AEW T/F
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Name: ______________________________

Collaborators: ______________________________

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1 Miscellaneous

Determine if the following statements are true(T) or false(F). Mark the correct answer. No justification needed.

(a) T **F** $\lim_{\rho\to 0} \frac{\rho\sin(\phi)\cos(\theta)\rho\sin(\phi)\sin(\theta)\rho\cos(\phi)}{\rho^2}$ in spherical coordinates does not exist.
False. The limit does exist and approaches a limiting value of 0. Notice the above limit is equivalent to

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = \lim_{\rho\to 0} \frac{\rho\sin(\phi)\cos(\theta)\rho\sin(\phi)\sin(\theta)\rho\cos(\phi)}{\rho^2} = \lim_{\rho\to 0} \frac{\rho^3}{\rho^2}\sin^2(\phi)\cos(\phi)\cos(\theta)\sin(\theta)$$

The reason the limit only goes to 0 for 3 dimensions and up is that the top will be proportional to ρ^n (where n is the number of dimensions) while the bottom (sort of by definition) is proportional to ρ^2. The limit is only guaranteed to go to 0 when $\rho \to 0$ if $n > 2$. It should be noted that the trig-expression is bounded and cannot approach infinity based on squeeze theorem. https://www.youtube.com/watch?v=r81_zDPeylE

(b) **T** F If $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, then $\lim_{x\to 0} f(x,0) = 0$.
True. The condition establishes that regardless of your path the limit is 0 , so it's easy to see that the path $y = 0$, $x \to 0$ is also 0 , so true.

(c) T **F** If $\lim_{x\to 0} f(x,0) = 0$, and $\lim_{y\to 0} f(0,y) = 0$, then $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.
False. The condition is giving just particular paths not all the paths, so you cannot say that the limit exists, so false.

(d) **T** F If $\lim_{(x,y)\to(a,b)} f(x,y) = L$ where $a, b \neq 0$, then $x = a + r\cos\theta$ and $y = b + r\sin\theta$
True. If the multi-variable limit is not approaching the origin, one can still use polar coordinates! This requires shifting the bounds for x and y based on values a and b respectively (see "multivariable Limits [example 4, extra]" for an example).

2 Chapter 13 VECTOR GEOMETRY

Determine if the following statements are true(T) or false(F). Mark the correct answer. No justification needed.

(a) **T** F Suppose a vector v is defined as $v = \langle a_2 - a_1, b_2 - b_1\rangle$, then the slope is given by $\frac{b_2-b_1}{a_2-a_1}$ where a and b are non-zero constants.
True. Since the slope is defined as the change in "y" divided by the change in "x", we can use the vector components. Recall vectors have both magnitude and direction.

(b) T **F** For any vectors u and v in $\mathbb{R}^n$, $|u+v| = |u| + |v|$.
False, unless the vectors are pointing in the same direction.

(c) **T** F For any vectors u and v in $\mathbb{R}^n$, $|u+v| \leq |u| + |v|$.
True. It's called the Triangle Inequality, as any side of a triangle is shorter than the sum of the lengths of the other two sides

(d) **T** F For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$.
True. Visual intuition: up to sign, yes, because (with the zero vector) the triple product describes the volume of a parallelepiped.

(e) T **F** For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\times\mathbf{w}$
False. For example, $i \times (i \times j) = i \times k = -j$, but $(i \times i) \times j = 0 \times j = 0$.

3 Chapter 14 CALCULUS OF VECTOR-VALUED FUNCTIONS

Determine if the following statements are true(T) or false(F). Mark the correct answer. No justification needed.

(a) $\boxed{T}$ $\boxed{F}$ If there is no solution t to the equation $u_0 + t\mathbf{u} = v_0 + t\mathbf{v}$ then the lines given by $\{u_0 + t\mathbf{u} : t \in \mathbb{R}\}$ and $\{v_0 + t\mathbf{v} : t \in \mathbb{R}\}$ do not intersect.
False. This just means that the two particles moving along the curves do not collide. They might hit the same spot at different times.

(b) $\boxed{T}$ $\boxed{F}$ For any line in $\mathbb{R}^3$ and a point not on that line, there is exactly one plane that is normal to the line and contains the point.
True. If the plane is described by $n \cdot x = k$, then the plane containing the point p is given by $n \cdot x = n \cdot p$

(c) $\boxed{T}$ $\boxed{F}$ If $|\mathbf{r}(t)| = 1$ for all t, then $|\mathbf{r}'(t)|$ is constant.
False. Geometrically, the claim is that if you are moving on the surface of a sphere then your speed has to be constant. False since you could run on the surface at any speed you like.

4 Chapter 15 DIFFERENTIATION IN SEVERAL VARIABLES

Determine if the following statements are true(T) or false(F). Mark the correct answer. No justification needed.

(a) $\boxed{T}$ $\boxed{F}$ Suppose their exist an angle of inclination ψ and $z = f(x, y)$, then $\psi = \tan^{-1}\left(\left\|\nabla f_{(a,b)}\right\| \sin\theta\right)$
False. By definition, $\tan\psi = D_{\mathbf{u}} f(P) = \left\|\nabla f_{(a,b)}\right\| \cos\theta$ which implies that $\psi = \tan^{-1}\left(\left\|\nabla f_{(a,b)}\right\| \cos\theta\right)$

(b) $\boxed{T}$ $\boxed{F}$ If (x, y) is a local minimum of a function f then f is differentiable at (x, y) and $\nabla f(x, y) = 0$.
False. For example, $f(x, y) = |x| + |y|$

(c) $\boxed{T}$ $\boxed{F}$ If x is a minimum of f given the constraints $g(x) = h(x) = 0$ then $\nabla f(x) = \lambda\nabla g(x)$ and $\nabla f(x) = \mu\nabla h(x)$ for some scalars λ and μ
False. $\nabla f(x) = \lambda\nabla g(x) + \mu\nabla h(x)$ for some λ, μ.

(d) $\boxed{T}$ $\boxed{F}$ $f_y(a, b) = \lim_{y\to b} \frac{f(a,y)-f(a,b)}{y-b}$
True (by definition).

(e) $\boxed{T}$ $\boxed{F}$ If $f(x, y) = \sin x + \sin y$ then $-\sqrt{2} \le D_u f(x, y) \le \sqrt{2}$ for all unit vectors u.
True. $|D_u f(x, y)| = |\nabla f(x, y) \cdot u| \le |\nabla f(x, y)| \cdot |u| = |\nabla f(x, y)|$

(f) $\boxed{T}$ $\boxed{F}$ If $f_x(a, b)$ and $f_y(a, b)$ both exist then f is locally linear at (a, b)
False. The mere existence of partial derivatives does not imply local linearity (see textbook). Similarly, just because two directional derivatives exists doesn't mean the function is differentiable. For example, $f(x, y) = xy^2/(x^2 + y^4)$

(g) $\boxed{T}$ $\boxed{F}$ For any unit vector $\mathbf{u}$ and any point $\mathbf{a}$, $Df_{-\mathbf{u}}(\mathbf{a}) = -Df_{\mathbf{u}}(\mathbf{a})$.
True. This is because $\nabla f \cdot (-\mathbf{u}) = -\nabla f \cdot \mathbf{u}$ at any point for any vector

(h) $\boxed{T}$ $\boxed{F}$ There exists a function f with continuous second-order partial derivatives such that $f_x(x, y) = kx + y^2$ and $f_y(x, y) = x - y^2$ for constant k.
False. Since we would have $f_{xy}(x, y) = 2y$ but $f_{yx}(x, y) = 1$ which does not satisfy Clairaut's theorem for mixed variable partial derivatives.

5 Chapter 16 MULTIPLE INTEGRATION

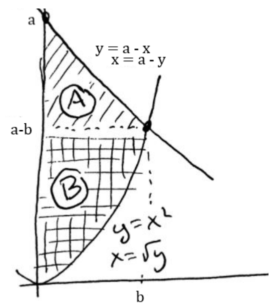
Determine if the following statements are true(T) or false(F). Mark the correct answer. No justification needed.

(a) $\boxed{T}$ $\boxed{F}$ $\int_0^1 \int_0^x \sqrt{x + y^2}\,dy\,dx = \int_0^x \int_0^1 \sqrt{x + y^2}\,dx\,dy$
False. The second integral isn't even well-defined on account of the $\int_0^x$ term! You have to be more careful when changing the limits of integration and make sure that your new limits specify the same geometric domain as the old ones.

(b) T F $\int_0^b \int_{x^2}^{a-x} f(x,y)\,dy\,dx = \int_0^{a-b} \int_0^{\sqrt{y}} f(x,y)\,dx\,dy + \int_{a-b}^{a} \int_0^{a-y} f(x,y)\,dx\,dy$ (assume $a > b$)

True. In this case, if we change the order of integration, we will need to split the region into two parts (i.e bounding curve changes. In the graph below where B represents the first integral and A the second integral).

$$\underbrace{\int_0^{a-b} \int_0^{\sqrt{y}} f(x,y)\,dx\,dy}_{(B)} + \underbrace{\int_{a-b}^{a} \int_0^{a-y} f(x,y)\,dx\,dy}_{(A)}$$



(c) T F If $f(x,y) = g(x)h(y)$, then $\iint_D f(x,y)\,dA = \left(\iint_D g(x)\,dA\right)\left(\iint_D h(y)\,dA\right)$

FALSE! you can split the integral as a product of two single-variable integrals IF the integral is over a rectangle

(d) T F $\lim_{m,n\to\infty} \left(\left(\frac{1}{n}\right)^\alpha + \left(\frac{2}{n}\right)^\alpha + \cdots + \left(\frac{n}{n}\right)^\alpha\right)\frac{1}{n} \cdot \left(\left(\frac{1}{m}\right)^{\beta+1} + \left(\frac{2}{m}\right)^{\beta+1} + \cdots + \left(\frac{m}{m}\right)^{\beta+1}\right)\frac{1}{m} = \int_0^1 \int_0^1 x^\alpha y^{\beta+1}\,dx\,dy$

True! This is tricky! This is the multiple integral definition as the limit of a Riemann sum so that $\frac{b-a}{n} = \frac{1-0}{n} = \Delta x$ and $\frac{d-c}{m} = \frac{1-0}{m} = \Delta y$. This implies that $A = \Delta x \cdot \Delta y = \frac{1}{n}\frac{1}{m}$. In this case, we can apply Fubini's theorem for separable multi-variable functions given the constant boundaries and independent functions (e.g., $f(x,y) = g(x) \cdot h(y)$ implying $\int_0^1 x^\alpha dx \cdot \int_0^1 y^{\beta+1} dy = \int_0^1 \int_0^1 x^\alpha y^{\beta+1}\,dx\,dy$).

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^\alpha \frac{1}{n} \cdot \lim_{m\to\infty} \sum_{j=1}^{m} \left(\frac{j}{m}\right)^{\beta+1} \frac{1}{m} = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\frac{k}{n}\right)^\alpha \left(\frac{j}{m}\right)^{\beta+1} \Delta x \Delta y = \int_0^1 \int_0^1 x^\alpha y^{\beta+1}\,dx\,dy$$

Note: The limit does exist and $x_k = 0 + k\Delta x = \frac{k}{n}$ and $y_j = 0 + j\Delta y = \frac{j}{m}$ see https://youtu.be/N6y_UJrZLAE

(e) T F $\int_{y=1}^{4} \int_{x=0}^{1} \left(x^2 + \sqrt{y}\right) \sin\left(x^2 y^2\right)\,dx\,dy \le 9$

True. In general if $f(x,y) \le K$ and a domain D has area A, $\iint_D f(x,y)\,dx\,dy \le K \cdot A$. Here, the domain is a rectangle with area 3 , so the trick is to show that $\left(x^2 + \sqrt{y}\right) \sin\left(x^2 y^2\right) \le 3$ for all (x,y) in the rectangle.

(f) T F $\int_{r=-1}^{1} \int_{\theta=0}^{1} e^{r^2+\theta^2}\,d\theta\,dr = \left[\int_{r=-1}^{1} e^{r^2}\,dr\right]\left[\int_{\theta=0}^{1} e^{\theta^2}\,d\theta\right]$

True. See Fubini's theorem.

(g) T F The integral $\int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{\rho=0}^{1} \rho^2 \sin\theta\,d\rho\,d\theta\,d\phi$ gives the volume of 1/4 of a sphere.

False. This problem can be completed without evaluating the integral. The bounds and integrand should suggest the answer is false. The $\sin\theta$ should be $\sin\phi$, and even with this change it only gives the volume of 1/8 of a sphere.

6 Chapter 17 LINE AND SURFACE INTEGRALS

Determine if the following statements are true(T) or false(F). Mark the correct answer. No justification needed.

(a) $\boxed{\text{T}}$ $\boxed{\text{F}}$ A surface integral is always a positive quantity.
False! When the field vectors are going the opposite direction as the vectors normal to the surface, the flux is negative. When the field vectors are orthogonal to the vectors normal to the surface, the flux is zero. Thus, a surface integral can be zero and does not have to be a positive quantity

(b) $\boxed{\text{T}}$ $\boxed{\text{F}}$ If a particle travels in a closed loop then the total work done on the particle over the loop is zero
False. The force may not be conservative (a racecar can accelerate on a circular track... non-zero work done)

(c) $\boxed{\text{T}}$ $\boxed{\text{F}}$ If there exists a closed curve C in D such that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ then $\mathbf{F}$ is conservative on D.
False. The condition is that this holds for all curves in D

(d) $\boxed{\text{T}}$ $\boxed{\text{F}}$ $\int_{-C} f ds = -\int_C f ds$
False!!! Not to be confused with the work done on a particle moving one direction along a curve versus moving in the opposite direction

(e) $\boxed{\text{T}}$ $\boxed{\text{F}}$ If S is the unit sphere $x^2 + y^2 + z^2 = 1$ and a, b, c are real numbers, then $\iint_S |ax + by + cz| \, dS \geq 0$
True. The absolute value function is always greater than or equal to zero. Note that surface area cannot be negative (i.e., integrand equals 1). If your curious, the integral evaluates to $2\pi\sqrt{a^2 + b^2 + c^2}$ solution

(f) $\boxed{\text{T}}$ $\boxed{\text{F}}$ If $F_Z(z) = P(X + Y \leq z) = \int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{z-y} f_{XY}(x, y) dx dy$, then $f_Z(z) = \int_{-\infty}^{+\infty} \left(\frac{\partial}{\partial z} \int_{-\infty}^{z-y} f_{XY}(x, y) dx \right) dy$ assume $Z = X + Y$ and the region $D_z : x + y \leq z$ is shaded.
True! This is tricky! Recall Leibnitz's differentiation rule from calculus 1 where $H(z) = \int_{a(z)}^{b(z)} h(x, z) dx$ then $\frac{dH(z)}{dz} = \frac{db(z)}{dz} h(b(z), z) - \frac{da(z)}{dz} h(a(z), z) + \int_{a(z)}^{b(z)} \frac{\partial h(x,z)}{\partial z} dx$. In our situation,we have the following $f_Z(z) = \int_{-\infty}^{+\infty} \left(f_{XY}(z - y, y) - 0 + \int_{-\infty}^{z-y} \frac{\partial f_{XY}(x,y)}{\partial z} dx \right) dy$

(g) $\boxed{\text{T}}$ $\boxed{\text{F}}$ If $S = \{(x, y, z) : f(x, y, z) = k\}$ is a level surface of a smooth function f with no critical points on S, then S must be orientable.
True. Take $\bar{n} = \frac{\nabla f}{\|\nabla f\|}$ which is a continuous normal vector since $\nabla f \perp S$

7 Chapter 18 FUNDAMENTAL THEOREMS OF VECTOR ANALYSIS

Determine if the following statements are true(T) or false(F). Mark the correct answer. No justification needed.

(a) $\boxed{\text{T}}$ $\boxed{\text{F}}$ The flux of $F = \langle x, 0, 0 \rangle$ across a sphere of radius 1 at the origin is strictly less than its flux across a sphere of radius 2 at the origin, where both are outwardly oriented.
True. Because flux $= \iiint_E div(F) dV = 2 \cdot Vol(E)$

(b) $\boxed{\text{T}}$ $\boxed{\text{F}}$ If S is be the unit sphere centered at the origin, oriented outwards with normal vector $\mathbf{n}$ and the integral $I = \iint_S D_{\mathbf{n}} f dS$ where $D_{\mathbf{n}}$ is the directional derivative along $\mathbf{n}$, then $I = \iiint_E div(\nabla f) dV$ where E is a solid sphere (assume f is a continuous function).
True. We have $D_{\bar{n}} f = \bar{n} \cdot \nabla f$ so the integral is $\iint_S \nabla f \cdot \bar{n} dS$ (i.e., the flux of ∇f across S). By divergence theorem, $I = \iiint_E div(\nabla f) dV$.

(c) $\boxed{\text{T}}$ $\boxed{\text{F}}$ If $\vec{F} = \left(x - \frac{2}{3}x^3, \frac{-4}{3}y^3, \frac{-8}{3}z^3 \right)$ and $\mathcal{J} = \iint_S \vec{F} \cdot \vec{n} dS$, then $\mathcal{J}$ is maximized with surface S described as $1 = 2x^2 + 4y^2 + 8z^2$
True! This is tricky! By divergence theorem, $\mathcal{J} = \iiint_V \left[1 - 2x^2 - 4y^2 - 8z^2 \right] dV$ which is maximized when the integrand is always non-negative. The surface S above gives an integrand of 0, and any surface bigger than that starts to contribute negative integrands. See maximizing a surface integral and maximizing a double integral

Note: To maximize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where the integrand is negative