AEW Worksheet 4

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MATH 1920

Name: ______________________

Collaborators: ______________________

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1

Determine if the following statements are true(T) or false(F). Mark the correct answer. No justification needed.

(a) T **F** If there is no solution t to the equation $u_0 + t\mathbf{u} = v_0 + t\mathbf{v}$ then the lines given by $\{u_0 + t\mathbf{u} : t \in \mathbb{R}\}$ and $\{v_0 + t\mathbf{v} : t \in \mathbb{R}\}$ do not intersect.
False. This just means that the two particles moving along the curves do not collide. They might hit the same spot at different times.

(b) **T** F For any line in $\mathbb{R}^3$ and a point not on that line, there is exactly one plane that is normal to the line and contains the point.
True. If the plane is described by $n \cdot x = k$, then the plane containing the point p is given by $n \cdot x = n \cdot p$

(c) T **F** If $|\mathbf{r}(t)| = 1$ for all t, then $|\mathbf{r}'(t)|$ is constant.
False. Geometrically, the claim is that if you are moving on the surface of a sphere then your speed has to be constant. False since you could run on the surface at any speed you like.

2

Sketch and describe the surface given below.

(a) $\rho^2 - 3\rho + 2 = 0$

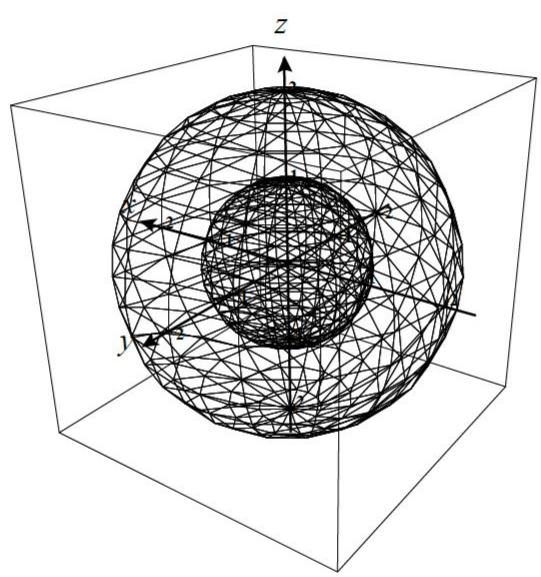
Solution

As suggested by the equation's format, we factor out ρ similar to that of a quadratic equation. Proceeding from this, we solve for two possible real values that can satisfy the equation and provide us potential surfaces.

$$(\rho - 1)(\rho - 2) = 0$$

$$\rho = 1 \text{ or } \rho = 2$$

From these values we have two possible surfaces that we must sketch:
$\rho = 1$ is a sphere with radius 1 and centered at origin
$\rho = 2$ is a sphere with radius 2 and centered at origin



There are many possible ways to sketch this such as cross-sections, overlap between shapes, etc.. The important part is to illustrate the $x, y,$ and z axes accordingly. The surface can be described as a pair of two spheres.

3

Explain in words the difference between colliding and intersecting for vector-valued functions.

(b) Collide -

(b) Intersect -

Solution

For the particles to collide:

(a) the curves along which they move must intersect

(b) the particles arrive at the intersection points at the same time

(c) if two particles collide, they must intersect

Essentially we are asking ourselves if there is some t value for which the particles are at the same point. When doing calculations, verify that the specific t value satisfies all the given equations.

For the particles to intersect:

(a) the particles merely touch at some value

(b) if two particles intersect, they don't necessarily collide (they might each pass through different points at different times)

If the particles do not collide, then assign one of the vector functions a value s and check if they collide accordingly with the other function's value of t.

4

Let E be the surface $x^2/9 + y^2/4 + z^2 = 1$, P be the plane $z = Ax + By$, and C be the intersection of E and P.

(a) Is C an ellipse for all values of A and B? Explain.

(b) Sketch and interpret the situation in which $A = 0$ and $B \neq 0$

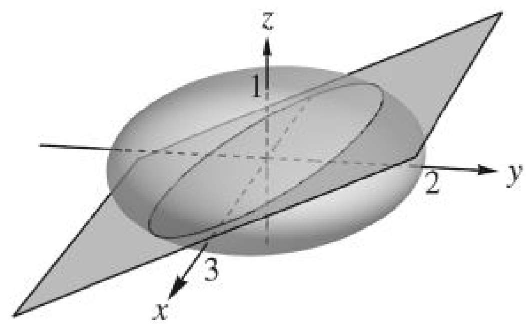
(c) Find an equation of the projection of C on the xy -plane.

(d) Assume $A = \frac{1}{6}$ and $B = \frac{1}{2}$. Find a parametric description of C as a curve in $\mathbb{R}^3$ (Hint: Assume C is described by $\langle a\cos t + b\sin t, c\cos t + d\sin t, e\cos t + f\sin t\rangle$ therefore find $a, b, c, d, e,$ and f)

Solution

(a) Yes. The ellipsoid E is centered at the origin which is on the plane P, so the intersection of E and P is an ellipse in the plane P.

(b) In this case one of the axes of symmetry for the ellipse C is the x -axis.



(c) Because $z = Ax + By$, any point (x, y, z) on C satisfies $\frac{x^2}{9} + \frac{y^2}{4} + (Ax + By)^2 = 1$, which gives the equation of the projection of C on the xy -plane.

(d) The equation:

$$\frac{x^2}{9} + \frac{y^2}{4} + \left(\frac{x}{6} + \frac{y}{2}\right)^2 = 1,$$

can be transformed as follows:

$$\frac{x^2}{9} + \frac{y^2}{4} = \left(\frac{x}{3} + \frac{y}{2}\right)^2 - \frac{xy}{3}$$

Since $A^2 + B^2 = (A + B)^2 - 2AB$

$$\left(\frac{x}{6} + \frac{y}{2}\right)^2 = \left(\frac{x}{6} - \frac{y}{2}\right)^2 + \frac{xy}{3}$$

Therefore:

$$\left(\frac{x}{3} + \frac{y}{2}\right)^2 + \left(\frac{x}{6} - \frac{y}{2}\right)^2 = 1$$

So the parameterization is:

$$\frac{x}{3} + \frac{y}{2} = \cos t$$

$$\frac{x}{6} - \frac{y}{2} = \sin t$$

So we conclude that:

$$x = 2\cos t + 2\sin t$$

$$y = \frac{2}{3}\cos t - \frac{4}{3}\sin t$$

$$z = \frac{x}{6} + \frac{y}{2} = \frac{2}{3}\cos t - \frac{1}{3}\sin t$$

All of these come from fraction manipulation and explicitly solving for x and y in both equations.

5

Prove that the following equations below is true.

$$\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

Solution

$$u \times v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$u \times v = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

$$\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) =$$

$$\frac{d}{dt} \langle u_2v_3 - v_2u_3, u_3v_1 - v_3u_1, u_1v_2 - u_2v_1 \rangle$$

$$= \langle u_2'v_3 + u_2v_3' - (v_2'u_3 + v_2u_3'), u_3'v_1 + u_3v_1' - (u_1'v_3 + u_1v_3'), u_1'v_2 + u_1v_2' - (v_1'u_2 + v_1u_2') \rangle$$

$$= \langle u_2'v_3 - v_2u_3' + u_2v_3' - v_2'u_3, u_3v_1' - u_1v_3' + u_3'v_1 - u_1'v_3, u_1'v_2 - v_1u_2' + u_1v_2' - v_1'u_2 \rangle$$

$$= \langle (u_2'v_3 - v_2u_3'), (u_3v_1' - u_1v_3'), (u_1'v_2 - v_1u_2') \rangle + \langle (u_2v_3' - v_2'u_3), (u_3'v_1 - u_1'v_3), (u_1v_2' - v_1'u_2) \rangle$$

$$= (\mathbf{u}'(t) \times \mathbf{v}(t)) + (\mathbf{u}(t) \times \mathbf{v}'(t))$$