Math 1920

Multivariable Calculus for Engineers

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Academic Excellence Workshops
Cornell University

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Learning Objectives

<u>By the end of this lesson, learners should be able to:</u>

- [Place Holder]
- [Place Holder]
- [Place Holder]
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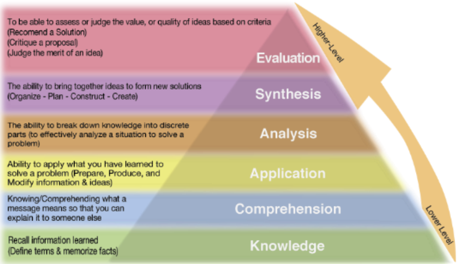
Bloom's Taxonomy's 6 Levels of Learning

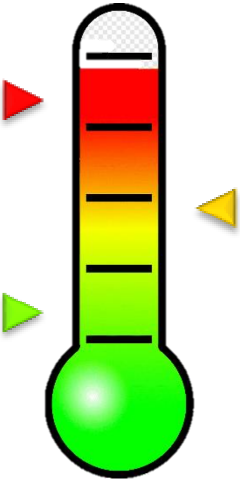


Goals:

- **Icebreaker**
  - [Place Holder]
  - [Place Holder]
  - [Place Holder]
  - [Place Holder]
- **Lecture Material**
  - [Place Holder]
  - [Place Holder]
  - [Place Holder]
- **Breakout-room and Worksheet**
  - Group-based problem solving
- **Review Problems**
  - Questions? Concerns?

Problem Difficulty Levels

HARD
Workshop Problems



MEDIUM
Prelim Problems

EASY
Example Problems

The Course

- **Textbook:** Calculus: Early Transcendentals Multivariable Fourth Edition by Jon Rogawski (Author), Colin Adams (Author), Robert Franzosa (Author)
- **Chapters Covered:** 13 – 18
- **Plotting Software:** CalcPlot3D, GeoAlgebra, Math3d, MATLAB, Maple
- **Topics Covered:** 3D Space, Vectors, Functions of Several Variables, Limits/Derivatives/Integrals, Volumes, Center of Mass, Curves/Trajectories, Surfaces, Vector Fields, Flow/Flux

Class Norms

- **Kind, Inclusive, Brave/Failure Tolerant**
- **Ask any questions (e.g., safe space)**
- **Adopt Growth-Mindset**
- **Active Participation (Workshops should be fun lol)**
- **Maintain Attendance**

General Exam Tips

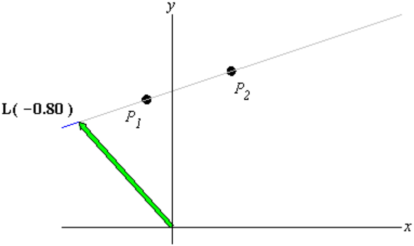
- Start with definitions/principles and then consider simplifications, complexities, or constraints
- Every 'hard problem' has some type of principle or technique from a more basic problem
- Problems may have interconnected/overlapping concepts; therefore, identify overlap in content areas
- Questions may often ask for algebraic answers instead of numerical answers. These questions are no different from questions with numbers, they instead usually require an extra or creative step
- "Mind your numbers" – ensure that there are NO 'dumb' mistakes since these are *fatal*. One can check his/her work one goes and focus on accuracy while having a reasonable speed. EX: When checking your work, check for individual characters or letters or check for common mistakes made during practice exams
    - Engine Mode – mindset that I will not make any mistakes
- Organize your work to show logic.
- Take a 5 second breather by not looking at the screen for 5 seconds
- Play to your strengths (do section/problem that you feel strongest about first and collect those points!)
- If stuck re-read questions for hints or details/specific wording.
- It is okay to not be able to answer a question when you first see it (do not panic). When you come back to the problem you will often find that you are now able to solve it.

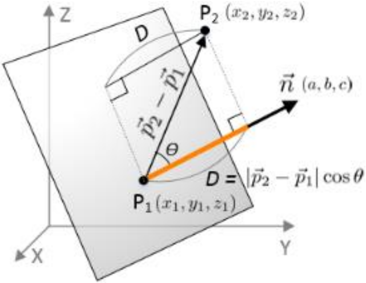
Resource List:

1. Professor Leonard (YouTube) – The GOAT
2. Paul's Math Notes (Online)
3. Textbook Problems/Explanations (TAs)
4. ELI Tutors
5. Math Support Center
6. List of Practice Problems
   - Links are in drive folder!

START

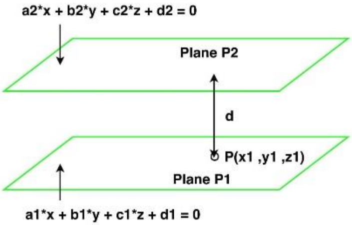
Planes in 3-space

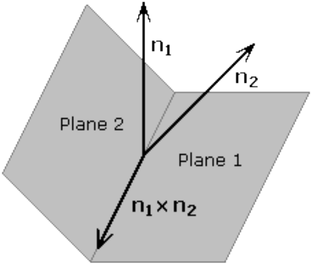




- Planes need **two things**:
  1. A point
  2. A $\perp$ vector to the plane
- Given one point on the plane and a normal vector to the plane, we can define a specified plane
  - **3 points define 2 vectors**, and 2 vectors can be used to find a normal vector
- The **intersection of a plane P** with a **coordinate plane** or a plane parallel to a coordinate plane is called a *trace*
  - *Important For:* Sketching Quadric surfaces, Multivariable Functions, Iterated Integrals

Planes in 3-space





- **Notice:**
    1. Two planes are || if normal vectors are ||
    2. Two planes are ⊥ if their normal vectors are ⊥
    3. Angle b/t planes = angle b/t normal vectors
- **Two Forms:**
    1. Vector form: $\boldsymbol{n} \cdot \langle x, y, z \rangle = d$
    2. Scalar form: $ax + by + cz = d$
- **Typical Types of Plane Problems:**
    - Angle between line and a plane
    - Distance between point and a plane
    - Distance between two parallel planes

Lecture Question

Show that $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$

Lecture Question

Show that $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2$

By definition: $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$

$$|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta = |\mathbf{a}|^2|\mathbf{b}|^2\left(1-\cos^2\theta\right)$$

$$= |\mathbf{a}|^2|\mathbf{b}|^2 - (|\mathbf{a}||\mathbf{b}|\cos\theta)^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2$$

Show that $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$

By definition: $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 \sin^2\theta = |\mathbf{a}|^2|\mathbf{b}|^2 \left(1 - \cos^2\theta\right)$$

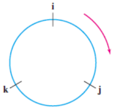
$$= |\mathbf{a}|^2|\mathbf{b}|^2 - (|\mathbf{a}||\mathbf{b}|\cos\theta)^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$$

Important Formulas:

$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$$

$$\mathbf{v} \times \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$$

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$$

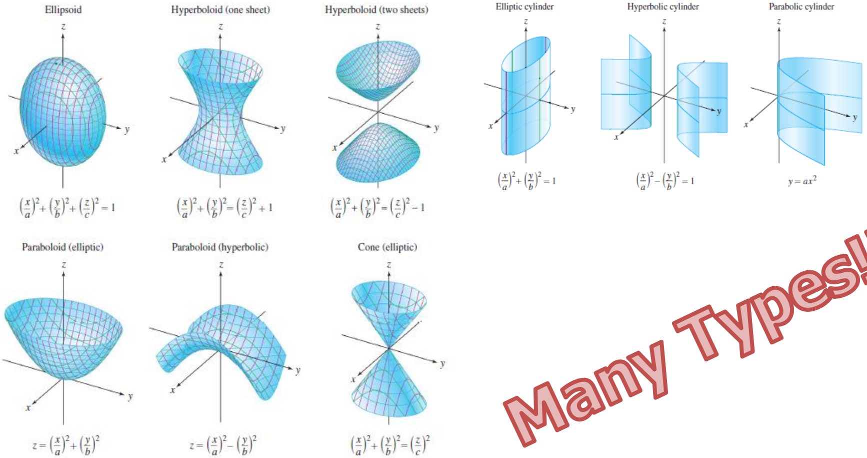


The parallelogram spanned by v and w has area $\|v \times w\|$.

The triangle spanned by $\mathbf{v}$ and $\mathbf{w}$ has area $\frac{\|\mathbf{v}\times\mathbf{w}\|}{2}$

The parallelepiped spanned by u, v , and w has volume $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$

Quadric Surfaces



Many Types!!

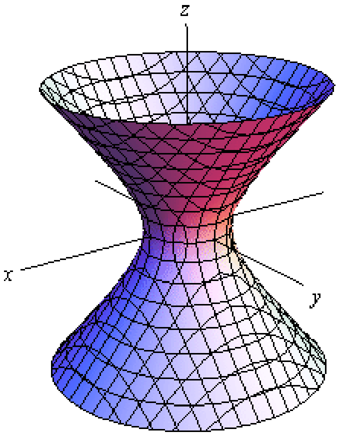
General Surfaces

1. Have 3 variables
2. Traces occur on coordinate plates and/or on planes || to coordinate planes
3. Still directed along an axis, but the trace changes along the axes

General Surfaces Steps:

1. Determine the type of surface
2. Determine direction axis
3. Find traces on coordinate planes
4. Find at-least two other traces on direction axis

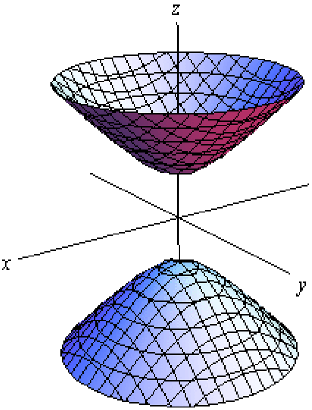
1-Sheet Hyperboloids:



1. Equation?

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

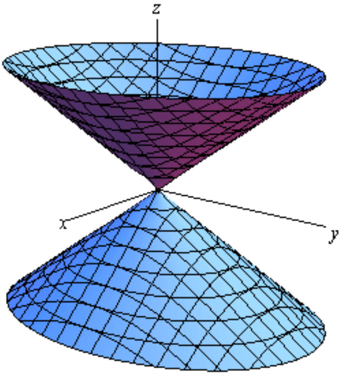
2. How do you tell?
   (a) one (-)
   (b) all power "2" (SQ^2)
   (c) has a constant
3. Notes
   (a) Always directed along axis with the minus (-)
   (b) Set (-) var = 0 and = $\pm\sqrt{DENOM}$ to get three traces
4. Visualize its shape
   (a) Circle or Ellipse



1. Equation?

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

2. How do you tell?
   (a) two minuses (or alternatively, one (+))
   (b) all power "2" (SQ^2)
   (c) has a constant
3. Notes
   (a) Always directed along axis with the plus (+)
   (b) Set (+) var = 0 and get nothing
   (c) Set both (-) var = 0 to get axis intercept
   (d) Set (+) var = to #'s divisible by DENOM (Remember it's always the odd-one-out)
4. Visualize its shape
   (a) Circle or Ellipse

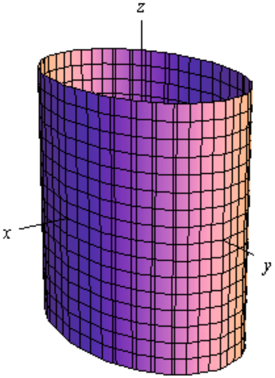


1. Equation?

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

2. How do you tell?
   (a) one (-)
   (b) all power "2" (SQ^2)
   (c) no constant
3. Notes
   (a) Always directed along axis with the minus (-)
   (b) Set (-) var = 0 and get nothing
   (c) Plug in values for (-) var that are divisible by DENOM
4. Visualize its shape
   (a) Circle or Ellipse

Cylinder

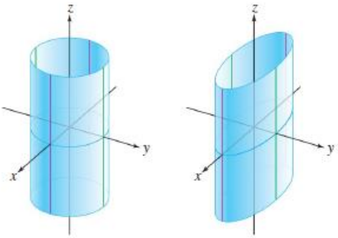


1. Equations with cylinders have only two variables - these equations give a trace of the curve on whatever coordinate plane denoted by the given variables

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

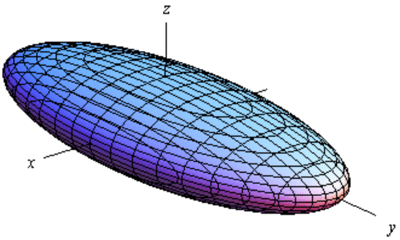
$$x^2 + y^2 = r^2$$

2. Curve is directed along the axis of the missing variable
3. The curve/trace does not change along the direction axes *If $a = b$ we have a cylinder whose cross section is a circle



$x^2 + y^2 = r^2$
Right-circular cylinder of radius r

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
Elliptic cylinder

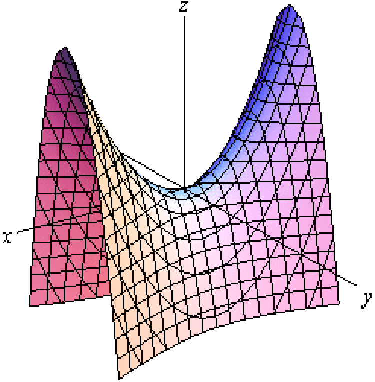


1. Equation?

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

2. How do you tell?
   (a) all (+)
   (b) all power "2" (SQ^2)
   (c) has a constant
3. Notes
   (a) Intercepts: $x = \pm a$, $y = \pm b$, $z = \pm c$

Hyperbolic Paraboloid

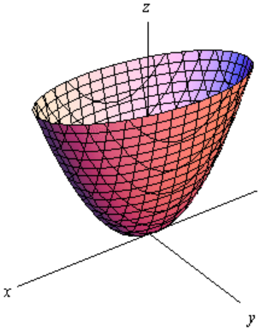


1. Equation?

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

2. How do you tell?
   (a) 3 variables with "2" (SQ^2)
   (b) one (SQ^2) has (-)
3. Notes
   (a) Degree one var gives direction axis
   (b) plug in both (+) and (-) into degree one var

Elliptic Paraboloid



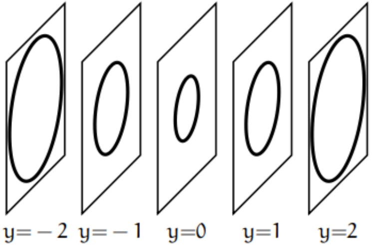
1. Equation?

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

2. How do you tell?
   (a) 3 variables with "2" (SQ^2)
   (b) variable with (SQ^2) are (+)
3. Notes
   (a) Opens along axis with degree one var
   (b) Coefficient of degree one var gives direction
      i. (+) opens towards (+) axis
      ii. (-) opens towards (-) axis
   (c) Set degree one var = 0 to get trace on coordinate plane (if parabola shifted)
      i. Example: (z+4) means shift down 4 on z-axis
         *If there are no shifts then plug in values into z to get good #'s
      ii. Plug in $x = y = 0$ to test if it's opening downward or upward (if you have a shift)

Lecture Question

Shown below are "slices" of a quadric surface taken at a few different values of y. Use this to identify the type of quadric surface and circle your answer from the following list.

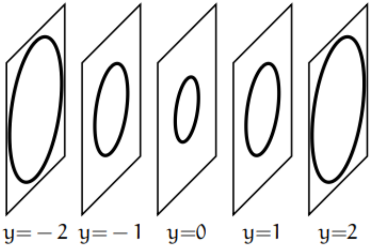


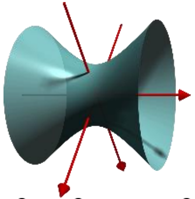
Answer Choices:

A. Ellipsoid
B. Hyperboloid of one sheet
C. Hyperboloid of two sheets
D. Cone
E. Elliptic paraboloid
F. Hyperbolic paraboloid

Lecture Question (Solution)

Shown below are "slices" of a quadric surface taken at a few different values of y. Use this to identify the type of quadric surface and circle your answer from the following list.





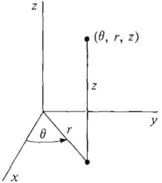
$x^2 + z^2 = 1 + y^2$

Answer Choices:

A. Ellipsoid
B. Hyperboloid of one sheet
C. Hyperboloid of two sheets
D. Cone
E. Elliptic paraboloid
F. Hyperbolic paraboloid

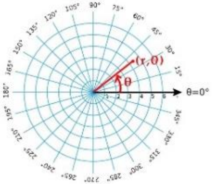
Cylindrical Coordinates:

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$
$$y = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$



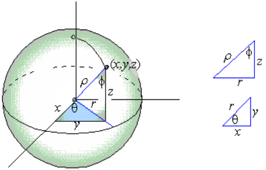
Polar Coordinates:

$$x = r\cos\theta \quad \& \quad y = r\sin\theta$$

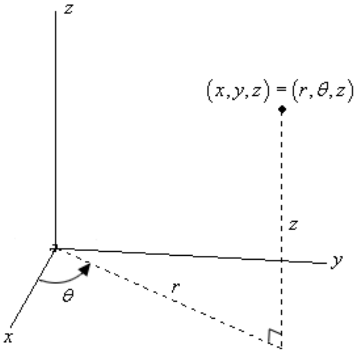


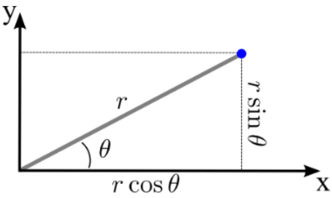
Spherical Coordinates:

$$r = \rho\sin\phi$$
$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$
$$\rho^2 = x^2 + y^2 + z^2$$



Cylindrical Coordinates

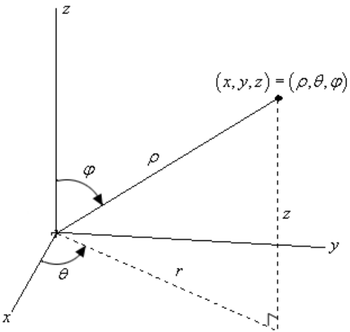


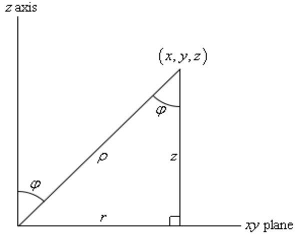


- Polar coordinates with 'z' component
- r is the distance from the origin to the projection of a point on the xy-plane
- The x and y values are derived from a triangle
- Allows for easier calculations

$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$

Spherical Coordinates





- In spherical there's a "p" (think rho) and "s" (think sinϕ)
- ρ is the distance from the origin to the point and ρ ≥0.
- θ is same as cylindrical, it is angle between the positive x-axis
- φ or ϕ is the angle from the positive z axis and 0≤φ≤π.
- Can view as an extension of polar

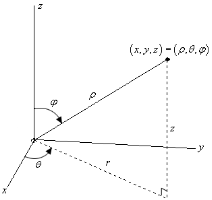
$$r = \rho \sin \varphi$$
$$x = \rho \sin \varphi \cos \theta$$
$$y = \rho \sin \varphi \sin \theta$$
$$z = \rho \cos \varphi$$
$$\rho^2 = x^2 + y^2 + z^2$$

A point has rectangular coordinates (-5,-7,4) and spherical coordinates (ρ, θ, ϕ). Find the rectangular coordinates of the point with spherical coordinates $(\rho, \theta, -\phi)$.

Answer Choices:

A. (-5,-7,4)
B. (5,-7,4)
C. (5,-7,-4)
D. (5,7,4)
E. (-5,7,4)

A point has rectangular coordinates (-5,-7,4) and spherical coordinates (ρ, θ, ϕ). Find the rectangular coordinates of the point with spherical coordinates $(\rho, \theta, -\phi)$.



We have that

$$-5 = \rho \sin\phi \cos\theta$$
$$-7 = \rho \sin\phi \sin\theta$$
$$4 = \rho \cos\phi$$

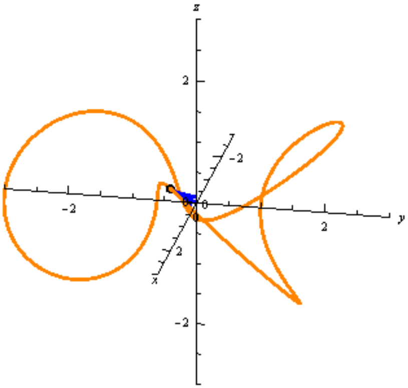
$$\rho \sin(-\phi) \cos\theta = -\rho \sin\phi \cos\theta = 5$$
$$\rho \sin(-\phi) \sin\theta = -\rho \sin\phi \sin\theta = 7$$
$$\rho \cos(-\phi) = \rho \cos\phi = 4$$

so the rectangular coordinates are $(5, 7, 4)$

Answer Choices:

A. (-5,-7,4)
B. (5,-7,4)
C. (5,-7,-4)
D. (5,7,4)
E. (-5,7,4)

Vector-Valued Functions



$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

- A vector function is a function that takes one or more variables and returns a vector
- a vector function is a parametrically defined function where the terminal points of our vector(s) trace(s) a curve in 3D
- The *domain* of a vector function is the set of all t's for which all the component functions are defined

Sketching:

1. Identify: "x", "y", & "z"
2. Use one or more components to get a curve or surface (get rid of "t")
   (a) for 2 components sketch on a plane
   (b) for 3 components, the curve is on a surface
3. Use values of "t" to find points and orientation
4. More than one surface is possible so stick to ones your famliar with. The "unused" component gives the curve **ON** the surface you made
   (a) See $\vec{r} = \langle t, t,^2, t^3 \rangle$

Vector-Valued Functions Example

Sketch the following vector function: $\vec{r}(t) = \langle 2\cos t, 4\sin t, t\rangle$ t $\in [0, 2\pi]$

$$x = 2\cos t\,,\; y = 4\sin t,\; z = t$$

$$\implies \frac{x}{2} = \cos(t) \qquad \cos^2 t + \sin^2 t = 1$$

$$\implies \frac{y}{4} = \sin(t) \qquad \frac{x^2}{4} + \frac{y^2}{16} = 1$$

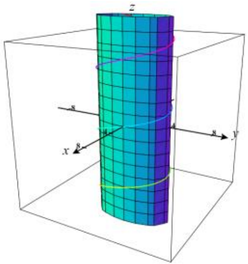
Therefore, it is a cylinder along the z. We can locate a trace on the xy-plane. Additionally, z = t is **ON** the cylinder.

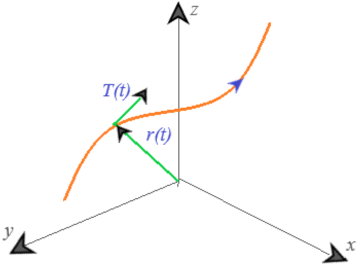
$$t = 0 : (2, 0, 0)$$

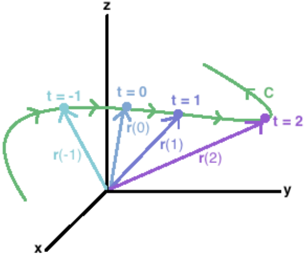
$$t = \frac{\pi}{2} : (0, 4, \frac{\pi}{2})$$

$$t = \pi : (-2, 0, \pi)$$

$$t = 2\pi : (2, 0, 2\pi)$$



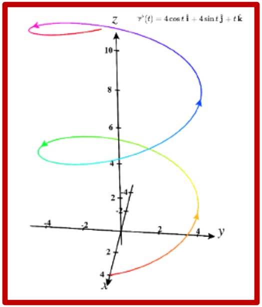


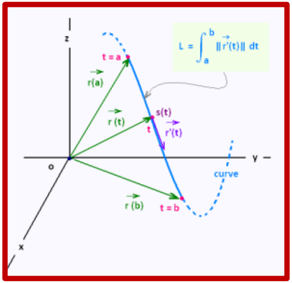


A curve C in three-dimensions represents by a vector-valued function **r(t)**, where sample values t=-1, t=0, t=1, and t=2 are arbitrarily plotted.

- Derivatives of vector functions $r(t)$ can represent a tangent vector $r'(t)$ to a curve through space
    - Tangent vectors can provide orientation of vector functions
    - Tangent lines need a point of tangency and a tangent vector
- Limits of vector functions can represent approaching a specific point
- Integrals of vector functions can have more physical representations (e.g., integral of velocity vector)
- All calculus operations are performed component-wise

Arclength and Speed





- <u>Given the position vector $r(t)$:</u>
  - **Position:** $r(t)$
    - $\vec{r}(t) = \int \vec{v}(t)\ dt + C$
  - **Velocity:** $\vec{v}(t) = r'(t)$
    - $\vec{v}(t) = \int \vec{a}(t)\ dt + C$
  - **Speed:** $||\vec{v}(t)|| = ||r'(t)||$
  - **Acceleration:** $\left(\vec{v}(t)\right)' = \vec{a}(t) = r''(t)$

- **Arclength** is a defined length provided by the magnitude of the *tangent vector*
  - $s = \int_a^b ||r'(t)||dt =$
    $\int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}\, dt$
  - Some problems may require identifying factorizations, simplifying, using trig. Identities, making fractions with the same denominator and crossing out items, etc.

Lecture Question

At what time t does the speed of the particle moving in space with its position function

$$\mathbf{r}(t) = \left\langle t^2, 3t, t^2 - 8t \right\rangle$$

have its minimum value?

Answer Choices:

A. $t = 1$
B. None of the below
C. $t = 3$
D. $t = 5$
E. $t = 2$
F. None of the above

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$$\mathbf{r}(t) = \left\langle t^2, 3t, t^2 - 8t \right\rangle$$

have its minimum value?

$$r'(t) = \langle 2t, 3, 2t - 8 \rangle$$

$$\left|r'(t)\right| = \sqrt{(2t)^2 + (3)^2 + (2t - 8)^2}$$

$$\left|r'(t)\right| = \sqrt{8t^2 - 32t + 73}$$

To maximize, we look under the square root.

$$f(t) = 8t^2 - 32t + 73$$

$$f' = 16t - 32 \overset{set}{=} 0$$

$$\implies \boxed{t = 2}$$

Answer Choices:

A. $t = 1$
B. None of the below
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Convert the point $(-1, 1, -\sqrt{2})$ from Cartesian to spherical coordinates.

Recall that (ρ, θ, ϕ) are spherical coordinates. First, we find ρ and then ϕ:

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 1 + 2} = 2$$

Ensure that ϕ is in the allowed range:

$$z = \rho \cos\phi \quad \Rightarrow \quad \cos\phi = \frac{z}{\rho} = \frac{-\sqrt{2}}{2} \quad \Rightarrow \quad \phi = \cos^{-1}\left(\frac{-\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$

Now find θ:

$$\sin\theta = \frac{y}{\rho \sin\varphi} = \frac{1}{2\left(\frac{\sqrt{2}}{2}\right)} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{4} \text{ or } \theta = \frac{3\pi}{4}$$

We have two possible choices, looking at our coordinates from above we know that the points are in the second quadrant. Therefore, we choose the appropriate angle.

The spherical coordinates of this point are then $\left(2, \frac{3\pi}{4}, \frac{3\pi}{4}\right)$

Lecture Question (Extra)

A point has rectangular coordinates (x, y, z) and spherical coordinates $\left(2, \frac{8\pi}{7}, \frac{2\pi}{9}\right)$.

Find the spherical coordinates of the point with rectangular coordinates $(x, y, -z)$.

Enter your answer in the form (ρ, θ, ϕ), where $\rho > 0, 0 \leq \theta < 2\pi$, and $0 \leq \phi \leq \pi$

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Enter your answer in the form (ρ, θ, ϕ), where $\rho > 0, 0 \leq \theta < 2\pi$, and $0 \leq \phi \leq \pi$

We have that

$$x = \rho \sin \tfrac{2\pi}{9} \cos \tfrac{8\pi}{7}$$
$$y = \rho \sin \tfrac{2\pi}{9} \sin \tfrac{8\pi}{7}$$
$$z = \rho \cos \tfrac{2\pi}{9}$$

Recall: $\cos(\pi - x) = -\cos x$ & $\sin(\pi - x) = \sin x$

We want to negate the z -coordinate. We can accomplish this by replacing $\frac{2\pi}{9}$ with $\pi - \frac{2\pi}{9} = \frac{7\pi}{9}$

$$\rho \sin \tfrac{7\pi}{9} \cos \tfrac{8\pi}{7} = \rho \sin \tfrac{2\pi}{9} \cos \tfrac{8\pi}{7} = x$$
$$\rho \sin \tfrac{7\pi}{9} \sin \tfrac{8\pi}{7} = \rho \sin \tfrac{2\pi}{9} \sin \tfrac{8\pi}{7} = y$$
$$\rho \cos \tfrac{7\pi}{9} = -\rho \cos \tfrac{2\pi}{9} = -z$$

Thus, the spherical coordinates of (x, y, z) are $\left(2, \frac{8\pi}{7}, \frac{7\pi}{9}\right)$

Two particles travel along the space curves

$$\mathbf{r}_1(t) = \left\langle t, t^2, t^3 \right\rangle \quad \mathbf{r}_2(t) = \langle 1+2t, 1+6t, 1+14t \rangle$$

Do the particles collide? Do their paths intersect?

Vector-valued Functions (Extra)

The particles collide provided $\mathbf{r}_1(t) = \mathbf{r}_2(t)$

$$\left\langle t, t^2, t^3 \right\rangle = \langle 1+2t, 1+6t, 1+14t \rangle$$

$$t = 1+2t, t^2 = 1+6t, \text{ and } t^3 = 1+14t$$

The first equation gives $t = -1$, but this does not satisfy the other equations

The particles do not collide

For the paths to intersect, we need to find a value for t and a value for s where

$$\mathbf{r}_1(t) = \mathbf{r}_2(s)$$

$$\left\langle t, t^2, t^3 \right\rangle = \langle 1+2s, 1+6s, 1+14s \rangle$$

$$t = 1+2s, t^2 = 1+6s, \text{ and } t^3 = 1+14s$$

$$\left\langle t, t^2, t^3 \right\rangle = \langle 1+2s, 1+6s, 1+14s \rangle$$

$$t = 1+2s, t^2 = 1+6s, \text{ and } t^3 = 1+14s$$

Substituting the first equation into the second gives:

$$(1+2s)^2 = 1+6s \Rightarrow$$

$$4s^2 - 2s = 0 \Rightarrow 2s(2s-1) = 0$$

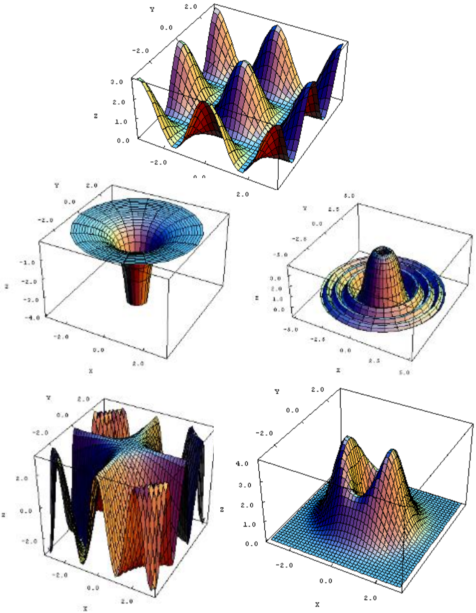
$$s = 0 \text{ or } s = \frac{1}{2}$$

From the first equation, $s = 0 \Rightarrow t = 1$ and $s = \frac{1}{2} \Rightarrow t = 2$

Checking, we see that both pairs of values satisfy the third equation.

Thus the paths intersect twice, at the point $(1,1,1)$ when $s = 0$ and $t = 1$ and at $(2,4,8)$ when $s = \frac{1}{2}$ and at $t = 2$

Multivariable Functions



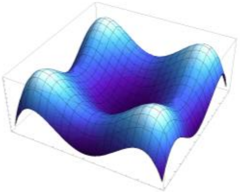
- Multivariable functions contain two or more input values (typically $x, y, or\ z$) and produce an output (typically function height or number) based on the inputs
- To graph a function, you must have one dimension more than the # of independent variables
  - $f(x) = x \rightarrow 2D$
  - $f(x,y) = z = x^2 + y^2 \rightarrow 3D$
  - $f(x,y,z) = w = \frac{x^2+y^2}{z-3} \rightarrow 4D$
- To graph a domain of a function, you must have the same # of dimensions as the indep. variable (need an axis for each indep. variable)
  - $f(x) \rightarrow \boldsymbol{D}: 1-D$
  - $f(x,y) \rightarrow \boldsymbol{D}: 2-D$
  - $f(x,y,z) \rightarrow \boldsymbol{D}: 3-D$

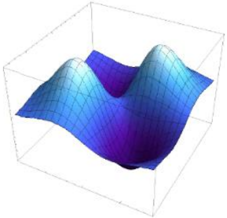
Lecture Question

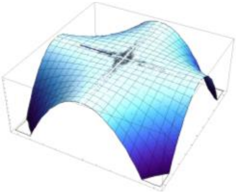
For each function label its graph from the options below (you may only use each function once, two of the graphs should remain unlabelled):

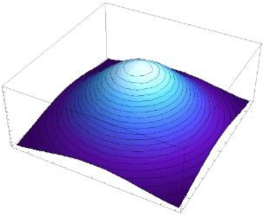
A) $f(x,y) = -\pi x^4 + 2\pi x^2 - \pi y^4 + 2\pi y^2$

B) $f(x,y) = \frac{1}{\pi} e^{-x^2-y^2}$









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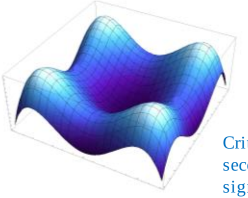
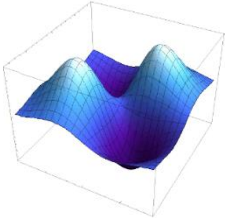


A

Critical points using second derivative test. The sign of both f_{xx} and f_{yy}. Traces at $x = 0$, or $y = 0$



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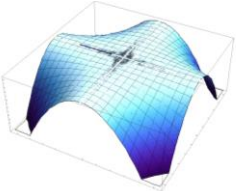

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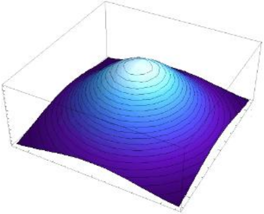
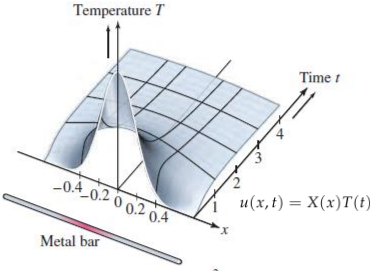
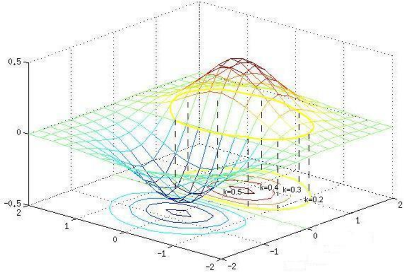


B

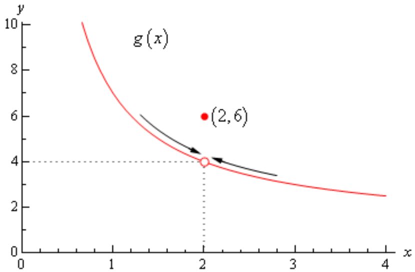
Polar symmetry provided by $f(r, \theta) = \frac{1}{\pi} e^{-r^2}$

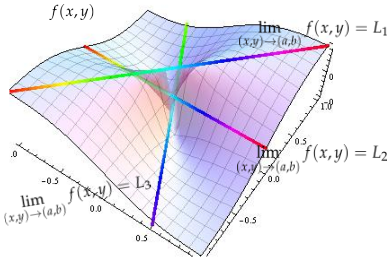
Traces & Level Curves





- **Level Curves:**
  - The shape we get when a plane intersects our surface at different levels along the axis of the dependent variable
  - A map of level curves is a contour plot
  - To find level curves set "f" equal to "k"
  - Traces allow for "slices" of a surface so that we can see how a function behaves
- **Contour Plot:**
  - Altitude does not change when moving across a level curve
  - Altitude does change when moving from one level curve to another
  - Closer level curves indicates steeper ascent



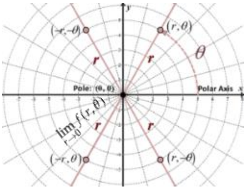


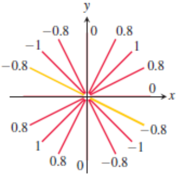
- **1-Var:**
  - This is a curve (a certain path). Limit is in two directions (left and right) to approach a point/value
- **2-Var:**
  - This is a surface (many paths). Limit has ∞ number of paths along a surface that approach our point in 3D space.
  - To prove that a limit exist, we must prove along all paths we approach the same point.
  - Therefore, we must focus on proving that a limit does not exist!
    - Can show that along 2 paths, we get a different value as we approach the same point $(x, y) = (a, b)$
    - May use squeeze theorem to prove a limit exist

General Strategy (Multivariable Limits)

General Strategy

- Check along paths $x = 0, y = 0, y = x$, or $y = mx$ (change with slope)
- If the paths above do not work, choose another path
  - a) Be certain that the point (a,b) is on your path (always plug in your point first)
  - b) Try substituting so that the number of degrees of the numerator & denominator are equal (e.g $y = x$, $y = (x-1)$, $x = y + 2$)
  - c) Always use either $x = 0$ or $y = 0$ as one path (check if $P = (0{,}0)$)
  - d) Plug in limiting values (where the numerator is zero)
  - e) May be forced to apply L'Hôpital's rule (L'H $= \frac{0}{0}$ or $= \frac{\infty}{\infty}$)
- Convert to polar coordinates and evaluate the limit (usually @ (0,0))
  - Easiest when function contains some part of $x^2 + y^2$ (or similar)
  - Evaluate the limit as $r \to 0$ and simplify $\sin\theta$ or $\cos\theta$
  - Can use squeeze theorem
  - Personal favorite! (deal with a lot of potential errors/mistakes, handles creative problems well, organized and neat format, etc.)
  - If not at (0,0) but instead (a,b) can substitute $x = a + r\cos\theta$, and $y = b + \sin\theta$





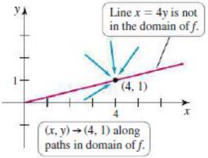
Multivariable Limits (example 1)

Evaluate $\displaystyle \lim_{(x,y)\to(4,1)} \frac{xy - 4y^2}{\sqrt{x} - 2\sqrt{y}}$

$$\text{Evaluate} \quad \lim_{(x,y)\to(4,1)} \frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}}$$

Points satisfy $x \geq 0$, $y \geq 0$, and $x \neq 4y$ (denominator).

$$\lim_{(x,y)\to(4,1)} \frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}} = \lim_{(xy)\to(4,1)} \frac{(xy-4y^2)\left(\sqrt{x}+2\sqrt{y}\right)}{(\sqrt{x}-2\sqrt{y})(\sqrt{x}+2\sqrt{y})}$$ **→ multiply by conjugate**



$$= \lim_{(xy)\to(4,1)} \frac{y(x-4y)(\sqrt{x}+2\sqrt{y})}{x-4y}$$ **→ simplify**

$$= \lim_{(x,y)\to(4,1)} y(\sqrt{x}+2\sqrt{y})$$ **→ evaluate**

$$= 4$$

Along all other paths to $(4,1)$ the function values approach 4.

Multivariable Limits (example 2)

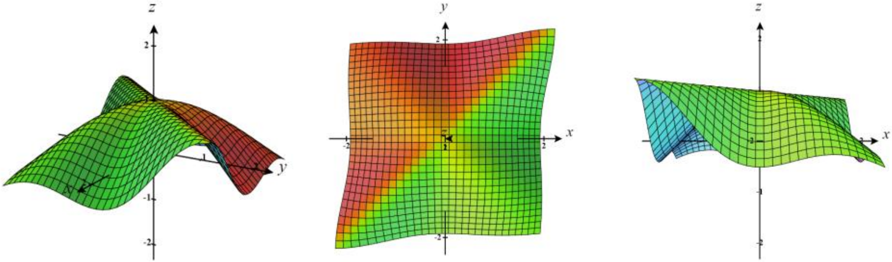
$$\text{Evaluate } \lim_{(x,y)\to(0,0)} \cos\left(\frac{x^3 - y^3}{x^2 + y^2}\right)$$

$$\text{Evaluate} \lim_{(x,y)\to(0,0)} \cos\left(\frac{x^3-y^3}{x^2+y^2}\right)$$

$$x = r\cos\theta, y = r\sin\theta$$

$$\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^3-y^3}{x^2+y^2}\right) = \lim_{r\to 0}\cos\left(\frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2}\right)$$

$$= \lim_{r\to 0}\cos\left(\frac{r\left(\cos^3\theta - \sin^3\theta\right)}{1}\right) = \cos 0 = 1$$



Multivariable Limits (example 3)

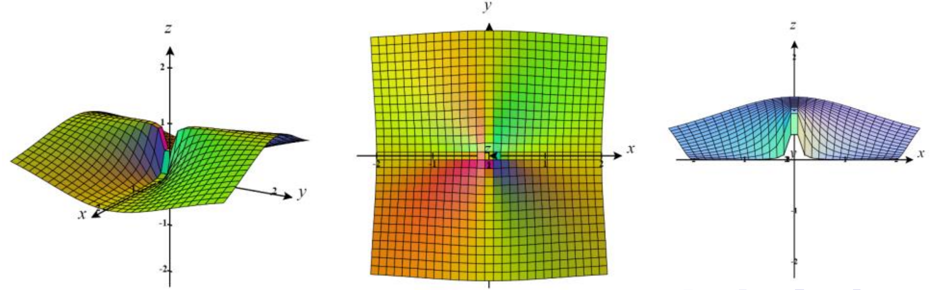
$$\text{Evaluate} \quad \lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2}$$

$$\text{Evaluate} \quad \lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\sin^2\theta}{r^2}$$

$$= \lim_{r\to 0}\left(\sin^2\theta\right) = \sin^2\theta$$

The limit does not exist since $\sin^2\theta$ is between 0 and 1 depending on θ.
Recall that $\theta = \tan^{-1}(\frac{y}{x})$, where $\frac{y}{x} = m$ which is slope.



Multivariable Limits (example 4, extra)

Evalute $\lim\limits_{(x,y)\to(-3,-2)} \left(x^2y^3 + 4xy\right)$ in polar and cartesian coordinates.

Cartesian:

$$\lim_{(x,y)\to(-3,-2)} \left(x^2y^3 + 4xy\right) = 9(-8) + 4(-3)(-2) = -72 + 24 = -48$$

Polar:

$$x = a + r\cos\theta, y = b + r\sin\theta$$

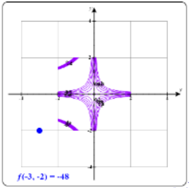
$$\implies x = -3 + r\cos\theta, y = -2 + r\sin\theta$$

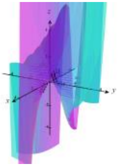
$$\lim_{r\to 0}\left((-3 + r\cos(\theta))^2(-2 + r\sin(\theta))^3 + 4(-3 + r\cos(\theta))(-2 + r\sin(\theta))\right)$$

$$= (-3 + 0\cdot\cos(\theta))^2(-2 + 0\cdot\sin(\theta))^3 + 4(-3 + 0\cdot\cos(\theta))(-2 + 0\cdot\sin(\theta))$$

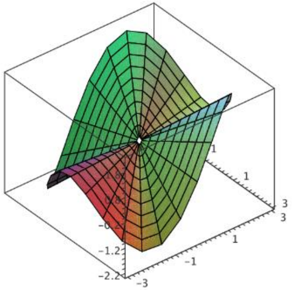
$$= \left((-3)^2(-2)^3 + 4(-3)(-2)\right)$$

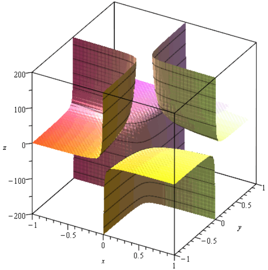
$$= -48$$





Multivariable Limits (continuity)





- A function is continuous at any point on the region for which it is defined (domain)
  - Polynomials are continuous everywhere
  - Rational functions are continuous everywhere where their denominators are not equal to zero
  - The composition of continuous functions are continuous
  - A function f of two variables is *continuous* at $P = (a, b)$ if
    $$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$
  - the definition of continuity applies at boundary points as well as interior points of the domain of f
  - Sums, differences, constant multiples, products, quotients, and powers of continuous functions are continuous where defined.

Multivariable Limits (continuous, example 1)

Determine the points at which the following function is continuous.

$$f(x,y) = \begin{cases} \frac{3xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

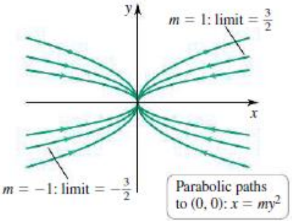
Determine the points at which the following function is continuous.

$$f(x,y) = \begin{cases} \frac{3xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

The function above is a rational function, so it is continous everywhere except the origin by defintion. We must check the origin so that $f(0,0) = 0$ and the limit exists.

$$\lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^4}$$

Along $y = mx$ the limit exists. However, along $x = my^2$



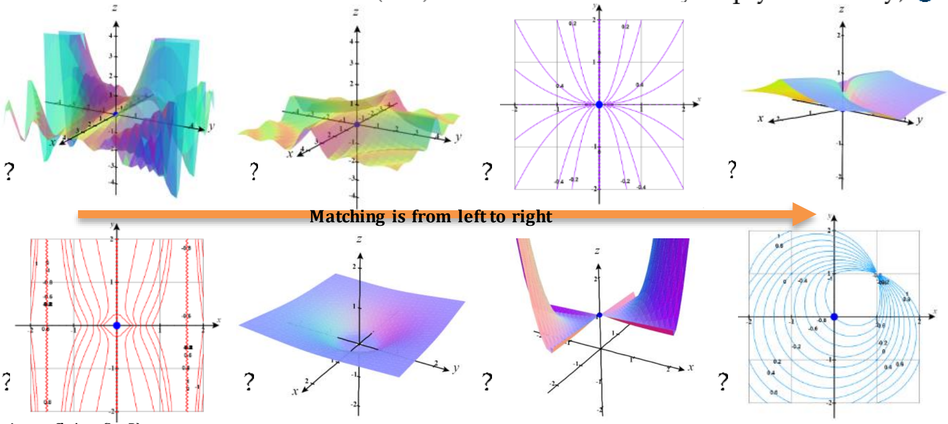
$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^4} &= \lim_{y\to 0} \frac{3\left(my^2\right)y^2}{\left(my^2\right)^2+y^4} \\ &= \lim_{y\to 0} \frac{3my^4}{m^2y^4+y^4} \\ &= \lim_{y\to 0} \frac{3m}{m^2+1} \\ &= \frac{3m}{m^2+1} \end{aligned}$$

Note: Here, we can use polar! (simpler)

The limit depends on the approach path (i.e. slope m) so f is not continuous @ $(0,0)$.

Lecture Question

Below is the contour maps and graphs of several functions. Determine if the limit does exist (Y), does not (N), or cannot tell (C) as $\lim_{(x,y)\to(0,0)} f(x,y)$ (the blue ball shows the location $(0,0)$ - does not necessarily imply continuity)

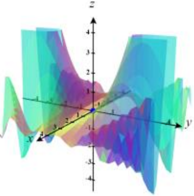


Answer Choices: (L → R)

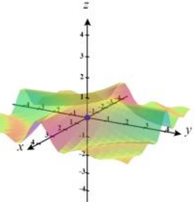
- **A.** **Row 1:** Y,Y,Y,C **Row 2:** C,C,N,Y
- **B.** **Row 1:** C,Y,C,N **Row 2:** Y,C,N,Y
- **C.** **Row 1:** Y,Y,N,N **Row 2:** Y,N,C,Y
- **D.** **Row 1:** C,Y,C,N **Row 2:** C,N,Y,Y
- **E.** **Row 1:** C,Y,C,C **Row 2:** C,C,C,Y
- **F.** **Other.**

Lecture Question (Solution)

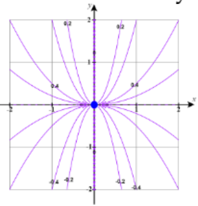
Below is the contour maps and graphs of several functions. Determine if the limit does exist (Y), does not (N), or cannot tell (C) as $\lim_{(x,y)\to(0,0)} f(x,y)$ (the blue ball shows the location $(0,0)$ - does not necessarily imply continuity)



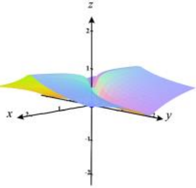
Y



Y

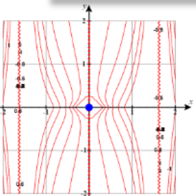


N

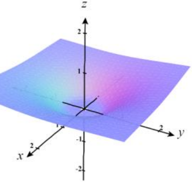


N

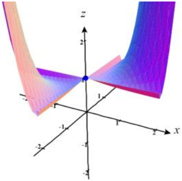
Matching is from left to right



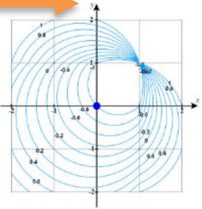
Y



N



C

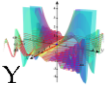


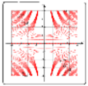
Y

Answer Choices: (L → R)

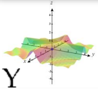
- **A.** **Row 1:** Y,Y,Y,C **Row 2:** C,C,N,Y
- **B.** **Row 1:** C,Y,C,N **Row 2:** Y,C,N,Y
- **C.** **Row 1: Y,Y,N,N Row 2: Y,N,C,Y**
- **D.** **Row 1:** C,Y,C,N **Row 2:** C,N,Y,Y
- **E.** **Row 1:** C,Y,C,C **Row 2:** C,C,C,Y
- **F.** **Other.**

Lecture Question (Detailed Solution)

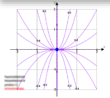


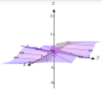


The function appears to have some type of symmetry. Moreover, the function may be viewed in terms of a level curve on the xy-plane ($k = 0$). It may also be viewed in terms of traces across various regions, all which indicate the function approaches a certain value at the origin.
(Main Idea: View the limit in terms of contour plots and level curves)

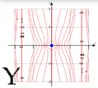


The function appears to have a height which is above the blue sphere. More so, the function appears to contain some type of symmetry (i.e. polar symmetry) about the origin with no abrupt changes.
(Main Idea: Does the graph have polar symmetry? Does the function appear defined?)

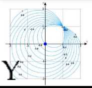




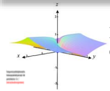
The function appears to have 'jumps' as indicated by the heights on the contour plot. More so, the function does not appear symmetrical about the xy-plane. Across the path $y = x$ the limit appears to exist, but does not exist across $y = \pm x^2$ or $x = \pm y^2$
(Main Idea: Check all paths. Read carefully the contour height and determine the limit.)



The function appears to be well-defined near the origin. In addition, the contours appear to approach a certain value at the origin. The function appears to have an increase as $(x, y) = (a, b)$ approaches the origin. Likewise, the vertical 'slice' near the origin appears to cross the origin if one looks very closely.
(Main Idea: Identify patterns such as increases or decreases around certain region)

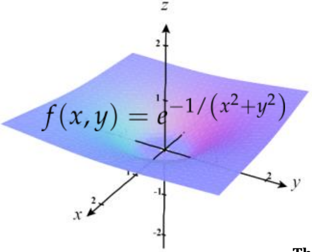


This function appears to approach a certain value at the origin. This is a resounding "yes" if one views the various altitudes, regardless of the ascent (the function does not appear to have a drastic variation in steepness)
(Main Idea: View the other altitudes on the contour map and determine steepness relative to each ascent)

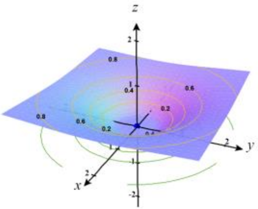


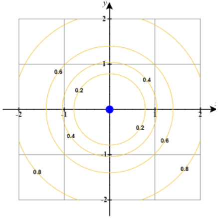
If one thinks in terms of polar coordinates, the function appears to have variations in the slope (i.e. θ) near the origin. This graph is oddly similar to example 3 in the powerpoint.
(Main Idea: Think in terms of polar coordinates)

Lecture Question (Detailed Solution)



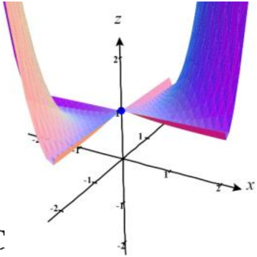
$\Longrightarrow$





N

This is a difficult one! If an individual looks closely, the function is not continuous at the origin (there is a 'white patch' which is not defined). This function would only have a limit, if one defines it piecewise such $(x, y) = (0, 0)$ at the origin.



C

$f(x,y) = (2xy)^{xy}$

This one is *extremely* tricky! Based solely on the graph of the multivariable function, we cannot tell whether the limit does or does not exist, since it leads to ambiguity. Further analysis is required to make a correct judgement. Although ambiguous, the answer choice "yes" will also be considered accepted, despite the inability to determine the limit based on the graph of the function. Through the process of elimination and the answer choices, one can still decide upon the correct answer choice. Below is the actual limit value across r and θ (note: the focus for determining if it exist or not depends on r)

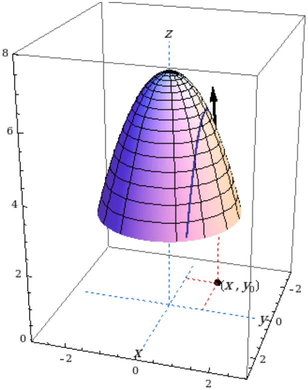
$$\lim_{r\to 0} \left(2r^2\cos(\theta)\sin(\theta)\right)^{r^2\cos(\theta)\sin(\theta)} = 1$$

$$\lim_{\theta\to 0} \left(2r^2\cos(\theta)\sin(\theta)\right)^{r^2\cos(\theta)\sin(\theta)} = 1$$

$$\lim_{\theta\to -\infty} \left(2r^2\cos(\theta)\sin(\theta)\right)^{r^2\cos(\theta)\sin(\theta)} = (r^2 - 1 \text{ to } 1)^{r^2 - 1 \text{ to } 1}$$

$$\lim_{\theta\to \infty} \left(2r^2\cos(\theta)\sin(\theta)\right)^{r^2\cos(\theta)\sin(\theta)} = (r^2 - 1 \text{ to } 1)^{r^2 - 1 \text{ to } 1}$$

Derivatives of Multivariable Functions



$f(x, y)$

is a surface in 3-D

- We're finding the slope of a tangent line to a surface
- To find the slope of tangent line in the "x-direction" we must contain the tangent line in a plane parallel to the xz plane (since it contains the x-axis)
    - Same assumption for the "y-direction"
- **Idea:** We're treating one variable as constant and thereby ensuring that the tangent line is in the direction of the other variable. This is called a *partial derivative*.
    - One variable is *changing* while the other(s) is/are *fixed*.

Partial Derivatives

Calculus 1

$$f(x) \quad \Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \frac{df}{dx}$$

Calculus 3

$$f(x,y) \Rightarrow$$

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h} = \frac{\partial f}{\partial y}$$

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} = \frac{\partial f}{\partial x}$$

- Essentially an extension of the calculus one limit definition, but here we instead hold one variable constant and the other changes
- In **calculus 1** the derivative of a function represents the rate of change of the function as *one variable* changes on a **2D plane**
- In **calculus 3** the derivative of a multivariable function represents the rate of change of *one variable* while holding the *others constant* or fixed across a **3D surface**

First Order

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f(x,y)) = z_x = \frac{\partial z}{\partial x} = D_x f$$
$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(f(x,y)) = z_y = \frac{\partial z}{\partial y} = D_y f$$

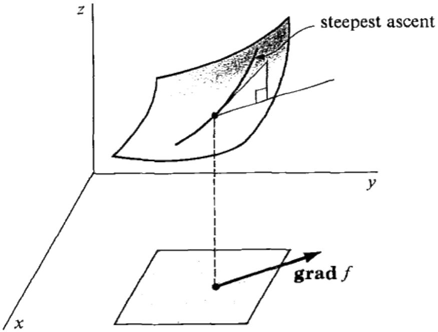
Higher Order

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \qquad f_{xyx} = (f_{xy})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial x \partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \qquad f_{yxx} = (f_{yx})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial^3 f}{\partial x^2 \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$



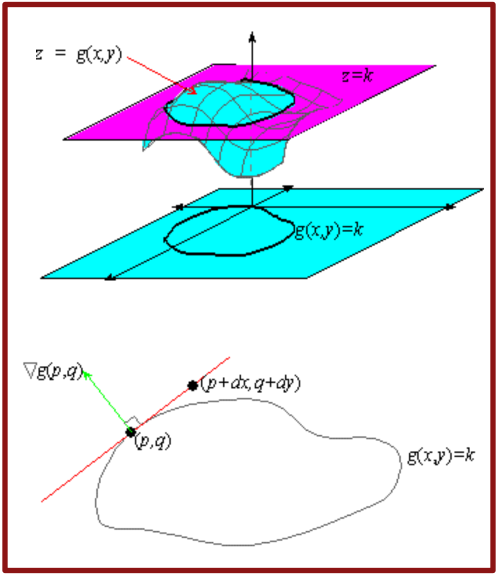
$$\nabla f = \langle f_x, f_y \rangle \quad \text{or} \quad \nabla f = \langle f_x, f_y, f_z \rangle$$

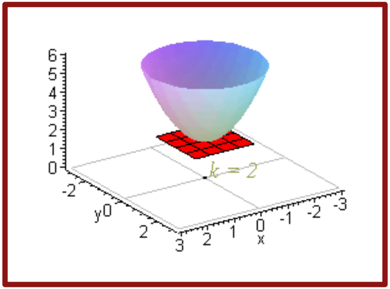
the gradient vector $\nabla f\,(x_0, y_0)$ is orthogonal (or perpendicular) to the level curve $f(x,y) = k$ at the point (x_0, y_0)

- The gradient is a vector quantity that relates "grade" (climb) of a surface
- The gradient gives the vector for the steepest grade of a surface at a point
    - Provides direction to put your tangent line so that you can get the maximum slope
- The gradient is one dimension lower than the function/surface
- The gradient vector is always orthogonal, or *normal*, to the surface at a point.

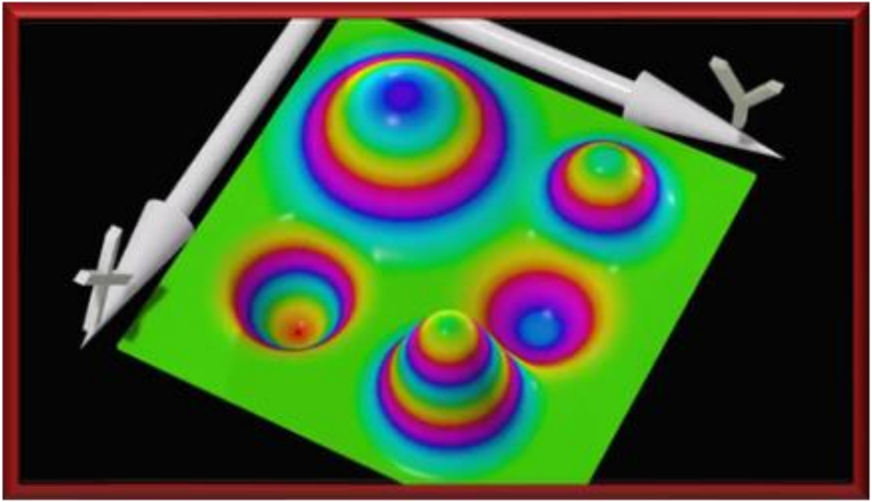
the gradient vector $\nabla f\,(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x,y,z) = k$ at the point (x_0, y_0, z_0)

Gradient & Level Curve (Visual)



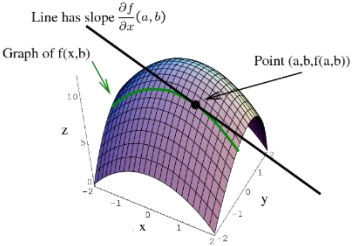


Partial Derivative/Gradient Visualization

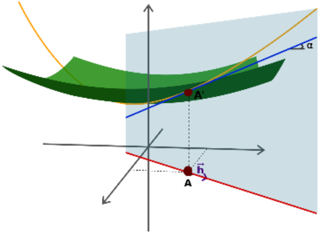


Directional Derivatives

- A **partial derivative** is not ‘coordinate free’
- A **partial derivative** is, in effect, a directional derivative in the “increasing” direction along the appropriate axis.
- A **partial derivative** is the rate of change of f(x,y), which can be thought of the slope of the function at a point (x0,y0).
- **Directional derivative**, on the other hand, is coordinate free
- **Directional derivative** is the instantaneous rate of change (which is a scalar) of f(x,y) in the direction of the unit vector u
- **Directional derivative** measures the rate of change in some direction from a point, and that direction could be any unit vector



Partial Derivative



Directional Derivative

Partial Derivative

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

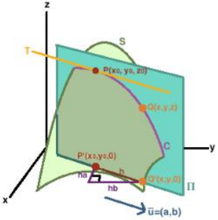
$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Directional Derivative

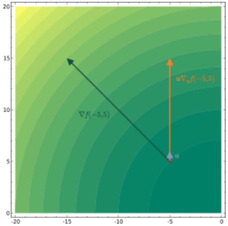
$$\text{unit vector } \vec{u} = \langle a, b \rangle$$

$$D_u f(x,y) = \lim_{h \to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h}$$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} \quad = \langle f_x, f_y \rangle \cdot \langle a, b \rangle$$



- **Directional derivative definition** is like that of the partial derivative definition but *different*.
- **Directional derivatives** consist of a *dot product* between the gradient vector and a unit vector
- **Directional derivative** is also a slope which is a scalar



Properties of Directional Derivatives & Gradient

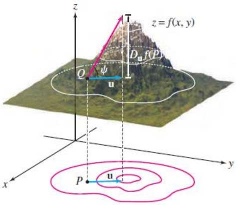
1. if $\nabla f = \vec{0}$, then $D_{\vec{u}} f = 0$ for any $\vec{u}$

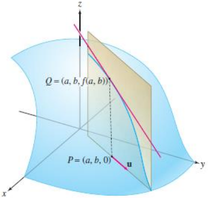
2. $D_{\vec{u}} f$ has its maximum value of $\|\nabla f\|$ and this happens when $\theta = 0$
   since $D_{\vec{u}} f = \nabla f \cdot \vec{u} = \|\nabla f\| \|\vec{u}\| \cos\theta = \|\nabla f\| \cos\theta$ hence $\vec{u} = \lambda \nabla f$
   parallel vectors for some scalar multiple λ
   Any other θ takes a fraction of ∇f so $D_{\vec{u}} f$ becomes less steep

3. $D_{\vec{u}} f$ has its minimum value of $-\|\nabla f\|$ and this happens when $\theta = \pi$

4. ∇f gives the vector for the steepest 'grade' of a surface at a point

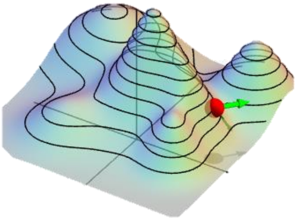
5. if $\vec{u}$ is not parallel to ∇f, think of $\vec{u}$ as turning $D_{\vec{u}} f$ from the direction of steepest climb ∇f

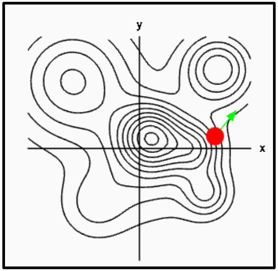




Lecture Question

Let the unit vector in the plot be $u = \langle 0.64, 0.77 \rangle$. From the graph and contour plot, what is the sign of the directional derivative at the red dot (point a)?

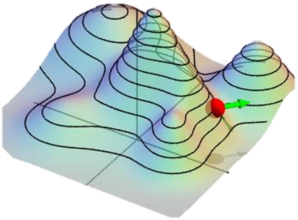


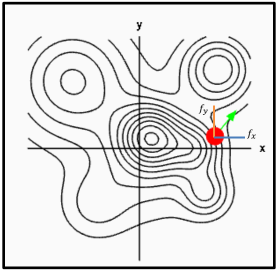


$$D_{\mathbf{u}} f(\mathbf{a}) > 0 \qquad D_{\mathbf{u}} f(\mathbf{a}) = 0 \qquad D_{\mathbf{u}} f(\mathbf{a}) < 0$$

Lecture Question

Let the unit vector in the plot be $u = \langle 0.64, 0.77 \rangle$. From the graph and contour plot, what is the sign of the directional derivative at the red dot (point a)?





$D_{\mathbf{u}} f(\mathbf{a}) > 0$ $\quad$ $D_{\mathbf{u}} f(\mathbf{a}) = 0$ $\quad$ $D_{\mathbf{u}} f(\mathbf{a}) < 0$

Both f_x and f_y are negative based on the contour plot and graph. By definition, we have that $D_u f(x,y) = \nabla f \cdot u = \langle (-), (-) \rangle \cdot \langle (+), (+) \rangle = (-)$

Directional Derivative (example)

Find each of the directional derivatives

$D_{\vec{u}} f(x,y,z)$ where $f(x,y,z) = x^2 z + y^3 z^2 - xyz$ in the direction of $\vec{v} = \langle -1, 0, 3 \rangle$

Find each of the directional derivatives
$D_{\vec{u}}f(x,y,z)$ where $f(x,y,z) = x^2z + y^3z^2 - xyz$ in the direction of $\vec{v} = \langle -1, 0, 3\rangle$

$$\|\vec{v}\| = \sqrt{1+0+9} = \sqrt{10} \neq 1$$

$$\vec{u} = \frac{1}{\sqrt{10}}\langle -1,0,3\rangle = \left\langle -\frac{1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}}\right\rangle$$

$$\nabla f = \langle 2xz - yz,\ 3y^2z^2 - xz,\ x^2 + 2y^3z - xy\rangle$$

$$D_{\vec{u}}f(x,y,z) = \left(-\frac{1}{\sqrt{10}}\right)(2xz - yz) + (0)\left(3y^2z^2 - xz\right) + \left(\frac{3}{\sqrt{10}}\right)\left(x^2 + 2y^3z - xy\right)$$

$$= \frac{1}{\sqrt{10}}\left(3x^2 + 6y^3z - 3xy - 2xz + yz\right)$$

Directional Derivative (example)

Find each of the directional derivatives

$D_{\vec{u}}f(2,0)$ where $f(x,y) = x\mathbf{e}^{xy} + y$ and $\vec{u}$ is the unit vector in the direction of $\theta = \dfrac{2\pi}{3}$

Directional Derivative (example)

Find each of the directional derivatives

$D_{\vec{u}}f(2,0)$ where $f(x,y) = x\mathbf{e}^{xy} + y$ and $\vec{u}$ is the unit vector in the direction of $\theta = \dfrac{2\pi}{3}$

$$\vec{u} = \langle \cos\theta, \sin\theta \rangle$$

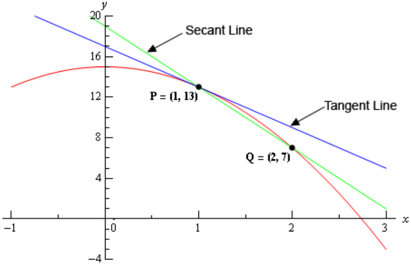
$$\vec{u} = \left\langle \cos\left(\frac{2\pi}{3}\right), \sin\left(\frac{2\pi}{3}\right) \right\rangle = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

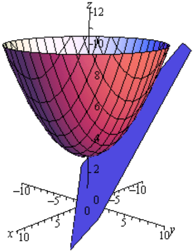
$$\nabla f = \langle \mathbf{e}^{xy} + xy\mathbf{e}^{xy}, x^2 e^{xy} + 1 \rangle$$

$$D_{\vec{u}}f(x,y) = \left(-\frac{1}{2}\right)(\mathbf{e}^{xy} + xy\mathbf{e}^{xy}) + \left(\frac{\sqrt{3}}{2}\right)\left(x^2\mathbf{e}^{xy} + 1\right)$$

$$D_{\vec{u}}f(2,0) = \left(-\frac{1}{2}\right)(1) + \left(\frac{\sqrt{3}}{2}\right)(5) = \frac{5\sqrt{3}-1}{2}$$

Tangent Planes





$$\mathbf{n} = \mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & f_y(a,b) \\ 1 & 0 & f_x(a,b) \end{vmatrix} = \langle f_x(a,b), f_y(a,b), -1 \rangle$$

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

1-VAR:

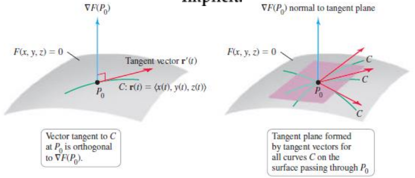
- **Tangent Line** is a line that just touches the graph of the function at the point in question and is "parallel" (in some way) to the graph at that point, involves slope

2-VAR:

- **Tangent plane** is a plane that touches the surface at a point and is "parallel" to the surface at the point, involves partial derivatives
  - Defined by a cross-product with two vectors with respect to f_x and f_y to a 3D surface that touches the surface at a point
  - Plug in the point to identify location
  - *Types:* Implicit and Explicit

Types of Tangent Planes

Implicit:



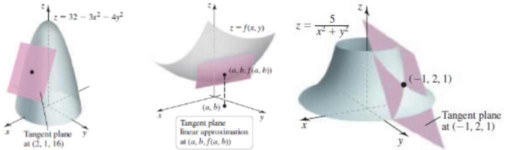
$$\frac{d}{dt}(F(x(t), y(t), z(t))) = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt}$$

$$= \underbrace{\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle}_{\nabla F(x,y,z)} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle}_{\mathbf{r}'(t)}$$

$$= \nabla F(x, y, z) \cdot \mathbf{r}'(t)$$

$$\implies \nabla F(x, y, z) \cdot \mathbf{r}'(t) = 0$$
$$\implies \nabla F(a, b, c) \cdot \langle x - a, y - b, z - c\rangle = 0$$
$$\implies F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

Explicit:



Explicit:

- Equation is defined by $z = f(x, y)$, or otherwise $F(x, y, z) = z - f(x,y) = 0$
- ∇f gives the normal to a level curve at a point
- Special case of implicit surface

Implicit:

- Equation is defined by $F(x,y,z) = 0$
- May view implicit surface as a level surface $F(x,y,z) = K$
- Tangent plane formed by tangent vectors, which involves chain rule for paths
- $r(t)$ is a vector function that represents a curve
- $r'(t)$ gives a slope of tangent vector to level curve
- When dot product is zero, the two vectors are orthogonal $\theta = \frac{\pi}{2}$
- ∇f gives the normal to a level curve at a point

Explicit Equation: $z = f(a, b) + f_x(a, b)(x - a) + f_y\ (a,b)(y - b)$

Implicit Equation: $F_x(a,b,c)\,(x - a) + F_y(a,b,c)(y - b) + F_z(a,b,c)(z - c) = 0$

Find an equation of the plane tangent to the paraboloid

$$z = f(x, y) = 32 - 3x^2 - 4y^2 \text{ at } (2,1,16)$$

Tangent Plane (explicit, example)

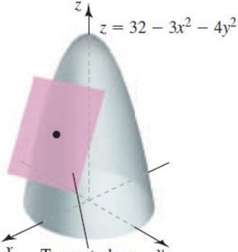
Find an equation of the plane tangent to the paraboloid

$$z = f(x,y) = 32 - 3x^2 - 4y^2 \text{ at } (2,1,16)$$

$$f_x = -6x \text{ and } f_y = -8y$$

$$f_x(2,1) = -12 \text{ and } f_y(2,1) = -8$$

$$\begin{aligned} z &= f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b) \\ &= -12(x-2) - 8(y-1) + 16 \\ &= \boxed{-12x - 8y + 48} \end{aligned}$$



Tangent plane at (2, 1, 16)

Explicit Equation: $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Tangent Plane (implicit, example)

Consider the ellipsoid, $F(x, y, z) = \frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1 = 0$

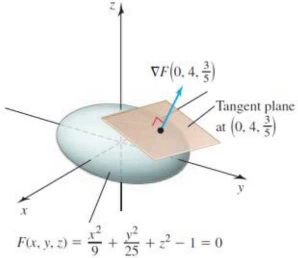
Find an equation of the plane tangent to the ellipsoid at $\left(0, 4, \frac{3}{5}\right)$.

Consider the ellipsoid, $F(x,y,z) = \dfrac{x^2}{9} + \dfrac{y^2}{25} + z^2 - 1 = 0$

Find an equation of the plane tangent to the ellipsoid at $\left(0, 4, \dfrac{3}{5}\right)$.

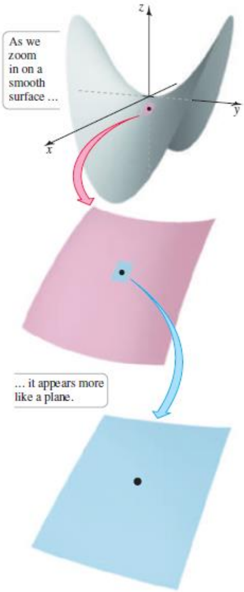
$$\nabla F(x,y,z) = \left\langle \frac{2x}{9}, \frac{2y}{25}, 2z \right\rangle \implies \nabla F\left(0,4,\frac{3}{5}\right) = \left\langle 0, \frac{8}{25}, \frac{6}{5} \right\rangle$$

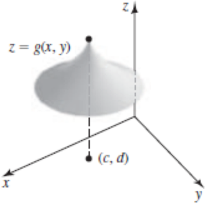
$$\implies 0\cdot(x-0) + \frac{8}{25}(y-4) + \frac{6}{5}\left(z - \frac{3}{5}\right) = 0 \implies \boxed{4y + 15z = 25}$$



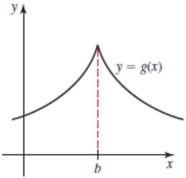
Implicit Equation: $F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0$

Differentiability



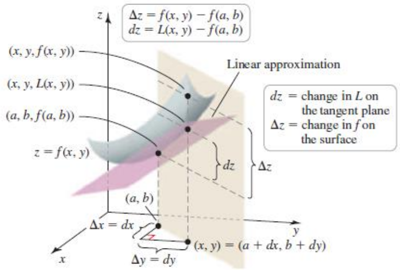


g not differentiable at $(c, d) \Longrightarrow$ no tangent plane at $(c, d, g(c, d))$



g not differentiable at $b \Longrightarrow$ no tangent line at $(b, g(b))$

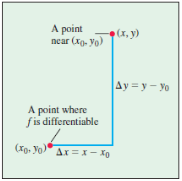
- **Locally linear** - if its surface (graph) looks flatter and flatter as we zoom in on a point
- **Differentiability** can involve looking at a subset or tiny region on a surface
  - Differentiability at a *point* is defined as having a tangent plane at that point (as we zoom in on the surface near the point, it behaves as a plane)
- **Partial derivatives** must exist and be continuous functions



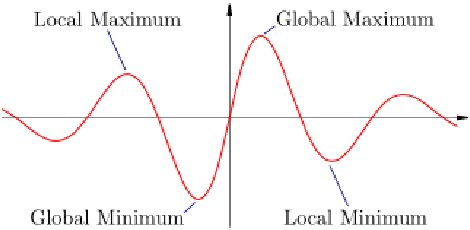
$$\Delta f \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y$$

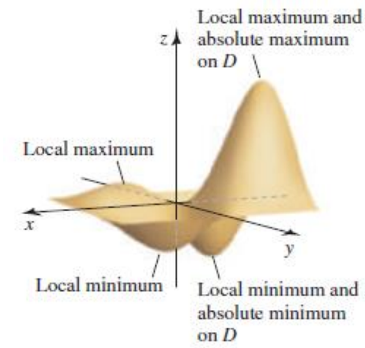
$$df = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$df = f_x(x,y,z)dx + f_y(x,y,z)dy + f_z(x,y,z)dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$



- Can use tangent plane to estimate changes in the function (i.e. linear approximation or differentials)
- Demonstrates tiny changes in the function value at a point
- Extension of calculus 1 differentials
- Δf is the actual change across a point P to a point Q
- df is the change in height from "P" to a point on the tangent plane
- Important for approximations in many fields, given the numerical accuracy





1-VAR:

- Relative max is a peak of a *curve*
- Given by critical points: $f'(x) = 0$ or where $f'(x)$ is undefined
- Critical points could be inflection points

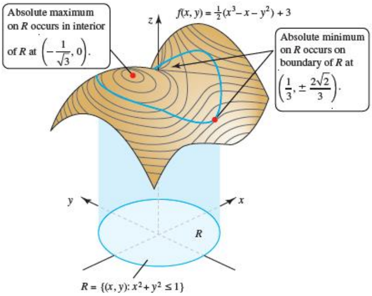
2-VAR:

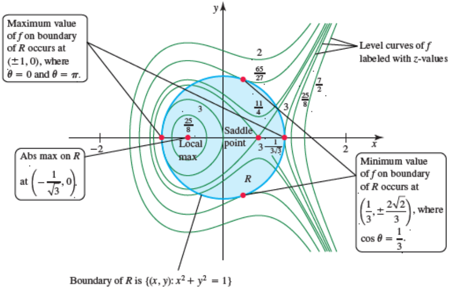
- Relative max is a peak of a *surface*
- Given by critical points, but BOTH f_x and f_y must equal zero at the *same point*
- Critical points can be undefined or saddle points
- All relative extrema are found at critical points or the boundary

BOTH:

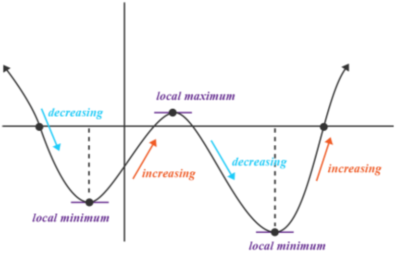
- Can have more than one extrema
- Relative minimum works the same

Optimization in Several Variables



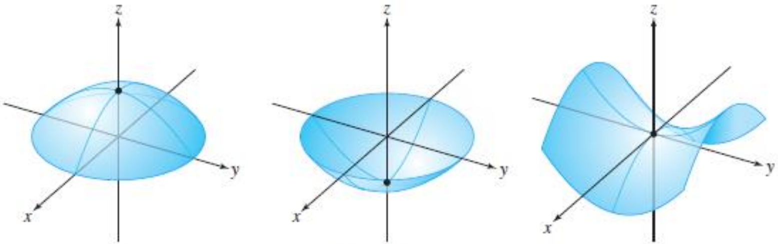


- Where both f_x and f_y are increasing (+) to decreasing (-) → **relative maximum**
- Where both f_x and f_y are decreasing (-) to increasing (+) → **relative minimum**
- Where one partial is increasing (+) and other is decreasing (-) or vice versa → **saddle point**
- Second derivative test relates critical points to a discriminant (often in determinant form)
- *Steps:*
  1. Find $f_x = 0$ and $f_y = 0$
  2. Find critical points
  3. Find $D(x, y)$
  4. Plug in critical points



$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

$$D = D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}^2(a,b)$$



(A) Local maximum (B) Local minimum (C) Saddle

$D(a,b) < 0 \Rightarrow$ saddle point
$D(a,b) = 0 \Rightarrow$ test inconclusive
$D(a,b) > 0, \quad f_{xx}(a,b) < 0 \Rightarrow f(a,b)$ is a local maximum
$D(a,b) > 0, \quad f_{xx}(a,b) > 0 \quad \Rightarrow \quad f(a,b)$ is a local minimum

Lecture Question

Find a value of k for which there exists a function $f(x, y)$ such that $f_x = kx + 6y$, $f_y = kx - 6y$, and find such a function.

Answer Choices:

A. $k = 1$
B. $k = 2$
C. $k = 3$
D. $k = 4$
E. $k = 5$
F. $k = 6$

Lecture Question

Find a value of k for which there exists a function $f(x,y)$ such that $f_x = kx + 6y$, $f_y = kx - 6y$, and find such a function.

From Clairaut's theorem, $k = f_{yx} = f_{xy} = 6$

A function such as the one below will work:

$f = 3x^2 + 6xy - 3y^2$ will do

Clairaut's Theorem

Suppose that f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are continuous on this disk then,

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Answer Choices:

- A. $k = 1$
- B. $k = 2$
- C. $k = 3$
- D. $k = 4$
- E. $k = 5$
- F. $k = 6$

Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0, y = 0,$ and $y = 9 - x$

Find the absolute maximum and minimum values of

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines

$$x = 0, y = 0, \text{ and } y = 9 - x$$

Check Interior points: $f_x = 2 - 2x = 0, \quad f_y = 4 - 2y = 0 \implies (x,y) = (1,2)$

$f(1,2) = \boxed{7}$

Check Boundary points: (i) On the segment OA, $y = 0$

$f(x,y) = f(x,0) = 2 + 2x - x^2$ where $0 \le x \le 9$ extreme values can be at end points!

$$x = 0 \quad \text{where} \quad f(0,0) = \boxed{2}$$

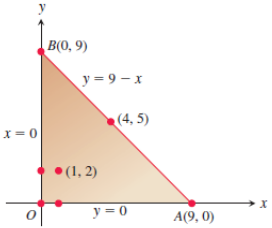
$$x = 9 \quad \text{where} \quad f(9,0) = 2 + 18 - 81 = \boxed{-61}$$

On the interior:

$$f'(x,0) = 2 - 2x = 0$$

$$f'(x,0) = 0 \;@\; x = 1$$

$$f(x,0) = f(1,0) = \boxed{3}$$



Find the absolute maximum and minimum values of

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines

$$x = 0, y = 0, \text{ and } y = 9 - x$$

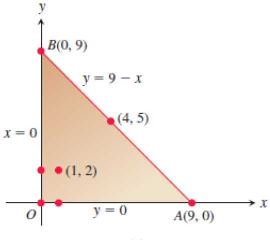
Check Boundary points: (ii) On the segment OB, $x = 0$

$f(x,y) = f(0,y) = 2 + 4y - y^2$ where $0 \leq y \leq 9 \implies f'(0,y) = 0$

$f'(0,y) = 4 - 2y$ On the interior: $f'(0,y) = 0 \; @ \; y = 2$

Checking interior and endpoints:

$$f(0,0) = 2, \quad f(0,9) = \boxed{-43}, \quad f(0,2) = \boxed{6}$$



Find the absolute maximum and minimum values of

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines

$$x = 0, y = 0, \text{ and } y = 9 - x$$

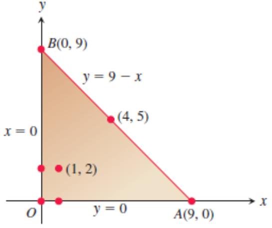
Check Boundary points: (iii) On the segment AB, $y = 9 - x$

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$

$$f(x, 9-x) = 2 + 2x + 4(9-x) - x^2 - (9-x)^2 = -43 + 16x - 2x^2$$

$$f'(x, 9-x) = 16 - 4x = 0 \implies x = 4$$

$$y = 9 - 4 = 5 \quad \text{and} \quad f(x,y) = f(4,5) = \boxed{-11}$$



Find the absolute maximum and minimum values of

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$

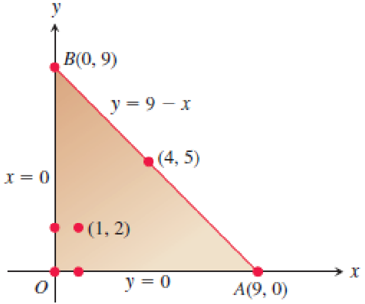
on the triangular region in the first quadrant bounded by the lines

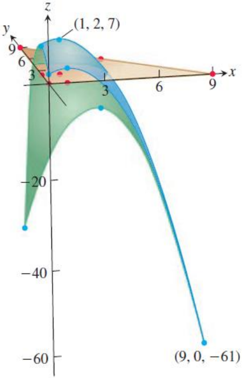
$$x = 0, y = 0, \text{ and } y = 9 - x$$

Summary: the function value candidates: $7, 2, -61, 3, -43, 6, -11$

The maximum is 7. $f(1,2) = \boxed{7}$

The minimum is -61. $f(9,0) = \boxed{-61}$

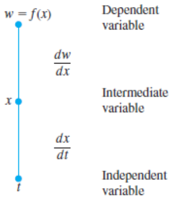




Chain Rule

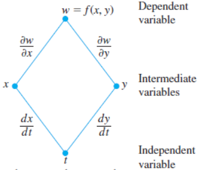
To find dw/dt, we read down the route from w to t, multiplying derivatives along the way.

Chain Rule



To remember the Chain Rule, picture the diagram below. To find dw/dt, start at w and read down each route to t, multiplying derivatives along the way. Then add the products.

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

1-VAR:

- One independent variable (i.e t)
- Involves "intermediate variable" with composition of functions
  - $w(t) = f(g(t))$
  - $\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$
  - *Intermediate variable:* $x = g(t)$

2-VAR:

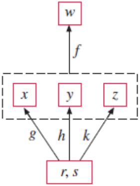
- Many independent variables
- Many forms and types
  - The types and/or formats depend on both independent and intermediate variables
  - May be defined implicitly or explicitly (as noted in the textbook)
- Many intermediate variables
  - Intermediate variables provide a pathway from the dependent variable to the independent variable(s)

Chain Rule for Two Independent Variables and Three Intermediate Variables:

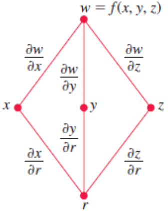
Dependent variable

Intermediate variables

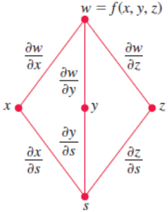
Independent variables



$$w = f(g(r,s), h(r,s), k(r,s))$$

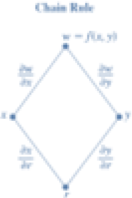


$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\delta y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$



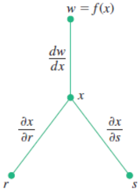
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Chain Rule for two intermediate variables:



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

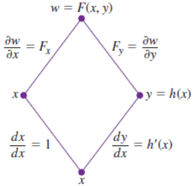
Chain Rule for one intermediate variable:



$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

Implicit Differentiation:



$$\frac{dw}{dx} = F_x \cdot 1 + F_y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

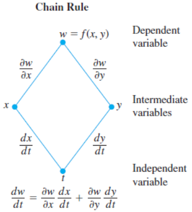
Multivariable Chain Rule (example 1)

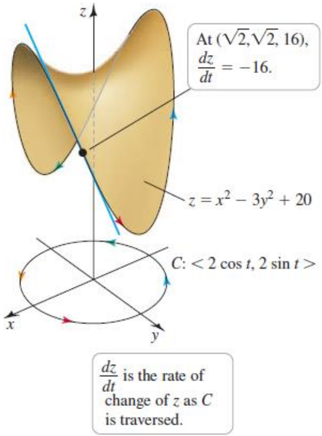
Let $z = x^2 - 3y^2 + 20$ where $x = 2\cos t$ and $y = 2\sin t$.

Find $\dfrac{dz}{dt}$ and evaluate it at $t = \pi/4$

Let $z = x^2 - 3y^2 + 20$ where $x = 2\cos t$ and $y = 2\sin t$.

Find $\dfrac{dz}{dt}$ and evaluate it at $t = \pi/4$





$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \underbrace{(2x)}_{\frac{\partial z}{\partial x}}\underbrace{(-2\sin t)}_{\frac{dx}{dt}} + \underbrace{(-6y)}_{\frac{\partial z}{\partial y}}\underbrace{(2\cos t)}_{\frac{dy}{dt}}$$

$$= -4x\sin t - 12y\cos t$$
$$= -8\cos t\sin t - 24\sin t\cos t$$
$$= -16\sin 2t$$

$$t = \pi/4 \text{ gives } \left.\frac{dz}{dt}\right|_{t=\pi/4} = -16$$

Lecture Question

Given that $f(x,y) = g(u(x,y), v(x,y))$ and the following information find $f_x(1,2)$

(a,b)	$g(a,b)$	$g_u(a,b)$	$g_v(a,b)$	$u(a,b)$	$u_x(a,b)$	$u_y(a,b)$	$v(a,b)$	$v_x(a,b)$	$v_y(a,b)$
$(1,2)$	-2	-1	3	2	1	3	-3	-1	2
$(2,1)$	-1	2	-3	4	-1	4	2	5	7
$(2,-3)$	3	4	-2	-3	1	-5	2	-1	-2
$(1,-2)$	1	-3	2	2	-1	2	-3	4	6

Answer Choices:

A. $\boldsymbol{f_x = 6}$

B. $f_x = -4$

C. $f_x = 4$

D. $f_x = -6$

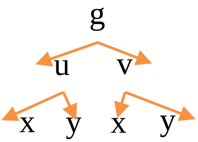
Lecture Question

Given that $f(x,y) = g(u(x,y), v(x,y))$ and the following information find $f_x(1,2)$

(a,b)	$g(a,b)$	$g_u(a,b)$	$g_v(a,b)$	$u(a,b)$	$u_x(a,b)$	$u_y(a,b)$	$v(a,b)$	$v_x(a,b)$	$v_y(a,b)$
$(1,2)$	-2	-1	3	2	1	3	-3	-1	2
$(2,1)$	-1	2	-3	4	-1	4	2	5	7
$(2,-3)$	3	4	-2	-3	1	-5	2	-1	-2
$(1,-2)$	1	-3	2	2	-1	2	-3	4	6

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x}$$

$$\begin{aligned} f_x(1,2) &= g_u(u(1,2), v(1,2))u_x(1,2) + g_v(u(1,2), v(1,2))v_x(1,2) \\ &= g_u(2,-3)\cdot 1 \quad + g_v(2,-3)\cdot(-1) \\ &= 4\cdot 1 + (-2)\cdot(-1) = 6 \end{aligned}$$



Answer Choices:

A. $f_x = 6$

B. $f_x = -4$

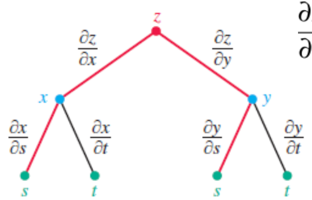
C. $f_x = 4$

D. $f_x = -6$

Let $z = \sin 2x \cos 3y$ where $x = s + t$ and $y = s - t$. Evaluate $\partial z/\partial s$ and $\partial z/\partial t$
Evaluate $\partial z/\partial s$ and $\partial z/\partial t$.

Let $z = \sin 2x \cos 3y$ where $x = s + t$ and $y = s - t$. Evaluate $\partial z/\partial s$ and $\partial z/\partial t$

Evaluate $\partial z/\partial s$ and $\partial z/\partial t$.

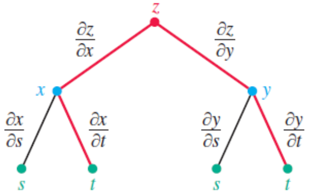


$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= 2\cos 2x \cos 3y \cdot 1 + (-3\sin 2x \sin 3y) \cdot 1$$

$$= 2\cos(\underbrace{2(s+t)}_{x})\cos(\underbrace{3(s-t)}_{y}) - 3\sin(\underbrace{2(s+t)}_{x})\sin(3\underbrace{(s-t)}_{y})$$



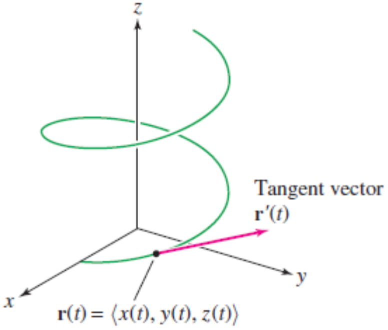
$$\boxed{\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}}$$

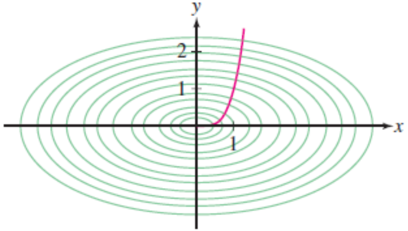
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= 2\cos 2x \cos 3y \cdot 1 + (-3\sin 2x \sin 3y) \cdot -1$$

$$= 2\cos(\underbrace{2(s+t)}_{x})\cos(\underbrace{3(s-t)}_{y}) + 3\sin(\underbrace{2(s+t)}_{x}))\sin(3(\underbrace{s-t}_{y}))$$

Chain Rule for Paths





- Application of the gradient and involves vector valued functions
  - Deals with composition of functions $f\big(\boldsymbol{r}(t)\big)$
  - $r(t)$ *represents* the path of endpoints of the vectors
  - $r'(t)$ *represents* the tangent or velocity vector
  - Requires dot product of tangent vector to gradient vector
  - ***Note:*** *do not confuse the Chain Rule for Paths with the more elementary Chain Rule for Gradients*

Chain Rule for Paths (example 1)

The temperature at a point (x,y) in the plane is $T(x, y)°C$.
If a bug crawls on the plane so that its position at time t (in minutes) is given by

$$x(t) = \sqrt{1+t}, \quad y(t) = 5 + \frac{1}{3}t$$

determine how fast the temperature is rising on the bug's path after 3 minutes, if

$$T_x(2,6) = 20, \quad T_y(2,6) = 3$$

Chain Rule for Paths (example 1)

The temperature at a point (x,y) in the plane is $T(x,y)^\circ C$.
If a bug crawls on the plane so that its position at time t (in minutes) is given by

$$x(t) = \sqrt{1+t}, \quad y(t) = 5 + \frac{1}{3}t$$

determine how fast the temperature is rising on the bug's path after 3 minutes, if

$$T_x(2,6) = 20, \quad T_y(2,6) = 3$$

The temperature on the bug's path is $S(t) = T(x(t), y(t))$

Thus by the Chain Rule for Paths

$$\frac{dS}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

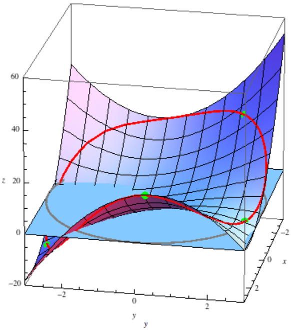
$$\frac{dx}{dt} = \frac{1}{2\sqrt{1+t}}, \quad \frac{dy}{dt} = \frac{1}{3}$$

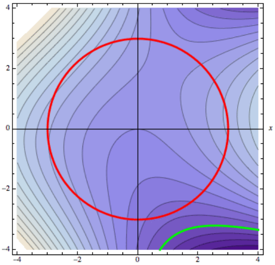
$$\implies \frac{dT}{dt} = \frac{1}{2\sqrt{1+t}}\frac{\partial T}{\partial x} + \frac{1}{3}\frac{\partial T}{\partial y}$$

$$(2,6) \quad @ \quad t = 3$$

$$T_x(2,6) = 20, \quad T_y(2,6) = 3$$

$$\frac{dT}{dt} = \frac{20}{4} + \frac{1}{3} \times 3 = \boxed{6^\circ C/\min}$$

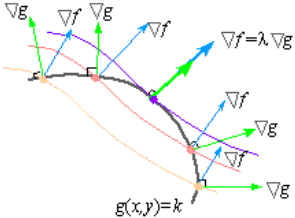




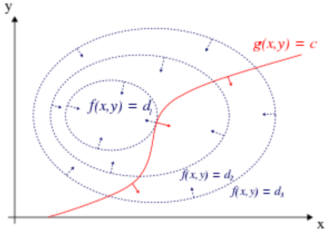
<u>Idea:</u>

- we have a function f(x,y) = z (a **surface** with <u>*many*</u> level curves)
- we have a constraint g(x,y) = c, a <u>*specific*</u> **level curve** to a surface (g(x,y)=z)
- The **intersection** between these two gives a <u>constrained maximum and minimum</u>
- If 2 curves intersect at only one point, they share a <u>common tangent</u> and tangents are <u>scalar multiples</u>
- The **<u>normal vectors</u>** of the level curves are scalar multiples

Lagrange Multipliers (cont.)



$$\begin{aligned} \text{Normal}_1 &= k \cdot \text{Normal}_2 \\ \nabla f &= \lambda \nabla g \\ \lambda &= \text{scalar multiple} \\ &= \text{lagrange multiplier} \end{aligned}$$



- **Idea:**
  - The **level curves** have parallel tangents therefore the level curves have a common or parallel normal
  - If the two graphs are tangent at that point, then their normal vectors must be parallel
    - Normal vectors must be **scalar multiples**
  - The **contour plot** allows us to visualize whether the surface is at a max. or min. based on the specific constraint
  - Normal to level curves of f(x,y) = Normal to level curves of g(x,y)
  - The **gradient** provides normal vectors to level curves
  - At the **optimum**, the gradient of the function is parallel to the gradient of the constraint

Lagrange Multipliers (cont.)

General Strategy:

1. **Identify the function and the constraint**
    - Write the constraint in terms of $g(x,y) = 0$
        - Move constants to one side with the variables and leave the other side equal to zero
2. **Write out the appropriate relationship depending on function**
    - If the function is of two variables such as $f(x,y)$ then the gradient will be $\nabla f(x,y)$, thus two components.
    - If the function is of three variables such as $f(x,y,z)$ then the gradient will be $\nabla f(x,y,z)$, thus three components. (etc.)
        - Model the number of variables in the constraint according to the number of variables in the function
        - Associate λ to the constraint function ($g(x,y) = c$ or $g(x,y,z)=c$)
3. **Solve for λ and then solve for x & y (generally)**
    - There are cases where tricky manipulation is required such as multiplying equations derived from $\nabla f= \lambda\nabla g$ in order to plug in values for the constraint
    - Some cases require solving for one equation derived from $\nabla f= \lambda\nabla g$ and using it by plugging that equation into the remaining equations

3 variable function:

$$\boxed{\begin{gathered}\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ \langle f_x, f_y, f_z \rangle = \lambda \langle g_x, g_y, g_z \rangle = \langle \lambda g_x, \lambda g_y, \lambda g_z \rangle \\ f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z\end{gathered}}$$

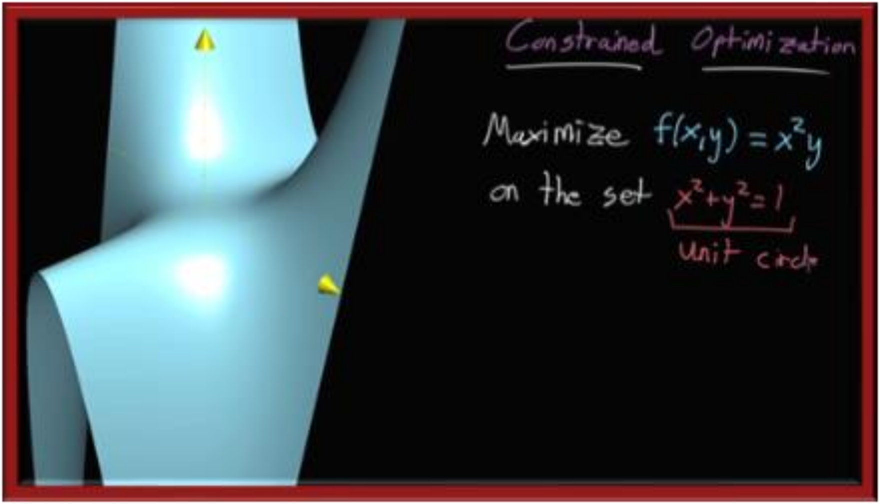
Gradient in 3D on xyz-planes (3D vector)
higher dimensional surface

2 variable function:

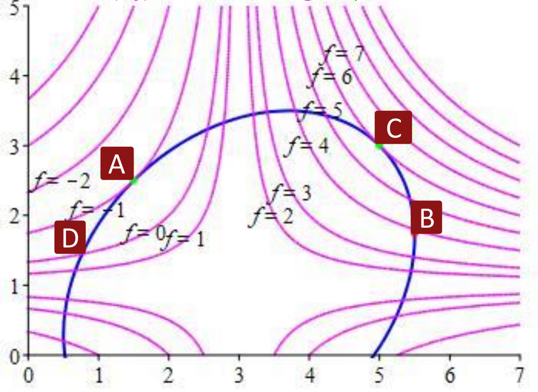
$$\boxed{\begin{gathered}\nabla f(x,y) = \lambda \nabla g(x,y) \\ \langle f_x, f_y \rangle = \lambda \langle g_x, g_y \rangle = \langle \lambda g_x, \lambda g_y \rangle \\ f_x = \lambda g_x \qquad f_y = \lambda g_y\end{gathered}}$$

Gradient in 2D on xy-plane (2D vector)
3D surface

Lagrange Multipliers (Visual)



At what points is the level curve tangent to the constraint? (the constraint, g(x,y)=C, is shown in blue and the level curves of the extremal, f(x,y), are shown in magenta.)



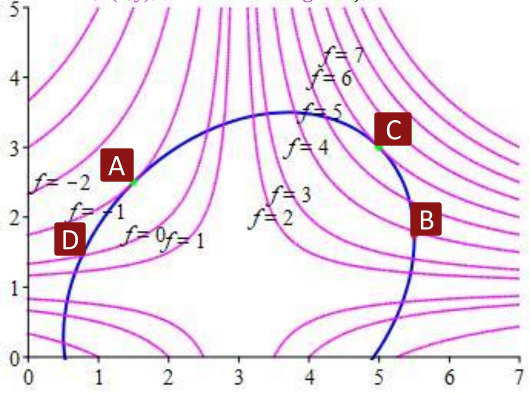
Answer Choices:

A. Points: D & C
B. Points: B & A
C. Points: A & C
D. Points: D & B

At what points is the level curve tangent to the constraint? (the constraint, $g(x,y)=C$, is shown in blue and the level curves of the extremal, $f(x,y)$, are shown in magenta.)

The minimum (Point A) and maximum (Point C) must occur at points where the level curve is tangent to the constraint.

At point B and D, the level set is not parallel. Since the level curve crosses the constraint, you can always move one way or the other and increase or decrease the value of the function while remaining on the constraint. So they cannot be the maximum or minimum!



Note: Although in principle, we are solving for λ. In fact, we don't really care about its value. It is merely a mathematical construct that helps us solve for the extrema for the Lagrange multiplier method.

Answer Choices:

A. Points: D & C
B. Points: B & A
C. Points: A & C
D. Points: D & B

Lagrange Multipliers (example 1)

Find the maximum and minimum of $f(x,y) = 5x - 3y$
subject to the constraint $x^2 + y^2 = 136$

Lagrange Multipliers (example 1)

Find the maximum and minimum of $f(x,y) = 5x - 3y$
subject to the constraint $x^2 + y^2 = 136$

Goal: Find λ

We solve the system for a max or min
(guaranteed by the Extreme Value Theorem)

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ f(x,y) &= 5x - 3y \\ g(x,y) &= x^2 + y^2 - 136 = 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} 5 &= 2\lambda x \\ -3 &= 2\lambda y \implies \lambda \neq 0 \\ x^2 + y^2 &= 136 \end{aligned}$$

Solving for x and y, we have:

$$x = \frac{5}{2\lambda} \qquad y = -\frac{3}{2\lambda}$$

Plugging these into the constraint:

$$\frac{25}{4\lambda^2} + \frac{9}{4\lambda^2} = \frac{17}{2\lambda^2} = 136$$

Solve for lambda:

$$\lambda^2 = \frac{1}{16} \quad \Rightarrow \quad \lambda = \pm\frac{1}{4}$$

Lagrange Multipliers (example 1)

Find the maximum and minimum of $f(x,y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$

$$x = \frac{5}{2\lambda} \qquad y = -\frac{3}{2\lambda}$$

If $\lambda = -\frac{1}{4}$: $\quad x = -10 \qquad y = 6$

If $\lambda = \frac{1}{4}$: $\quad x = 10 \qquad y = -6$

Testing values:

$f(-10,6) = -68$ Minimum at $(-10,6)$

$f(10,-6) = 68$ Maximum at $(10,-6)$

Important Note: plugging values into our function allows us to determine if its a maximum or minimum. Remember to plug in values to check.

Extreme Value Theorem: allows for max. or min. since we are concerned with a closed and bounded region (guarantees existence).

Lagrange Multipliers (example 2)

Find the maximum and minimum values of $f(x, y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \leq 4$

Lagrange Multipliers (example 2)

Find the maximum and minimum values of $f(x,y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \leq 4$

First: find all the critical points that are in the disk (i.e. satisfy the constraint)

$$\nabla f = \langle 0,0 \rangle: \quad \begin{array}{lllll} f_x = 8x & \Rightarrow & 8x = 0 & \Rightarrow & x = 0 \\ f_y = 20y & \Rightarrow & 20y = 0 & \Rightarrow & y = 0 \end{array}$$

The only critical point is (0,0) and it does satisfy the inequality.

$$\begin{array}{lcr} \nabla f = \lambda \nabla g & & 8x = 2\lambda x \\ f(x,y) = 4x^2 + 10y^2 & \Longrightarrow & 20y = 2\lambda y \\ g(x,y) = x^2 + y^2 - 4 = 0 & & x^2 + y^2 = 4 \end{array}$$

From the first equation we get:

$$2x(4-\lambda) = 0 \quad \Rightarrow \quad x = 0 \text{ or } \lambda = 4$$

If we have $x = 0$ then the constraint gives us $y = \pm 2$.

If we have $\lambda = 4$ the second equation gives us

$$20y = 8y \quad \Rightarrow \quad y = 0$$

The constraint then tells us that $x = \pm 2$

Lagrange Multipliers gives us four points to check: $(0,2), (0,-2), (2,0)$, and $(-2,0)$

Lagrange Multipliers (example 2)

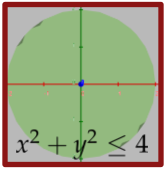
Find the maximum and minimum values of $f(x,y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \leq 4$

To find the maximum and minimum we need to simply plug these four points along with the critical point in the function.

$f(0,0) = 0$	Minimum
$f(2,0) = f(-2,0) = 16$	
$f(0,2) = f(0,-2) = 40$	Maximum

In this case, the minimum was interior to the disk and the maximum was on the boundary of the disk.

Important Note: notice how this region is shaded since we have the greater than or equal to sign



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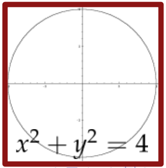
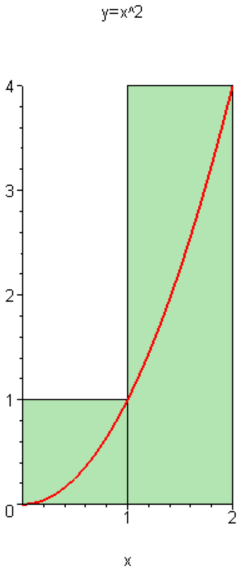


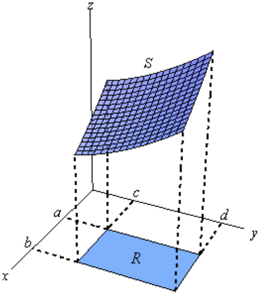
Double Integrals (intro)



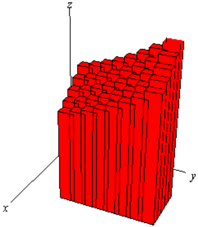
- **Calc 1:**
  - Take [a,b] and divide into "n" equal parts
    $$\frac{b-a}{n} = \Delta x$$
  - Find height at each
    $$f\left(x_i^*\right)$$
  - Find area of each rectangle
    $$f\left(x_i^*\right)\Delta x$$
  - Add them up
    $$A = \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$
  - Take a limit to get an exact value

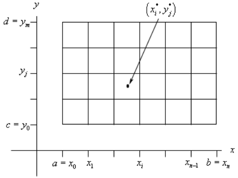
$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \int_a^b f(x)\, dx$$

Double Integrals (intro)



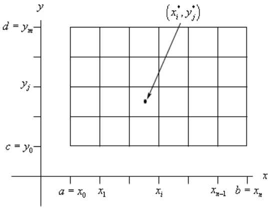
$$R = [a,b] \times [c,d]$$



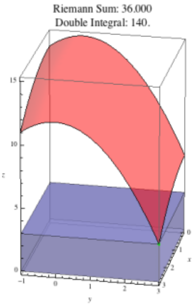


- **<u>Calc 3:</u>**
  - Take x from [a,b] and y from [c,d] and divide into "n" and "m" equal parts
    $$\frac{b-a}{n} = \Delta x \quad \& \quad \frac{d-c}{m} = \Delta y$$
  - Find height at each
    $$f\left(x_i^*, y_i^*\right)$$
  - Area of each rectangle is:
    $$\square = A = \Delta x \cdot \Delta y$$
  - Therefore volume is:
    $$V = \text{Area} \cdot \text{Height}$$
    $$V = f\left(x_i^*, y_i^*\right) \cdot \Delta x \cdot \Delta y$$
  - Approximate Volume: (sum in both x and y thru. the grid)
    $$V \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(x_i^*, y_j^*\right) \Delta A$$

Double Integrals (intro)



$$R = [a, b] \times [c, d]$$



Calc 3:

- Take a limit to get an exact value:

$$V = \lim_{n,\, m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(x_i^*, y_j^*\right) \Delta A = \iint_R f(x, y)\, dA$$

- *Double integrals* are about finding a **volume** under a surface as bounded by a region *R*
- The **most important part** of double integrals is about learning how to set up a region *R* and integrating properly across 'general regions' (non-rectangular)
    - Discussed later in CH. 16.2+
- We have two integrals to denote the that we are dealing with a two-dimensional region

Estimate

$$\iint\limits_R (3-x)(3-y)^2\,dA$$

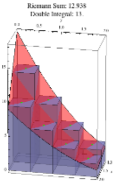
Use a midpoint Riemann sum with subrectangles of size 0.5×0.5 where $R = [1,2] \times [0,2]$.

Double Integrals (intro - example)

Estimate
$$\iint_R (3-x)(3-y)^2\,dA$$
Use a midpoint Riemann sum with subrectangles of size 0.5×0.5 where $R = [1,2] \times [0,2]$.

$\Delta x = (2-1)/0.5 = 2 \implies x$ subintervals, then, are $[1, 1.5]$ and $[1.5, 2]$
$\Delta y = (2-0)/0.5 = 4 \implies y$ subintervals are $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, and $[1.5, 2]$
$\text{Midpoint}_x : \bar{x}_1 = 1.25$ and $\bar{x}_2 = 1.75$
$\text{Midpoint}_y : \bar{y}_1 = 0.25, \bar{y}_2 = 0.75, \bar{y}_3 = 1.25,$ and $\bar{y}_4 = 1.75$
$\Delta A = 0.5 \cdot 0.5 = 0.25$

$$\iint_R (3-x)(3-y)^2\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{4}(3-\bar{x}_i)(3-\bar{y}_j)^2 \cdot 0.25$$



$$\begin{aligned}&\approx 13.23 \cdot 0.25 + 8.86 \cdot 0.25 + 5.36 \cdot 0.25 + 2.73 \cdot 0.25\\ &\quad + 9.45 \cdot 0.25 + 6.33 \cdot 0.25 + 3.83 \cdot 0.25 + 1.95 \cdot 0.25\\ &\approx 12.9\end{aligned}$$

Double Integrals (properties)

Volume:

$$V = \iint_{\mathcal{D}} (z_2(x,y) - z_1(x,y))\, dA$$

Average Value:

$$\bar{f} = \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} f(x,y)dA = \frac{\iint_{\mathcal{D}} f(x,y)dA}{\iint_{\mathcal{D}} 1dA}$$

Constant Function:

For any constant C, $\iint_{\mathcal{D}} C dA = C \cdot \text{Area}(\mathcal{D})$

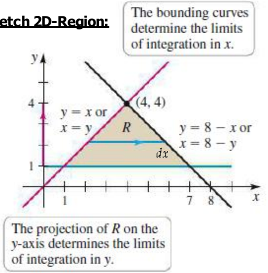
Fubini's Theorem:

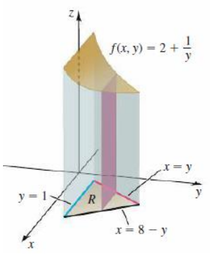
$$\iint_{\mathcal{R}} f(x,y)dA = \int_{x=a}^{b} \int_{y=c}^{d} f(x,y)dydx = \int_{y=c}^{d} \int_{x=a}^{b} f(x,y)dxdy$$

Double Integrals (example 1)

Find the volume of the solid below the surface $f(x,y) = 2 + \frac{1}{y}$ and above the region R in the xy-plane bounded by the lines $y = x$, $y = 8 - x$, and $y = 1$

1. Sketch 2D-Region:





2. Integrate in x first to stay between two function:

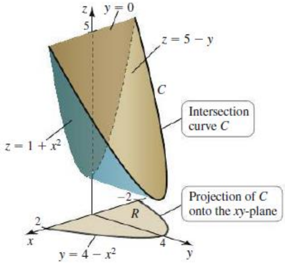
$$
\begin{aligned}
\iint_R \left(2 + \frac{1}{y}\right) dA &= \int_1^4 \int_y^{8-y} \left(2 + \frac{1}{y}\right) dxdy \\
&= \int_1^4 \left(2 + \frac{1}{y}\right) x \Big|_y^{8-y} dy \\
&= \int_1^4 \left(2 + \frac{1}{y}\right)(8 - 2y)dy \\
&= \int_1^4 \left(14 - 4y + \frac{8}{y}\right) dy \\
&= \left(14y - 2y^2 + 8\ln|y|\right)\Big|_1^4 \\
&= 12 + 8\ln 4 \approx 23.09
\end{aligned}
$$

Remark: We could integrate in terms of y first, but this would require two separate integrals. This is due to integration in y first forcing us to switch between two separate functions. The arrow visualization trick allows us to determine which direction to integrate first, assuming one has drawn a region in 2D.

Double Integrals (example 2)

Find the volume of the solid bounded by the parabolic cylinder $z = 1 + x^2$ and the planes $z = 5 - y$ and $y = 0$.

1. Sketch 2D-Region:



Remark: In this example we used symmetry to simplify calculations for our double integral. To use symmetry to simplify a double integral, you must check that *both* the region of integration and the integrand have the same symmetry.

1. Find Intersection between Solids:

$$5 - y = 1 + x^2 \implies y = 4 - x^2$$

$$R = \left\{(x,y) : 0 \le y \le 4 - x^2, -2 \le x \le 2\right\}$$

$$2\int_0^2 \int_0^{4-x^2} \underbrace{(5-y)}_{f(x,y)} - (\underbrace{(1+x^2)}_{g(x,y)})dydx$$

$$=2\int_0^2 \int_0^{4-x^2} \left(4 - x^2 - y\right) dydx$$

$$= 2\int_0^2 \left(\left(4 - x^2\right)y - \frac{y^2}{2}\right)\Big|_0^{4-x^2} dx$$

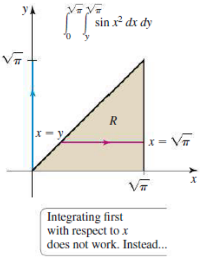
$$= \int_0^2 \left(x^4 - 8x^2 + 16\right) dx$$

$$= \left(\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right)\Big|_0^2$$

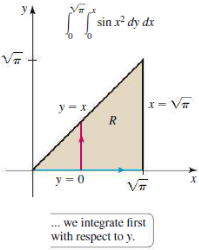
$$= \frac{256}{15}$$

Consider the iterated integral $\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin(x^2)\, dxdy$ evaluate the integral.

1. Sketch 2D-Region:



(a)



(b)

2. Must Switch Integration Order:

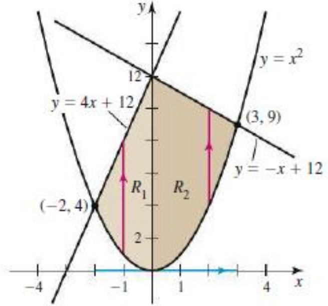
$$
\begin{aligned}
\iint_R \sin x^2 dA &= \int_0^{\sqrt{\pi}} \int_0^{x} \sin x^2 dydx \\
&= \int_0^{\sqrt{\pi}} y \sin x^2 \Big|_0^x \, dx \\
&= \int_0^{\sqrt{\pi}} x \sin x^2 dx \\
&= -\frac{1}{2} \cos x^2 \Big|_0^{\sqrt{\pi}} \\
&= 1
\end{aligned}
$$

Remark: Due to the nasty integrand, we are forced to switch orders of integrations. In these scenarios, one should switch based upon a sketched region and change the bounds accordingly. This requires usage of bounds and regions to re-evaluate a nicer integral. Some questions may require switching orders of integration and having an integral define by the sum of two separate regions.

Double Integrals (example 4)

Find the area of the region R bounded by $y = x^2, y = -x + 12$, and $y = 4x + 12$

1. Sketch 2D-Region:



Issue: The region R in its entirety is bounded neither above and below by two curves, nor on the left and right by two curves. We are forced to decompose the region into a sum of two separate regions, based on the drawing and the bounds. We must locate intersection points based on the bounds (depicted in diagram).

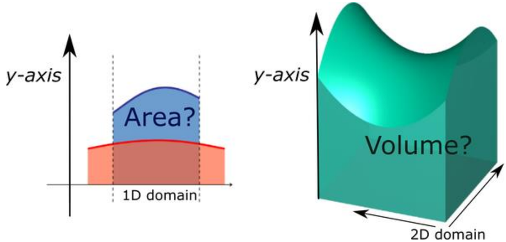
$$A = \iint_{R_1} 1dA + \iint_{R_2} 1dA$$
$$= \int_{-2}^{0}\int_{x^2}^{4x+12} 1dydx + \int_{0}^{3}\int_{x^2}^{-x+12} 1dydx$$
$$= \int_{-2}^{0} (4x + 12 - x^2)\, dx + \int_0^3 (-x + 12 - x^2)\, dx$$
$$= \left(2x^2 + 12x - \frac{x^3}{3}\right)\Big|_{-2}^{0} + \left(-\frac{x^2}{2} + 12x - \frac{x^3}{3}\right)\Big|_{0}^{3}$$
$$= \frac{40}{3} + \frac{45}{2} = \frac{215}{6}$$

$$\int f(x)dx \implies$$ Area under $f(x)$ (2D) **or** mass of straight wire (1D)

$$\iint_R f(x,y)dA \implies$$ Volume under $f(x,y)$ (3D) **or** mass of thin plate (2D)

$$\iiint_E f(x,y,z)dV \implies$$ Some measurement of a region under $f(x,y,z)$ (4-D) **or** mass of plate (region) with thickness (3D)

Note: The mass idea work only if $f(x)$, $f(x,y)$, or $f(x,y,z)$ is the mass density function.



Connection Between Double & Triple Integral

- A **double integral** represents the *signed volume* of the three-dimensional region
- The graph of a function f(x, y, z) of three variables lives in *four-dimensional space*, and thus, a **triple integral** represents a *four-dimensional volume*.
- However, triple integrals can represent several quantities (average value, probability, center of mass, total mass)

$$\int f(x)dx \Longrightarrow A = \iint_{\mathcal{R}} 1dA \quad \text{and} \quad \iint_{\mathcal{R}} f(x,y)dA \Longrightarrow \quad V = \iiint_{\mathcal{W}} 1dV$$

- The connection between double and triple integrals are provided through volume.
    - Similarly the connection between a single integral and a double integral is provided through area

$$V = \iiint_{\mathcal{W}} 1dV = \iint_{\mathcal{D}} \left(h_2(x,y) - h_1(x,y)\right) dA = \iint_{\mathcal{D}} f(x,y)dA$$

example: if a z-simple region between $z = h_1(x,y)$ and $z = h_2(x,y)$

Double Integral: 1^{st} step was to define $'R'$ between two curves.

$$\iint_D f(x,y)dA \implies dA = dx\,dy$$

Triple Integral: 1^{st} step is to define $'\mathcal{W}'$ between two surfaces.

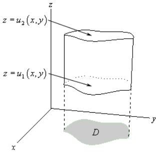
$$\iiint_{\mathcal{W}} f(x,y,z)dV \implies dV = dx\,dy\,dz$$

This allows you to deal with the **first** $\int$ and leaves $\iint_D$, **BUT** $'D'$ could be on an xy, xz, or yz plane

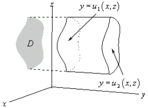
- **Two Main Tasks:**
  1. A region in 3D (region between surfaces)
     a) 6 different possible orders for the triple integral. Pick the easiest integration order.
     b) Essentially reduce it to a double integral
  2. Then a very specific 2D region on a *coordinate plane*
     a) Always draw the region!

Triple Integrals (cont.)

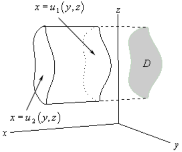
- **z-simple:** define 3D region between two "z=f(x,y)" functions therefore "D" will be on the xy-plane



- **y-simple:** define 3D region between two "y=f(x,z)" functions therefore "D" will be on the xz-plane



- **x-simple:** define 3D region between two "x=f(y,z)" functions therefore "D" will be on the yz-plane



Triple Integrals (cont.)

$$\iiint_{\mathcal{W}} f(x,y,z)\,dV$$

3D (Two Variables)

2D (One Variable)

Number Line (Constants)

$\iiint_{\mathcal{W}}$ $\mathcal{W}$

$\iint_{\mathcal{D}}$ $\mathcal{D}$

Note: Here, we are paying attention to the **integral(s)** and **respected regions**, not necessarily the function.
This is also a general rule of thumb. It can vary case by case. The variable number will depend on the problem.

Triple Integrals (example 1)

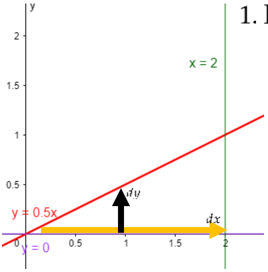
Calculate triple integral of: $f(x,y,z) = z$

$\mathcal{W}: x^2 + z^2 = 4, x = 2y, y = 0, z = 0, x \geq 0$

R-3: z-simple, $0 \leq z \leq \sqrt{4-x^2}, z = 0, z = \sqrt{4-x^2}$

(x-simple is bad since we would be between two functions with variables $z \& y$)

R: xy-plane, $0 \leq y \leq \frac{1}{2}x, 0 \leq x \leq 2 \implies \iint_{\mathcal{D}} \int_{z=0}^{z=\sqrt{4-x^2}} z\, dzdydx$



1. Put in plot anything that involves x's and y's from given
2. Since we're on xy-plane $z = 0$.

$$x^2 + z^2 = 4 \implies x^2 = 4 \implies x = \pm 2 \quad \text{But } x \geq 0$$

3. Integrate in dy first $\uparrow$ since lower function is y = 0

$$\int_{x=0}^{x=2} \int_{y=0}^{y=\frac{1}{2}x} \int_{z=0}^{z=\sqrt{4-x^2}} z\, dzdydx = \int_0^2 \int_0^{\frac{1}{2}x} \left(\frac{-x^2+4}{2} \right) dydx = 1$$

Notice we have upper and lower bounds that explicitly show us which variable we are integrating with.

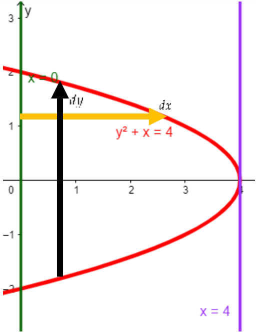
When integrating in a coordinate plane, try to stay between 'two functions'. Do not switch between two functions.

Find the volume of $\mathcal{W}: x = 4 - y^2, x + z = 4, x = 0, z = 0$

R-3: z-simple, $0 \leq z \leq 4 - x, z = 0, z = 4 - x$
(x-simple is bad since we would be between two functions with variables $z \& y$)

R: xy-plane, $0 \leq x \leq 4 - y^2, -2 \leq y \leq 2 \implies \iint_{\mathcal{D}} \int_{z=0}^{z=4-x} 1 dz dx dy$

1. Put in plot anything that involves x's and y's from given
2. Since we're on xy-plane $z = 0$. However, little info. is derived from it.



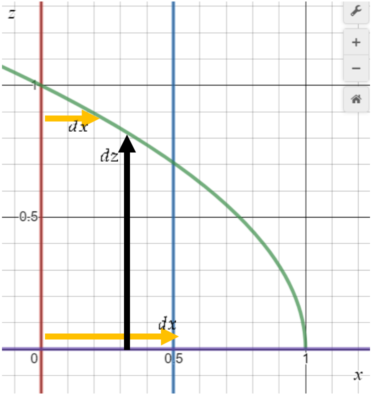
3. Find intersection between x and y, given the info.

$$x = 4 - y^2 \text{ since } x = 0 \implies 0 = 4 - y^2 \implies y = \pm 2$$

4. Integrate in dx first so we do not switch between two functions. We want to stay between two functions the whole time.

$$\int_{y=-2}^{y=2} \int_{x=0}^{x=4-y^2} \int_{z=0}^{z=4-x} 1 dz dx dy = \int_{-2}^{2} \int_{0}^{4-y^2} (-x+4) dx dy$$
$$= \frac{128}{5}$$

Triple Integrals (example 3)



$x = 0$

$z = (1-x)^{\frac{1}{2}}$

$x = \frac{1}{2}$

$z = 0$

R-3: y-simple **R**: xz-plane

$\mathcal{W}: x = 0, x = \frac{1}{2}, z = 0, z = \sqrt{1-x}, y = 0, y = 4 - z$

$$\iint_D \int_{y=0}^{y=4-z} 1 dydzdx$$

$$\implies \int_{x=0}^{x=\frac{1}{2}} \int_{z=0}^{z=\sqrt{1-x}} \int_{y=0}^{y=4-z} 1 dydzdx$$

$$= \int_0^{\frac{1}{2}} \int_0^{\sqrt{1-x}} (-z+4) dzdx$$

$$= \frac{119}{48} - \frac{2\sqrt{2}}{3} \approx 1.536$$

dx makes us switch functions between:

RED-to-GREEN & RED to BLUE $\rightarrow$

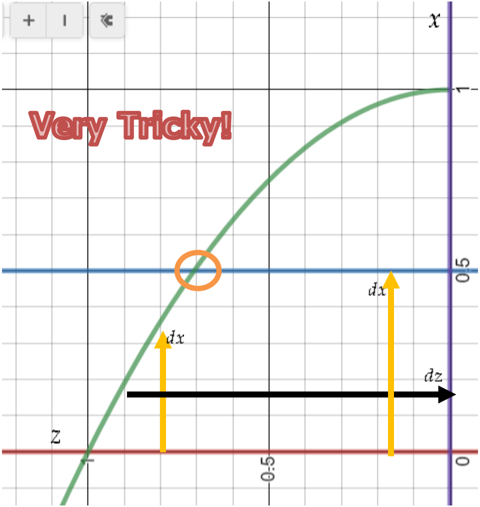
dz makes us stay between two functions:

PURPLE-to-GREEN $\uparrow$

Thus integrate in dz first.

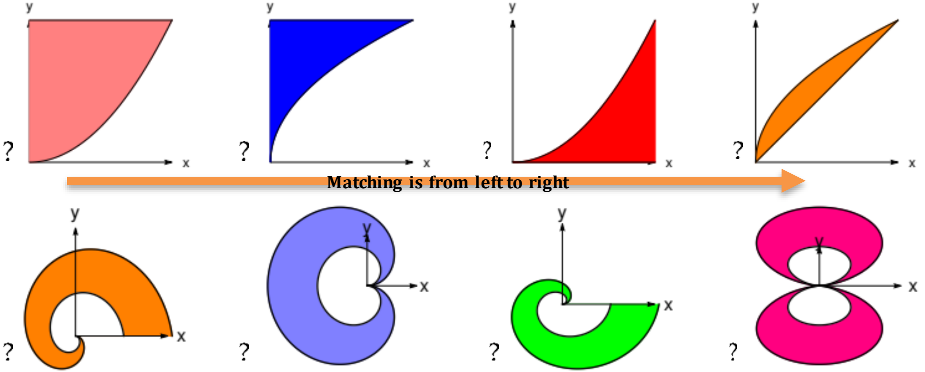
Triple Integrals (example 3, cont.)

- We would be forced to switch to x in terms of z and have two integrals
    - e.g $x = 1 - z^2$
    - One integral from RED-to-GREEN ↑
    - Another from RED to BLUE ↑
        - BLUE is x = .5
    - Diff. Integrals denoted with orange circle
- Then integrate with z as a constant
- Remember always draw the region and decide the **easiest integration order**!



Lecture Question

Match the integrals with their graphs.



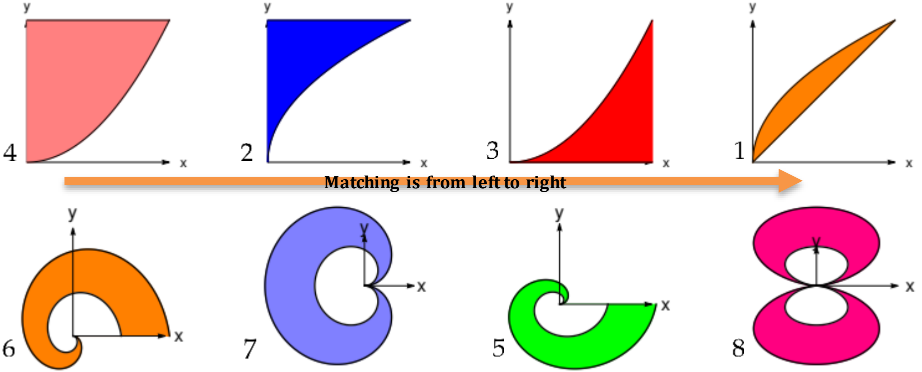
Answer Choices: (L → R)

- **A.** **Row 1:** 4,2,3,1 **Row2:** 5,7,6,8
- **B.** **Row 1:** 2,4,3,1 **Row2:** 6,7,5,8
- **C.** **Row 1:** 4,3,2,1 **Row2:** 8,5,7,6
- **D.** **Row 1:** 3,1,2,4 **Row2:** 7,6,5,8
- **E.** **Row 1:** 4,2,3,1 **Row2:** 6,7,5,8

1	$\int_0^1 \int_x^{\sqrt{x}} 1\; dydx$
2	$\int_0^1 \int_{\sqrt{x}}^1 1\; dydx$
3	$\int_0^1 \int_{\sqrt{y}}^1 1\; dxdy$
4	$\int_0^1 \int_0^{\sqrt{y}} 1\; dxdy$

5	$\int_0^{2\pi} \int_\theta^{2\theta} r\; drd\theta$
6	$\int_0^{2\pi} \int_{\pi-\theta/2}^{2\pi-\theta} r\; drd\theta$
7	$\int_0^{2\pi} \int_{\sin(\theta/2)}^{2\sin(\theta/2)} r\; drd\theta$
8	$\int_0^{2\pi} \int_{\sin^2(\theta)}^{2\sin^2(\theta)} r\; drd\theta$

Match the integrals with their graphs.



Answer Choices: (L → R)

- **A.** **Row 1:** 4,2,3,1 **Row2:** 5,7,6,8
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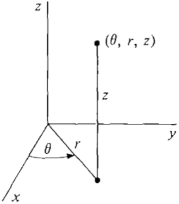
- **E.** **Row 1: 4,2,3,1 Row 2: 6,7,5,8**

1	$\int_0^1 \int_x^{\sqrt{x}} 1\, dydx$
2	$\int_0^1 \int_{\sqrt{x}}^1 1\, dydx$
3	$\int_0^1 \int_{\sqrt{y}}^1 1\, dxdy$
4	$\int_0^1 \int_0^{\sqrt{y}} 1\, dxdy$

5	$\int_0^{2\pi} \int_\theta^{2\theta} r\, drd\theta$
6	$\int_0^{2\pi} \int_{\pi-\theta/2}^{2\pi-\theta} r\, drd\theta$
7	$\int_0^{2\pi} \int_{\sin(\theta/2)}^{2\sin(\theta/2)} r\, drd\theta$
8	$\int_0^{2\pi} \int_{\sin^2(\theta)}^{2\sin^2(\theta)} r\, drd\theta$

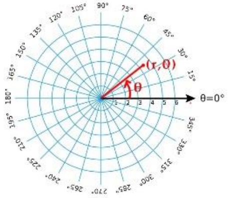
Cylindrical Coordinates:

$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$



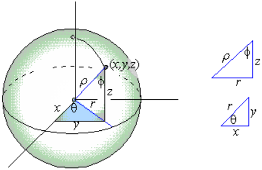
Polar Coordinates:

$$x = r\cos\theta \quad \& \quad y = r\sin\theta$$

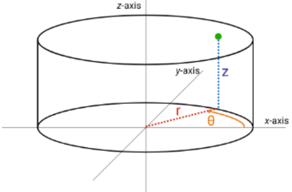


Spherical Coordinates:

$$r = \rho\sin\phi$$
$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$
$$\rho^2 = x^2+y^2+z^2$$

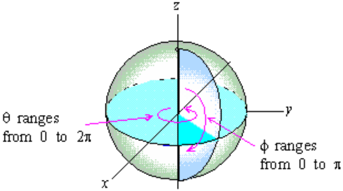


Cylindrical Integration (*Triple Int.*):



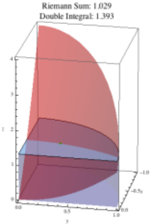
$$\int_{\theta_1}^{\theta_2}\int_{r=r_1(\theta)}^{r_2(\theta)}\int_{z=z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

Spherical Integration (*Triple Int.*):



$$\int_{\theta_1}^{\theta_2}\int_{\phi=\phi_1}^{\phi_2}\int_{\rho=\rho_1(\theta,\phi)}^{\rho_2(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Polar Integration (*Double Int.*):



$$\iint_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r=r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Differential Element:

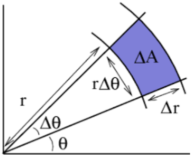
$$dA = r\,dr\,d\theta \quad \text{(Polar)}$$

$$dV = r\,dz\,dr\,d\theta \quad \text{(Cylindrical)}$$

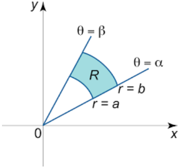
$$dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \quad \text{(Spherical)}$$

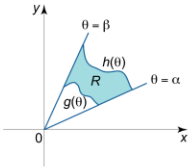
General Strategy

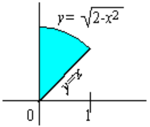
- Identify patterns in the bounds, function, integration, or general regions
  - $r^2 = x^2 + y^2$
  - $\rho^2 = x^2 + y^2 + z^2$
  - $\rho^2 = r^2 + z^2$
- Manipulate functions or bounds according to properties
  - $-x^2 - y^2 = -(x^2 + y^2) = -r^2$
  - $-x^2 - y^2 - z^2 = -(x^2 + y^2 + z^2) = -\rho^2$
- Always ask: "Can I change it to polar, cylindrical, or spherical?"
  - Convert to desired system that allows for the easiest integration
- Apply the correct differential element (dA, dV) according to the coordinate system (polar, cylindrical, or spherical)
- Visualization of the shape and intersection of curves can allow for insight into the specific problem
  - (e.g. is it a quadric surface, spherical cap, cylinder, cone, cardioid, torus, etc.)
- Does the integral require a specific integration technique? (substitution, integration by parts/DI (tabular) method, improper integration, partial fractions, trig. identity, etc.)



$$\Delta A \approx \triangle r \cdot r\Delta\theta \implies dA = r\,dr\,d\theta$$







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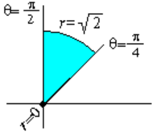


Polar

$$\iint f(r,\theta)r\,dr\,d\theta$$

- Used for **circular, axial,** and **rotational objects**
- Involves **two angles** and **two radii**
  - Radii and angles depends on a <u>specific region</u>
- It always has the **differential element** as:
  - $dA = r\,dr\,d\theta \quad \text{(Polar)}$
- <u>Regions/Curves</u> may require <u>completing the square</u> to turn them into polar
- <u>Move like **r**:</u>
  - Start at origin and ask what curve do you immediately intersect:
    - If it starts at the origin within a non-hollow region, r = 0
    - If it begins at a curve, then r = f(θ)
  - Now look for second intersection curve
  - <u>Lower bound</u> is *first curve*, <u>upper bound</u> is *second curve*
- <u>Move like θ:</u>
  - Look at region and its quadrant and determine which angles it is between
  - If the region is in <u>all four quadrants</u>, then full rotation which means $\theta \in$ **[0, 2π]**
  - If the region is in <u>one, two or three total quadrants</u>, look carefully for the values of θ
- <u>Process:</u>
  - Draw picture of the region
  - Change to polar (function and regions)
  - Pay attention to bounds

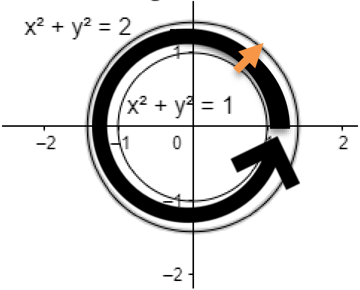
Polar (example 1)

Find volume below $z = \frac{y^2}{x^2+y^2}$, above xy plane and between cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$

Change the function: $\frac{r^2 \sin^2\theta}{r^2} = \sin^2\theta$

Find the bounds: $x^2 + y^2 = 1 \implies r = 1$, $x^2 + y^2 = 2 \implies r = \sqrt{2}$

Draw the region: $0 \le \theta \le 2\pi, 1 \le r \le \sqrt{2}$



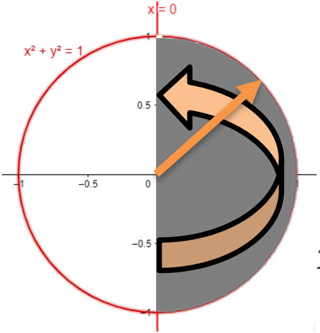
$$V = \iint_D z\,dA = \iint_D \frac{y^2}{x^2+y^2}\,dA$$
$$= \int_0^{2\pi}\int_1^{\sqrt{2}} \sin^2(\theta)\,r dr d\theta = \frac{\pi}{2} \approx 1.5708$$

Find the value of: $\mathcal{I} = \int_{y=-1}^{1} \int_{x=0}^{\sqrt{1-y^2}} \frac{1}{1+x^2+y^2} dxdy$

Change the function: $\frac{1}{1+x^2+y^2} = \frac{1}{1+r^2}$

Find the bounds: $x = 0, x = \sqrt{1-y^2} \implies x^2 + y^2 = 1 \implies r = 1, r = 0$

Draw the region: $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq 1$



$$y = 1 = r\sin\theta \implies \sin\theta = 1 \implies \theta = \frac{\pi}{2}$$
$$y = -1 = r\sin\theta \implies \sin\theta = -1 \implies \theta = -\frac{\pi}{2}$$

Note: we cannot multiply by 2 then integrate since we don't know if the function is symmetrical.

Note: We cannot use $\frac{3\pi}{2}$ to $\frac{\pi}{2}$ since we must go counterclockwise so in the positive direction.

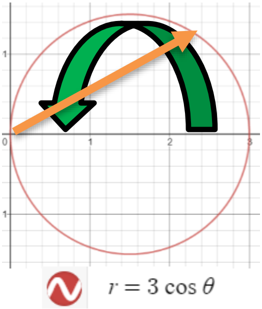
$$\mathcal{I} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \frac{1}{1+r^2} r dr d\theta = \frac{\pi}{2}\ln(2) \approx 1.08879$$

Find the area of $\mathcal{R}$ where $'\mathcal{R}'$ is the region bounded by $r = 3\cos\theta$

Change the function: $r = 3\cos\theta \implies r^2 = 3r\cos\theta \implies r^2 = 3x \implies$
$x^2 + y^2 = 3x \implies$ complete the square $\implies (x - \frac{3}{2})^2 + y^2 = \frac{9}{4}$

Find the bounds and draw the region: $0 \le r \le 3\cos\theta\,, 0 \le \theta \le \dfrac{\pi}{2}$

$$A = \iint_{\mathcal{D}} 1\,dA \quad = 2 \cdot \int_0^{\frac{\pi}{2}} \int_0^{3\cos\theta} r\,dr d\theta = \frac{9\pi}{4} \approx 7.06858$$



OR $0 \le r \le 3\cos\theta\ , 0 \le \theta \le \pi$

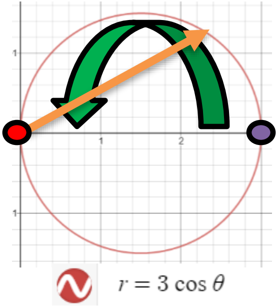
$$A = \iint_{\mathcal{D}} 1\,dA = \int_0^{\pi} \int_0^{3\cos\theta} r\,dr d\theta = \frac{9\pi}{4} \approx 7.06858$$

Note: $A_c = \pi r^2 = \pi \left(\dfrac{3}{2}\right)^2 = \dfrac{9\pi}{4}$

Note: The integral gives us the net signed area and here we want the area. A constant function 1 is an even function so it is symmetric with respect to the y-axis. Thus we can multiply by two and then integrate.

Even Functions: $f(-x) = f(x)$

The circle is completed at less than 2π



Values for θ : $r = 3\cos\theta$

$\theta = 0 \implies r = 3 \implies$

$\theta = \frac{\pi}{2} \implies r = 0 \implies$

$\theta = \pi \implies r = -3 \implies$ opposite side of :

(2nd quad)

$\frac{\pi}{2}$ to π is negative so we get a full rotation from

0 to π

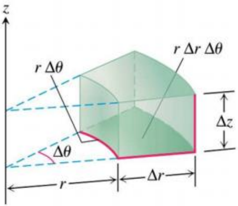
From 0 to 2π it would either double it or get zero.

We get half a circle from 0 to $\frac{\pi}{2}$

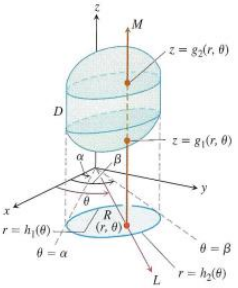
We get a full circle from 0 to π

It should be noted that $\cos\theta$ is mapped to x so $r = 3\cos\theta$ is shifted across the x.

Integration in Cylindrical Coordinates



$$\Delta V \approx \Delta z \cdot \Delta r \cdot r\Delta\theta \Longrightarrow dV = r\,dz\,dr\,d\theta$$



$$\iiint f(r,\theta,z)rdrd\theta dz$$

- An extension of polar coordinates, but instead we have a 'z-component'
  - (**r**, θ , **z**)
- Involves a triple integral instead of a double integral
- It always has the **differential element** as:
  - $dV = r\,dz\,dr\,d\theta$ (Cylindrical)
    - Derived from polar coordinates
- Some questions may require a 'like-cylindrical' integration if the component is not 'z'
  - Therefore, modify to account for xz or yz plane
  - e.g, $y = x^2 + z^2 \Longrightarrow y = r^2$
- The approach is to follow the same rules as *polar coordinates*, but instead account for the z-component
  - R-3 simple region

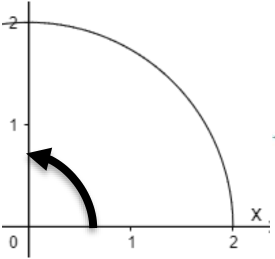
Cylindrical (example 1)

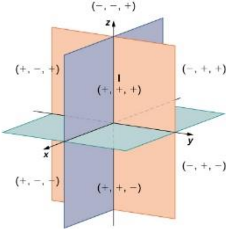
Calculate: $\iiint_{\mathcal{W}} y\, dV$, where $\mathcal{W}: z = 4 - x^2 - y^2$ in the first octant

R-3: z-simple, octant so $x = y = z = 0$, $0 \leq z \leq 4 - x^2 - y^2 \implies 0 \leq z \leq 4 - r^2$

R: xy-plane, $z = 0 \implies x^2 + y^2 = 4 \implies r = 2$. Since first octant $0 \leq \theta \leq \frac{\pi}{2}$

Find the bounds and draw the region:





Change the function: $y = r\sin\theta$

$$\iiint_{\mathcal{W}} y\, dV$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{4-r^2} y\, dz\, r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{4-r^2} r\sin(\theta) dz r dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{4-r^2} r^2 \sin(\theta) dz dr d\theta$$

$$= \frac{64}{15}$$

Cylindrical (example 2)

Calculate: $\iiint_{\mathcal{W}} \sqrt{x^2+z^2}\,dV$, where $\mathcal{W}: y_1 = x^2+z^2,\ y_2 = 8 - x^2 - z^2$

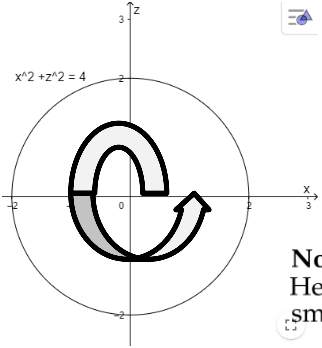
R-3: y-simple, $x^2+z^2 \le y \le 8 - x^2 - z^2 \implies r^2 \le y \le 8 - r^2$

R: xz-plane. Find intersection provided by two surfaces and get a bounded region

If $y_2 = y_1 \implies x^2+z^2 = 8 - x^2 + z^2 \implies x^2+z^2 = 4$

To find the top or bottom function, plug in random points.

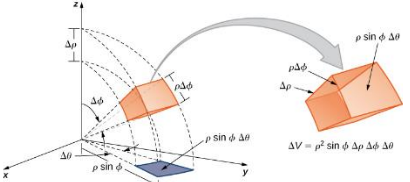
Find the bounds and draw the region: $0 \le \theta \le 2\pi, 0 \le r \le 2$



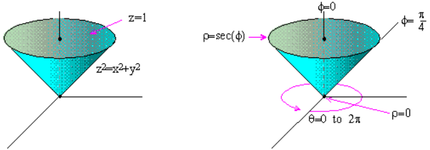
Change the function: $\sqrt{x^2+z^2} = \sqrt{r^2} = r$

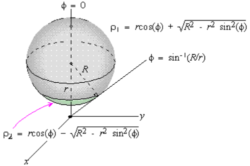
$$\begin{aligned}
\iiint_{\mathcal{W}} \sqrt{x^2+z^2}\,dV &= \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} r\,dy\,r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} r^2 dy dr d\theta \\
&= \frac{256\pi}{15}
\end{aligned}$$

Note : The surface is extending in the y direction. Here we are essentially subtracting the larger and smaller function to find the volume.



$$\text{A.L} = \text{Arclength} = (\text{radius}) \cdot (\text{angle})$$
$$\Delta V = (\text{A.L along } \theta) \cdot (\text{A.L along } \phi) \cdot (\text{depth})$$
$$\Delta V = (r\Delta\theta) \cdot (\rho\Delta\phi) \cdot (\Delta\rho)$$
$$\Delta V = (\rho \sin(\phi) \cdot \Delta\theta) \cdot (\rho\Delta\phi) \cdot (\Delta\rho)$$
$$\Delta V = \rho^2 \sin(\phi)\, \Delta\rho\Delta\phi\Delta\theta$$
$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$





- Used for **cones, spheres, ice cream cones, air balloons, etc.**
- Involves **centrally simple objects**
  - 'Rho', 'phi', and 'theta' will depend on <u>specific regions</u>
- It always has the **differential element** as:
  - $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \quad \text{(Spherical)}$
- It is important to understand the meaning of **ρ, ϕ, and** θ
- Integration for an object may require '**superposition**' (sum of parts/components for a total volume)
- <u>Move like **ρ**:</u>
  - Bounds: $\boldsymbol{\rho \geq 0}$
  - Set x = 0 (trace on yz-plane)
  - Draw yz trace according to given region $\mathcal{W}$
  - $\mathcal{W}$ and the **drawn region** will determine the bounds for the specific problem

<u>Move like **ϕ**:</u>
  - Bounds: $\boldsymbol{\phi \in [0, \pi]}$
  - Set x = 0 (trace on yz-plane)
  - Look at the angle rotation starting at the positive z-axis towards the ending curve

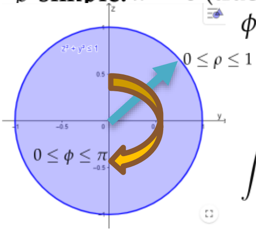
<u>Move like θ:</u>
  - Bounds: $\boldsymbol{\theta \in [0, 2\pi]}$
  - Set z = 0 (trace on xy-plane) or look for an intersection of two z surfaces

- <u>Process:</u>
  - Draw the associated regions for the yz and xy plane
  - Change to spherical coordinates
  - Pay attention to the bounds (especially on **ϕ** and **ρ**)

Spherical (example 1)

Calculate: $\iiint_{\mathcal{W}} \sqrt{x^2+y^2+z^2}\, dV$, where $\mathcal{W} : x^2+y^2+z^2 \leq 1$

ρ-simple: $x = 0$ (trace on yz plane) $\implies y^2 + z^2 \leq 1\,, 0 \leq \rho \leq 1$

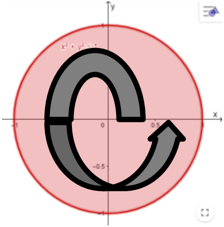


ϕ-simple: Because of the possible bounds on ϕ and full rotation $\implies 0 \leq \phi \leq \pi$

Change the function: $\sqrt{x^2+y^2+z^2} = \sqrt{\rho^2} = \rho$

$$\iiint_{\mathcal{W}} \sqrt{x^2+y^2+z^2}\, dV \implies \int \int_0^{\pi} \int_0^1 \rho \cdot \rho^2 \sin(\phi) d\rho d\phi d\theta$$

θ-simple: $z = 0$ (trace on xy plane) $\implies x^2 + y^2 \leq 1\,, 0 \leq \theta \leq 2\pi$

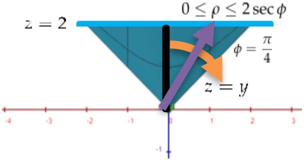


$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho \cdot \rho^2 \sin(\phi) d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^3 \sin(\phi) d\rho d\phi d\theta$$

$$= \pi \approx 3.14159$$

Calculate the volume where $\mathcal{W}: z = 2$ and $z = \sqrt{x^2 + y^2}$

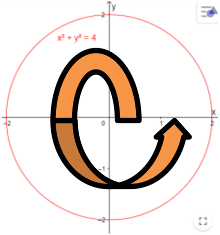
ρ-simple: $z = 2 \implies \rho\cos\phi = 2 \implies \rho = 2\sec\phi\ , 0 \le \rho \le 2\sec\phi$



$x = 0 \implies z = \sqrt{y^2} = y$

ϕ-simple: $\tan\phi = 1 \implies 0 \le \phi \le \frac{\pi}{4}$

θ-simple: $z = 0$, doesn't work $\implies z = 2 = \sqrt{x^2 + y^2} \implies 4 = x^2 + y^2$



$$\implies 0 \le \theta \le 2\pi$$

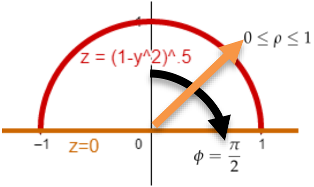
$$V = \iiint_{\mathcal{W}} 1 dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{2\sec\phi} \rho^2 \sin(\phi) d\rho d\phi d\theta$$

$$= \frac{8\pi}{3}$$

Calculate: $\iiint_{\mathcal{W}} y\,dV$, where $\mathcal{W}: z = \sqrt{1-x^2-y^2}$ and xy-plane

ρ-simple: $z = \sqrt{1-x^2-y^2} \implies z^2 = 1-x^2-y^2 \implies x^2+y^2+z^2 = 1$

$\implies \rho = 1$,$0 \leq \rho \leq 1$,xy-plane $\implies z = 0$ and $x = 0 \implies z = \sqrt{1-y^2}$

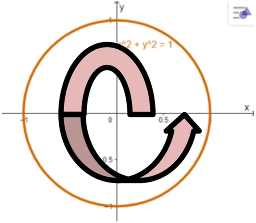


ϕ-simple: Because its from the positive z axis to z = 0

$$\implies 0 \leq \phi \leq \frac{\pi}{2}$$

Change the function: $y = \rho \sin\phi \sin\theta$

θ-simple: $z = 0 \implies 0 = \sqrt{1-x^2-y^2} \implies x^2+y^2 = 1$,$0 \leq \theta \leq 2\pi$



$$\iiint_{\mathcal{W}} y\,dV = \iiint_{\mathcal{W}} \rho \sin\phi \sin\theta\, \rho^2 \sin(\phi) d\rho d\phi d\theta$$

$$\implies \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 \rho \sin(\phi) \sin(\theta) \rho^2 \sin(\phi) d\rho d\phi d\theta = 0$$

The answer here is **zero**. However, if you get a zero on a triple integral re-check your math.

Try the following if you get zero on a triple integral on $\theta \in [0, 2\pi]$:

Double the triple integral and restrict $\theta \in [0, \pi]$**:**

$$2 \cdot \int_0^{\pi} \int \int f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \cdot \rho^2 \sin(\phi) d\rho d\phi d\theta$$

$\implies$ if the region is symmetrically it should work and give a value of **zero**

Quadruple the triple integral and restrict $\theta \in [0, \frac{\pi}{2}]$**:**

$$4 \cdot \int_0^{\frac{\pi}{2}} \int \int f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \cdot \rho^2 \sin(\phi) d\rho d\phi d\theta$$

$\implies$ generally, this **should not** give you a value of **zero**

Note: This technique works specifically with $\cos(x)$ and $\cos(x)\sin(x)$.

$$\int_{\phi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \underbrace{\left(\cos\theta \sin^2\phi \cos\phi + \cos\theta \sin\theta \sin^3\phi\right.}_{\text{Integral over } \theta \text{ is zero}} \left. + \cos\phi \sin\phi\right) d\theta d\phi$$

Since the integrals of $\cos\theta$ and $\cos\theta \sin\theta$ over $[0, 2\pi]$ are both zero

$$\int_0^{2\pi} \cos(x) dx = 2 \cdot \int_0^{\pi} \cos(x) dx = \int_0^{2\pi} \cos(x)\sin(x) dx = 2 \cdot \int_0^{\pi} \cos(x)\sin(x) dx = 0$$

Very-Important-Note: When trying this make sure to integrate θ *last*. This would be the very very last step you do if your numerical result is zero (as a method to check your work). A value of zero can suggest symmetry and cancelation of volumes. Recall fubini's theorem.

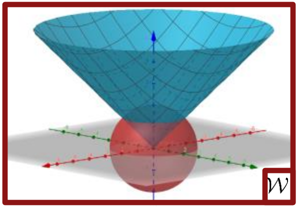
Spherical (example 4, extra)

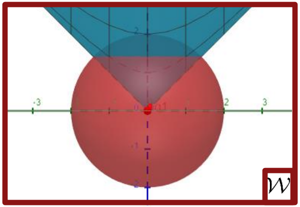
Calculate: $\iiint_{\mathcal{W}} xz\,dV$, where $\mathcal{W} : x^2+y^2+z^2=4$ and $z=\sqrt{x^2+y^2}$

ρ-simple: $0 \le \rho \le 2$
$x = 0$

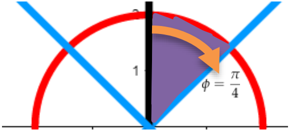
ϕ-simple: $0 \le \phi \le \frac{\pi}{4}$
$x = 0$

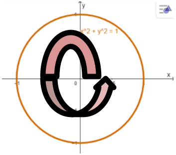
θ-simple: $0 \le \theta \le 2\pi$
$z = 0$

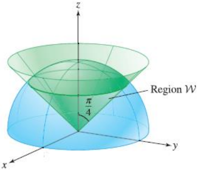




$$\iiint_{\mathcal{W}} xz\,dV = \iiint_{\mathcal{W}} (\rho\sin\phi\cos\theta)\cdot(\rho\cos\phi)\cdot\rho^2\sin(\phi)d\rho d\phi d\theta$$
$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{2}(\rho\sin\phi\cos\theta)\cdot(\rho\cos\phi)\cdot\rho^2\sin(\phi)d\rho d\phi d\theta$$
$$= 0$$







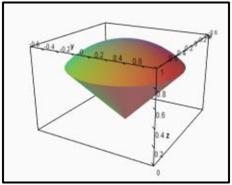
Lecture Question

Determine the correct integration region W for each of the following integrals. Write your answer in the space provided.

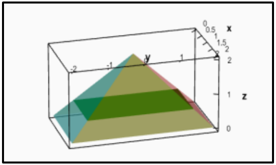
$$\text{(i)} \quad \int_0^1 \int_0^{2(1-z)} \int_0^{3(1-x/2-z)} f(x,y,z)\, dydxdz \quad \_\_$$

$$\text{(ii)} \quad \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1/2-x^2}}^{\sqrt{1/2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z) dzdydx \quad \_\_$$

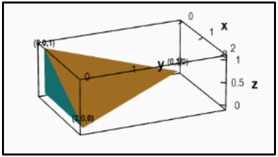
A



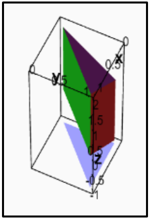
B



C



D

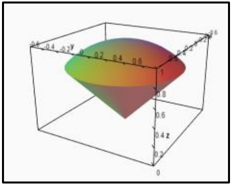


Determine the correct integration region W for each of the following integrals. Write your answer in the space provided.

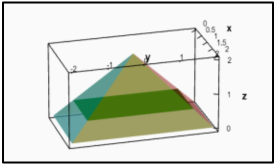
(i) $\displaystyle\int_0^1 \int_0^{2(1-z)} \int_0^{3(1-x/2-z)} f(x,y,z)\, dydxdz$ **C**

(ii) $\displaystyle\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1/2-x^2}}^{\sqrt{1/2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z) dzdydx$ **A**

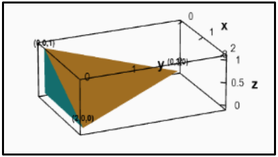
A



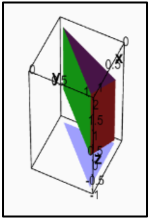
B



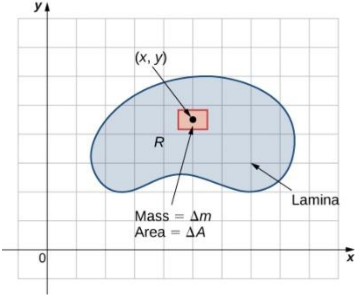
C



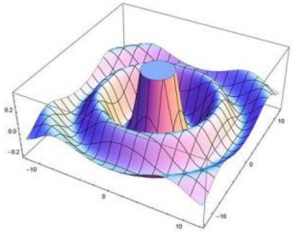
D



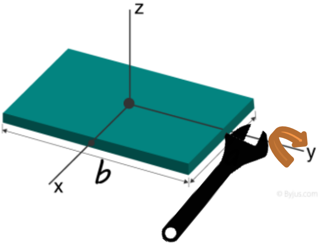
Total Amount and Mass

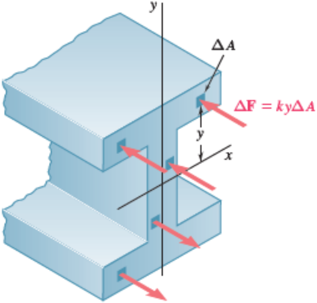


Total mass: $M = \iint_D \delta(x,y)\,dA \quad M = \iiint_W \delta(x,y,z)\,dV$



- **"Total Amount"** is an additive property based on the material or given circumstance
- The **density function** $\delta(x,y)$ has units of "amount per unit area" (or per unit volume).
  - The units may vary based on the problem and/or situation at hand
- The **mass of a solid** whose mass density at each point is given by a function δ is equal to the integral of δ over the region occupied by the solid
  - Can extend to two or three variables based on the density function





- **Moments** calculate the tendency for an object or lamina to *rotate* about an axis. In vector calculus (Math 1920), this moment is a <u>scalar value</u>.
- M_x denotes the tendency for an object/lamina to rotate about the **x-axis**
  - Multiply by y since it is perpendicular to rotation axis (signed distance)
- M_y denotes the tendency for an object/lamina to rotate about the **y-axis**
  - Multiply by x since it is perpendicular to rotation axis (signed distance)
- If the distance x *or* y is further away, it means the object is more likely to rotate, since the value of the moment increases
- Application to Mechanical Engineering (e.g., Statics and Mechanics of Solids)
  - $\sum M = 0, \sum F = 0$

$$M_x = \iint_D y\delta(x,y)\,dA$$
$$M_y = \iint_D x\delta(x,y)\,dA$$

$$M_{yz} = \iiint_W x\delta(x,y,z)\,dV$$
$$M_{xz} = \iiint_W y\delta(x,y,z)\,dV$$
$$M_{xy} = \iiint_W z\delta(x,y,z)\,dV$$

Center of Mass

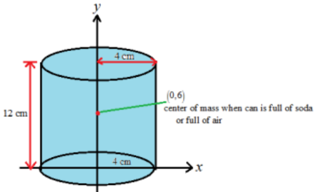
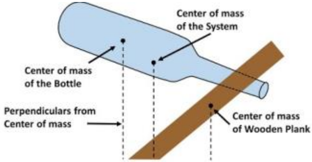
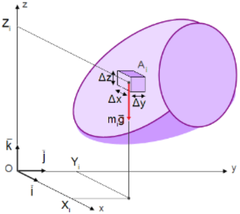


Figure 1





- **Center of mass** can be viewed as a **"weighted averaged"** based on a moment and total mass
- **Center of mass** is a **location**, not necessarily a value
- The **center of mass** for an object can be represented in two or three coordinates:
  - **2D:**
    - M_y means moment about the y-axis
    - M_x means moment about the x-axis
    - Points $(\bar{x}, \bar{y})$
  - **3D:**
    - M_{yz} means moment about the yz-plane
    - M_{xz} means moment about the xz-plane
    - M_{xy} means moment about the xy-plane
    - Points $(\bar{x}, \bar{y}, \bar{z})$

$$x_{\text{CM}} = \frac{M_y}{M}, \quad y_{\text{CM}} = \frac{M_x}{M}$$

$$x_{\text{CM}} = \frac{M_{yz}}{M}, \quad y_{\text{CM}} = \frac{M_{xz}}{M}, \quad z_{\text{CM}} = \frac{M_{xy}}{M}$$

	In $\mathbb{R}^2$	In $\mathbb{R}^3$
Total mass	$M = \iint_{\mathcal{D}} \delta(x, y)\, dA$	$M = \iiint_{\mathcal{W}} \delta(x, y, z)\, dV$
Moments	$M_x = \iint_{\mathcal{D}} y\delta(x, y)\, dA$ $M_y = \iint_{\mathcal{D}} x\delta(x, y)\, dA$	$M_{yz} = \iiint_{\mathcal{W}} x\delta(x, y, z)\, dV$ $M_{xz} = \iiint_{\mathcal{W}} y\delta(x, y, z)\, dV$ $M_{xy} = \iiint_{\mathcal{W}} z\delta(x, y, z)\, dV$
Center of mass	$x_{\mathrm{CM}} = \frac{M_y}{M}$, $y_{\mathrm{CM}} = \frac{M_x}{M}$	$x_{\mathrm{CM}} = \frac{M_{yz}}{M}$, $y_{\mathrm{CM}} = \frac{M_{xz}}{M}$, $z_{\mathrm{CM}} = \frac{M_{xy}}{M}$
Moments of inertia	$I_x = \iint_{\mathcal{D}} y^2\delta(x, y)\, dA$ $I_y = \iint_{\mathcal{D}} x^2\delta(x, y)\, dA$ $I_0 = \iint_{\mathcal{D}} (x^2 + y^2)\delta(x, y)\, dA$ $(I_0 = I_x + I_y)$	$I_x = \iiint_{\mathcal{W}} (y^2 + z^2)\delta(x, y, z)\, dV$ $I_y = \iiint_{\mathcal{W}} (x^2 + z^2)\delta(x, y, z)\, dV$ $I_z = \iiint_{\mathcal{W}} (x^2 + y^2)\delta(x, y, z)\, dV$

Let R be the unit square, $R = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Suppose the density of R is given by the function

$$\delta(x,y) = \frac{1}{y+1}$$

Find the center of mass $(\bar{x}, \bar{y})$.

Center of Mass (example 1)

Let R be the unit square, $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$. Suppose the density of R is given by the function

$$\delta(x,y) = \frac{1}{y+1}$$

Find the center of mass $(\bar{x}, \bar{y})$.

$$x_{\text{CM}} = \frac{M_y}{M}, \quad y_{\text{CM}} = \frac{M_x}{M} \qquad M = \iint_{\mathcal{D}} \delta(x,y)\,dA \qquad \begin{aligned} M_x &= \iint_{\mathcal{D}} y\delta(x,y)\,dA \\ M_y &= \iint_{\mathcal{D}} x\delta(x,y)\,dA \end{aligned}$$

Find the mass:

$$M = \iint_R \frac{1}{y+1} dA = \int_0^1 \int_0^1 \frac{1}{y+1} dy dx = \int_0^1 \ln(y+1)\Big|_0^1 dx = \int_0^1 \ln 2 dx = \ln 2 = 0.693147\ldots$$

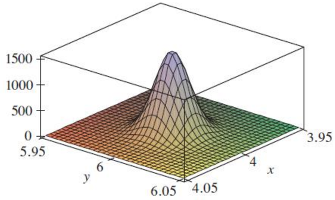
Find the moments:

$$M_x = \iint_R \frac{y}{y+1} dA = \int_0^1 \int_0^1 \left(1 - \frac{1}{y+1}\right) dy dx = \int_0^1 (y - \ln(y+1))\Big|_0^1 dx = 1 - \ln 2 = 0.306852819\ldots$$

$$M_y = \iint_R \frac{x}{y+1} dA = \int_0^1 \int_0^1 \frac{x}{y+1} dy dx = \int_0^1 x \ln 2 dx = \frac{1}{2} x^2 \ln 2\Big|_0^1 = \frac{1}{2} \ln 2 = 0.346573590\ldots$$

Find the center of mass:

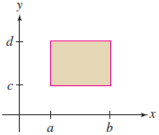
$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{\frac{1}{2}\ln 2}{\ln 2}, \frac{1 - \ln 2}{\ln 2}\right) = \left(\frac{1}{2}, 0.442095\ldots\right)$$



- A non-negative function which integrates to 1 is called a **probability density function (pdf)**
- The **integral of the pdf** over a region is supposed to give the **probability** that the *outcome* of the random experiment lies in that region
- A ***random variable* X** is defined as the outcome of an experiment
- **Two conditions on a pdf:** $p(x) \geq 0$ and $p(x)$ satisfies $\int_{-\infty}^{\infty} p(x)\, dx = 1$
- Double integration involves "**joint probabilities**" of two random variables **X** and **Y**
- A **joint probability density function** is a function of two variables $p(x, y)$
- **Normalization condition** requires that the total probability is one over a joint pdf's domain.
- **Two conditions of a joint probability density function:** $p(x, y) \geq 0$ and $p(x)$ satisfies $\iint_{-\infty\,-\infty}^{\infty\;\;\infty} p(x, y)\; dy\; dx$

If the joint density function for X and Y is given by

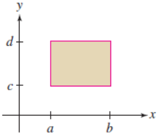
$$p(x,y) = \begin{cases} C(x+2y) & \text{if } 0 \leq x \leq 10, 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$



find the value of the constant C. Then find $P(X \leq 7, Y \geq 2)$.

If the joint density function for X and Y is given by

$$p(x,y) = \begin{cases} C(x+2y) & \text{if } 0 \le x \le 10, 0 \le y \le 10 \\ 0 & \text{otherwise} \end{cases}$$



find the value of the constant C. Then find $P(X \le 7, Y \ge 2)$.

Find the value of C by ensuring that the double integral of f is equal to 1:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = \int_0^{10}\int_0^{10} C(x+2y)dydx = C\int_0^{10}\left[xy+y^2\right]_{y=0}^{y=10} dx$$

$$= C\int_0^{10}(10x+100)dx = 1500C$$

Therefore $1500C = 1$ and so $C = \frac{1}{1500}$

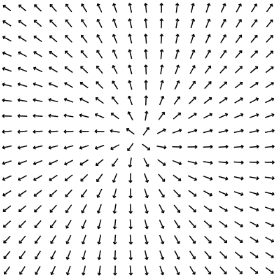
Compute the probability that X is at most 7 and Y is at least 2:

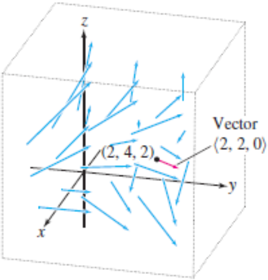
$$P(X \le 7, Y \ge 2) = \int_{-\infty}^{7}\int_2^{\infty} p(x,y)dydx = \int_0^7\int_2^{10}\frac{1}{1500}(x+2y)dydx$$

$$= \frac{1}{1500}\int_0^7\left[xy+y^2\right]_{y=2}^{y=10} dx = \frac{1}{1500}\int_0^7(8x+96)dx$$

$$= \frac{868}{1500} \approx 0.5787$$

Note: can use $P(A) = 1 - P\left(A^C\right)$

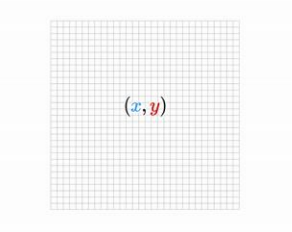




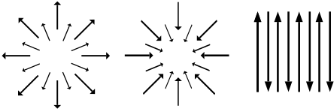
- A ***vector field*** assigns a vector to each point in a domain
- A **vector field** can represent velocities or quantities such as winds, currents, fields, or fluid flows
- A vector field can be two or three dimensional (or more!)
  - $F(x,y,z) = F_1\,\hat{\imath} + F_2\,\hat{\jmath} + F_3\,\hat{k}$
  - $F(x,y) = F_1\,\hat{\imath} + F_2\,\hat{\jmath}$
- A **unit vector field** is a vector field **F** such that $\|F(P)\| = 1$ for all points P.

$$\mathbf{e}_r = \left\langle \frac{x}{r}, \frac{y}{r} \right\rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

$$\mathbf{e}_r = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$



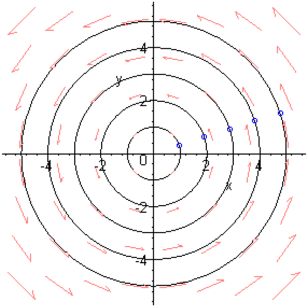
- **Divergence** and **curl** are two characteristics of how "*flow*" is "*behaving*" on a vector field in a small neighborhood around a given point 'P'
- **Divergence** is the measurement of how much fluid/flow *enters* the neighborhood around P compared to how much *exits*
  - If <u>more fluid/flow</u> **ENTERS** the neighborhood than **leaves** it, then divergence will be **negative (-)** at P → fluid gathering at a point
  - If the <u>**SAME**</u> amount of fluid/flow enters as leaves, then divergence will be **zero (0)** at P → *incompressible*
  - If <u>more fluid/flow</u> **LEAVES** then **enters**, then divergence will be **positive (+)** at P → think of fluid leaving at that point (i.e., diverging/divergent)



$\partial/\partial x(\mathbf{V}_x) > 0$	$\partial/\partial x(\mathbf{V}_x) < 0$	$\partial/\partial x(\mathbf{V}_x) = 0$
$\partial/\partial y(\mathbf{V}_y) > 0$	$\partial/\partial y(\mathbf{V}_y) < 0$	$\partial/\partial y(\mathbf{V}_y) = 0$
$\nabla\cdot(\mathbf{V}) > 0$	$\nabla\cdot(\mathbf{V}) < 0$	$\nabla\cdot(\mathbf{V}) = 0$

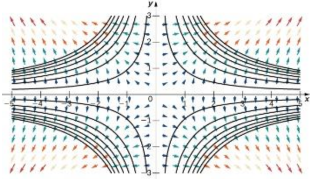
$$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Operations on Vector Fields (Curl)



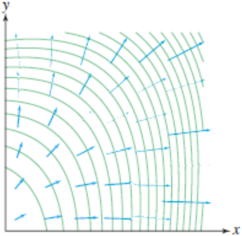
- **Curl** is the measurement of *rotation* of the vector field in the neighborhood around P – how much the paddle spins in a vector field (spininess)
  - If **curl (+)** at a point, then flow/fluid will spin **counterclockwise**
  - If **curl (-)** at a point rotate **clockwise**
  - If **curl (0)** at a point no rotation (**irrotational**)

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$



$$\mathbf{F} = \langle F_1, F_2 \rangle$$
$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

$$\mathbf{F} = \langle F_1, F_2, F_3 \rangle$$
$$\text{curl}(\mathbf{F}) = 0$$



- A vector field '**F**' is conservative <u>iff</u> **F = ∇f** for some function ' **f** '
- ' **f** ', the function, is the antiderivative of **∇f,** the gradient of the function
- A field is conservative if and only if it's a gradient field
- The curl of a gradient vector is zero
  ∇ x ∇f = 0
- Any conservative field is *irrotational*
- Any **irrotational field**, which is differentiable on a domain <u>with no holes</u>, is conservative
- A **conservative vector field** is *orthogonal* to the <u>level curves</u> of the *potential function*.
- **Conservative vector fields** are important in later chapters due to a certain feature called "path independence"

Show that the divergence of a curl is zero $\vec{\nabla} \cdot \vec{\nabla} \times \vec{V} = 0$ for a vector $\vec{V}(x, y, z)$.
$\vec{V} = \langle V_x, V_y, V_z \rangle$

Show that the divergence of a curl is zero $\vec{\nabla} \cdot \vec{\nabla} \times \vec{V} = 0$ for a vector $\vec{V}(x, y, z)$.
$\vec{V} = \langle V_x, V_y, V_z \rangle$

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{V}(x, y, x) = \vec{\nabla} \cdot \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \vec{\nabla} \cdot \left(\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \hat{i} - \left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z} \right) \hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \hat{k} \right)$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) - \frac{\partial}{\partial y} \left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)$$

$$= \frac{\partial^2 V_z}{\partial x \partial y} - \frac{\partial^2 V_y}{\partial x \partial z} + \frac{\partial^2 V_x}{\partial y \partial z} - \frac{\partial^2 V_z}{\partial y \partial x} + \frac{\partial^2 V_y}{\partial z \partial x} - \frac{\partial^2 V_x}{\partial z \partial y}$$

$$= 0$$

Cross-Partial Condition: $\frac{\partial^2 f_z}{\partial x \partial y} = \frac{\partial^2 f_z}{\partial y \partial x}$

Three Categories of Line Integrals:

1. Scalar Line Integrals (not over vector fields)
2. Flux Line Integrals
3. Line Integrals over Vector Fields
    - Conservative Vector Fields
    - Non-conservative Vector Fields
    - Line integrals over vector fields where your travel connects (you stop and start at the same point)

Summary of the types of ways to write the work integral:

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$$
$$\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$$
$$a \leq t \leq b$$

$$\boxed{W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \quad \text{(The definition)}}$$

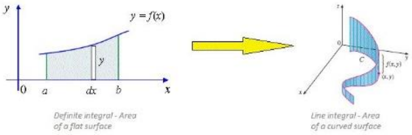
$$\boxed{W = \int_C \mathbf{F} \cdot d\mathbf{r} \quad \text{(Vector differential form)}}$$

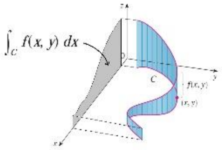
$$\boxed{W = \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt \quad \text{(Parametric vector evaluation)}}$$

$$\boxed{W = \int_a^b \left(Mg'(t) + Nh'(t) + Pk'(t)\right) dt \quad \text{(Parametric scalar evaluation)}}$$

$$\boxed{W = \int_C M dx + N dy + P dz \quad \text{(Scalar differential form)}}$$

Scalar Line Integrals (Summary)

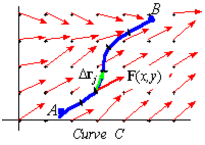




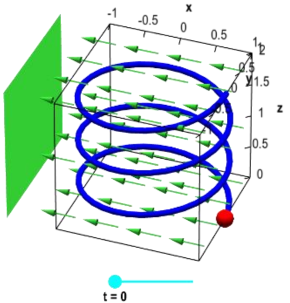
$$f(x,y)\,ds = \text{height} \cdot \text{length} = \text{Area}$$

$$\int_C 1 ds = \text{length of } C$$

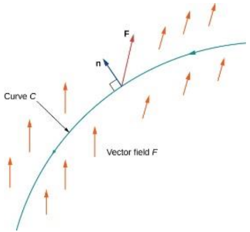
$$\int_C f(x,y)ds$$



- **Scalar line integrals**, in general, focus on boundaries instead of general regions, where these boundaries are called **'curves'**
- $\boldsymbol{f(x,y)}$ is a surface that gives a **height** above each point on the curve C
- The arc-length differential $\boldsymbol{ds}$ gives a **tiny length** of the curve.
- **<u>Main Ideas:</u>**
  1. Take curve and project the curve onto some surface $f(x,y)$
  2. For each point on the curve, there will be different heights corresponding to $f(x,y)$
  3. Use an integral as a sum along the curve. This will allow for us to get the area under $f(x,y)$, but only along the curve
     - ➢ **Goal:** finding the area between curve on the xy-plane and the curve projected onto the surface, which corresponds to the function's height $f(x,y)$. Thus, we are obtaining the area between the "projected curve" and the "curve on the xy-plane".
     - ➢ **Concept:** the line integral will gives us the area of the 'ribbon/fence' between the curve C in the xy-plane and the projection of C onto the surface $f(x,y)$.
     - ➢ **Other:** can represent mass

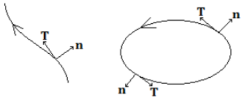


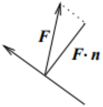
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

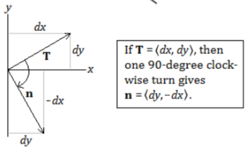


- **Vector line integrals** involve integrating over the tangential component $\mathrm{F} \cdot T$ along the curve C
  - We must parameterize our vector field and position vector accordingly
  - Involves a dot product with the vector field to the tangent vector $r'(t)$ due to integrating over tangential component
- It can be thought of as a measure of **overall alignment** between the oriented curve and the vector field
- It represents the **work** it takes to move a particle in the presence of a force **F** along the path, where motion with the field counts as positive and against the field as negative
- **Vector line integrals** depend on the *direction* along the curve (i.e., orientation)

Flux Line Integrals (Summary)







flux across $C = \int_C (\mathbf{F}\cdot\mathbf{n})ds = \int_a \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{N}(t)dt$

- **Flux** can be considered the amount of **'stuff'** from the vector field that goes through the curve
- **Flux will depend on certain factors:** scenario, clockwise/counterclockwise orientation, inward/outward flux, upward/downward orientation
- **Flux line integrals** are integrals over the <u>normal component</u> $F \cdot n$ of a vector field
  - Must define normal component correctly so that it is normal to the tangent vector $r'(t) = \langle x'(t), y'(t)\rangle$
  - <u>**Two Possible Options:**</u>
    - $N(t) = \langle y'(t), -x'(t)\rangle$
    - $N(t) = \langle -y'(t), x'(t)\rangle$
      - **Idea:** rotate vector $90°$ so that dot product with tangent vector is zero

Normal 1 Dot Product:

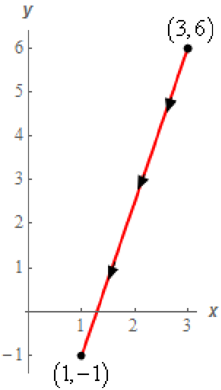
$$\mathbf{r}'(t)\cdot\mathbf{N}(t) = \langle x'(t), y'(t)\rangle \cdot \langle y'(t), -x'(t)\rangle = x'(t)y'(t) - x'(t)y'(t) = 0$$

Normal 2 Dot Product:

$$\mathbf{r}'(t)\cdot\mathbf{N}(t) = \langle x'(t), y'(t)\rangle \cdot \langle -y'(t), x'(t)\rangle = -x'(t)y'(t) + x'(t)y'(t) = 0$$

Scalar Line Integral (example 1)

Evaluate $\int_C 3x^2 - 2y\,ds$ where C is the line segment from $(3,6)$ to $(1,-1)$



Evaluate $\int_C 3x^2 - 2y\,ds$ where C is the line segment from $(3,6)$ to $(1,-1)$

Parameterize downward:

$$\vec{r}(t) = (1-t)\langle 3,6\rangle + t\langle 1,-1\rangle = \langle 3-2t, 6-7t\rangle \qquad 0 \le t \le 1$$

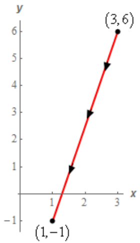
$$\vec{r}'(t) = \langle -2,-7\rangle \qquad \|\vec{r}'(t)\| = \sqrt{(-2)^2+(-7)^2} = \sqrt{53}$$

Parameterize Integrand:

$$3x^2 - 2y = 3(3-2t)^2 - 2(6-7t) = 3(3-2t)^2 - 12 + 14t$$

$$\int_C 3x^2 - 2y\,ds = \int_0^1 \left(3(3-2t)^2 - 12 + 14t\right)\sqrt{53}\,dt$$

$$= \sqrt{53}\left[-\frac{1}{2}(3-2t)^3 - 12t + 7t^2\right]\Big|_0^1 = \boxed{8\sqrt{53}}$$



Scalar Line Integral (example 2)

Evaluate $\int_C ydx + xdy + zdz$ where C is given by $x = \cos t, y = \sin t, z = t^2, 0 \leq t \leq 2\pi$

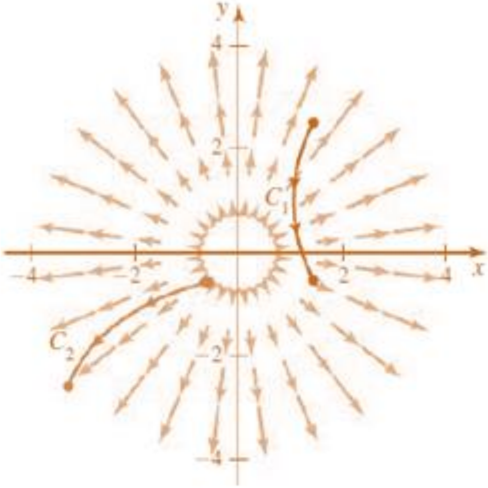
Scalar Line Integral (example 2)

Evaluate $\int_C ydx + xdy + zdz$ where C is given by $x = \cos t, y = \sin t, z = t^2, 0 \le t \le 2\pi$

$$
\begin{aligned}
\int_C y\,dx + x\,dy + z\,dz &= \int_C y\,dx + \int_C x\,dy + \int_C z\,dz \\
&= \int_0^{2\pi} \sin t\,(-\sin t)\;dt + \int_0^{2\pi} \cos t\,(\cos t)\;dt + \int_0^{2\pi} t^2\,(2t)\;dt \\
&= -\int_0^{2\pi} \sin^2 t\,dt + \int_0^{2\pi} \cos^2 t\,dt + \int_0^{2\pi} 2t^3\,dt \\
&= -\frac{1}{2}\int_0^{2\pi} (1-\cos(2t))\;dt + \frac{1}{2}\int_0^{2\pi} (1+\cos(2t))\;dt + \int_0^{2\pi} 2t^3\,dt \\
&= \left(-\frac{1}{2}\left(t - \frac{1}{2}\sin(2t)\right) + \frac{1}{2}\left(t + \frac{1}{2}\sin(2t)\right) + \frac{1}{2}t^4\right)\Bigg|_0^{2\pi} \\
&= 8\pi^4
\end{aligned}
$$

Lecture Question

Determine whether $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the paths C1 and C2 shown in the following vector field is positive, negative or zero.



Determine whether $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the paths C1 and C2 shown in the following vector field is positive, negative or zero.

C1→ negative (-)

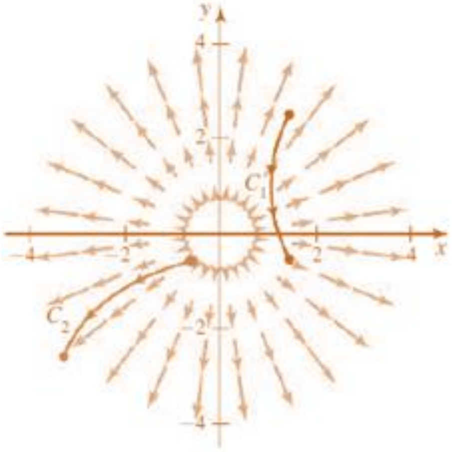
- For C1 the vector field points "against" the curve for most of its length, and with larger magnitude, so the integral is negative.

C2 → positive (+)

- For C2 the vector field points with the curve for its entire length, so the integral is positive.

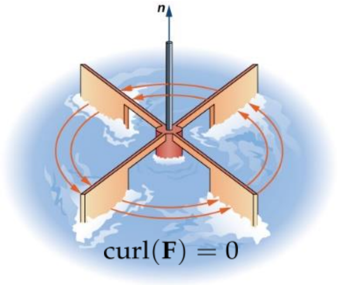
Further Explanation:

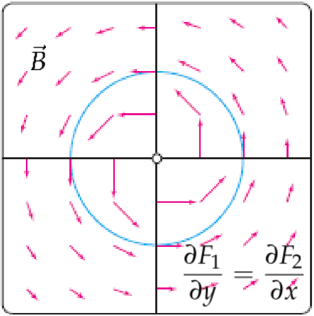
- Try to imagine the curve as tiny arrow moving along the vector field. If same direction, then positive. If opposite direction, then negative. If perpendicular, then zero. The more they are aligned (or oppose), the larger the weight of this positive or negative outcome.



Positive work - the work done on an object is said to be positive work when force (vector field) and displacement (change in position) are in same direction – here try to imagine the vector field as a giant river, where vectors specify the direction of the flowing water

Conservative Vector Fields





- A vector field '**F**' is conservative <u>iff</u> **F = ∇f** for some function ' **f** '
- ' **f** ', the function, is the antiderivative of **∇f,** the gradient of the function
- A field is conservative if and only if it's a gradient field
- The curl of a gradient vector is zero **∇ x ∇f = 0**
- Any conservative field is *irrotational*
- Any **irrotational field**, which is differentiable on a domain <u>with no holes</u>, is conservative

$$\vec{B} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

- **B** is *irrotational* but not conservative (note the hole at the origin); the curl allows us to understand how a field or a fluid behaves, but take intuition for "rotation" carefully, as shown by the vortex field

Vector Line Integrals:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \frac{d\mathbf{r}(t)}{dt}dt$$

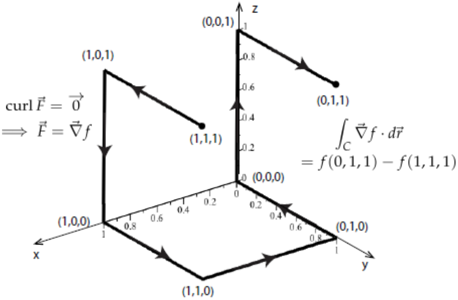
Conservative Vector Line Integrals:

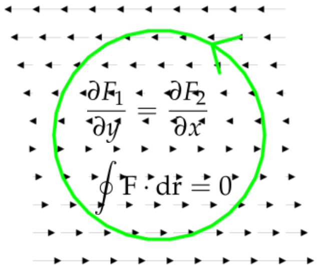
$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_A^B \nabla f\cdot d\mathbf{r} \quad (\text{Iff } \mathbf{F}=\nabla f)\\ &= \int_{t=a}^{t=b} \nabla f(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt\\ &= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))dt \quad (\text{CR})\\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \quad (\text{FTC})\\ &= f(B) - f(A)\end{aligned}$$

- **Line integrals** are not, in general, path independent. This means that for some vector fields **F**, the integral may be different along two different curves connecting the same pair of points
- For **conservative vector fields** we do not need to define a *curve* 'C'
- If it is a conservative vector field, the **line integral** is independent of path so the line integral will be the same no matter what *curve* we choose as the endpoints stay the same

CR = Chain Rule & FTC = Fundamental Theorem of Calculus I&II

Implications of FTC for Line Integrals





- *Line integrals* for <u>gradient fields</u> are <u>path independent</u>
- *Line integrals* of <u>gradient fields</u> around **closed curves** are *zero* $\oint$**F•dr** = 0
- To evaluate a line integral $\int$**F•dr** we can look for a function **f** whose gradient equals **F** and subtract its values at the endpoints (but this function might not exist – vortex field)
- Thus, if *the line integral is independent of path* it means that a parameterization of the path is not needed to evaluate line integrals of <u>conservative fields</u>.

Strategy for Line Integrals over Conservative Vector Fields:

- Show that 'F' is a conservative vector field
  - For $F(x,y) = \langle x_{comp}, y_{comp} \rangle \implies \frac{\partial(x_{comp})}{\partial y} = \frac{\partial(y_{comp})}{\partial x}$ **(Equal Mixed Partials)**
  - For $F(x,y,z) \implies curl(\vec{F}) = \vec{0}$ **(satisfies property of gradient $\nabla \times \nabla f = 0$)**
- Start with your gradient
- Take pieces and integrate
  - Integrate one given component (say x) and then take a derivative w.r.t. the other component (say y)
  - Set equal given component from the question to the derivative you have just taken w.r.t to (say y)
  - Now simplify and integrate the derivative of the other component (say y) to help find the potential function

Three Categories of Line Integrals:

1. Scalar Line Integrals (not over vector fields)
2. Flux Line Integrals
3. Line Integrals over Vector Fields
   - Conservative Vector Fields
   - Non-conservative Vector Fields
   - Line integrals over vector fields where your travel connects (you stop and start at the same point)

comp = component
w.r.t. = with respect to

General Overview (cont.)

Define an arbitrary vector field as the following: $F(x,y) = \langle x_{comp}, y_{comp} \rangle$

Show conservative:

$$\frac{\partial(x_{comp})}{\partial y} = \frac{\partial(y_{comp})}{\partial x}$$

If conservative, $\nabla f = F = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$

Start with x component and integrate:

$$f(x,y) = \int x_{comp}\, dx = \int \frac{\partial f}{\partial x}\, dx = \int f_x(x,y)\, dx = \mathcal{H}(x) + g_1(y)$$

$\mathcal{H}(x) =$ integrated x_{comp} & $g_1(y) =$ a constant in terms of y

Take partial deriv. w.r.t. y for $f(x,y)$ and set given y_{comp} equal to deriv. :

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial f}{\partial y} \implies y_{comp} = 0 + g_1'(y) \implies \text{simplify} \implies g_1'(y) = y_{comp}$$

Integrate in terms of y:

$$g_1(y) = \int g_1'(y)\, dy = g_2(y) + c \implies f(x,y) = \mathcal{H}(x) + g_2(y) + c$$

$c =$ is a numerical constant

Can extend to multiple components and variables

The x_{comp} is only in terms of x in this overview, but in general it has x's and y's, and the same is true for the y_{comp}, as we will see in the following examples.

Line Integral over Conserv. Vector Field (example 1)

Find the work done on $F(x,y) = \langle xe^{2y}, x^2e^{2y} \rangle$ from $A(0,0) \rightarrow B(-1,1)$

Find the work done on $F(x,y) = \langle xe^{2y}, x^2e^{2y}\rangle$ from $A(0,0) \to B(-1,1)$

Show conservative:

$$\langle F_1, F_2\rangle \implies \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \implies 2xe^{2y} = 2xe^{2y} \text{ conservative so, } \nabla f = F = \langle \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}\rangle$$

Start with x_{comp}:

$$f(x,y) = \int xe^{2y}dx = \frac{1}{2}x^2e^{2y} + g(y)$$

Take partial deriv. w.r.t. y for $f(x,y)$ and set given y_{comp} equal to deriv. :

$$\frac{\partial f(x,y)}{\partial y} = x^2e^{2y} + g'(y) \implies x^2e^{2y} + g'(y) = x^2e^{2y} \implies g'(y) = 0$$

Integrate in terms of y:

$$g(y) = \int g'(y)\,dy = c$$

Re-build function:

$$f(x,y) = \frac{1}{2}x^2e^{2y} + g(y) = \frac{1}{2}x^2e^{2y} + c$$

Compute:

$$w = \int_C \overrightarrow{\mathbf{F}} \cdot d\,\overrightarrow{\mathbf{r}} = \int_C \nabla f \cdot d\,\overrightarrow{\mathbf{r}} = \underbrace{f(-1,1) - f(0,0)}_{\text{Point2 - Point1}} = \frac{e^2}{2}$$

Note: the $'c'$ cancels out since we subtracted points

Line Integral over Conserv. Vector Field (example 2)

Find the work done on $F(x,y) = \langle x^2 + \frac{y}{x}, y^2 + \ln x \rangle$ from $A(1,0) \to B(e,1)$

Find the work done on $F(x,y) = \langle x^2 + \frac{y}{x}, y^2 + \ln x \rangle$ from $A(1,0) \to B(e,1)$

Show conservative:

$$\langle F_1, F_2 \rangle \implies \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \implies \frac{1}{x} = \frac{1}{x}$$ conservative so, $\nabla f = F = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$

Start with x_{comp}:

$$f(x,y) = \int \left(x^2 + \frac{y}{x} \right) dx = \frac{1}{3}x^3 + y\ln(x) + g(y)$$

Take partial deriv. w.r.t. y for $f(x,y)$ and set given y_{comp} equal to deriv. :

$$\frac{\partial f(x,y)}{\partial y} = \ln x + g'(y) \implies \ln x + g'(y) = y^2 + \ln x \implies g'(y) = y^2$$

Integrate in terms of y:

$$g(y) = \int y^2 dy = \frac{1}{3}y^3 + c$$

Re-build function:

$$f(x,y) = \frac{1}{3}x^3 + y\ln(x) + g(y) = \frac{1}{3}x^3 + y\ln(x) + \frac{1}{3}y^3 + c$$

Compute:

$$w = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f(e,1) - f(1,0) = \frac{e^3}{3} + 1$$

Note: the $'c'$ cancels out since we subtracted points

Find work done on $F(x,y,z) = \langle \cos y, z^2 - x \sin y, 2yz \rangle$ from $(1,0,0) \to (2,2\pi,1)$

Find work done on $F(x,y,z) = \langle \cos y, z^2 - x\sin y, 2yz \rangle$ from $(1,0,0) \to (2,2\pi,1)$

Show conservative: $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k} = \langle F_1, F_2, F_3 \rangle$

$$\text{curl}(\mathbf{F}) = \nabla \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos y & z^2 - x\sin y & 2yz \end{vmatrix}$$

$$= (2z - 2z)\hat{\imath} - (0-0)\hat{\jmath} + (-\sin y - (-\sin y))\hat{k}$$

$\text{curl}(\mathbf{F}) = \vec{0} = \langle 0,0,0 \rangle$ conservative so, $\nabla f = F = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

Start with x_{comp}:

$$f(x,y,z) = \int \cos y dx = x\cos y + g(y,z)$$

Take partial deriv. w.r.t. y for $f(x,y,z)$ and set given y_{comp} equal to deriv.:

$$\frac{\partial f}{\partial y} = -x\sin y + \frac{\partial g}{\partial y} \implies -x\sin y + \frac{\partial g}{\partial y} = z^2 - x\sin y \implies \frac{\partial g}{\partial y} = z^2$$

Integrate in terms of y:

$$g(y,z) = \int z^2 dy = yz^2 + h(z)$$

Line Integral over Conserv. Vector Field (example 3,cont.)

Find work done on $F(x,y,z) = \langle \cos y, z^2 - x\sin y, 2yz \rangle$ from $(1,0,0) \to (2, 2\pi, 1)$

Re-build function & find $h(z)$:

$$g(y,z) = yz^2 + h(z) \implies f(x,y,z) = x\cos y + g(y,z) = x\cos y + yz^2 + h(z)$$

Take partial deriv. w.r.t. z for $f(x,y,z)$ and set given z_{comp}

$$\frac{\partial f}{\partial z} = 2yz + h'(z) \implies 2yz + h'(z) = 2yz \implies h'(z) = 0$$

Integrate in terms of z:

$$h(z) = \int h'(z)\,dz = c$$

Re-build function:

$$f(x,y,z) = x\cos y + yz^2 + h(z) = x\cos y + yz^2 + c$$

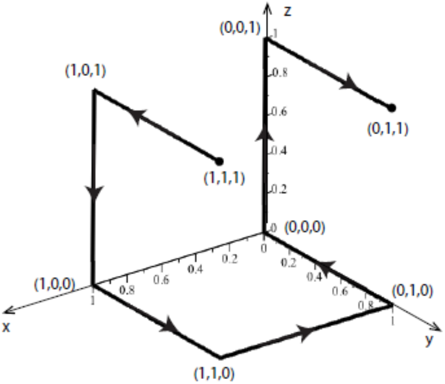
Compute:

$$w = \int_C \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{r}} = \int_C \nabla f \cdot d\overrightarrow{\mathbf{r}} = f(2, 2\pi, 1) - f(1,0,0) = 2\pi + 1$$

Note: Always check for both z and y components, even if $\frac{\partial f}{\partial y} = 0$ or $\frac{\partial f}{\partial z} = 0$ it is important to not skip steps here and instead follow the guideline.

Note: the $'c'$ cancels out since we subtracted points

Let $g(x, y, z) = x^2 \ln y + yz^2$, and let C be the curve shown in the image.

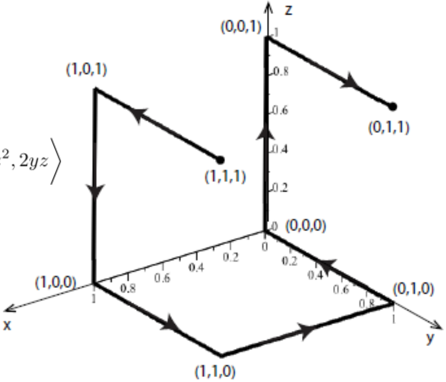


Calculate $\int_C \nabla g \cdot dr$.

Answer Choices:

A. 0
B. 1
C. 2
D. 3
E. 4

Let $g(x, y, z) = x^2 \ln y + yz^2$, and let C be the curve shown in the image.



$\vec{g}(x, y, z) = \left\langle 2x \ln y, \frac{x^2}{y} + z^2, 2yz \right\rangle$

$\operatorname{curl} \vec{g} = \vec{0}$

$\implies \vec{g} = \vec{\nabla} g$

$\int_C \vec{\nabla} g \cdot d\vec{r}$

$= g(0,1,1) - g(1,1,1)$

$= 1 - 1 = 0$

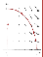
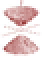
Calculate $\int_C \nabla g \cdot dr$.

Answer Choices:

A. 0
B. 1
C. 2
D. 3
E. 4

General Overview

Curves vs. Surfaces:

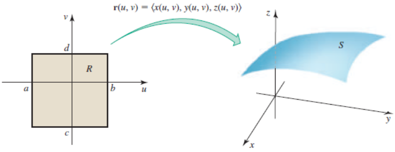
 Curves	Surfaces 
Arc length $s = \int_a^b \|\mathbf{r}'(t)\| \, dt$	Surface area $\text{Area}(\mathcal{S}) = \iint_{\mathcal{D}} \|\mathbf{N}(u,v)\| du dv$
Line Integrals (scalar vs. vector)	Surface Integrals (scalar vs. vector)
One-Parameter Description $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$	Two-Parameter Description $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$
$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$	$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{n}) dS$

Two Categories of Surface Integrals:

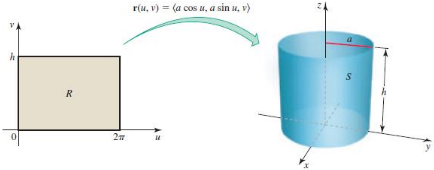
1. Scalar Surface Integrals (not over vector fields)
   - ➢ Used to calculate surface area, total charge, or gravitational potential
2. Surface Integrals over vector fields
   - ➢ Used to calculate flux (transport, heat, electric, or magnetic flux)

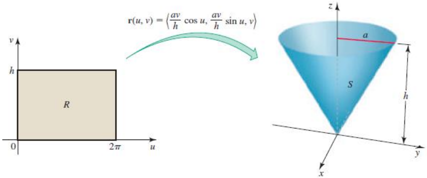
Note: r(u,v) = G(u,v), where G(u,v) is the book's notation

Parametric Surfaces

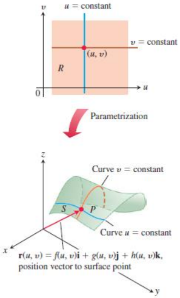


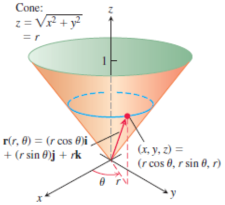
A rectangle in the uv-plane is mapped to a surface in xyz-space.



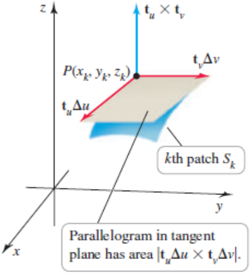


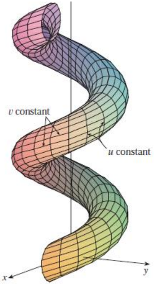
- Transforms x, y, and z components into a vector function in terms of u and v (two parameters)
- Parametrizing a surface amounts to finding a way to describe a location on the surface using a pair of numbers.
  - gives the position of a point on the surface as a vector function of two variables.





Surface Area

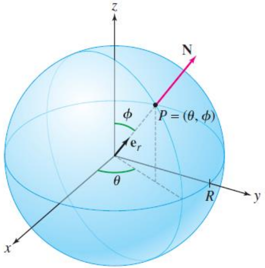




- Utilizes concept of a tangent plane (Chapter 15.4)
- $\boldsymbol{T_u}$ is a vector tangent to the surface corresponding to a change in u with v constant in the uv–plane.
- $\boldsymbol{T_v}$ is a vector tangent to the surface corresponding to a change in v with u constant in the uv-plane.
- Tiny increments in each corresponding vector allows for a summation, which provides surface area
- In surface integrals the value $||\boldsymbol{T_u} \times \boldsymbol{T_v}||$ plays an analogous role as in scalar line integrals with $||\boldsymbol{r}'(\boldsymbol{t})||\ dt$
- In computing surface area, we assume that $G(u, v)$ is one-to-one and regular so that our normal vector $N(u, v)$ is non-zero

General Notes:

- Derivation in the textbook
- Has application to a variety of problems
- Requires sphere to be centered at the origin
- Not necessary to re-derive each time



$$G(\theta, \phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$$

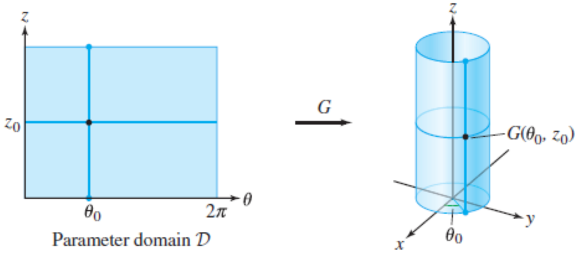
Unit radial vector: $\mathbf{e}_r = \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi \rangle$

Outward normal: $\mathbf{N} = \mathbf{T}_\phi \times \mathbf{T}_\theta = \left(R^2 \sin\phi\right) \mathbf{e}_r$

$$dS = \|\mathbf{N}\| d\phi d\theta = R^2 \sin\phi d\phi d\theta$$

General Notes:

- Derivation in the textbook
- Has application to a variety of problems
- Requires center to be centered at the origin
- Not necessary to re-derive each time



$$G(\theta, z) = (R\cos\theta, R\sin\theta, z)$$

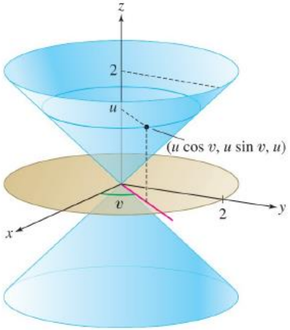
Outward normal: $\quad \mathbf{N} = \mathbf{T}_\theta \times \mathbf{T}_z = R\langle \cos\theta, \sin\theta, 0\rangle$

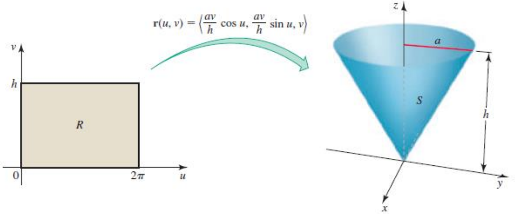
$$dS = \|\mathbf{N}\| d\theta dz = R d\theta dz$$

Parameterization of a Cone

General Notes:

- Can parameterize in terms of θ or u's and v's (same thing)
- Has a general formula in terms of radius r and height h
- Derivation in the textbook (example 4)





$$\{(r, \theta, z) : 0 \leq r \leq a, 0 \leq \theta \leq 2\pi, z = rh/a\}$$

$$x = r\cos\theta = \frac{az}{h}\cos\theta \quad \text{and} \quad y = r\sin\theta = \frac{az}{h}\sin\theta$$

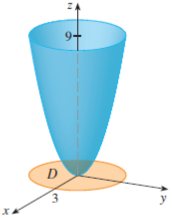
$$G(u, v) = (u\cos v, u\sin v, u)$$
$$G(\theta, t) = (t\cos\theta, t\sin\theta, t)$$

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle = \left\langle \frac{av}{h}\cos u, \frac{av}{h}\sin u, v \right\rangle$$

$$0 \leq u \leq 2\pi \text{ and } 0 \leq v \leq h$$

Parameterization of a Paraboloid (example 1)

Find parametric descriptions for the paraboloid. $z = x^2 + y^2$, for $0 \leq z \leq 9$



Parameterization of a Paraboloid (example 1)

Find parametric descriptions for the paraboloid. $z = x^2 + y^2$, for $0 \le z \le 9$

Think in terms of polar! $u = \theta$ and $v = \sqrt{z}$ so that $z = v^2$

v plays the role of the polar coordinate r.

$$x = v\cos\theta = v\cos u \text{ and } y = v\sin\theta = v\sin u$$

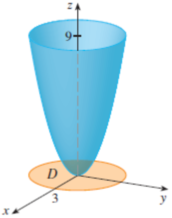
$$\mathbf{r}(u,v) = \left\langle v\cos u, v\sin u, v^2 \right\rangle$$

$$0 \le u \le 2\pi \text{ and } 0 \le v \le 3$$

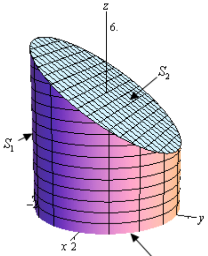
Altematively, we could choose $u = \theta$ and $v = z$.

$$\mathbf{r}(u,v) = \langle \sqrt{v}\cos u, \sqrt{v}\sin u, v \rangle$$

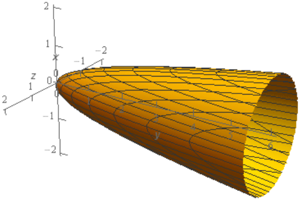
$$0 \le u \le 2\pi \text{ and } 0 \le v \le 9$$



Surface Integral



S_3 is the bottom of the cylinder

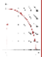
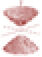


$$\iint_S f(x,y,z)dS = \iint_D f(G(u,v))\|\mathbf{N}(u,v)\|dudv$$

- Analogous to scalar line integrals, but instead we are focusing on a surface in 3D-space not a curve
- We may view G(u, v) as a mapping from the uv-plane to the xy-plane, and we find that $\|\mathbf{N}(u,v)\|$ is the Jacobian of this mapping
- Must consider the small differential surface element $dS = \| N(u,v) \| \, du \, dv$
- Can represent mass, if $f(x,y)$ is the mass density function
- The double integral integrates over a flat region, the surface integral can integrate over a non-flat region (i.e., projection of our surface 'S' onto a coordinate plane gives us the region 'R')

General Overview

Curves vs. Surfaces:

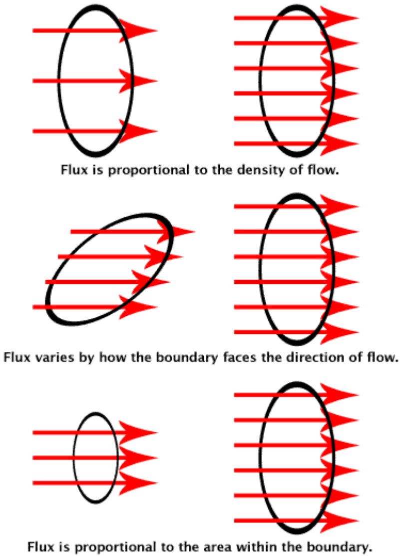
 Curves	Surfaces 
Arc length $s = \int_a^b \lVert \mathbf{r}'(t) \rVert \, dt$	Surface area $\text{Area}(\mathcal{S}) = \iint_{\mathcal{D}} \lVert \mathbf{N}(u,v) \rVert du dv$
Line Integrals (scalar vs. vector)	Surface Integrals (scalar vs. vector)
One-Parameter Description $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$	Two-Parameter Description $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$
$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$	$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{n}) dS$

Two Categories of Surface Integrals:

1. Scalar Surface Integrals (not over vector fields)
   - ➢ Used to calculate surface area, total charge, or gravitational potential
2. Surface Integrals over vector fields
   - ➢ Used to calculate flux (transport, heat, electric, or magnetic flux)

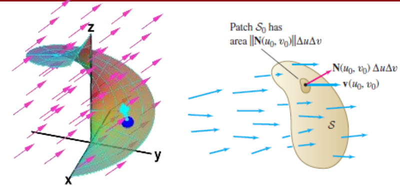
Note: r(u,v) = G(u,v), where G(u,v) is the book's notation

Concept of Flux

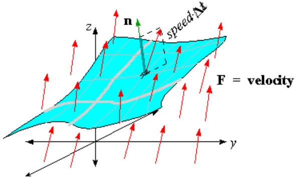


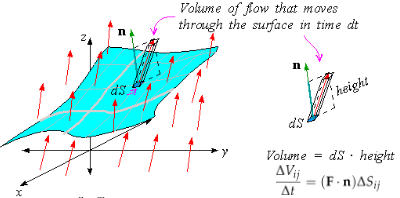
- **Flux** can be considered the **amount of 'stuff'** from the vector field that goes through a curve or surface.
- **Flux**, in vector calculus, is a **scalar quantity**, defined as the surface integral of the perpendicular component of a vector field over a surface
    - In transport phenomena flux can be a vector quantity
- **Flux** will depend on the <u>strength</u> of the vector field, the <u>size</u> of the surface area it passes through, and on how the area is <u>oriented</u> with respect to the field
- If **F**, a vector field, contains a surface "**S**", then **F** can describe the velocity of the flow/fluid at any point across the surface. Thus, the rate of flow (amount/volume of flow across the surface) is the flux.

Concept of Flux (cont. – fluid flux)



Flow rate through $S_0 \approx \mathbf{v}(u_0,v_0)\cdot\mathbf{N}(u_0,v_0)\,\Delta u\Delta v$





$$\frac{\Delta V}{\Delta t} = \lim_{h\to 0}\sum_{j=1}^{n}\sum_{k=1}^{m}(\mathbf{F}\cdot\mathbf{n})\Delta S_{ij} = \iint_S \mathbf{F}\cdot d\mathbf{S}$$

- The normal component(s) from the surface goes through the 'stuff' (or the fluid)
- The vector field is the flow of the fluid
- A dot product between the <u>normal component</u> and the <u>vector field</u> is the amount of flow times the size of that tiny region.
  - **Recall:** geometrically, the dot product tells you what amount of one vector goes in the direction of another vector (alignment)
- We must use an integral to add up all the flows across the 'little bits' of surface, allowing for a vector surface integral

$$\iint_S \underbrace{\mathbf{F}\cdot d\mathbf{S}}_{\text{2 vectors}} = \iint_S \underbrace{(\mathbf{F}\cdot\mathbf{n})dS}_{\text{dot product}}$$

Note: Pay attention to the S (bold vs italicized)

Unit Normal:

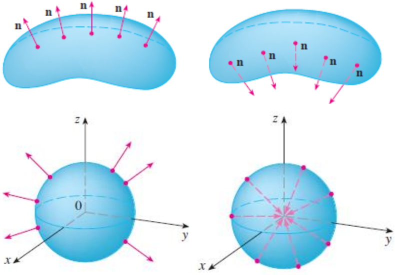
$$\mathbf{n} = \frac{\mathbf{N}(u,v)}{\|\mathbf{N}(u,v)\|}$$

Vector Surface Element:

$$d\mathbf{S} = \mathbf{N}(u,v)\,du\,dv$$

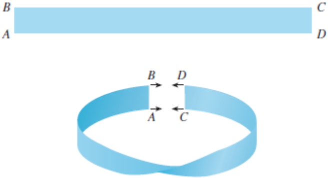
$$dS = \|\mathbf{N}(u,v)\|\,du\,dv$$

Orientable Surfaces:



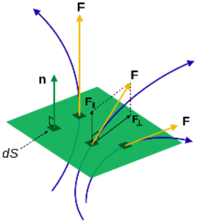
Positive orientation **Negative orientation**

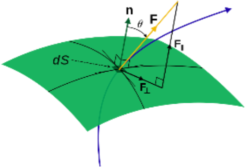
Non-Orientable Surfaces:

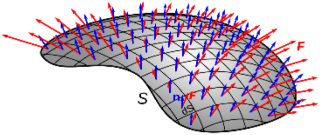


- **Orientable surfaces** are surfaces that we can tell the difference between the inside and outside of the surface. These surfaces are two-sided, unlike the non-orientable **Möbius strip** (only one side)
- If we can define a 'unit normal vector' at everyone point so that this 'vector' varies continuously over a surface S, then S is an oriented surface with two possible orientations
  - **positive orientation** is the one for which the normal vectors point *outward*
  - *inward*-pointing normal vectors give the **negative orientation**
  - More specifically we have the following orientations: upwards, downwards, inwards, and outwards

Surface Integrals of Vector Fields







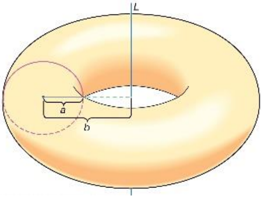
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(G(u,v)) \cdot \mathbf{N}(u,v)\,du\,dv$$

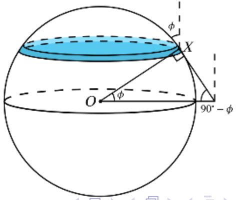
- The vector surface integral is the integral of the normal component across an orientable surface
- If the orientation of the surface is reversed, then the surface integral changes sign
- We must parametrize the surface in terms of two parameters. It is also necessary for us to change our vector field according to the surface's parametrization ('a sort of jacobian mapping')
    - ~ a way to describe a location on the surface using a pair of numbers
- **IMPORTANT:** In surface integrals, the Jacobian factor (whether that be r, or $\rho^2 \sin\phi$) is incorporated into the magnitude of the normal component **N** but **NOT** the unit vector **n**

  e.g $dr\,d\theta$, and not $r\,dr\,d\theta$

Strategy for Surface Integrals over Vector Fields:

1. Parametrize the surface (if not already given) in terms of two parameters of the seven given coordinate functions (ℓ) – it might require converting into functions of u's & v's ($\mathcal{J}$ displays some possible outputs)
$$\ell = [x, y, z, r, \theta, \rho, \phi] \rightarrow \mathcal{J} = (\,(u,v), (\theta,\phi), (r,\theta), (x,y), (\theta,x), (x,z)\,)$$
2. Find partial derivatives of the parametric surface with respect to two of your chosen parameters (allowing for two tangent vectors)
3. Find the normal vector **N** by the cross product of the two tangent vectors
4. Re-define the vector field **F** with your two chosen parameters
5. Calculate the dot product between the normal vector and the vector field, or **F • N**
6. Determine the appropriate bounds for the integral based on your two chosen parameters, the domain $\boldsymbol{D}$, and the surface $\boldsymbol{S}$
7. Evaluate the surface integral





Fluid Flux (example 1)

Let $v(x, y, z) = \langle 2x, 2y, z \rangle$ represent a velocity field (with units of meters/second) of a fluid with constant density 80 kg$/m^3$. Let S be hemisphere $x^2 + y^2 + z^2 = 9$ with $z \leq 0$ such that S is oriented outward.
Find the mass flow rate of the fluid across S

We use spherical coordinates:

$$x = R\cos\theta\sin\phi, \quad y = R\sin\theta\sin\phi, \quad z = R\cos\phi$$

For a sphere, we have the following:

$$\text{Outward normal:} \quad \mathbf{N} = \mathbf{T}_\phi \times \mathbf{T}_\theta = \left(R^2\sin\phi\right)\mathbf{e}_r$$

$$\text{Unit radial vector: } \mathbf{e}_r = \langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$$

$$x^2 + y^2 + z^2 = 9 \implies R^2 = 9$$

$$\vec{N} = \langle 9\cos\theta\sin^2\phi, 9\sin\theta\sin^2\phi, 9\sin\phi\cos\phi\rangle$$

However, direct computation provides us the same result:

$$\vec{G}(\phi,\theta) = \langle 3\cos\theta\sin\phi, 3\sin\theta\sin\phi, 3\cos\phi\rangle, \ 0 \le \theta \le 2\pi, \ 0 \le \phi \le \pi/2$$

$$\mathbf{T}_\theta = \langle -3\sin\theta\sin\phi, 3\cos\theta\sin\phi, 0\rangle$$

$$\mathbf{T}_\phi = \langle 3\cos\theta\cos\phi, 3\sin\theta\cos\phi, -3\sin\phi\rangle$$

$$\mathbf{N} = \mathbf{T}_\phi \times \mathbf{T}_\theta = \langle 9\cos\theta\sin^2\phi, 9\sin\theta\sin^2\phi, 9\sin\phi\cos\phi\rangle$$

$$v(x,y,z) = \langle 2x, 2y, z\rangle$$

$$v(x,y,z) \implies v(r(\phi,\theta))$$

$$v(r(\phi,\theta)) = \langle 6\cos\theta\sin\phi, 6\sin\theta\sin\phi, 3\cos\phi\rangle$$

$$v(r(\phi,\theta)) \cdot \mathbf{N} = 54\sin^3\phi + 27\cos^2\phi\sin\phi$$

Since we want the mass flowrate then $\iint_S \mathbf{v} \cdot d\mathbf{S} \implies \iint_S \rho v \cdot dS, \ \rho = 80kg/m^3$

Fluid Flux (example 1, cont.)

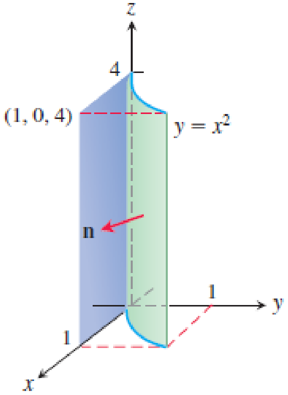
Computation:

$$
\begin{aligned}
\iint_S \rho v \cdot dS &= 80 \int_0^{2\pi} \int_0^{\pi/2} v(r(\phi,\theta)) \cdot (t_\phi \times t_\theta)\, d\phi\, d\theta \\
&= 80 \int_0^{2\pi} \int_0^{\pi/2} \langle 6\cos\theta \sin\phi,\, 6\sin\theta \sin\phi,\, 3\cos\phi\rangle \cdot \langle 9\cos\theta \sin^2\phi,\, 9\sin\theta \sin^2\phi,\, 9\sin\phi\cos\phi\rangle\, d\phi\, d\theta \\
&= 80 \int_0^{2\pi} \int_0^{\pi/2} 54\sin^3\phi + 27\cos^2\phi \sin\phi\, d\phi\, d\theta \\
&= 80 \int_0^{2\pi} \int_0^{\pi/2} 54(1-\cos^2\phi)\sin\phi + 27\cos^2\phi\sin\phi\, d\phi\, d\theta \\
&= 80 \int_0^{2\pi} \int_0^{\pi/2} 54\sin\phi - 27\cos^2\phi\sin\phi\, d\phi\, d\theta \\
&= 80 \int_0^{2\pi} \Big[-54\cos\phi + 9\cos^3\phi\Big]_{\phi=0}^{\phi=2\pi} d\theta \\
&= 80 \int_0^{2\pi} 45\, d\theta \\
&= 7200\pi.
\end{aligned}
$$

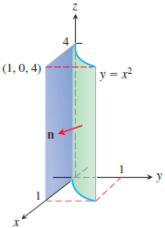
The mass flow rate is 7200π kg/sec/m^2

Surface Integrals over Vector Fields (example 2)

Find the flux of $\mathbf{F} = \langle yz, x, -z^2 \rangle$ through the parabolic cylinder $y = x^2$, $0 \leq x \leq 1, 0 \leq z \leq 4$ in the direction n indicated in the image.



Find the flux of $\mathbf{F} = \langle yz, x, -z^2 \rangle$ through the parabolic cylinder $y = x^2$, $0 \le x \le 1, 0 \le z \le 4$ in the direction n indicated in the image.



$$r(x,z) = G(x,z) = (x, f(x,z), z)$$

$$x = x, y = x^2, \text{ and } z = z$$

$$G(x,z) = (x, x^2, z)$$

$$\mathbf{N} = \mathbf{T}_x \times \mathbf{T}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} - \mathbf{j}$$

$$\mathbf{n} = \frac{\mathbf{N}(x,y)}{\|\mathbf{N}(x,y)\|} = \frac{2x\mathbf{i} - \mathbf{j}}{\sqrt{4x^2+1}}$$

Since $y = x^2$ on the surface, we have:

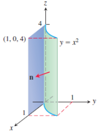
$$\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k} = x^2z\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}$$

$$\mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{4x^2+1}} \left(\left(x^2 z\right)(2x) + (x)(-1) + \left(-z^2\right)(0) \right)$$

$$= \frac{2x^3z - x}{\sqrt{4x^2+1}}$$

Note: From the image the unit normal vectors are pointing outward

Find the flux of $\mathbf{F} = \langle yz, x, -z^2\rangle$ through the parabolic cylinder $y = x^2$, $0 \le x \le 1, 0 \le z \le 4$ in the direction n indicated in the image.



$$d\mathbf{S} = \mathbf{N}(x,z)dxdz$$

$$\mathbf{N} = \mathbf{T}_x \times \mathbf{T}_z = 2x\mathbf{i} - \mathbf{j}$$

$$\mathbf{n} = \frac{\mathbf{N}(x,z)}{\|\mathbf{N}(x,z)\|} = \frac{2x\mathbf{i} - \mathbf{j}}{\sqrt{4x^2+1}}$$

Recall:

$$\iint_S \underbrace{\mathbf{F}\cdot d\mathbf{S}}_{\text{2 vectors}} = \iint_S \underbrace{(\mathbf{F}\cdot\mathbf{n})dS}_{\text{dot product}}$$

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_R \mathbf{F}\cdot \frac{\mathbf{T}_x \times \mathbf{T}_z}{||\mathbf{T}_x \times \mathbf{T}_z||}\,||\mathbf{T}_x \times \mathbf{T}_z||\,dxdz = \iint_R \mathbf{F}\cdot(\mathbf{T}_x \times \mathbf{T}_z)\,dxdz$$

$$\mathbf{F}\cdot\mathbf{n} = \frac{2x^3z - x}{\sqrt{4x^2+1}}$$

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^4 \int_0^1 \frac{2x^3z - x}{\sqrt{4x^2+1}} \sqrt{4x^2+1}dxdz = 2$$

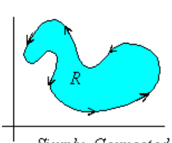
However, with the simpler method we get the same result:

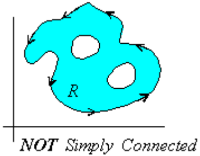
$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(G(x,z))\cdot\mathbf{N}(x,z)dxdz$$

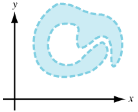
$$\mathbf{F}\cdot(\mathbf{T}_x \times \mathbf{T}_z) = \left(x^2z\right)(2x) + (x)(-1) = 2x^3z - x$$

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^4 \int_0^1 \left(2x^3z - x\right) dxdz = 2$$

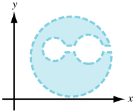
Connected Regions



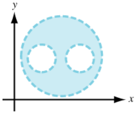




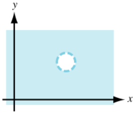
(a) A simply connected domain



(b) A simply connected domain

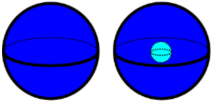


(c) A multiply connected domain

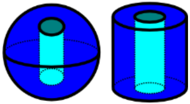


(d) A multiply connected domain

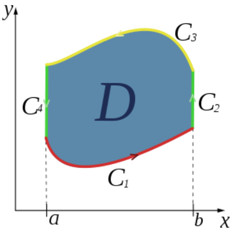
Simply connected

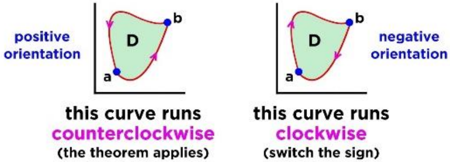


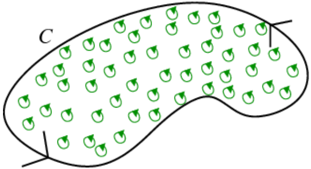
Non-simply connected



- A region D is **open** if it doesn't contain any of its boundary points.
- A region D is **connected** if we can connect any two points in the region with a path that lies completely in D. (going from one point to another without having to go outside a region)
- **Simply-connected** regions have no holes and its connected
- **Multiply-connected** regions have holes, but they are unions of simply-connected regions with 'cuts'
- A path C is **simple** if it doesn't cross itself.
- A path C is called **closed** if its initial and final points are the same point.





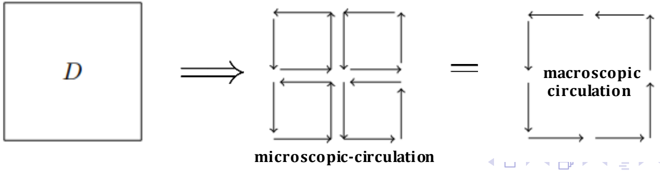


- **Green's theorem** deals with line integrals of a simple, closed curve, over non-conservative vector fields
- It relates a *line integral* for a simple, closed curve to a *double integral* over the region that the curve contains (only for the case where it is a closed curve – if it is an open curve do not use!)
- **Positive orientation** is denoted through a counter-clockwise direction (move through the path and domain is on the left)
- **Negative orientation** is denoted through a clockwise direction

Concept of Green Theorem (cont.)

- The **line integral** portion of the equation relates to how much the vector field circulates around a closed curve (macroscopic circulation)
- The **double integral** portion of the equation relates a summation based on the two-dimensional curl (microscopic circulation)
  - Recall: curl essentially measures rotation
- Macroscopic Circulation = Sum of Microscopic Circulation
- Thus, we have two possible methods of calculation, given by Green's Theorem (if we have a closed curve!)
- $\boldsymbol{\partial D}$ denotes the boundary of the domain (it means 'the boundary of D')

$$\oint_{\partial D} F_1 dx + F_2 dy = \iint_D \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix} dA = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$



Area via Green's Theorem:

$$\oint_{\partial D} F_1 dx + F_2 dy = \iint_D \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix} dA = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

$$A = \iint_D dA$$

$$\underbrace{\partial_x(F_2) - \partial_y(F_1)}_{\text{Integrand } = 1} = 1$$

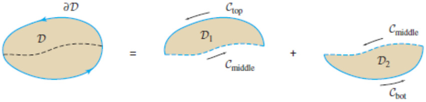
$$\begin{array}{lll} F_1 = 0 & F_1 = -y & F_1 = -\frac{y}{2} \\ F_2 = x & F_2 = 0 & F_2 = \frac{x}{2} \end{array}$$

$$\text{Area}(\mathcal{D}) = \oint_C x\,dy = \oint_C -y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

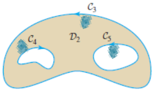
Additivity of Circulation:

$$\oint_{\partial \mathcal{D}_1} \mathbf{F}\cdot d\mathbf{r} + \oint_{\partial \mathcal{D}_2} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{C}_{\text{top}}} \mathbf{F}\cdot d\mathbf{r} + \int_{\mathcal{C}_{\text{bot}}} \mathbf{F}\cdot d\mathbf{r} = \oint_{\partial \mathcal{D}} \mathbf{F}\cdot d\mathbf{r}$$

$$\partial \mathcal{D}_2 = \mathcal{C}_3 + \mathcal{C}_4 - \mathcal{C}_5$$



the region lies to the left as the curve is traversed in the direction specified by the orientation



(B) Oriented boundary of $\mathcal{D}_2$ is $\mathcal{C}_3 + \mathcal{C}_4 - \mathcal{C}_5$

Right-Foot-Rule

Right foot on the curve
D should be on your left

Vector Form of Green's Theorem

- In the **vector form**, we are integrating over the **normal component** (**F** • **n**) instead of the *tangential component* (**F** • **T** from *scalar form*)
- The **vector form** relates to the **divergence** of a vector field (div(**F**)), instead of the two-dimensional *curl* of a vector field (curl(**F**)) from the *scalar form*
- The **vector form** is based on the outward or inward **flux** given by the line integral, instead of the *circulation* from the line integral via the *scalar form*

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathbf{n}ds &= \int_a^b (\mathbf{F}\cdot\mathbf{n})(t)\,\|\mathbf{r}'(t)\|\,dt \\ &= \int_a^b \left[\frac{F_1y'(t)}{\|\mathbf{r}(t)\|} - \frac{F_2x'(t)}{\|\mathbf{r}(t)\|}\right]\|\mathbf{r}(t)\|dt \\ &= \int_a^b F_1y'(t)dt - F_2x'(t)dt \\ &= \int_a^b F_1dy - F_2dx\end{aligned}$$

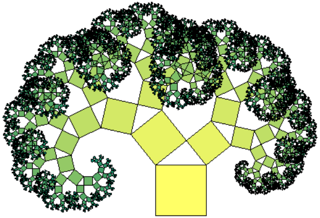
$$\begin{aligned}&\int_{\partial D} F_1dy - F_2dx \\ &= \iint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dA \\ &= \iint_D \operatorname{div}(\mathbf{F})dA\end{aligned}$$

Pythagoras tree theorem states that the generation n Pythagorean tree has area $n+1$.
In each generation, new squares are added along a right angle triangle.
If a curve $\mathcal{C}$ traces each of the α square boundaries counter clockwise, what is the value of the line integral, given a vector field below and a $(k-n)!$ generation Pythagoras tree:

$$\vec{F}(x,y) = \langle -y + x^8, x - y^9 \rangle$$

Recall:

$n\,! = n(n-1)(n-2)\ldots(2)(1)$



α square boundaries

Answer Choices:

A. 0
B. k! + 1
C. 2(k-n)! + 2
D. (k-n-1)! +1
E. It cannot be determined.

Lecture Question (Solution)

Pythagoras tree theorem states that the generation n Pythagorean tree has area $n+1$.
In each generation, new squares are added along a right angle triangle.
If a curve C traces each of the α square boundaries counter clockwise, what is the value of the line integral, given a vector field below and a $(k-n)!$ generation Pythagoras tree:

$$\vec{F}(x,y) = \langle -y + x^8, x - y^9 \rangle$$

$$\oint_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

$$curl_z(\vec{F}) = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = \frac{\partial}{\partial x}\left(x - y^9\right) - \frac{\partial}{\partial y}\left(-y + x^8\right) = 2$$

The curl of $\vec{F}$ is constant 2 .

$$A = n+1$$

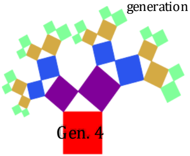
$$A = \iint_D dA = (k-n)! + 1$$

$$\oint_{\partial D} F_1 dx + F_2 dy = \iint_D 2 dA$$

$$= 2 \cdot \iint_D dA = 2 \cdot ((k-n)! + 1) = 2(k-n)! + 2$$

Reasoning: We can apply Green's Theorem since the α square boundaries form a simple, closed curve regardless of the generation, size, or iteration (although some squares will overlap in future generations, regardless we still will have a closed curve on the boundary). Conceptually, a constant curl means that the 'rotation' is the same regardless of location.

Extra Info: In generation 2, we have added 4 new squares which together have area 1 so that the tree now has area 3. In generation 3, we have added 8 squares of total area 1 so that the generation tree has area 4. The picture to the far-right shows generation 7. Its area of all its (partly overlapping) leaves is 8.



Answer Choices:

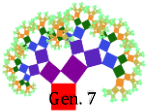
A. 0

B. k! + 1

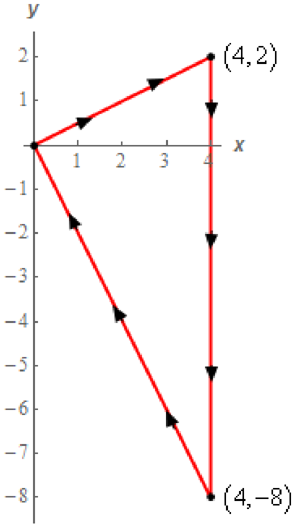
C. **2(k-n)! + 2 (correct answer)**

D. (k-n-1)! +1

E. It cannot be determined.



Use Green's Theorem to evaluate $\int_C x^2y^2dx + \left(yx^3 + y^2\right) dy$ where C is shown below.



Green's Theorem (example 1)

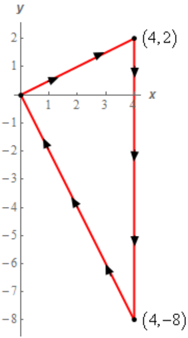
Use Green's Theorem to evaluate $\int_C x^2y^2dx + \left(yx^3 + y^2\right) dy$ where C is shown below.

Since the curve has a clockwise direction (-), we can't use Green's Theorem to evaluate the given integral.

If C has the negative orientation then $-C$ will have the positive orientation

$$\int_C x^2y^2\,dx + \left(yx^3 + y^2\right)\,dy = -\int_{-C} x^2y^2\,dx + \left(yx^3 + y^2\right)\,dy$$

Thus we can use Green's Theorem to compute the value of the following integral:



$$\int_{-C} x^2y^2\,dx + \left(yx^3 + y^2\right)\,dy$$

$$F_1 = x^2y^2 \qquad F_2 = yx^3 + y^2$$

Apply Green's Theorem:

$$\int_{-C} x^2y^2\,dx + \left(yx^3 + y^2\right)\,dy = \iint_D 3yx^2 - 2yx^2\,dA = \iint_D yx^2\,dA$$

Using the point slope formula: $y - y_0 = m\left(x - x_0\right)$

$$0 \le x \le 4$$

$$-2x \le y \le \frac{1}{2}x$$

Use Green's Theorem to evaluate $\int_C x^2y^2 dx + \left(yx^3 + y^2\right) dy$ where C is shown below.

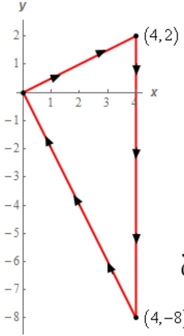
$$\int_{-C} x^2y^2\,dx + \left(yx^3 + y^2\right)\,dy = \iint_D yx^2\,dA$$

$$= \int_0^4 \int_{-2x}^{\frac{1}{2}x} yx^2\,dy\,dx$$

$$= \int_0^4 \frac{1}{2}y^2x^2 \Big|_{-2x}^{\frac{1}{2}x} dx$$

$$= \int_0^4 -\frac{15}{8}x^4\,dx$$

$$= -\frac{3x^5}{8}\Big|_0^4 = -384$$



Using the relationship between the integrals over C and $-C$:

$$\int_C x^2y^2\,dx + \left(yx^3 + y^2\right)\,dy = -\int_{-C} x^2y^2\,dx + \left(yx^3 + y^2\right)\,dy = 384$$

Green's Theorem (example 2)

Calculate the outward flux of the vector field $\mathbf{F}(x, y) = 2e^{xy}\mathbf{i} + y^3\mathbf{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$

Green's Theorem (example 2)

Calculate the outward flux of the vector field $\mathbf{F}(x,y) = 2e^{xy}\mathbf{i} + y^3\mathbf{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$

$$F_1 = 2e^{xy} \quad F_2 = y^3$$

$$\text{Closed curve!} \implies \quad \text{div}\,\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 2ye^{xy} + 3y^2$$

$$\oint_{\partial D} \mathbf{F}\cdot\mathbf{n}ds = \iint_D \text{div}(\mathbf{F})dA = \int_{-1}^{1}\int_{-1}^{1}\left(2ye^{xy} + 3y^2\right)dxdy$$

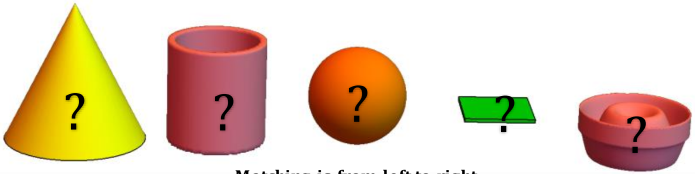
$$= \int_{-1}^{1}\left(2e^{y} + 6y^2 - 2e^{-y}\right)dy = \left[2e^{y} + 2y^3 + 2e^{-y}\right]_{-1}^{1} = 4$$

Remarks:

- We can apply Green's Theorem since it is a closed curve
- Calculating the flux with a line integral would take four integrations, one for each side of the square
- By using a double integral instead a line integral, we can simplify calculations

Lecture Question

We teach an AI to recognize objects and ask to match the pictures with the parametric surface. Do the job for the AI and match the parametrizations which best suit the surfaces. In each case, the parameters (u, v) range over some region R which are not given as it is irrelevant for the matching task here.



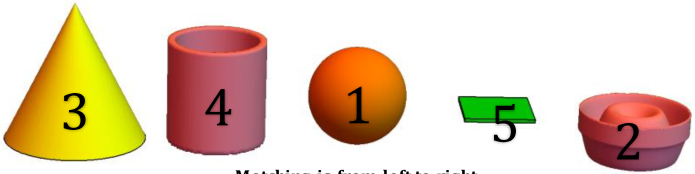
Matching is from left to right

1) $\vec{G}(u,v) = (\sin(v)\cos(u), \sin(v)\sin(u), \cos(v))$
2) $\vec{G}(u,v) = (u, v, \sin(u^2+v^2))$
3) $\vec{G}(u,v) = ((1-u)\cos(v), (1-u)\sin(v), u)$
4) $\vec{G}(u,v) = (\cos(v), \sin(v), (1-u))$
5) $\vec{G}(u,v) = ((1-u), v, 1)$

Answer Choices: (L → R)

A. 2,4,3,5,1
B. 3,2,1,5,4
C. 1,4,5,2,3
D. 3,4,1,5,2
E. 1,2,3,4,5

We teach an AI to recognize objects and ask to match the pictures with the parametric surface. Do the job for the AI and match the parametrizations which best suit the surfaces. In each case, the parameters (u, v) range over some region R which are not given as it is irrelevant for the matching task here.



Matching is from left to right

~ Parametrization of a sphere → 1) $\vec{G}(u,v) = (\sin(v)\cos(u), \sin(v)\sin(u), \cos(v))$

2) $\vec{G}(u,v) = (u, v, \sin(u^2+v^2))$ ← **Polar/cylindrical symmetry, think $r^2 = x^2 + y^2$**

3) $\vec{G}(u,v) = ((1-u)\cos(v), (1-u)\sin(v), u)$ ← **Process of Elimination or plug (u,v)=(0,0)**

~ Parametrization of a cylinder → 4) $\vec{G}(u,v) = (\cos(v), \sin(v), (1-u))$

5) $\vec{G}(u,v) = ((1-u), v, 1)$ ← **Constant function in the 3rd comp., think z = 1**

Answer Choices: (L → R)

A. 2,4,3,5,1

B. 3,2,1,5,4

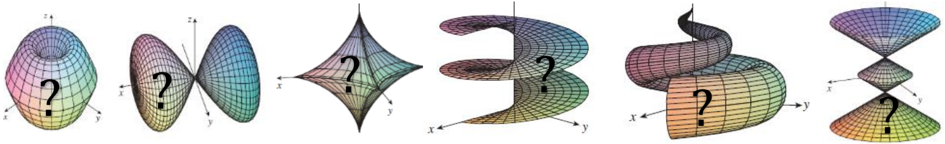
C. 1,4,5,2,3

D. 3,4,1,5,2 (correct answer)

E. 1,2,3,4,5

Lecture Question

After reinforcement learning, the AI can successfully match a range of equations. However, the AI has difficulty matching complex equations. Again, do the job for the AI and match the parametrizations.



Matching is from left to right

1) $\vec{G}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}$

2) $\vec{G}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + \sin u\mathbf{k}, \quad -\pi \le u \le \pi$

3) $\vec{G}(u,v) = \sin v\mathbf{i} + \cos u\sin 2v\mathbf{j} + \sin u\sin 2v\mathbf{k}$

4) $\vec{G}(u,v) = (1-u)(3+\cos v)\cos 4\pi u\mathbf{i} + (1-u)(3+\cos v)\sin 4\pi u\mathbf{j} + (3u + (1-u)\sin v)\mathbf{k}$

5) $\vec{G}(u,v) = \cos^3 u\cos^3 v\mathbf{i} + \sin^3 u\cos^3 v\mathbf{j} + \sin^3 v\mathbf{k}$

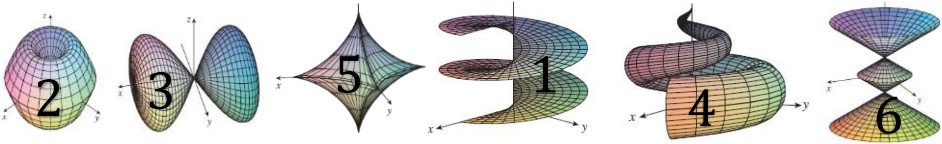
6) $\vec{G}(u,v) = (1-|u|)\cos v\mathbf{i} + (1-|u|)\sin v\mathbf{j} + u\mathbf{k}$

Answer Choices: (L $\to$ R)

A. 1,2,3,4,5,6
B. 2,5,4,3,6,1
C. 3,1,4,6,5,2
D. 1,5,3,2,6,4
E. 6,2,1,3,5,4
F. 2,3,5,1,4,6

Lecture Question (Solution)

After reinforcement learning, the AI can successfully match a range of equations. However, the AI has difficulty matching complex equations. Again, do the job for the AI and match the parametrizations.



Matching is from left to right

1) $\vec{G}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}$

2) $\vec{G}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + \sin u\mathbf{k}, \quad -\pi \le u \le \pi$

3) $\vec{G}(u,v) = \sin v\mathbf{i} + \cos u\sin 2v\mathbf{j} + \sin u\sin 2v\mathbf{k}$

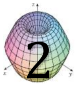
4) $\vec{G}(u,v) = (1-u)(3+\cos v)\cos 4\pi u\mathbf{i} + (1-u)(3+\cos v)\sin 4\pi u\mathbf{j} + (3u + (1-u)\sin v)\mathbf{k}$

5) $\vec{G}(u,v) = \cos^3 u\cos^3 v\mathbf{i} + \sin^3 u\cos^3 v\mathbf{j} + \sin^3 v\mathbf{k}$

6) $\vec{G}(u,v) = (1-|u|)\cos v\mathbf{i} + (1-|u|)\sin v\mathbf{j} + u\mathbf{k}$

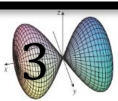
Answer Choices: (L → R)

A. 1,2,3,4,5,6
B. 2,5,4,3,6,1
C. 3,1,4,6,5,2
D. 1,5,3,2,6,4
E. 6,2,1,3,5,4
F. **2,3,5,1,4,6 (Correct Answer)**



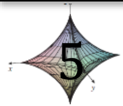
If $u = u_0$ is held constant, then $x = u_0 cos(v), y = u_0 sin(v)$ so each grid is a circle of radius $|u_0|$ in the horizontal plane z = $\sin(u_0)$. If $v = v_0$ is constant, *then* $x = u\,cos(v_0), y = u\,sin(v_0)$ *thus* $y = tan(v_0)x$ so the grid curves lie in vertical planes $y = kx$ through the z-axis. Since x and y are constant multiple of u and $z = sin(u)$, each trace is a sine wave. **(Main Idea: Hold both variables constant, see traces on grid curves, and notice trig. waves + bounds)**

$\vec{G}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + \sin u\mathbf{k}, \quad -\pi \le u \le \pi$



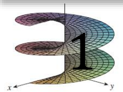
If $v = v_0$ is fixed, then $x = sin(v_0)$ is constant and $y = sin(2v_0)cos(u)$ and $z = sin(2v_0)sin(u)$ describe a circle of radius $|\sin(2v_0)|$ so each grid curve is a circle contained in the vertical plane $x = sin(v_0)$ parallel to the yz-plane. The lengthwise grid curves correspond to holding u constant such that $y = cos(u_0)sin(2v), z = sin(u_0)\,sin(2v)$ so $z = tan(u_0)y$ lying in a plane $z = ky$ that includes the x-axis. **(Main Idea: Hold both variables constant and see traces on grid curves)**

$\vec{G}(u,v) = \sin v\mathbf{i} + \cos u\sin 2v\mathbf{j} + \sin u\sin 2v\mathbf{k}$



If $v = v_0$ is held constant, then z is constant, so the grid curves lie in a horizontal plane. This graph does not have curves that are circles nor straight lines. The vertical grid curves we see on the surface correspond to $u = u_0$ held constant, so the corresponding grid curve lies in the vertical plane $y = \tan^3(u_0)\,x$ through the z-axis. **(Main Idea: Use intuition about the given graph (does it appear to be circular? Sinusoidal? Symmetric?))**

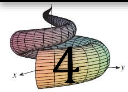
$\vec{G}(u,v) = \cos^3 u\cos^3 v\mathbf{i} + \sin^3 u\cos^3 v\mathbf{j} + \sin^3 v\mathbf{k}$



Looking at grid curves, if we fix v, then x and y parametrize a straight line in the plane $z = v$ which intersects the z-axis. If u is held constant, the projection onto the xy-plane is circular; with $z = v$, each grid is a helix. Thus the graph appears to be a spiraling ramp.

(Main Idea: How does the graph correspond to u = v? Do traces of the graph highlight a specific feature?)

$\vec{G}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}$

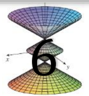


When u = 0: $x = 3 + \cos(v), y = 0$, and $z = \sin(v)$ which describes a circle with radius 1 in the xz-plane centered at (3,0,0). When u = .5, then y = 0 and we have circle with radius .5 in the xz-plane centered at (1.5, 0, 1.5).

When u = 1: $x = y = 0$ and $z = 3$, which gives the topmost point in the graph. This suggest that grid curves with u constant are the vertically oriented circles visible on the surface. The spiraling grid curves correspond to keeping v constant.

(Main Idea: Plug in 'LIMITING VALUES' to see how the points correspond/plot on the graph. (0,0), (u,0), (0,v) are always a good point to plug)

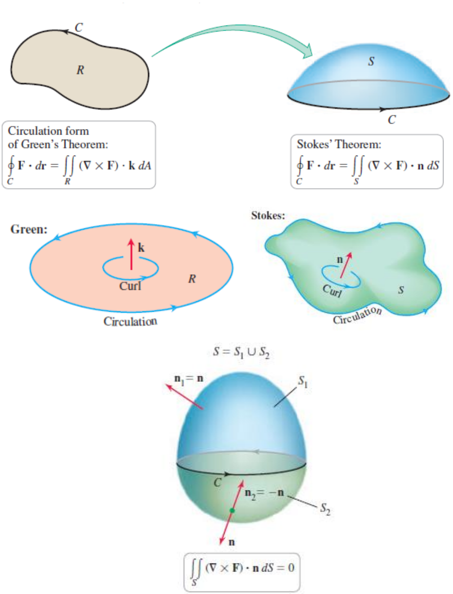
$\vec{G}(u,v) = (1-u)(3+\cos v)\cos 4\pi u\mathbf{i} + (1-u)(3+\cos v)\sin 4\pi u\mathbf{j} + (3u + (1-u)\sin v)\mathbf{k}$



Since $x^2 + y^2 = (1-|u|)^2$, so if u is held constant each grid curve is a circle of radius $(1-|u|)$ in the horizontal plane $z = u$. If v is held constant, so $v = v_0$, then $x = (1-|u|)\cos(v_0)$ and $y = (1-|u|)\sin(v_0)$. *Then* $y = \tan(v_0)x$, so the grid curves we see running vertically along the surface in the planes $y = kx$ correspond to keeping v constant.

(Main Idea: Often times components (x and y) may form shapes such as circles. Hold values constant and see corresponding traces)

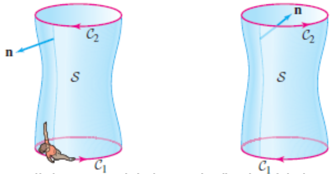
$\vec{G}(u,v) = (1-|u|)\cos v\mathbf{i} + (1-|u|)\sin v\mathbf{j} + u\mathbf{k}$



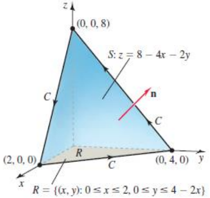
- The **double integral** side of Stokes' theorem represents *integrated circulation density*, so Stokes' theorem is just saying that integrating circulation density gives the total circulation
  - Relates the work done by a vector field along a closed curve to a double integral of the curl(**F**)
- Use **Stokes' theorem** when you see a 2D region bounded by a closed curve, which may contain a surface in 3D-space
- **Stokes' theorem** implies that if a surface S is closed, then the flow through the surface of the curl of any vector field is equal to **0**
- ∂S is the boundary of the surface
- **Green's Theorem** is a special case of Stokes' Theorem in the xy-plane
  - Stokes' Theorem generalizes circulation to 3D space

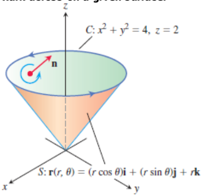
$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$$

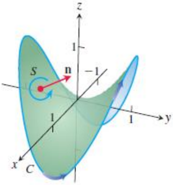
Stokes' Theorem (Orientation & Boundaries)

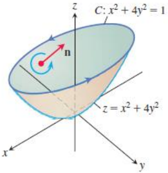


For a cylinder, top curve is in the opposite direction of the bottom curve, since we want the normal vector to point in the same direction (outward or inward normal) regardless of which curve we walk across on a given surface.

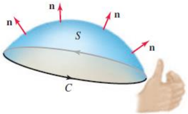


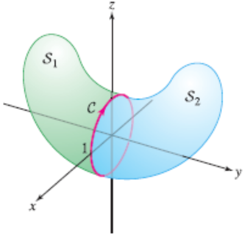


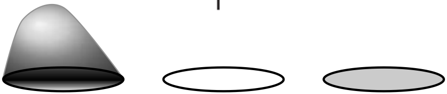




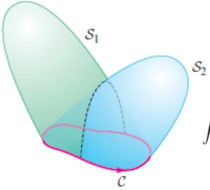
- The orientation of ∂D is the direction for which the surface is on your left as you walk around the boundary in that direction.
- The number of boundary curves will depend on the surface
- Orientation of a curve and surface must be compatible
- If we 'walk' along a curve positively and our surface is on the left, then we have a positive orientation.
- One can use the right-hand-rule where the thumb could be your normal vector to a surface (many variations) – visualize your thumb as your 'head' walking across a curve







$$\iint_R (\nabla \times \vec{\mathbf{F}}) \bullet \hat{\mathbf{n}}\, dS \;=\; \oint_{\partial R = \partial D} \vec{\mathbf{F}} \bullet d\vec{\mathbf{p}} \;=\; \iint_D (\nabla \times \vec{\mathbf{F}}) \bullet \hat{\mathbf{n}}\, dS$$

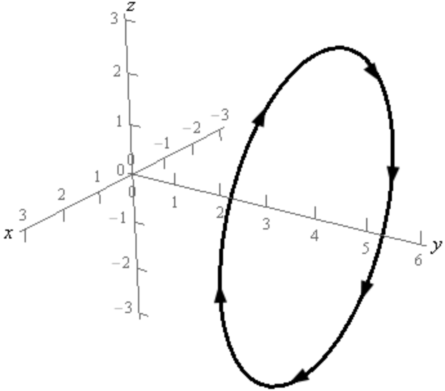


If $\mathbf{F} = \mathrm{curl}(\mathbf{A})$

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{r}$$

- A **scalar potential** has some function g such that $\vec{\nabla} g = \vec{g}$, where g is a scalar potential.
- Likewise, a **vector potential** has some vector $\vec{A}$ such that $curl(\vec{A}) = \vec{F}$, where $\vec{A}$ is a vector potential and $\vec{F}$ is our original vector field
    - The function g is <u>not unique</u>, as it can *differ by a constant* (say +C)
    - Similarly, vector $\vec{A}$ <u>is not unique</u>, as it can *differ by addition with another vector field* (say $\vec{B}$)
- Thus we can relate two different surfaces (S_1 & S_2), given the same boundary curve and $curl(\vec{A}) = \vec{F}$ (intuitively, the line integrals will be equal, and the flux/circulation will depend on the boundary, not the surface itself)

Use Stokes' Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = -yz\,\vec{i} + (4y+1)\,\vec{j} + xy\,\vec{k}$ and C is is the circle of radius 3 at $y = 4$ and perpendicular to the y -axis. C has a clockwise rotation if you are looking down the y -axis from the positive y -axis to the negative y -axis.

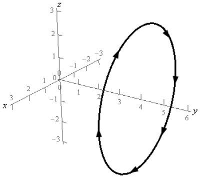


Stokes' Theorem (example 1)

Use Stokes' Theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = -yz\,\vec{i} + (4y+1)\,\vec{j} + xy\,\vec{k}$ and C is is the circle of radius 3 at $y = 4$ and perpendicular to the y -axis.

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -yz & 4y+1 & xy \end{vmatrix} = x\vec{i} - y\vec{j} + z\vec{k} - y\vec{j} = x\vec{i} - 2y\vec{j} + z\vec{k}$$



Plan of Attack:

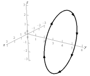
- Must find a surface, with a proper orientation, based on the boundary curve
- From the given information, the boundary must be a circle, so we need surfaces with circular cross-sections (Spheres, Cones, or Elliptic Paraboloids)
- **What is the best choice?**
  - A. Sphere
  - B. Cone
  - C. Elliptic Paraboloid
  - D. Cylinder

Stokes' Theorem (example 1)

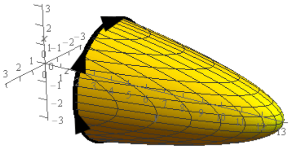
Use Stokes' Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = -yz\,\vec{i} + (4y+1)\,\vec{j} + xy\,\vec{k}$ and C is is the circle of radius 3 at $y = 4$ and perpendicular to the y -axis.

Recall: we must plug our chosen surface into the vector field and then take the dot product of this with normal vector (which will also come from the surface)

- **Sphere:**
  - *Grunge, grimy*, and *messy* calculations which may tend to many errors
- **Cone:**
  - Generally, these aren't difficult to compute but for this problem it requires that the boundary curve is in the y-axis so $y = \sqrt{ax^2 + bz^2}$. This will lead to a messy integrand
- **Elliptic Paraboloids**
  - Simple equations and simple normal vectors
- **Cylinder:**
  - Would be difficult since a cylinder, whether circular or not, requires two boundary curves.
- **Remarks:**
  - For some vector fields the curl may end up being very simple with a sphere or cone, so it depends on the scenario and problem



Use Stokes' Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = -yz\,\vec{i} + (4y+1)\,\vec{j} + xy\,\vec{k}$ and C is is the circle of radius 3 at $y = 4$ and perpendicular to the y -axis.



Opens in -y

Equation Derivation:

$$y = a - x^2 - z^2$$
Since $y = 4$
$$\implies 4 = a - x^2 - z^2$$
$$\implies x^2 + z^2 = a - 4 = 9$$
$$\implies a = 13$$
$$y = 13 - x^2 - z^2$$

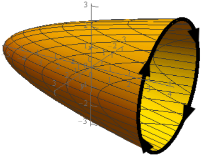
Two Possible Choices:

With Constraints:

- $y = 4$
- $x^2 + z^2 = 9$

Either choice could work. However, we must correctly pick a normal vector for either surface. What gives normal vectors to a surface? The gradient vector does!

For simplicity, we will pick the surface on the left opening in the –y direction so that we are moving inside of the cup/bowl/surface. Thus, the normal vector will point in towards the region enclosed by the surface (the normal vector will need a negative y component). Meanwhile, x and z can be either negative or positive depending on our location on the surface's interior



Opens in +y

Equation Derivation:

$$y = x^2 + y^2 + a$$
Since $y = 4$
$$\implies 4 = x^2 + z^2 + a$$
$$\implies x^2 + z^2 = 4 - a = 9$$
$$\implies a = -5$$
$$y = x^2 + z^2 - 5$$

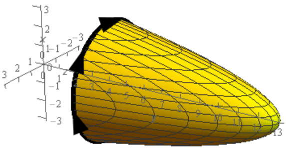
Stokes' Theorem (example 1)

Use Stokes' Theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = -yz\,\vec{i} + (4y+1)\,\vec{j} + xy\,\vec{k}$ and C is is the circle of radius 3 at $y = 4$ and perpendicular to the y -axis.

$$f(x,y,z) = 13 - x^2 - z^2 - y = 0$$

$$\operatorname{curl}\vec{F} = x\vec{i} - 2y\vec{j} + z\vec{k}$$

$$\vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle -2x, -1, -2z\rangle}{\|\nabla f\|}$$



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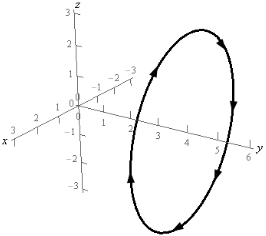


$$\operatorname{curl}\vec{F}\cdot\vec{n} = \left\langle x, -2\left(13 - x^2 - z^2\right), z\right\rangle \cdot \frac{\langle -2x, -1, -2z\rangle}{\|\nabla f\|}$$

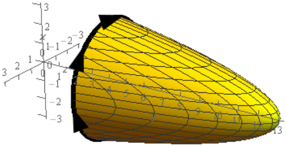
$$= \frac{1}{\|\nabla f\|}\left(-2x^2 + 2\left(13 - x^2 - z^2\right) - 2z^2\right)$$

$$= \frac{1}{\|\nabla f\|}\left(26 - 4x^2 - 4z^2\right)$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

Use Stokes' Theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = -yz\,\vec{i} + (4y+1)\,\vec{j} + xy\,\vec{k}$ and C is is the circle of radius 3 at $y=4$ and perpendicular to the y-axis.

Apply Stokes': $\int_C \vec{F}\cdot d\vec{r} = \iint_S \mathrm{curl}\vec{F}\cdot d\vec{S}$

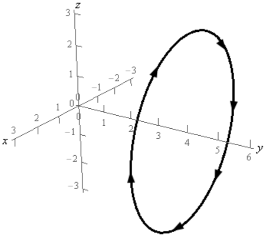


Opens in -y

$$= \iint_S \frac{1}{\|\nabla f\|}\left(26 - 4x^2 - 4z^2\right) dS$$

$$= \iint_D \frac{1}{\|\nabla f\|}\left(26 - 4x^2 - 4z^2\right) \|\nabla f\|\, dA$$

$$= \iint_D 26 - 4x^2 - 4z^2 dA$$



$x = r\cos\theta \qquad z = r\sin\theta \qquad x^2+z^2=r^2$

$0 \le \theta \le 2\pi$

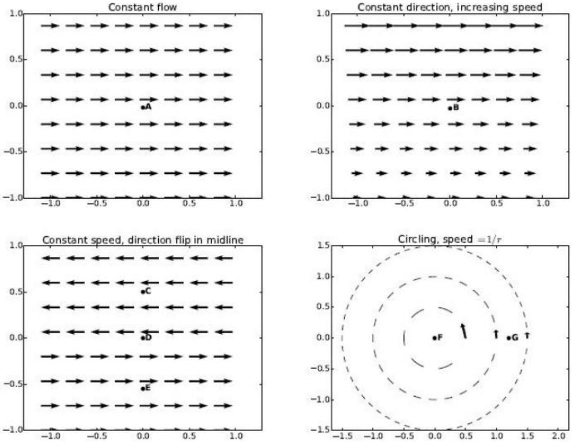
$0 \le r \le 3$

$$\iint_S \mathrm{curl}\,\vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^3 \left(26-4r^2\right) r\,dr\,d\theta = 72\pi$$

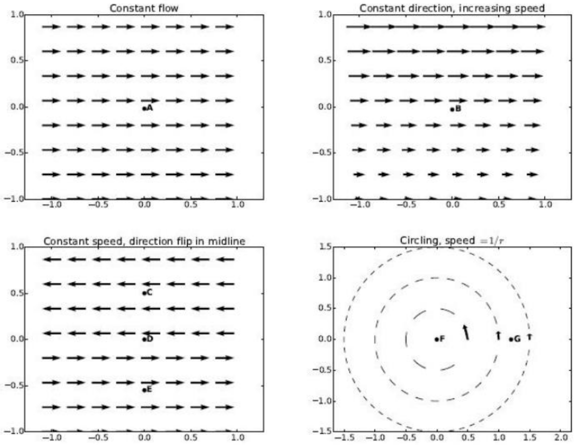
Lecture Question

The following four plots show vector fields or flows. It is assumed that the vector field has no z-component and that the flow is the same in all planes parallel to the x-y plane. Therefore the only component of the curl that can be nonzero is the z-component.

The z-axis is perpendicular to the plane of the paper and pointing towards the sky. For each of the **points A to G**, figure out if the **z-component of the curl** is **negative, zero, or positive**. Read the titles of each plot carefully. It is important that D is located exactly on the midline, where the flow flips direction. Likewise, it is significant that F is the center.



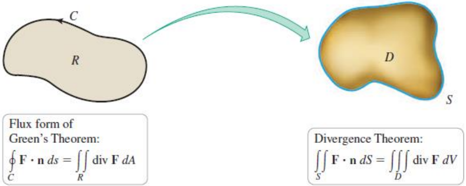
Lecture Question

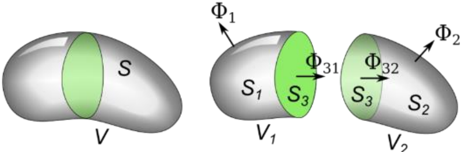


A: zero.
B: negative.
C: zero.
D: positive and infinite.
E: zero.
F: positive and infinite.
G: zero.

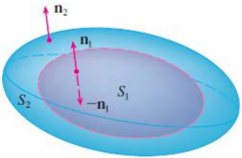
Note: For these types of problems, it is important to consider loops of an appropriate shape and think in terms of Stokes' theorem. One should estimate curl or divergence by looking at circulation and flux for tiny, convenient regions.

Divergence Theorem





$$\Phi(V_1) + \Phi(V_2) = \Phi_1 + \Phi_{31} + \Phi_2 + \Phi_{32}$$
$$\Phi(V_1) + \Phi(V_2) = \Phi(V)$$



$$\iiint_W \operatorname{div}(\mathbf{F})dV = -\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{S}$$

- Generalizes Green's Theorem for flux (vector form) to 3D-space
  - closed curve → oriented closed surface
  - Flux line integral & double integral → surface integral & triple integral
- The **triple integral** side of Divergence theorem represents the *cumulative expansion/contraction* of the solid region
- The **surface integral** side of Divergence theorem represents *the flux of the vector field* across the boundary (i.e. surface)
  - Outer boundary (surface) = solid region (volume)
- Use **Divergence theorem** when you see a closed surface in 3D-space which encloses a volume (solid region)
- Can extend to "*hollow-regions*"
- **Divergence** of a curl is always <u>zero</u>
  - $div(curl(F)) = \nabla \cdot (\nabla \times F) = 0$

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_W \operatorname{div}(\mathbf{F})dV$$

Divergence Theorem (confusion)

Question:

When verifying the divergence theorem in this question, we had to consider the two pieces of the disc and the paraboloid itself in our surface integral. *When do we have to calculate the two pieces of the boundary?* There was a similar question to this one (surfaces review #2) and I tried calculating the two pieces of the boundary, but the answer was just the paraboloid. I'm a little confused about boundaries, so I'd really appreciate some *clarification* **on when** we need to split up the boundary.

Answer:

I believe this depends on if you're dealing with an integral over a *surface* or the *integral over a volume*. If you have a surface integral and they describe the paraboloid as the *surface*, then you just need to integrate over the paraboloid. On the other hand, question 8 is about divergence theorem. When you go from the *triple integral* to the *surface integral*, you are not just interpreting the same equation as a surface instead if a **volume.** You need to think about every boundary of the solid (Prof … analogy was thinking about a mole poking its head out of a contained solid of dirt). The *paraboloid* and *circle* are both boundaries of the solid. Does that help?

~ An instructor (Reyer Sjamaar) endorsed this answer ~

Divergence Theorem (example 1)

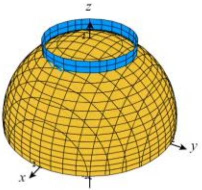
Let T be the solid which consists of the points satisfying

$$z \geq 0, \quad x^2 + y^2 \geq 1, \quad \text{and} \quad x^2 + y^2 + z^2 \leq 4$$

$$I = \iint_{\partial T} \left\langle e^{-x} + xz, \arctan(x+z), ze^{-x} + \sin\left(y^2\right) \right\rangle \cdot \mathbf{n} dS$$

Find the following flux across the boundary of the solid T,

(this is the top half of a sphere of radius 2 which has a hole drilled through the middle)



Let T be the solid which consists of the points satisfying

$$z \geq 0, \quad x^2 + y^2 \geq 1, \quad \text{and} \quad x^2 + y^2 + z^2 \leq 4$$

$$I = \iint_{\partial T} \left\langle e^{-x} + xz, \arctan(x+z), ze^{-x} + \sin\left(y^2\right) \right\rangle \cdot \mathbf{n} dS$$

Find the following flux across the boundary of the solid T,

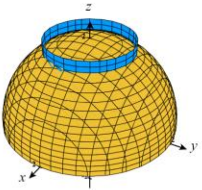
(this is the top half of a sphere of radius 2 which has a hole drilled through the middle)

$$I = \iiint_T \left(\frac{\partial}{\partial x}\left(e^{-x} + xz\right) + \frac{\partial}{\partial y}\left(\arctan(x+z)\right) + \frac{\partial}{\partial z}\left(ze^{-x} + \sin\left(y^2\right)\right) \right) \mathrm{d}V$$
$$= \iiint_T \left(\left(-e^{-x} + z\right) + (0) + \left(e^{-x} + 0\right)\right) \mathrm{d}V = \iiint_T z \, \mathrm{d}V$$

Use cylindrical coordinates. This is symmetrical around the z!

$\implies 0 \leq \theta \leq 2\pi \implies 1 \leq r \leq 2$ (drilled hole to outer radius of sphere)

$\implies z^2 = 4 - \left(x^2 + y^2\right) = 4 - r^2 \implies 0 \leq z \leq \sqrt{4 - r^2}$



$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W \operatorname{div}(\mathbf{F}) dV$$

Divergence Theorem (example 1)

Let T be the solid which consists of the points satisfying

$$z \geq 0, \quad x^2 + y^2 \geq 1, \quad \text{and} \quad x^2 + y^2 + z^2 \leq 4$$

$$I = \iint_{\partial T} \left\langle e^{-x} + xz, \arctan(x+z), ze^{-x} + \sin\left(y^2\right) \right\rangle \cdot \mathbf{n} dS$$

Find the following flux across the boundary of the solid T,

(this is the top half of a sphere of radius 2 which has a hole drilled through the middle)

$$\iiint_T z\,\mathrm{dV} = \int_0^{2\pi} \int_1^2 \int_0^{\sqrt{4-\mathrm{r}^2}} \mathrm{zrdzdrd}\theta$$

$$= \int_0^{2\pi} \int_1^2 \frac{1}{2}\mathrm{z}^2\mathrm{r}\Big|_{z=0}^{z=\sqrt{4-\mathrm{r}^2}} \mathrm{drd}\theta$$

$$= \int_0^{2\pi} \int_1^2 \frac{1}{2}\mathrm{r}\left(4 - \mathrm{r}^2\right) \mathrm{drd}\theta$$

$$= \int_0^{2\pi} \int_1^2 \left(2\mathrm{r} - \frac{1}{2}\mathrm{r}^3\right) \mathrm{drd}\theta$$

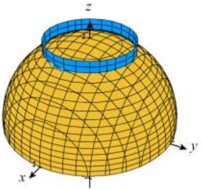
$$= \int_0^{2\pi} \left(\mathrm{r}^2 - \frac{1}{8}\mathrm{r}^4\right)\Big|_{\mathrm{r}=1}^{\mathrm{r}=2} \mathrm{d}\theta$$

$$= \int_0^{2\pi} \left((4-2) - \left(1 - \frac{1}{8}\right)\right) \mathrm{d}\theta$$

$$= \frac{9}{8}\theta\Big|_{\theta=0}^{\theta=2\pi}$$

$$= \frac{9}{8}(2\pi)$$

$$= \boxed{\frac{9}{4}\pi}$$

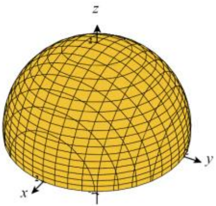


Note: the word "solid" can indicate usage of divergence theorem (pay attention to it!)

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F} = \left(z^2 x + e^{z^2 - y^2}\right)\mathbf{i} + \left(\frac{1}{3}y^3 + x^2 y + \sin\left(z + x^2\right)\right)\mathbf{j} + x^2\mathbf{k}$$

and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$ oriented upward.



Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

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and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$ oriented upward.

S is not closed since the disc $x^2 + y^2 \leq 1, z = 0$ is not part of S.

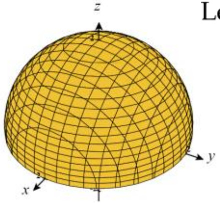
Let D be this disc, oriented downwards.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \underbrace{\iint_{S \cup D} \mathbf{F} \cdot d\mathbf{S}}_{(1)} - \underbrace{\iint_D \mathbf{F} \cdot d\mathbf{S}}_{(2)}$$

In (1), the surface $S \cup D$ is closed and hence a prime target for the divergence theorem.

$$\operatorname{div} \mathbf{F} = z^2 + x^2 + y^2$$

Let E be the half-ball enclosed by $S \cup D$; then,



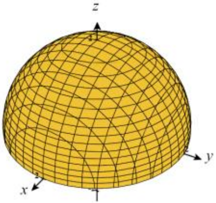
Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where

$$\mathbf{F} = \left(z^2x + e^{z^2-y^2}\right)\mathbf{i} + \left(\frac{1}{3}y^3 + x^2y + \sin\left(z+x^2\right)\right)\mathbf{j} + x^2\mathbf{k}$$

and S is the top half of the sphere $x^2+y^2+z^2=1$ oriented upward.

Let E be the half-ball enclosed by $S \cup D$; then,

$$\begin{aligned}\iint_{S\cup D} \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,dV\\ &= \iiint_E x^2+y^2+z^2\,dV\\ &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \rho^2\rho^2\sin(\phi)\,d\rho\, d\theta\, d\phi\\ &= \int_0^{\pi/2}\sin(\phi)\,d\phi\int_0^{2\pi}d\theta\int_0^1\rho^4\,d\rho\\ &= (1)(2\pi)(1/5) = \frac{2\pi}{5}\end{aligned}$$



Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where

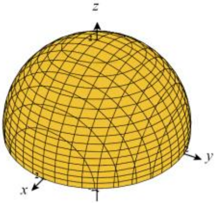
$$\mathbf{F} = \left(z^2x + e^{z^2-y^2}\right)\mathbf{i} + \left(\frac{1}{3}y^3 + x^2y + \sin\left(z+x^2\right)\right)\mathbf{j} + x^2\mathbf{k}$$

and S is the top half of the sphere $x^2+y^2+z^2=1$ oriented upward.
S is not closed since the disc $x^2+y^2\le 1, z=0$ is not part of S.
We compute (2) explicitly. The normal vector pointing downwards is $\langle 0,0,-1\rangle$

Graph of $z=g(x,y)$: $\begin{array}{l} G(x,y) = (x,y,g(x,y)) \\ \mathbf{N} = \mathbf{T}_x \times \mathbf{T}_y = \langle -g_x, -g_y, 1\rangle \end{array}$

Since a disk: $g(x,y)=z=0 \implies \nabla g = \langle 0,0,-1\rangle$ (two possible normals)

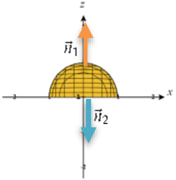
$$\iint_D \mathbf{F}\cdot d\mathbf{S} = \iint_D \left\langle e^{-y^2}, y^3/3 + x^2y + \sin\left(x^2\right), x^2\right\rangle \cdot \langle 0,0,-1\rangle dA$$



$$= \iint_D -x^2 dA$$

$$= \int_0^1\int_0^{2\pi} -r^3\cos^2(\theta)d\theta dr$$

$$= -\frac{\pi}{4}$$



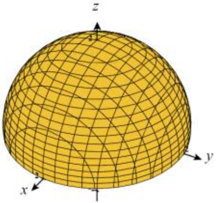
Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where

$$\mathbf{F} = \left(z^2x + e^{z^2-y^2}\right)\mathbf{i} + \left(\frac{1}{3}y^3 + x^2y + \sin\left(z+x^2\right)\right)\mathbf{j} + x^2\mathbf{k}$$

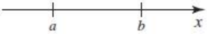
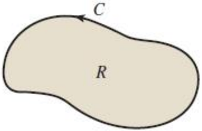
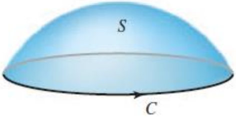
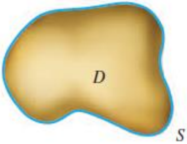
and S is the top half of the sphere $x^2+y^2+z^2=1$ oriented upward.

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \underbrace{\iint_{S\cup D} \mathbf{F}\cdot d\mathbf{S}}_{(1)} - \underbrace{\iint_D \mathbf{F}\cdot d\mathbf{S}}_{(2)}$$

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \frac{2\pi}{5} + \frac{\pi}{4} = \boxed{\frac{13}{20}\pi}$$



Moral of the Story: Pay attention to the problem statement and application of divergence theorem. *If not closed*, ensure the surface becomes closed (if one desires to use this theorem, considering the other theorems applicable). *If a closed solid region*, apply divergence theorem properly.

Fundamental Theorem of Calculus	$\int_a^b f'(x)\,dx = f(b) - f(a)$	
Fundamental Theorem for Line Integrals	$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$	
Green's Theorem (Circulation form)	$\iint_R (g_x - f_y)\,dA = \oint_C f\,dx + g\,dy$	
Stokes' Theorem	$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$	
Divergence Theorem	$\iiint_D \nabla \cdot \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot \mathbf{n}\,dS$	

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