AEW Worksheet 11
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MATH 1920

Name: ______________________

Collaborators: ______________________

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1

Determine if the following statements are true(T) or false(F). Mark the correct answer. No justification needed.

(a) T F If S is the unit sphere $x^2 + y^2 + z^2 = 1$ and a, b, c are real numbers, then $\iint_S |ax + by + cz| \, dS \geq 0$
True. The absolute value function is always greater than or equal to zero. Note that surface area cannot be negative (i.e., integrand equals 1). If your curious, the integral evaluates to $2\pi\sqrt{a^2 + b^2 + c^2}$ solution

(b) T F If $S = \{(x, y, z) : f(x, y, z) = k\}$ is a level surface of a smooth function f with no critical points on S, then S must be orientable.
True. Take $\bar{n} = \frac{\nabla f}{\|\nabla f\|}$ which is a continuous normal vector since $\nabla f \perp S$

(c) T F If $F_Z(z) = P(X + Y \leq z) = \int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{z-y} f_{XY}(x, y) dx dy$, then $f_Z(z) = \int_{-\infty}^{+\infty} \left(\frac{\partial}{\partial z} \int_{-\infty}^{z-y} f_{XY}(x, y) dx \right) dy$ assume $Z = X + Y$ and the region $D_z : x + y \leq z$ is shaded.
True! This is tricky! Recall Leibnitz's differentiation rule from calculus 1 where $H(z) = \int_{a(z)}^{b(z)} h(x, z) dx$ then $\frac{dH(z)}{dz} = \frac{db(z)}{dz} h(b(z), z) - \frac{da(z)}{dz} h(a(z), z) + \int_{a(z)}^{b(z)} \frac{\partial h(x,z)}{\partial z} dx$. In our situation,we have the following $f_Z(z) = \int_{-\infty}^{+\infty} \left(f_{XY}(z - y, y) - 0 + \int_{-\infty}^{z-y} \frac{\partial f_{XY}(x,y)}{\partial z} dx \right) dy$

2

Suppose R is a positive real number. Let S be the cone given by the equation $z = \sqrt{x^2 + y^2}$ with $0 \leq z \leq R$, oriented downward. Compute the flux of $G = \langle xz, yz, xy \rangle$ across S.

Solution

Warning (Future Reference): You cannot use the divergence theorem here unless you also compute the flux integral across the "missing" top of our cone. We parameterize the cone by

$$\mathbf{r}(u, v) = \langle u \cos(v), u \sin(v), u \rangle$$

for $0 \leq v \leq 2\pi$ and $0 \leq u \leq R$. We compute

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \cos(v), \sin(v), 1 \rangle \times \langle -u \sin(v), u \cos(v), 0 \rangle = \langle u \cos(v), u \sin(v), -u \rangle$$

This is oriented down (e.g., evaluate at $u = v = 0$ to get $-\vec{k}$), hence correctly. We compute

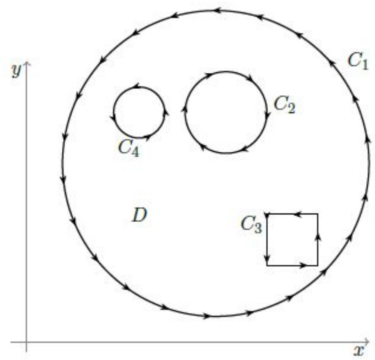
$$\begin{aligned}
\iint_S \mathbf{G} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^R \left\langle u^2 \cos(v), u^2 \sin(v), u^2 \sin(v) \cos(v) \right\rangle \cdot \langle u \cos(v), u \sin(v), -u \rangle du dv \\
&= \int_0^{2\pi} \int_0^R u^3 (1 - \sin(v) \cos(v)) du dv \\
&= R^4/4 \int_0^{2\pi} (1 - \sin(v) \cos(v)) dv \\
&= \boxed{R^4 \pi/2}
\end{aligned}$$

3

Suppose that D is the bounded region in the plane that has boundary given by the oriented simple closed piecewise smooth curves C_1, C_2, C_3, and C_4 as in the picture. Suppose $\mathbf{F} = \langle P, Q, 0 \rangle : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field and P and Q have continuous partial derivatives on $\mathbb{R}^2$. If the following is true,

$$\oint_{C_k} \mathbf{F} \cdot d\mathbf{r} = 2^k$$

find $\iint_D (Q_x - P_y)\, dA = \iint_D (\nabla \times \mathbf{F}) \cdot \hat{k} dA$



Solution

Apply Green's Theorem (or Stokes' Theorem both work here). The surface D is oriented upwards (this is the implicit assumption in Green's Theorem). Using the right foot rule (right foot on the curve and domain should be on your left), we see that the the given orientations of C_3 and C_4 are opposite to those required by Green's Theorem, while the remaining curves are oriented correctly for Green's Therorem. Thus,

$$\begin{aligned}
\iint_D (Q_x - P_y)\, dA &= \iint_D (\nabla \times \mathbf{F}) \cdot \hat{k} dA \\
&= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} - \int_{C_3} \mathbf{F} \cdot d\mathbf{r} - \int_{C_4} \mathbf{F} \cdot d\mathbf{r} \\
&= 2 + 4 - 8 - 16 \\
&= \boxed{-18}
\end{aligned}$$

4

Define $\mathbf{G} = \left\langle 2zxe^{x^2-y^2}, -2zye^{x^2-y^2}, e^{x^2-y^2} + 2z \right\rangle$, $\mathbf{H} = \langle 0, x, -y \rangle$ and $\mathbf{F} = \mathbf{G} + \mathbf{H}$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the line segment from (1,2,4) to (-1,1,1).

Solution

Approach: Notice that we cannot use Green's theorem! Thus, we must calculate the line integrals for G and H separately. Use a different method for each integral. The vector field G is conservative. We look for a potential:

$$f_x = 2zxe^{x^2-y^2} \Rightarrow f = ze^{x^2-y^2} + g(y,z) \Rightarrow f_y = -2zye^{x^2-y^2} + g_y(y,z)$$

Then $g_y = 0$ giving $g(y,z) = h(z)$, so

$$f = ze^{x^2-y^2} + h(z) \Rightarrow f_z = e^{x^2-y^2} + h'(z)$$

Then $h'(z) = 2z$ giving $h = z^2 + c$, where c is a constant. A potential for G is

$$f(x,y,z) = ze^{x^2-y^2} + z^2$$

By the fundamental theorem of line integrals

$$\int_C \mathbf{G} \cdot d\mathbf{r} = f(-1,1,1) - f(1,2,4) = -14 - 4e^{-3}$$

For **H** we evaluate the integral directly. A parametrization of C

$$\begin{aligned} \mathbf{r}(t) &= \langle 1,2,4\rangle + t\langle -2,-1,-3\rangle \\ &= \langle 1-2t, 2-t, 4-3t\rangle \quad t \in [0,1] \end{aligned}$$

$$\begin{aligned} \int_C \mathbf{H} \cdot d\mathbf{r} &= \int_0^1 \langle 0, 1-2t, -2+t\rangle \cdot \langle -2,-1,-3\rangle dt = \int_0^1 5 - t dt \\ &= 5 - \frac{1}{2} \end{aligned}$$

Thus,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -9 - \frac{1}{2} - 4e^{-3} = \boxed{-\frac{19}{2} - 4e^{-3}}$$

5 (Challenge)

Prove that for the vector field,

$$\mathbf{F} = \langle my + g(x,y), nx + h(x,y)\rangle$$

and the positively oriented curve C around any isosceles right triangle (that is, right triangle having legs of equal length), the following must be true,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = a^2$$

where a is the length of the legs of the triangle. That is, show that there is a condition for which the statement above holds true, provided we have an isosceles right triangle and the given vector field. Note that you should not pick a specific triangle or value of a when completing this problem. Likewise, do not pick specific values of $g(x,y)$ and $h(x,y)$ (neither m nor n).

Solution

Recall Green's Theorem for area, we need that $\mathbf{F} = \langle F_1, F_2\rangle$ such that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$. In other words,

$$\vec{\nabla} \times \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = C = 1$$

$$\iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_{\mathcal{D}} 1 dA = \text{area}(\mathcal{D})$$

For this problem, we must take a similar approach. For a triangle, in general, $A = \frac{1}{2}bh$. Using our intuition about Green's Theorem for area, we need the following equal to some constant C.

$$\vec{\nabla} \times \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = C$$

Computing values we have that

$$\begin{aligned} \frac{\partial F_1}{\partial y} &= m + g_y(x,y) \\ \frac{\partial F_2}{\partial x} &= n + h_x(x,y) \end{aligned}$$

Subtracting values and rearranging provides us,

$$\vec{\nabla} \times \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$= (n - m) + (h_x(x, y) - g_y(x, y)) = \mathcal{K}_1 + \mathcal{K}_2 = C$$

Therefore $h_x(x, y) - g_y(x, y)$ and $(n - m)$, must equal some constants (e.g $\mathcal{K}_1$ or $\mathcal{K}_2$, respectively) for this to be true. Since we have an isosceles right triangle, then $A = \frac{1}{2}bh = \frac{1}{2}a^2$. From this equation, $2A = a^2$, which is equal to our desired line integral. Thus to satisfy this equation, we must have that $(h_x(x, y) - g_y(x, y)) = \mathcal{K}_1$, $(n - m) = \mathcal{K}_2$, and $C = 2$. More formally, **the entire integrand must equal 2**.

$$2A = 2\,\text{area}(\mathcal{D}) = 2\iint_{\mathcal{D}} 1\,dA = \iint_{\mathcal{D}} 2\,dA = a^2$$

$$= \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \vec{\nabla} \times \vec{F}\,dA = \iint_R 2\,dA$$

Here, the condition is that the integrand must be equal to 2. If the integrand was not equal to 2, this condition would not hold for an isosceles right triangle and the given vector field. From these facts, the integral $\oint_C \mathrm{F} \cdot \mathrm{dr}$ is twice the area of the triangle, which is $A = \frac{1}{2}a^2$. This shows that our desired result:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = a^2$$

Example: As there are many possible values for $h(x, y)$ and $g(x, y)$, below is a possible option with consideration to partial derivatives.

$$\mathcal{K}_1 = h_x(x, y) - g_y(x, y) = 0$$

$$\mathcal{K}_2 = (n - m) = 4 - 2 = 2$$

$$\mathcal{K}_1 + \mathcal{K}_2 = C = 2$$

$$h_x(x, y) = g_y(x, y) = 3xy$$

$$g(x, y) = \frac{3}{2}xy^2$$

$$h(x, y) = \frac{3}{2}x^2y$$

Thus,

$$\mathbf{F} = \left\langle 2y + \frac{3}{2}xy^2, 4x + \frac{3}{2}x^2y \right\rangle$$

By Green's Theorem, we know that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dA$$

Here we have $\frac{\partial F_1}{\partial y} = 2 + 3xy$ and $\frac{\partial F_2}{\partial x} = 4 + 3xy$, so $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2$ such that $2A = a^2$

6 (Challenge)

Evaluate the following line integral,

$$\int_{\mathcal{C}} (e^{\sinh(\ln(\ln(-x^{2020} \cdot e^{-x^{2021}} + 2022)))} + y)dx + (3x + y)dy$$

on the non-closed path C connecting $M(0, 0)$ to $N(2, 2)$, then to $P(2, 4)$, and then to $Q(0, 6)$.

Solution

The path is not closed nor is it path independent. However, we must close the path in order to use Green's Theorem ($F_1 = P, F_2 = Q$). Therefore,

$$\int_C \mathbf{F}\cdot d\mathbf{r} + \int_{C_1} \mathbf{F}\cdot d\mathbf{r}_1 = \iint_R (Q_x - P_y)\, dA$$

where C_1 is the path from (0,6) back to the origin. Re-arranging we have,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_R (Q_x - P_y)\, dA - \int_{C_1} \mathbf{F}\cdot d\mathbf{r}_1$$

Thus,

$$\begin{aligned}\iint_R (Q_x - P_y)\, dA &= \iint_R 2dA = 2\iint_R dA = 2(\text{Area of } R)\\ &= 2(2+2+4) = 16\end{aligned}$$

As for the path C_1,

$$\begin{matrix} x = 0 \\ y = -t \end{matrix} \quad -6 \le t \le 0 \quad \mathbf{r}(t) = \langle 0, -t\rangle$$

$$d\mathbf{r} = \langle \mathbf{0}, -1\rangle dt$$

$$\mathbf{F} = \langle e^{\sinh(\ln(\ln(-x^{2020}\cdot e^{-x^{2021}}+2022)))} + y, 3x + y\rangle$$

$$\mathbf{F}_{\text{on } c_1} = \langle[\text{Something}], -t\rangle$$

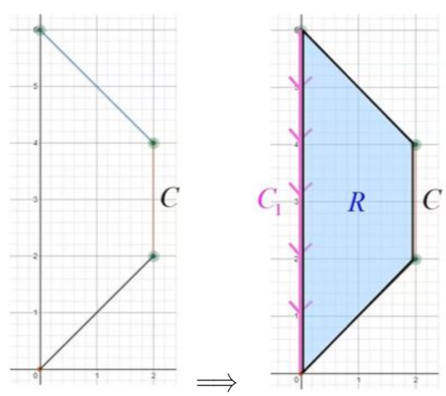
[Something] is immaterial since the position vector will cancel it out.

$$\int_{C_1} F\cdot d\mathbf{r}_1 = \int_{-6}^{0} t dt = \left[\frac{t^2}{2}\right]_{-6}^{0} = [0 - 18] = -18$$

Therefore,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_R (Q_x - P_y)\, dA - \int_{C_1} \mathbf{F}\cdot d\mathbf{r}_1 = 16 - (-18) = \boxed{34}$$

$$\Longrightarrow$$



Very rarely problems will require closing off a curve or replacing a closed curve - be prepared to tackle both types.