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Mathematik – Übung

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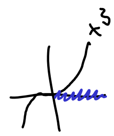
Grundlagen zu Ableitungen und Integralen

* **Ableitungen.** Geben Sie für die folgenden Funktionen den Definitionsbereich und die Ableitung an:

1. $f(x) = \sqrt[4]{x^3}$ Wegen $\sqrt[4]{\ldots}$: $x^3 \geq 0$ 

für $x \geq 0$: $D = [0, \infty[= \mathbb{R}_0^+$

$f'(x) = \left(x^{\frac{3}{4}}\right)' = \frac{3}{4} x^{\frac{3}{4}-1} = \frac{3}{4} x^{-\frac{1}{4}} = \frac{3}{4} \frac{1}{\sqrt[4]{x}}$.

2. $f(x) = \frac{x^2}{\sqrt[3]{x}}$ $D = \mathbb{R} \setminus \{0\}$ $\left(f(x) = x^{2-\frac{1}{3}} = x^{\frac{5}{3}} = \sqrt[3]{x^5} \quad D = \mathbb{R} \right)$

$f'(x) = \left(x^{\frac{5}{3}}\right)' = \frac{5}{3} x^{\frac{5}{3}-1} = \frac{5}{3} x^{\frac{2}{3}} = \frac{5}{3} \sqrt[3]{x^2}$.

3. $f(x) = -10x^4 + 2x^3 - 2$ Polynom $D = \mathbb{R}$

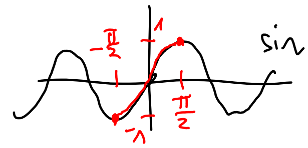
$f'(x) = -40x^3 + 6x^2$

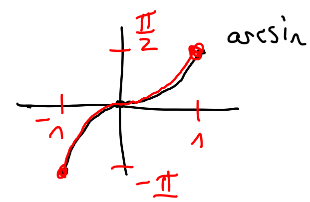
4. $h(x) = a\cos(x) - x^2 + e^x + 1$, $D = \mathbb{R}$
Konstante/feste Zahl

$h'(x) = a(-\sin x) - 2x + e^x = -a\sin x - 2x + e^x$

5. $f(x) = 2x \cdot \ln x$ $D = \mathbb{R}^+ =]0, \infty[$
Produktregel

$f'(x) = 2 \cdot \ln x + 2x \cdot \frac{1}{x} = 2\ln x + 2$

6. $f(x) = x^2 \arcsin(x)$ $D = [-1, 1]$ 



Produktregel

$f'(x) = 2x \cdot \arcsin(x) + x^2 \cdot \underbrace{(\arcsin(x))'}_{\frac{1}{\sqrt{1-x^2}}}$ ($\sim$ Vorlesung)

7. $f(x) = \sin x \cos x$ $\quad D = \mathbb{R}$

Produktregel

$$f'(x) \overset{\text{Produktregel}}{=} \cos x \cos x + \sin x\,(-\sin x) = \cos^2 x - \sin^2 x$$

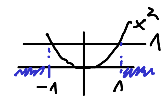
8. $f(x) = \frac{5x^2-6x+1}{x^2+2x+1}$ $\quad$ NST Nenner: $x^2+2x+1 = (x+1)^2$ NST $x_{1,2} = -1$, $D = \mathbb{R}\setminus\{-1\}$

Quotientenregel

$$f'(x) \overset{\text{Quotientenregel}}{=} \frac{\underbrace{(x^2+2x+1)}_{(x+1)^2}\cdot(10x-6) - (5x^2-6x+1)\underbrace{(2x+2)}_{2\cdot(x+1)}}{\underbrace{(x^2+2x+1)^2}_{(x+1)^4}} = \frac{\cancel{(x+1)}\cdot\left[(x+1)(10x-6) - \overbrace{(5x^2-6x+1)}^{5(x-\frac{6}{5})(x-1)}\cdot 2\right]}{(x+1)^{\cancel{4}\,3}} = \ldots = \frac{16x-8}{(x+1)^3}$$

9. $f(x) = \arccos(\sqrt{x^2-1})$: $\underbrace{-1 \le \sqrt{x^2-1}}_{\text{erfüllt!}} \le 1$ $\quad \& \quad \underbrace{x^2-1 \ge 0}_{}$

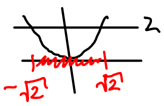
$\underbrace{\sqrt{x^2-1}}_{(*)} \ge 0$



$x^2 \ge 1 \iff x \le -1 \vee x \ge 1 \iff \underbrace{|x| \ge 1}_{(*)}$

Bleibt noch: $\sqrt{x^2-1} \le 1 \overset{()^2}{\iff} x^2-1 \le 1 \iff x^2 \le 2$

$\sqrt{x^2-1} \ge 0$, $1 \ge 0$ str. mo. wa. auf $[0,\infty[$



$-\sqrt{2}$ $\quad \sqrt{2}$

Insgesamt:

$$D = [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$$

$$f'(x) = \left(\arccos(\sqrt{x^2-1})\right)' \overset{\text{Kettenregel}}{=} -\frac{1}{\sqrt{1-(\sqrt{x^2-1})^2}}\,\underbrace{(\sqrt{x^2-1})'}_{\frac{1}{\cancel{2}\sqrt{x^2-1}}(x^2-1)' \;=\; \frac{1}{\cancel{2}\sqrt{x^2-1}}\cancel{2}x}$$

$$= -\frac{x}{\underbrace{\sqrt{1-x^2+1}}_{2}\,\sqrt{x^2-1}}.$$

5^5, $\left(\frac{1}{2}\right)^{\frac{1}{2}} = \sqrt{\frac{1}{2}}$ ✓

0^0 ↯

-3^{-3}

$\left(-\frac{1}{2}\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{-\frac{1}{2}}}$ ↯

10. $f(x) = x^x = \begin{cases} e^{x\ln x} & x>0 \\ 0^0 \text{ ↯} & x=0 \\ \text{i.A. nicht definiert ↯} & \end{cases}$ $\quad \lim_{x\to 0} e^{x\ln x} = e^0 = 1$ stetig fortsetzbar $\quad D = \mathbb{R}^+$

$$f'(x) = (e^{x\ln x})' = \underbrace{e^{x\ln x}}_{x^x}\,\underbrace{(x\ln x)'}_{\ln x + x\cdot\frac{1}{x}} = x^x(\ln x + 1)$$

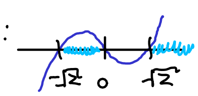
11. $f(x) = (\frac{x}{1+x})^n \quad (n\in\mathbb{N})$

$D = \mathbb{R}\setminus\{-1\}$

$$f'(x) = \left(\left(\frac{x}{1+x}\right)^n\right)' \overset{\text{Kettenregel}}{=} n\left(\frac{x}{1+x}\right)^{n-1}\underbrace{\left(\frac{x}{1+x}\right)'}_{\frac{(1+x)-x}{(1+x)^2}} = n\left(\frac{x}{1+x}\right)^{n-1}\cdot\frac{1}{(1+x)^2}$$

12. $f(x) = 2\ln(x^3 - 2x)$

$\underbrace{x^3-2x}_{x(x^2-2)} > 0$ NST $0, \pm\sqrt{2}$:



$-\sqrt{2}$ $\quad 0 \quad$ $\sqrt{2}$

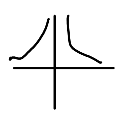
$D = \;]-\sqrt{2}, 0[\; \cup \;]\sqrt{2}, \infty[$

$$f'(x) \overset{\text{Kettenregel}}{=} 2\cdot\frac{1}{x^3-2x}(x^3-2x)' = 2\cdot\frac{3x^2-2}{x^3-2x}.$$

13. $f(x) = 5e^{-x^2}$ $\quad D = \mathbb{R}$

$$f'(x) \overset{\text{Kettenregel}}{=} 5\cdot e^{-x^2}\underbrace{(-x^2)'}_{(-2x)} = -10x\cdot e^{-x^2}$$

14. $x(t) = \ln(\underbrace{\tfrac{1}{t^2}}_{>0}) + \ln(\underbrace{\tfrac{t+4}{t}}_{>0})$ $\quad \frac{1}{t^2} > 0$



✓

$$\frac{t+4}{t} > 0 \iff \begin{cases} \cdot\, t>0 \\ \iff t+4>0 \iff t>-4 : t>0 \\ \cdot\, t<0 \\ \iff t+4<0 \iff t<-4 : t<-4 \end{cases}$$

$D = \;]-\infty, -4[\; \cup \;]0, \infty[$

$$x'(t) = \frac{1}{\frac{1}{t^2}}\underbrace{\left(\frac{1}{t^2}\right)'}_{(t^{-2})' = -2\cdot t^{-3}} + \frac{1}{\frac{t+4}{t}}\underbrace{\left(\frac{t+4}{t}\right)'}_{\frac{t-(t+4)}{t^2}} = -2\frac{t^2}{t^3} + \frac{t}{t+4}\cdot\frac{-4}{t^2}$$

$$= -\frac{2}{t} - \frac{4}{(t+4)\cdot t}$$

ODER: $\ln\left(\frac{1}{t^2}\right) = \ln\left(t^{-2}\right) = -2\ln t \xrightarrow{\text{Ableiten}} -\frac{2}{t}$

ODER: $\ln(A) + \ln(B) = \ln(A \cdot B) \xrightarrow{\text{Ableiten}} \ldots$

Eigener Lösungsversuch.

1. $f(x) = \sqrt[4]{x^3}$

2. $f(x) = \frac{x^2}{\sqrt[3]{x}}$

3. $f(x) = -10x^4 + 2x^3 - 2$

4. $h(x) = a\cos(x) - x^2 + e^x + 1$

5. $f(x) = 2x \ln x$

6. $f(x) = x^2 \arcsin(x)$

7. $f(x) = \sin x \cos x$

8. $f(x) = \frac{5x^2-6x+1}{x^2+2x+1}$

9. $f(x) = \arccos(\sqrt{x^2 - 1})$

10. $f(x) = x^x$

11. $f(x) = (\frac{x}{1+x})^n \quad (n \in \mathbb{N})$

12. $f(x) = 2\ln(x^3 - 2x)$

13. $f(x) = 5e^{-x^2}$

14. $x(t) = \ln(\frac{1}{t^2}) + \ln(\frac{t+4}{t})$

Zweite Ableitungen. Berechnen Sie die zweiten Ableitungen:

1. $f(x) = x \cdot \sin x$

   $f'(x) = 1 \cdot \sin x + x \cdot \cos x$

   $f''(x) = \cos x + 1 \cdot \cos x + x\,(-\sin x) = 2\cos x - x \cdot \sin x.$

2. $f(x) = x^2 + \ln x$

   $f'(x) = 2x + \frac{1}{x}$

   $\left(\frac{1}{x}\right)' = \left(x^{-1}\right)' = -x^{-2} = -\frac{1}{x^2}.$

   $f''(x) = 2 - \frac{1}{x^2}$

3. $f(x) = \cos(x^2)$

   $f'(x) = -\sin(x^2) \cdot 2x = -2x \cdot \sin(x^2)$

   $f''(x) = -2\sin(x^2) + (-2x) \cdot \underbrace{\left(\sin(x^2)\right)'}_{\cos(x^2) \cdot 2x} = -2\sin(x^2) - 4x^2\cos(x^2).$

Eigener Lösungsversuch.

1. $f(x) = x \sin x$

2. $f(x) = x^2 + \ln x$

3. $f(x) = \cos(x^2)$

Polynom dritten Grades. Gesucht ist ein Polynom $p(x)$ mit

$$p(3) = 4, \quad p'(3) = -2, \quad p''(3) = 7, \quad p'''(3) = -3.$$

Lösung.

Ansatz: $p(x) = ax^3 + bx^2 + cx + d$

$p'(x) = 3ax^2 + 2bx + c$

$p''(x) = 6ax + 2b$

$p'''(x) = 6a$

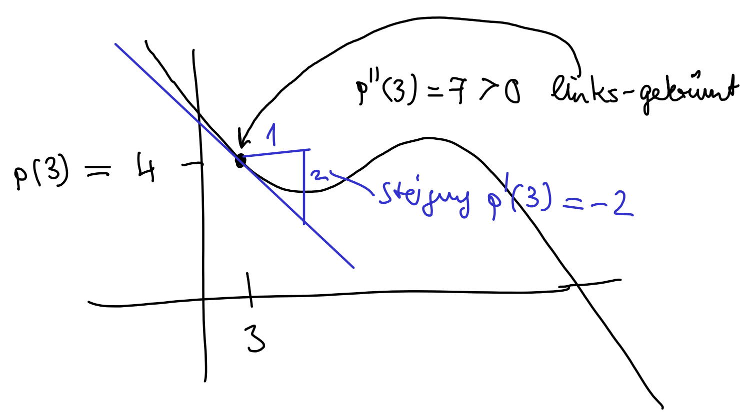
$p(3) = 27a + 9b + 3c + d \overset{!}{=} 4 \Rightarrow d = 55$

$p'(3) = 27a + 6b + c \overset{!}{=} -2 \Rightarrow c = -\frac{73}{2}$

$p''(3) = 18a + 2b \overset{!}{=} 7 \Rightarrow b = 8$

$p'''(3) = 6a \overset{!}{=} -3 \Rightarrow a = -\frac{1}{2}$

$$\Rightarrow p(x) = -\frac{1}{2}x^3 + 8x^2 - \frac{73}{2}x + 55$$



Eigener Lösungsversuch.

* **Integrale.** Berechnen Sie:

Unbestimmte Integrale

1. $\int 3\cos(x)dx = 3\sin(x) + c$ mit $c \in \mathbb{R}$ ← Menge aller Stammfkten = Unbestimmtes Integral

$$\boxed{\int x^n\,dx \overset{*}{=} \begin{cases} \frac{x^{n+1}}{n+1} & n \neq -1 \\ \ln|x| & n = -1 \end{cases}}$$ $n \in \mathbb{R}$!

2. $\int \frac{2}{x^2}dx = 2\int x^{-2}\,dx \overset{*}{=} 2\cdot\frac{x^{-2+1}}{-2+1} + c = 2\cdot\frac{x^{-1}}{-1} + c = -2\cdot\frac{1}{x} + c$

3. $\int 2x^3 + 4x dx \overset{*}{=} 2\frac{x^4}{4} + 4\frac{x^2}{2} + c = \frac{1}{2}x^4 + 2x^2 + c$

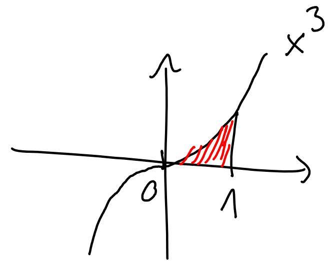
4. $\int e^t - 3t^2 dt \overset{*}{=} e^t - 3\frac{t^3}{3} + c = e^t - t^3 + c$

$\int e^t dt = e^t + c$

5. $\int \frac{x^2\cdot\sqrt{x}}{\sqrt[3]{5}}dx = \frac{1}{\sqrt[3]{5}}\cdot\int \underbrace{x^{2+\frac{1}{2}}}_{x^{\frac{5}{2}}}\,dx \overset{*}{=} \frac{1}{\sqrt[3]{5}}\,\frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + c = \frac{1}{\sqrt[3]{5}}\,\frac{x^{\frac{7}{2}}}{\frac{7}{2}} = \frac{2}{7\cdot\sqrt[3]{5}}\cdot\sqrt{x^7} + c$

bestimmte Integrale = Fläche unter dem Graphen

6. $\int_0^1 x^3 dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4}$



7. $\int_0^1 (e^t + 2t)dt = \left[e^t + t^2\right]_0^1 = (e^1 + 1^2) - (\underbrace{e^0 + 0^2}_{1}) = e \approx 2{,}71\ldots$

8. $\int_0^2 \frac{x^2}{1+x^3}dx = \frac{1}{3}\int_0^2 \frac{1}{1+x^3}\,3x^2\,dx \overset{\text{Kettenregel}}{\underset{\text{Substitutionsregel (→ später)}}{=}} \left[\frac{1}{3}\cdot\ln|1+x^3|\right]_0^2 = \frac{1}{3}\Big(\ln\underbrace{|1+2^3|}_{9} - \ln\underbrace{|1+0^3|}_{0}\Big) = \frac{1}{3}\ln 9$

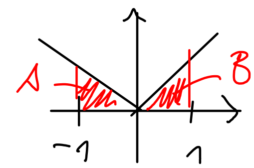
⋮

Hier Spezialfall der Substitutionsregel: Logarithmische Integration $\boxed{\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c}$

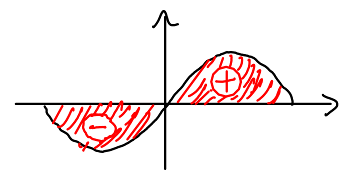
9. $\int\limits_0^1 x^2 + 1dx = \left[\frac{x^3}{3} + x\right]_0^1 = \left(\frac{1^3}{3}+1\right) - \underbrace{\left(\frac{0^3}{3}+0\right)}_{0} = \frac{4}{3}$

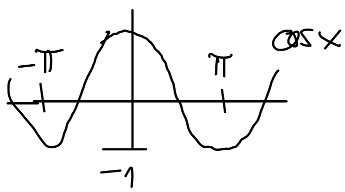
10. $\int\limits_{-1}^1 |x|dx = \underbrace{\int\limits_{-1}^{0} -x\,dx}_{A} + \underbrace{\int\limits_{0}^{1} x\,dx}_{B} = \left[-\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^1 = \left(0+\frac{1}{2}\right) + \left(\frac{1}{2}-0\right) = 1$

⚠ Vorsicht bei abschnittsweise definierten Funktionen: $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$



11. $\int\limits_{-\pi}^{\pi} \sin(x)dx = \left[-\cos(x)\right]_{-\pi}^{\pi} = -\underbrace{\cos(\pi)}_{-1} + \underbrace{\cos(-\pi)}_{-1} = 0$





Eigener Lösungsversuch.

1. $\int 3\cos(x)dx$

2. $\int \frac{2}{x^2}dx$

3. $\int 2x^3 + 4xdx$

4. $\int e^t - 3t^2dt$

5. $\int \frac{x^2\cdot\sqrt{x}}{\sqrt[3]{5}}dx$

6. $\int\limits_0^1 x^3dx$

7. $\int\limits_0^1 (e^t + 2t)dt$

8. $\int\limits_0^2 \frac{x^2}{1+x^3}dx$

9. $\int_{0}^{1} x^2 + 1dx$

10. $\int_{-1}^{1} |x|dx$

11. $\int_{-\pi}^{\pi} \sin(x)dx$