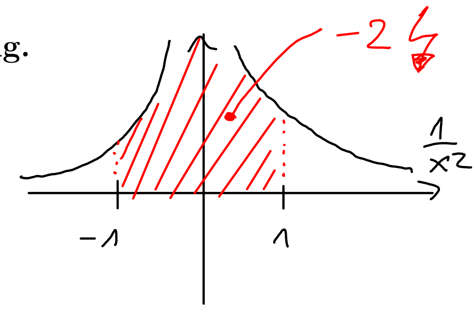
Integrale

* **Fehlerteufel.** Wo steckt der Fehler? $\int\limits_{-1}^{1} \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -\frac{1}{1} - \left(-\frac{1}{-1}\right) = -2$

Lösung.



Fehler: $\frac{1}{x^2}$ hat $D = \mathbb{R} \setminus \{0\}$.

Es wurde über die Definitionslücke $x = 0$ integriert!

$$\int_{-1}^{1} \frac{1}{x^2}\,dx \overset{\text{symmetrisch}}{=} 2 \cdot \int_{0}^{1} \frac{1}{x^2}\,dx = 2 \cdot \lim_{\varepsilon \to 0^+} \underbrace{\underbrace{\int_{\varepsilon}^{1} \frac{1}{x^2}\,dx}_{\left[-\frac{1}{x}\right]_{\varepsilon}^{1}}}_{-1 + \underbrace{\frac{1}{\varepsilon}}_{\to \infty}} = \infty$$

ist die Fläche!

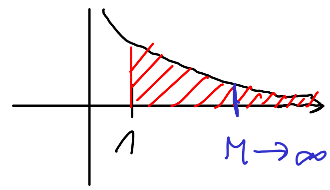
Eigener Lösungsversuch.

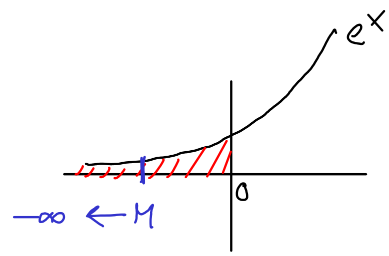
Uneigentliche Integrale. Berechnen Sie:

* 1. $\int\limits_1^\infty \frac{2}{x^3}dx$, 2. $\int\limits_{-\infty}^0 e^x dx$, 3. $\int\limits_{-\infty}^\infty \frac{1}{1+x^2}dx$.

Lösung.

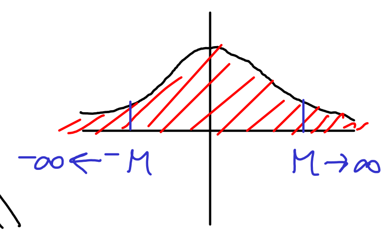
1. $$\int_1^\infty \frac{2}{x^3}\,dx = \lim_{M\to\infty} \underbrace{\int_1^M \frac{2}{x^3}\,dx}_{\left[\frac{\cancel{2}}{-\cancel{2}x^2}\right]_1^M} = \lim_{M\to\infty}\Big(-\underbrace{\frac{1}{M^2}}_{\to 0} + \frac{1}{1^2}\Big) = 1$$

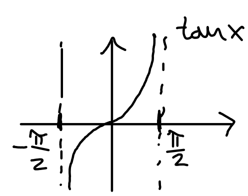


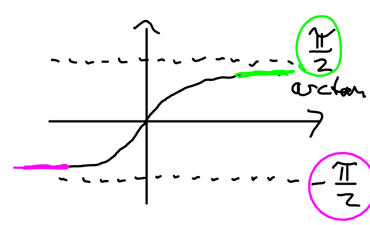


2. $$\int_{-\infty}^0 e^x\,dx = \lim_{M\to-\infty} \underbrace{\int_M^0 e^x\,dx}_{\left[e^x\right]_M^0} = \lim_{M\to-\infty}\big(\underbrace{e^0}_{1} - \underbrace{e^M}_{\downarrow\, 0}\big) = 1$$

3. $$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \lim_{M\to\infty} \underbrace{\int_{-M}^M \frac{1}{1+x^2}\,dx}_{\left[\arctan x\right]_{-M}^M} = \lim_{M\to\infty}\big(\underbrace{\arctan(M)}_{\downarrow\, \frac{\pi}{2}} - \underbrace{\arctan(-M)}_{\downarrow\, -\frac{\pi}{2}}\big) = \pi$$







Eigener Lösungsversuch.

Partielle Integration. Berechnen Sie die folgenden Integrale:

* 1. $\int xe^x dx$,

2. $\int e^x(2-x^2)dx$,

3. $\int x^2 \ln x\, dx$,

4. $\int_1^e \ln x\, dx$,

Lösung.

1. $$\int \underset{g}{x}\,\underset{f}{e^x}\,dx = \underset{F}{e^x}\cdot \underset{g}{x} - \int \underset{F}{e^x}\cdot \underset{g'}{1}\,dx = xe^x - e^x + C.$$

2. $$\int \underset{f}{e^x}\,\underset{g}{(2-x^2)}\,dx = \underset{F}{e^x}\,\underset{g}{(2-x^2)} - \int \underset{F}{e^x}\cdot\underset{g'}{(-2x)}\,dx = e^x(2-x^2) + \underbrace{\int \underset{f}{e^x}\cdot \underset{g}{2x}\,dx}_{\underset{F}{e^x}\cdot\underset{g}{2x} - \int \underset{F}{e^x}\cdot\underset{g'}{2}\,dx}$$

$$= e^x(2-x^2) + e^x\cdot 2x - 2e^x + C$$

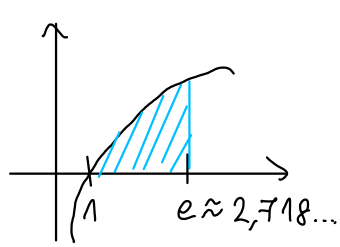
$$= e^x(-x^2+2x) + C$$

3. $$\int \underset{f}{x^2}\,\underset{g}{\ln x}\,dx = \underset{F}{\frac{x^3}{3}}\cdot\underset{g}{\ln x} - \int \underbrace{\underset{F}{\frac{x^3}{3}}\cdot\underset{g'}{\frac{1}{x}}}_{\frac{x^2}{3}}\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

$$= \frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + C$$

4. $$\int_1^e \underset{g}{\ln x}\cdot\underset{f}{1}\,dx = \left[\underset{F}{x}\cdot\underset{g}{\ln x}\right]_1^e - \int_1^e \underbrace{\underset{F}{x}\cdot\underset{g'}{\frac{1}{x}}}_{1}\,dx = \left[x\ln x - x\right]_1^e = \underbrace{(e\underbrace{\ln e}_{1} - e)}_{0} - (1\cdot\underbrace{\ln 1}_{0} - 1)$$

$$= 1.$$



Eigener Lösungsversuch.

Substitutionsregel. Berechnen Sie die folgenden Integrale:

* 1. $\int_0^{\frac{\pi}{2}} \sin(2x)\,dx$,

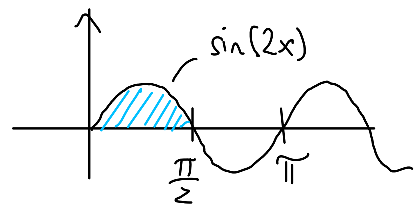
2. $\int \sin(x)(\cos(x))^3 dx$,

3. $\int \frac{x}{5+x^2} dx$,

4. $\int_0^2 \frac{t^2}{1+t^3} dt$.

Lösung.

1. $\frac{1}{2}\cdot\int_0^{\pi/2} \underbrace{\sin(2x)}_{f(u(x))}\cdot \underbrace{2}_{u'(x)}\,dx = \frac{1}{2}\cdot\Big[\underbrace{-\cos(2x)}_{F(u(x))}\Big]_0^{\frac{\pi}{2}} = \frac{1}{2}\cdot\Big(-\underbrace{\cos(\pi)}_{-1} + \underbrace{\cos(0)}_{1}\Big) = 1$



2. $-\int (\cos(x))^3\cdot(-\sin(x))\,dx = -\frac{(\cos x)^4}{4} + C$

3. $\frac{1}{2}\int \frac{1}{5+x^2}\cdot 2x\,dx = \frac{1}{2}\cdot \ln|5+x^2| + C$

4. $\frac{1}{3}\int_0^2 \frac{1}{1+t^3}\,3t^2\,dx = \frac{1}{3}\cdot\Big[\ln|1+t^3|\Big]_0^2 = \frac{1}{3}\cdot\Big(\underbrace{\ln|1+8|}_{\ln 9} - \underbrace{\ln|1|}_{0}\Big) = \frac{1}{3}\ln \underbrace{9}_{3^2}$

$= \frac{2}{3}\ln 3.$

3./4. mit logarithmischer Integration $\int \frac{f'}{f}\,dx = \ln|f| + C$

3. $\frac{1}{2}\int \frac{\overset{f'}{2x}}{\underset{f}{5+x^2}}\,dx = \frac{1}{2}\ln|5+x^2| + C$

4. $\frac{1}{3}\int \frac{\overset{f'}{3t^2}}{\underset{f}{1+t^3}}\,dx = \frac{1}{3}\ln|1+t^3| + C$

Eigener Lösungsversuch.

Integrale-Mix. Berechnen Sie die folgenden Integrale:

1. $\int 4\sin x + \frac{2}{\sqrt{1-x^2}}dx$, 2. $\int_0^2 t\sqrt{t^2+4}dt$, 3. $\int x^2 e^x dx$

Lösung.

1. $$\int 4\sin x + \frac{2}{\sqrt{1-x^2}}\,dx \overset{\int \text{linear}}{=} 4\underbrace{\int \sin x\,dx}_{-\cos x + c} + 2\underbrace{\int \frac{1}{\sqrt{1-x^2}}\,dx}_{\arcsin x + C} = -4\cos x + 2\arcsin x + C$$

2. $$\frac{1}{2}\int_0^2 (t^2+4)^{\frac{1}{2}}\, 2t\, dt \overset{\text{Subst.}}{=} \frac{1}{2}\left[\frac{(t^2+4)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\left[\underbrace{(\underbrace{2^2+4}_{8})^{\frac{3}{2}}}_{(\sqrt{8})^3,\ \underbrace{\sqrt{8}}_{\underbrace{2\cdot 4}_{2\sqrt{2}}}} - \underbrace{(\underbrace{0^2+4}_{4})^{\frac{3}{2}}}_{8}\right]$$
$$= \frac{1}{3}\left(16\sqrt{2} - 8\right) = \frac{8}{3}\left(2\sqrt{2} - 1\right)$$

3. $$\int \underset{g}{x^2}\, \underset{f}{e^x}\, dx \overset{\text{part. Int}}{=} \underset{F\cdot g}{e^x\cdot x^2} - \underbrace{\int \underset{F\cdot g'}{e^x\cdot 2x}\, dx}_{\left[e^x\cdot 2x - \underbrace{\int e^x\cdot 2\, dx}_{2e^x + c}\right]} = \underbrace{x^2e^x - 2xe^x + 2e^x}_{e^x(x^2 - 2x + 2)} + C$$

Eigener Lösungsversuch.