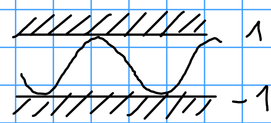
Lösungsvorschlag Probe-Prüfung Analysis

1. a) $$\frac{6n^4-3n^2+7}{3n^4-2n} = \frac{\cancel{n^4}\left(6-\overset{\nearrow 0}{\frac{3}{n^2}}+\overset{\nearrow 0}{\frac{7}{n^4}}\right)}{\cancel{n^4}\left(3-\underset{\searrow 0}{\frac{2}{n^3}}\right)} \xrightarrow{n\to\infty} \frac{6}{3} = \underline{2}$$

b) $$\lim_{x\to 0}\frac{\sin x}{x} \overset{l'H}{\underset{„\frac{0}{0}“}{=}} \lim_{x\to 0}\frac{\cos x}{1} = \frac{\cos 0}{1} = \underline{1}$$

c) $$\lim_{x\to\infty}\frac{\cos x}{e^x} \underset{„\frac{\text{beschränkt}}{\infty}“}{=} \underline{0}$$ (da $\cos x \in [-1,1]$ 

)

d) $$\sum_{k=0}^{\infty}\frac{2^k}{k!} = e^2 \neq \cos(2)$$ ODER: $\cos(2) < 0$ & $\sum_{k=0}^{\infty}\frac{2^k}{k!} = 1+\frac{2}{1!}+\frac{2^2}{2!}+\ldots > 1$

e) Berechne den Konvergenzradius r der Potenzreihe für die Koeffizienten $a_k = \frac{3k}{(k+1)!}$:

$$r = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty}\frac{\frac{3k}{(k+1)!}}{\frac{3(k+1)}{(k+2)!}} = \lim_{k\to\infty}\frac{\frac{\cancel{3}k}{\cancel{(k+1)!}}}{\frac{\cancel{3}(k+1)}{(k+2)\cancel{!}}} = \lim_{k\to\infty}\frac{k(k+2)}{k+1}$$

$$= \lim_{k\to\infty}\frac{\overset{\infty}{k^{\cancel{2}}}\left(1+\frac{2}{k}\right)}{\cancel{k}\left(1+\frac{1}{k}\right)} = \infty$$

$$\underbrace{\frac{\left(1+\frac{2}{k}\right)}{\left(1+\frac{1}{k}\right)}}_{\to 1}$$

Also konvergiert die Reihe für $x \in \left]2-\underset{\infty}{\underset{\shortparallel}{r}}, 2+\underset{\infty}{\underset{\shortparallel}{r}}\right[= \mathbb{R}$.

2. a) Brauche Ableitungen von $h(x) = \ln x$:

$$h'(x) = \frac{1}{x} = x^{-1}, \quad h''(x) = -x^{-2} = -\frac{1}{x^2}, \quad h'''(x) = 2x^{-3} = \frac{2}{x^3}$$

$$T_3(x) = \sum_{k=0}^{3} \frac{h^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$= h(1) + h'(1)(x-1) + \frac{h''(1)}{2}(x-1)^2 + \frac{h'''(1)}{6}(x-1)^3$$

$$= \underbrace{\ln(1)}_{0} + \underbrace{\frac{1}{1}}_{1}(x-1) + \underbrace{\frac{-1/1^2}{2}}_{-\frac{1}{2}}(x-1)^2 + \underbrace{\frac{2/1^3}{6}}_{\frac{1}{3}}(x-1)^3$$

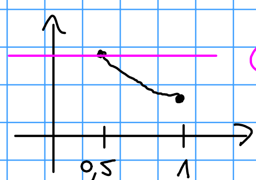
$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

$$h(0,5) \approx T_3(0,5) = \underbrace{(0,5-1)}_{-0,5} - \frac{1}{2}\underbrace{(0,5-1)^2}_{0,25} + \frac{1}{3}\underbrace{(0,5-1)^3}_{-0,125} = -0,\overline{6}$$

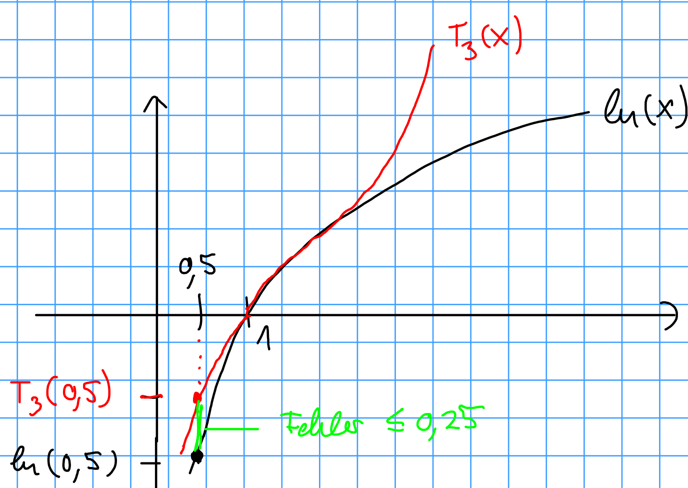
absoluter Fehler der Approximation ist:

$$|R_3(0,5)| \leq C \cdot \frac{|x-x_0|^4}{4!} = C \cdot \frac{|0,5-1|^4}{4!} = 96 \cdot \frac{0,0625}{24} = 0,25$$

zu C: $\quad |h^{(4)}(x)| = |-6x^{-4}| = \left|-\frac{6}{x^4}\right| = \frac{6}{x^4}$ in $[0,5;1]$ streng mo. fa:



$$C = \max_{x\in[0,5;1]} \frac{6}{x^4} = \frac{6}{0,5^4} = 96$$



3. $g(x) = x^2 e^x$

a) $\lim\limits_{x \to -\infty} x^2 e^x = \lim\limits_{x \to -\infty} \frac{x^2}{e^{-x}} \underset{\text{"}\frac{\infty}{\infty}\text{"}}{\overset{l'H}{=}} \lim\limits_{x \to -\infty} \frac{2x}{-e^{-x}} \underset{\text{"}\frac{-\infty}{-\infty}\text{"}}{\overset{l'H}{=}} \lim\limits_{x \to -\infty} \frac{2}{\underbrace{e^{-x}}_{\to \infty}} = 0.$

$\lim\limits_{x \to \infty} \underbrace{x^2}_{\downarrow \infty} \underbrace{e^x}_{\downarrow \infty} = \infty.$

b) NST: $x^2 \underbrace{e^x}_{>0} = 0 \iff x^2 = 0 \iff x = 0.$

Lokale Extrema: $g'(x) = x^2 e^x + 2x e^x = (x^2 + 2x) \underbrace{e^x}_{>0} \overset{!}{=} 0 \iff$

$\iff \underbrace{x^2 + 2x}_{x(x+2)} = 0 \iff x = 0 \ \vee \ x = -2.$

$g''(x) = (x^2 + 2x) e^x + (2x+2) e^x = (x^2 + 4x + 2) e^x$

$g''(0) = 2 > 0$ ☺ an Stelle $x = 0$ Min.

$g''(-2) = ((-2)^2 + 4(-2) + 2) \frac{1}{e^2} = -2 \frac{1}{e^2} < 0$ ☹ an Stelle $x = -2$ Max

WP: $g''(x) = 0 \overset{e^x > 0}{\iff} x^2 + 4x + 2 = 0 \iff x_{1,2} = \frac{-4 \pm \sqrt{16 - 4 \cdot 2}}{2} = \frac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}$

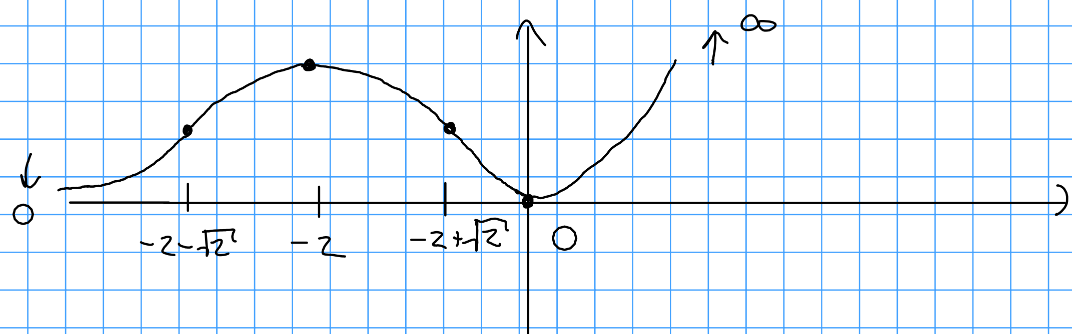
$g'''(x) = (x^2 + 4x + 2) e^x + (2x + 4) e^x = (x^2 + 6x + 6) e^x$

$g'''(-2 + \sqrt{2}) = ((-2+\sqrt{2})^2 + 6(-2+\sqrt{2}) + 6) e^{-2+\sqrt{2}} \approx 1{,}57 \cdot e^{\cdots} \neq 0$

$g'''(-2 - \sqrt{2}) = ((-2-\sqrt{2})^2 + 6(-2-\sqrt{2}) + 6) \underbrace{e^{-2-\sqrt{2}}}_{>0} \approx -0{,}09 \cdot \underbrace{e^{\cdots}}_{>0} \neq 0$

$\Rightarrow$ zwei WP an den Stellen $x = -2 \pm \sqrt{2}$

c)

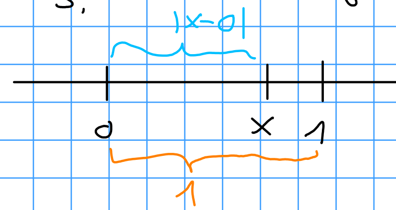


d) $T_2(x) = g(0) + g'(0)\cdot x + \frac{g''(0)}{2}\, x^2$

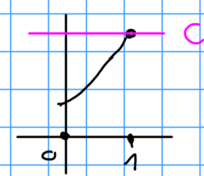
$= 0 + 0\cdot x + \frac{2}{2}\, x^2 = x^2$

e) absoluter Fehler $= |R_2(x)| \le C \cdot \frac{|x-0|^3}{3!} \le 13\cdot e\, \frac{1^3}{6} \approx \underline{5{,}88961\ldots}$

- $|x-0| \le 1$ für $x \in [0,1]$

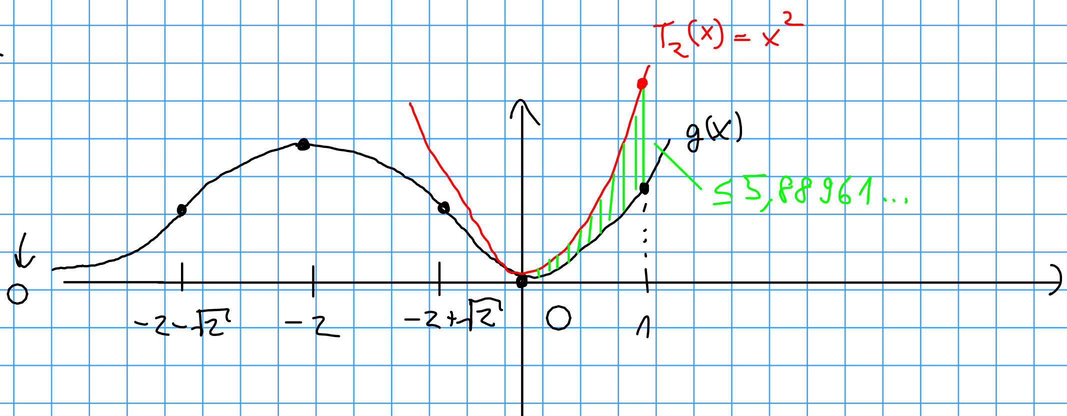


- C: $|g'''(x)| = \Big|\underbrace{\underbrace{(x^2+6x+6)}_{\text{str. mo. wa in } [0,1]}\ \underbrace{e^x}_{\text{str. mo wa}}}_{\text{str. mo wa in } [0,1]}\Big|$, also



$\Rightarrow C = \max_{x\in[0,1]} |g'''(x)| = |g'''(1)| = |(\underbrace{1^2+6\cdot 1+6}_{13})\, e^1| = \underline{13\, e \approx 35{,}33766\ldots}$

<u>Also:</u>



4. $f(x) = x^{-2} e^{x^2} = \frac{e^{x^2}}{x^2}$.

a) $D = \mathbb{R} \setminus \{0\}$ wegen Nenner $x^2 \neq 0$.

b) $\lim\limits_{x \to \pm\infty} \frac{e^{x^2}}{x^2} \underset{\text{„}\frac{\infty}{\infty}\text{"}}{\overset{l'H}{=}} \lim\limits_{x \to \pm\infty} \frac{2x \cdot e^{x^2}}{2x} = \lim\limits_{x \to \pm\infty} e^{x^2} = \infty$

$\lim\limits_{x \to 0} \frac{e^{x^2}}{x^2} \underset{\text{„}\frac{1}{0}\text{"}}{=} \infty$.

c) f ist stetig hebbar, falls $\lim\limits_{x \to 0} \frac{e^{x^2}}{x^2} \in \mathbb{R}$ existieren würde. Also <u>nein</u>!

d) NST: $\frac{e^{x^2}}{x^2} = 0 \overset{\cdot x^2}{\Rightarrow} \underbrace{e^{x^2}}_{>0} = 0$ ↯ keine NST!

e) $\underline{\underline{f'(x)}} = \left(\frac{e^{x^2}}{x^2}\right)' = \frac{x^2 \cdot 2x \cdot e^{x^2} - e^{x^2} \cdot 2x}{(x^2)^2} = \frac{2x e^{x^2} \cdot (x^2 - 1)}{x^4} = \underline{\underline{\frac{2e^{x^2} \cdot (x^2-1)}{x^3}}}$

$\underline{\underline{f''(x)}} = \frac{x^3 \cdot [2 \cdot 2x \cdot e^{x^2} \cdot (x^2 - 1) + 2e^{x^2} \cdot (2x)] - 2e^{x^2} \cdot (x^2 - 1) \cdot 3x^2}{(x^3)^2} =$

$= \frac{2x^2 e^{x^2} \cdot \{x \cdot [2x(x^2-1) + 2x] - (x^2-1) \cdot 3\}}{x^6} = \underline{\underline{\frac{2e^{x^2} \cdot \{2x^4 - 3x^2 + 3\}}{x^4}}}$

f) <u>Lokale Extrema:</u> $f'(x) = 0 \iff \underbrace{2e^{x^2}}_{>0}(x^2-1) = 0 \iff x^2-1=0$

$\iff x^2 = 1 \iff \underline{x = \pm 1}$

$$f''(\pm 1) = \frac{2e^{(\pm1)^2}\cdot\left(2(\pm1)^4 - 3(\pm1)^2+3\right)}{(\pm1)^4} = \frac{2e\,(2-3+3)}{1} = 4e > 0$$

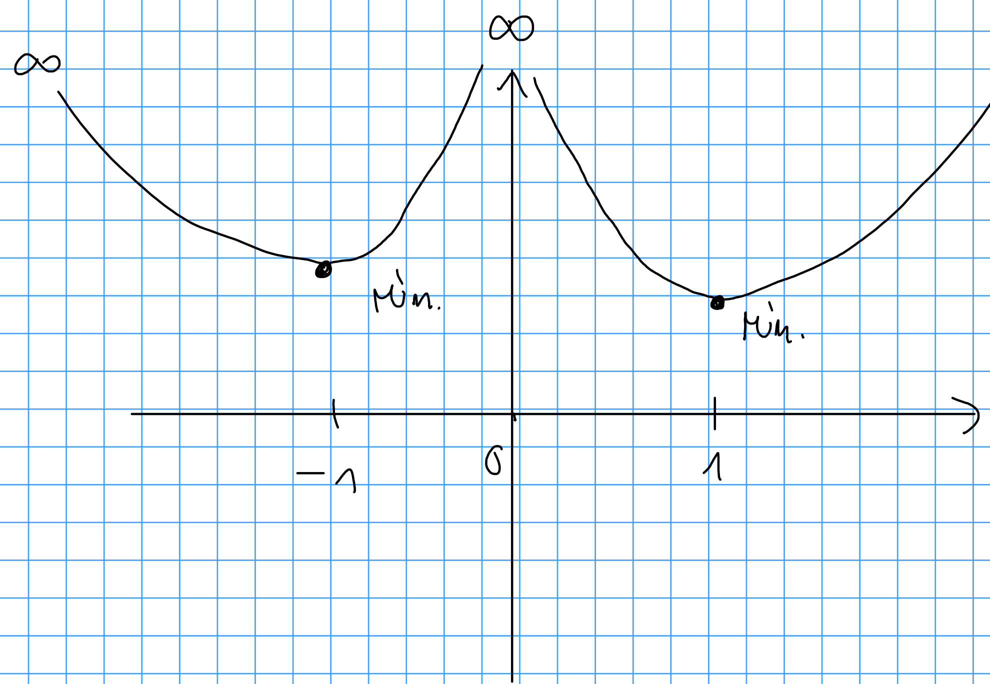
$\Rightarrow$ an Stellen $x = \pm 1$ ist ein Minimum.

<u>Wendepunkte:</u> $f''(x) = 0 \iff \underbrace{2e^{x^2}}_{>0}(2x^4-3x^2+3) = 0 \iff 2x^4-3x^2+3 = 0$

Substitution $u = x^2$: $\quad 2u^2 - 3u + 3 = 0 \Rightarrow x_{1/2} = \frac{3 \pm \sqrt{9-4\cdot 2\cdot 3}}{\dots}$ <0 ↯

$\Longrightarrow$ keine Wendepunkte!

g)



h) $T_2(x) = f(1) + f'(1)\cdot(x-1) + \frac{f''(1)}{2}(x-1)^2$

$$= \frac{e^{1^2}}{1^2} + \frac{2e^{1^2}(1^2-1)}{1^3}(x-1) + \frac{\frac{2e^{1^2}(2\cdot 1^4 - 3\cdot 1^2+3)}{1^4}}{2}(x-1)^2$$

$$= e + 0 + 2\cdot e\,(x-1)^2$$

$$= e + 2e(x-1)^2.$$

i) $f(1{,}01) \approx T_2(1{,}01) = e + 2e\cdot 0{,}01^2 \approx 2{,}718825484824737$

$f(1{,}01) = \frac{e^{1{,}01^2}}{1{,}01^2}$ $\overset{TR}{\approx} 2{,}718823740130507$

5. a) $$\int_0^2 z\sqrt{z^2+4}\,dz = \frac{1}{2}\int_0^2 2z\sqrt{z^2+4}\,dz = \frac{1}{2}\left[\frac{(z^2+4)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2$$

$$= \frac{1}{\cancel{2}}\left[\frac{\sqrt{2^2+4}^{\,3}}{\frac{3}{\cancel{2}}} - \frac{\sqrt{4}^{\,3}}{\frac{3}{\cancel{2}}}\right] = \frac{1}{3}\cdot\left(\sqrt{8}^{\,3} - \underbrace{2^3}_{8}\right)$$

<u>ODER</u>: $$\int_0^2 z\sqrt{z^2+4}\,dz \underset{\substack{u=z^2+4\\ \frac{du}{dz}=2z\\ dz=\frac{du}{2\cdot z}}}{=} \int_{0^2+4}^{2^2+4}\frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_4^8 = \ldots$$

b) $$\int \underset{f}{x}\,\underset{g'}{e^x}\,dx = \underset{f}{x}\cdot\underset{g}{e^x} - \int \underset{f'}{1}\cdot \underset{g}{e^x}\,dx = xe^x - e^x + C = e^x(x-1)+C$$

c) $$\int x e^{x^2}\,dx = \frac{1}{2}\int 2x e^{x^2}\,dx = \frac{1}{2}e^{x^2}+C$$

<u>ODER</u>: $$\int x e^{x^2}\,dx \underset{\substack{u=x^2\\ \frac{du}{dx}=2x\\ dx=\frac{du}{2x}}}{=} \int \frac{1}{2}e^u\,du = \frac{1}{2}e^u + c = \frac{1}{2}e^{x^2}+c.$$

d) $$\int \underset{f'}{\sin(x)}\,\underset{g}{(2-x)}\,dx = \underset{f}{-\cos(x)}\cdot\underset{g}{(2-x)} - \int \underset{f}{(-\cos(x))}\cdot\underset{g'}{(-1)}\,dx = -\cos(x)\cdot(2-x) - \sin(x) + C$$

e) $$\int \frac{x+1}{2x^2+4x}\,dx = \frac{1}{4}\int\frac{4x+4}{2x^2+4x}\,dx = \frac{1}{4}\ln\left|2x^2+4x\right| + c$$

$(2x^2+4x)' = 4x+4$

f) nicht echt-rational → Polynomdivision:

$$\begin{array}{l} x^2 : (x^2-2x+1) = 1 + \frac{2x-1}{(x-1)^2}. \\ \underline{-(x^2-2x+1)} \\ \quad 2x-1 \end{array}$$

$\uparrow$ doppelte NST des Nenners

$$\frac{2x-1}{(x-1)^2} \overset{!}{=} \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2} \Rightarrow \underline{2x-1 = A(x-1)+B}$$

$x=1$ einsetzen liefert: $1 = B$

$x=0$ einsetzen liefert: $-1 = A\cdot(-1)+1 \Rightarrow A=2$

$$\Rightarrow \underline{\int \frac{x^2}{x^2-2x+1}\,dx} = \int 1 + \frac{2}{x-1} + \frac{1}{(x-1)^2}\,dx =$$

$$= x + 2\ln|x-1| + \underbrace{\int 1\cdot(x-1)^{-2}\,dx}_{\frac{(x-1)^{-2+1}}{-2+1}+c} =$$

$$= \underline{x + 2\ln|x-1| - \frac{1}{x-1} + c}$$

g) $$\underline{\int \frac{2x+8}{x^2+4x+5}\,dx} = \int \frac{2x+4}{x^2+4x+5} + \frac{4}{x^2+4x+5}\,dx =$$

$\uparrow$ $(x^2+4x+5)' = 2x+4$

$$= \ln|x^2+4x+5| + 4\cdot\underbrace{\int 1\cdot\frac{1}{(x+2)^2+1}\,dx}_{\arctan(x+2)+c} =$$

$$= \underline{\ln|x^2+4x+5| + 4\cdot\arctan(x+2) + c.}$$