AEW
Auxiliary Problems II

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MATH 1920

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1

Determine the minimum non-negative integer m such that both

$$\lim_{(x,y)\to(0,0)} x^{\frac{m}{3}}|x-y|$$

$$\lim_{(x,y)\to(0,0)} \frac{x^{\frac{m}{3}}|x-y|}{\sqrt{x^2+y^2}}$$

are real numbers.

Solution

For the first limit, if we convert to polar coordinates, we get

$$\lim_{r\to 0^+} r^{\frac{m+3}{3}} \cos\theta |\cos\theta - \sin\theta|$$

This is defined and real when $\frac{m+3}{3} > 0$, so $m > -3$.

Similarly for the second limit, we get

$$\lim_{r\to 0^+} r^{\frac{m}{3}} |\cos\theta - \sin\theta|$$

This is defined and real when $\frac{m}{3} > 0$, so $m > 0$.

Since we want $m > -3$ and $m > 0$, we have $\boxed{m = 1}$ is the minimum integer solution.

2

Let $p = (\alpha, \beta, \gamma)$ be a point in which the function

$$f(x, y, z) = 4x + 2y - z^2$$

with the restriction $x^2 + y^2 + z^2 = 16$, takes the global minimum value. Find an expression for the sum $\alpha + 2\beta + 2\gamma$.

Solution

Since the given function is

$$f(x, y, z) = 4x + 2y - z^2$$

then the gradient is

$$\nabla f(x, y, z) = \langle 4, 2, -2z \rangle$$

Similarly,

$$g(x, y, z) = x^2 + y^2 + z^2 - 16$$

$$\nabla g(x, y, z) = \langle 2x, 2y, 2z \rangle$$

Applying Lagrange multipliers,

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

$$\langle 4, 2, -2z \rangle = \lambda \langle 2x, 2y, 2z \rangle$$

Thus, we have the following set of equations

$$4 = 2\lambda x$$

$$2 = 2\lambda y$$
$$-2z = 2\lambda z$$
$$x^2 + y^2 + z^2 = 16$$

simplifying, we have

$$2 = \lambda x$$
$$1 = \lambda y$$
$$(\lambda + 1)z = 0$$
$$x^2 + y^2 + z^2 = 16$$

Case 1, $\lambda = -1$:

$$x = -2$$
$$y = -1$$
$$16 = x^2 + y^2 + z^2 = 4 + 1 + z^2 = 5 + z^2$$
$$z^2 = 11$$
$$z = \pm\sqrt{11}$$

Case 2, $z = 0$:

$$x^2 + y^2 = 16$$
$$5 = 4 + 1 = 2^2 + 1^1 = (\lambda x)^2 + (\lambda y)^2 = \lambda^2(x^2 + y^2) = 16\lambda^2$$
$$\lambda = \pm\frac{\sqrt{5}}{4}$$
$$x = \pm\frac{8\sqrt{5}}{5}$$
$$y = \pm\frac{4\sqrt{5}}{5}$$
$$z = 0$$

Now we must check the following set of coordinates,

$$(x, y, z) \in \left\{ \left(-2, -1, \sqrt{11}\right), \left(-2, -1, -\sqrt{11}\right), \left(\frac{8\sqrt{5}}{5}, \frac{4\sqrt{5}}{5}, 0\right), \left(-\frac{8\sqrt{5}}{5}, -\frac{4\sqrt{5}}{5}, 0\right) \right\}$$

and find which one gives the minimum value. We know that

$$z^2 = 16 - x^2 - y^2$$

so

$$f(x, y, z) = 4x + 2y - 16 + x^2 + y^2 = (x + 2)^2 + (y + 1)^2 - 21 \geq -21 \Rightarrow \min f(x, y, z) = -21$$

for

$$x = -2, y = -1, z = \pm\sqrt{11}$$

Thus, we have two possible expressions for $\alpha + 2\beta + 2\gamma$ since $P = (-2, -1, \pm\sqrt{11})$, where the sum is $\boxed{2.633}$ or $\boxed{-10.633}$.

3

Let $\mathbf{F}(x, y) = \langle F_1, F_2 \rangle$ where $F_1 = e^{8xy}$ and $F_2 = -\ln\left(\cos^2(x + y) + \pi^x y^{100}\right)$. Let $\mathcal{C}$ be the curve parametrized by

$$\mathbf{r}(t) = \begin{cases} \langle \cos t, \sin t \rangle & \text{for } 0 \leq t \leq \pi, \\ \langle \cos t, -\sin t \rangle & \text{for } \pi \leq t \leq 2\pi \end{cases}$$

Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$

Solution

The curve $\mathcal{C}$ follows a semi-circle $\mathcal{C}_1$ from $(1,0)$ to $(-1,0)$ and and then returns to $(1,0)$ along a curve $\mathcal{C}_2$, which follows the same semi-circle in the reverse direction. Accordingly, the integral splits into two cancelling parts, giving the answer 0 :

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} &= \int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r}\\ &= \int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} - \int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r}\\ &= \boxed{0}\end{aligned}$$

4

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function defined by

$$f(x,y) = \begin{cases} \frac{x^2y^3}{x^4+y^6}, \text{ if } x \neq 0, y \in \mathbb{R},\\ 0, \text{ if } x = 0, y \in \mathbb{R}.\end{cases}$$

(a). Find all $(a, b) \in \mathbb{R}^2 \setminus \{(0,0)\}$ such that f has a nonzero directional derivative at (0, 0) with respect to the direction (a, b).
(b) Is f continuous at (0, 0)? Justify your answer.
Hint: Part (a) requires using the limit definition.

Solution

(a)
Let $a = 0, b \neq 0$. Then the directional derivative at $(0,0)$ with respect to the direction (a,b) is

$$\lim_{\epsilon\to 0} \frac{f(\epsilon a, \epsilon b) - f(0,0)}{\epsilon} = \lim_{\epsilon\to 0} \frac{f(0,\epsilon b)}{\epsilon} = \lim_{\epsilon\to 0} 0 = 0.$$

Let $a \neq 0$. Then the directional derivative at $(0,0)$ with respect to the direction (a,b) is

$$\begin{aligned}\lim_{\epsilon\to 0} \frac{f(\epsilon a, \epsilon b) - f(0,0)}{\epsilon} &= \lim_{\epsilon\to 0} \frac{f(\epsilon a, \epsilon b)}{\epsilon} =\\ = \lim_{\epsilon\to 0} \frac{\epsilon^5 a^2 b^3}{\epsilon\left(\epsilon^4 a^4 + \epsilon^6 b^6\right)} &= \lim_{\epsilon\to 0} \frac{a^2b^3}{a^4+\epsilon b^6} = \frac{b^3}{a^2}.\end{aligned}$$

f has a nonzero directional derivative at $(0,0)$ with respect to the direction (a,b) if and only if $a \neq 0$ and $b \neq 0$.
(b) If f is continuous at $(0,0)$, then for any sequence $(x_n, y_n) \to 0$ we have $f(x_n, y_n) \to 0$. Let $x_n = \frac{1}{n^3}, y_n = \frac{1}{n^2}$. Then

$$f(x_n, y_n) = \frac{n^{-6}n^{-6}}{n^{-12}+n^{-12}} = \frac{1}{2} \nrightarrow 0.$$

So, f is not continuous at $(0,0)$.

5

For a fixed vector $\vec{p}$, define the vector field $\vec{F} = \overrightarrow{p} \times \vec{r}$, where $\vec{r}(x,y,z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is the usual radial vector field. Find a non-zero scalar multiple λ such that

$$\vec{\nabla} \times \vec{F} = \lambda \overrightarrow{p}.$$

Solution

Let $\overrightarrow{p} = \langle a, b, c \rangle$. We then compute that

$$\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\hat{\mathbf{i}} + (cx - az)\hat{\mathbf{j}} + (ay - bx)\hat{\mathbf{k}}.$$

Calculating the curl

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (bz - cy) & (cx - az) & (ay - bx) \end{vmatrix} = 2a\hat{\imath} + 2b\hat{\mathbf{j}} + 2c\hat{\mathbf{k}} = 2\overrightarrow{\mathbf{p}}.$$

Thus, $\boxed{\lambda = 2}$

6

The stream function $\vec{\Psi}$ for a particular flow is given by $\vec{\Psi} = \vec{F} + \vec{G}$ with

$$\vec{F}(r, \theta, z) = \left(1 - \frac{1}{r^2}\right) r \sin\theta \hat{\mathbf{k}}, \vec{G}(r, \theta, z) = -\ln r \hat{\mathbf{k}}$$

where (r, θ, z) are the usual cylindrical coordinates. The velocity vector is then defined by $\overrightarrow{u} = \vec{\nabla} \times \Psi$. Also, let

$$\varphi = \left(1 + \frac{1}{r^2}\right) r \cos\theta$$

(a)Compute $\vec{\nabla} \times \vec{F}$ and $\vec{\nabla} \times \vec{G}$.

(b) Show that $\vec{\nabla} \times \overrightarrow{\mathbf{u}} = \overrightarrow{0}$. Give proper justification, and indicate any theorem you might be using.

Solution

(a)
Note that in Cartesean coordinates

$$\vec{F} = y - \frac{y}{x^2 + y^2}, G = -\ln\sqrt{x^2 + y^2}.$$

One can check by direct computation that

$$\begin{aligned} \vec{\nabla} \times \vec{F} &= \left(1 + \frac{y^2 - x^2}{(x^2 + y^2)^2}\right)\hat{\imath} - \frac{2xy}{(x^2 + y^2)^2}\hat{\jmath} \\ \vec{\nabla} \times \vec{G} &= -\frac{y}{x^2 + y^2}\hat{\mathbf{i}} + \frac{x}{x^2 + y^2}\hat{\mathbf{j}}. \end{aligned}$$

(b)
$\vec{\nabla} \times \vec{G}$ is indeed a familiar vector field, and $\operatorname{curl}(\vec{\nabla} \times \vec{G}) = \overrightarrow{0}$.On the other hand $\vec{\nabla} \times \vec{F} = \vec{\nabla}\varphi$, and by a theorem proved in the textbook, $\operatorname{curl}(\vec{\nabla}\varphi) = \overrightarrow{0}$ Since $\overrightarrow{u} = \vec{\nabla} \times \vec{F} + \vec{\nabla} \times \vec{G}$, clearly $\vec{\nabla} \times \overrightarrow{u} = \overrightarrow{0}$.

7

Show that if $\vec{u} = \langle u_1, \cdots, u_n \rangle$ and $\vec{v} = \langle v_1, \cdots, v_n \rangle$, then

$$|\vec{u}|^2 |\vec{v}|^2 - |\vec{u} \cdot \vec{v}|^2 = \sum_{i<j} (u_i v_j - u_j v_i)^2.$$

Solution

Note that

$$\begin{aligned}
|\vec{u}|^2|\vec{v}|^2 &= \left(\sum_{i=1}^{n} |u_i|^2\right)\left(\sum_{j=1}^{n} |v_j|^2\right) \\
&= \sum_{i,j} |u_i|^2 |v_j|^2 \\
&= u_1^2v_1^2 + u_2^2v_2^2 + \cdots + u_n^2v_n^2 + \sum_{i<j} \left(u_i^2v_j^2 + v_i^2u_j^2\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|\vec{u}\cdot\vec{v}|^2 &= \left(\sum_{k=1}^{n} u_k v_k\right)^2 \\
&= u_1^2v_1^2 + u_2^2v_2^2 + \cdots + u_n^2v_n^2 + 2\sum_{i<j} u_i v_i u_j v_j
\end{aligned}$$

So subtracting the two, we see that

$$\begin{aligned}
|\vec{u}|^2|\vec{v}|^2 - |\vec{u}\cdot\vec{v}|^2 &= \sum_{i<j} \left(u_i^2v_j^2 + v_i^2u_j^2 - 2u_iv_iu_jv_j\right). \\
&= \sum_{i<j} \left(u_iv_j - u_jv_i\right)^2.
\end{aligned}$$

Note: Do the cases $n = 2, 3$, to get a feel for what is happening. The notations means that we are summing over all indices i and j from 1 to n with $i < j$. Without the restriction $i < j$, the right hand side would be larger by a factor of two.

8

Collinearity is defined as a set of points lying on a single line in coordinate space. Given the coordinate points,

$$(8, 3, -3)$$

$$(-1, 6, 3)$$

$$(2, 5, c)$$

For what integer value(s) of c, do the given points lie in a straight line? Note that a calculator may be helpful for algebra and computation!

Solution

Given three points A, B, and C in $\mathbb{R}^n$, we can determine if they are colinear if

$$|AB| = |BC| + |AC|$$

Therefore, we must compute the distances between each pair of points and see if one is the sum of the other two. Let $A = (8, 3, -3)$, $B = (-1, 6, 3)$, and $C = (2, 5, c)$. By the distance formula,

$$|AB| = \sqrt{(8+1)^2 + (3-6)^2 + (-3-3)^2} = \sqrt{81+9+36} = \sqrt{126}$$

$$|BC| = \sqrt{(-1-2)^2 + (6-5)^2 + (3-c)^2} = \sqrt{19 - 6c + c^2}$$

$$|AC| = \sqrt{(8-2)^2 + (3-5)^2 + (-3-c)^2} = \sqrt{49 + 6c + c^2}$$

After trial and error in determining the longest segment based on the distance formula, we can note that $|AB| = |BC| + |AC|$, which is required for the points to be colinear. To see this, and solve for c, consider squaring the right hand side,

$$126 = \left(\sqrt{19 - 6c + c^2} + \sqrt{49 + 6c + c^2}\right)^2$$
$$\sqrt{19 - 6c + c^2} + \sqrt{49 + 6c + c^2} - 3\sqrt{14} = 0$$

Hence, $\boxed{c = 1}$ is the only integer value.

9

Find each of the following limits or show that it does not exist:

$$\lim_{t\to\infty} \left\langle e^{2t}/\cosh^2 t, t^{2012}e^{-t}, e^{-2t}\sinh^2 t\right\rangle$$

Solution

We find the limit in a similar manner as above. Here, $x(t) = e^{2t}/\cosh^2 t, y(t) = t^{2012}e^{-t}, z(t) = e^{-2t}\sinh^2 t$. First, $\cosh(t) = (e^t + e^{-t})/2$ and $\sinh(t) = (e^t - e^{-t})/2$. From these it can be shown that $d/dt \cosh(t) = \sinh(t)$ and $d/dt \sinh(t) = \cosh(t)$. The limits are

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{4e^{2t}}{(e^t + e^{-t})^2} = \lim_{t\to\infty} \frac{4e^{2t}}{e^{2t} + 2 + e^{-2t}} = \lim_{t\to\infty} \frac{4}{1 + 2e^{-2t} + e^{-4t}} = 4$$
$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{t^{2012}}{e^t} = \lim_{t\to\infty} \frac{2012t^{2011}}{e^t} = \lim_{t\to\infty} \frac{2012 * 2011t^{2010}}{e^t} = \ldots = \lim_{t\to\infty} \frac{2012!}{e^t} = 0$$
$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \frac{(e^t - e^{-t})^2}{4e^{2t}} = \lim_{t\to\infty} \frac{e^{2t} - 2 + e^{-2t}}{4e^{2t}} = \lim_{t\to\infty} \left(1/4 - 1/2e^{-2t} + 1/4e^{-4t}\right) = 1/4$$

where repeated L'Hopitals rule has been used in finding the limit of $y(t)$. Thus,

$$\lim_{t\to\infty} \left\langle e^{2t}/\cosh^2 t, t^{2012}e^{-t}, e^{-2t}\sinh^2 t\right\rangle = \boxed{\langle 4, 0, 1/4\rangle}$$

10

Find the most general vector function whose n^{th} derivative vanishes, $\mathbf{r}^{(n)}(t) = \mathbf{0}$, in an interval.

Solution

Suppose that $\mathbf{r}^{(n)}(t) = 0$ for some positive integer n. Then $\mathbf{r}^{(n)}(t)$ is integrable and

$$\mathbf{r}^{(n-1)}(t) = \int \mathbf{r}^{(n)}(t)dt = n!\mathbf{c}_n$$

for some constant vector $n!\mathbf{c}_n$!. Now, $\mathbf{r}^{(n-1)}(t)$ is integrable, and

$$\mathbf{r}^{(n-2)}(t) = \int \mathbf{r}^{(n-1)}(t)dt = tn!/1!\mathbf{c}_n + (n-1)!\mathbf{c}_{n-1}$$

for some constant vector $(n-1)!\mathbf{c}_{n-1}$. Then $\mathbf{r}^{(n-2)}(t)$ is integrable, and

$$\mathbf{r}^{(n-3)}(t) = \int \mathbf{r}^{(n-2)}(t)dt = t^2 n!/2!\mathbf{c}_n + t(n-1)!/1!\mathbf{c}_{n-1} + (n-2)!\mathbf{c}_{n-2}$$

for some constant vector $(n-2)!c_{n-2}$. Continuing in this fashion, we reach

$$\boxed{\mathbf{r}(t) = t^{n-1}\mathbf{c}_n + t^{n-2}c_{n-1} + \ldots + t\mathbf{c}_2 + \mathbf{c}_1}$$

for constant vectors $\mathbf{c}_k, k = 1, 2, \ldots, n$.

11 (Challenge)

Find and sketch the domain of each of the following function:

$$f(x, y) = \text{sign}(\sin x \sin y)$$

Note that here $\text{sign}(a)$ is the sign function, it has the values 1 and 1 for positive and negative a, respectively.

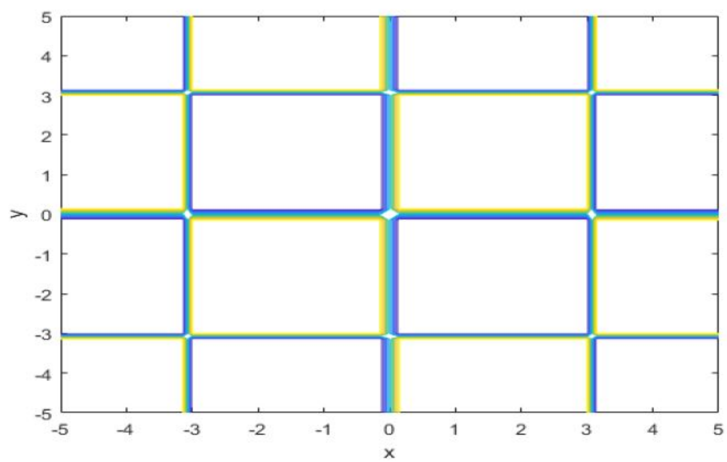
Solution

Here, the level sets are given by $\text{sign}(\sin x \sin y) = k$ for $k = -1, 0, 1$. Recall that $\sin(x) = 0$ for $x = n\pi$ where n is an integer. Thus $\text{sign}(\sin x \sin y)$ partitions $\mathbb{R}^2$ with grid lines at $x = n\pi$ and $y = n\pi$. Focus on the plot with $0 \leq x \leq 2\pi$ and $0 \leq y \leq 2\pi$. This region is subdivided into four squares (by the lines $x = \pi$ and $y = \pi$).

In the lower left ($0 < x < \pi$ and $0 < y < \pi$), $\sin x$ and $\sin y$ are both positive, so $\text{sign}(\sin x \sin y) = 1$. Next, in the lower right ($\pi < x < 2\pi$ and $0 < y < \pi$) square, $\sin x$ is negative whereas $\sin y$ is positive, so $\text{sign}(\sin x \sin y) = -1$.

In the upper left ($0 < x < \pi$ and $\pi < y < 2\pi$) square, $\sin x$ is positive whereas $\sin y$ is negative, so $\text{sign}(\sin x \sin y) = -1$.

Finally, in the upper right ($\pi < x < 2\pi$ and $\pi < y < 2\pi$) square, $\sin y$ is negative and $\sin y$ is negative, so $\text{sign}(\sin x \sin y) = 1$. Due to the periodic nature of sin, this plot may be tessellated to cover all of $\mathbb{R}^2$. The level sets are shown as a contour plot below:



In the diagram above, all white spaces should either be shaded yellow/blue, depending on the border of the square. Teal corresponds to 0.

12 (Challenge)

Find the limit, if it exists, or show that the limit does not exist

$$\lim_{r \to \infty} \frac{\ln\left(x^2 y^2 z^2\right)}{x^2 + y^2 + z^2}$$

Hint: Consider the limits along the curves $x = y = z = t$ and $x = e^{-t^2}, y = z = t$

Solution

Step 1: The continuity argument does not apply because f is not defined at $\mathbf{r}_0$.

Step 2: No substitution is possible to transform the limit to a one-variable limit.

Step 3: First let $x(t) = y(t) = z(t) = t$. Then the limit becomes

$$\lim_{\mathbf{r}\to\infty} \frac{\ln\left(x^2y^2z^2\right)}{x^2+y^2+z^2} = \lim_{t\to\infty} \frac{\ln\left(t^6\right)}{3t^2} = \lim_{t\to\infty} \frac{2\ln t}{t^2} = \lim_{t\to\infty} \frac{2(1/t)}{2t} = 0$$

Now let $x(t) = e^{-t^2}$ and $y(t) = z(t) = t$. Then the limit becomes

$$\begin{aligned}\lim_{\mathbf{r}\to\infty} \frac{\ln\left(x^2y^2z^2\right)}{x^2+y^2+z^2} &= \lim_{t\to\infty} \frac{\ln\left(e^{-2t^2}t^4\right)}{e^{-2t^2}+2t^2} = \lim_{t\to\infty} \frac{-2t^2+4\ln(t)}{e^{-2t^2}+2t^2} = \lim_{t\to\infty} \frac{-4t+4/t}{-4te^{-2t^2}+4t} \\ &= \lim_{t\to\infty} \frac{-t^2+1}{-t^2e^{-2t^2}+t^2} = \lim_{t\to\infty} \frac{-2t}{-2te^{-2t^2}+4t^3e^{-2t^2}+2t} \\ &= \lim_{t\to\infty} \frac{-1}{-e^{-2t^2}+2^3e^{-2t^2}+1} = -1\end{aligned}$$

therefore the limit $\boxed{\text{does not exist}}$ because it is path dependent.

13 (Challenge)

$$\lim_{\mathbf{r}\to\infty} \frac{e^{3x^2+2y^2+z^2}}{\left(x^2+2y^2+3z^2\right)^{2012}}$$

Hint: Consider the inequality $1+u \le e^u$

Solution

Step 1: The continuity argument does not apply because f is not defined at $\mathbf{r}_0$.

Step 2: No substitution is possible to transform the limit to a one-variable limit.

Step 3: As a guess, let $x(t) = y(t) = z(t) = t$. Then the limit becomes

$$\lim_{\mathbf{r}\to\infty} \frac{e^{3x^2+2y^2+z^2}}{\left(x^2+2y^2+3z^2\right)^{2012}} = \lim_{t\to\infty} \frac{e^{6t^2}}{\left(6t^2\right)^{2012}} = \lim_{u\to\infty} \frac{e^u}{u^{2012}} = \ldots = \lim_{u\to\infty} \frac{e^u}{2012!} = \infty$$

where the substitution $u = 6t^2$ has been used, and l'Hospital's rule has been used many times.

Step 4: Consider the function $g(x,y,z) = \left(1+3x^2+2y^2+\ z^2\right) / \left(x^2+2y^2+3z^2\right)$. Owing to the known inequality $1+u \le e^u$, we have that

$$g \le f(x,y,z) = e^{3x^2+2y^2+z^2} / \left(x^2+2y^2+3z^2\right)$$

It suffices to show that $\lim_{\mathbf{r}\to\infty} g(\mathbf{r}) = \infty$, due to the squeeze theorem. Let $R = \sqrt{x^2+y^2+z^2}$. For limits at infinity we need that $g(\mathbf{r}) > M$ for every $M > 0$ if $\|\mathbf{r}\| > \delta$.

14

Find the repeated limits

$$\lim_{x\to1}\left(\lim_{y\to0} \log_x(x+y)\right) \text{ and } \lim_{y\to0}\left(\lim_{x\to1} \log_x(x+y)\right)$$

What can be said about the corresponding two-variable limit?

Solution

SOLUTION: First recall the change of base formula

$$\log_a b = \frac{\ln b}{\ln a}$$

(proof: set $\log_a b = k$ then $b = a^k$, and $\ln b = \ln a^k = k \ln a$. So $k = \ln b / \ln a$). The repeated limits are computed below:

$$\lim_{x \to 1} \left(\lim_{y \to 0} \frac{\ln(x+y)}{\ln x} \right) = \lim_{x \to 1} \left(\frac{\ln(x+0)}{\ln x} \right) = \lim_{x \to 1} (1) = 1$$
$$\lim_{y \to 0} \left(\lim_{x \to 1} \frac{\ln(x+y)}{\ln x} \right) = \lim_{y \to 0} \left(\frac{\ln(1+y)}{\ln 1} \right) = \infty$$

So the multivariable limit $\boxed{\text{does not exist}}$.

15 (Challenge)

Find the specified partial derivatives of each of the following function:

$$f(x, y, z) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{r}), \text{ where } \mathbf{a} \text{ and } \mathbf{b} \text{ are constant vectors; } \mathbf{r} \text{ is the radial vector field}$$

Solution

We have, by the product rule,

$$f'_u = \mathbf{a}'_u \cdot (\mathbf{b} \times \mathbf{r}) + \mathbf{a} \cdot (\mathbf{b} \times \mathbf{r})'_u = \mathbf{a} \cdot (\mathbf{b}'_u \times \mathbf{r} + \mathbf{b} \times \mathbf{r}'_u) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{r}'_u)$$

since $\mathbf{a}'_u = \mathbf{b}'_u = 0$ (because they are constant). Moreover, $\mathbf{r}'_x = \langle 1, 0, 0 \rangle$, $\mathbf{r}'_y = \langle 0, 1, 0 \rangle$, and $\mathbf{r}'_z = \langle 0, 0, 1 \rangle$. So, $\mathbf{b} \times \mathbf{r}'_x = \langle 0, b_3, -b_2 \rangle$, $\mathbf{b} \times \mathbf{r}'_y = \langle -b_3, 0, b_1 \rangle$, and $\mathbf{b} \times \mathbf{r}'_z = \langle b_2, -b_1, 0 \rangle$ Thus we have

$$f'_x = a_2 b_3 - a_3 b_2 = (\mathbf{a} \times \mathbf{b})_x$$
$$f'_y = -a_1 b_3 + a_3 b_1 = (\mathbf{a} \times \mathbf{b})_y$$
$$f'_z = a_1 b_2 - a_2 b_1 = (\mathbf{a} \times \mathbf{b})_z$$

Another approach, calculate the cross-product first

$$\langle \; b_1, \quad b_2, \quad b_3 \; \rangle \times \langle \; x, \quad y, \quad z \; \rangle$$

which is equivalent to

$$\langle b_2 z - b_3 y, \quad b_3 x - b_1 z, \quad b_1 y - b_2 x \rangle$$

then calculate the dot product,

$$f(x, y, z) = \langle \; a_1, \quad a_2, \quad a_3 \; \rangle \cdot \langle \; b_2 z - b_3 y, \quad b_3 x - b_1 z, \quad b_1 y - b_2 x \; \rangle$$
$$= a_1 (b_2 z - b_3 y) + a_2 (b_3 x - b_1 z) + a_3 (b_1 y - b_2 x)$$

calculating partial derivatives, we have

$$\boxed{\begin{aligned} f'_x &= a_2 b_3 - a_3 b_2 = (\mathbf{a} \times \mathbf{b})_x \\ f'_y &= -a_1 b_3 + a_3 b_1 = (\mathbf{a} \times \mathbf{b})_y \\ f'_z &= a_1 b_2 - a_2 b_1 = (\mathbf{a} \times \mathbf{b})_z \end{aligned}}$$

the same answer as in the first approach.

16

Find the integral of $f(x, y, z) = z \left(x^2 + y^2 + z^2\right)^{-7/4}$ over the half-ball $x^2 + y^2 + z^2 \leq 1, z \geq 0$, if it exists.

Solution

The function is non-negative in the region of integration and singular at the origin. Therefore if the improper integral of f exists for a particular regularization, it exists for any regularization and has the same value. Let us regularize the improper integral in question. Put

$$E_\varepsilon = \left\{(x, y, z) \mid \varepsilon^2 \leq x^2 + y^2 + z^2 \leq 1, z \geq 0\right\}$$

The region E_ε is the image of the rectangular box $E'_\varepsilon = [\varepsilon, 1] \times [0, \pi/2] \times [0, 2\pi]$ in spherical coordinates. By converting the integral of f over E_ε to spherical coordinates and using Fubini's theorem to evaluate it,

$$\begin{aligned}
\iiint_{E_\varepsilon} f\,dV &= \iiint_{E'_\varepsilon} \rho\cos\phi \cdot \rho^{-7/2} \cdot \rho^2 \sin\phi\,dV' \\
&= \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\phi\cos\phi\,d\phi \int_\varepsilon^1 \rho^{-1/2}\,d\rho \\
&= 2\pi \cdot \left(\frac{1}{2}\sin^2\phi\Big|_0^{\pi/2}\right) \cdot \left(2\rho^{1/2}\Big|_\varepsilon^1\right) \\
&= 2\pi(1 - \sqrt{\varepsilon}).
\end{aligned}$$

Taking the limit $\varepsilon \to 0^+$,

$$\iiint_E f\,dV = \lim_{\varepsilon\to 0^+} \iiint_{E_\varepsilon} f\,dV = \lim_{\varepsilon\to 0^+} 2\pi(1 - \sqrt{\varepsilon}) = \boxed{2\pi}$$

17

Two spacecraft are following paths in space given by $\mathbf{r}_1 = \langle \sin t, t, t^2 \rangle$ and $\mathbf{r}_2 = \langle \cos t, 1 - t, t^3 \rangle$. If the temperature for points in space are given by $T(x, y, z) = x^2 y(1 - z)$, use the chain rule to determine the rate of change of the difference D in the temperatures the two spacecraft experience at time $t = \pi$.

Solution

Let T_1 describe the temperature experienced by the first spaceship and T_2 the temperature experienced by the second spaceship. Then the difference in temperatures between the two spacecraft is given by $D = T_1 - T_2$. The rate of change of this difference is $\frac{dD}{dt}$. Using the sum rule for derivatives we know

$$\begin{aligned}
\frac{dD}{dt} &= \frac{d}{dt}(T_1 - T_2) \\
&= \frac{dT_1}{dt} - \frac{dT_2}{dt}.
\end{aligned}$$

So we want to compute the two derivatives in the last line. To compute each of these derivatives, we will use the chain rule.

Note that the path $\mathbf{r}_1$ tells us about x, y and z as a function of t, for the first space craft. In particular, $x_1(t) = \sin t, y_1(t) = t$ and $z_1(t) = t^2$. By the chain rule we have

$$\frac{dT_1}{dt} = \frac{\partial T_1}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial T_1}{\partial y_1}\frac{\partial y_1}{\partial t} + \frac{\partial T_1}{\partial z_1}\frac{\partial z_1}{\partial t} = \nabla T_1\left(\mathbf{r}_1(t)\right) \cdot \mathbf{r}'_1(t)$$

Taking the partial derivatives we get

$$\frac{dT_1}{dt} = 2x_1 y_1\left(1 - z_1\right)(\cos t) + x_1^2\left(1 - z_1\right)(1) + \left(-x_1^2 y_1\right)(2t)$$

However, we still need to put x_1, y_1, and z_1 in terms of t. Making this substitution we obtain

$$\frac{dT_1}{dt} = 2(\sin t)(t)\left(1 - t^2\right)(\cos t) + (\sin t)^2\left(1 - t^2\right)(1) + \left(-(\sin t)^2 t\right)(2t)$$

Lastly, we want to determine what this derivative is specifically when $t = \pi$. Note that $\sin\pi = 0$ and every term in our expression is a multiple of $\sin\pi$. This implies $\frac{dT_1}{dt} = 0$.

Now, we need to compute the derivative of T_2 with respect to t. As before, we will do so using the chain rule. By the chain rule,

$$\frac{dT_2}{dt} = \frac{\partial T_2}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial T_2}{\partial y_2}\frac{\partial y_2}{\partial t} + \frac{\partial T_2}{\partial z_2}\frac{\partial z_2}{\partial t} = \nabla T_2\left(\mathbf{r}_2(t)\right)\cdot \mathbf{r}_2'(t)$$

Taking these derivatives and then making the substitutions for x_2, y_2 and z_2 as functions of t we obtain

$$\begin{aligned}\frac{dT_2}{dt} &= 2x_2y_2\left(1-z_2\right)\left(-\sin t\right) + x_2^2\left(1-z_2\right)\left(-1\right) + \left(-x_2^2y_2\right)\left(3t^2\right)\\ &= 2(\cos t)(1-t)\left(1-t^3\right)\left(-\sin t\right) + (\cos t)^2\left(1-t^3\right)\left(-1\right) + \left(-(\cos t)^2(1-t)\right)\left(3t^2\right).\end{aligned}$$

At time $t = \pi$, the first term in the expression of our derivative becomes 0 . The rest simplifies as follows:

$$\begin{aligned}\frac{dT_2}{dt} &= (\cos\pi)^2\left(1-(\pi)^3\right)(-1) + \left(-(\cos\pi)^2(1-\pi)\right)\left(3(\pi)^2\right)\\ &= -\left(1-\pi^3\right) - (1-\pi)\left(3\pi^2\right)\\ &= 4\pi^3 - 3\pi^2 - 1.\end{aligned}$$

Combining this with the result of our first derivative, we obtain

$$\frac{dD}{dt} = 0 - \left(4\pi^3 - 3\pi^2 - 1\right) = \boxed{-4\pi^3 + 3\pi^2 + 1}.$$

18

If $u(x, y)$ is a solution to the Laplace Equation in the plane, what is the value of the line integral

$$\int_{\partial D} u_y\,dx - u_x\,dy$$

when C is a simple closed curve oriented counterclockwise? Assume that $u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad \text{for all } (x,y)\in D$.

Solution

Any function f satisfying Laplace's equation $u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad \text{for all } (x,y)\in D$ can be used as either a potential function for a conservative vector field or a stream function for a source free vector field. From the statement of the problem, we know that we can apply Green's Theorem to $\int_{\partial D} u_y\,dx - u_x\,dy$. We set $P = u_y$ and $Q = u_x$. We obtain

$$\int_{\partial D} u_y\,dx - u_x\,dy = \int_D -u_{xx} - u_{yy}\,dx dy = -\int_D u_{xx} + u_{yy}\,dx dy = 0$$

Where we have set the integral equal to zero since the integrand is zero everywhere in the disk. By Green's theorem, the integral value is $\boxed{0}$.

19

Find the specified partial derivatives of the function $f(\mathbf{r}) = \exp(\mathbf{a}\cdot\mathbf{r})$, where $\mathbf{a}\cdot\mathbf{a} = 1$ and $\mathbf{r}\in\mathbb{R}^m$, $f''_{x_1x_1} + f''_{x_2x_2} + \ldots + f''_{x_mx_m} = f$

Solution

First calculate the necessary partial derivatives as follows:

$$f''_{x_ix_i} = \left(\exp(\mathbf{a}\cdot\mathbf{r})\left[\mathbf{a}'_{x_i}\cdot\mathbf{r} + \mathbf{a}\cdot\mathbf{r}'_{x_i}\right]\right)'_{x_i} = a_i(\exp(\mathbf{a}\cdot\mathbf{r}))'_{x_i} = a_i^2\exp(\mathbf{a}\cdot\mathbf{r})$$

for $i = 1, 2, \ldots, m$. Note that $\exp(x) = e^x$. Then we have that

$$a_1^2\exp(\mathbf{a}\cdot\mathbf{r}) + a_2^2\exp(\mathbf{a}\cdot\mathbf{r}) + \ldots + a_m^2\exp(\mathbf{a}\cdot\mathbf{r}) = \exp(\mathbf{a}\cdot\mathbf{r})\sum_{k=1}^m a_k^2 = [\exp(\mathbf{a}\cdot\mathbf{r})](\mathbf{a}\cdot\mathbf{a}) = f$$

where it is assumed that $\mathbf{a}\cdot\mathbf{a} = 1$.

20

(a) Let $\mathbf{a} = s\hat{\mathbf{u}} + \hat{\mathbf{v}}$ and $\mathbf{b} = \hat{\mathbf{u}} + s\hat{\mathbf{v}}$ where the angle between unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ is $\pi/3$. Find the values of s for which the dot product $\mathbf{a} \cdot \mathbf{b}$ is maximal, minimal, or zero if such values exist. Do you notice anything special about these values?

(b) Let $\mathbf{a} = s\hat{\mathbf{u}} + w\hat{\mathbf{v}}$ and $\mathbf{b} = w\hat{\mathbf{u}} + s\hat{\mathbf{v}}$ where the angle between unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ is $\pi/3$. Find values of s and w for which the dot product $\mathbf{a} \cdot \mathbf{b}$ is maximal, minimal, or zero if such values exist. Do you notice anything special about these values?

Solution

(a) First let us compute $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}}$. By the geometric properties of the dot product,

$$\begin{aligned} \hat{\mathbf{u}} \cdot \hat{\mathbf{v}} &= \frac{\cos\theta}{\|\hat{\mathbf{u}}\|\|\hat{\mathbf{v}}\|} \\ &= \frac{\cos\pi/3}{(1)(1)} = \frac{1}{2} \end{aligned}$$

Note that since the dot product is commutative $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = \hat{\mathbf{v}} \cdot \hat{\mathbf{u}}$, and that $\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = \|\hat{\mathbf{u}}\|$. Next we will compute $\mathbf{a} \cdot \mathbf{b}$ as follows

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (s\hat{\mathbf{u}} + \hat{\mathbf{v}}) \cdot (\hat{\mathbf{u}} + s\hat{\mathbf{v}}) \\ &= s\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} + s\hat{\mathbf{u}} \cdot s\hat{\mathbf{v}} + \hat{\mathbf{v}} \cdot \hat{\mathbf{u}} + \hat{\mathbf{v}} \cdot s\hat{\mathbf{v}} \\ &= s\|\hat{\mathbf{u}}\| + \left(s^2 + 1\right) \hat{\mathbf{u}} \cdot \hat{\mathbf{v}} + s\|\hat{\mathbf{v}}\| \\ &= s + \left(s^2 + 1\right)\left(\frac{1}{2}\right) + s \\ &= \frac{1}{2}\left(s^2 + 4s + 1\right) = \frac{1}{2}\left(s^2 + 4s + 4 - 3\right) = \frac{1}{2}(s+2)^2 - \frac{3}{2} \end{aligned}$$

The function $f(s) = \frac{1}{2}(s+2)^2 - \frac{3}{2}$ is a parabola opening upwards. Hence there is no maximum value. There is a single minimum occurring at the vertex, when $s = -2$. The zeroes occur when $f(s) = 0$,

$$\begin{aligned} \frac{1}{2}(s+2)^2 - \frac{3}{2} &= 0 \\ (s+2)^2 &= 3 \\ s + 2 &= \pm\sqrt{3} \\ s &= \boxed{-2 \pm \sqrt{3}} \end{aligned}$$

(b)
As noted in part (a), by the geometric properties of the dot product, the value of $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = \frac{1}{2}$. We can compute $\mathbf{a} \cdot \mathbf{b}$ as follows

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (s\hat{\mathbf{u}} + w\hat{\mathbf{v}}) \cdot (w\hat{\mathbf{u}} + s\hat{\mathbf{v}}) \\ &= s\hat{\mathbf{u}} \cdot w\hat{\mathbf{u}} + s\hat{\mathbf{u}} \cdot s\hat{\mathbf{v}} + w\hat{\mathbf{v}} \cdot w\hat{\mathbf{u}} + w\hat{\mathbf{v}} \cdot s\hat{\mathbf{v}} \\ &= sw\|\hat{\mathbf{u}}\| + s^2\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} + w^2\hat{\mathbf{v}} \cdot \hat{\mathbf{u}} + ws\|\hat{\mathbf{v}}\| \\ &= sw + \frac{1}{2}s^2 + \frac{1}{2}w^2 + sw \\ &= 2sw + \frac{1}{2}s^2 + \frac{1}{2}w^2 \end{aligned}$$

The function $f(s, w) = 2sw + \frac{1}{2}s^2 + \frac{1}{2}w^2$ is a hyperbolic paraboloid opening upwards. Hence, there is no maximum value. There is a single value occuring at the vertex when $f(s, w) = 0$. To classify critical points formally, allow for the substitution $x = s$ and $y = w$

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{x^2}{2} + 2xy + \frac{y^2}{2}\right) &= x + 2y \\ \frac{\partial}{\partial y}\left(\frac{x^2}{2} + 2xy + \frac{y^2}{2}\right) &= 2x + y \end{aligned}$$

Next, we solve the system $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases}$. The system has the following real solution: $(x, y) = (0, 0)$. By the discriminant, $D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial y \partial x}\right)^2 = -3$. For completeness the partial derivatives are

$$\frac{\partial^2}{\partial x^2}\left(\frac{x^2}{2} + 2xy + \frac{y^2}{2}\right) = 1$$
$$\frac{\partial^2}{\partial y \partial x}\left(\frac{x^2}{2} + 2xy + \frac{y^2}{2}\right) = 2$$
$$\frac{\partial^2}{\partial y^2}\left(\frac{x^2}{2} + 2xy + \frac{y^2}{2}\right) = 1$$

Since $D(0, 0) = -3$ is less than 0 , it can be stated that $(0, 0)$ is a saddle point at which $f(s, w) = 0$. Thus, the dot product is zero when $\boxed{s = w = 0}$.