AEW
Auxiliary Problems

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MATH 1920

Name: ______________________________

Collaborators: ______________________________

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1

Is there a number b such that

$$\lim_{(x,y)\to(0,0)} \left(\frac{b(xy-2)^2 + 15(xy-2) + 15 + b}{(xy-2)^2 + (xy-2) - 2} \right)$$

exists? If so, find the value of b and the value of the limit.

2

Evaluate each of the following limits or state that it does not exist. Justify your answer.

$$\lim_{(x,y)\to(\infty,\infty)} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2}, \qquad \lim_{(x,y)\to(-\infty,-\infty)} \frac{e^{-3\left(x^2+y^2\right)} - 2e^{8\left(x^2+y^2\right)}}{9e^{8\left(x^2+y^2\right)} - 7e^{-3\left(x^2+y^2\right)}}$$

3 (Challenge)

The equation below describes an unique function defined by an exponential diophantine equation. In this problem, the equation is defined implicitly by a parameter C to generate level surfaces. For this problem you should assume that C is a multiple of 2^n where n is a positive integer.

$$f(x, y, z) = 2^x + 4^y + 8^z = C$$

(a) Describe the level surfaces of $f(x, y, z)$ in the scenario where $C = 2^n$ for values $n = 3, 5, 7$

(b) Sketch the graph of $f(x, y, z)$ in the scenario where $C = 2^n$, $y = 0$ and $n = 3$

(c) Given $x, y, z \in \mathbb{N}$, find any ordered triple of x, y, z that satisfies the equation $f(x, y, z) = C = 328$.

4

Fill in the table below:

Name	Equation	Vertical Trace	Horizontal Trace	Which Way is Up
	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	Ellipse		
Elliptic Paraboloid			Ellipse	
	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}$	Parabola		the unsquared variable is "up". The variable which is negative has downward parabolic traces in that direction.
Hyperboloid (One Sheet)			Ellipse	
Hyperboloid (Two Sheets)		Hyperbola		
	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$	Hyperbola or two lines		The lonely one (here, z)
Elliptic Cylinder		Two parallel lines	Ellipse	

5

Use the limit definition of partial derivative to compute the partial derivatives of the functions at the specified points.

$$f(x, y) = \sqrt{2x + 3y - 1}, \quad \frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} \quad \text{at } (-2, 3)$$

6

The directional derivative is usually given by

$$D_{\vec{u}} f\,(x, y) = \nabla f \cdot \vec{u}$$

Suppose the second directional derivative of $f(x, y)$ is

$$D^2_{\mathbf{u}} f(x, y) = D_{\mathbf{u}}\left[D_{\mathbf{u}} f(x, y)\right]$$

(a) If $f(x, y) = x^3 + 5x^2y + y^3$ and $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$, calculate $D^2_u f(2, 1)$

(b) If $\mathbf{u} = \langle a, b \rangle$ is a unit vector and f has continuous second partial derivatives, show that

$$D^2_u f = f_{xx} a^2 + 2f_{xy} ab + f_{yy} b^2$$

(c) Find the second directional derivative of $f(x, y) = xe^{2y}$ in the direction of $\mathbf{v} = \langle 4, 6 \rangle$

7

Find the critical points of function

$$f(x, y) = \sqrt{4y^2 - 9x^2 + 24y + 36x + 36}$$

8

A cat-food company makes its food from chicken, which costs 25 cents per ounce, and beef, which costs 20 cents per ounce. Chicken has 10 grams of protein and 4 grams of fat per ounce, while beef has 5 grams of protein and 8 grams of fat per ounce. Each package of food must weigh between 10 and 16 ounces, and it must also have at least 95 grams of protein and at least 80 grams of fat. How much chicken and beef should the company use in each package to minimize the total cost while also satisfying these requirements?

9

(a) Find the equation of the tangent plane to the surface

$$x^{1/3} + y^{1/3} + z^{1/3} = 1$$

at (x_0, y_0, z_0). Show that the surn of the square root of the $x-$, $y-$, and z -intercepts of any tangent plane is 1.

(b) Assume that $x^{1/3} + y^{1/3} + z^{1/3} = 1$. Find the partial derivatives of z with respect x and y. Approximate the value of z when $x = 1.01$ and $y = .97$.

(c) Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex on the surface $x^{1/3} + y^{1/3} + z^{1/3} = 1$.

10 (Challenge)

Given two equivalent functions below use their gradients to find values for α and β. For full credit the gradient must be used here. Assume the following (no typos here):

$$f(y, x) = 2y + \tan^{-1}(\alpha\beta x)$$

$$f(x, y) = \alpha\pi y + \alpha\beta \sin^{-1}(\frac{x}{\sqrt{1+x^2}})$$

11

Ant-ony the ant is currently on a metal sheet. Ant-ony is very particular about the temperature of where he is standing, and in particular if the temperature, as measured by the Romer scale, is not at 20 then Ant-ony will move in the direction from his current position which has the greatest rate of temperature change that will get Ant-ony closer to 20. Given that the temperature at a point (x, y) on the metal sheet is given by:

$$T(x, y) = \frac{40y}{1+x^2} - 2xy$$

Determine how Ant-ony will react if placed at the following points. (Namely, will he stay put, or move. If Ant-ony moves, give a vector indicating which way he initially moves (it does not have to be a unit vector).)

(a) (x, y) = (1, 2)

(b) (x, y) = (2, 5)

(c) (x, y) = (3, 6)

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Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration

$$d\rho\, d\phi\, d\theta$$

$$d\phi\, d\rho\, d\theta$$

13

Let C be the curve

$$r(t) = \left\langle \cos\left(\pi t^2\right) + t^{137}, e^{t(1-t)} - \sin(\pi t/2) \right\rangle \quad t \in [0, 1]$$

and let

$$\mathbf{F} = \langle 3y^2 + \cos(x+y), -\cos(x+y) \rangle$$

Find $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$.

14

For each of the following vector fields F, decide whether it is conservative or not by computing curl F. Find the potential function if conservative. Assume the potential function has a value of zero at the origin.

(a)

$$F(x, y, z) = -3x\hat{\imath} - 2y\hat{\jmath} + \hat{k}$$

(b)

$$F(x, y, z) = -3x^2\hat{\imath} + 5y^2\hat{\jmath} + 5z^2\hat{k}$$

15 (Challenge)

The heat flow vector field for conducting objects is $\mathbf{F} = -k\nabla T$, where $T(x, y, z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material. Compute the outward flux of across the following surface S for the given temperature distribution (assume k = 1)

$$T(x, y, z) = 100e^{-x-y}$$

where S consists of the faces of the cube $|x| \leq 1, |y| \leq 1, |z| \leq 1$

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A hot air balloon known as the **TARDIS** (for Tethered Aerial Release Developed InStyle) has the shape of the surface S given by the part of ellipsoid $2x^2 + 2y^2 + z^2 = 9$ with $-1 \leq z \leq 3$. The hot gases that the balloon uses to fly have a velocity vector field given by $\mathbf{v} = \nabla \times \mathbf{F}$, where $\mathbf{F}(x, y, z) = \langle -y, x, xy + z^2 \rangle$. The rate at which the gases escape from the balloon is equal to the flux of $\mathbf{v}$ across the surface of the balloon, given by

$$\iint_S v \cdot d\vec{S}$$

where S is given the outward orientation (away from the z -axis). Calculate the rate at which the gases escape from the balloon.

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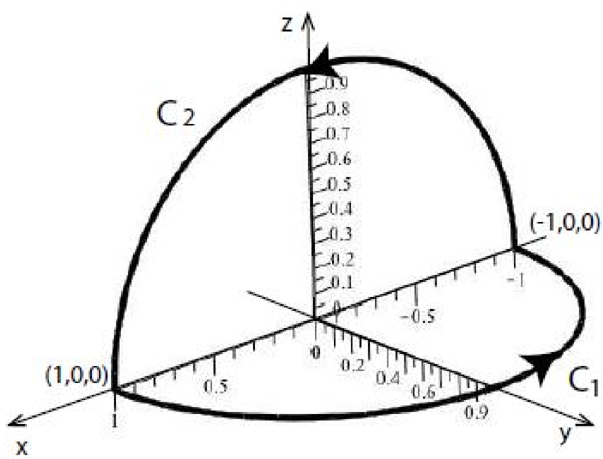
Let C_1 be the curve from the point (1,0,0) to the point (-1,0,0) along the circle $x^2 + y^2 = 1$ on the xy -plane, and let C_2 be the curve from the point (-1,0,0) to the point (1,0,0) along the circle $x^2 + z^2 = 1$ on the xz -plane. The curves are shown in the picture. Let C be the union of C_1 and C_2. Evaluate the work done by the vector field

$$\vec{F}(x, y, z) = x\vec{i} + (x - 2yz)\vec{j} + \left(x^2 + z^4\right)\vec{k}$$

in moving a particle along C.



18

Suppose that S_1 is the set of points on the sphere $x^2+y^2+z^2 = 1$ which are not inside the sphere $x^2+y^2+(z+1)^2 = 1$ and suppose that S_2 is the set of points on the sphere $x^2 + y^2 + (z + 1)^2 = 1$ which are not inside the sphere

$x^2 + y^2 + z^2 = 1$. We may interpret S_1 and S_2 as surfaces carrying an orientation from the inside to the outside. Find

$$\iint_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{A} \quad \text{and} \quad \iint_{S_2} \nabla \times \mathbf{F} \cdot d\mathbf{A}$$

where $\mathbf{F} = \langle yz, x, e^{xyz} \rangle$.

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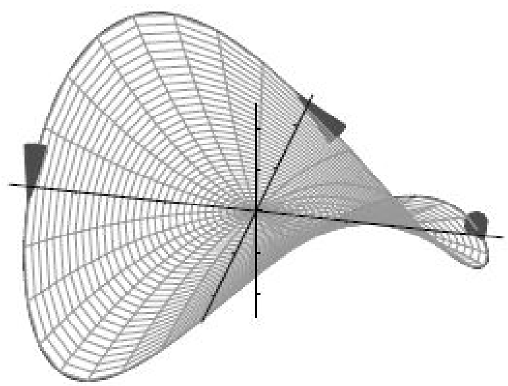
Find the circulation of the field,

$$\mathbf{F} = (3xz - y)\mathbf{i} + (xz + yz)\mathbf{j} + \left(x^2 + y^2\right)\mathbf{k}$$

along the boundary of the Pringles potato chip (i.e., the part of the surface $z = xy$ contained inside the cylinder $x^2 + y^2 = 1$) oriented as shown.

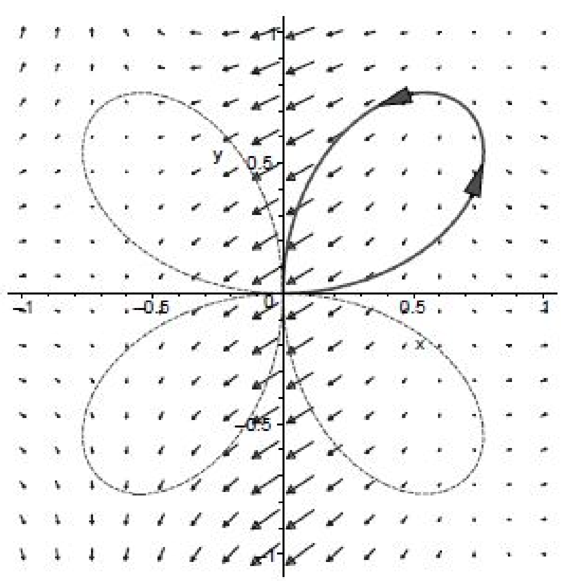


20

Find the circulation of the vector field

$$\mathbf{F} = (\cos y + 2\ln x)\mathbf{i} + \left(\ln x + y^3 - x \sin y\right)\mathbf{j}$$

over the first-quadrant petal of a four-petal rose given in polar coordinates by the equation $r = \sin(2\theta)$.



21 (Challenge)

Let R be the region defined by $x^2+y^2+z^2 \leq 1$. Use the divergence theorem to evaluate

$$\iiint_R z^2 \, dV$$

22

Evaluate the following integral:

$$I = \int_0^1 \int_0^1 \sin\left(\max\left\{x^2, y^2\right\}\right) dx dy$$

23

Evaluate the integral

$$I = \iint_S z^{2018} dS, \quad S = \left\{x^2+y^2+z^2 = 1\right\}$$

24

Set up, but do NOT evaluate, the following double integral.

$$f(x,y) = |16xy|, \quad x^2+y^2 \leq 25$$

Note: For this problem consider the four coordinates.

25

Determine all $a \in \mathbb{R}$ such that

$$1 = \int_e^{a^e} \frac{dx}{x \int_a^{ax} \frac{dy}{y}}$$

26

Find the minimum value of $f(x,y)$ subject to the constraint $g(x,y) = 0$.

$$f(x,y) = x^2+y^2, \quad g(x,y) = (y-1)^3 - x^2.$$

Do you notice anything special about this problem?

27 (Challenge)

In linear algebra, the quadratic form $Q(\mathbf{x}) = (\mathbf{x}, \mathbf{A}\mathbf{x})$ is said to be positive definite when $Q(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$, where $\mathbf{x}$ is a defined vector. Suppose, there exists such a function $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ where

$$Q(\mathbf{x}) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2$$

Show that there exists a maximum and minimum within the unit disk via multivariable calculus.

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Let $f(x, y)$ be a continuous function, a and b non-negative constants, and $\mathbb{D}$ be a triangular region in the first quadrant bounded by the lines $x = 0, y = 0$, and $y = 9 - x$. Assume that $b > a$ for all of $\mathbf{R}^2$.

$$f(x, y) = a + ax + by - x^2 - y^2$$

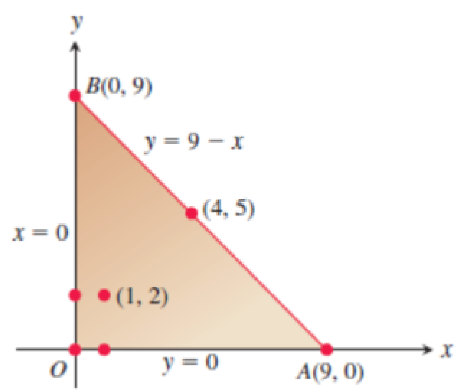
(a) Describe and sketch the domain $\mathbb{D}$ of $f(x, y)$.

(b) Find the critical point(s) of $f(x, y)$ on $\mathbf{R}^2$ (if any), where a and b are a non-negative constants. (The answers could depend on a and b).

(c) Classify the critical points from part (b). That is, determine which of them are local maxima, which are local minima, and which are saddle points. If there are no critical points, explain why.

(d) *(Challenge)* Using (a) and (b), or otherwise, find the absolute maximum and minimum values of $f(x, y)$ on D.

(e) *(Challenge)* For what value(s) of a and b (if any) does $f(x, y)$ take on the coordinates in the plot below if the function value candidates are $7, 2, -61, 3, -43, 6, -11$?



29

Write in spherical coordinates, the parametric equations for x, y, and z for a graph with a sphere with center (a, b, c) and radius ρ.

30 (Challenge)

Let $f(x, y) = g(x, y) - g(y, x)$, where g is a continuously differentiable function which satisfies

$$g_y(x, y) - g_x(x, y) = e^{\beta}$$

If an observer is moving through $\mathbf{R}^2$ along the path (assume both α and β are non-zero constants)

$$\mathbf{r}(t) = \langle \alpha \cos(\pi t), \alpha \sin(\pi t) \rangle$$

For what value(s) of t is the rate of the change of the value of f measured by the observer equal to zero?