Classication of SPD matrices using a Riemannian-based kernel

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Motivation

- Structural connectome can be seen as symmetric adjacency matrix

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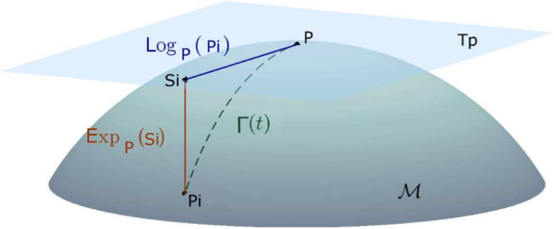
- Take into account geometry of the data

Basic notations

- $\mathcal{M}(n)$ — space of $n \times n$ real matrices
- $\{S \in \mathcal{M}(n) : S^T = S\}$ — space of $n \times n$ symmetric matrices
- $\{P \in S(n) : u^T P u > 0 \; \forall u \in \mathbb{R}^n\}$ — space of $n \times n$ SPD matrices
- $\exp(S) = C \operatorname{diag}(\exp(\lambda_1), \dots, \exp(\lambda_n)) \; C^T \in P(n)$
- $\log(P) = C \operatorname{diag}(\log(\lambda_1), \dots, \log(\lambda_n)) \; C^T \in S(n)$
- $P^{\frac{1}{2}}$ — symmetric matrix A, s.t. $AA = P$

Riemannian manifold

- Smooth manifold
- Tangent finite-dimensional Euclidean space at every point with inner product g_P
- g_P, Riemannian metric, varies smoothly from point to point



Exponential map

Exponential mapping

$$\exp_P(S_i) = P_i = P^{\frac{1}{2}} \exp(P^{-\frac{1}{2}} S_i P^{-\frac{1}{2}}) P^{\frac{1}{2}}$$

Logarithmic mapping

$$\log_P(P_i) = S_i = P^{\frac{1}{2}} \log(P^{-\frac{1}{2}} P_i P^{-\frac{1}{2}}) P^{\frac{1}{2}}$$

Riemannian metrics

Inner product

$$\langle S_1, S_2 \rangle_P = \mathrm{Tr}(S_1 P^{-1} S_2 P^{-1})$$

Norm

$$\|S\|_P^2 = \mathrm{Tr}(S P^{-1} S P^{-1})$$

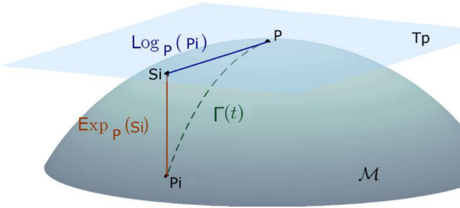
$P = I$:

$$\|S\|_I^2 = \mathrm{Tr}(S^2) = \mathrm{Tr}(S^T S) = \|S\|_F^2 = \|\mathrm{vect}(S)\|_2^2$$

$$\mathrm{vect}(S) = \left[S_{1,1}, \sqrt{2} S_{1,2}, S_{2,2}, \sqrt{2} S_{1,3}, \sqrt{2} S_{2,3}, S_{3,3}, \ldots, S_{n,n} \right]^T$$

Riemannian Geodesic distance

The geodesic — shortest path between two points
Riemannian distance — length of the geodesic



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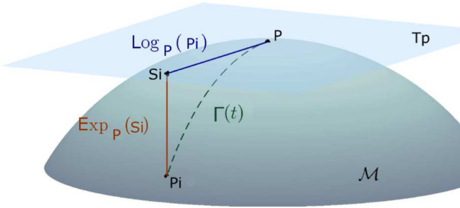
$$\Gamma(t) : [0,1] \to P(n)$$

$$L(\Gamma(t)) = \int_0^1 \|\dot{\Gamma(t)}\|_{\Gamma(t)} dt$$

S_i can be seen as derivative of the geodesic between P and P_i at point $t = 1$



Riemannian distance

$$\sigma_R(P_1, P_2) = \|\log(P_1^{-1}P_2)\|_F = \left(\sum \log^2 \lambda_i\right)^{\frac{1}{2}}$$

Classification in the Riemannian manifold

- Minimum distance to Riemanian Mean

Geometric mean

$$P^* = \arg\min_{P} \sum_{i=1}^{m} \sigma_R^2(P, P_i)$$

- kNN with Riemannian distances

Kernel Approach

Mapping function

$$\phi(P_i) = \log_{P^*}(P_i)$$

Riemannian base kernel

$$k_R(P_i, P_j) = \langle \phi(P_i), \phi(P_j) \rangle_{P^*} = \mathrm{Tr}(\log_{P^*}(P_i) P^{*-1} \log_{P^*}(P_j) P^{*-1}) =$$

$$= \mathrm{Tr}(\log(P^{*-\frac{1}{2}} P_i P^{*-\frac{1}{2}}) \log(P^{*-\frac{1}{2}} P_j P^{*-\frac{1}{2}})) = \mathrm{Tr}(\tilde{P}_i \tilde{P}_j) =$$

$$= \langle \tilde{P}_i, \tilde{P}_j \rangle_F = \mathrm{vect}(\tilde{P}_i)^T \mathrm{vect}(\tilde{P}_j) = \langle \mathrm{vect}(\tilde{P}_i), \mathrm{vect}(\tilde{P}_j) \rangle_2$$

Reference point

$$k_R(P_i, P_j) = \mathrm{Tr}(\log(P^{*-\frac{1}{2}} P_i P^{*-\frac{1}{2}}) \log(P^{*-\frac{1}{2}} P_j P^{*-\frac{1}{2}})) = \mathrm{Tr}(\tilde{P}_i \tilde{P}_j)$$

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log-Euclidean kernel

$$P^* = I$$

$$k_{LE}(P_i, P_j) = \mathrm{Tr}(\log(P_i) \log(P_j))$$

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Geometric mean

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From SI to SPD

1. Laplacian of the SI adjacency matrix — SPSD matrix

$$L_S = D_S - S$$

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$$L_S = D_S - S$$

2. Add regularization

$$P = L_S + \lambda I, \ \lambda > 0$$

References

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The End