AEW
Auxiliary Problems II

Ave Kludze (akk86)

MATH 1920

Name: ______________________________

Collaborators: ______________________________

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1

Determine the minimum non-negative integer m such that both

$$\lim_{(x,y)\to(0,0)} x^{\frac{m}{3}}|x-y|$$

$$\lim_{(x,y)\to(0,0)} \frac{x^{\frac{m}{3}}|x-y|}{\sqrt{x^2+y^2}}$$

are real numbers.

2

Let $p = (\alpha, \beta, \gamma)$ be a point in which the function

$$f(x, y, z) = 4x + 2y - z^2$$

with the restriction $x^2+y^2+z^2 = 16$, takes the global minimum value. Find an expression for the sum $\alpha+2\beta+2\gamma$.

3

Let $\mathbf{F}(x, y) = \langle F_1, F_2\rangle$ where $F_1 = e^{8xy}$ and $F_2 = -\ln\left(\cos^2(x+y) + \pi^x y^{100}\right)$. Let $\mathcal{C}$ be the curve parametrized by

$$\mathbf{r}(t) = \begin{cases} \langle \cos t, \sin t\rangle & \text{for } 0 \le t \le \pi, \\ \langle \cos t, -\sin t\rangle & \text{for } \pi \le t \le 2\pi \end{cases}$$

Evaluate $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$

4

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function defined by

$$f(x, y) = \begin{cases} \frac{x^2y^3}{x^4+y^6}, \text{ if } x \neq 0, y \in \mathbb{R}, \\ 0, \text{ if } x = 0, y \in \mathbb{R}. \end{cases}$$

(a). Find all $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ such that f has a nonzero directional derivative at $(0, 0)$ with respect to the direction (a, b).
(b) Is f continuous at $(0, 0)$? Justify your answer.
Hint: Part (a) requires using the limit definition.

5

For a fixed vector $\vec{p}$, define the vector field $\vec{F} = \vec{p} \times \vec{r}$, where $\vec{r}(x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is the usual radial vector field. Find a non-zero scalar multiple λ such that

$$\vec{\nabla} \times \vec{F} = \lambda\vec{p}.$$

6

The stream function $\vec{\Psi}$ for a particular flow is given by $\vec{\Psi} = \vec{F} + \vec{G}$ with

$$\vec{F}(r, \theta, z) = \left(1 - \frac{1}{r^2}\right) r \sin\theta \hat{\mathbf{k}}, \vec{G}(r, \theta, z) = -\ln r \hat{\mathbf{k}}$$

where (r, θ, z) are the usual cylindrical coordinates. The velocity vector is then defined by $\overrightarrow{u} = \vec{\nabla} \times \Psi$. Also, let

$$\varphi = \left(1 + \frac{1}{r^2}\right) r \cos\theta$$

(a)Compute $\vec{\nabla} \times \vec{F}$ and $\vec{\nabla} \times \vec{G}$.

(b) Show that $\vec{\nabla} \times \overrightarrow{\mathbf{u}} = \overrightarrow{0}$. Give proper justification, and indicate any theorem you might be using.

7

Show that if $\vec{u} = \langle u_1, \cdots, u_n \rangle$ and $\vec{v} = \langle v_1, \cdots, v_n \rangle$, then

$$|\vec{u}|^2 |\vec{v}|^2 - |\vec{u} \cdot \vec{v}|^2 = \sum_{i<j} (u_i v_j - u_j v_i)^2 .$$

8

Collinearity is defined as a set of points lying on a single line in coordinate space. Given the coordinate points,

$$(8, 3, -3)$$

$$(-1, 6, 3)$$

$$(2, 5, c)$$

For what integer value(s) of c, do the given points lie in a straight line? Note that a calculator may be helpful for algebra and computation!

9

Find each of the following limits or show that it does not exist:

$$\lim_{t\to\infty} \left\langle e^{2t}/\cosh^2 t, t^{2012} e^{-t}, e^{-2t} \sinh^2 t \right\rangle$$

10

Find the most general vector function whose n^{th} derivative vanishes, $\mathbf{r}^{(n)}(t) = \mathbf{0}$, in an interval.

11 (Challenge)

Find and sketch the domain of each of the following function:

$$f(x, y) = \text{sign}(\sin x \sin y)$$

Note that here $\text{sign}(a)$ is the sign function, it has the values 1 and 1 for positive and negative a, respectively.

12 (Challenge)

Find the limit, if it exists, or show that the limit does not exist

$$\lim_{r\to\infty} \frac{\ln\left(x^2y^2z^2\right)}{x^2+y^2+z^2}$$

Hint: Consider the limits along the curves $x=y=z=t$ and $x=e^{-t^2}, y=z=t$

13 (Challenge)

$$\lim_{r\to\infty} \frac{e^{3x^2+2y^2+z^2}}{(x^2+2y^2+3z^2)^{2012}}$$

Hint: Consider the inequality $1+u\leq e^u$

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Find the repeated limits

$$\lim_{x\to 1}\left(\lim_{y\to 0}\log_x(x+y)\right) \text{ and } \lim_{y\to 0}\left(\lim_{x\to 1}\log_x(x+y)\right)$$

What can be said about the corresponding two-variable limit?

15 (Challenge)

Find the specified partial derivatives of the function:

$$f(x,y,z)=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{r}),\ \text{where } \mathbf{a} \text{ and } \mathbf{b} \text{ are constant vectors; } \mathbf{r} \text{ is the radial vector field}$$

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Find the integral of $f(x,y,z)=z\left(x^2+y^2+z^2\right)^{-7/4}$ over the half-ball $x^2+y^2+z^2\leq 1, z\geq 0$, if it exists.

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Two spacecraft are following paths in space given by $\mathbf{r}_1=\left\langle \sin t, t, t^2\right\rangle$ and $\mathbf{r}_2=\left\langle \cos t, 1-t, t^3\right\rangle$. If the temperature for points in space are given by $T(x,y,z)=x^2y(1-z)$, use the chain rule to determine the rate of change of the difference D in the temperatures the two spacecraft experience at time $t=\pi$.

18

If $u(x,y)$ is a solution to the Laplace Equation in the plane, what is the value of the line integral

$$\int_{\partial D} u_y\,dx - u_x\,dy$$

when C is a simple closed curve oriented counterclockwise? Assume that $u_{xx}(x,y)+u_{yy}(x,y)=0, \quad$ for all $(x,y)\in D$.

19

Find the specified partial derivatives of the function $f(\mathbf{r})=\exp(\mathbf{a}\cdot\mathbf{r})$, where $\mathbf{a}\cdot\mathbf{a}=1$ and $\mathbf{r}\in\mathbb{R}^m$, $f''_{x_1x_1}+f''_{x_2x_2}+\ldots+f''_{x_mx_m}=f$

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(a) Let $\mathbf{a} = s\hat{\mathbf{u}} + \hat{\mathbf{v}}$ and $\mathbf{b} = \hat{\mathbf{u}} + s\hat{\mathbf{v}}$ where the angle between unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ is $\pi/3$. Find the values of s for which the dot product $\mathbf{a} \cdot \mathbf{b}$ is maximal, minimal, or zero if such values exist. Do you notice anything special about these values?

(b) Let $\mathbf{a} = s\hat{\mathbf{u}} + w\hat{\mathbf{v}}$ and $\mathbf{b} = w\hat{\mathbf{u}} + s\hat{\mathbf{v}}$ where the angle between unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ is $\pi/3$. Find values of s and w for which the dot product $\mathbf{a} \cdot \mathbf{b}$ is maximal, minimal, or zero if such values exist. Do you notice anything special about these values?