5. Übung

5.1 a)

$$1 \overset{!}{=} \int_{-\infty}^{\infty} f(x)\,dx = c\int_2^3 (x-2)\,dx = c\left[\tfrac{1}{2}x^2 - 2x\right]_2^3 = c\,(4.5-6-2+4)$$

$$= \frac{c}{2} \quad \Rightarrow \quad c = 2$$

$$F(x) = \begin{cases} 0 & , \text{ falls } x \le 2 \\ 2\int_2^x (t-2)\,dt & , \text{ falls } 2 \le x < 3 \\ 1 & , \text{ falls } x \ge 3 \end{cases}$$

$$\int f(t)\,dt = F(t) + C$$

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

$$2\int_{2}^{x} (t-2)\,dt = 2\left[\tfrac{1}{2}t^2 - 2t\right]_2^x = x^2 - 4x \underbrace{-\,2(2-4)}_{+4} = (x-2)^2$$

b) $P(2.1 < X < 2.8) = F(2.8) - F(2.1) = 0.8^2 - 0.1^2 = 63\%$

c) $E[X] = 2\int_2^3 x\,(x-2)\,dx = 2\left[\tfrac{1}{3}x^3 - x^2\right]_2^3 = \frac{8}{3}$

$$Var[X] = E[X^2] - (E[X])^2 = 2\int_2^3 x^2(x-2)\,dx - \frac{64}{9} =$$

$$= 2\left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3\right]_2^3 - \frac{64}{9} = \frac{43}{6} - \frac{64}{9} = \frac{1}{18}$$

Median: Gesucht ist x_M mit $F(x_M) = 0.5$

$$\Leftrightarrow \underbrace{(x_M - 2)^2}_{F(x_M)} = \frac{1}{2} \quad \underset{2 < x_M < 3}{\Leftrightarrow} \quad x_M = 2 + \frac{1}{\sqrt{2}}$$

5.3

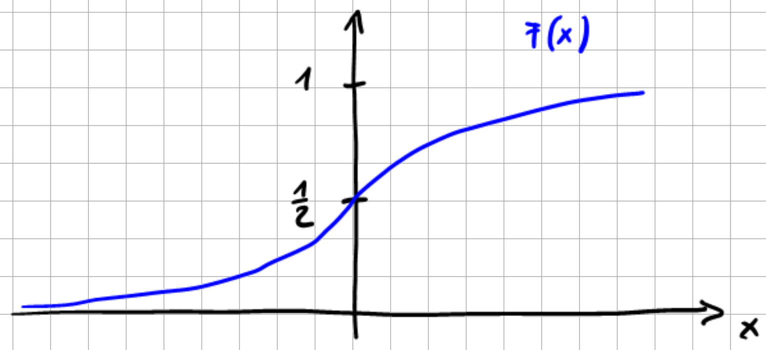
$$F(x) = \frac{e^x}{1+e^x} = \frac{(e^x+1)-1}{(e^x+1)} = 1 - \frac{1}{e^x+1}$$

$$f(x) = F'(x) = \frac{e^x(1+e^x) - e^x \cdot e^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2} > 0$$

$\Rightarrow$ $F(x)$ ist str. monoton wachsend und damit umkehrbar

$$\lim_{x \to -\infty} F(x) = 0 \quad , \quad \lim_{x \to \infty} F(x) = 1 \quad , \quad F(0) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$F(0) = \frac{e^0}{1+e^0} = \frac{1}{2}$$



Umkehrfunktion

$x_p = F^{-1}(p)$ durch Auflösen von

$F(x_p) = p$ nach x_p

$$\frac{e^{x_p}}{1+e^{x_p}} = p \quad \Big| \cdot (1+e^{x_p})$$

$$e^{x_p} = p(1+e^{x_p}) \iff e^{x_p} - pe^{x_p} = p$$

$$\iff e^{x_p}(1-p) = p$$

$$\iff e^{x_p} = \frac{p}{1-p} \quad \Big| \ln()$$

$$\Rightarrow x_p = \ln \frac{p}{1-p} = F^{-1}(p)$$

$p = 0.25$: $\quad x_{0.25} = \ln \dfrac{\frac{1}{4}}{\frac{3}{4}} = \ln \frac{1}{3} \approx -1.0986$

$= \ln 3^{-1} = -\ln 3$

Median: $p = 0.5$: $\quad x_{0.5} = \ln 1 = 0$

$p = 0.75$: $\quad x_{0.75} = \ln 3 \approx 1.0986$

5.3

X : Ankunftszeit mit Dichte $f(x) = \begin{cases} \frac{1}{90} & , \ 0 \leq x \leq 90 \\ 0 & , \ \text{sonst} \end{cases}$

Y: Wartezeit in sek

$Y = g(X)$ mit $g(x) = \begin{cases} 0 & , \ 0 \leq x \leq 25 \\ 90 - x & , \ 25 \leq x \leq 90 \end{cases}$

Mittlere Wartezeit

Ges.: $E[Y] = E[g(X)] = \int_0^{90} g(x) \cdot f(x) \, dx = \frac{1}{90} \int_0^{90} (90 - x) \, dx$

$= \frac{1}{90} \left[90x - \frac{1}{2} x^2 \right]_0^{90} = \frac{845}{36} \approx 23.5 \text{ sek}$