

# MATH1064 Study Notes

University of Sydney  
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MATH1064



Happy studying!

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# 1 Introduction to Discrete Maths and Set Theory

## 1.1 Introduction to Discrete Maths

Discrete maths is the study of "discrete structures", this includes objects which are:

- Countable or listable
- Distinct and unconnected.

There are two objectives of MATH1064:

1. Develop mathematical reasoning skills. This involves using **logic and proofs**, and **rigorously** (exhausting all possibilities) finding solutions to problems.
2. Study **discrete structures** and their properties including:
  - Sets and functions
  - Prime numbers and modular arithmetic
  - Graphs and networks
  - Counting and probability.

## 1.2 Introduction to Set Theory

### 1.2.1 Definitions

**Definition 1** (Set). A **set**  $S$  is a collection of objects, called **elements** of  $S$ .

- If  $x$  is an element of  $S$ ,  $x \in S$
- If not, then  $x \notin S$ .

Sets can be finite or infinite:

- Example (finite):  $S = \{2, 3, 5\}$
- Example (infinite):  $S = \{0, 1, 2, \dots\} = \mathbb{N}$

**Definition 2** (Set Equivalence). Two sets  $S$  and  $T$  are said to be equal if they contain the same elements, regardless of **order** or **repetition**.

- Example 1:

$$S = \{1, 1, 5, 3\}$$

$$T = \{1, 3, 5\}$$

$$S = T$$

- Example 2:

$$S = \{-1, 0, 1, \dots\}$$

$$T = \{0, 1, \dots\}$$

$$S \neq T$$

**Definition 3** (Empty Set). The **empty set** is a unique set containing no elements.  $\emptyset = \{\}$ .

**Definition 4** (Singleton Set). A **singleton set** has only one element, e.g.  $S = \{x\}$  or  $S = \{x, x, x\}$

### 1.2.2 Unpacking Sets

Sets can contain other sets as elements, inner sets are considered distinct elements even if their contents are the same as other elements in the outer set. This is because when we unpack sets, we only remove outer curly braces  $\{ \}$ .

$$S = \{1, 2, \{1\}\}$$

$$|S| = 3 \text{ distinct elements}$$

### 1.2.3 Sets with Properties

We can describe sets using **set builder notion** which indicates the properties of a set.

$$A = \{x \in S \mid P(x)\}$$

"The set A consists of all elements  $x$  of  $S$  such that  $x$  has property  $P$ ".

Examples:

$$\{x \in \mathbb{N} \mid 3 \leq x \leq 5\} = \{3, 4, 5\}$$

$$\{y \in \mathbb{Z} \mid y = 2k \text{ for some } k \in \mathbb{Z}\} = \{\dots, -2, 0, 2, \dots\}$$

$$\{2z + 1 \mid z \in \mathbb{N}\} = \{1, 3, 5, \dots\}$$

### 1.2.4 Russel's Paradox

Define a set  $T = \{S, \text{set} \mid S \notin S\}$ . The set  $T$  contains any set  $S$  which does not contain itself.

Consider if  $T \in T$ :

- If  $T \in T$ :  $T$  does not satisfy the condition.
- If  $T \notin T$ :  $T$  does satisfy the condition, thus  $T \in T$  according to our definition.

This induces a contradiction, hence demonstrating that we need **axioms** which **rigorously** state how to define and build sets.

### 1.2.5 Operations on Sets

**Definition 5 (Union).** Given two sets  $S$  and  $T$ , the **union** of  $S$  and  $T$  is the set containing all elements from  $S$  and  $T$ . This is written as  $S \cup T$  where  $x \in S$  **OR**  $x \in T$ .

- Example 1:  $\{1, 2, 3\} \cup \{2, 5\} = \{1, 2, 3, 5\}$
- Example 2:  $\{0, 1, 2, \dots\} \cup \{0, -1, -2, \dots\} = \mathbb{Z}$

**Definition 6 (Intersection).** Given two sets  $S$  and  $T$ , the **intersection** of  $S$  and  $T$  is the set of elements belonging to both  $S$  and  $T$ . This is written as  $S \cap T$  where  $x \in S$  **AND**  $x \in T$ .

- Example 1:  $\{1, 2, 3\} \cap \{2, 5\} = \{2\}$
- Example 2:  $\{1, 2, 3, \dots\} \cap \{-1, -2, -3, \dots\} = \emptyset$

Multiple unions and intersections can be taken at a time.

$$\begin{aligned} \bigcup_{i=1}^3 \{i, 2i\} &= \{1, 2\} \cup \{2, 4\} \cup \{3, 6\} \\ &= \{1, 2, 3, 4, 6\} \end{aligned}$$

$$\begin{aligned} \bigcap_{i=1}^3 \{i, i+1, i+2\} &= \{1, 2, 3\} \cap \{2, 3, 4\} \cap \{3, 4, 5\} \\ &= \{3\} \end{aligned}$$

Formally, for sets  $A_1, A_2, \dots, A_i$  we define the set  $A_i$  as:

$$\bigcup_{i=1}^{\infty} A_i = \{x \mid x \in A_k \text{ for SOME } k \geq 1\}$$

That is for an infinite series of unions,  $x$  is an element that appears **at least once** in the sets  $A_k$  for  $k \geq 1$ .

And similarly for intersections, we define the set  $A_i$  as:

$$\bigcap_{i=1}^{\infty} A_i = \{x \mid x \in A_k \text{ for ALL } k \geq 1\}$$

That is, for an infinite series of intersections,  $x$  is an element which appears in **all** sets  $A_k$  for  $k \geq 1$ .

### 1.2.6 Subsets

**Definition 7** (Subsets). A set  $S$  is a **subset** of another set  $T$  if every element of  $S$  is an element of  $T$ . This is written as  $S \subset T$ .

Additionally, if  $S$  is a subset of  $T$  but is not equal to  $T$ , then it is considered a **proper subset**, denoted as  $S \subsetneq T$ .

For example,

$$\begin{aligned} S &= \{2, 4, 6\} \\ T &= \{2, 4, 6, 8\} \\ S &\subsetneq T, \text{ since } 8 \notin S \end{aligned}$$

### 1.2.7 Proving Subset Relationships

To prove  $S \subseteq T$ , we need to:

1. Take an arbitrary element of  $S$ , which we call  $x$
2. Show that  $x \in T$

**Example:** Let  $S$  and  $T$  be sets where,

$$\begin{aligned} S &= \{2n \mid n \in \mathbb{N}, n \geq 1\} \\ T &= \{2^m \mid m \in \mathbb{N}\} \end{aligned}$$

*Proof.* Let  $x \in S$ , by definition  $x = 2^n$  for some  $n \geq 1$ .

$$\begin{aligned} x &= 2^n \\ x &= 2(2^{n-1}), \text{ rewriting } x \end{aligned}$$

Since  $n \geq 1$ ,  $n - 1 \geq 0$  meaning that  $n - 1 \in \mathbb{N} \implies 2^{n-1} \in \mathbb{N}$ . Because  $2^{n-1}$  is a natural number, we can rewrite  $x = 2m$  where  $m = 2^{n-1}$  as we know  $m \in \mathbb{N}$ ,  $x \in T \implies S \subseteq T$ .  $\square$

### 1.2.8 Proving Equality Relationships

To prove that  $S = T$ , we need a "**double containment proof**" which shows that both sets have the same elements, that is:

1. Every  $x \in S$  also satisfies  $x \in T$
2. Every  $x \in T$  also satisfies  $x \in S$

**Example:** Let  $S$  and  $T$  be sets where,

$$\begin{aligned} S &= \{2m + 1 \mid m \in \mathbb{Z}\} \\ T &= \{2r - 1 \mid r \in \mathbb{Z}\} \end{aligned}$$

*Proof. Show  $S \subseteq T$ . Let  $x \in S$ , then  $x = 2m + 1$  for some  $m \in \mathbb{Z}$ .*

Let  $r = m + 1$ .

$$x = 2m + 1$$

$$x = 2m + 2 - 1, \text{ rewriting } x$$

$$x = 2(m + 1) - 1, \text{ notice } m + 1 = r$$

$$x = 2r - 1, \text{ this is the same as } x \in T$$

$x = 2r - 1$  for some  $r \in \mathbb{Z}$ . Thus,  $x \in T \implies S \subseteq T$ . □

*Proof. Show  $T \subseteq S$ . Let  $x \in T$ , then  $x = 2r - 1$  for some  $r \in \mathbb{Z}$ .*

Let  $m = r - 1$ .

$$x = 2r - 1$$

$$x = 2r - 2 + 1$$

$$x = 2(r - 1) + 1$$

$$x = 2m + 1$$

$x = 2m + 1$  for some  $m \in \mathbb{Z}$ . Thus,  $x \in S \implies T \subseteq S$ . □

Since  $S \subseteq T$  and  $T \subseteq S$ , the two sets must have the same elements and are equal,  $S = T$ .

## 1.3 More Set Theory

### 1.3.1 Cardinality

The **cardinality** of a set  $S$  in a rough sense refers to the size of  $S$ , i.e. the no. of elements in  $S$ .

- If  $S$  is finite, then  $|S|$  is the number of distinct elements in  $S$ .
- If  $S$  is infinite, then we write  $|S| = \infty$ .

Note that there can be **different infinite cardinalities**, or sizes of infinities. A basic example of this is the set of natural numbers  $\mathbb{N}$  compared to the set of real numbers  $\mathbb{R}$ .

### 1.3.2 Set Differences

**Definition 8** (Set Difference). Given two sets  $S$  and  $T$ , the **set difference** is the set of elements  $x \in S$  and  $x \notin T$ . This is written as  $S \setminus T$  or  $S - T$ .

- Example 1:  $\{1, 2, 3\} \setminus \{2, 5\} = \{1, 3\}$
- Example 2:  $\{0, 1, 2, \dots\} \setminus \{0, -1, -2, \dots\} = \{1, 2, \dots\}$
- Example 3:  $\mathbb{N} \setminus \mathbb{Z} = \emptyset$



### 1.3.3 Universal Set

**Definition 9** (Universal Set). Let  $\mathbb{U}$  be some **universal set** which is the set containing all elements of which other sets are subsets. The universal set is context dependent.

- $\mathbb{U} = \mathbb{Z}$ , if working with number theory
- $\mathbb{U} = \mathbb{R}^2$ , if working with plane geometry

### 1.3.4 Complementary Set

**Definition 10** (Complement). For a set  $S \subseteq \mathbb{U}$ , the **complement** of  $S \in \mathbb{U}$  is given by  $x \in \mathbb{U}$  where  $x \notin S$ . This is written as  $\bar{S}$  or  $S^c$ .

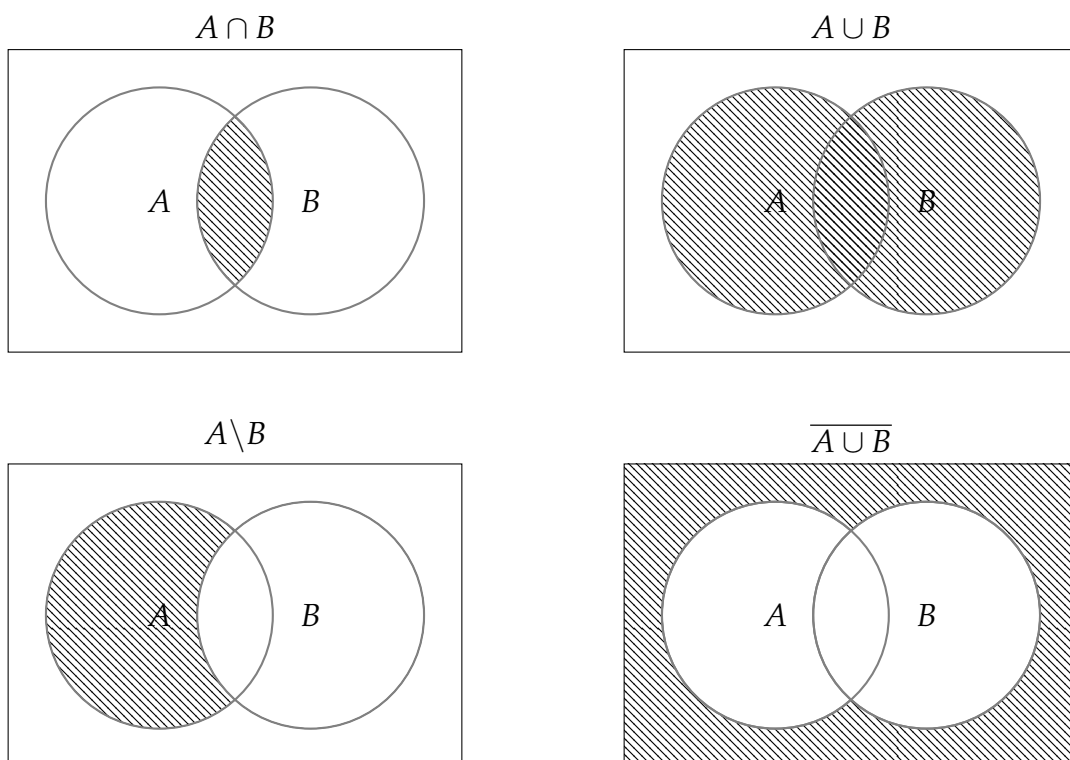
In other words,  $\bar{S}$  is the set containing all elements not in  $S$ , where the range of elements considered is constrained by  $\mathbb{U}$ . This resolves the Russel Paradox.

For example, if  $\mathbb{U} = \mathbb{Z}$ :

- $\overline{\{1, 2, 3\}} = \{\dots, -1, 0, 4, \dots\}$
- $\overline{\{x \in \mathbb{Z} \mid x > 0\}} = \{x \in \mathbb{Z} \mid x < 0\}$
- $\overline{\mathbb{Z}} = \emptyset$

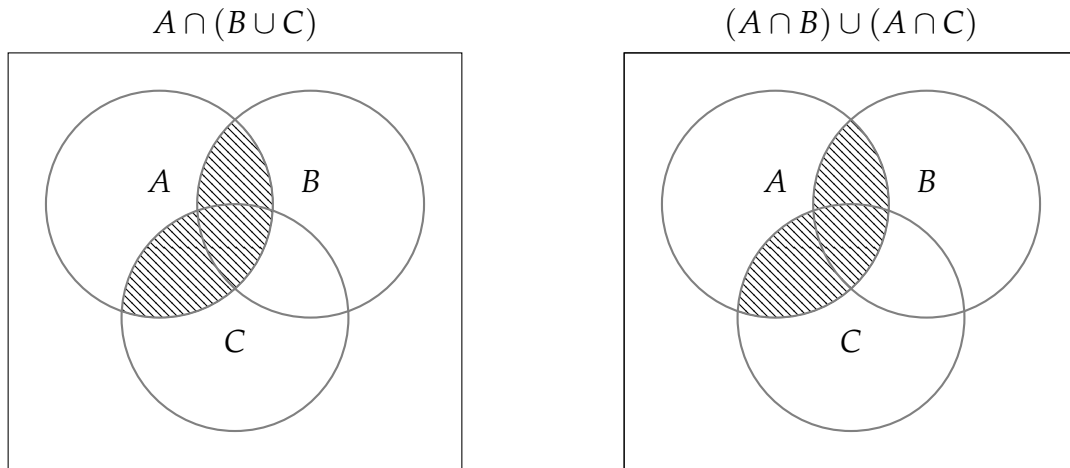
### 1.3.5 Venn Diagrams

**Venn Diagrams** are tools used to visualise relationships between sets.



Venn diagrams are useful for intuition and can be used to guide proofs, but are not sufficient as rigorous proofs in of themselves.

**Example:** Prove if  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



*Proof. Double Containment.* First, show  $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ .

Let  $x \in A \cap (B \cup C)$ :

$$\begin{aligned}
 &\text{Then } x \in A \text{ and } x \in (B \cup C) \\
 &x \in (B \cup C) \implies x \in B \text{ or } x \in C \\
 &\text{if } x \in B, x \in (A \cap B) \implies x \in (A \cap B) \cup (A \cap C) \\
 &\text{if } x \in C, x \in (A \cap C) \implies x \in (A \cap C) \cup (A \cap B) \\
 &\implies A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)
 \end{aligned}$$

This proof works because we know that if  $x \in (A \cap B)$ , then suppose we expand the set  $(A \cap B)$  by including another arbitrary set, say  $(A \cap C)$ . We can implicitly assume  $x$  is an element of  $(A \cap B) \cup (A \cap C)$  since we are simply expanding the scope of the set.

Second, show  $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ .

Let  $x \in (A \cap B) \cup (A \cap C)$ :

$$\begin{aligned}
 &\text{Then } x \in A \text{ and } x \in B \\
 &\implies x \in (B \cup C) \\
 &\implies x \in A \cap (B \cup C)
 \end{aligned}$$

Then  $x \in A$  and  $x \in C$

$$\implies x \in (C \cup B)$$

$$\implies x \in A \cap (C \cup B)$$

Thus,  $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ , indicating that both sets are equal.

□

### 1.3.6 Set Identities

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- $A \cap (A \cup B) = A$

## 2 Week 2

### 2.1 Power Sets and Cartesian Planes

#### 2.1.1 Power Set

**Definition 11** (Power Set). For a set  $S$  the **power set** of  $S$  is the set which contains all possible subsets of  $S$ . This is denoted as  $P(S) = \{A \mid A \subseteq S\}$ .

For example,

- $S = \{a, b\} \implies P(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
- $S = \emptyset \implies P(S) = \{\emptyset\}$
- $S = \{1, 2, 3\} \implies P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

**Theorem 1** (Cardinality of the Power Set). Let  $S$  be a finite set with  $|S| = n$ , then  $|P(S)| = 2^n$

#### 2.1.2 Cartesian Product

**Definition 12** (Cartesian Product). For two sets  $A$  and  $B$ , the cartesian product  $A \times B$  is given by the set of ordered pairs,  $A \times B = \{(a, b) \mid a \in A, b \in B\}$ .

**Note:** in ordered pairs, the order of elements, and repetition matter (like cartesian coordinates).

**Example:**  $A = a, b, B = 1, 2$

- $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$
- $B \times A = \{(1, a), (1, b), (2, a), (2, b)\}$

**Theorem 2** (Cardinality of Cross Product). The cardinality of the cross product of two sets  $A$  and  $B$  is given by  $|n \times m|$  where  $|A| = n$  and  $|B| = m$ .

The cross product of an empty set is always the empty set. For example, if  $A = \emptyset$ ,  $B = \{1, 2\}$ , then  $A \times B = \emptyset$ .

$A \times B = B \times A \iff A = B$  or when  $A$  or  $B = \emptyset$ .

*Proof.* If  $A$  or  $B = \emptyset$ , then there is nothing to prove.

Otherwise, assume  $A \times B = B \times A$ .

**Show**  $A \subseteq B$ .

Let  $x \in A$  and choose some  $y \in B$ :

$$\begin{aligned} \text{Then } (x, y) &\in A \times B \\ A \times B = B \times A &\implies (x, y) \in B \times A \\ &\implies x \in B \\ &\implies A \subseteq B \end{aligned}$$

**Show**  $B \subseteq A$ .

$$\begin{aligned} (y, x) &\in B \times A \\ A \times B = B \times A &\implies (y, x) \in A \times B \\ &\implies y \in A \\ &\implies B \subseteq A \end{aligned}$$

Therefore,  $A = B$  by double containment.

□

### 2.1.3 Functions

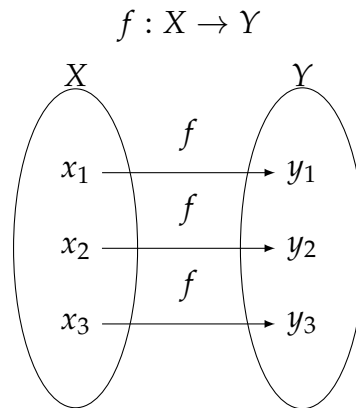
**Definition 13** (Function). Given two sets  $X$  and  $Y$ , the function  $f$  from  $X$  to  $Y$ , written as  $f : X \rightarrow Y$ , maps each  $x \in X$  to exactly one  $y \in Y$  (see vertical line test).

**Important Criteria:**

- Every  $x \in X$  maps somewhere, i.e.  $X \rightarrow Y$ .
- Every  $x \in X$  maps to **at most one**  $y \in Y$ .

**Examples:**

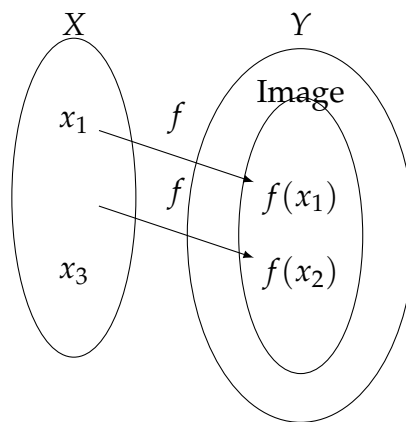
- $f : \mathbb{N} \rightarrow \mathbb{N}, f(n) = n!$  is a function
- $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = \ln x$  is not a function as  $f(x)$  is undefined for  $x < 0$
- $f : \mathbb{Q} \rightarrow \mathbb{N}, f(\frac{n}{m}) = n$  is not a function as there are multiple ways to write the same input with different outputs, e.g.  $\frac{3}{2} = 3$  but  $\frac{6}{4} = 6$ .



#### 2.1.4 Function Terminology

Let  $f : X \rightarrow Y$  be a function. Then we say:

- $X$  is called the **domain** of  $f$
- $Y$  is called the **co-domain** of  $f$
- $f(x) \in Y$  is called the **image** of  $x$  (also called range).



Similarly, the **pre-image** of a  $y$  is the set of all input values which map to the co-domain  $Y$ . This is written as  $f^{-1}(y) = \{x \in X \mid f(x) = y\} \subseteq X$ . In essence, it explains where does  $y$  come from under the function  $f(x)$ .

#### 2.1.5 Identity Function

**Definition 14** (Identity Function). The **identity function** is a function which maps a set to itself. For any set  $S$ ,  $i_x : X \rightarrow X$  is defined by  $i_x(a) = a, a \in X$ .

### 2.1.6 Floors and Ceilings

**Definition 15** (Floor). For  $x \in \mathbb{R}$ , the floor of  $x$  is the unique integer  $n$  such that  $n \leq x \leq n + 1$ . In essence, it is "rounding down"  $x$  in terms of value. It is written as  $\lfloor x \rfloor$ .

**Definition 16** (Ceiling). For  $x \in \mathbb{R}$ , the ceiling of  $x$  is the unique integer  $m$  such that  $m - 1 \leq x \leq m$ . In essence, it is "rounding up"  $x$  in terms of value. It is written as  $\lceil x \rceil$ .

**Note:** both floor and ceiling are functions  $f : \mathbb{R} \rightarrow \mathbb{Z}$ . Additionally, for  $x < 0$ , the rounding up/down of these functions are reversed.

**Examples:**

- $\lfloor 2.5 \rfloor = 2$
- $\lfloor -2.5 \rfloor = -3$
- $\lceil 2.5 \rceil = 3$
- $\lceil -2.5 \rceil = -2$

## 2.2 Properties of Functions

### 2.2.1 Equality of Functions

**Definition 17** (Equal Functions). Two functions,  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  are said to be equal if  $f(x) = g(x)$  for all  $x \in X$ .

**Example:**  $f$  and  $g$  are equal.

- $f(x) = n!$
- $g(x) = \frac{(n+1)!}{n+1} = \frac{(n+1)n!}{n+1} = n!$

**Example:**  $f$  and  $g$  are not equal (not same co-domain).

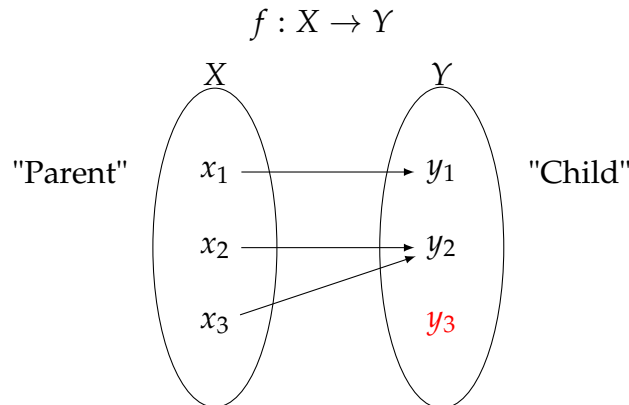
- $f : \mathbb{N} \rightarrow \mathbb{N}, f(n) = n^2$
- $g : \mathbb{N} \rightarrow \mathbb{R}, f(n) = n^2$

### 2.2.2 Surjective/Onto

**Definition 18.** A function  $f : X \rightarrow Y$  is considered **onto** or **surjective** if for all  $y \in Y$ , there exists some  $f(x) = y$ ,  $x \in X$ . In other words, the co-domain has no orphans; every  $y$  is the image of at least one  $f(x)$ .

**Example:**

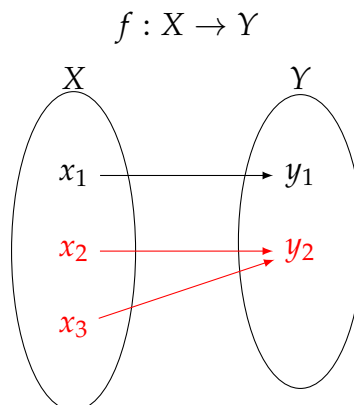
The below function is **not** a surjective function since  $y_3$  is "orphaned".



### 2.2.3 Injective/One-to-One

**Definition 19.** A function  $f : X \rightarrow Y$  is considered **one-to-one** or **injective** if  $f(x_1) = f(x_2)$  implies  $x_1 = x_2$  for all  $x_1, x_2 \in X$ . In other words, different elements in  $X$  must map to different images in  $Y$  (two elements can't map to same place).

**Example:** The function below is **not** an injective function since  $x_2$  and  $x_3$  map to  $y_2$ , and  $x_2 \neq x_3$ .



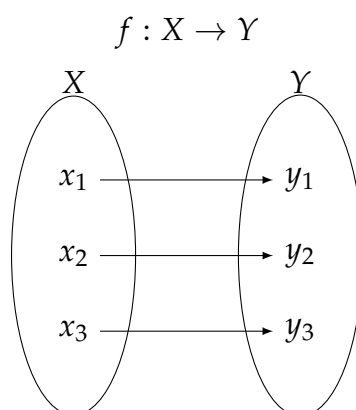
### 2.2.4 Bijective

**Definition 20.** A function is said to have a **one-to-one correspondence** or **bijective** if it is both injective and surjective. In other words, if all elements  $x \in X$  are paired



with a unique  $y \in Y$ .

**Example:** The below function is bijective.



### 2.2.5 Proving Properties of Functions

Property	Proving	Disproving
<b>Surjective</b>	Take any arbitrary $y \in Y$ , construct some $x \in X$ so that $f(x) = y$	Find a counterexample, $y \in Y$ and show $y \neq f(x)$ for any $x \in X$
<b>Injective</b>	Assume $f(x_1) = f(x_2)$ and deduce $x_1 = x_2$	Find a counterexample, $x_1, x_2 \in X$ , where $x_1 \neq x_2$ but $f(x_1) = f(x_2)$
<b>Bijective</b>	Prove both	Disprove at least one

**Example:** Let  $f, g : \mathbb{N} \rightarrow \mathbb{N}$ . Prove if  $f$  and  $g$  are injective and surjective.

$f(n) = n^2$	
<b>Surjective?</b>	No, the function is not surjective as the image of $x$ only covers squares, whilst the codomain is $y \in \mathbb{N}$ . For example, let $y$ be 3, then $f(n) = 3$ , $n = \pm\sqrt{3}$ , but $\pm\sqrt{3} \notin \mathbb{N}$ .
<b>Injective?</b>	<p>Yes the function is injective according to the proof.</p> $\begin{aligned} \text{Assume } f(n) &= f(m) \\ \implies n^2 &= m^2 \\ n^2 - m^2 &= 0 \\ (n + m)(n - m) &= 0 \\ \implies n &= -m \text{ or } m = -n \\ \text{but } n &\in \mathbb{N}, \text{ so } n \neq -m \end{aligned}$ <p>Since <math>n</math> can only equal <math>m</math>, we know that the function is injective.</p>

$g(n) =  n - 1 $	
<b>Surjective?</b>	<p>Yes, the function is surjective according to the proof.</p> $\begin{aligned} \text{Let } y &\in \mathbb{N} \\ \text{If } y &\in \mathbb{N}, \text{ then } y + 1 \in \mathbb{N} \\ g(y + 1) &=  (y + 1) - 1  \\ &=  y  \\ &= y \end{aligned}$
<b>Injective?</b>	No, the function is not injective, consider $g(n) = 1$ , then $g(0)$ and $g(2)$ both satisfy $f(n) = 1$ , but $0 \neq 2$ .

### 2.2.6 Functions and Finite Sets

Let  $X, Y$  be two finite sets, and  $f : X \rightarrow Y$ .

- If  $f$  is **injective**, then  $|X| \leq |Y|$
- If  $f$  is **surjective**, then  $|X| \geq |Y|$
- If  $f$  is **bijective**, then  $|X| = |Y|$

### 2.2.7 Compositions of Functions

**Definition 21** (Composite Function). Let  $f : X \rightarrow Y$   $g : Y \rightarrow Z$  be two functions. The composition is written  $g \circ f : X \rightarrow Z$

**Example:** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) = xy$  and  $g : \mathbb{R} \rightarrow \mathbb{R}^2, g(x) = (\sin x, x)$  be functions.

$$\begin{aligned} f \circ g : \mathbb{R} &\rightarrow \mathbb{R} \\ f(g(x)) &= f((\sin x, x)) \\ &= \sin x \times x \end{aligned}$$

$$\begin{aligned} g \circ f : \mathbb{R} &\rightarrow \mathbb{R}^2 \\ g(f(x)) &= g(xy) \\ &= (\sin xy, xy) \end{aligned}$$

## 2.3 Sequences

### 2.3.1 Sequences

**Definition 22** (Sequence). A **sequence** is an ordered list of data. That is, given a set  $S$ , a sequence in  $S$  is a function  $f : I \rightarrow S$  where  $I \subseteq \mathbb{Z}$  (we can assign a numerical order for elements in the set  $S$ ).

**Notation:** we use letters and subscripts to indicate elements of a sequence, for example  $a_1 = f(1), a_2 = f(2), a_n = f(n)$ . The terms of a sequence are its elements.

### 2.3.2 Equality of Sequences

To prove that two sequences,  $p_n$  and  $b_n$  are equal sequences, we need  $b_n$  in closed form, and  $p_n$  as a recursive sequence. Then we need to show that  $b_n$  satisfies the recursion relation and initial condition.

**Example:**

- Let  $(p_n) = (2p_{n-1} + 1)$ , where  $p_1 = 1$
- Let  $(b_n)_{n \geq 1} = (2^n - 1)$

**Prove**  $p_n = b_n$

**Initial condition:**  $b_1 = (2^1 - 1) = 1 = p_1 = 1$

**Recursive relation:** Sub  $b_{n-1}$  into  $p_n$  if it equals  $b_n$  then it is a recursive relation.

$$\begin{aligned} 2(b_{n-1}) + 1 &= 2(2^{n-1} - 1) + 1 \\ &= 2^{n-1+1} - 2 + 1 \\ &= 2^n - 1 \\ &= b_n \end{aligned}$$

## Summation

**Definition 23.** The summation of a sequence  $a_n$  is given by  $\sum_{i=1}^N a_n$ . In discrete math, we are interested in finite summations and the closed formulas for them

**Example:** Find a closed formula for  $\sum_{i=1}^N i = 1 + 2 + \cdots + N$ .

**Claim:**  $\sum_{i=1}^N i = \frac{N(N+1)}{2}$

*Proof.* Let  $M = \sum_{i=1}^N i$ .

Then,

$$\begin{aligned} 2M &= 2 \sum_{i=1}^N i \\ &= \sum_{i=1}^N i + \sum_{k=1}^N k \end{aligned}$$

$$\begin{aligned} \sum_{k=1}^N k &= N + (N-1) + (N-2) + \cdots + 1 \\ &= \sum_{k=1}^N (N - k + 1) \end{aligned}$$

$$\begin{aligned} 2M &= \sum_{i=1}^N i + \sum_{k=1}^N (N - k + 1) \\ &= \sum_{i=1}^N i = 1^N N + 1 = N(N+1) \end{aligned}$$

$$M = \frac{N(N+1)}{2}$$

Note<sub>1</sub>: we changed variable  $i$  to  $k$  in the second summation.

Note<sub>2</sub>: the variables  $i$  and  $k$  are equal to each other, so we add them in the final  $2M$  calculation. □

### 3 Week 3

## 4 Week 4

## 5 Week 5