5. Cálculo de predicados

Os seguintes exercícios e problemas são adaptados a partir do livro *Lógica e Aritmética*, de Augusto Franco de Oliveira, 3ª edição, Gradiva.

1. **(I)** Efectue as seguintes derivações (no sistema **DNQ**) indicadas no texto, exibindo as dependências de hipóteses:
   (a) $\forall x(Px \to Qx), Pa \vdash Qa$;
   (b) $\forall x(Px \to Qx), \neg Qa \vdash \neg Pa$;
   (c) $\forall x(Px \to Qx), \forall x(Qx \to Rx) \vdash \forall x(Px \to Rx)$;
   (d) $\forall x(Px \to Qx), \forall xPx \vdash \forall xQx$;
   (e) $\forall x(Px \land Qx) \vdash \forall xPx$;
   (f) $\forall x(Px \land Qx) \dashv\vdash \forall xPx \land \forall xQx$;
   (g) $\forall x(Px \lor Qx), \forall x\neg Px \vdash \forall xQx$;
   (h) $\forall x(Px \to Qx), Pa \vdash \exists xQx$;
   (i) $\forall x(Px \to Qx), \exists xPx \vdash \exists xQx$;
   (j) $\forall xPx \dashv\vdash \forall yPy$;
   (k) $\exists xPx \dashv\vdash \exists yPy$;
   (l) $\forall xyQxy \dashv\vdash \forall yxQxy$;
   (m) $\exists xyQxy \dashv\vdash \exists yxQxy$;
   (n) $\exists xQx \dashv\vdash \exists xyPx \land Py$;
   (o) $\exists x\forall yRxy \vdash \forall y\exists xRxy$;
   (p) $\forall xPx \dashv\vdash \neg\exists x\neg Px$;
   (q) $\exists xPx \dashv\vdash \neg\forall x\neg Px$;
   (r) $\forall x\big(\theta \to \phi(x)\big) \dashv\vdash \theta \to \forall x\phi(x)$;
   (s) $\forall x\big(\theta \land \phi(x)\big) \dashv\vdash \theta \land \forall x\phi(x)$;
   (t) $\forall x\big(\theta \lor \phi(x)\big) \dashv\vdash \theta \lor \forall x\phi(x)$;
   (u) $\exists x\big(\theta \lor \phi(x)\big) \dashv\vdash \theta \lor \exists x\phi(x)$;
   (v) $\exists x\big(\phi(x) \to \theta\big) \dashv\vdash \forall x\phi(x) \to \theta$;
   (w) $\forall x\big(\phi(x) \to \theta\big) \dashv\vdash \exists x\phi(x) \to \theta$;

2. **(I)** Mostre que
   (a) $\forall x(Px \to Qx) \vdash \forall xPx \to \forall xQx$;
   (b) $\forall x(Px \to Qx) \vdash \exists xPx \to \exists xQx$;
   (c) $\forall x(Px \land Qx) \dashv\vdash \forall xPx \land \forall xQx$;
   (d) $\exists x(Px \lor Qx) \dashv\vdash \exists xPx \lor \exists xQx$;
   (e) $\exists x(Px \land Qx) \vdash \exists xPx \land \exists xQx$;
   (f) $\forall xPx \lor \forall xQx \vdash \forall x(Px \lor Qx)$.

3. **(I)** Mostre, por meio de contra-exemplos, que não se pode substituir "$\vdash$" por "$\dashv$" nas alíneas (2a), (2b), (2e) e (2f) acima.

4. **(I)** Mostre, por meio de contra-exemplos, que as sentenças seguintes não são universalmente válidas:
   (a) $\exists xyRxy \to \exists xRxx$;
   (b) $\exists xPx \land \exists xQx \to \exists x(Px \land Qx)$.

5. **(I)** Efectue as seguintes derivações (no sistema **DNQ**, com regras de inferência para a igualdade) indicadas no texto, exibindo as dependências de hipóteses:
   (a) $\vdash \exists x(x \doteq t)$ (onde t é um termo fechado);
   (b) $\vdash \exists x(x \doteq x)$;
   (c) $\vdash \forall x(x \doteq x)$;
   (d) $\vdash \forall xy(x \doteq y \to y \doteq x)$;
   (e) $\vdash \forall xyz(x \doteq y \land y \doteq z \to x \doteq z)$;
   (f) $\vdash \forall xyuv\big(x \doteq u \land y \doteq v \to (Rxy \leftrightarrow Ruv)\big)$;
   (g) $\vdash \forall xyuv(x \doteq u \land y \doteq v \to fxy \doteq fuv)$;
   (h) $Pa \dashv\vdash \exists x(x \doteq a \land Px)$;
   (i) $\forall x(Px \to x \doteq c \lor x \doteq d), \exists x(Px \land Qx) \vdash Qc \lor Qd$;
   (j) $Pa \dashv\vdash \forall x(x \doteq a \to Px)$;
   (k) $\vdash \forall xyz(x \doteq z \land y \doteq z \to x \doteq y)$ [*lei de Euclides*];