Challenge Problem:
Pushing The Limit

Ave Kludze (akk86)

MATH 1920

Name: ______________________________

Collaborators: ______________________________

____

1

Below is the limit of a multivariable function.

$$\lim_{(x,y)\to(0,0)} \left(\frac{\sin(xy)}{xy}\right)^{\left(\frac{1}{1-\cos(xy)}\right)}$$

(a) Show that along the path $y = x$ the multivariable limit below is equal to e^{α}. Find α.

(b) Given the constraint below, evaluate the limit or determine that it does not exist.

$$\lim_{\substack{(x,y)\to(0,0) \\ x\neq y}} \left(\frac{\sin(xy)}{xy}\right)^{\left(\frac{1}{1-\cos(xy)}\right)}$$

Caution: Be warned, this is one of the hardest limit problems (above challenging difficulty)! Nonetheless, it is has an interesting solution!

Solution

Method 1:
(a)

$$\lim_{(x,y)\to(0,0)} \left(\frac{\sin(xy)}{xy}\right)^{\left(\frac{1}{1-\cos(xy)}\right)}$$

First set $y = x$ and reduce the limit to a single variable as suggested.

$$\lim_{(x,y)\to(0,0)} \left(\frac{\sin(xy)}{xy}\right)^{\left(\frac{1}{1-\cos(xy)}\right)}$$

$$\lim_{(x,x)\to(0,0)} \left(\frac{\sin(x^2)}{x^2}\right)^{\left(\frac{1}{1-\cos(x^2)}\right)}$$

Now allow $x^2 = u$ to reduce to a single variable limit and allow for cleaner calculations.

$$\lim_{u\to 0} \left(\frac{\sin(u)}{u}\right)^{\left(\frac{1}{1-\cos(u)}\right)}$$

Since this limit is an indeterminate form, we need to re-arrange this limit to some nicer form.

$$\lim_{u\to 0} \left(\frac{\sin(u)}{u}\right)^{\left(\frac{1}{1-\cos(u)}\right)} = L$$

$$\lim_{u\to 0} \ln\left(\frac{\sin(u)}{u}\right)^{\left(\frac{1}{1-\cos(u)}\right)} = \ln L$$

From this, we understand that our solution should be in the form of:

$$\boxed{L = e^{\lim_{u\to 0} f(u)}}$$

Applying logarithm rules and L'Hospital's rule, we get the following:

$$\lim_{u \to 0} \left(\left(\frac{1}{1 - \cos(u)} \right) \ln \left(\frac{\sin(u)}{u} \right) \right)$$

$$\lim_{u \to 0} \left(\frac{\ln \left(\frac{\sin(u)}{u} \right)}{1 - \cos(u)} \right)$$

$$= \frac{0}{0}$$

$$= \lim_{u \to 0} \left(\frac{\frac{-1 + u \cot(u)}{u}}{\sin(u)} \right)$$

$$= \lim_{u \to 0} \left(\frac{-1 + u \cot(u)}{u \sin(u)} \right)$$

Note: Compute the limit in the numerator via manipulation, where:

$$u \cot(u) = \frac{u}{\frac{1}{\cot(u)}} = \lim_{u \to 0} \left(\frac{u}{\frac{1}{\cot(u)}} \right)$$

Apply L'Hopital's Rule to the numerator limit

$$\lim_{u \to 0} \left(\frac{1}{\sec^2(u)} \right) = 1$$

For the entire limit:

$$= \frac{0}{0}$$

$$= \lim_{u \to 0} \left(\frac{-u \csc^2(u) + \cot(u)}{\sin(u) + u \cos(u)} \right)$$

$$= \frac{0}{0}$$

$$= \lim_{u \to 0} \left(\frac{2u \csc^2(u) \cot(u) - 2 \csc^2(u)}{2 \cos(u) - u \sin(u)} \right)$$

Note:

$$2u \csc^2(u) \cot(u) - 2 \csc^2(u) = 2u \csc^2(u) \cot(u) \left(1 - \frac{2 \csc^2(u)}{2u \csc^2(u) \cot(u)} \right)$$

$$\lim_{u \to 0} \left(\frac{2u \csc^2(u) \cot(u) \left(1 - \frac{2 \csc^2(u)}{2u \csc^2(u) \cot(u)} \right)}{2 \cos(u) - u \sin(u)} \right)$$

Simplify.

$$\frac{2u \csc^2(u) \cot(u) \left(1 - \frac{2 \csc^2(u)}{2u \csc^2(u) \cot(u)} \right)}{2 \cos(u) - u \sin(u)} = \frac{2 \csc^2(u)(u \cot(u) - 1)}{2 \cos(u) - u \sin(u)}$$

$$= 2 \cdot \lim_{u \to 0} \left(\frac{\csc^2(u)(u\cot(u) - 1)}{2\cos(u) - u\sin(u)} \right)$$

$$= 2 \cdot \frac{\lim_{u \to 0} \left(\csc^2(u)(u\cot(u) - 1)\right)}{\lim_{u \to 0}(2\cos(u) - u\sin(u))}$$

Noting that:

$$\lim_{u \to 0} \left(\csc^2(u)(u\cot(u) - 1)\right) = -\frac{1}{3}$$

$$\lim_{u \to 0}(2\cos(u) - u\sin(u)) = 2$$

Then,

$$= 2 \cdot \frac{-\frac{1}{3}}{2} = \frac{-1}{3} = \ln(y)$$

Thus,

$$L = e^{\frac{-1}{3}}$$

$$\boxed{\alpha = \frac{-1}{3}}$$

Note: For this limit below, express as cosines and sines, apply L'H Rule twice and solve the limit

$$\lim_{u \to 0} \left(\csc^2(u)(u\cot(u) - 1)\right) = -\frac{1}{3}$$

$$= \lim_{u \to 0} \left(\frac{u\cos(u) - \sin(u)}{\sin^3(u)} \right)$$

Apply L'Hopital's Rule

$$= \lim_{u \to 0} \left(\frac{-u\sin(u)}{3\sin^2(u)\cos(u)} \right)$$

$$= \lim_{u \to 0} \left(-\frac{2u}{3\sin(2u)} \right)$$

Apply L'Hopital's Rule

$$= \lim_{u \to 0} \left(\frac{-2}{6\cos(2u)} \right)$$

(b)

To evaluate this limit, we must change into polar coordinates (any other option would be tedious and impossibly difficult). We are also told that $y \neq x$, so we cannot use this path to determine the value of the limit (the path is blocked). However, we have the following through polar coordinates.

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\lim_{\substack{(x,y) \to (0,0) \\ x \neq y}} \left(\frac{\sin(xy)}{xy} \right)^{\left(\frac{1}{1-\cos(xy)}\right)}$$

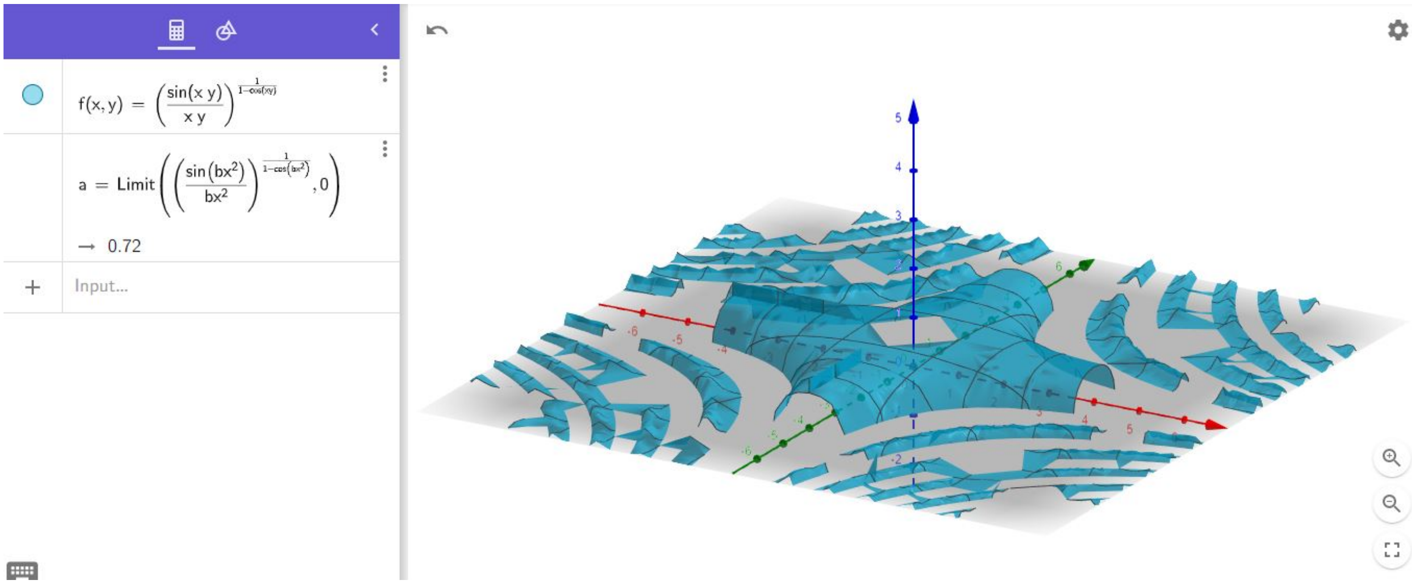
$$\implies \lim_{r \to 0} \left(\frac{\sin\left(r^2 \sin(\theta)\cos(\theta)\right)}{r^2 \cos(\theta)\sin(\theta)} \right)^{\frac{1}{1-\cos\left(r^2\cos(\theta)\sin(\theta)\right)}}$$

Since $\cos\theta$ and $\sin\theta$ range from -1 to 1, we can set their product equal to a constant b, which reduces the arithmetic.

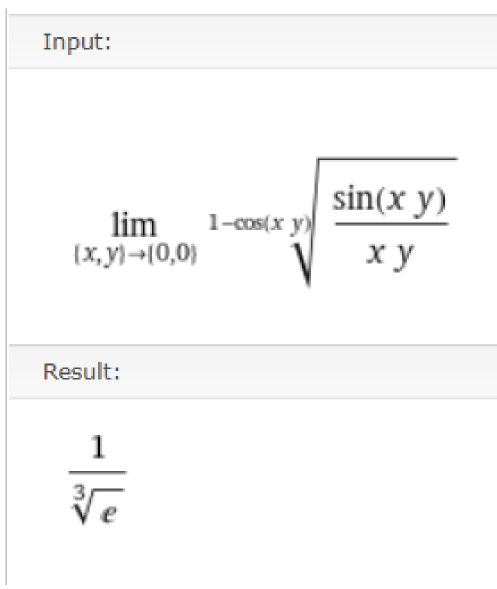
$$b = (\sin(\theta)\cos(\theta)) \quad b \in [-1, 1]$$

$$\lim_{r \to 0} \left(\frac{\sin\left(r^2 b\right)}{r^2 b} \right)^{\frac{1}{1-\cos\left(r^2 b\right)}}$$

Following the same approach as above, we arrive at the same exact value in polar coordinates, $e^{\frac{-1}{3}}$. Therefore, the limit does exist. Below is a graph of the function and the limit value with GeoGebra. Observe that the limit does not depend on a specific value of b.



The limit is confirmed by Wolfram-Alpha as well.



Method 2:

Rather than repeatedly applying L'H rule, one can expand each function at $x = 0$ and keep the lowest non-vanishing terms prior to taking a limit. With this approach, one can state that $x = u$. The variables are kept in x below for simplicity.

Here $O(x^n)$ notation is applied, which means we are omitting terms that have x^n power or greater. For instance, the full series expansion for $\sin(x)$ is approximately $x - x^3/6 + x^5/120 + ...$, but here we are considering values of x very close to 0, where the higher powers of x^n are considerably smaller than the first few terms of the series, and in fact will not matter when taking the limit. Note that

$$\sin(x) = x - \frac{x^3}{6} + O\left(x^5\right)$$

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{6} + O\left(x^4\right)$$

Since

$$\ln(1+x) = x + O\left(x^2\right) \quad \text{for} \quad |x| < 1$$

we have for x close to 0 that

$$\ln\left(\frac{\sin(x)}{x}\right) = \ln\left(1 - \frac{x^2}{6} + O\left(x^4\right)\right) = -\frac{x^2}{6} + O\left(x^4\right)$$

Now we have,

$$\cos(x) = 1 - \frac{x^2}{2} + O\left(x^4\right)$$

so

$$1 - \cos(x) = \frac{x^2}{2} + O\left(x^4\right) = \frac{1}{2}x^2\left(1 + O\left(x^2\right)\right)$$

from

$$\frac{1}{1+x} = 1 + O(x)$$

we have

$$\frac{1}{1-\cos(x)} = \frac{1}{\frac{1}{2}x^2\left(1 + O\left(x^2\right)\right)} = \frac{2}{x^2}\frac{1}{\left(1 + O\left(x^2\right)\right)} = \frac{2}{x^2}\left(1 + O\left(x^2\right)\right)$$

so so

$$\ln\left(\frac{\sin(x)}{x}\right)\frac{1}{1-\cos(x)} = \left(-\frac{x^2}{6} + O\left(x^4\right)\right)\left(\frac{2}{x^2}\left(1 + O\left(x^2\right)\right)\right) = -\frac{1}{3} + O\left(x^2\right)$$

which implies that

$$\lim_{x\to 0}\ln\left(\frac{\sin(x)}{x}\right)\frac{1}{1-\cos(x)} = \lim_{x\to 0}\left(-\frac{1}{3} + O\left(x^2\right)\right) = -\frac{1}{3}$$

hence

$$\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)^{\frac{1}{1-\cos(x)}} = e^{\lim_{x\to 0}\left[\ln\left(\frac{\sin(x)}{x}\right)\frac{1}{1-\cos(x)}\right]}$$
$$= e^{-\frac{1}{3}}$$

Note: This problem worked out nicely because the leading order term of $\ln(\frac{\sin(x)}{x})$ and $1-\cos(x)$ are both of order x^2, so they cancel to give you a finite limit. If for example, you consider the limit as $x \to 0$ of $\frac{\sin(x)}{(1-\cos(x))}$, you'll find that this diverges to infinity. This is because the leading order term of $\sin(x)$ is x, while $1-\cos(x)$ is x^2, so while they both go to 0, $1-\cos(x)$ goes to zero faster than $\sin(x)$.

See on "Infinitesimal asymptotic". Big O Notation