Mathematical Biostatistics Boot Camp 2: Lecture 8, Chi-Squared Tests

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Chi-squared testing

- An alternative approach to testing equality of proportions uses the chi-squared statistic
$$\sum \frac{(\text{Observed} - \text{Expected})^2}{\text{Expected}}$$
- "Observed" are the observed counts
- "Expected" are the expected counts under the null hypothesis
- The sum is over all four cells
- This statistic follows a Chi-squared distribution with 1 df
- The Chi-squared statistic is exactly the square of the difference in proportions Score statistic

Example

Trt	Side Effects	None	Total
X	44	56	100
Y	77	43	120
	121	99	220

- p_1 and p_2 are the cure rates
- $H_0 : p_1 = p_2$

- The χ^2 statistic is $\sum \frac{(O-E)^2}{E}$
- $O_{11} = 44,\ E_{11} = \frac{121}{220} \times 100 = 55$
- $O_{21} = 77,\ E_{21} = \frac{121}{220} \times 120 = 66$
- $O_{12} = 56,\ E_{12} = \frac{99}{220} \times 100 = 45$
- $O_{22} = 43,\ E_{22} = \frac{99}{220} \times 120 = 54$

$$\chi^2 = \frac{(44-55)^2}{55} + \frac{(77-66)^2}{666} + \frac{(56-45)^2}{45} + \frac{(43-54)^2}{54}$$

Which turns out to be 8.96. Compare to a χ^2 with one degree of freedom (reject for large values).

```
pchisq(8.96, 1, lower.tail = FALSE)
#result is 0.002
```

R code

```
dat <- matrix(c(44, 77, 56, 43), 2)
chisq.test(dat)
chisq.test(dat, correct = FALSE)
```

Notation reminder

$n_{11} = x$	$n_{12} = n_1 - x$	$n_1 = n_{1+}$
$n_{21} = y$	$n_{22} = n_2 - y$	$n_2 = n_{2+}$
n_{+1}	n_{+2}	

Notes

- Reject if the statistic is too large
- Alternative is two sided
- Do not divide α by 2
- A small χ^2 statistic implies little difference between the observed values and those expected under H_0
- The χ^2 statistic and approach generalizes to other kinds of tests and larger contingency tables
- Alternative computational form for the χ^2 statistic

$$\chi^2 = \frac{n(n_{11}n_{22} - n_{12}n_{21})^2}{n_{+1}n_{+2}n_{1+}n_{2+}}$$

Notes

- Notice that the statistic:

$$\chi^2 = \frac{n(n_{11}n_{22} - n_{12}n_{21})^2}{n_{+1}n_{+2}n_{1+}n_{2+}}$$

  does not change if you transpose the rows and the columns of the table
- Surprisingly, the χ^2 statistic can be used
  - the rows are fixed (binomial)
  - the colums are fixed (binomial)
  - the total sample size is fixed (multinomial)
  - none are fixed (Poisson)
- For a given set of data, any of these assumptions results in the same value for the statistic

Testing independence

- Maternal age versus birthweight[1]
- Cross-sectional sample, only the total sample size is fixed

	Birthweight		
Mat. Age	$< 2500g$	$\geq 2,500g$	Total
$< 20y$	20	80	100
$\geq 20y$	30	270	300
Total	50	350	400

- H_0 : MA is independent of BW
- H_a : MA is not independent of BW

[1] From Agresti Categorical Data Analysis second edition

Continued

- Under H_0 (est) $P\,(\text{MA} < 20) = \frac{100}{400} = .25$
- Under H_0 (est) $P\,(\text{BW} < 2500) = \frac{50}{400} = .125$
- Under H_0 (est)

$$P\,(\text{MA} < 20 \text{ and BW} < 2,500) = .25 \times .125$$

- Therefore
  - $E_{11} = \frac{100}{400} \times \frac{50}{400} \times 400 = 12.5$
  - $E_{12} = \frac{100}{400} \times \frac{350}{400} \times 400 = 87.5$
  - $E_{21} = \frac{300}{400} \times \frac{50}{400} \times 400 = 37.5$
  - $E_{22} = \frac{300}{400} \times \frac{350}{400} \times 400 = 262.5$
  - $\chi^2 = \frac{(20-12.5)^2}{12.5} + \frac{(80-87.5)^2}{87.5} \frac{(30-37.5)^2}{37.5} + \frac{(270-262.5)^2}{262.5} = 6.86$
- Compare to critical value
  `qchisq(.95, 1)=3.84`
- Or calculate P-value
  `pchisq(6.86, 1, lower.tail = F)=.009`

Chi-squared testing cont'd

	Alcohol use		
Group	High	Low	Total
Clergy	32	268	300
Educators	51	199	250
Executives	67	233	300
Retailers	83	267	350
Total	233	967	1,200

[2]

[2] From Agresti's Categorical Data Analysis second edition

- Interest lies in testing whether or not the proportion of high alcohol use is the same in the four occupations
- $H_0 : p_1 = p_2 = p_3 = p_4 = p$
- H_a : at least two of the p_j are unequal
- $O_{11} = 32, \; E_{11} = 300 \times \frac{233}{1200}$
- $O_{12} = 268, \; E_{12} = 300 \times \frac{967}{1200}$
- ...
- Chi-squared statistic $\sum \frac{(0-E)^2}{E} = 20.59$
- $df = (Rows - 1)(Columns - 1) = 3$
- Pvalue `pchisq(20.59, 3, lower.tail = FALSE)` ≈ 0

Word distributions

	Book			
Word	1	2	3	Total
a	147	186	101	434
an	25	26	11	62
this	32	39	15	86
that	94	105	37	236
with	59	74	28	161
without	18	10	10	38
Total	375	440	202	1017

[3]

[3] From Rice Mathematical Statistics and Data Analysis, second edition

- H_0 : The probabilities of each word are the same for every book
- H_a : At least two are different
- $O_{11} = 147$ $E_{11} = 375 \times \frac{434}{1017}$
- $O_{12} = 186$ $E_{12} = 440 \times \frac{434}{1017}$
- ...
- $\sum \frac{(O-E)^2}{E} = 12.27$
- $df = (6-1)(3-1) = 10$

Testing independence

	Wife's Rating				
Husband	N	F	V	A	Tot
N	7	7	2	3	19
F	2	8	3	7	20
V	1	5	4	9	19
A	2	8	9	14	33
	12	28	18	33	91

N=never, F=fairly often, V=very often, A=almost always [4]

[4] From Agresti's Categorical Data Analysis second edition

Independence cont'd

- H_0 : H and W ratings are independent
- H_a : not independent
- $P(H = N \ \& \ W = A) = P(H = N)P(W = A)$
- $stat = \sum \frac{(O-E)^2}{E}$
- $O_{11} = 7$ $E_{11} = 91 \times \frac{19}{91} \times \frac{12}{91} = 2.51$
- $E_{ij} = n_{i+}n_{+j}/n$
- $df = (Rows - 1)(Cols - 1)$

Independence cont'd

```
x<-matrix(c(7,7,2,3,
            2,8,3,7,
            1,5,4,9,
            2,8,9,14),4)
chisq.test(x)
```

- $\sum \frac{(O-E)^2}{E} = 16.96$
- $df = (4-1)(4-1) = 9$
- $p - value = .049$
- Cell counts might be too small to use large sample approximation

Notes

- Equal distribution and independence test yield the same results
- Same test results if
    - row totals are fixed
    - column totals are fixed
    - total ss is fixed
    - none are fixed
- Note that this is common in statistics; mathematically equivalent results are applied in different settings, but result in different interpretations

- Chi-squared result requires large cell counts
- df is always $(Rows - 1)(Columns - 1)$
- Generalizations of Fishers exact test can be used or continuity corrections can be employed

Exact permutation test

- Reconstruct the individual data

  ```
  W:NNNNNNNFFFFFFVVAAANNFFFFFFFFF ...
  H:NNNNNNNNNNNNNNNNNNNFFFFFFFFFF ...
  ```

- Permute either the `W` or `H` row
- Recalculate the contingency table
- Calculate the χ^2 statistic for each permutation
- Percentage of times it is larger than the observed value is an exact P-value

  ```
  chisq.test(x, simulate.p.value = TRUE)
  ```

Chi-squared goodness of fit

Results from R's RNG

	$[0, .25)$	$[.25, .5)$	$[.5, .75)$	$[.75, 1)$	Total
Count	254	235	267	244	1000
TP	.25	.25	.25	.25	1

- $H_0: \; p_1 = .25, \; p_2 = .25, \; p_3 = .25, \; p_4 = .25$
- H_a : any $p_i \neq$ it's hypothesized value

Continued

- $O_1 = 254 \;\; E_1 = 1000 \times .25 = 250$
- $O_2 = 235 \;\; E_2 = 1000 \times .25 = 250$
- $O_3 = 267 \;\; E_3 = 1000 \times .25 = 250$
- $O_4 = 244 \;\; E_4 = 1000 \times .25 = 250$
- $\sum \frac{(O-E)^2}{E} = 2.264$
- $df = 3$
- $P - value = .52$

Testing Mendel's hypothesis

	Phenotype		
	Yellow	Green	Total
Observed	6022	2001	8023
TP	.75	.25	1
Expected	6017.25	2005.75	8023

- $H_0 : p_1 = .75, \; p_2 = .25$
- $\sum \frac{(0-E)^2}{E} = \frac{(6022-6017.25)^2}{6017.25} + \frac{(2001-2005.75)^2}{2005.75} = .015$

Continued

- $df = 1$
- P-value $= .90$
- Fisher combined several of Mendel's tables
- $\sum \chi^2_{v_i} \sim \chi^2_{\sum v_i}$
- Statistic 42, $df = 84$, P-value $= .99996$
- Agreement with theoretical counts is perhaps too good?

Notes on GOF

- Test of whether or not observed counts equal theoretical values
- Test statistic is $\sum \frac{(0-E)^2}{E}$
- TS follows χ^2 distribution for large n
- df is the number of cells minus 1
- Undirected alternative is problematic
- Especially useful for testing RNGs
- Kolmogorov/Smirnov test is an alternative test that does not require discretization but often has low power