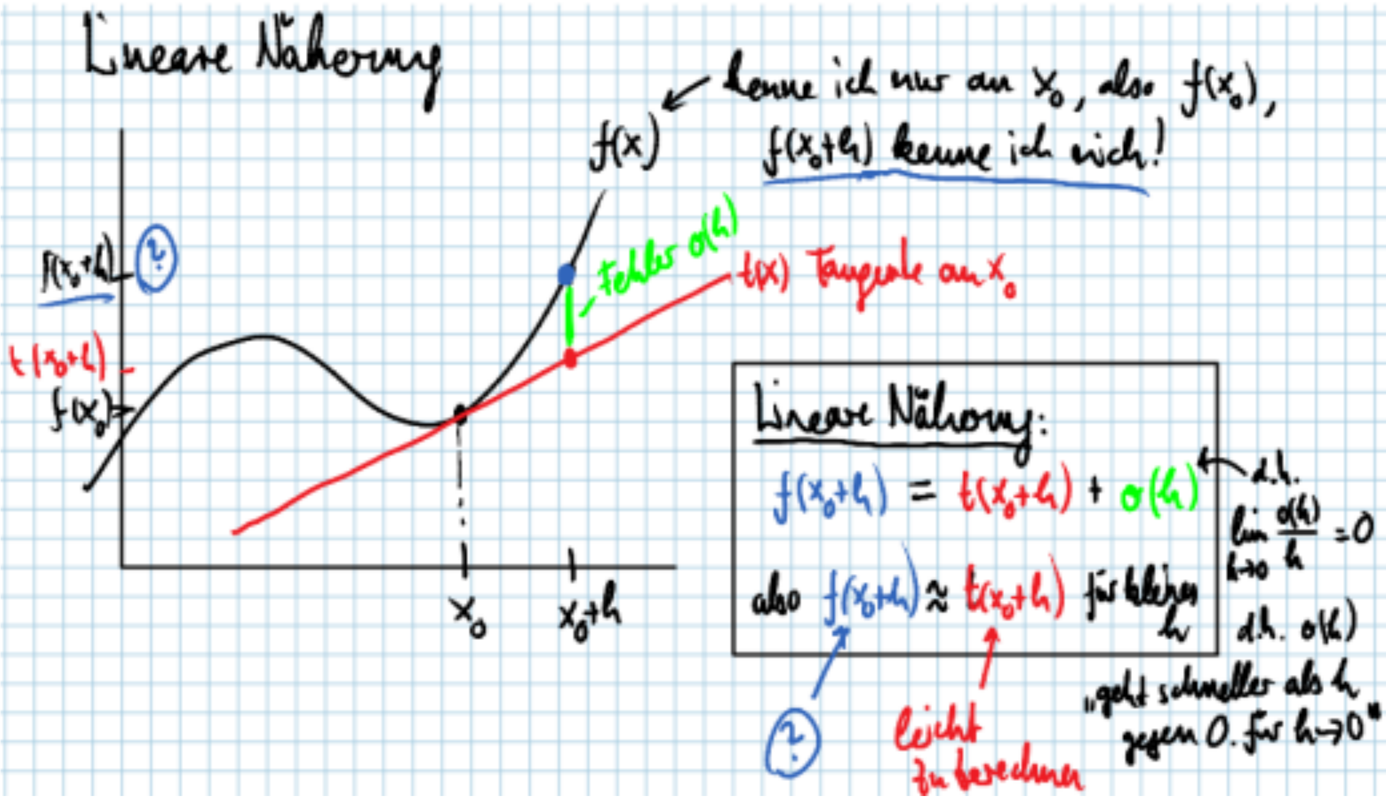
Prof. Dr. M. Helbig
Mathematatik – Übung

Lineare Näherung

Fragen?

Wiederholung



Wie lautet nochmal die Funktionsgleichung $t(x)$ der **Tangente an x_0 von** f?

$$t(x) = f'(x_0)\cdot(x - x_0) + f(x_0)$$

oder mit $x = x_0 + h$ äquivalent zu

$$t(x_0 + h) = f'(x_0)\cdot h + f(x_0)$$

lineare Näherung von Funktionswerten. Berechnen Sie folgende Werte durch lineare Näherung (ohne Taschenrechner!). Schätzen Sie den Fehler ab!

* a) $f(x) = \ln(x)$, $f(1,01) = ?$

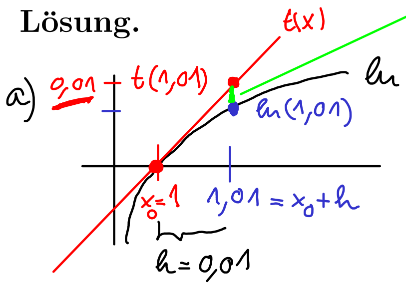
b) $f(x) = \sqrt{x}$, $f(0,98) = ?$

c) $f(x) = \sin(x)$, $f(-0,02) = ?$

d) $f(x) = \sin(x)$, $f(3) = ?$ (Hinweis: Benutzen Sie $\pi = 3,14159$)

Lösung.

a)



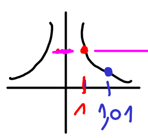
$$\ln(1{,}01) \overset{\text{lin. Näherung}}{\approx} t(1{,}01) = \underbrace{f'(x_0)}_{\frac{1}{1}} \cdot \underbrace{h}_{0{,}01} + \underbrace{f(x_0)}_{\underbrace{\ln(1)}_{0}} = 0{,}01$$

[Vgl TR: $\ln(1{,}01) \overset{TR}{\approx} 0{,}00995\ldots$ Fehler $\approx 0{,}00005\ldots$]

Fehlerabschätzung: absolute Fehler $$|\text{Fehler}| \le \frac{\max\limits_{x\in[1;\,1{,}01]} |f''(x)|}{2} \cdot \underbrace{h^2}_{(0{,}01)^2} = \frac{1}{2}\, 0{,}0001 = 0{,}00005$$

$\max\limits_{x\in[1;\,1{,}01]} |f''(x)| = 1$

$$|f''(x)| = \left|-\frac{1}{x^2}\right| = \frac{1}{x^2}$$

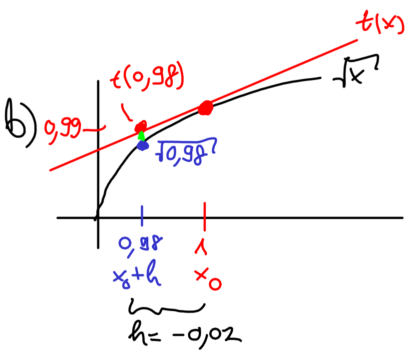


$\frac{1}{1^2} = 1$

d.h. $\ln(1{,}01) \in [0{,}01 - 0{,}00005;\ \cancel{0{,}01 + 0{,}00005}]$

$\ln(1{,}01) \le \underbrace{0{,}01}_{t(1{,}01)}$

b)



$$\sqrt{0{,}98} \overset{\text{lin. N.}}{\approx} t(0{,}98) = \frac{1}{2\sqrt{1}}(-0{,}02) + \sqrt{1} = 0{,}99$$

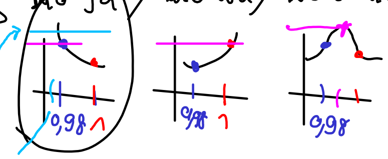
Fehlerabsch.: $$|\text{Fehler}| \le \frac{\max\limits_{x\in[0{,}98;\,1]} |f''(x)|}{2} \cdot \underbrace{h^2}_{(-0{,}02)^2} \le \frac{\frac{1}{3{,}528}}{2} \cdot 0{,}0004 \approx 0{,}00005669\ldots$$

$$|f''(x)| = \left|\left(x^{\frac{1}{2}}\right)''\right| = \left|\left(\frac{1}{2}x^{-\frac{1}{2}}\right)'\right| = \left|\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot x^{-\frac{3}{2}}\right| = \left|-\frac{1}{4\sqrt{x^3}}\right| \overset{x\ge 0}{=} \frac{1}{4\sqrt{x^3}} \quad \text{in } [0{,}98,\ 1]$$

mo. fa.

drei Fälle:

mo fa / mo wa / nicht mo



$\frac{1}{4\sqrt{0{,}98^3}}$

$$\frac{1}{4\sqrt{0{,}98^3}} = \frac{1}{4\cdot 0{,}98\cdot \underbrace{\sqrt{0{,}98}}_{\ge 0{,}81}} \le \frac{1}{4\cdot 0{,}98 \cdot \underbrace{\sqrt{0{,}81}}_{0{,}9}} = \frac{1}{3{,}528}$$

?

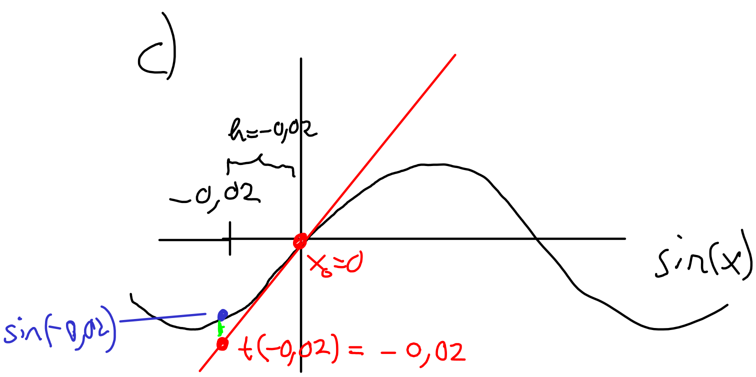
Trick: Suche Zahl, wo ich Wurzel kenne, so dass d.h. $\sqrt{0{,}98} \in [0{,}99 - 0{,}0005\ldots,\ 0{,}99 + \cancel{0{,}0005\ldots}]$

oberhalb von $\sqrt{x}$

da $\sqrt{0{,}98} \leq t(0{,}98) = 0{,}99$

~~Eigener Lösungsversuch.~~

c)

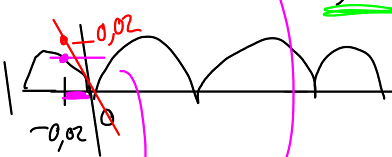


$$\sin(-0{,}02) \underset{\text{lin. N.}}{\approx} t(-0{,}02) = \underbrace{\cos(0)}_{1}(-0{,}02) + \underbrace{\sin(0)}_{0} = -0{,}02$$

Fehlerabsch.:

$$|\text{Fehler}| \leq \frac{\max\limits_{x\in[-0{,}02;0]} |f''(x)|}{2} h^2 \leq \frac{0{,}02}{2} \cdot 0{,}0004 = 0{,}000004$$

$$|f''(x)| = |-\sin(x)| = |\sin(x)|$$

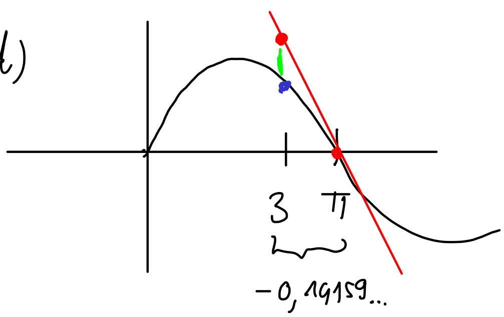


$|\underbrace{\sin(-0{,}02)}| \leq 0{,}02$

$t(-0{,}02) \leq \sin(-0{,}02)$

$$\sin(-0{,}02) \in [-0{,}02 - \cancel{0{,}000004},\ -0{,}02 + 0{,}000004]$$

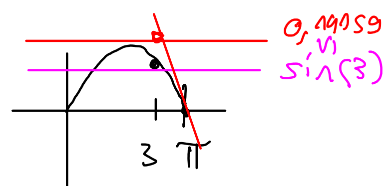
d)



$$\sin(3) \approx t(3) = \underbrace{\cos(\pi)}_{-1}(-0{,}14159\ldots) + \underbrace{\sin(\pi)}_{0} = 0{,}14159$$

Fehlerabsch.: $|\text{Fehler}| \leq \frac{\max |f''(x)|}{2} \cdot h^2 \leq \frac{0{,}14159}{2}(0{,}14159)^2 = 0{,}00141 9\ldots$

$$|f''| = |\sin x|$$



$$\sin(3) \in [0{,}14159 - 0{,}0014\ldots,\ 0{,}14159 + \cancel{0{,}0014\ldots}]$$

$\sin 3 \leq t(3)$

Wolfram-α: plot sin x, $\underbrace{-x+\pi}$ from 2.5 to 3.6

$t(x) = (-1)\cdot(x-\pi) + 0$