Wdh. Analysis

1. Differential-/Integralrechng : Grenzwerte/Stetigkeit
2. — " — : Differentialr. / Kurvendiskussion
3. — " — : Integralr.
4. Taylorpolynom

$\boxed{Ü}$ Berechnen Sie den Grenzwert ($n \to \infty$) folgender Folgen:

a) $a_n = \frac{2n+1}{4n}$, b) $b_n = \frac{n^2+4}{n}$, c) $c_n = \frac{n^2+4n-1}{n^2-3n}$

a) $a_n = \frac{\cancel{n}(2+\frac{1}{n})}{\cancel{n}\cdot 4} \xrightarrow{n\to\infty} \frac{2+0}{4} = \frac{1}{2}$

b) $b_n = \frac{n^{\cancel{2}}(1+\frac{4}{n^2})}{\cancel{n}} = \underbrace{\underbrace{n}_{\to\infty} \underbrace{(1+\underbrace{\frac{4}{n^2}}_{\to 0})}_{\to 1}} \xrightarrow{n\to\infty} \infty$

c) $c_n = \frac{\cancel{n^2}(1+\frac{4}{n}-\frac{1}{n^2})}{\cancel{n^2}(1-\frac{3}{n})} \longrightarrow \frac{1+0-0}{1-0} = 1$

$\boxed{Ü}$ Berechnen Sie folgende Funktionsgrenzwerte:

a) $\lim\limits_{x\to 0} \frac{x^2-2x}{x^2+3x}$, b) $\lim\limits_{x\to 0} \frac{\sqrt{1+x^2}-1}{x}$, c) $\lim\limits_{x\to 2} \frac{(x-2)(3x+1)}{4x-8}$

a) $\frac{x^2-2x}{x^2+3x} = \frac{\cancel{x}(x-2)}{\cancel{x}(x+3)} \xrightarrow{x\to 0} \frac{0-2}{0+3} = \frac{-2}{3}$

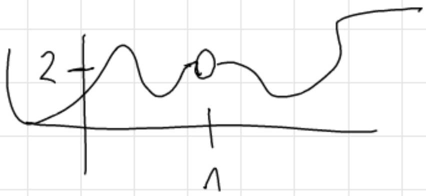
oder l'H: $\lim\limits_{x\to 0} \frac{x^2-2x}{x^2+3x} \overset{l'H}{\underset{„\frac{0}{0}“}{=}} \lim\limits_{x\to 0} \frac{2x-2}{2x+3} = \frac{2\cdot 0-2}{2\cdot 0+3} = -\frac{2}{3}$

b) $\lim\limits_{x\to 0} \frac{\sqrt{1+x}-1}{x} \overset{l'H}{\underset{"\frac{0}{0}"}{=}} \lim\limits_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}}\cdot 1}{1} = \frac{\frac{1}{2\sqrt{1+0}}\cdot 1}{1} = \frac{1}{2}.$

c) $\frac{(x-2)(3x+1)}{4x-8} = \frac{\cancel{(x-2)}(3x+1)}{4\cancel{(x-2)}} \xrightarrow{x\to 2} \frac{3\cdot 2+1}{4} = \frac{7}{4}.$ (oder mit l'H)

Ü Zeigen Sie, dass die Fkt. $f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{für } x \neq 1 \\ 2 & \text{für } x = 1 \end{cases}$

an der Stelle $x_0 = 1$ stetig ist.



Zz: $\lim\limits_{\substack{x\to 1\\(x\neq 1)}} f(x) = f(1).$

$\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \frac{x^2-1}{x-1} = \lim\limits_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = 1+1 = 2 = f(1)$ ✓

d.h. $\frac{x^2-1}{x-1}$ ist an $x_0 = 1$ stetig hebbar!

Ü Lassen sich die Definitionslücken von $f(x) = \frac{x^2-x}{x^3-x^2+x-1}$ stetig heben? Geben Sie ggf. die stetige Fortsetzung an!

Def. Lücken: $x^3 - x^2 + x - 1 \overset{!}{=} 0$ Rate: Teiler von $1: \pm 1$
$\to$ NST $+1$

Weitere NST? Pol.-Div.:

$$\begin{array}{l} (x^3 - x^2 + x - 1) : (x-1) = \underline{\underline{x^2+1}} \\ \underline{-(x^3 - x^2)} \\ \quad - \quad - \quad (x-1) \\ \qquad\qquad \underline{-(x-1)} \\ \qquad\qquad\quad - \quad - \end{array}$$

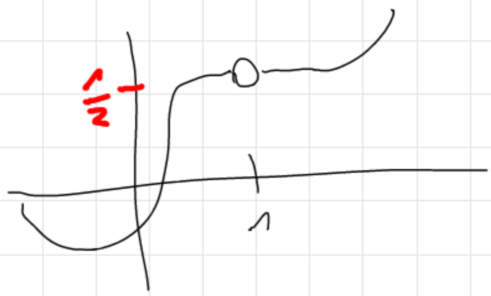
keine NST in $\mathbb{R}$

$\Rightarrow f(x) = \frac{x^2-x}{(x-1)(x^2+1)}$ hat 1 Def.Lücke $\underline{\underline{x_0 = 1}}$.

Stetig hebbar?

stetige Fortsetzung

$$g(x) = \begin{cases} f(x) & x \neq 1 \\ 2 \quad \frac{1}{2} & x = 1 \end{cases}$$



$$\lim_{x\to 1} f(x) = \lim_{x \to 1} \frac{x^2 - x}{(x-1)(x^2+1)} = \lim_{x\to 1} \frac{x\cancel{(x-1)}}{\cancel{(x-1)}(x^2+1)} = \frac{1}{(1^2+1)} = \frac{1}{2}$$

[Weiteres Bsp: $e^{-\frac{1}{x^2}}$ ist stetig fortsetzbar]

$\boxed{\text{Ü}}$ Bestimmen Sie lokale Extremwerte (f', f''), Wendepunkte (f'', f''') sowie das Verhalten für $x \to \pm\infty$ der folgenden Fkten. Skizzieren Sie diese dann:

a) $f(x) = x e^{-x}$

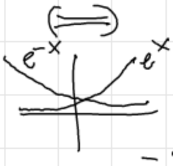
b) $g(x) = \frac{\ln(x^2)}{x}$, $D = \mathbb{R} \setminus \{0\}$, $\lim_{x \to 0} g(x) = ?$ ($x \searrow 0$, $x \nearrow 0$)

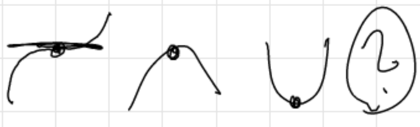
a) $\underline{f'(x)} = 1 \cdot e^{-x} + x \cdot \underbrace{(e^{-x})'}_{= e^{-x} \cdot \underbrace{(-x)'}_{-1} = -e^{-x}} = e^{-x} - x e^{-x} \left(= \underline{e^{-x}(1-x)} \right)$

$$\underline{f''(x)} = -e^{-x} - (x e^{-x})' = -e^{-x} - \left[e^{-x} - x e^{-x} \right]$$
$$= -2e^{-x} + x e^{-x} = \underline{e^{-x}(x-2)}$$

$$\underline{f'''(x)} = +2e^{-x} + (x e^{-x})' = 2e^{-x} + e^{-x} - x e^{-x}$$
$$= 3e^{-x} - x e^{-x} = \underline{e^{-x}(3-x)}.$$

Extremwerte: $f'(x) \stackrel{!}{=} 0 \stackrel{e^{-x} \neq 0}{\iff} 1 - x = 0 \iff \underline{x = 1}$





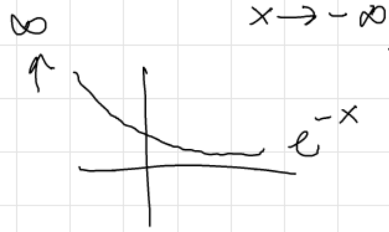
$f''(1) = e^{-1}(1-2) = -\frac{1}{e} < 0$ Max

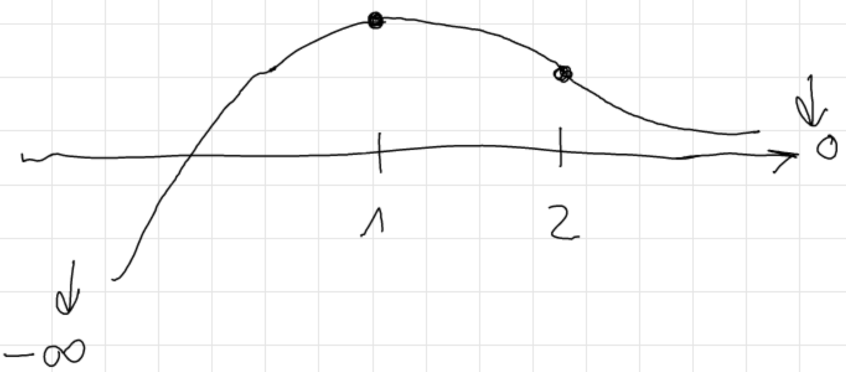
WP: $f''(x) = 0 \overset{e^{-x} \neq 0}{\iff} x - 2 = 0 \iff \underline{x = 2}$

$$f'''(2) = e^{-2}\underbrace{(3-2)}_{1} = \frac{1}{e^2} \underset{\neq}{>} 0 \quad \text{, d.h. } \underline{x = 2 \text{ WP}}$$

$x \to \pm\infty$: $\lim\limits_{x\to\infty} x e^{-x} = \lim\limits_{x\to\infty} \frac{x}{e^x} \overset{l'H}{\underset{\text{"}\frac{\infty}{\infty}\text{"}}{=}} \lim\limits_{x\to\infty} \frac{1}{\underbrace{e^x}_{\to\infty}} = 0$

$$\lim_{x\to-\infty} \underbrace{x}_{\to-\infty}\underbrace{e^{-x}}_{\to\infty} = \cancel{\lim_{x\to-\infty} \frac{x}{e^x} = \frac{-\infty}{0}} \left(\lim_{x\to-\infty} (-1)\underbrace{(-x e^{-x})}_{\to\infty}\right) = \underline{-\infty}$$





b) $g(x) = \frac{\ln(x^2)}{x}$, $\quad \left(\frac{\text{NAZ} - \text{ZAN}}{N^2}\right)$

$$g'(x) = \frac{x(\ln(x^2))' - \ln(x^2)\cdot x'}{x^2} = \frac{x\cdot\left[\frac{1}{x^2}2x\right] - \ln(x^2)}{x^2} = \frac{2 - \ln(x^2)}{x^2}$$

$$g''(x) = \frac{x^2(2-\ln(x^2))' - (2-\ln(x^2))\cdot 2x}{x^4} = \frac{x^2\cdot\left[-\frac{1}{x^2}2x\right] - (2-\ln(x^2))\cdot 2x}{x^4}$$

$$= \frac{-2x - (2-\ln(x^2))2x}{x^4} = \frac{-2 - 2(2-\ln(x^2))}{x^3} = \frac{-2(3-\ln(x^2))}{x^3}$$

$$g'''(x) = \frac{x^3(-2(3-\ln(x^2)))' + 2(3-\ln(x^2))\cdot 3x^2}{x^6} = \frac{\cancel{x}\cdot\left[x\cdot\left[+2\frac{1}{x^2}2x\right] + 6(3-\ln(x^2))\right]}{x^{\cancel{6}4}}$$

$$= \frac{4 + 6(3-\ln(x^2))}{x^4}$$

Extremwerte: $g'(x) = 0 \implies \frac{2-\ln(x^2)}{x^2} = 0 \overset{\cdot x^2}{\implies} 2 - \ln(x^2) = 0$

$$\implies 2 = \ln(x^2)$$

$$\overset{e^{\sqcup}}{\Rightarrow} \quad e^2 = x^2 \quad \overset{\sqrt{\ }}{\Rightarrow} \quad \underline{\underline{x}} = \pm\sqrt{e^2} = \underline{\underline{\pm e}}$$

$$\underline{x=e}: \quad g''(e) = \frac{-2(3-\ln(e^2))}{e^3} = \frac{-2(3-2)}{e^3} = \frac{-2}{e^3} < 0 \quad \underline{\underline{\text{Max}}}$$

$$\underline{x=-e}: \quad g''(-e) = \frac{-2(3-\ln((-e)^2))}{(-e)^3} = \frac{-2(3-2)}{-e^3} = \frac{-2}{-e^3} = \frac{2}{e^3} > 0 \quad \underline{\underline{\text{Min}}}$$

$$\underline{\underline{\text{WP}}}: \quad g''(x) = 0 \Rightarrow -2(3-\ln(x^2)) = 0 \overset{:(-2)}{\Rightarrow} 3-\ln(x^2) = 0$$

$$\Rightarrow 3 = \ln(x^2) \overset{e^{\sqcup}}{\Rightarrow} e^3 = x^2 \overset{\sqrt{\ }}{\Rightarrow} x = \pm\sqrt{e^3}$$

$$x = \sqrt{e^3}: \quad g'''(\sqrt{e^3}) = \frac{4+6(3-\ln(e^3))}{e^6}$$

$$x = -\sqrt{e^3}: \quad g'''(-\sqrt{e^3}) = \frac{4+6(3-\ln(e^3))}{e^6}$$

$$= \frac{4+6(\overbrace{3-3}^{0})}{e^6} = \frac{4}{e^6} > 0 \quad \underline{\underline{\neq 0}}$$

beides ist $\underline{\underline{\text{WP}}}$

$$= \frac{4+6(3-\ln(x^2))}{x^4}$$

$\underline{\text{Grenzwerte:}}$

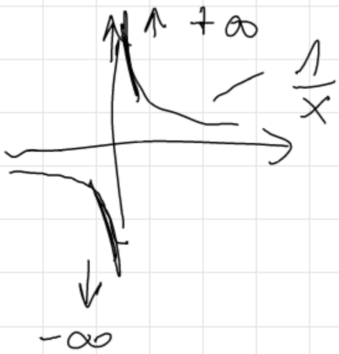
$$\lim_{x\to\infty} \frac{\ln(x^2)}{x} \underset{\text{„}\frac{\infty}{\infty}\text{“}}{\overset{l'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{x^2}\,2x}{1} = \lim_{x\to\infty} \frac{2}{x} = 0$$

$$\lim_{x\to-\infty} \frac{\ln(x^2)}{\cancel{x}} \underset{\frac{\infty}{-\infty}}{\overset{l'H}{=}} \lim_{x\to-\infty} \frac{2}{x} = 0$$

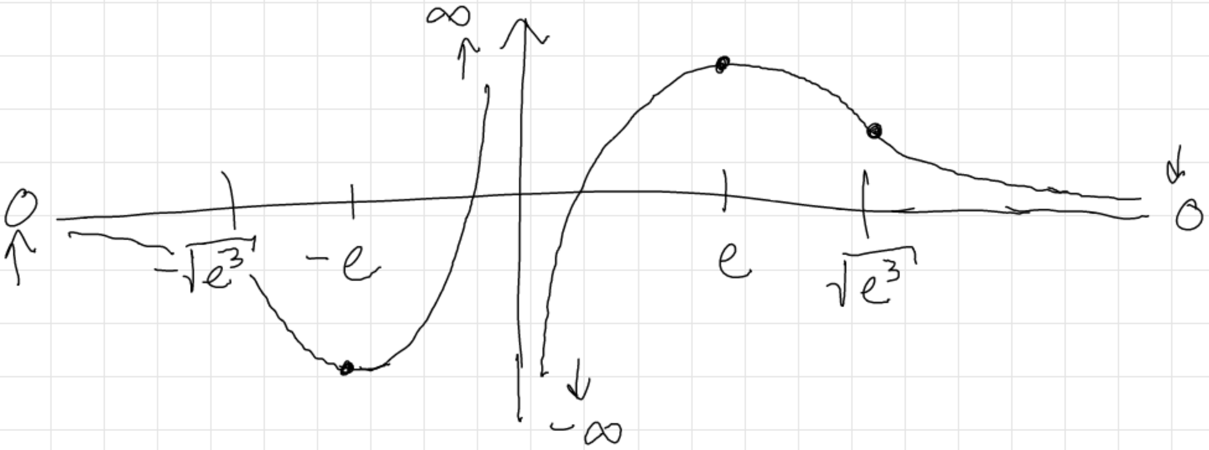
$$\lim_{x\to 0} \frac{\ln(x^2)}{x} \cancel{\underset{\frac{-\infty}{0}}{=}} \lim_{x\to 0} \underbrace{\frac{1}{x}}_{\to\pm\infty} \underbrace{\ln(x^2)}_{\to-\infty}$$

$\underline{\underline{\text{exist. nicht!}}}$

$$\lim_{x\searrow 0} \underbrace{\frac{1}{x}}_{\downarrow\, +\infty} \underbrace{\ln(x^2)}_{-\infty} = -\infty$$



$$\lim_{x\nearrow 0} \underbrace{\frac{1}{x}}_{\downarrow\, -\infty} \underbrace{\ln(x^2)}_{\downarrow\, -\infty} = +\infty,$$



Ü Berechnen Sie: a) $\int_0^{\frac{\pi}{2}} \cos^3(x)\sin(x)\,dx$

b) $\int x\sin(3x)\,dx$

c) Fläche zwischen $f(x)=x^2-2x-1$ & $g(x)=3x-1$.

Bestimmen Sie dazu zuvor die Schnittpunkte!

a) $-\int_0^{\frac{\pi}{2}} (\cos(x))^3(-\sin(x))\,dx = -\left[\frac{(\cos x)^4}{4}\right]_0^{\pi/2} = -\left(\frac{(\overbrace{\cos\frac{\pi}{2}}^{0})^4}{4} - \frac{(\overbrace{\cos 0}^{1})^4}{4}\right) = +\frac{1}{4}$

$$\int_a^b f(\varphi(x))\cdot\varphi'(x)\,dx = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx \overset{F'=f}{=} \left[F(\varphi(x))\right]_a^b$$

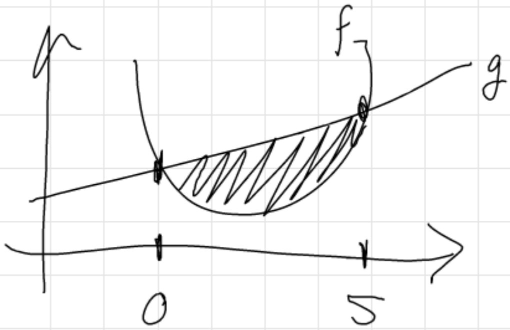
Unbestimmt!

b) $\int \underset{f}{x}\,\underset{g'}{\sin(3x)}\,dx = -\frac{1}{3}x\cos(3x) - \int 1\cdot\left(-\frac{1}{3}\cos(3x)\right)dx$

$\int fg' = f\cdot g - \int f'g$

$= -\frac{1}{3}x\cos(3x) + \frac{1}{3}\cdot\frac{1}{3}\cdot\sin(3x) + \underline{C}.$

c)



Schnittpkte: $f(x)=g(x)$

$\Rightarrow x^2-2x-1 = 3x-1$

$\Rightarrow x^2-5x=0 \Rightarrow x(x-5)=0$

$$\Rightarrow \underline{x=0 \lor x=5}$$

$$\int_0^5 g(x)-f(x)\,dx = \int_0^5 -x^2+5x\,dx = \left[-\frac{x^3}{3}+5\frac{x^2}{2}\right]_0^5$$

$$=\left(-\frac{5^3}{3}+5\frac{5^2}{2}\right)-\underbrace{\left(-\frac{0^3}{3}+5\frac{0^2}{2}\right)}_{0} = \frac{5^3}{2}-\frac{5^3}{3} = 5^3\left(\frac{1}{2}-\frac{1}{3}\right)$$

$$= 5^3\cdot\frac{1}{6} = \frac{125}{6} \approx 20{,}8\ldots$$

$$\boxed{\arctan'(x) = \frac{1}{1+x^2}}$$

Ü $\arctan: \mathbb{R} \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

a) Bestimmen Sie:
- Taylor-Reihe — an der Stelle $\underline{x_0 = 0}$
- Taylor-Polynome vom Grad 0, 1, 2, 3

Skizzieren Sie arctan & Taylor-Polynome

b) Berechnen Sie den Konvergenzradius der Taylor-Reihe.

c) Zeigen Sie: $\arctan(1) = \frac{\pi}{4}$ (Hinweis: $\tan^{-1} = \arctan$)

d) Berechnen Sie π näherungsweise durch $T_3(x)$ aus a) mit Hilfe von c).

e) Schätzen Sie den Fehler aus d) mittels einer Restglied-abschätzung ab.

f) Programmieren Sie das Taylorpolynom an $\underline{x=1}$ bis zu einem beliebigen Grad n in Java.

a) $$T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$$ Taylor-Reihe

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$$ Taylor-Polynom Grad n

$$\arctan^{(0)}(x) = \arctan(x) \overset{x=0}{=} 0$$

$$\arctan^{(1)}(x) = \frac{1}{1+x^2} \overset{x=0}{=} 1 = +0!$$

$$\arctan''(x) = \frac{\cancel{(1+x^2)\cdot 0} - 1\cdot 2x}{(1+x^2)^2} \overset{x=0}{=} 0$$

$$\arctan'''(x) = \frac{6x^2-2}{(1+x^2)^3} \overset{x=0}{=} -2 = -2!$$

$$\arctan^{(IV)}(x) = -\frac{24x(x^2-1)}{(x^2+1)^4} \overset{x=0}{=} 0$$

$$\arctan^{(5)}(x) = \frac{24(5x^4-10x^2+1)}{(x^2+1)^5} \overset{x=0}{=} +24 = +4!$$

$$\arctan^{(6)}(x) = -\frac{240x(3x^4-10x^2+3)}{(x^2+1)^6} \overset{x=0}{=} 0 \quad \ldots$$

$$\arctan^{(k)}(0) = \begin{cases} 0, & k = 2\cdot l \text{ gerade} \\ (-1)^l (2l)!, & k = 2\cdot l+1 \text{ ungerade} \end{cases}$$

$$\frac{(-1)^l (2l)!}{(2l+1)!}$$

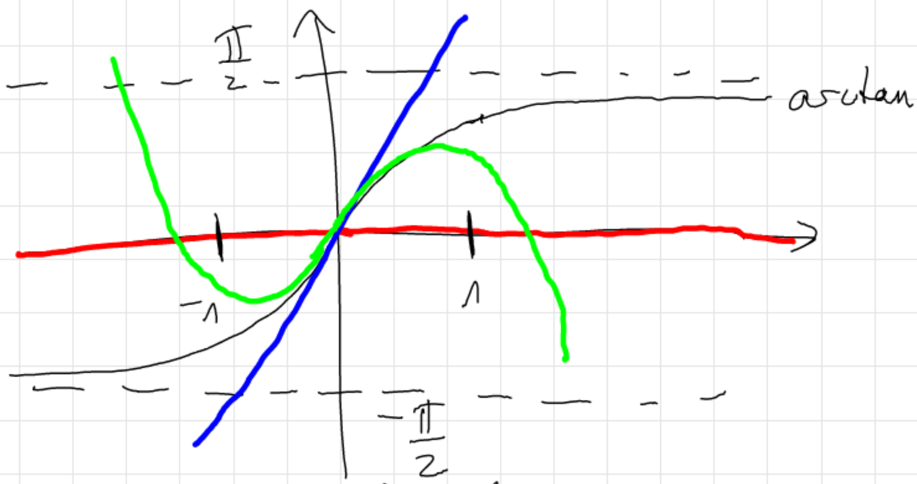
$$T(x) = \sum_{l=0}^{\infty} \underbrace{\frac{(-1)^l}{2l+1}}_{a_l} x^{2l+1} = \underset{l=0}{x} - \underset{l=1}{\frac{x^3}{3}} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} \pm \ldots$$

$T_0(x) = 0$

$T_1(x) = x$

$T_2(x) = x$

$T_3(x) = x - \frac{x^3}{3}$



b) Konvergenzradius: $$\lim_{l\to\infty} \left| \frac{a_l}{a_{l+1}} \right| = \lim_{l\to\infty} \left| \frac{(-1)^l / 2l+1}{(-1)^{l+1} / 2(l+1)+1} \right| =$$

$$= \lim_{\ell \to \infty} \frac{2(\ell+1)+1}{2\ell+1} = \lim_{\ell\to\infty} \frac{2\ell+3}{2\ell+1} = \frac{2}{2} = 1$$

c) $$\arctan(1) \overset{!}{=} \frac{\pi}{4} \overset{\tan(\sqcup)}{\Longleftrightarrow} 1 = \tan\left(\frac{\pi}{4}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)}$$ ✓

$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

d) $$\pi = 4 \cdot \underbrace{\arctan(1)}_{} \approx 4 \cdot 0,\overline{6} = \frac{8}{3} = 2,\overline{6}$$

$$\approx T_3(1) = 1 - \frac{1^3}{3} = 1 - \frac{1}{3} = \frac{2}{3} = 0,\overline{6}$$

e) $$\underline{\text{Fehler}} = |R_3(1)| \le \frac{C}{4!}|1-0|^4 \le \frac{4{,}7}{24} = \cancel{0,78\overline{3}} \quad 0,1958\ldots$$

C obere Schranke von $\left|f^{(4)}(x)\right| = \left|-\frac{24x(x^2-1)}{(x^2+1)^4}\right|$ im Bereich $[\overset{x_0}{0}, 1]$: Lokales Maximum bei $x = \sqrt{1-\frac{2}{\sqrt{5}}} \approx 0,325$ mit Wert ← Kurvendiskussion!

$$C = \frac{15}{16}\sqrt{\frac{25}{2} + \frac{11\sqrt{5}}{2}} \approx 4,6686 \le 4,7$$

f) $$T_n(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \pm \ldots + \frac{(-1)^n}{2n+1}$$

$$(-1)^n = \begin{cases} +1 & n \text{ gerade} \\ -1 & n \text{ ungerade} \end{cases}$$

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(n%2==0) ? 1 : -1
(n&1==0) ? 1 : -1
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→ Homepage!