Challenge Problem:
Pushing The Limit

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MATH 1920

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Below is the limit of a multivariable function.

$$\lim_{(x,y)\to(0,0)} \left(\frac{\sin(xy)}{xy}\right)^{\left(\frac{1}{1-\cos(xy)}\right)}$$

(a) Show that along the path $y = x$ the multivariable limit below is equal to e^{α}. Find α.

(b) Given the constraint below, evaluate the limit or determine that it does not exist.

$$\lim_{\substack{(x,y)\to(0,0) \\ x\neq y}} \left(\frac{\sin(xy)}{xy}\right)^{\left(\frac{1}{1-\cos(xy)}\right)}$$

Caution: Be warned, this is one of the hardest limit problems (above challenging difficulty)! Nonetheless, it is has an interesting solution!

Solution

(a)

$$\lim_{(x,y)\to(0,0)} \left(\frac{\sin(xy)}{xy}\right)^{\left(\frac{1}{1-\cos(xy)}\right)}$$

First set $y = x$ and reduce the limit to a single variable as suggested.

$$\lim_{(x,y)\to(0,0)} \left(\frac{\sin(xy)}{xy}\right)^{\left(\frac{1}{1-\cos(xy)}\right)}$$

$$\lim_{(x,x)\to(0,0)} \left(\frac{\sin(x^2)}{x^2}\right)^{\left(\frac{1}{1-\cos(x^2)}\right)}$$

Now allow $x^2 = u$ to reduce to a single variable limit and allow for cleaner calculations.

$$\lim_{u\to0} \left(\frac{\sin(u)}{u}\right)^{\left(\frac{1}{1-\cos(u)}\right)}$$

Since this limit is an indeterminate form, we need to re-arrange this limit to some nicer form.

$$\lim_{u\to0} \left(\frac{\sin(u)}{u}\right)^{\left(\frac{1}{1-\cos(u)}\right)} = L$$

$$\lim_{u\to0} \ln\left(\frac{\sin(u)}{u}\right)^{\left(\frac{1}{1-\cos(u)}\right)} = \ln L$$

From this, we understand that our solution should be in the form of:

$$\boxed{L = e^{\lim_{u\to0} f(u)}}$$

Applying logarithm rules and L'Hospital's rule, we get the following:

$$\lim_{u\to 0}\left(\left(\frac{1}{1-\cos(u)}\right)\ln\left(\frac{\sin(u)}{u}\right)\right)$$

$$\lim_{u\to 0}\left(\frac{\ln\left(\frac{\sin(u)}{u}\right)}{1-\cos(u)}\right)$$

$$=\frac{0}{0}$$

$$=\lim_{u\to 0}\left(\frac{\frac{-1+u\cot(u)}{u}}{\sin(u)}\right)$$

$$=\lim_{u\to 0}\left(\frac{-1+u\cot(u)}{u\sin(u)}\right)$$

Note: Compute the limit in the numerator via manipulation, where:

$$u\cot(u)=\frac{u}{\frac{1}{\cot(u)}}=\lim_{u\to 0}\left(\frac{u}{\frac{1}{\cot(u)}}\right)$$

Apply L'Hopital's Rule to the numerator limit

$$\lim_{u\to 0}\left(\frac{1}{\sec^2(u)}\right)=1$$

For the entire limit:

$$=\frac{0}{0}$$

$$=\lim_{u\to 0}\left(\frac{-u\csc^2(u)+\cot(u)}{\sin(u)+u\cos(u)}\right)$$

$$=\frac{0}{0}$$

$$=\lim_{u\to 0}\left(\frac{2u\csc^2(u)\cot(u)-2\csc^2(u)}{2\cos(u)-u\sin(u)}\right)$$

Note:

$$2u\csc^2(u)\cot(u)-2\csc^2(u)=2u\csc^2(u)\cot(u)\left(1-\frac{2\csc^2(u)}{2u\csc^2(u)\cot(u)}\right)$$

$$\lim_{u\to 0}\left(\frac{2u\csc^2(u)\cot(u)\left(1-\frac{2\csc^2(u)}{2u\csc^2(u)\cot(u)}\right)}{2\cos(u)-u\sin(u)}\right)$$

Simplify.

$$\frac{2u\csc^2(u)\cot(u)\left(1-\frac{2\csc^2(u)}{2u\csc^2(u)\cot(u)}\right)}{2\cos(u)-u\sin(u)}=\frac{2\csc^2(u)(u\cot(u)-1)}{2\cos(u)-u\sin(u)}$$

$$= 2 \cdot \lim_{u \to 0} \left(\frac{\csc^2(u)(u\cot(u) - 1)}{2\cos(u) - u\sin(u)} \right)$$

$$= 2 \cdot \frac{\lim_{u\to 0} \left(\csc^2(u)(u\cot(u) - 1)\right)}{\lim_{u\to 0}(2\cos(u) - u\sin(u))}$$

Noting that:

$$\lim_{u \to 0} \left(\csc^2(u)(u\cot(u) - 1)\right) = -\frac{1}{3}$$

$$\lim_{u \to 0} (2\cos(u) - u\sin(u)) = 2$$

Then,

$$= 2 \cdot \frac{-\frac{1}{3}}{2} = \frac{-1}{3} = \ln(y)$$

Thus,

$$L = e^{\frac{-1}{3}}$$

$$\boxed{\alpha = \frac{-1}{3}}$$

Note: For this limit below, express as cosines and sines, apply L'H Rule twice and solve the limit

$$\lim_{u \to 0} \left(\csc^2(u)(u\cot(u) - 1)\right) = -\frac{1}{3}$$

$$= \lim_{u \to 0} \left(\frac{u\cos(u) - \sin(u)}{\sin^3(u)} \right)$$

Apply L'Hopital's Rule

$$= \lim_{u \to 0} \left(\frac{-u\sin(u)}{3\sin^2(u)\cos(u)} \right)$$

$$= \lim_{u \to 0} \left(-\frac{2u}{3\sin(2u)} \right)$$

Apply L'Hopital's Rule

$$= \lim_{u \to 0} \left(\frac{-2}{6\cos(2u)} \right)$$

(b)
To evaluate this limit, we must change into polar coordinates (any other option would be tedious and impossibly difficult). We are also told that $y \neq x$, so we cannot use this path to determine the value of the limit (the path is blocked). However, we have the following through polar coordinates.

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \left(\frac{\sin(xy)}{xy} \right)^{\left(\frac{1}{1-\cos(xy)}\right)}$$

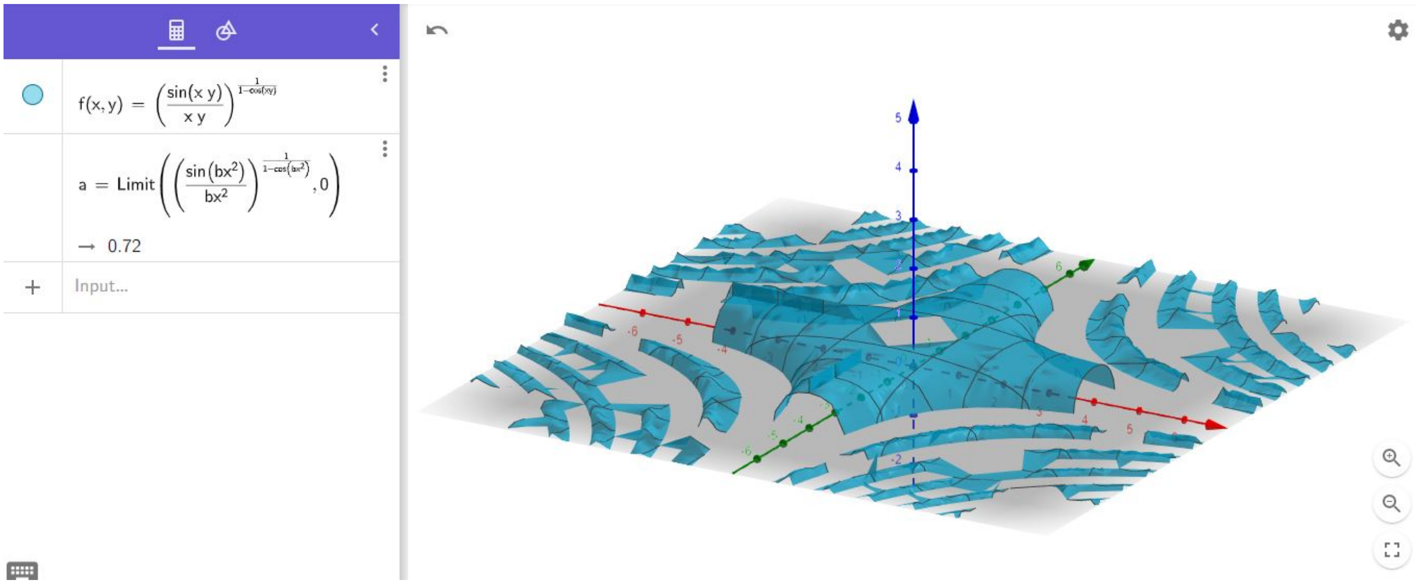
$$\implies \lim_{r \to 0} \left(\frac{\sin\left(r^2\sin(\theta)\cos(\theta)\right)}{r^2\cos(\theta)\sin(\theta)} \right)^{\frac{1}{1-\cos\left(r^2\cos(\theta)\sin(\theta)\right)}}$$

Since $\cos\theta$ and $\sin\theta$ range from -1 to 1, we can set their product equal to a constant b, which reduces the arithmetic.

$$b = (\sin(\theta)\cos(\theta)) \quad b \in [-1, 1]$$

$$\lim_{r \to 0} \left(\frac{\sin\left(r^2 b\right)}{r^2 b} \right)^{\frac{1}{1-\cos\left(r^2 b\right)}}$$

Following the same approach as above, we arrive at the same exact value in polar coordinates, $e^{\frac{-1}{3}}$. Therefore, the limit does exist. Below is a graph of the function and the limit value with GeoGebra. Observe that the limit does not depend on a specific value of b.



The limit is confirmed by Wolfram-Alpha as well.