THE EE EPP TEAM

ELECTRICAL ENGINEERING PRINCIPLES AND PRACTICE II

EE2111A NOTES (WEEK 1 - WEEK 6)

NATIONAL UNIVERSITY OF SINGAPORE

Introduction to AC Circuits

DC versus AC

In EPP1, you learnt about electrical circuits driven by **DC** (direct current) sources. When a load is connected to such a source, current flows in one particular direction. Batteries and photovoltaic (PV) modules are examples of practical DC sources used in electrical and electronic systems.

In AC circuits, the polarity of the source continuously alternates. If a load is connected to such a source, the direction of current alternates. The power sockets seen on the walls of our homes and offices provide AC voltage.

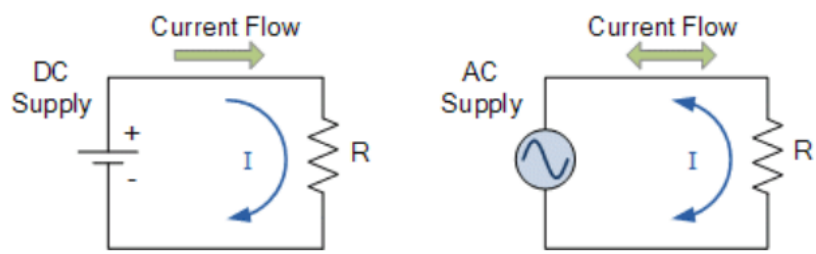


Figure 1: DC circuit: current flows in one direction, AC circuit: direction of current alternates.

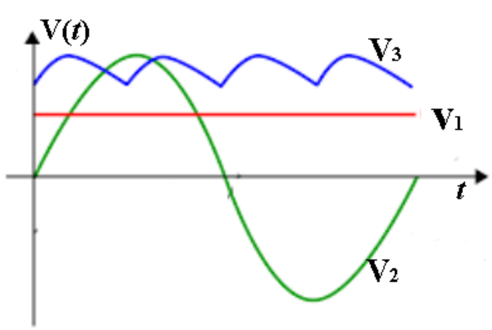


Figure 2: AC voltage or current is a time-varying quantity (V_2 in this image is AC voltage). DC voltage or current can be either constant or time-varying. Both V_1 and V_3 shown in the image are DC. V_1 is constant while V_3 is time varying.

When you observe a DC voltage using an oscilloscope, you see the voltage signal above or below the ground position. But when you observe an AC voltage, the voltage signal alternately goes above and below the ground position.

Waveform

Waveform is the graph showing a time-varying quantity (for example, voltage & current) versus time (Figure 2). A waveform is called **periodic** if it repeats the exact same shape again and again. Sinusoidal waveform, a waveform where the dependent variable varies according to the trigonometric function *sine* or *cosine*, is by far the most common waveform used in AC circuit analysis.

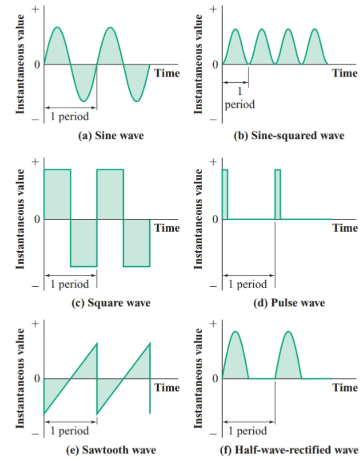


Figure 3: Some common periodic waves (Source of the image: electricalacademia.com)

Why Sinusoidal Waveform?

- Generators at power stations which are rotating machines produce sinusoidal voltage.
- Any periodic waveform can be reconstructed by adding a set of sinusoidal waves (Fourier series). You will learn more about Fourier series in higher level modules such as EE2023. A simple explanation of this concept is given with the help of graphical illustration at the end of this chapter.

Sinusoidal variation of voltage or current can be represented using either the sine function or the cosine function, for example,

$$\begin{aligned} v(t) &= V_m \sin(\omega t + \theta) \\ i(t) &= I_m \cos(\omega t + \theta). \end{aligned}$$

The meanings of the symbols ω, θ, V_m, and I_m are explained in the next few sub-sections.

Waveform parameter: Period and Frequency

A periodic waveform repeats the exact same shape. The **period** is the interval of that repetition. For the waveform shown in Figure 4, the period is 0.04 second or 40 milliseconds.

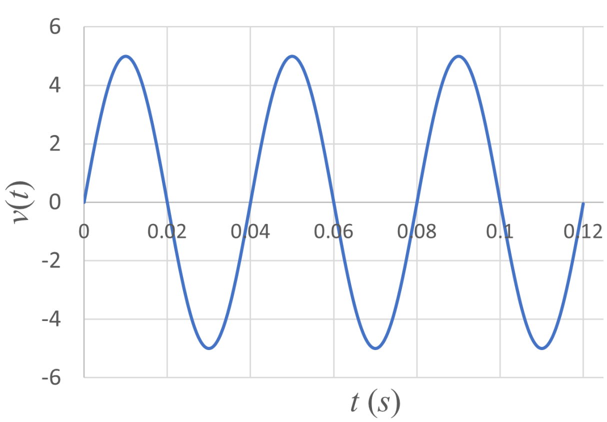


Figure 4: A periodic waveform with period of 40 millisecond or 0.04 s

Frequency f is the number of repetitions in unit time. If T is the period of a periodic waveform, then its frequency is

$$f = \frac{1}{T}.$$

Taking *second* as the unit of time, the unit of frequency is *cycles per second* which is also known as **hertz (Hz)**. The frequency of the waveform shown in Figure 4 is

$$f = \frac{1}{40 \times 10^{-3}\ s} = 25\ Hz.$$

Waveform parameter: Angular frequency

The functions $\sin(x)$ and $\cos(x)$ repeat themselves every 2π radians or $360°$, which is the period of these two functions with radian or degree as the unit of the variable x.

If a time-varying sinusoidal function $\sin(\omega t)$ or $\cos(\omega t)$ repeats itself every T seconds, then

$$\begin{aligned} \omega T &= 2\pi, \\ \omega &= \frac{2\pi}{T}. \end{aligned}$$

ω is the **angular frequency** of the sinusoidal waveform, and its unit is rad/s. Since $f = \frac{1}{T}$, the angular frequency can also be expressed in terms of f:

$$\omega = 2\pi f.$$

Points to note:

- Period and frequency are parameters of any periodic waveform regardless of its shape.
- Angular frequency is a parameter used only for a sinusoidal waveform.

Waveform parameter: Amplitude

The parameter V_m of

$$V_m \sin(\omega t)$$

is its **amplitude**. It represents the maximum deviation from the mean. The amplitude is 5 units for the waveform shown in Figure 4.

For a DC-shifted waveform

$$v(t) = 2 + 5\sin(\omega t)$$

shown in Figure 5, though the value of the variable v varies between 7 and -3, the amplitude is 5. The mean of this waveform is 2.

If you observe the waveform in Figure 4 and Figure 5,

$$v(t)|_{t=0} = V_{mean}.$$

In other words,

$$\begin{aligned} v(0) &= 5\sin(\omega \times 0) = 0 \quad for\ the\ first\ one, \\ v(0) &= 2 + 5\sin(\omega \times 0) = 2 \quad for\ the\ 2nd\ one, \end{aligned}$$

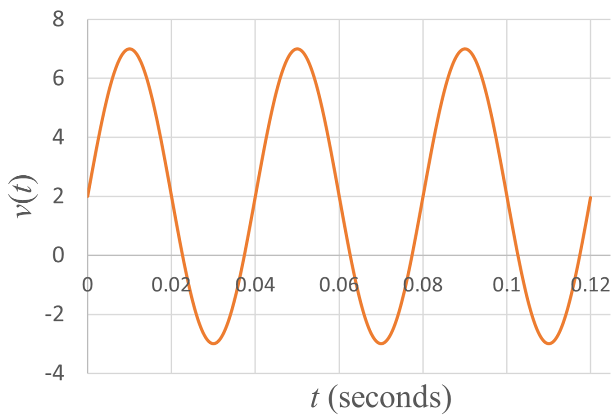


Figure 5: DC-shifted sinusoidal waveform

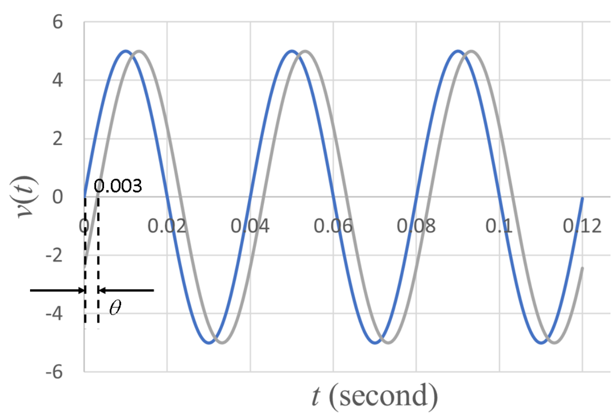


Figure 6: Phase-shifted sinusoidal waveform

Waveform parameter: Phase

If a sinusoidal waveform shows an offset in time-axis, we need another parameter called **phase** to describe that offset. In Figure 6, the blue colored graph represents the function $5\sin(\omega t)$. The grey colored graph is also a sine wave but with an offset in time-axis. The phase θ in

$$V_m \sin(\omega t + \theta)$$

describes the offset in time-axis.

For the grey-colored waveform in Figure 6, $\sin(0)$ occurs at $t = 0.003$ s. So,

$$\begin{aligned} 5\sin(0) &= 5\sin(\omega \times 0.003 + \theta), \\ \omega \times 0.003 + \theta &= 0, \\ \theta &= -0.003\omega, \\ \theta &= \frac{-0.003}{T} \times 2\pi. \end{aligned}$$

The grey-colored waveform can be expressed as

$$5\sin(\omega t - \frac{0.003}{T} \times 2\pi) = 5\sin(\omega t - 0.47).$$

This waveform has a negative phase and is said to be **lagging** the graph of $\sin(\omega t)$ by 0.47 rad.

If you plot the waveform

$$5\sin(\omega t + 0.5),$$

you will notice that this one appears earlier in time with respect to the waveform $5\sin(\omega t)$. The waveform $5\sin(\omega t + 0.5)$ is said to be **leading** the graph of $5\sin(\omega t)$ by 0.5 rad.

The unit of ω is rad/s and hence the unit of ωt is rad. In the two examples shown here, the unit of phase is also rad. However, many books and published literature use degrees (°) for phase.

Sine or Cosine?

Sinusoidal variation can be expressed using either *sine* or *cosine* as these two functions have identical shape but with offset in time axis that results in a phase shift of $\frac{\pi}{2}$ rad or 90°.

The waveform $\sin(\omega t)$ lags the waveform $\cos(\omega t)$ by $\frac{\pi}{2}$ rad.

$$\begin{aligned}\cos(\omega t - \frac{\pi}{2}) &= \cos(\omega t)\cos(\frac{\pi}{2}) + \sin(\omega t)\sin(\frac{\pi}{2}) \\ &= \sin(\omega t).\end{aligned}$$

The waveform $\cos(\omega t)$ leads the waveform $\sin(\omega t)$ by $\frac{\pi}{2}$ rad.

$$\begin{aligned}\sin(\omega t + \frac{\pi}{2}) &= \sin(\omega t)\cos(\frac{\pi}{2}) + \cos(\omega t)\sin(\frac{\pi}{2}) \\ &= \cos(\omega t).\end{aligned}$$

Summary

The general expression of a sinusoidal waveform is

$$v(t) = V_m \sin(\omega t + \theta)$$

or

$$v(t) = V_m \cos(\omega t + \theta)$$

- The **amplitude** V_m represents the magnitude of the maximum deviation from the mean
- ω is the angular frequency which is related to the frequency (in Hz) and the period of the waveform:

  $$\omega = 2\pi f = \frac{2\pi}{T}.$$

- The **phase** θ represents offset in time-axis with respect to another waveform.
  - If θ is positive, the waveform is said to be leading with respect to the reference waveform.
  - If θ is negative, the waveform is said to be lagging with respect to the reference waveform.

RMS voltage and RMS current

Although the voltage at the power socket varies with time, we do not use a time-varying function when we refer to these voltages. Instead, we say 230 V, 50 Hz (in Singapore, UK, Australia etc.) or 120 V, 60 Hz (in the USA or Canada) power supply.

Similarly, when an equipment is connected to such socket, the current drawn is also a time-varying quantity but we say 15 A or 3 A to refer to the current drawn.

These constant numbers used to describe the magnitude of AC voltages or currents are their **RMS** (root mean square) values. As the name suggests, the RMS value is the square **root** of the **mean** of the **square** of a time-varying quantity.

In case of a set of N discrete values $(x_1, x_2, x_3, \cdots x_N)$,

$$X_{rms} = \sqrt{\frac{x_1^2 + x_2^2 + x_3^2 + \cdots + x_N^2}{N}}.$$

For a continuous function $x(t)$,

$$X_{rms} = \lim_{T \to \infty} \sqrt{\frac{1}{T} \int_0^T x^2(t)dt}.$$

If the continuous function $x(t)$ is periodic with period T_p then,

$$X_{rms} = \sqrt{\frac{1}{T_p} \int_0^{T_p} x^2(t)dt}.$$

The RMS value of a sinusoidal waveform

$$V_m \sin(\omega t + \theta) \;\; or \;\; V_m \cos(\omega t + \theta)$$

is

$$V_{rms} = \frac{V_m}{\sqrt{2}}. \tag{1}$$

Refer to Appendix A-1 for proof.

AC Circuit Analysis

The circuit laws and device characteristics, *i.e.*. KVL, KCL and device property (Ohm's law for resistor) are the same in both DC circuit and AC circuit. What make the AC circuit analysis difficult and often challenging are

1. time-varying nature of voltage and current, and
2. properties of capacitor and inductor involving rate of change of voltage and current, respectively.

$$i_c = C\frac{dv_c}{dt}, \quad for\ C$$

$$v_L = L\frac{di_L}{dt}, \quad for\ L.$$

Let's consider the simple cases where only one circuit element (a resistor, an inductor, or a capacitor) is connected to a sinusoidal voltage source.

Load is a resistor

Instantaneous voltage $v_R(t)$ across the resistor is same as the source voltage:

$$v_R(t) = V_m \sin(\omega t).$$

Then the instantaneous current through the resistor (by Ohm's law) is

$$i_R(t) = \frac{V_m \sin(\omega t)}{R} = I_m \sin(\omega t),$$

where, the amplitude of the current waveform is

$$I_m = \frac{V_m}{R} \quad I_{rms} = \frac{V_{rms}}{R}.$$

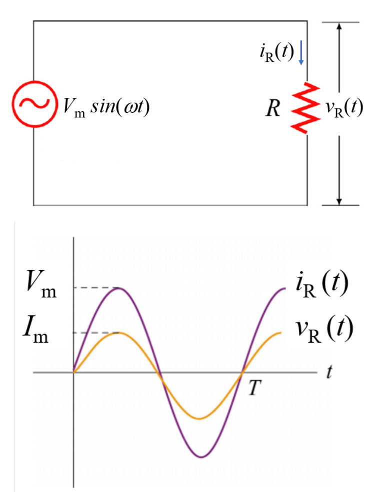


Figure 7: Top: A resistor connected to a sinusoidal voltage source, Bottom: Voltage waveform and current waveform

Load is an inductor

In the circuit of Figure 8,

$$v_L(t) = V_m \sin(\omega t).$$

For the inductor,

$$v_L(t) = L\frac{di_L}{dt}.$$

Therefore,

$$\frac{di_L}{dt} = \frac{v_L}{L} = \frac{V_m}{L}\sin(\omega t).$$

Integrating,

$$i_L = \frac{V_m}{L}\int \sin(\omega t)dt = -\frac{V_m}{\omega L}\cos(\omega t).$$

Using the trigonometric identity

$$\sin(\omega t - \frac{\pi}{2}) = -\cos(\omega t),$$

we get

$$i_L = \frac{V_m}{\omega L}\sin(\omega t - \frac{\pi}{2}) = I_m \sin(\omega t - \frac{\pi}{2}).$$

The product ωL is called the **reactance** of L, and is represented by

$$X_L = \omega L.$$

So,

$$I_m = \frac{V_m}{X_L}, \quad I_{rms} = \frac{V_{rms}}{X_L}.$$

The SI unit of X_L is **ohm** (Ω), similar to the unit of resistance. However, unlike resistance, the value of X_L varies linearly with varying angular frequency ω. Thus, for an inductor, the resistance to current flow increases with increasing frequency of the source voltage.

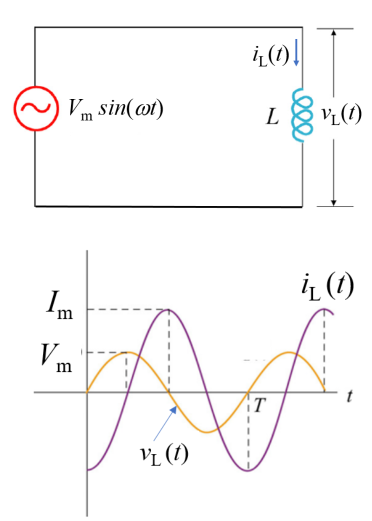


Figure 8: Top: An inductor connected to a sinusoidal voltage source, Bottom: Voltage waveform and current waveform

Load is a capacitor

In the circuit of Figure 9,

$$v_C(t) = V_m \sin(\omega t).$$

For a capacitor (C Farad),

$$i_C(t) = C\frac{dv_C}{dt}.$$

So,

$$i_C = C\frac{d}{dt}V_m \sin(\omega t) = \omega C V_m \cos(\omega t).$$

Using the trigonometric identity

$$\sin(\omega t + \frac{\pi}{2}) = \cos(\omega t),$$

we get

$$i_C = \omega C V_m \sin(\omega t + \frac{\pi}{2}) = I_m \sin(\omega t + \frac{\pi}{2}).$$

The reciprocal of the product ωC is called the **reactance** of C, and is represented by

$$X_C = \frac{1}{\omega C}.$$

So,

$$I_m = \frac{V_m}{X_C}, \quad I_{rms} = \frac{V_{rms}}{X_C}.$$

The SI units of X_C is **ohm** (Ω). The value of X_C is inversely proportional to the angular frequency ω. Thus, for a capacitor, the resistance to current flow decreases with increasing frequency of the source voltage.

These results for the applied voltage $V_m \sin(\omega t)$ are summarized in Table 1

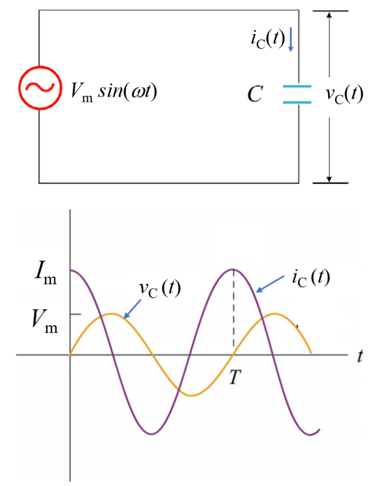


Figure 9: Top: A capacitor connected to a sinusoidal voltage source, Bottom: Voltage waveform and current waveform

	Resistor	Inductor	Capacitor
Current waveform	$I_m \sin(\omega t)$	$I_m \sin(\omega t - \frac{\pi}{2})$	$I_m \sin(\omega t + \frac{\pi}{2})$
Frequency of $i(t)$	ω	ω	ω
Resistance/ reactance	R	$X_L = \omega L$	$X_C = \frac{1}{\omega C}$
Current amplitude	$\frac{V_m}{R}$	$\frac{V_m}{X_L}$	$\frac{V_m}{X_C}$
RMS current	$\frac{V_{rms}}{R}$	$\frac{V_{rms}}{X_L}$	$\frac{V_{rms}}{X_C}$
Phase of current	in-phase with v	lags v by $\frac{\pi}{2}$	leads v by $\frac{\pi}{2}$

Table 1: For identical voltage waveform, the currents in R, L and C are different.

Circuit with multiple components of different types

If only one type of component is connected to a sinusoidal voltage source, as illustrated above, the resulting current can be found easily.

In this section, circuits with two elements (RC and RL) are considered first, followed by a series RLC circuit.

I. Series RC circuit with sinusoidal voltage source

Applying KVL to the circuit shown in Figure 10,

$$v_R(t) + v_C(t) = V_m \sin(\omega t).$$

But

$$v_R = Ri = RC\frac{dv_C}{dt}.$$

So,

$$RC\frac{dv_C}{dt} + v_C(t) = V_m \sin(\omega t),$$

$$\frac{dv_C}{dt} + \frac{1}{RC}v_C(t) = \frac{V_m}{RC}\sin(\omega t).$$

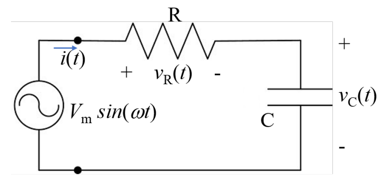


Figure 10: Series RC circuit with sinusoidal voltage source.

The solution of this non-homogeneous first-order ordinary differential equation (ODE) is the capacitor voltage $v_C(t)$. Once the capacitor voltage is known, we can determine the current $i(t)$ and the voltage $v_R(t)$. **Solving ODE is not under the purview of this module.** So, the equation is solved numerically, and different voltages and current in the steady state are shown in Figure 11.

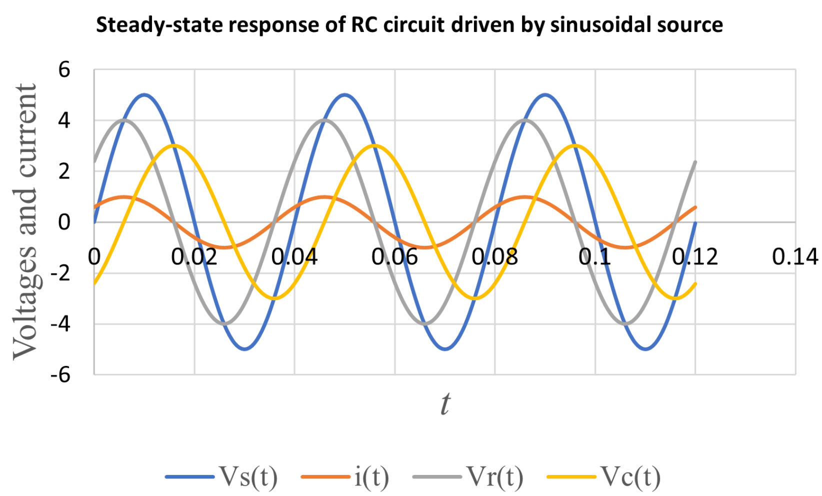


Figure 11: Steady-state voltages and current in an RC circuit driven by a sinusoidal voltage source.

The analytical steady-state solution of a series RC circuit driven by a sinusoidal voltage source is given in the appendix A-3 at the end of this chapter. Interested students may refer to that.

Points to note: In the steady state,

1. Frequency is same for all voltages and current
2. $i(t)$ is in-phase with $V_r(t)$
3. $i(t)$ leads $V_c(t)$ by $\frac{\pi}{2}$ rad
4. $i(t)$ leads $V_s(t)$ but the phase shift is less than $\frac{\pi}{2}$ rad

II. Series RL circuit with sinusoidal voltage source

Applying KVL to the circuit shown in Figure 12,

$$v_R(t) + v_L(t) = V_m \sin(\omega t).$$

Substituting

$$v_R = Ri, \;\; \& \;\; v_L = L\frac{di}{dt}$$

in the KVL equation,

$$\begin{aligned} Ri + L\frac{di}{dt} &= V_m \sin(\omega t), \\ \frac{di}{dt} + \frac{R}{L}i &= \frac{V_m}{L}\sin(\omega t). \end{aligned}$$

Different voltages and current in the steady state are shown in Figure 13.

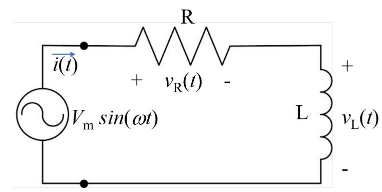


Figure 12: Series RL circuit with sinusoidal voltage source.

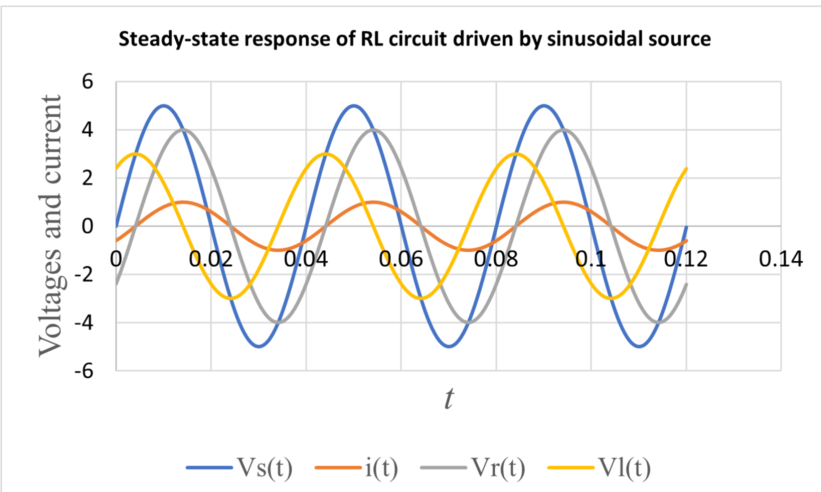


Figure 13: Steady-state voltages and current in an RC circuit driven by a sinusoidal voltage source.

Points to note: In the steady state

1. Frequency is same for all voltages and current
2. $i(t)$ is in-phase with $V_r(t)$
3. $i(t)$ lags $V_L(t)$ by $\frac{\pi}{2}$ rad
4. $i(t)$ lags $V_s(t)$ but the phase shift is less than $\frac{\pi}{2}$ rad

III. Series RLC circuit with sinusoidal voltage source

Applying KVL around the loop of Figure 14,

$$v_R + v_L + v_C = V_m \sin(\omega t).$$

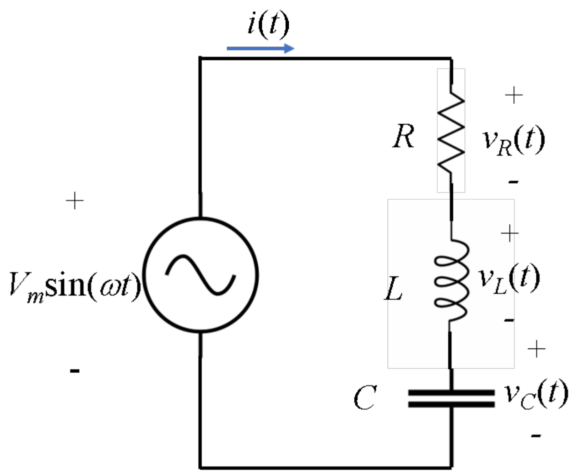


Figure 14: Series RLC circuit driven by a sinusoidal source.

Properties of the individual components:

$$v_R = Ri, \quad v_L = L\frac{di}{dt}, \quad i = C\frac{dv_C}{dt}.$$

So,

$$Ri + L\frac{di}{dt} + v_C = V_m \sin(\omega t).$$

Substituting $i = C\frac{dv_C}{dt}$,

$$RC\frac{dv_C}{dt} + LC\frac{d^2v_C}{dt^2} + v_C = V_m \sin(\omega t).$$

After rearranging the terms:

$$\frac{d^2v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC}v_C = \frac{V_m}{LC}\sin(\omega t).$$

The solution of this non-homogeneous second-order ODE is the capacitor voltage $v_C(t)$. Once it is known, all other voltages and current can then be derived.

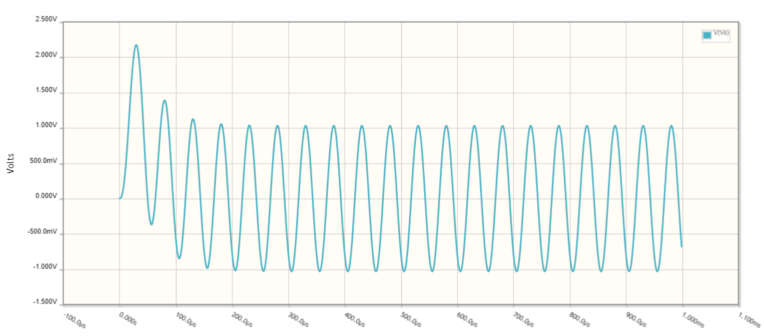


Figure 15: The capacitor voltage $v_C(t)$ in a series R-L-C circuit driven by a sinusoidal voltage source of amplitude 10V and frequency 20 kHz. The values of R, L, and C are 100 Ω, 1 mH, and 0.5 μF, respectively. Both transient and steady state responses are shown.

You will learn more about the phase relation between current and source voltage in series RLC circuit with the help of CircuitLab simulation and hands-on experiment in the studio sessions.

Similar to what we saw in RC and RL circuit,

- all voltages and currents have the same frequency,
- i is in-phase with v_R,
- i leads v_C by $\frac{\pi}{2}$ rad, and
- i lags v_L by $\frac{\pi}{2}$ rad.

The relationship between the phases of i and v_S depends on the relative amplitudes of v_L and v_C.

- If the amplitude of v_L is greater than the amplitude of the v_C, then i lags v_S,
- If the amplitude of v_L is less than the amplitude of the v_C, then i leads v_S, and
- If the amplitude of v_L is equal to the amplitude of the v_C, then i is in-phase with v_S.

In the analysis of AC circuits driven by sinusoidal sources, we are interested in the steady-state solution. The chapter **AC circuit analysis: Part 2** presents another method of finding the steady-state solution of ac circuit driven by a sinusoidal source. The method does not require the solution of ODE, rather it uses the concepts of **phasor** and **impedance**.

- The time domain circuit equation involving ODE is transformed into an algebraic problem when phasor and impedance are used.

Appendix

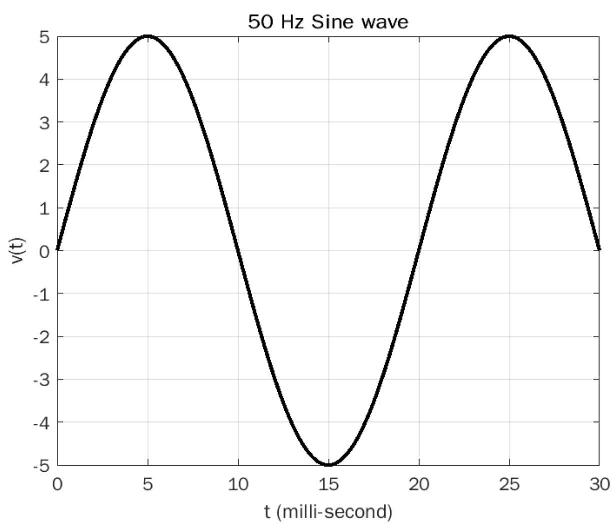
A-1: A periodic waveform can be reconstructed by adding sinusoidal waveform of a set of specific frequencies

A Fourier series is an expansion of a periodic function $f(t)$ in terms of an infinite sum of sines and cosines. The computation and study of Fourier series is known as harmonic analysis and is extremely useful as a way to break up an arbitrary periodic function into a set of simple terms that can be plugged in, solved individually, and then recombined to obtain the solution to the original problem or an approximation to it to whatever accuracy is desired or practical.

Harmonic analysis is beyond the scope of this module. Instead, the concept is explained by showing how adding sine waves of different frequencies leads to a square wave.

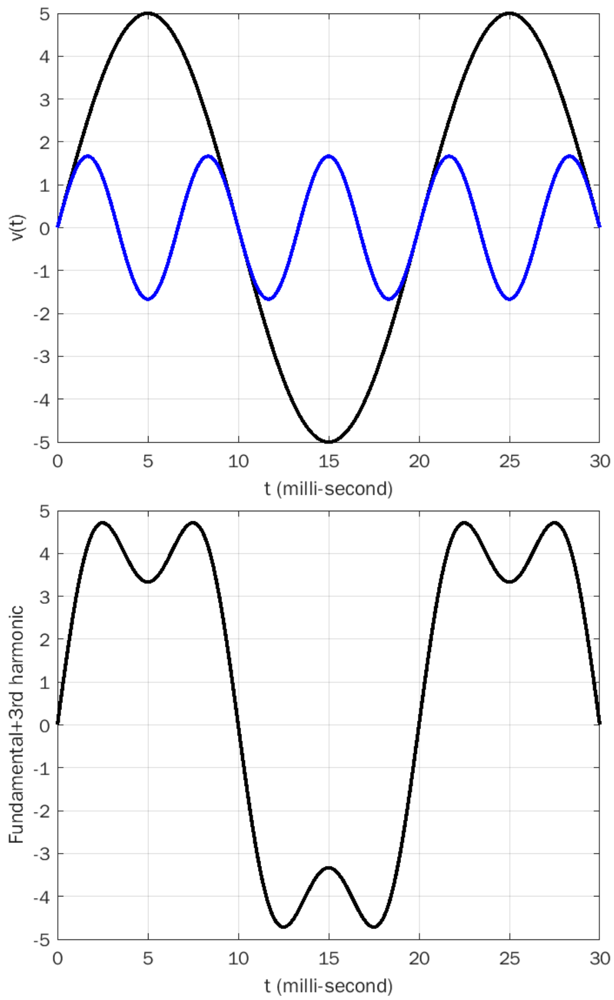
Consider a sine wave of amplitude 5 and frequency 50 Hz:

$$v_1(t) = 5\sin(2\pi \times 50t).$$



Now add to this function its 3rd-harmonic (frequency is three times the frequency of the first function) with amplitude one third of the first waveform, *i.e.*, $\frac{5}{3}$. These two functions and their sum are shown in the next two figures.

$$v_{3har}(t) = \frac{5}{3}\sin(2\pi \times 3 \times 50t).$$



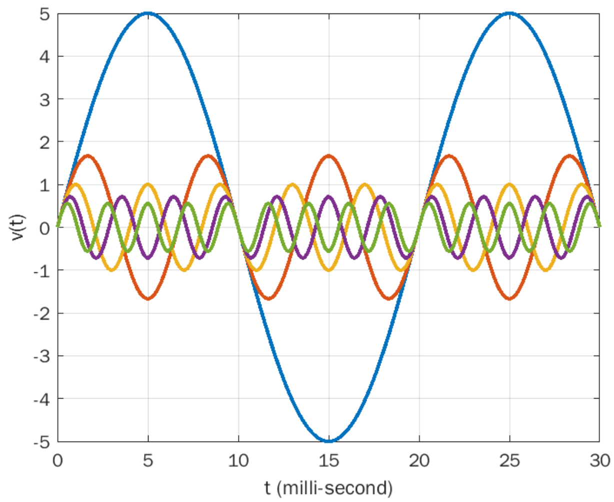
The resulting waveform is taking the shape of a square wave but a wriggly square instead of two constant voltage levels. The wriggles are diminished with increasing number of odd harmonics (with appropriate amplitude) added to it. The amplitudes of the harmonics decrease with increasing harmonic numbers:

$$v_{5har}(t) = \frac{5}{5}\sin(2\pi \times 5 \times 50t),$$

$$v_{7har}(t) = \frac{5}{7}\sin(2\pi \times 7 \times 50t),$$

$$v_{9har}(t) = \frac{5}{9}\sin(2\pi \times 9 \times 50t).$$

The odd harmonics up to the 9th are shown in the figure below.



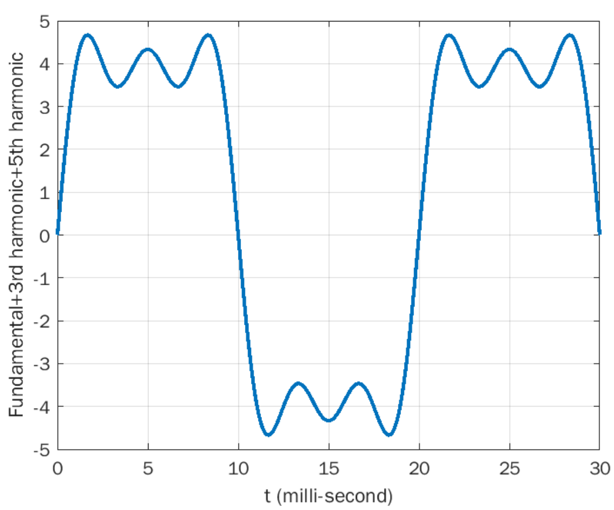
The next two figures shows the functions

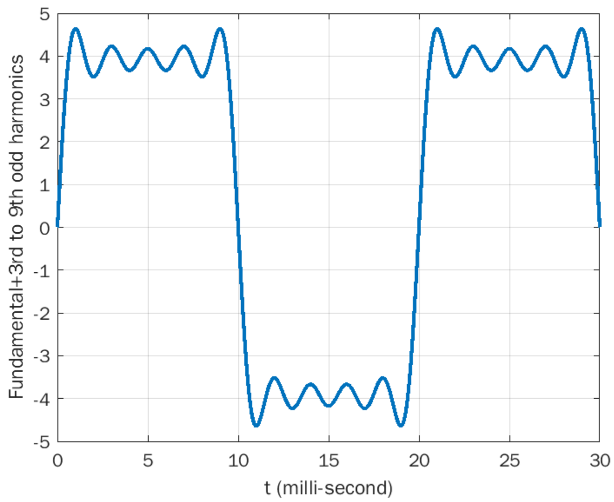
$$v(t) = v_1(t) + v_{3har}(t) + v_{5har}(t)$$

and

$$v(t) = v_1(t) + v_{3har}(t) + v_{5har}(t) + v_{7har}(t) + v_{9har}(t),$$

respectively.





Adding more and more odd harmonics with appropriate amplitude makes the waveform look more similar to a square wave.

Fourier analysis is a method for systematically identifying frequency components of a periodic waveform and the corresponding amplitude and phase. You will learn more about this in EE2023.

A-2: RMS value of sine function

$$x(t) = A\sin(\omega t)$$

If the period of the waveform is T seconds then

$$\omega T = 2\pi.$$

The RMS value of $x(t)$:

$$X_{rms} = \sqrt{\frac{1}{T}\int_0^T A^2 \sin^2(\omega t) dt}.$$

But,

$$\sin^2(\omega t) = \frac{1-\cos 2\omega t}{2}.$$

So,

$$\begin{aligned}
X_{rms} &= \sqrt{\frac{1}{T}\int_0^T \frac{A^2}{2}(1-\cos(2\omega t))dt} \\
&= \sqrt{\frac{A^2}{2T}\int_0^T (1-\cos(2\omega t))dt} \\
&= \sqrt{\frac{A^2}{2T}(\int_0^T dt - \int_0^T \cos(2\omega t)dt)} \\
&= \sqrt{\frac{A^2}{2T}((T-0)-(\frac{\sin 2\omega T - \sin 0}{2\omega}))} \\
&= \frac{A}{\sqrt{2}}.
\end{aligned}$$

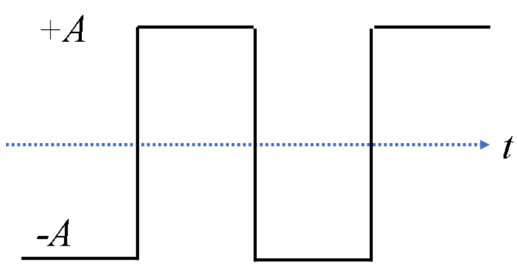
$\sin 2\omega T = \sin 4\pi = 0.$

RMS value depends on the amplitude of the waveform and not its frequency or phase. The functions $A\sin(100t)$, $A\cos(100t)$ and $A\sin(100t + 0.2\pi)$ have the same RMS value which is $\frac{A}{\sqrt{2}}$.

A-2: RMS value of non-sinusoidal periodic waveform

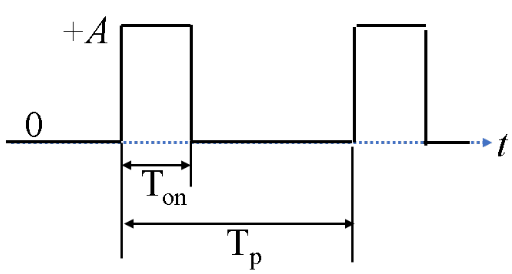
RMS values of some non-sinusoidal periodic functions are given below.

Bipolar square wave



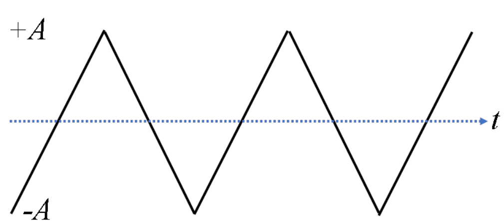
RMS value $= A$.

Series of square pulses



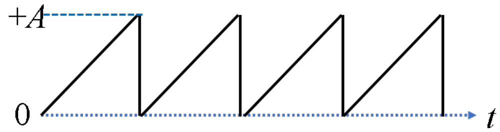
RMS value $= A\sqrt{\frac{T_{on}}{T_p}}$.

Triangular wave



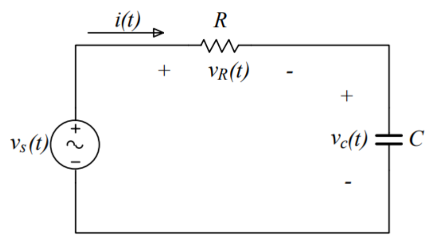
RMS value $= \frac{A}{\sqrt{3}}$.

Saw-tooth wave



RMS value $= \frac{A}{\sqrt{3}}$.

A-3: Steady state solution of series R-C circuit driven by sinusoidal source



Let the time-domain description of the source be

$$v_s = V_m \sin(\omega t).$$

Applying KVL around the loop, and using the current-voltage relationship of the resistor and the capacitor, we get

$$\tau \frac{dv_c}{dt} + v_c = V_m \sin(\omega t), \tag{2}$$

where $\tau = RC$ is the time constant.

This is a linear circuit and, therefore, the principle of superposition can be applied. We may view the effect of the source on the capacitor voltage (v_c) as the superposition of the response with the source set equal to zero (source-free or natural response$v_{ch}(t)$) and the forced response ($v_{cp}(t)$):

$$v_c(t) = v_{ch}(t) + v_{cp}(t).$$

In mathematical language these two responses are called the homogeneous solution (v_{ch}) and the particular solution (v_{cp}) of the ODE.

The homogeneous solution, *i.e.*, the solution of the equation

$$\tau \frac{dv_c}{dt} + v_c = 0$$

has the form

$$v_{ch}(t) = Be^{-\frac{t}{\tau}},$$

where B is a constant that needs to be determined using the initial conditions. The homogeneous solution is independent of the type of function that represents the voltage source.

However, the particular solution, *i.e.*, the solution of the equation

$$\tau \frac{dv_c}{dt} + v_c = V_m \sin(\omega t)$$

depends on the type of forcing function. As the forcing function in this case is sinusoidal, the particular solution is also a sinusoidal function with the same frequency:

$$v_{cp}(t) = A \sin(\omega t + \phi),$$

but the amplitude A and the phase ϕ are to be determined.

The complete solution is

$$v_C(t) = Be^{-\frac{t}{\tau}} + A \sin(\omega t + \phi). \tag{3}$$

We are interested in obtaining the steady state response. This is equivalent to saying that the source $v_S(t)$ was connected to the system long time ago and all transient phenomena are gone. In order to be more precise, if the time $t >> \tau$ then the exponential term in equation 3 would go to zero. In this case the observable and thus the important response is the steady state response which is given by

$$v_C(t) = A \sin(\omega t + \phi). \tag{4}$$

In order to determine the values of A and ϕ, we need to substitute this solution into equation 2.

Before making that substitution, let us expand the solution using trigonometric identity:

$$v_c(t) = A \cos\phi \sin\omega t + A \sin\phi \cos\omega t.$$

Therefore,

$$\frac{d}{dt} v_c = \omega A \cos\phi \cos\omega t - \omega A \sin\phi \sin\omega t.$$

Substituting this in equation 2,

$$\omega\tau A \cos\phi \cos\omega t - \omega\tau A \sin\phi \sin\omega t + A \cos\phi \sin\omega t + A \sin\phi \cos\omega t = V_m \sin\omega t.$$

$$(\omega\tau A \cos\phi + A \sin\phi) \cos\omega t + (A \cos\phi - \omega\tau A \sin\phi) \sin\omega t = V_m \sin\omega t.$$

Comparing the coefficients of $\sin\omega t$ and $\cos\omega t$ on both sides of the equation, we get

$$\begin{aligned} A \cos\phi - \omega\tau A \sin\phi &= V_m, \\ A \sin\phi + \omega\tau A \cos\phi &= 0. \end{aligned}$$

From the second equation,

$$\tan\phi = -\omega\tau \quad \Rightarrow \quad \phi = -\tan^{-1} \omega\tau.$$

From the first equation,

$$A = \frac{V_m}{\cos\phi - \omega\tau \sin\phi}.$$

AC Circuit Analysis using Phasors

Formulation of KVL or KCL in a circuit consisting of reactance (because of inductance and capacitance) leads to ODE. The order of the ODE and hence the complexity of analytical solution increases with increasing number of branches with reactance.

Finding the **steady-state solution** of a constant coefficient ODE with sinusoidal forcing function, for example, the 2nd-order case

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = V_m \sin(\omega t + \phi)$$

can be simplified by replacing the sinusoidal function with complex exponential $e^{j\omega t}$. This is true for ODE of any order. However, it should be noted that the voltage from a sinusoidal source is not a complex exponential functions. We use complex exponential for transforming the time-domain problem into another domain where finding the solution is easier.

According to **Euler's formula:**

$$e^{j\theta} = \cos\theta + j\sin\theta,$$

where

$$j = \sqrt{-1}$$

is the imaginary number. Therefore,

In your mathematics course, you might have used i as the symbol for representing the imaginary number. Since, i is used for current in electrical engineering, the symbol j is used as unit imaginary number.

$$\begin{aligned} A\cos(\omega t + \phi) &= \Re(Ae^{j(\omega t+\phi)}), \\ A\sin(\omega t + \phi) &= \Im(Ae^{j(\omega t+\phi)}). \end{aligned}$$

If the ac circuit is driven by a sinusoidal source,

1. First, substitute the sinusoidal function with complex exponential in the KVL/KCL equation. This makes it possible to find the steady-state solution by solving algebraic equations instead of solving ODEs.

2. However, the solution obtained is a complex exponential function.

   - If the driving function is a cos function, the real part of the complex exponential solution is the steady-state solution.
   - If the driving function is a sin function, the imaginary part of the complex exponential solution is the steady-state solution.

Phasor

Phasor is a useful tool for finding the steady-state solution of an ODE with sinusoidal forcing function. AC circuit with sinusoidal voltage (or current) source is one such case.

You have seen in the chapter **Introduction to AC Circuits**, that when a circuit element (R, L or C) is driven by a sinusoidal voltage source, the resulting steady-state current is also sinusoidal with the same frequency as the voltage's. That means, in a linear AC circuit driven by sinusoidal source, voltages at all nodes and currents in all branches will be sinusoidal functions with the same frequency.

The time domain KVL/KCL equations involve trigonometric functions *sine* and *cosine*. Arithmetic operations of sinusoidal functions involve use of trigonometric identities and, in general, are not convenient. An abstraction of sinusoidal functions as real (or imaginary) part of a rotating vector in the complex plane brings convenience to their arithmetic operations. This abstraction is the **phasor** representation of the sinusoidal function.

Consider the sinusoidal function

$$x(t) = A\cos(\omega t + \phi) \tag{5}$$

which can be expressed using complex exponential as

$$\begin{aligned} x(t) &= \Re A e^{j(\omega t+\phi)}, \\ &= \Re A e^{j\phi} e^{j\omega t}, \\ &= \Re \bar{X} e^{j\omega t}. \end{aligned}$$

- The sinusoidal function $x(t)$ is factored into a time-independent part $\bar{X} = Ae^{j\phi}$ and a time-varying part $e^{j\omega t}$.
- The time-varying part carries with it the information about the frequency that is same in the voltages and currents of a circuit.
- The time-independent part $Ae^{j\phi}$ that carries in it the information about amplitude and phase is the **phasor** representation of the sinusoidal function. This factor is different at different parts of the circuit.
  - For finding the steady-state voltages and currents in an ac circuit driven by sinusoidal source, we use the phasor only, and ignore the $e^{j\omega t}$ term while carrying out the calculations.

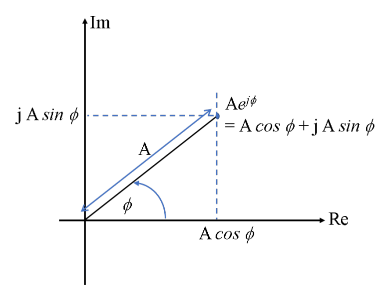


Figure 16: Argand diagram of the phasor $\bar{X} = Ae^{j\phi}$: its imaginary part is $A\sin\phi$ which is $x_i(t) = A\sin(\omega t + \phi)$ at $t = 0$ regardless of the value of ω. Similarly, the real part is the value of $x_r(t) = A\cos(\omega t + \phi)$ at $t = 0$.

Extracting phasor from the time-domain waveform:

There are two conventions for defining phasor:

$$x(t) = A\cos(\omega t + \phi) \quad \Rightarrow \quad \bar{X} = Ae^{j\phi} \tag{6}$$

and

$$x(t) = A\sin(\omega t + \phi) \quad \Rightarrow \quad \bar{X} = Ae^{j\phi}. \tag{7}$$

The first convention (equation 6), with RMS value as the magnitude, is used in the power engineering community:

$$x(t) = V_{rms}\sqrt{2}\cos(\omega t + \phi) \quad \Rightarrow \quad \bar{V} = V_{rms}e^{j\phi}.$$

The phasor $Ae^{j\phi}$ is a complex number with magnitude A and argument ϕ.

$$\begin{aligned} Ae^{j\phi} &= A(\cos\phi + j\sin\phi), \\ |A| &= A\sqrt{\cos^2\phi + \sin^2\phi} = A, \\ \angle A &= \tan^{-1}\frac{A\sin\phi}{A\cos\phi} = \phi. \end{aligned}$$

Three different notations are used interchangeably for the phasor $\bar{X}$:

$$Ae^{j\phi} = A\cos\phi + jA\sin\phi = A\angle\phi.$$

What is $e^{j\omega t}$?

$$\begin{aligned} e^{j\omega t} &= A(\cos\omega t + j\sin\omega t), \\ |e^{j\omega t}| &= \sqrt{\cos^2\omega t + \sin^2\omega t} = 1, \\ \angle e^{j\omega t} &= \tan^{-1}\frac{A\sin\omega t}{A\cos\omega t} = \omega t. \end{aligned}$$

It's a complex number whose magnitude is 1 and the argument (ωt) increases linearly as time passes. In the argand diagram, this is the locus of a point rotating counter-clockwise (CCW) keeping a distance of 1 unit from the origin.

Multiplying the phasor $\bar{X} = Ae^{j\phi}$ by $e^{j\omega t}$ gives a rotating vector whose initial position is $A\angle\phi$ at $t = 0$, and it rotates CCW in the complex plane at the rate of ω radians per second. Plotting the imaginary part of this rotating vector versus time will generate the plot of

$$A\sin(\omega t + \phi)$$

and plotting the real part will give

$$A\cos(\omega t + \phi).$$

Plotting the value of the imaginary part of the waveform

$$V_m \sin(\omega t + \phi)$$

as a function of time is illustrated in Figure 18.

Note: The same phasor $Ae^{j\phi}$ can be associated with both cos and sin waveform in time-domain depending whether we want to use real part of the complex exponential or the imaginary part.

- All voltages and currents in a linear AC circuit driven by a sinusoidal source are sinusoidal and have the same frequency. Therefore, all can be represented by phasors.
- The rotating vectors corresponding to all voltages and currents will be rotating CCW at the same rate (ω rad/s).

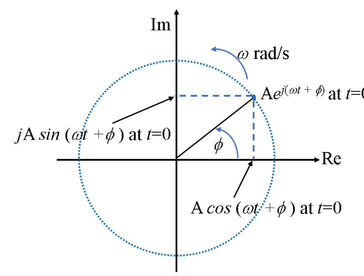


Figure 17: The argand diagram of $Ae^{j(\omega t+\phi)}$ is a rotating vector rotating counter clockwise at the rate of ω rad/s. The phasor $Ae^{j\phi}$ is that position of the rotating vector at $t = 0$.

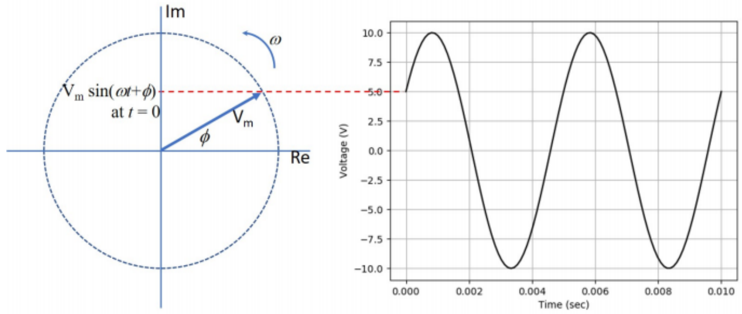


Figure 18: The phasor of $v(t) = V_m \sin(\omega t + \phi)$ is shown on the left and the corresponding waveform on the right for $V_m = 10$ and $\phi = \pi/6$. The phasor itself does not tell what the frequency is. From the waveform, you can determine the frequency which is $f = 1/0.005 = 200\ Hz$ or $\omega \approx 1257$ rad/s.

- If observed in a reference frame which itself is rotating at the same rate, all rotating vectors become standing still, *i.e.*, we can simply drop the time-varying component $e^{j\omega t}$, and represent the functions in terms of their phasors capturing the amplitude and phase.

- **This transforms the time-varying problem of AC circuit into a constant value (DC) problem.**

Example 1: Use of phasor to find sum of sinusoidal waveform

Consider three voltage sources

$$v_1 = 6\sin\omega t, \quad v_2 = 12\sin(\omega t + \frac{\pi}{2}), \quad v_3 = 4\sin(\omega t - \frac{\pi}{2})$$

connected in series. What is the overall voltage of the combination?
Solution:

Using the sine convention (equation 7)

$$v(t) = 6\sin\omega t + 12\sin(\omega t + \pi/2) + 4\sin(\omega t - \pi/2).$$

- The actual problem is in time-domain.

Do the addition after expressing each individual voltage as phasor,

$$\begin{aligned}
\bar{V} &= 6\angle 0 + 12\angle\frac{\pi}{2} + 4\angle -\frac{\pi}{2}, \\
&= (6 + j0) + (0 + j12) + (0 - j4), \\
&= 6 + j8, \\
&= \sqrt{6^2 + 8^2}\angle\tan^{-1}(8/6), \\
&= 10\angle 0.9.
\end{aligned}$$

- Numerical manipulations are done in phasor-domain.

Converting back into time-domain using the convention given in equation 7:

$$v(t) = 10\sin(\omega t + 0.9).$$

- Appropriate phasor-to-time domain transformation gives the solution in time-domain.

Using the cosine convention (equation 6)
At first, we need to express the voltages as cosine functions.

$$6\sin\omega t = 6\cos(\omega t - \frac{\pi}{2}).$$

$$12\sin(\omega t + \frac{\pi}{2}) = 12\cos(\omega t).$$
$$4\sin(\omega t - \frac{\pi}{2}) = -4\cos(\omega t).$$

So,

$$v(t) = 6\cos(\omega t - \frac{\pi}{2}) + 12\cos(\omega t) - 4\cos(\omega t.$$

Do the addition in phasor domain:

$$\begin{aligned}
\bar{V} &= 6\angle - \frac{\pi}{2} + 12\angle 0 - 4\angle 0, \\
&= (0 - j6) + (12 + j0) - (4 + j0), \\
&= 8 - j6, \\
&= \sqrt{6^2 + 8^2}\angle \tan^{-1}\frac{-6}{8}, \\
&= 10\angle - 0.6.
\end{aligned}$$

Converting back into time-domain using the convention given in equation 6:

$$v(t) = 10\cos(\omega t - 0.6).$$

To verify whether the two solutions represent the same waveform, let's convert this cosine function into a sine function:

$$10\cos(\omega t - 0.6) = 10\sin(\omega t - 0.6 + \frac{\pi}{2}) = 10\sin(\omega t + 0.9).$$

- In the example above, the numbers are converted into Cartesian form before adding them. Addition/subtraction of complex numbers are done easily on paper if the numbers are given in the Cartesian form.
- Product and division are done easily in polar coordinate representation. However, most scientific calculators nowadays can do complex number operations directly in any form of number representation.

Which phasor convention should we use?

As seen in the example above, the two conventions give identical results.

In this note, we have chosen trigonometric function $A\sin(\omega t + \phi)$ to represent the time-domain waveform and the corresponding phasor is $A\angle\phi$.

Complex Impedance

Let $\bar{V}$ and $\bar{I}$, respectively, be the phasors of the sinusoidal voltage across a component/load and the resulting sinusoidal current through it. Then the complex impedance of the component/ load is

$$Z = \frac{\bar{V}}{\bar{I}}.$$

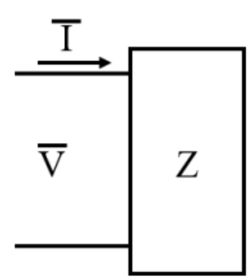


Figure 19: Impedance of a load is the ratio of voltage to current, both expressed as phasor.

In the following subsections, simple one element loads, *i.e.*, resistor, inductor, and capacitor are considered to find their impedances.

1. Impedance of ideal resistor

For a resistor, the voltage waveform and current waveform (Figure 20) are in-phase:

$$\begin{aligned} v_R(t) &= V_m \sin(\omega t), \\ i_R(t) &= \frac{V_m}{R} \sin(\omega t), \\ &= I_m \sin(\omega t). \end{aligned}$$

Corresponding phasors are:

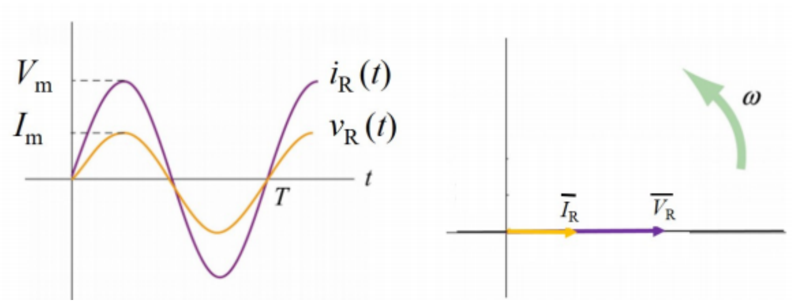


Figure 20: Voltage waveform and current waveform of a resistor (left) and the phasor diagram (right)

$$\bar{V}_R = V_m \angle 0, \quad \bar{I}_R = I_m \angle 0.$$

The impedance is

$$Z_R = \frac{V_m \angle 0}{I_m \angle 0} = \frac{V_m}{I_m} = R.$$

Impedance of R: $Z_R = R\angle 0 = R + j0$

Impedance of a resistor is a real number and is equal to the value of its resistance. The voltage phasor and the current phasor of a resistor are shown in the **phasor diagram** in Figure 20 (right).

2. Impedance of ideal inductor

For ideal inductor, voltage waveform and current waveform are out of phase by $\frac{\pi}{2}$ radians, with **the current lagging the voltage**:

$$\begin{aligned} v_L(t) &= V_m \sin(\omega t), \\ i_L(t) &= -\frac{V_m}{\omega L} \cos(\omega t), \\ &= I_m \sin(\omega t - \pi/2). \end{aligned}$$

The phasors are:

$$\bar{V}_L = V_m \angle 0, \quad \bar{I}_L = I_m \angle -\frac{\pi}{2}.$$

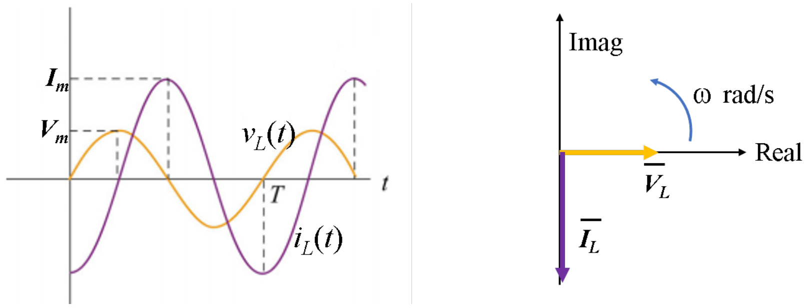


Figure 21: Voltage waveform and current waveform of an inductor (left) and the phasor diagram (right)

Impedance of L:
$Z_L = \omega L \angle \frac{\pi}{2} = 0 + j\omega L$

The impedance of ideal inductor is

$$Z_L = \frac{V_m \angle 0}{I_m \angle -\frac{\pi}{2}} = \frac{V_m}{I_m} \angle \frac{\pi}{2} = \omega L \angle \frac{\pi}{2}.$$

Impedance of ideal inductor is a positive imaginary number with magnitude equal to ωL, the reactance of the inductor. The voltage phasor and the current phasor of an inductor are shown in the phasor diagram in Figure 21 (right).

3. *Impedance of a capacitor*

Voltage waveform and current waveform are out of phase by 90° or $\frac{\pi}{2}$ radians with **the current leading the voltage**:

$$\begin{aligned} v_C(t) &= V_m \sin(\omega t), \\ i_C(t) &= \omega C V_m \cos(\omega t), \\ &= I_m \sin(\omega t + \pi/2). \end{aligned}$$

The phasors are:

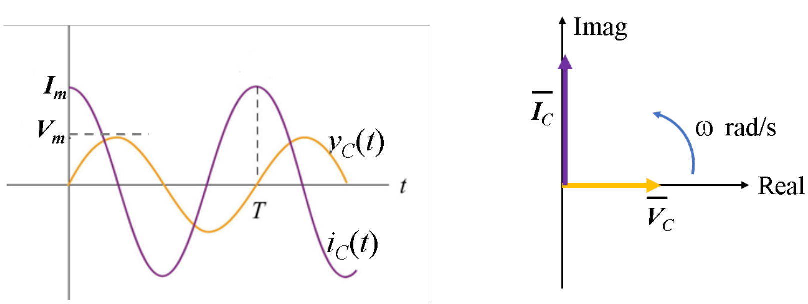


Figure 22: Voltage waveform and current waveform of a capacitor (left) and the phasor diagram (right)

$$\bar{V}_C = V_m \angle 0, \quad \bar{I}_C = \omega C V_m \angle + \frac{\pi}{2}.$$

Impedance of C:
$Z_C = \frac{1}{\omega C} \angle -\frac{\pi}{2} = 0 - j\frac{1}{\omega C}$

The impedance of a capacitor is

$$Z_C = \frac{V_m \angle 0}{I_m \angle \frac{\pi}{2}} = \frac{V_m}{I_m} \angle - \frac{\pi}{2} = \frac{1}{\omega C} \angle - \frac{\pi}{2}.$$

Impedance of a capacitor is a negative imaginary number with magnitude equal to $\frac{1}{\omega C}$, the reactance of the capacitor. The voltage phasor and the current phasor of a capacitor are shown in the phasor diagram in Figure 22 (right).

Impedance of load consisting of multiple components

- When two or more components are connected in series, the equivalent impedance is equal to the sum of individual impedances:

$$Z_{eq} = Z_1 + Z_2 + Z_3 + \cdots\cdots\cdots + Z_n.$$

- When components are connected in parallel, the reciprocal of equivalent impedance is equal to the sum of the reciprocals of individual impedance:

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots\cdots\cdots + \frac{1}{Z_n}.$$

  Reciprocal of impedance is known as **admittance**,

$$Y = \frac{1}{Z}.$$

  When components are connected in parallel, the equivalent admittance is sum of admittances:

$$Y_{eq} = Y_1 + Y_2 + \cdots\cdots\cdots + Y_n.$$

1. Impedance of R-L and R-C

Impedance of a series R-L branch is

$$Z_{RL} = Z_R + Z_L = R + j\omega L.$$

In polar form:

$$\begin{aligned} Z_{RL} &= \sqrt{R^2 + (\omega L)^2} \angle \tan^{-1} \frac{\omega L}{R}, \\ &= |Z_{RL}| \angle \theta_{RL}. \end{aligned}$$

Since, ω, R, and L are all non-zero, positive real numbers, $\theta_{RL} = \tan^{-1} \frac{\omega L}{R}$ varies between 0 and $+\pi/2$:

$$0 < \theta_{RL} < +\frac{\pi}{2}.$$

Can you show that the condition $0 < \theta_{RL} < +\frac{\pi}{2}$ is also true when R and L are connected in parallel?

Impedance of a series R-C branch is

$$Z_{RC} = Z_R + Z_C = R - j\frac{1}{\omega C}.$$

In polar form:

$$\begin{aligned} Z_{RC} &= \sqrt{R^2 + \frac{1}{(\omega C)^2}} \angle \tan^{-1} \frac{-\frac{1}{\omega C}}{R}, \\ &= |Z_{RC}| \angle \theta_{RC}. \end{aligned}$$

Since, ω, R, and C are all non-zero, negative real numbers, $\theta_{RC} = \tan^{-1} \frac{-\frac{1}{\omega C}}{R}$ varies between 0 and $-\pi/2$:

$$0 > \theta_{RC} > -\frac{\pi}{2}.$$

Can you show that the condition $0 > \theta_{RC} < -\frac{\pi}{2}$ is also true when R and C are connected in parallel?

Impedance of R-L load or R-C load is a complex number in the form of

$$Z_{RL} = R + j\omega L = |Z_{RL}| \angle \theta_{RL}, \quad 0 < \theta_{RL} < \frac{\pi}{2},$$

$$Z_{RC} = R - j\frac{1}{\omega C} = |Z_{RC}| \angle \theta_{RC}, \quad 0 > \theta_{RC} > -\frac{\pi}{2}.$$

Impedance of R-L-C

Consider a series R-L-C load:

$$Z_{RLC} = R + j\omega L - j\frac{1}{\omega C} = R + jX_L - jX_C.$$

Depending on the relative values of X_L and X_C, there are three possible characteristics of the equivalent impedance:

1. **Inductive** ($X_L > X_C$):

   $$Z_{RLC} = R + jX,$$

   where $X = X_L - X_C$.

2. **Capacitive** ($X_L < X_C$):

   $$Z_{RLC} = R - jX.$$

3. **Resistive** ($X_L = X_C$):

   $$Z_{RLC} = R.$$

   The condition $X = X_L - X_C = 0$ in an RLC circuit is called **resonance**. At resonance, a series RLC load behaves like a resistor. Keeping the amplitude of the source voltage constant, if the frequency is varied, the amplitude of the current waveform in the series-RLC will be maximum at the resonant frequency.

   At the resonant frequency ($\omega_r = 2\pi f_r$) of an RLC circuit,

   $$X_L = X_C \quad \Rightarrow \quad \omega_r L = \frac{1}{\omega_r C}.$$

   $$\omega_r = \frac{1}{\sqrt{LC}},$$

   $$f_r = \frac{1}{2\pi\sqrt{LC}}.$$

- Can you find the resonant frequency for parallel RLC circuit?
- How does the amplitude of current vary with varying frequency for the case of parallel RLC?

The impedance of an ac circuit varies with frequency. For a given input voltage amplitude, the current through the circuit will be different if the frequency of the input voltage is changed. You will learn more about this when you study about **filters**.

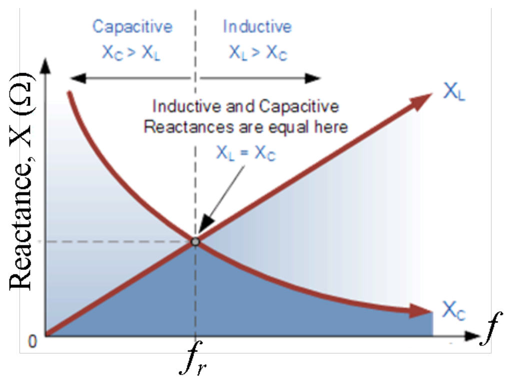


Figure 23: Reactance of a capacitor decreases with increasing frequency but the reactance of an inductor increases.

AC circuit analysis using phasor and impedance

1. Replace the time-domain description of the sources with the corresponding phasor

$$V_m \sin(\omega t + \phi) \quad \Rightarrow \quad V_m \angle \phi.$$

   - You can do this for all sources provided their frequencies are the same.
   - If there are sources with different frequencies, analyze the circuit for one frequency at a time and then apply principle of superposition.

2. Replace each component with its complex impedance.

3. Because of the steps mentioned above, the ac circuit is transformed into a time-independent circuit with constant-valued sources. Analyze the circuit using any technique you have learnt for DC circuit analysis, but you need to perform calculations using complex numbers.

4. The solution you get will be in the phasor-domain.

   - If you are interested to know the magnitude of voltage/ current, and its phase shift with respect to the source, you can get it directly from the phasor.

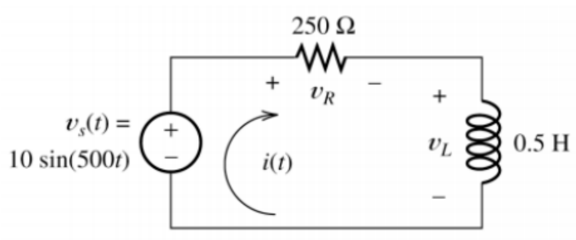
$$A\angle\gamma \quad \Rightarrow \quad Amplitude = A, \;\; Phase = \gamma.$$

   - If the time-domain expression is desired, transform the phasor into trigonometric function.

$$A\angle\gamma \quad \Rightarrow \quad A\sin(\omega t + \gamma).$$

Example 3: AC circuit analysis using phasor and impedance?

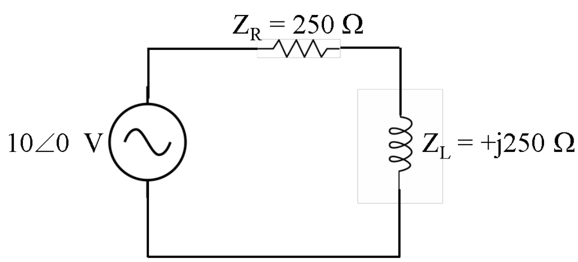
Find the RMS current for the circuit shown below and the phase of the current. Also find the time-domain expressions for the voltages v_R and v_L.



Solution:

$$
\begin{aligned}
v_S(t) &= 10\sin(500t) \quad \Rightarrow \quad \bar{V}_S = 10\angle 0 \;\; volt, \\
\omega &= 500\, rad/s, \\
Z_R &= 250\,\Omega, \\
Z_L &= +j\omega L = +j250\,\Omega.
\end{aligned}
$$

Redraw the circuit with phasors and impedances:



Total impedance:

$$Z = 250 + j250 = 250 \times \sqrt{2}\angle\frac{\pi}{4}\ \Omega$$

.

Phasor of the current:

$$\bar{I} = \frac{10\angle 0 \;\; volt}{250 \times \sqrt{2}\angle\frac{\pi}{r}\ \ \Omega} = 28.3 \times 10^{-3}\angle - \frac{\pi}{4}\ \ ampere.$$

$$I_m = 28.3 \times 10^{-3}\, ampere.$$

$$I_{rms} = \frac{28.3 \times 10^{-3}}{\sqrt{2}} = 20.0 \times 10^{-3}\, ampere.$$

RMS current is **20.0 mA** and current waveform lags input voltage waveform by $\frac{\pi}{4}$ radians.

$$\bar{V}_R = \bar{I} \times Z_R = 28.3 \times 10^{-3}\angle - \frac{\pi}{4} \times 250 = 7.07\angle - \frac{\pi}{4}\, volt.$$

$$v_R(t) = 7.07\sin(500t - \frac{\pi}{4})\, volt.$$

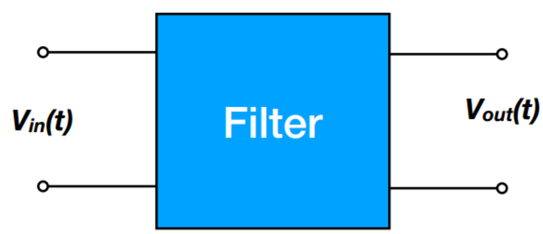
$$\bar{V}_L = \bar{I} \times Z_L = 28.3 \times 10^{-3} \angle -\frac{\pi}{4} \times 250 \angle \frac{\pi}{2} = 7.07 \angle +\frac{\pi}{4} \; volt.$$

$$v_L(t) = 7.07 \sin(500t + \frac{\pi}{4}) \; volt.$$

Filters

What is a filter

A filter is a 2-port device that takes in a signal $V_{in}(t)$ at the input port, modifies it in some way, and sends out the modified signal $V_{out}(t)$ on the output port. Signals are often simply time-varying voltages.



Potential divider as a filter

Let us start with a very simple example. Consider the potential divider circuit shown in Figure 24 as a filter.

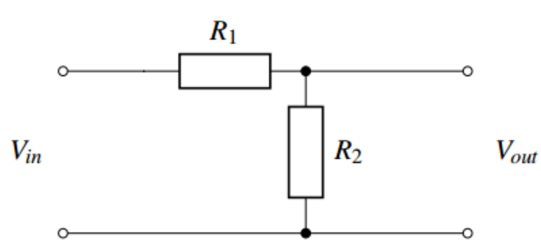


Figure 24: Very simple filter

The output V_{out} is simply related to the input V_{in} as:

$$V_{out} = \frac{R_2}{R_1 + R_2} V_{in}. \tag{8}$$

So the output signal is simply a scaled version of the input signal. If the input signal is time-varying, the output signal will also vary with time:

$$V_{out}(t) = \frac{R_2}{R_1 + R_2} V_{in}(t). \tag{9}$$

The scaling factor G is what we call the **Gain** of the filter.

$$V_{out}(t) = GV_{in}(t), \tag{10}$$
$$G = \frac{V_{out}}{V_{in}} = \frac{R_2}{R_1 + R_2}. \tag{11}$$

The gain G here is independent of the input signal $V_{in}(t)$. This is not generally the case for other circuits, as we see next.

RC low-pass filter

Let us next consider a circuit that is very similar to the potential divider we just studied, but one of the resistors is replaced with a capacitor. If the input

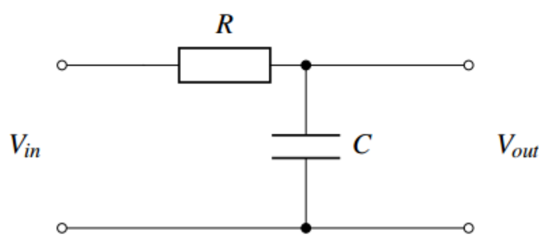


Figure 25: RC low-pass filter

voltage V_{in} is not time-varying, the capacitor acts as an open circuit, and the output voltage $V_{out} = V_{in}$ in the steady-state. The gain G of this circuit is then just 1.

However, things get more interesting with a time-varying input voltage $V_{in}(t)$. We already know how to analyze linear AC circuits, where the voltages are sinusoidally time-varying with constant frequency f. To simplify notation, we write our analysis in terms of the angular frequency $\omega = 2\pi f$.

Let the time-varying input voltage be represented by $V_{in}\angle 0$ where V_{in} is the amplitude of the sinusoid of angular frequency ω rad/s and phase angle is 0 (we choose the time origin suitably). The time-varying output voltage $V_{out}\angle\phi_{out}$ then has an amplitude V_{out} and phase angle ϕ_{out}. To analyze this circuit, we use the impedance $Z_R = R$ and $Z_C = \frac{1}{j\omega C}$ for resistor R and capacitor C, respectively. The gain G now depends on ω:

$$G = \frac{V_{out}}{V_{in}} = \frac{Z_C}{Z_R + Z_C} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}. \tag{12}$$

This circuit has a different gain G for different frequencies! To study this further, let us plot $|G|$ against f for some example values of R (1 k Ω) and C (5 nF) which is shown in Figure 26.

We see that this filter has a gain of $\approx$ 1 at low frequencies, and the gain decreases with increasing frequency. By 50 kHz, the gain is 0.5, and by 200 kHz, the gain drops to less than 0.2. This filter allows low frequencies to pass through, but attenuates high frequencies. Such a filter is known as a *low-pass filter*.

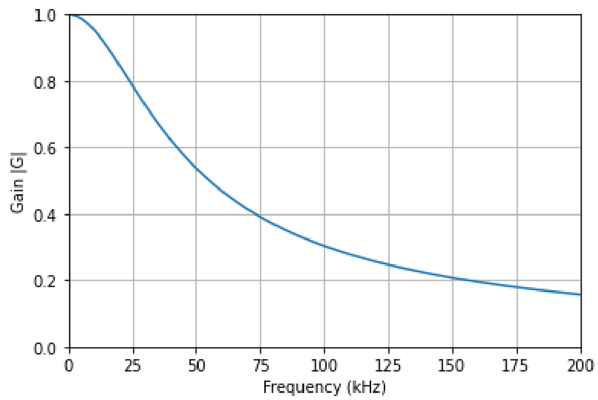


Figure 26: Gain versus frequency plot for RC low pass filter

It is customary to plot the gain and frequency on a logarithmic scale (Figure 27). The units for gain in the logarithmic scale is called decibel (dB), and is defined as $G_{dB} = 20\log_{10}(|G|)$.

The frequency at which the response drops to -3 dB (0.707 in linear scale) is known as the *cutoff frequency*. For an RC low-pass filter, this is given by

$$f_{cutoff} = \frac{1}{2\pi RC}.$$

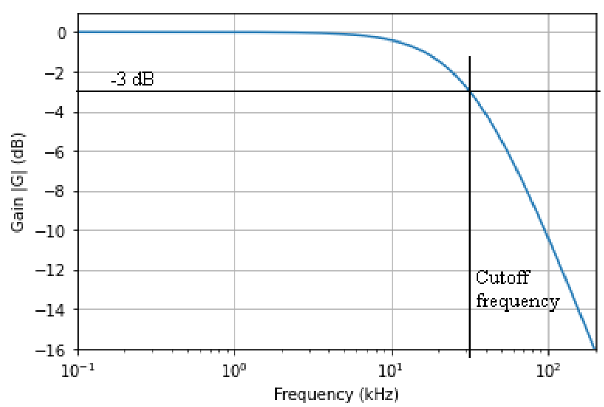


Figure 27: Frequency response of RC low pass filter: Gain (dB) versus frequency plot

RC high-pass filter

What happens if we take the same circuit as above, but exchange the positions of the resistor and the capacitor? If the input voltage V_{in} is not time-varying, the capacitor acts as an open circuit, and the output voltage $V_{in} = 0$. The gain G of this circuit is then just 0. However, as the frequency increases, the impedance of the capacitor starts to decrease and the gain increases.

Let us analyze the circuit as before. We again use the impedance $Z_R = R$ and $\frac{1}{j\omega C}$ for resistor R and capacitor C, respectively. The gain G depends on

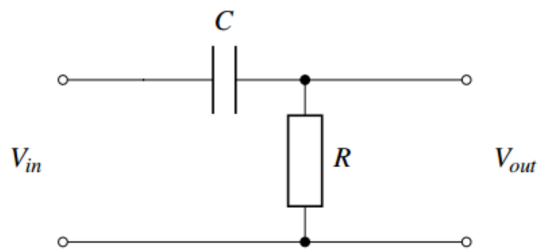


Figure 28: RC high-pass filter

ω:

$$G(\omega) = \frac{V_{out}}{V_{in}} = \frac{Z_R}{Z_C + Z_R} = \frac{R}{\frac{1}{j\omega C} + R} = \frac{j\omega RC}{1 + j\omega RC}.$$

$$|G(\omega)| = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} \tag{13}$$

Let us plot $20 \log_{10} |G(\omega)|$ against f for the same example values of R (1 kΩ) and C (5 nF) (Figure 29).

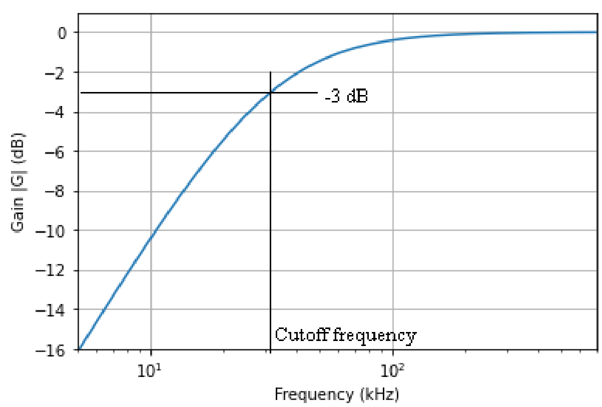


Figure 29: Frequency response of RC high pass filter: Gain (dB) versus frequency plot

We now see that this forms a high-pass filter, allowing high frequencies to pass through, and attenuating the low frequencies. The cutoff frequency of this filter is the same $f_{cutoff} = \frac{1}{2\pi RC}$.

Bandpass filter

Once we know how the basic building blocks of RC filter work, we can combine high- and low-pass filters to create more complex filters. For example, if we want a filter that only allows frequencies between f_1 and f_2 to pass through, we can combine a RC high-pass filter with a RC low-pass filter. Such filters are called *band-pass* filters. The difference $f_2 - f_1$ is known as the *bandwidth* of the filter. The *frequency response* of the above filter, *i.e.*, change in gain of the filter as function of frequency, for $R1 = 1\ k\Omega$, $C1 = 0.5\ nF$, $R2 = 10\ k\Omega$ and $C2 = 5\ nF$ is shown in Figure 31.

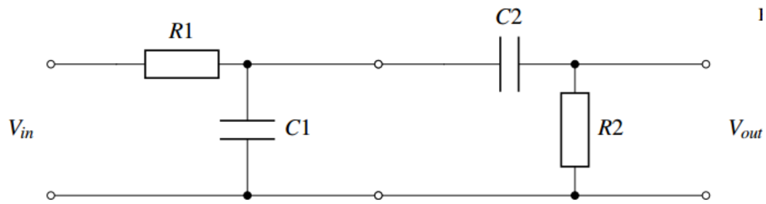


Figure 30: RC band-pass filter

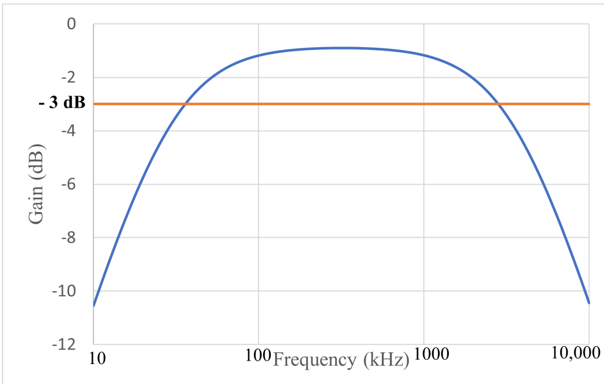


Figure 31: Frequency response of a band pass filter

- Note that the lower and higher cutoff frequencies of the RC bandpass filter are not simply the cutoff frequencies of the high-pass and the low-pass sections of the filter. Why is that? Can you analyze the RC bandpass filter circuit to compute it's gain?
- Can you think of what other types of filters could you create by cascading various RC filters?
- Can you think of how to build a band-pass filter using one resistor, one capacitor and one inductor?

What if my signal is not sinusoidal?

So far, we have studies how our filters respond to time-invariant signals (DC voltages), and to sinusoidally time-varying signals. But what if the signal is time-varying but not sinusoidal? Surely real world signals are not always just sinusoidal!

You're right, real-world signals can have arbitrary time variations. But the mathematics of **Fourier series** tells us that any signal can be represented as a sum of many sinusoidal signals with different frequencies, different amplitudes and different phases. Since our filter circuit is linear, if we know how it behaves for each of the component frequencies, we also know how it behaves for the sum (simply sum up the outputs). Hence it is sufficient to know the frequency response of the filter to know how exactly it'll modify any input signal! This is a very powerful idea, and the basis of most signal processing techniques.

Note: You will learn more about Fourier series in **EE2023 Signals and Systems**.

Applications

Okay, so now we understand how to build filters – but what are they used for?

Filters appear everywhere in Engineering. Take an example of some sensor (say the antenna of your cellphone) that generates a tiny voltage that we are interested in measuring. Chances are that the output of the sensor is contaminated by a 50 Hz noise being picked up from power transmission lines that are oscillating with a AC voltage at 50 Hz and radiating electromagnetic waves. This noise may even be much larger in amplitude than the signal from the sensor we want to measure! So what do we do? Create a filter that'll filter away the 50 Hz noise! Your cellphone probably operates at the 900 MHz band, and so a high-pass filter with cutoff frequency just below this frequency can easily filter away the 50 Hz noise.

Take another example of a sensor – say a light sensor that is measuring ambient light to automatically switch on your porch light at night. Again, if its output is contaminated by the 50 Hz power-line noise, you don't want it to turn your porch light on/off many times a second. We know that the ambient light does not vary rapidly, so we can setup a low-pass filter with a cutoff frequency of 0.1 Hz (or even lower) at the output of your sensor to filter off the 50 Hz noise. This will give you a stable output that can be used to drive a relay to turn on/off your porch light.

- Do you recall the AC/DC coupling option on the Oscilloscope channel? What kind of a filter do you think that uses?