Mathematical Biostatistics Boot Camp 2: Lecture 2, Power

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Power

- Power is the probability of rejecting the null hypothesis when it is false
- Ergo, power (as it's name would suggest) is a good thing; you want more power
- A type II error (a bad thing, as its name would suggest) is failing to reject the null hypothesis when it's false; the probability of a type II error is usally called β
- Note Power $= 1 - \beta$

Notes

- Consider our previous example involving RDI
- $H_0 : \mu = 30$ versus $H_a : \mu > 30$
- Then power is

$$P\left(\frac{\bar X - 30}{s/\sqrt{n}} > t_{1-\alpha,n-1} \mid \mu = \mu_a\right)$$

- Note that this is a function that depends on the specific value of μ_a!
- Notice as μ_a approaches 30 the power approaches α

Calculating power

Assume that n is large and that we know σ

$$
\begin{aligned}
1-\beta &= P\left(\frac{\bar X - 30}{\sigma/\sqrt{n}} > z_{1-\alpha} \mid \mu = \mu_a\right) \\
&= P\left(\frac{\bar X - \mu_a + \mu_a - 30}{\sigma/\sqrt{n}} > z_{1-\alpha} \mid \mu = \mu_a\right) \\
&= P\left(\frac{\bar X - \mu_a}{\sigma/\sqrt{n}} > z_{1-\alpha} - \frac{\mu_a - 30}{\sigma/\sqrt{n}} \mid \mu = \mu_a\right) \\
&= P\left(Z > z_{1-\alpha} - \frac{\mu_a - 30}{\sigma/\sqrt{n}} \mid \mu = \mu_a\right)
\end{aligned}
$$

Example continued

- Suppose that we wanted to detect a increase in mean RDI of at least 2 events / hour (above 30). Assume normality and that the sample in question will have a standard deviation of 4; what would be the power if we took a sample size of 16?
- $Z_\alpha = 1.645$ and $\frac{\mu_a - 30}{\sigma/\sqrt{n}} = 2/(4/\sqrt{16}) = 2$
- $P(Z > 1.645 - 2) = P(Z > -0.355) = 64\%$

Example continued

- What n would be required to get a power of 80%
- I.e. we want
$$0.80 = P\left(Z > z_{1-\alpha} - \frac{\mu_a - 30}{\sigma/\sqrt{n}} \mid \mu = \mu_a\right)$$
- Set $z_{1-\alpha} - \frac{\mu_a - 30}{\sigma/\sqrt{n}} = z_{0.20}$ and solve for n

Notes

- The calculation for $H_a : \mu < \mu_0$ is similar
- For $H_a : \mu \neq \mu_0$ calculate the one sided power using $\alpha/2$ (this is only approximately right, it excludes the probability of getting a large TS in the opposite direction of the truth)
- Power goes up as α gets larger
- Power of a one sided test is greater than the power of the associated two sided test
- Power goes up as μ_1 gets further away from μ_0
- Power goes up as n goes up

Power for the T test

- Consider calculating power for a Gossett's T test for our example
- The power is
$$P\left(\frac{\bar X - 30}{S/\sqrt{n}} > t_{1-\alpha,n-1} \mid \mu = \mu_a\right)$$
- Notice that this is equal to

$$\begin{aligned} &= P\left(\sqrt{n}(\bar X - 30) + > t_{1-\alpha,n-1} S \mid \mu = \mu_a\right) \\ &= P\left(\frac{\sqrt{n}(\bar X - 30)}{\sigma} + > t_{1-\alpha,n-1} \frac{S}{\sigma} \mid \mu = \mu_a\right) \end{aligned}$$

Continued

- Continued

$$P\left(\frac{\sqrt{n}(\bar X - \mu_a)}{\sigma} + \frac{\sqrt{n}(\mu_a - 30)}{\sigma} > \frac{t_{1-\alpha,n-1}}{\sqrt{n-1}} \times \sqrt{\frac{(n-1)S^2}{\sigma^2}}\right)$$

  (where we ommitted the conditional on μ_a part for space)

- This is now equal to

$$P\left(Z + \frac{\sqrt{n}(\mu_a - 30)}{\sigma} > \frac{t_{1-\alpha,n-1}}{\sqrt{n-1}}\sqrt{\chi^2_{n-1}}\right)$$

  where Z and χ^2_{n-1} are independent standard normal and chi-squared random variables

- While computing this probability is outside the scope of the class, it would be easy to approximate with Monte Carlo

Example

Let's recalculate power for the previous example using the T distribution instead of the normal; here's the easy way to do it. Let $\sigma = 4$ and $\mu_a - \mu_0 = 2$

```
##the easy way
power.t.test(n = 16, delta = 2 / 4,
             type = "one.sample",
             alt = "one.sided")
##result is 60%
```

Example

Using Monte Carlo

```
nosim <- 100000
n <- 16
sigma <- 4
mu0 <- 30
mua <- 32
z <- rnorm(nosim)
xsq <- rchisq(nosim, df = 15)
t <- qt(.95, 15)
mean(z + sqrt(n) * (mua - mu0) / sigma >
     t / sqrt(n - 1) * sqrt(xsq))
##result is 60%
```

Comments

- Notice that in both cases, power required a true mean and a true standard deviation
- However in this (and most linear models) the power depends only on the mean (or change in means) divided by the standard deviation