AEW Worksheet 12

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MATH 1920

Name: ______________________

Collaborators: ______________________

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1

Determine if the following statements are true(T) or false(F). Mark the correct answer. No justification needed.

(a) $\boxed{T}$ $\boxed{F}$ The flux of $F = \langle x, 0, 0\rangle$ across a sphere of radius 1 at the origin is strictly less than its flux across a sphere of radius 2 at the origin, where both are outwardly oriented.

(b) $\boxed{T}$ $\boxed{F}$ If S is be the unit sphere centered at the origin, oriented outwards with normal vector $\mathbf{n}$ and the integral $I = \iint_S D_{\mathbf{n}} f dS$ where $D_{\mathbf{n}}$ is the directional derivative along $\mathbf{n}$, then $I = \iiint_E div(\nabla f)dV$ where E is a solid sphere (assume f is a continuous function).

(c) $\boxed{T}$ $\boxed{F}$ If $\vec{F} = \left(x - \frac{2}{3}x^3, \frac{-4}{3}y^3, \frac{-8}{3}z^3\right)$ and $\mathcal{J} = \iint_S \vec{F}\cdot\vec{n}dS$, then $\mathcal{J}$ is maximized with surface S described as $1 = 2x^2 + 4y^2 + 8z^2$

2

True or False? If *true*, provide a justification and an example. If *false*, explain why or provide a counterexample.

(a) Suppose we are given a closed surface S and a vector field with the property such that $\vec{F} = \text{curl}(\vec{F})$. Then via Stokes' and Divergence Theorem,

$$\oint_{\partial S} \mathbf{F}\cdot d\mathbf{r} = \iint_{\mathcal{S}} \text{curl}(\mathbf{F})\cdot d\mathbf{S} = \iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \iiint_{\mathcal{W}} \text{div}(\mathbf{F})dV$$

(b) Suppose we are given a simple closed curve $\mathcal{C}$ and a constant vector field $\vec{F}$. Then via Green's and Divergence Theorem,

$$\oint_{\partial\mathcal{D}} \mathbf{F}\cdot\mathbf{n}\, ds = \iint_{\mathcal{D}} \text{div}(\mathbf{F})dA = \iiint_{\mathcal{W}} \text{div}(\mathbf{F})dV$$

(c) Suppose we are given a simple closed curve $\mathcal{C}$ and a vector field with the property such that $\vec{F} = \text{curl}(\vec{F})$. Then via Green's, Stokes', and Divergence Theorem,

$$\oint_{\partial\mathcal{D}} \mathbf{F}\cdot d\mathbf{r} = \iint_{\mathcal{D}} \text{curl}_z(\mathbf{F})dA = \oint_{\partial S} \mathbf{F}\cdot d\mathbf{r} = \iint_{\mathcal{S}} \text{curl}(\mathbf{F})\cdot d\mathbf{S} = \iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \iiint_{\mathcal{W}} \text{div}(\mathbf{F})dV$$

(d) If $\vec{F}$ is a non-conservative vector field, then $\vec{F}$ does not influence the previous scenarios.

(e) The specific orientation(s) of the surface or curve in the previous scenarios (a - c) does not have any influence on any of the statements.

3

Let S_r denote the sphere of radius r with the center at the origin, with outward orientation. Suppose that $\mathbf{E}$ is a vector field well-defined on all of $\mathbb{R}^3$ and such that

$$\iint_{S_r} \mathbf{E}\cdot d\mathbf{S} = ar + b$$

for some fixed constants a and b.

(a) Compute in terms of a and b the following integral:

$$\iiint_D \operatorname{div} \mathbf{E} dV$$

where $D = \left\{(x, y, z) \mid 25 \leq x^2 + y^2 + z^2 \leq 49\right\}$

(b) Suppose that in the above situation $\mathbf{E} = \operatorname{curl} \mathbf{F}$ for some vector field $\mathbf{F}$. What conditions, if any, does this place on the constants a and b?

4

Consider the vector field

$$\vec{F}(x, y, z) = \left\langle z^2 + y \sin(yz), 2xze^{z^2} - y - z, x^2 + y^2 + z \right\rangle$$

It is known that $\vec{F} = \operatorname{curl} \vec{G}$ for some vector field $\vec{G}(x, y, z)$. (You do not need to check this fact). Now consider the sphere $x^2 + y^2 + z^2 = a^2$. Suppose, the plane $z = a^2$ divides the sphere into two parts (given that a is a real number). Let S be the part that is below the plane. Evaluate the integral

$$\iint_S \vec{F} \cdot d\vec{S}$$

where S is oriented "downward". For what value(s) of a (if any) does the integral above have its maximum or minimum value?

5 (Challenge)

Suppose S is the portion of the surface $z = 1 - x^2 - y^2$ above the $xy-$plane and the vector field is given below.

$$\vec{F} = \langle e^{x+y+z}, -e^{x+y+z}, x^2 + y^2 \rangle$$

(a) Find I given the following:

$$I = \iint_S \vec{F} \cdot \vec{n} \, dA$$

where $\vec{n}$ is in the upwards direction.

(b) If $\vec{F} = \vec{\nabla} \times \vec{G}$, where

$$\vec{G}(x, y, z) = \langle G_1, G_2, G_3 \rangle$$

Find possible values of G_1, G_2, G_3 and show that

$$\oint_C \vec{G} \cdot d\vec{r} = 2 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \frac{2}{9} - \cdots$$

6 *(Challenge)*

Consider the vector field

$$\vec{F}(x, y, z) = \langle x + y, y + e^z, y^5 \rangle$$

Calculate the value of $\iint_S \operatorname{curl}(F) \cdot d\vec{S}$, given that fact that S is the portion of the sphere $x^2 + y^2 + z^2 = a^2 + b^2$ *strictly* between the planes $z = a^2$ and $z = b^2$ (without the top and bottom), oriented outwards, where a and b are constants such that $a > b$. For what value(s) of a and b (if any) does the integral $\iint_S \operatorname{curl}(F) \cdot d\vec{S}$ have its maximum or minimum value (if any), if a and b are constrained such that $a^2 + b^2 \leq \frac{3}{4}$?