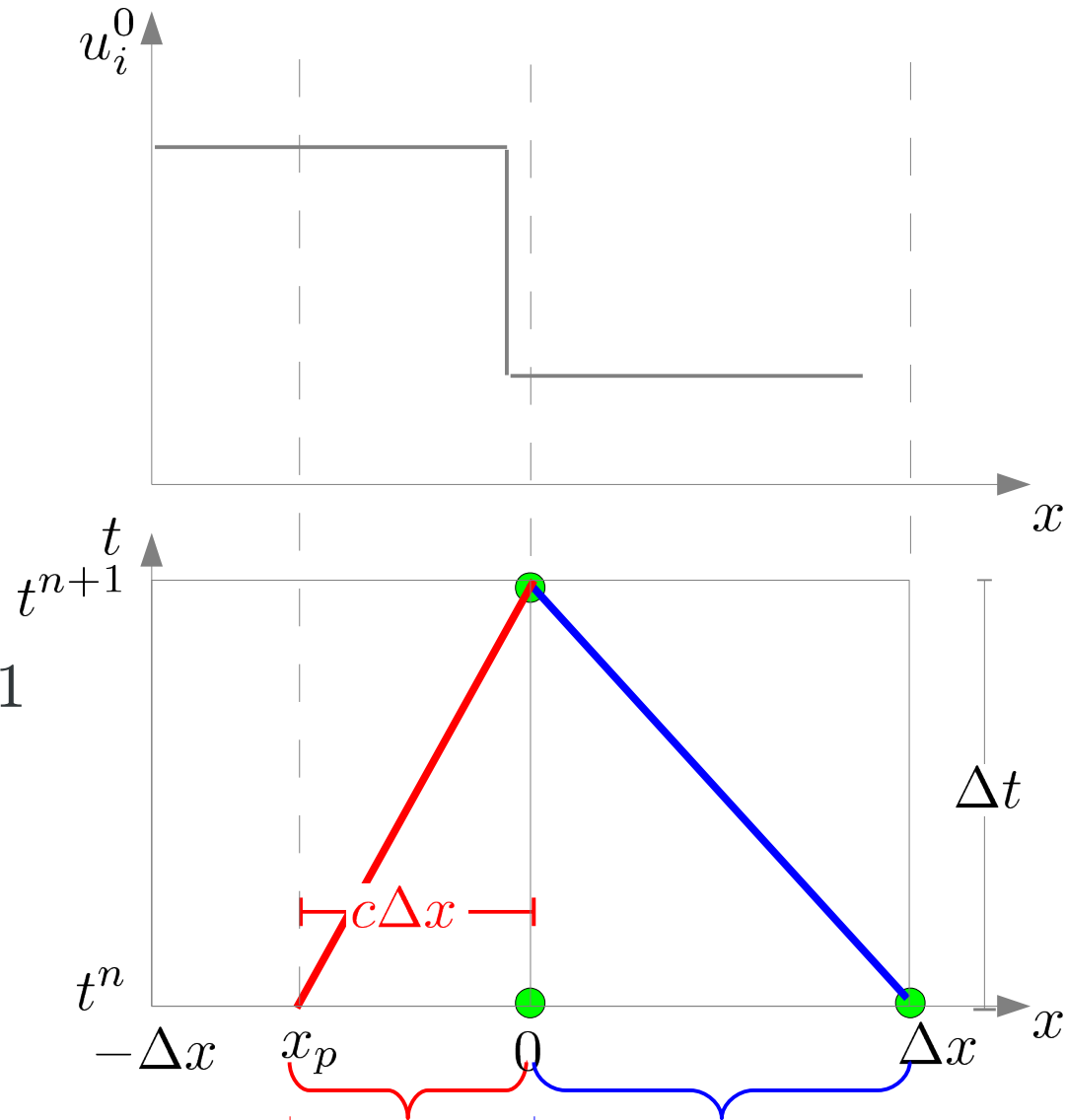
- True solution at  $x = 0$  is  $a_0^1 = 1$

- Downwind method:

$$a_i^{n+1} = a_i^n - C(a_{i+1}^n - a_i^n)$$

gives:

$$a_0^1 = a_0^0 - C(a_1^0 - a_0^0) = 0$$



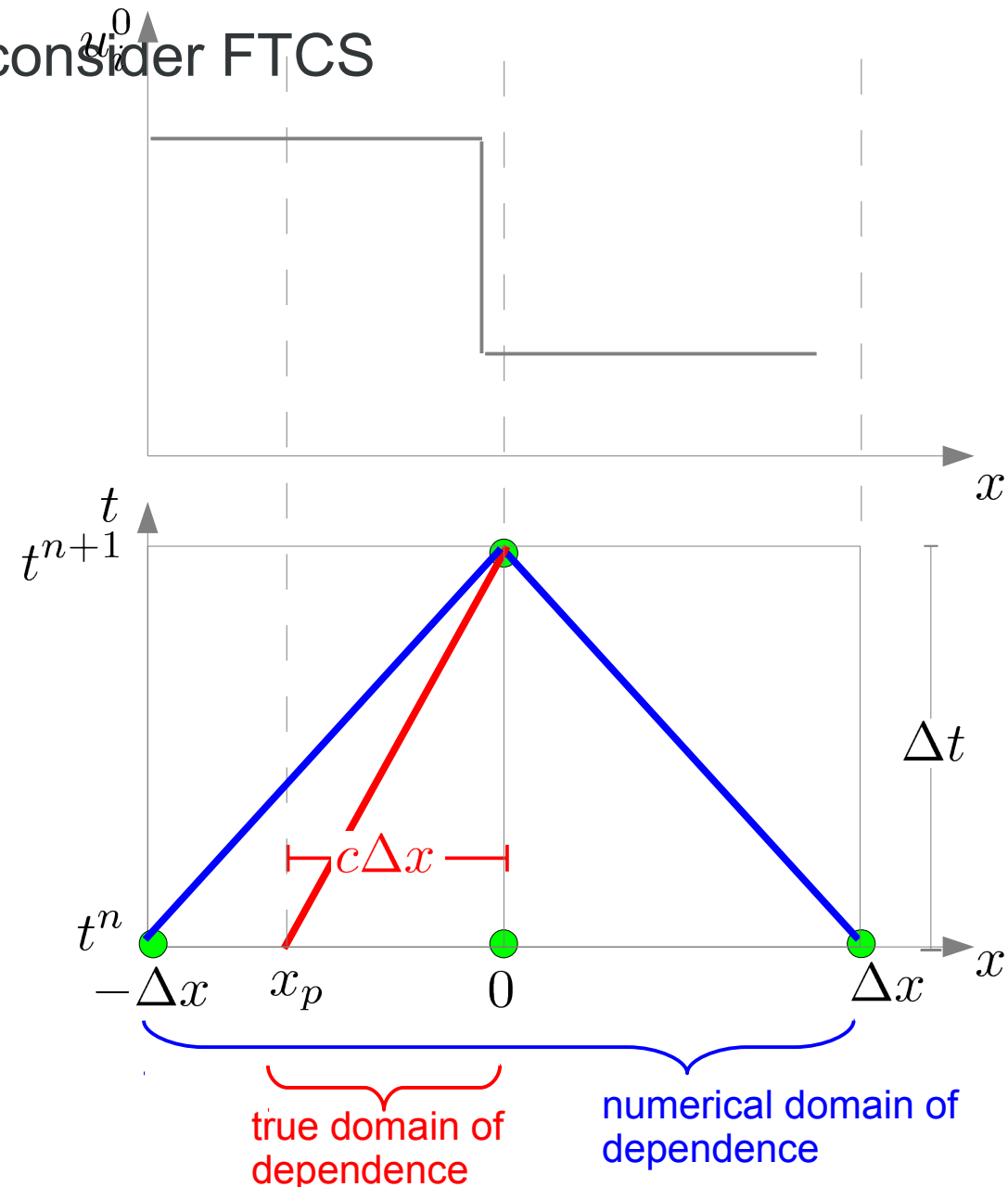
The numerical domain of dependence doesn't include the true domain of dependence, and we get the wrong answer.

true domain of dependence

numerical domain of dependence

# A Little More Insight on Stability

- Necessary, but not sufficient—consider FTCS



# Implicit Differencing

- If we discretize the spatial derivative using information at the new time, then we get an implicit method
  - Unconditionally stable
  - You'll explore this on your homework

# Other Finite-Difference Methods

- There are many other finite-difference methods that build off of what we already saw
- **Lax-Friedrichs:**
  - Start with FTCS:  $a_i^{n+1} = a_i^n + \frac{C}{2}(a_{i+1}^n - a_{i-1}^n)$
  - Simple change, replace:  $a_i^n \rightarrow \frac{1}{2}(a_{i-1}^n + a_{i+1}^n)$
  - We get:
$$a_i^{n+1} = \frac{1}{2}(1 + C)a_{i-1}^n + \frac{1}{2}(1 - C)a_{i+1}^n$$
  - This is stable for  $C < 1$ , but notice that it doesn't contain the point we are updating! (**Let's explore this...**)
    - Can suffer from odd-even decoupling
    - Can be more diffusive than upwinding (and gets worse for smaller  $C$ )
- You'll explore **Lax-Wendroff** in your homework—it is second order in space and time. Works well, but can oscillate...

# Measuring Convergence

- As we move to higher-order methods, we will want to measure the convergence of our methods
  - We have data at  $N$  points for each time level
  - We need a norm that operates on the gridded data. A popular choice is the L2 norm:

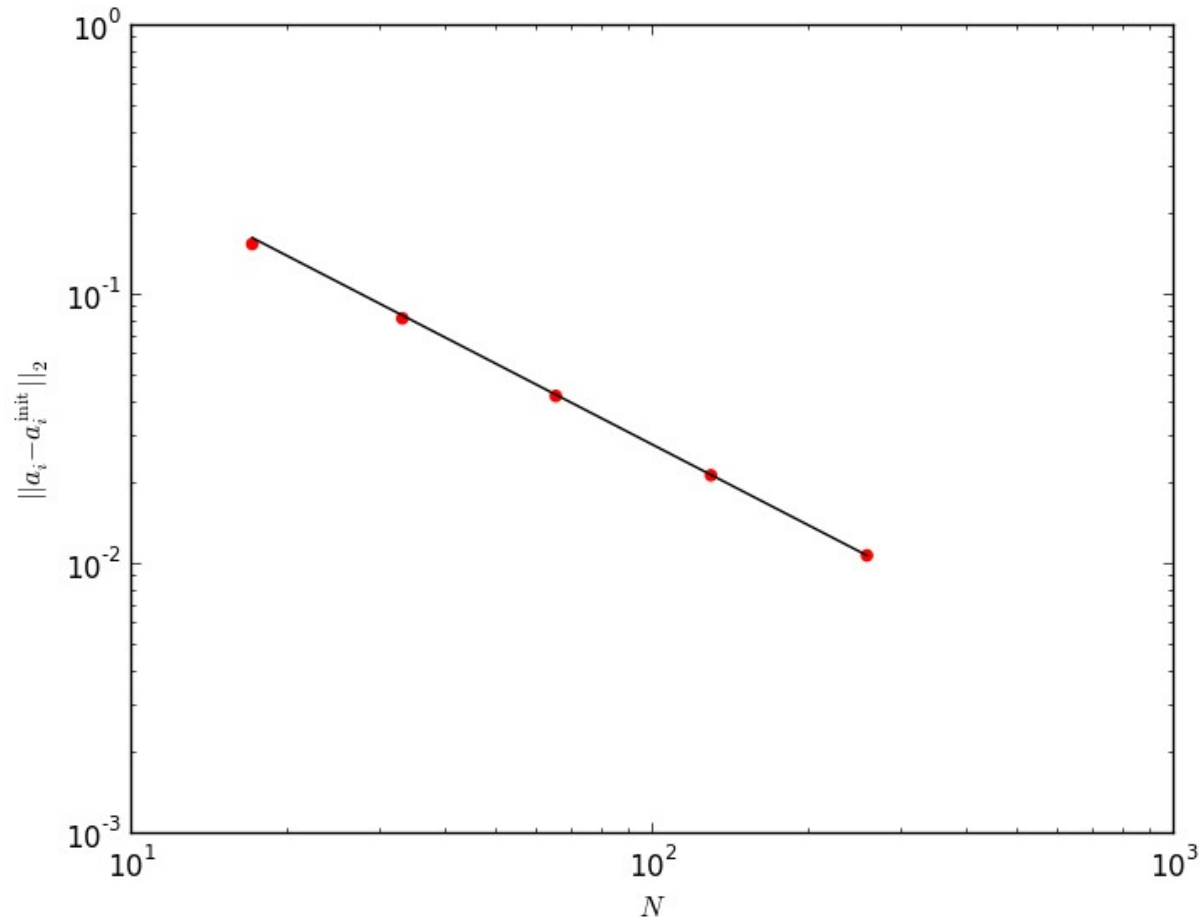
$$\|\phi\|_2 = \left( \Delta x \sum_{i=0}^{N-1} \phi_i^2 \right)^{1/2}$$

- Using the grid width ensure that this measure is resolution independent
- We can take the error of our method to be:

$$\epsilon = \|a_i - a^{\text{exact}}(x_i)\|_2$$

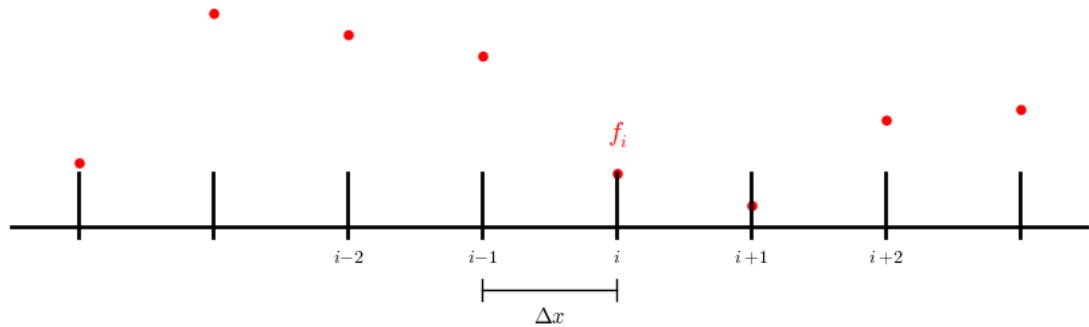
# Measuring Convergence

- For advection with periodic BCs, compare to the initial conditions
  - Note: you need to be careful to end always at the same time (e.g. exactly on a period)

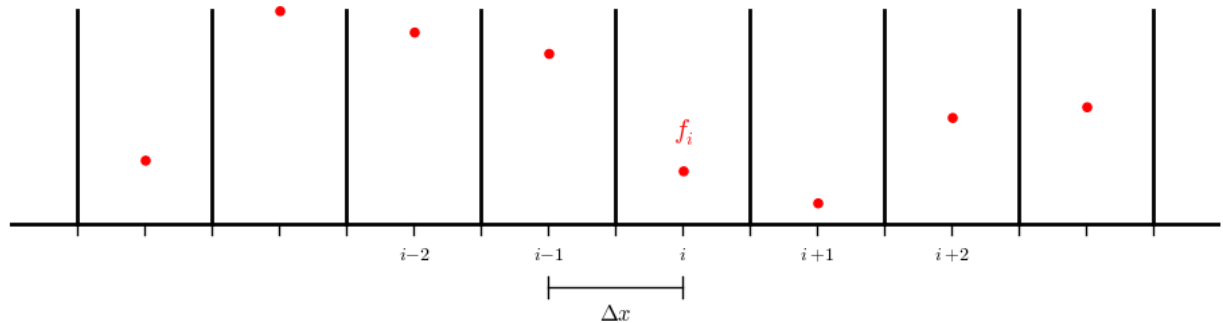


# Grid Types

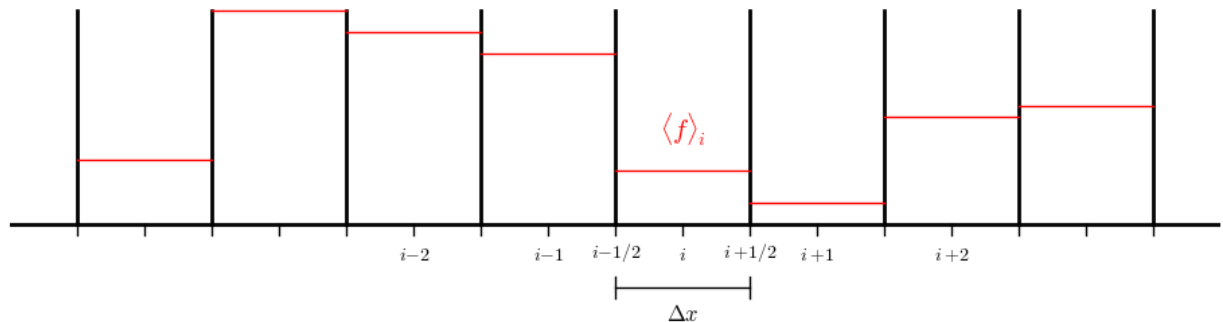
“Regular” finite-difference grid. Data is associated with nodes spaced  $\Delta x$  apart. Note that here we can have a point exactly on the boundary



Cell-centered finite-difference grid. Here we consider cells of width  $\Delta x$  and associate the data with a point at the center of the cell. Note that data will be  $\Delta x/2$  inside the boundary



Finite-volume grid—similar to the cell-centered grid, we divide the domain into cells/zones. Now we store the average value of the function in each zone.



# Grid Types

- Finite-difference
  - Some methods use staggered grids: some variables are on the nodes (cell-boundaries) some are at the cell centers
  - Boundary condition implementation will differ depending on the centering of the data
  - For cell-centered f-d grids: ghost cells implement the boundary conditions
- Finite-volume
  - This is what we'll focus on going forward
  - Very similar in structure to cell-centered f-d, but the interpretation of the data is different.



# Conservation Laws

- Many systems appear as conservation laws:

$$U_t + [F(U)]_x = 0$$

- Linear advection:  $a_t + [ua]_x = 0$

- Burger's equation:  $u_t + \left[ \frac{1}{2} u^2 \right]_x = 0$

- Hydrodynamics

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} \quad F(U) = \begin{pmatrix} \rho u \\ \rho u u + p \\ \rho u E + u p \end{pmatrix}$$

Understanding advection is key to each of these systems

# Finite-Volume Approximation

- Finite-volume methods are designed for conservation laws
  - Integrate our hyperbolic equation over a control volume

$$\frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} U_t = - \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial}{\partial x} F(U) dx$$

$$\frac{\partial}{\partial t} U_i = - \frac{1}{\Delta x} \{ [F(U)]_{i+1/2} - [F(U)]_{i-1/2} \}$$

- Here we write the cell-averages without the  $\langle \rangle$
- We still need to discretize in time
- **Telescoping property**
  - Note that the flux of  $U$  that leaves one volume enters the adjacent one—guaranteeing conservation

# Finite-Volume Approximation

- A second-order approximation for our advection equation is

$$\frac{a_i^{n+1} - a_i^n}{\Delta t} = - \frac{[f(a)]_{i+1/2}^{n+1/2} - [f(a)]_{i-1/2}^{n+1/2}}{\Delta x}$$

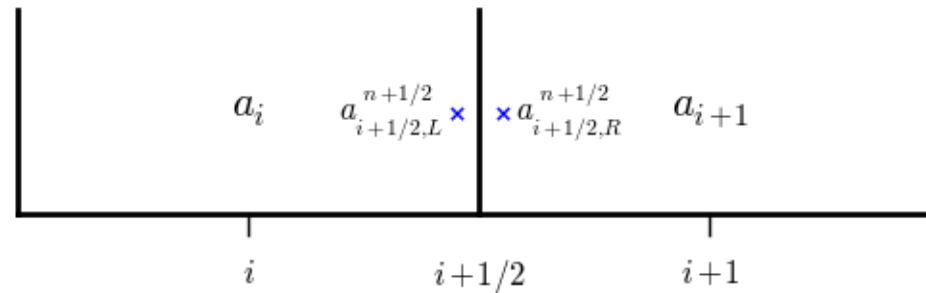
- Notice that the RHS is centered in time
- To complete this method, we need to compute the fluxes on the edges
  - We can do this in terms of the interface state

$$[f(a)]_{i+1/2}^{n+1/2} = f(a_{i+1/2}^{n+1/2})$$

- Now we need to find the interface state. There are two possibilities: coming from the left or right of the interface

# First-Order Advection

- Our interface states



- Simplest choice:

$$a_{i+1/2,L}^{n+1/2} = a_i^n; \quad a_{i+1/2,R}^{n+1/2} = a_{i+1}^n$$

- Now we need to resolve the degeneracy of the states. This requires knowledge of the equation
  - **Riemann problem**: two states separated by an interface
  - **For advection, we do upwinding**

# First-Order Advection

- Riemann problem (upwinding):

$$\mathcal{R}(a_{i+1/2,L}^{n+1/2}, a_{i+1/2,R}^{n+1/2}) = \begin{cases} a_{i+1/2,L}^{n+1/2} & u > 0 \\ a_{i+1/2,R}^{n+1/2} & u < 0 \end{cases}$$

- This is simple enough that we can write out the resulting update (for  $u > 0$ ):

$$\frac{a_i^{n+1} - a_i^n}{\Delta t} = - \frac{u a_i^n - u a_{i-1}^n}{\Delta x}$$

- This is identical to the first-order upwind finite-difference discretization we already studied

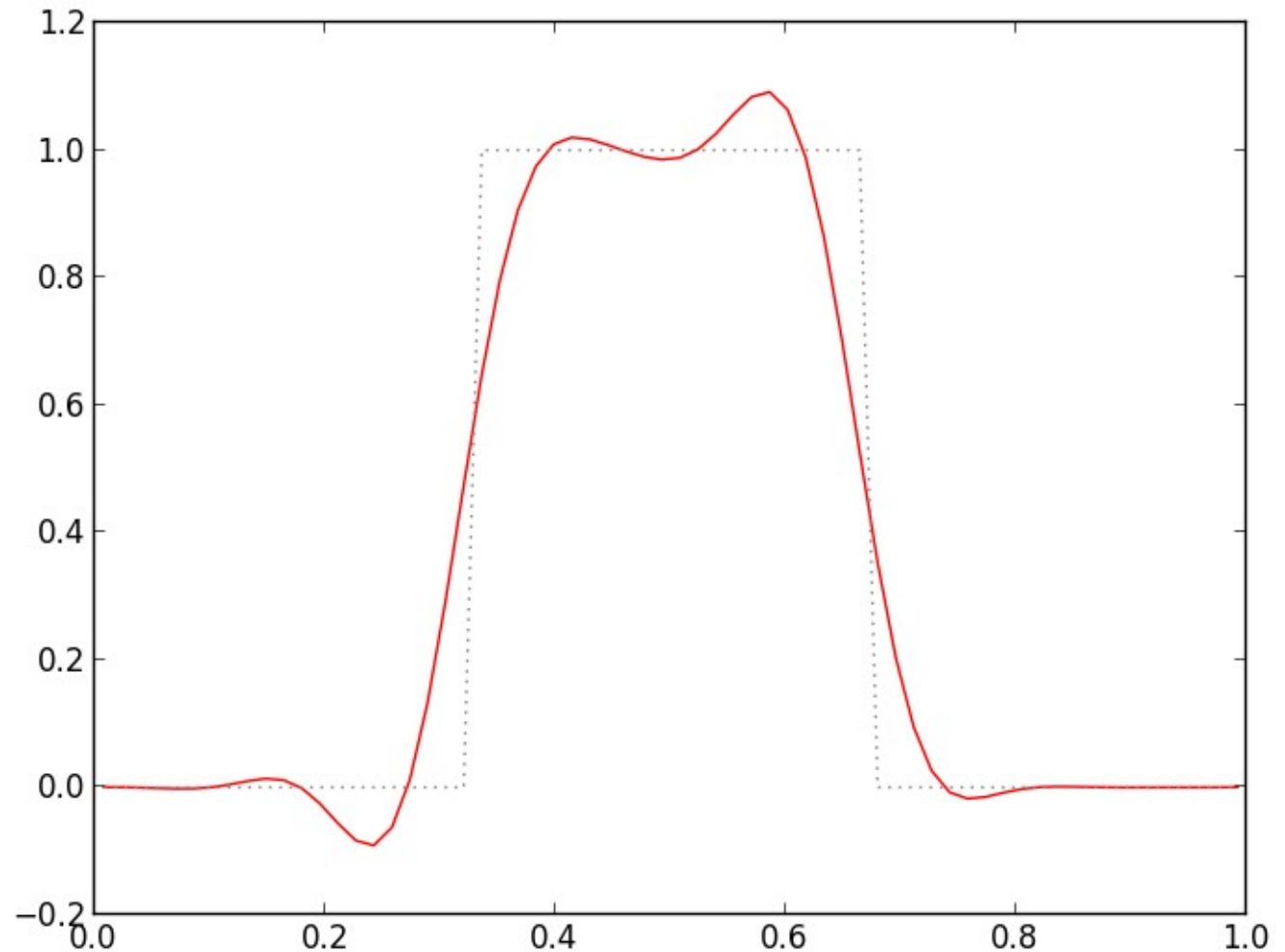
# Second-Order Advection

- Going to second-order requires making the interface states second-order in space and time
  - We will use a **piecewise linear reconstruction** of the cell-average data to approximate the true functional form
  - We Taylor expand in space and time to find the time-centered, interface state
- **Let's derive this on the blackboard...**

# Second-Order Advection

- Tophat with 64 zones,  $C = 0.8$

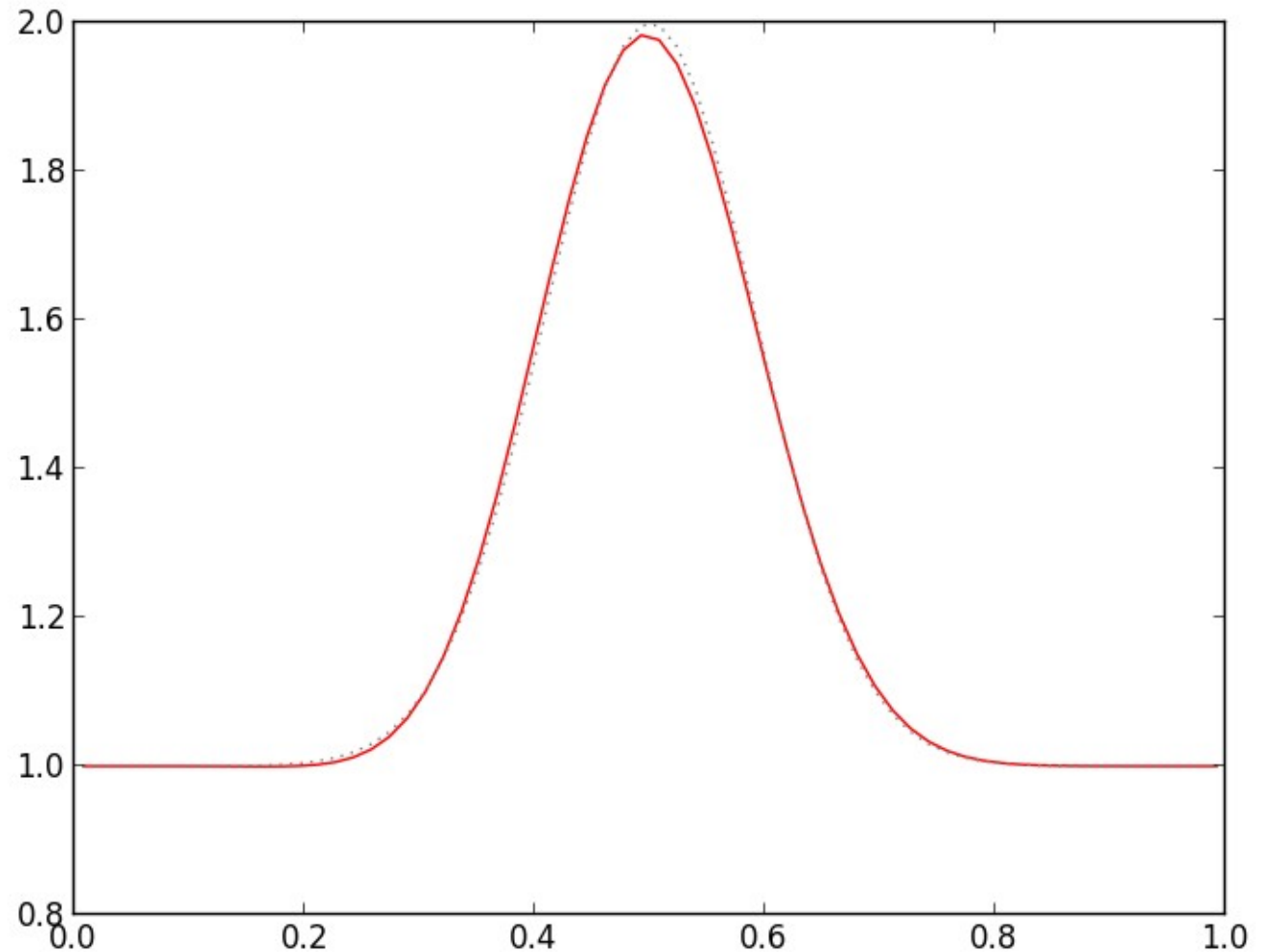
Notice the oscillations—they seem to be associated with the steep jumps



# Second-Order Advection

- Gaussian with 64 zones,  $C = 0.8$

This looks nice and smooth

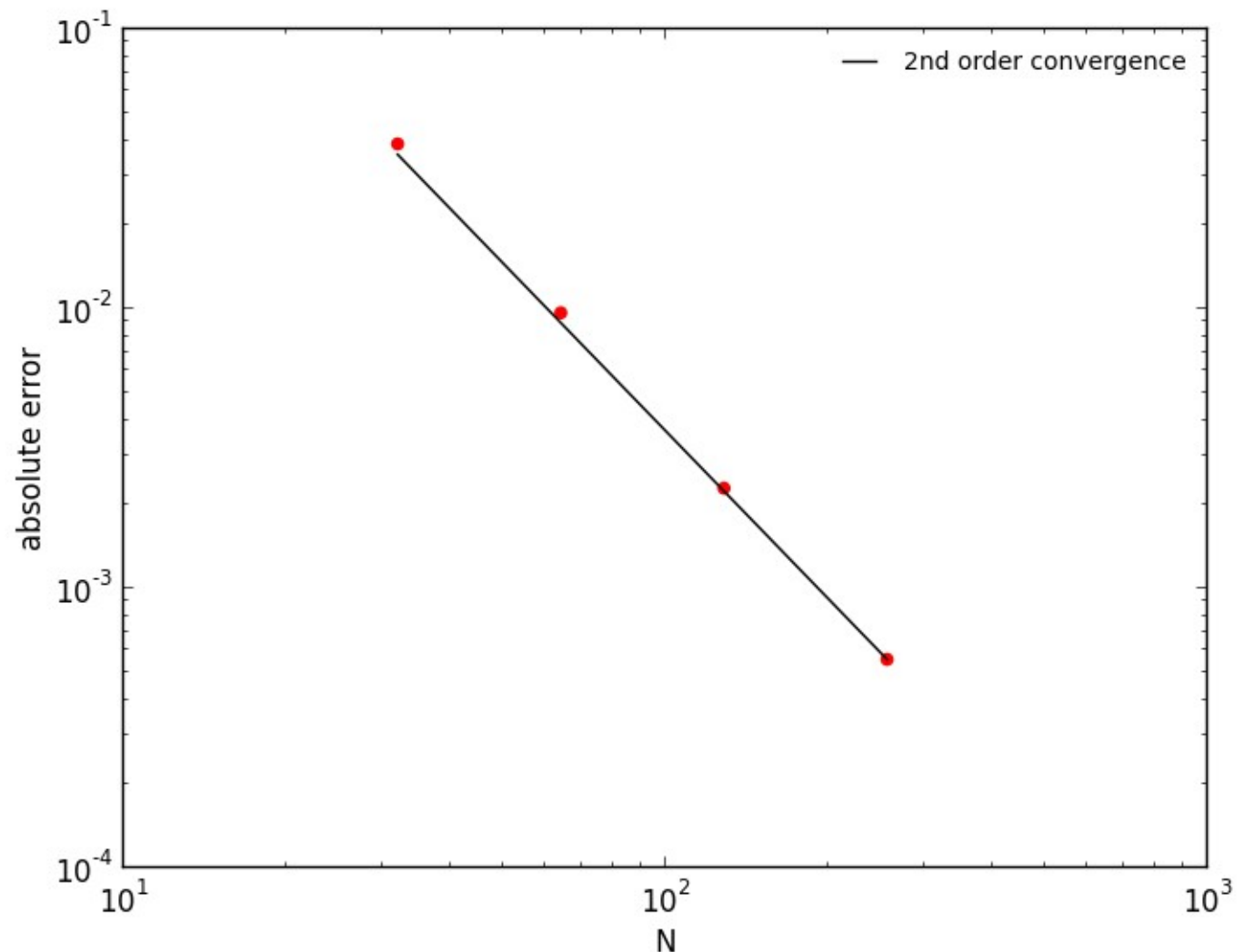




# Second-Order Advection

- Remember—you should always look at the convergence of your code—if you do not get what you are supposed to get, you probably have a bug (or you don't understand the method well enough...)

Convergence  
for the  
Gaussian

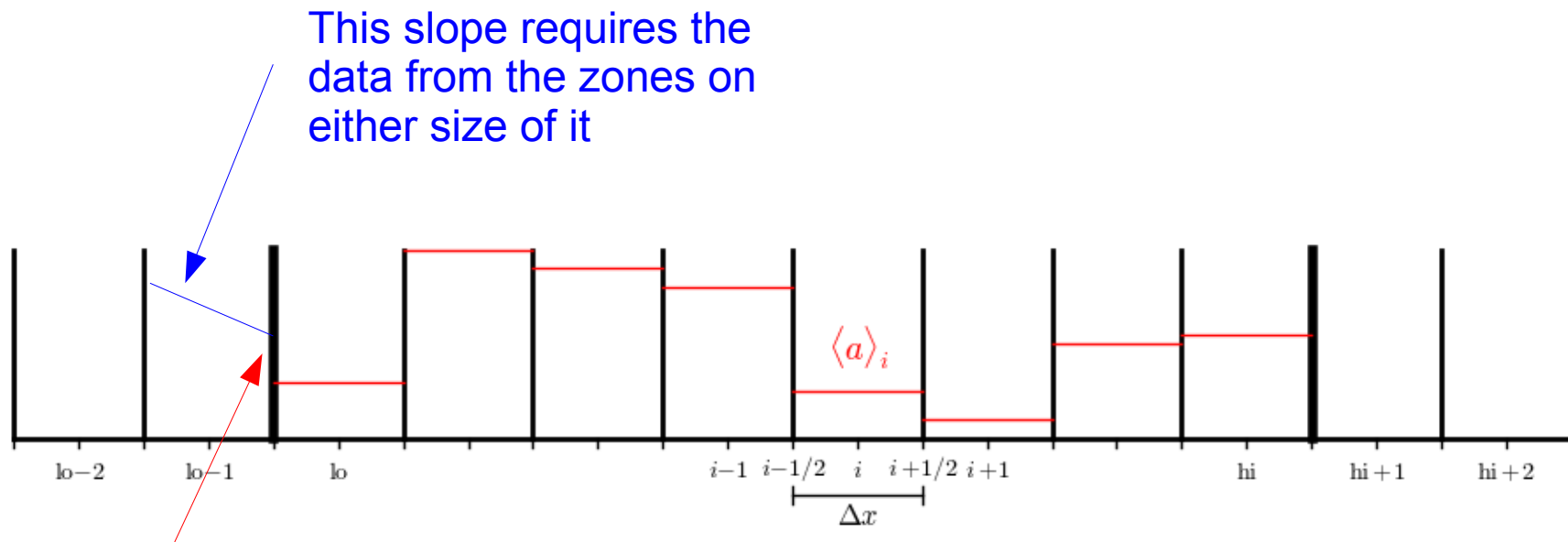


# Second-Order Advection

- General notes:
  - We derived both the left and right state at each interface—for advection, we know that we are upwinding, so we only really need to do the upwinded state
  - But, for a general conservation law, the upwind direction can change from zone to zone and timestep to timestep AND there can be multiple waves (e.g. systems), so for the general problem, we need to compute both states
- If you set the slopes to 0, you reduce to the first-order method
- Let's look at the code...

# Second-Order Advection

- We need 2 ghost cells on each end:



The state here depends on the slope in zone  $l_0-1$

# Design of a General Solver

- Simulation codes for solving conservation laws generally follow the following flow:
  - Set initial conditions
  - Main evolution loop (loop until final time reached)
    - Fill boundary conditions
    - Get the timestep
    - Compute the interface states
    - Solve the Riemann problem
    - Do conservative update
- We'll see that this same procedure can be applied to nonlinear problems and systems (like the equations of hydrodynamics)

# REA

- An alternate way to think about these methods is as a Reconstruct-Evolve-Average process
  - We can demonstrate this on the blackboard...

