AEW Worksheet 9
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MATH 1920

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1

Determine if the following statements are true(T) or false(F). Mark the correct answer. No justification needed.

(a) $\boxed{\text{T}}$ $\boxed{\text{F}}$ $\lim_{m,n\to\infty}\left(\left(\frac{1}{n}\right)^{\alpha}+\left(\frac{2}{n}\right)^{\alpha}+\cdots+\left(\frac{n}{n}\right)^{\alpha}\right)\frac{1}{n}\cdot\left(\left(\frac{1}{m}\right)^{\beta+1}+\left(\frac{2}{m}\right)^{\beta+1}+\cdots+\left(\frac{m}{m}\right)^{\beta+1}\right)\frac{1}{m}=\int_0^1\int_0^1 x^{\alpha}y^{\beta+1}dxdy$

True! This is tricky! This is the multiple integral definition as the limit of a Riemann sum so that $\frac{b-a}{n}=\frac{1-0}{n}=\Delta x$ and $\frac{d-c}{m}=\frac{1-0}{m}=\Delta y$. This implies that $A=\Delta x\cdot\Delta y=\frac{1}{n}\frac{1}{m}$. In this case, we can apply Fubini's theorem for separable multi-variable functions given the constant boundaries and independent functions (e.g., $f(x,y)=g(x)\cdot h(y)$ implying $\int_0^1 x^{\alpha}dx\cdot\int_0^1 y^{\beta+1}dy=\int_0^1\int_0^1 x^{\alpha}y^{\beta+1}dxdy$).

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{\alpha}\frac{1}{n}\cdot\lim_{m\to\infty}\sum_{j=1}^{m}\left(\frac{j}{m}\right)^{\beta+1}\frac{1}{m}=\lim_{n,m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{k}{n}\right)^{\alpha}\left(\frac{j}{m}\right)^{\beta+1}\Delta x\Delta y=\int_0^1\int_0^1 x^{\alpha}y^{\beta+1}dxdy$$

Note: The limit does exist and $x_k=0+k\Delta x=\frac{k}{n}$ and $y_j=0+j\Delta y=\frac{j}{m}$ see https://youtu.be/N6y_UJrZLAE

(b) $\boxed{\text{T}}$ $\boxed{\text{F}}$ $\int_{y=1}^{4}\int_{x=0}^{1}\left(x^2+\sqrt{y}\right)\sin\left(x^2y^2\right)dxdy\le 9$

True. In general if $f(x,y)\le K$ and a domain D has area A, $\iint_D f(x,y)dxdy\le K\cdot A$. Here, the domain is a rectangle with area 3 , so the trick is to show that $(x^2+\sqrt{y})\sin\left(x^2y^2\right)\le 3$ for all (x,y) in the rectangle.

(c) $\boxed{\text{T}}$ $\boxed{\text{F}}$ $\int_{r=-1}^{1}\int_{\theta=0}^{1}e^{r^2+\theta^2}d\theta dr=\left[\int_{r=-1}^{1}e^{r^2}dr\right]\left[\int_{\theta=0}^{1}e^{\theta^2}d\theta\right]$

True. See Fubini's theorem.

(d) $\boxed{\text{T}}$ $\boxed{\text{F}}$ The integral $\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{\rho=0}^{1}\rho^2\sin\theta\, d\rho d\theta d\phi$ gives the volume of 1/4 of a sphere.

False. This problem can be completed without evaluating the integral. The bounds and integrand should suggest the answer is false. The $\sin\theta$ should be $\sin\phi$, and even with this change it only gives the volume of 1/8 of a sphere.

2

(a) Evaluate

$$\iint_D\frac{1}{(x^2+y^2)^{n/2}}dA,$$

where n is an integer and D is the region bounded by the circles with center the origin and radii r and R, $0<r<R$.

(b) For what values of n does the integral have a limit as $r\to 0^+$?

(c) Find

$$\iiint_E\frac{1}{(x^2+y^2+z^2)^{n/2}}dV,$$

where E is the region bounded by the spheres with center the origin and radii r and R, $O<r<R$

(d) For what values of n does the integral in part (b) have a limit as $r\to 0^+$?

Solution

(a)

$$\iint_D \frac{1}{(x^2+y^2)^{n/2}}\,dA = \int_0^{2\pi}\int_r^R \frac{1}{(t^2)^{n/2}}\,t\,dt\,d\theta = 2\pi\int_r^R t^{1-n}\,dt$$
$$= \begin{cases} \frac{2\pi}{2-n}t^{2-n}\Big]_r^R = \frac{2\pi}{2-n}\left(R^{2-n}-r^{2-n}\right) & \text{if } n\neq 2 \\ 2\pi\ln(R/r) & \text{if } n=2 \end{cases}$$

(b)
The integral in part (a) has a limit as $r\to 0^+$ for all values of n such that $2-n>0 \Leftrightarrow n<2$.
(c)

$$\iiint_E \frac{1}{(x^2+y^2+z^2)^{n/2}}\,dV = \int_r^R\int_0^{\pi}\int_0^{2\pi} \frac{1}{(\rho^2)^{n/2}}\rho^2\sin\phi\,d\theta\,d\phi\,d\rho = 2\pi\int_r^R\int_0^{\pi} \rho^{2-n}\sin\phi\,d\phi\,d\rho$$

$$= \begin{cases} \frac{4\pi}{3-n}\rho^{3-n}\Big]_r^R = \frac{4\pi}{3-n}\left(R^{3-n}-r^{3-n}\right) & \text{if } n\neq 3 \\ 4\pi\ln(R/r) & \text{if } n=3 \end{cases}$$

(d)
As $r\to 0^+$, the above integral has a limit, provided that $3-n>0 \Leftrightarrow n<3$

3

Given the triple integrals below:

$$\pi\int_0^1\int_{\sqrt[3]{z}}^1\int_0^{\ln 3} \frac{\pi e^{2x}\sin\left(\pi y^2\right)}{y^2}\,dx\,dy\,dz = 2a+b$$
$$\int_b^a\int_0^{\pi/4}\int_0^{\sec\phi} (\rho\cos\phi)\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta = \frac{\pi}{4}$$

Find the values of a and b.

Solution

For the first integral:

$$\int_0^1\int_{\sqrt[3]{z}}^1\int_0^{\ln 3} \frac{\pi e^{2x}\sin\pi y^2}{y^2}\,dx\,dy\,dz = \int_0^1\int_{\sqrt[3]{z}}^1 \frac{4\pi\sin\left(\pi y^2\right)}{y^2}\,dy\,dz$$

We must change the order of integration:

$$= \int_0^1\int_0^{y^3} \frac{4\pi\sin\left(\pi y^2\right)}{y^2}\,dz\,dy$$

$$= \int_0^1 4\pi y\sin\left(\pi y^2\right)\,dy = \left[-2\cos\left(\pi y^2\right)\right]_0^1 = -2(-1)+2(1) = 4$$
$$\int_0^1\int_{\sqrt[3]{z}}^1\int_0^{\ln 3} \frac{\pi e^{2x}\sin\left(\pi y^2\right)}{y^2}\,dx\,dy\,dz = 4$$

Thus

$$4\pi = 2a+b$$

For the second integral:

$$\int_b^a \int_0^{\pi/4} \int_0^{\sec\phi} \rho^3 \sin\phi\cos\phi d\rho d\phi d\theta = \frac{1}{4}\int_b^a \int_0^{\pi/4} \tan\phi \sec^2\phi d\phi d\theta =$$

$$= \frac{1}{4}\int_b^a \left[\frac{1}{2}\tan^2\phi\right]_0^{\pi/4} d\theta$$

$$= \frac{1}{8}\int_b^a d\theta = \frac{a-b}{8}$$

$$\int_b^a \int_0^{\frac{\pi}{4}} \int_0^{\sec(\phi)} \rho\cos(\phi)\rho^2\sin(\phi) d\rho d\phi d\theta = \frac{a-b}{8}$$

Thus

$$\frac{a-b}{8} = \frac{\pi}{4}$$

So we have

$$\frac{a-b}{8} = \frac{\pi}{4}$$

and

$$2a + b = 4\pi$$

A system of equations:

$$\begin{bmatrix} 2a + b = 4\pi \\ \frac{a-b}{8} = \frac{\pi}{4} \end{bmatrix}$$

Solving we get

$$\boxed{a = 2\pi, b = 0}$$

4

Radium 223 decays with a half-life of 11.43 days; Radium 224, with a half life of 3.632 days As a result, the probability that an atom of Radium 223 will decay at a time x days has a density function $p(x) = me^{-mx}$, where $m = 0.06064$ and the probability that an atom of Radium 224 will decay at a time y days has a density function $q(y) = ne^{-ny}$, where $n = 0.1908$

(a) Assuming that the decay times of the two atoms is independent, find the probability that an atom of Radium 223 will decay before an atom of Radium 224.

Solution

We know that the joint distribution function, assuming that the decay of the two substances is independent, is $P(x, y) = p(x) \cdot q(y)$ The probability that the first substance decays first is given by the case $x < y$. Thus the first substance decays first on the (x, y) domain $0 \leq y < \infty, 0 \leq x < y$. The probability of this occurring is then:

$$\int_0^\infty \int_0^y mne^{-(mx+ny)} dx dy$$

$$= \int_0^\infty - ne^{-(mx+ny)}\Big|_{x=0}^{x=y} dy$$

$$= \int_0^\infty -n\left(e^{-(m+n)y} - e^{-ny}\right) dy$$

$$= \lim_{a\to\infty}\left(\frac{n}{m+n}e^{-(m+n)y} - e^{-ny}\right)\Big|_{y=0}^{y=a}$$

$$= 1 - \frac{n}{m+n} = \frac{m}{m+n} = \boxed{0.2411}$$

So there is an approximately 24 chance that an atom of Radium 223 will decay before an atom of Radium 224.