AEW Worksheet 10

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MATH 1920

Name: ______________________

Collaborators: ______________________

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1

Determine if the following statements are true(T) or false(F). Mark the correct answer. No justification needed.

(a) T **F** If a particle travels in a closed loop then the total work done on the particle over the loop is zero
False. The force may not be conservative (a racecar can accelerate on a circular track... non-zero work done)

(b) T **F** If there exists a closed curve C in D such that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ then $\mathbf{F}$ is conservative on D.
False. The condition is that this holds for all curves in D

(c) T **F** $\int_{-C} f ds = -\int_C f ds$
False!!! Not to be confused with the work done on a particle moving one direction along a curve versus moving in the opposite direction

2

Evaluate

$$I = \int_C x^2 y \, dx + (x - 2y) \, dy$$

over the parts of the parabola $y = x^2$ from (0,0) to (1,1).

Solution

First, parametrize the curve:

$$x = t, y = t^2, \quad 0 \le t \le 1.$$

Note, we specified the range of t to get exactly the part of the curve we wanted. Next, compute the differentials of x and y :

$$dx = dt, \quad dy = 2t dt$$

Finally substitute everything in the integral and compute the standard single variable integral:

$$I = \int_0^1 t^2 \left(t^2\right) dt + \left(t - 2t^2\right) 2t dt = \int_0^1 t^4 + 2t^2 - 4t^3 dt = \boxed{-\frac{2}{15}}.$$

3

Sketch the gradient vector field for $f(x, y) = x^2 + y^2$ as well as several contours for this function.

Solution

Recall that the contours for a function are nothing more than curves defined by,

$$f(x, y) = k$$

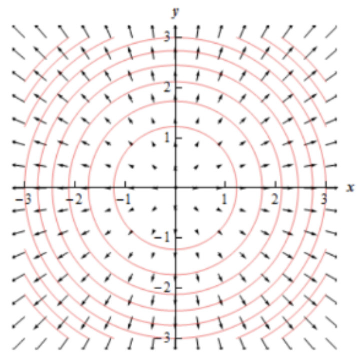
for various values of k. So, for our function the contours are defined by the equation,

$$x^2 + y^2 = k$$

and so they are circles centered at the origin with radius $\sqrt{k}$. Here is the gradient vector field for this function.

$$\nabla f(x, y) = 2x\vec{i} + 2y\vec{j}$$

Here is a sketch of several of the contours as well as the gradient vector field.



4

(a) Find a constant a such that the vector field

$$F(x, y) = \left\langle ax^2y - y^3, 3x^2 - 3xy^2 \right\rangle$$

is conservative or else show that there is no such constant a. If conservative, find a potential function.

(b) Find constants a and b such that the vector field

$$\vec{F} = \left\langle ay^2, 2xy + 2yz, by^2 + z^2 \right\rangle$$

is conservative or else show that there is no such constants. If conservative, find a potential function.

(c) Using part b, find the integral $\int_C \vec{F} \cdot d\vec{r}$, where the curve C is parametrized by $x = e^t - te^t, y = 2t^2, z = 3t, 0 \leq t \leq 1$

(d) Using part (b) and (c), give an equation of the surface S that contains all points P so that $\int_O^P \vec{F} \cdot d\vec{r} = 1$, where $O = (0, 0, 0)$ is the origin.

Solution

(a)

$$\text{If } a \text{ is constant then}$$
$$\frac{\partial P}{\partial y} = ax^2 - 3y^2 \quad \text{and} \quad \frac{\partial Q}{\partial x} = 6x - 3y^2$$
$$\text{If F is conservative then } \partial P/\partial y = \partial Q/\partial x \text{ but } \boxed{\text{no constant } a} \text{ can make these two things equal.}$$
$$\text{Thus, there is no such constant } a \text{ making the vector field conservative.}$$

(b)

$$\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$
$$\text{curl}\vec{F} = \nabla \times \vec{F} = \langle 2by - 2y, 0 - 0, 2y - 2ay \rangle = \vec{0}$$

$$\boxed{a = 1, \quad b = 1}$$

$$\vec{F} = \left\langle y^2, 2xy + 2yz, y^2 + z^2 \right\rangle$$

After solving, the potential function is then

$$\boxed{f(x, y, z) = xy^2 + y^2 z + \frac{z^3}{3}}$$

(c)

$$\int_c \vec{F} \cdot d\vec{r} = f(0, 2, 3) - f(1, 0, 0)$$

$$= 2^2 \times 3 + \frac{3^3}{3} - 0$$

$$= \boxed{21}$$

(d)

$$\left.\begin{aligned} f(P) - f(O) &= 1 \\ f(O) &= 0 \\ f(P) &= f(x, y, z) \end{aligned}\right\}$$

$$\Rightarrow f(x, y, z) = 1$$

Thus, we have

$$\boxed{xy^2 + y^2 z + \frac{z^3}{3} = 1}$$

5

(a) Prove that the rotation field

$$\mathbf{F} = \frac{\langle -y, x \rangle}{|\mathbf{r}|^p},$$

where $\mathbf{r} = \langle x, y \rangle$ is not conservative for $p \neq 2$

(b) For $p = 2$, show that F is conservative on any region not containing the origin.

(c) Find a potential function for $\mathbf{F}$ when $p = 2$

Solution

(a)
This field is

$$\mathbf{F} = \langle -y, x \rangle \left(x^2 + y^2\right)^{-p/2},$$

and we have

$$\frac{\partial}{\partial y}\left(-y\left(x^2 + y^2\right)^{-p/2}\right)$$

$$= -\left(x^2 + y^2\right)^{-p/2} + py^2 \frac{\left(x^2 + y^2\right)^{-p/2}}{x^2 + y^2}$$

$$= -\left(x^2 + y^2\right)^{-p/2} + py^2 \left(x^2 + y^2\right)^{-1-p/2}$$

$$= \frac{-x^2 + (p-1)y^2}{\left(x^2 + y^2\right)^{1+p/2}}$$

$$\frac{\partial}{\partial x}\left(x\left(x^2+y^2\right)^{-p/2}\right)$$

$$= \left(x^2+y^2\right)^{-p/2} - px^2\frac{\left(x^2+y^2\right)^{-p/2}}{x^2+y^2}$$

$$= \left(x^2+y^2\right)^{-p/2} - px^2\left(x^2+y^2\right)^{-1-p/2} = \frac{-(p-1)x^2+y^2}{\left(x^2+y^2\right)^{1+p/2}}$$

For the force field to be conservative, these two would have to be equal, or alternatively:

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

However, their difference is

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$= 2\left(x^2+y^2\right)^{-p/2} - p\left(x^2+y^2\right)\left(x^2+y^2\right)^{-1-p/2}$$

$$= 2\left(x^2+y^2\right)^{-p/2} - p\left(x^2+y^2\right)^{-p/2}$$

$$= (2-p)\left(x^2+y^2\right)^{-p/2}$$

which is in general nonzero.
b) From the above formula, if $p = 2$, then the mixed partials are equal, so that $\mathbf{F}$ is conservative. For

$$p = 2, \mathbf{F} = \frac{1}{x^2+y^2}\langle -y, x\rangle.$$

c) Integrating the x component of F with respect to x gives $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$