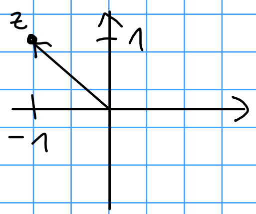
Lösungsvorschlag Probe-Prüfung Lineare Algebra

1. $z = i - 1$

a)

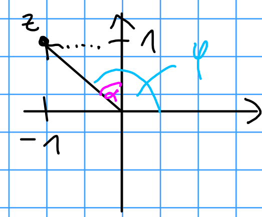


b) $r = |z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\alpha = \arctan\left(\frac{1}{1}\right) = 45° \hat{=} \frac{\pi}{4}$

$\varphi = 90° + \alpha = 135° \hat{=} \frac{3\pi}{4}$

$\Rightarrow \underline{z = \sqrt{2}\, e^{i\frac{3\pi}{4}}}$



c) $z^{-1} = \frac{1}{z \cdot \bar{z}}\, \bar{z} = \frac{1}{(-1)^2 + 1^2}(-i - 1) = \frac{1}{2}(-i - 1)$

<u>ODER</u>: $z^{-1} = \frac{1}{\sqrt{2}\, e^{i\frac{3\pi}{4}}} = \frac{1}{\sqrt{2}}\, e^{-i\frac{3\pi}{4}}$

d) $u^4 = z = \sqrt{2}\, e^{i\frac{3\pi}{4}}$ besitzt Lösungen:

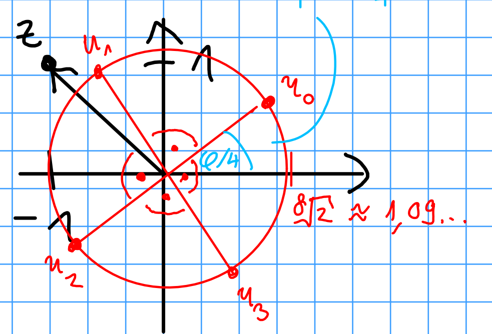
$u_0 = \sqrt[4]{\sqrt{2}}\; e^{i\frac{3\pi/4}{4}}$

$u_1 = \sqrt[4]{\sqrt{2}}\; e^{i\left(\frac{3\pi/4}{4} + 1 \cdot \frac{2\pi}{4}\right)}$

$u_2 = \sqrt[4]{\sqrt{2}}\; e^{i\left(\frac{3\pi/4}{4} + 2 \cdot \frac{2\pi}{4}\right)}$

$u_3 = \sqrt[4]{\sqrt{2}}\; e^{i\left(\frac{3\pi/4}{4} + 3 \cdot \frac{2\pi}{4}\right)}$

$\frac{\varphi}{4} = \frac{135°}{4} = 33{,}75°$

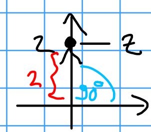


2. $w = 2i$

a) $\left(\frac{1}{2}w\right)^{1024} = i^{1024} = (\underbrace{i^2}_{-1})^{512} = (-1)^{512} = 1.$

b) $w^{-1} = \frac{1}{w \cdot \bar{w}} \cdot \bar{w} = \frac{1}{0^2 + 2^2}(-2i) = -\frac{1}{2}i$

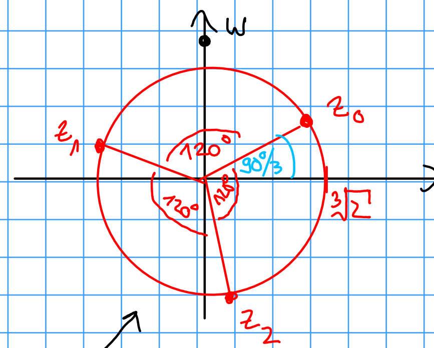
c) $z^3 = w = 2i = 2e^{i\frac{\pi}{2}}$

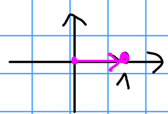


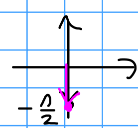
$z_0 = \sqrt[3]{2}\; e^{i\frac{\pi/2}{3}}$

$z_1 = \sqrt[3]{2}\; e^{i\left(\frac{\pi/2}{3} + 1 \cdot \frac{2\pi}{3}\right)}$

$z_2 = \sqrt[3]{2}\; e^{i\left(\frac{\pi/2}{3} + 2 \cdot \frac{2\pi}{3}\right)}$



d) a) 

b) 

c)

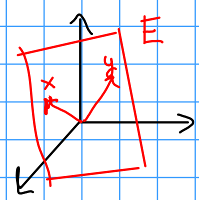
3. $(A|b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 2 & 2a & -1 \end{array}\right) \sim_{II - 2\cdot I} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -2 & 2a-6 & -3 \end{array}\right)$

unabhängig von a immer lösbar & da x_3 frei immer ∞-viele Lösungen,

d.h. $\forall a \in \mathbb{R}$: LGS besitzt ∞-viele Lösungen.

4.

a) $E: \quad \lambda x + \mu y = \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 6 \\ 3 \end{pmatrix}$



b) $n = x \times y = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} -12 \\ -9 \\ 10 \end{pmatrix}$

ODER: $n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ mit $n \cdot x = 0$ & $n \cdot y = 0$

$$n \cdot x = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = n_1 + 2n_2 + 3n_3 \overset{!}{=} 0$$

$$n \cdot y = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 6 \\ 3 \end{pmatrix} = -2n_1 + 6n_2 + 3n_3 \overset{!}{=} 0$$

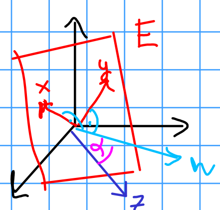
LGS: $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -2 & 6 & 3 & 0 \end{array}\right) \underset{II+2\cdot I}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 10 & 9 & 0 \end{array}\right)$ $n_3 = \alpha$ frei!

$II: 10n_2 + 9\alpha = 0 \Rightarrow n_2 = -\frac{9}{10}\alpha$

$I: n_1 + 2 \cdot \left(-\frac{9}{10}\alpha\right) + 3\alpha = 0 \Rightarrow n_1 = -\frac{12}{10}\alpha$

z.B. $n_3 = \alpha = 10$ liefert $n = \begin{pmatrix} -12 \\ -9 \\ 10 \end{pmatrix}$.

c)



$$\alpha = \sphericalangle(z, n) = \arccos\left(\frac{z \cdot n}{|z| \cdot |n|}\right) = \arccos\left(\frac{\begin{pmatrix} -12 \\ -9 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}}{\sqrt{12^2+9^2+10^2} \cdot \sqrt{1^2+2^2+4^2}}\right)$$

$$= \arccos\left(\frac{34}{5\sqrt{13} \cdot \sqrt{21}}\right) \approx 65{,}7^\circ$$

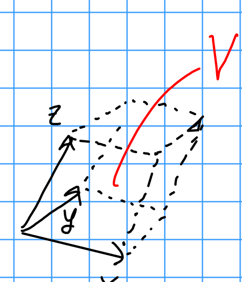
$$\sphericalangle(z, E) = 90^\circ - 65{,}7^\circ = 24{,}3^\circ.$$

d)

$$V = \left| \det(x|y|z) \right| = \left| \det \begin{pmatrix} 1 & -2 & -1 \\ 2 & 6 & 2 \\ 3 & 3 & 4 \end{pmatrix} \right| =$$

$$= \left| 24 + (-12) + (-6) - (-18) - 6 - (-16) \right|$$

$= 34$ (ODER: $V = |(x \times y) \cdot z| = |n \cdot z| = 34$ siehe c))



5. a)

$$x_2 = \frac{\begin{vmatrix} 1 & 1 & 3 \\ 0 & -1 & 2 \\ 0 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}} \overset{\text{Sarrus}}{=} \frac{-1-4}{1} = -5$$

b) EW:

$$\chi_A(\lambda) = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3, \text{ d.h. } \lambda_{1,2,3} = 1.$$

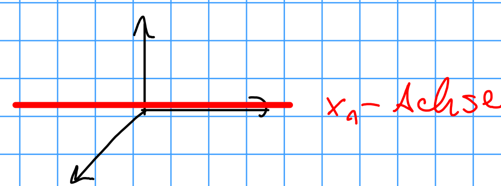
EV: $\lambda = 1$:

$$(A-\lambda E_3 \mid 0) = \left(\begin{array}{ccc|c} 1-1 & 2 & 3 & 0 \\ 0 & 1-1 & 2 & 0 \\ 0 & 0 & 1-1 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 = \alpha$ frei

II: $2x_3 = 0 \Rightarrow x_3 = 0$, III: $2x_2 = 0 \Rightarrow x_2 = 0$.

$$\Rightarrow \quad x = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \underline{\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$



6. $B = \begin{pmatrix} 1 & 2b \\ 3b & 6 \end{pmatrix}$

a) B invertierbar $\Longleftrightarrow$ $\det(B) \neq 0$

$\det(B) = 6 - 6b^2 = 6(1-b^2)$, also falls $1-b^2 \neq 0 \Leftrightarrow \underline{b \neq \pm 1}$

Dann ist $B^{-1} = \frac{1}{6(1-b^2)} \begin{pmatrix} 6 & -2b \\ -3b & 1 \end{pmatrix}$.

b) B besitzt EW $\Longleftrightarrow$ $\chi_B(\lambda)$ besitzt NST.

$$\chi_B(\lambda) = \begin{vmatrix} 1-\lambda & 2b \\ 3b & 6-\lambda \end{vmatrix} = \underbrace{(1-\lambda)(6-\lambda)}_{\lambda^2 - 7\lambda + 6} - 6b^2 = \lambda^2 - 7\lambda + (6-6b^2)$$

besitzt NST, falls die Diskriminante $7^2 - 4 \cdot 1 \cdot (6-6b^2) \geq 0$

$$\Longleftrightarrow 49 \geq 24(1-b^2)$$

$$\Longleftrightarrow \frac{49}{24} \geq 1 - b^2$$

$$\Longleftrightarrow \underbrace{\frac{49}{24} - 1}_{\frac{25}{24}} \geq -b^2$$

$$\overset{\cdot(-1)}{\Longleftrightarrow} -\frac{25}{24} \leq \underbrace{b^2}_{\geq 0} \quad \forall b \in \mathbb{R} \text{ erfüllt!}$$

d.h. $\forall b \in \mathbb{R}$: B besitzt EW.