AEW Worksheet 6

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MATH 1920

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Collaborators: ____________________

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1

Determine if the following statements are true(T) or false(F). Mark the correct answer. No justification needed.

(a) $\boxed{T}$ $\boxed{F}$ If $\lim_{(x,y)\to(a,b)} f(x,y) = L$ where $a, b \neq 0$, then $x = a + r\cos\theta$ and $y = b + r\sin\theta$
True. If the multi-variable limit is not approaching the origin, one can still use polar coordinates! This requires shifting the bounds for x and y based on values a and b respectively (see "multivariable limits [example 4, extra]" for an example).

(b) $\boxed{T}$ $\boxed{F}$ There exists a function f with continuous second-order partial derivatives such that $f_x(x,y) = kx + y^2$ and $f_y(x,y) = x - y^2$ for constant k.
False. Since we would have $f_{xy}(x,y) = 2y$ but $f_{yx}(x,y) = 1$ which does not satisfy Clairaut's theorem for mixed variable partial derivatives.

(c) $\boxed{T}$ $\boxed{F}$ Suppose their exist an angle of inclination ψ and $z = f(x,y)$, then $\psi = \tan^{-1}\left(\left\|\nabla f_{(a,b)}\right\|\sin\theta\right)$
False. By definition, $\tan\psi = D_{\mathbf{u}}f(P) = \left\|\nabla f_{(a,b)}\right\|\cos\theta$ which implies that $\psi = \tan^{-1}\left(\left\|\nabla f_{(a,b)}\right\|\cos\theta\right)$

2

Consider the discriminant,

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -12\pi x^2 + 2\pi a & b \\ b & -12\pi y^2 + 2\pi a \end{vmatrix}$$

where $D(x,y)$ is in the determinant form.

(a) Compute the determinant in only terms of x, y, a and b.

(b) By considering the second derivative test, describe the classification of critical points, if $b = 0$, only terms of x, y, and a.

(c) Find a function $f(x,y)$ that could represent this discriminant, if $a = 2$ and $b = 0$. Assume that the function behaves such that $f(0,0) = 0$, $f(1,0) = \pi$, and $f(0,1) = \pi$.

(d) Find critical points of f and determine whether the points are local minima, maximum or saddle points.

Solution

(a) The computation is fairly simple. It will be useful to keep the values in a factored form.

$$D(x,y) = \left(-12\pi x^2 + 2\pi a\right)\left(-12\pi y^2 + 2\pi a\right) - b^2$$

$$= \boxed{144\pi^2x^2y^2 - 24\pi^2ax^2 - 24\pi^2ay^2 + 4\pi^2a^2 - b^2}$$

(b)
We may treat f_{xx} and f_{yy} as one variable functions to determine where they are positive or negative, as a result of the factored format.

$$D(x,y) = \left(-12\pi x^2 + 2\pi a\right)\left(-12\pi y^2 + 2\pi a\right)$$

If $D = 0$, the test is inconclusive.

$$\underbrace{\left(-12\pi x^2 + 2\pi a\right)}_{=0}\underbrace{\left(-12\pi y^2 + 2\pi a\right)}_{=0} = 0$$

$$x = \sqrt{\frac{a}{6}},\ x = -\sqrt{\frac{a}{6}}, \quad y = \sqrt{\frac{a}{6}},\ y = -\sqrt{\frac{a}{6}}$$

If $D > 0$ and $f_{xx}(x, y) > 0$, then $f(x, y)$ is a local minimum.

$$\underbrace{\left(-12\pi x^2 + 2\pi a\right)}_{+}\underbrace{\left(-12\pi y^2 + 2\pi a\right)}_{+} > 0$$

$$-\frac{\sqrt{a}}{\sqrt{6}} < x < \frac{\sqrt{a}}{\sqrt{6}}, \quad -\frac{\sqrt{a}}{\sqrt{6}} < y < \frac{\sqrt{a}}{\sqrt{6}}$$

If $D > 0$ and $f_{xx}(x, y) < 0$, then $f(x, y)$ is a local maximum.

$$\underbrace{\left(-12\pi x^2 + 2\pi a\right)}_{-}\underbrace{\left(-12\pi y^2 + 2\pi a\right)}_{-} > 0$$

$$x > \frac{\sqrt{a}}{\sqrt{6}} \text{ or } x < -\frac{\sqrt{a}}{\sqrt{6}}, \quad y > \frac{\sqrt{a}}{\sqrt{6}} \text{ or } y < -\frac{\sqrt{a}}{\sqrt{6}}$$

If $D < 0$, then f has a saddle point at (x, y)

$$\underbrace{\left(-12\pi x^2 + 2\pi a\right)}_{-}\underbrace{\left(-12\pi y^2 + 2\pi a\right)}_{+} < 0$$

$$x > \frac{\sqrt{a}}{\sqrt{6}} \text{ or } x < -\frac{\sqrt{a}}{\sqrt{6}} \quad -\frac{\sqrt{a}}{\sqrt{6}} < y < \frac{\sqrt{a}}{\sqrt{6}}$$

$$\underbrace{\left(-12\pi x^2 + 2\pi a\right)}_{+}\underbrace{\left(-12\pi y^2 + 2\pi a\right)}_{-} < 0$$

$$-\frac{\sqrt{a}}{\sqrt{6}} < x < \frac{\sqrt{a}}{\sqrt{6}} \quad y > \frac{\sqrt{a}}{\sqrt{6}} \text{ or } y < -\frac{\sqrt{a}}{\sqrt{6}}$$

Notice that if $D > 0$, then $f_{xx}(x, y)$ and $f_{yy}(x, y)$ must have the same sign, so the sign of $f_{yy}(x, y)$ also determines whether $f(x, y)$ is a local minimum or a local maximum.

(c)

The best approach is to perform partial integrate with respect to the partial derivatives. This will force us to have a function in terms of x and y, where we can guess a possible solution.

$$f_{xx} \implies \int f_{xx}\,dx = \int -12\pi x^2 + 4\pi\,dx = -4\pi x^3 + 4\pi x + C = f_x$$

$$f_x \implies \int f_x\,dx = \int -4\pi x^3 + 4\pi x + C\,dx = -\pi x^4 + 2\pi x^2 + C_1 x + h(y) = f_1(x, y)$$

$$f_{yy} \implies \int f_{yy}\,dy = \int -12\pi y^2 + 4\pi\,dy = -4\pi y^3 + 4\pi y + C = f_y$$

$$f_y \implies \int f_y\,dy = \int -4\pi y^3 + 4\pi y + C\,dy = -\pi y^4 + 2\pi y^2 + C_2 y + g(x) = f_2(x, y)$$

We must now consider $f_{yx} = f_{xy}$.

$$f_{yx} = f_{xy} = 0 \implies f_y = C_3 \text{ or } f_x = C_4$$

$$f_3(x, y) = C_3 y + z(x)$$
$$f_4(x, y) = C_4 x + z(y)$$

Therefore, we have four possible function in terms of x and y. The main issue we must consider is that if the possible functions represent the determinant. From this fact, we need a combination of $f_1(x, y)$ and $f_2(x, y)$, preferably with their higher order terms and coefficients. Since the values of $C_1, C_2, C_3,$ and C_4 are indeterminable, we can guess that their values are zero through trial and error. The same approach follows for $h(y), g(x), z(x)$ and $z(y)$, through guess-work. It should be noted that applying the initial conditions for $f(x, y)$ is another possible solution. It is also possible to relate the four functions and their first partial derivatives (an approach similar to finding a potential function for a conservative vector field - 17.3).

$$f_1(x, y) = -\pi x^4 + 2\pi x^2 + C_1 x + h(y)$$
$$f_2(x, y) = -\pi y^4 + 2\pi y^2 + C_2 y + g(x)$$
$$f_3(x, y) = C_3 y + z(x)$$
$$f_4(x, y) = C_4 x + z(y)$$

With the previous information we have,

$$\boxed{f(x, y) = -\pi x^4 + 2\pi x^2 - \pi y^4 + 2\pi y^2}$$

By checking our guess-function against the information presented it appears to represent the determinant.

(d)

$$f(x, y) = -\pi x^4 + 2\pi x^2 - \pi y^4 + 2\pi y^2$$
$$f_x(x, y) = -4\pi x^3 + 4\pi x = 0$$
$$f_y(x, y) = -4\pi y^3 + 4\pi y = 0$$

Therefore,

$$x = 0,\ x = -1,\ x = 1$$
$$y = 0,\ y = -1,\ y = 1$$

From the second derivative test, the **maximum values** are

$$2\pi \text{ at } (x, y) = (-1, -1)$$
$$2\pi \text{ at } (x, y) = (-1, 1)$$
$$2\pi \text{ at } (x, y) = (1, -1)$$
$$2\pi \text{ at } (x, y) = (1, 1)$$

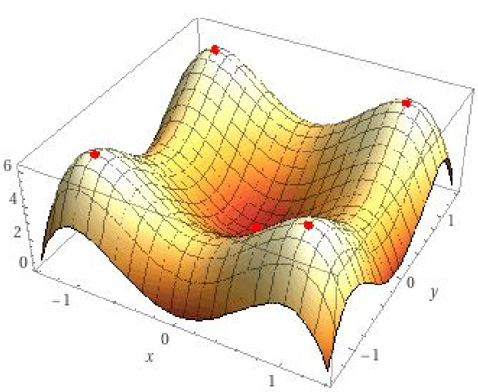
The **minimum value** is

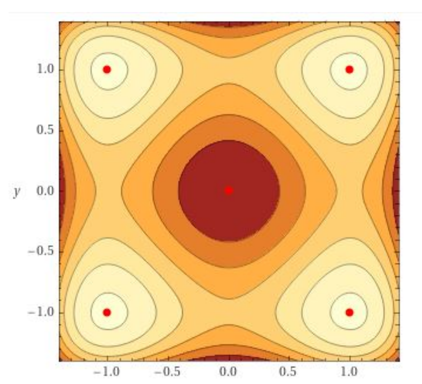
$$0 \text{ at } (x, y) = (0, 0)$$

The **saddle points** are

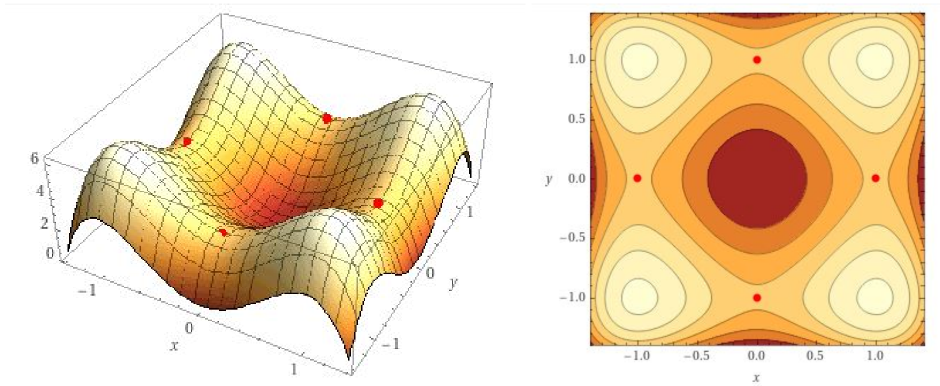
$$\pi \text{ at } (x, y) = (-1, 0)$$
$$\pi \text{ at } (x, y) = (0, 1)$$
$$\pi \text{ at } (x, y) = (0, -1)$$
$$\pi \text{ at } (x, y) = (1, 0)$$

Graph of Maximum and Minimum:





Graph of Saddle Points:



3

The N corporation (maker of the finest Y) has recently merged with the J (maker of the finest Z). Currently Y sell for three dollars each and Z sell for nine dollars each. By combining their production the new company enjoys economy of scope and is now able to produce y Y and z Z at a cost of $10 + \frac{1}{2}y^2 + \frac{1}{3}z^3 - yz$ dollars. Determine how many Y and Z respectively should be made in order to maximize profit. Also, verify that your answer is a maximum by using the second derivative test. **Note:** Profit = Revenue − Cost

Solution

The revenue can be modeled by $3y + 9z$ (three dollars for each of the Y and nine dollars for each of the Z). Using the equation for profit,

$$P(y,z) = \underbrace{(3y+9z)}_{\text{Revenue}} - \underbrace{\left(10 + \frac{1}{2}y^2 + \frac{1}{3}z^3 - yz\right)}_{\text{cost}} = -\frac{1}{2}y^2 - \frac{1}{3}z^3 + yz + 3y + 9z - 10$$

To find the critical point, we set the partial derivatives to 0

$$\frac{\partial P}{\partial y}(y,z) = -y + z + 3 = 0$$
$$\frac{\partial P}{\partial z}(y,z) = -z^2 + y + 9 = 0$$

From the first equation we get $z = y - 3$, which if we substitute into the second equation becomes

$$0 = -(y-3)^2 + y + 9 = -\left(y^2 - 6y + 9\right) + y + 9 = -y^2 + 7y = -y(y-7)$$

This gives us either $y = 0$ or $y = 7$. However, $y \neq 0$ since it means that $z = -3$. Therefore, $\boxed{\text{y=7 and z=4}}$. Since this is the only critical point this must be the correct production level, so we should produce 7 Y and 4 Z. To verify that this is indeed a maximum.

$$P_{yy}P_{zz} - (P_{yz})^2 = (-1)(-2z) - (1)^2 = 2z - 1$$

Since at $y = 7$ and $z = 4$ we have that this is positive (i.e. 7) we are at a max or a min

$$\implies P_{yy} = -1 < 0$$

We can conclude that this is indeed at a maximum.

4

Suppose that $f(x,y)$ is differentiable and $f\left(t^3 - t + 1, 2 - t^2\right) = t^4 - 4t^3 + 4t + 6$

$$\text{Find } \frac{\partial f}{\partial x}(1,1) \text{ and } \frac{\partial f}{\partial y}(1,1)$$

Solution

In this case we have that f is a function of x and y and that $x = t^3 - t + 1$. and $y = 2 - t^2$, i.e., x and y are functions of t. So we can use the chain rule to compute the derivative of f with respect to t.

$$\frac{d}{dt}f(\mathbf{c}(t)) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle x'(t), y'(t) \rangle = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Therefore:

$$4t^3 - 12t^2 + 4 = \frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = \frac{\partial f}{\partial x} \cdot \left(3t^2 - 1\right) + \frac{\partial f}{\partial y} \cdot (-2t)$$

Now some good choices for t are $t = -1$ and $t = 1$, because these both correspond to the point $(1, 1)$. Plugging these in we get:

$$t = -1 : -12 = 2\frac{\partial f}{\partial x}(1,1) + 2\frac{\partial f}{\partial y}(1,1)$$

$$t = 1 : \quad -4 = 2\frac{\partial f}{\partial x}(1,1) - 2\frac{\partial f}{\partial y}(1,1)$$

Adding these two equations we get:

$$-16 = 4\frac{\partial f}{\partial x}(1,1)$$

$$\boxed{\frac{\partial f}{\partial x}(1,1) = -4}$$

Subtracting these two equations we get:

$$-8 = 4\frac{\partial f}{\partial y}(1,1)$$

$$\boxed{\frac{\partial f}{\partial y}(1,1) = -2}$$

5

Suppose we know the following:

$$\frac{d}{dt}f(\mathbf{c}(t)) = 2 \quad \text{if} \quad \vec{c}(t) = \langle t, t \rangle$$

$$\frac{d}{dt}f(\mathbf{c}(t)) = 3 \quad \text{if} \quad \vec{c}(t) = \langle t, -t \rangle$$

Find $\nabla f(0, 0)$.

Solution

By the chain rule, we have:

$$\frac{d}{dt}f(\mathbf{c}(t)) = \nabla f_{\mathbf{c}(t)} \cdot \mathbf{c}'(t)$$

$$\frac{d}{dt}f(\mathbf{c}(t)) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle x'(t), y'(t) \rangle = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\text{If } \vec{c}(t) =< t, t >, \text{ then } c'(t) =< 1, 1 >$$

$$\text{If } \vec{c}(t) =< t, -t >, \text{ then } c'(t) =< 1, -1 >$$

Thus $\nabla f(0,0)$ can be found since $c'(t)$ is independent of t. We have two equation and two unknowns:

$$\frac{\partial f}{\partial x} * 1 + \frac{\partial f}{\partial y} * 1 = 2$$

$$\frac{\partial f}{\partial x} * 1 + \frac{\partial f}{\partial y} * (-1) = 3$$

Solving we get the following:

$$\frac{\partial f}{\partial x} = \frac{5}{2}$$

$$\frac{\partial f}{\partial y} = \frac{-1}{2}$$

Thus

$$\boxed{\nabla f(0,0) = \langle \frac{5}{2}, \frac{-1}{2} \rangle}$$