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DEPARTMENT OF MATHEMATICS

A SIMPLE EXAMPLE OF AN ILL-CONDITIONED MATRIX.

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The following example has been found useful in courses of Numerical Analysis, for illustrating the concept of ill-conditioning.

Consider the matrix A of order 3:

$$A = \begin{bmatrix} 11 & 10 & 14 \\ 12 & 11 & -13 \\ 14 & 13 & -66 \end{bmatrix}, \tag{1}$$

which is non-singular:

$$\det(A) = 1, \tag{2}$$

$$A^{-1} = \begin{bmatrix} -557 & 842 & -284 \\ 610 & -922 & 311 \\ 2 & -3 & 1 \end{bmatrix}. \tag{3}$$

However, A is "nearly singular", in the sense that

$$A_{22} = \text{cofactor of } a_{22} = -922, \tag{4}$$

and hence det(A) would become zero if a_{22} were increased by $\frac{1}{922}$.

The equation

$$Ax = b, \tag{5}$$

where

$$b = \begin{bmatrix} 1.001 \\ 0.999 \\ 1.001 \end{bmatrix}, \tag{6}$$

has the solution:

$$x = A^{-1}b = \begin{bmatrix} -0.683 \\ 0.843 \\ 0.006 \end{bmatrix}. \tag{7}$$

Let the elements of b be rounded, to give:

$$b + \delta b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{8}$$

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Then, the perturbed equation

$$Az = b + \delta b \tag{9}$$

has the solution

$$z = x + \delta x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \tag{10}$$

which is approximately -1.4 times x.

Indeed,

$$A\delta x = \delta b, \tag{11}$$

and hence

$$\delta x = A^{-1}\delta b = \begin{bmatrix} -557 & 842 & -284 \\ 610 & -922 & 311 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} -0.001 \\ 0.001 \\ -0.001 \end{bmatrix} = \begin{bmatrix} 1.683 \\ -1.843 \\ -0.006 \end{bmatrix}. \tag{12}$$

Thus, perturbing each of the elements of b by 0.1%, we get perturbations of c240% in the larger elements of the solution x of Ax = b.