MIT integration bee - 2011

Notación y Consideraciones Generales

A lo largo del documento vamos a utilizar la siguiente notación, en algunos problemas más complejos utilizaremos la notación usual:

$$\{\ f(x)\ \}_a^b := \int_a^b f(x)\ dx \ \ , \ \ [\ f(x)\ ] := \frac{d}{dx} f(x)$$

Recursos obtenidos de: https://math.mit.edu/~yyao1/integrationbee.html

Herramientas que te pueden ser útiles:

- Calculadora gráfica: https://www.desmos.com/calculator?lang=es
- Calculadora de integrales: https://mathdf.com/es/
- Calculadora de integrales: https://www.wolframalpha.com/
- Motor de busqueda para fórmulas en latex https://approach0.xyz/search/
- Te ayuda con ideas: https://chat.deepseek.com

Ejercicio 1

$$\int \frac{x^6-1}{x^4+x^3-x-1} dx$$

$$---\ \text{Demostración}\ ---$$

$$\begin{aligned} I &= \{\ x^2-x+1\ \} \\ &= \frac{x^3}{3} - \frac{x^2}{2} + x + C \end{aligned}$$

Nota:

1. Si tenemos una funcion racional $\frac{p(x)}{q(x)}$ donde $\deg p(x) \geq \deg q(x)$, entonces por el algoritmo de la división para polinomios tenemos que $p(x) = s(t)q(x) + r(x)$ donde $\deg r(x) < \deg q(x)$.

Ejercicio 2

$$\int \left(2\ln x + (\ln x)^2\right) dx$$

$$---\ \text{Demostración}\ ---$$

$$\begin{aligned} I &= \{\ 2te^t + t^2e^t\ \} && (1) \\ &= \{\ 2te^t + t^2e^t\ \} && (1) \\ &= t^2e^t + C && (2) \end{aligned}$$

- (1): $\begin{cases} x &= e^t \\ dx &= e^t dt \end{cases}$
- (2): $[e^t f(t)] = e^t(f(t) + f'(t))$

Ejercicio 3

$$\int \frac{2x}{\sqrt{1-x^4}}dx$$

$$--- \text{ Demostración } ---$$

$$\begin{aligned} I &= \left\{ \frac{1}{\sqrt{1-t^2}} \right\} && (1) \\ &= \arcsin(t) + C \end{aligned}$$

- (1): $\begin{cases} t &= x^2 \\ dt &= 2xdx \end{cases}$

Ejercicio 4

$$\int \frac{x^2+1}{x+1}dx$$

$$--- \text{ Demostración } ---$$

$$\begin{aligned} I &= \left\{ x - 1 + \frac{2}{x+1} \right\} \\ &= \frac{x^2}{2} - x + 2\ln|x+1| + C \end{aligned}$$

Ejercicio 5

$$\int \frac{\sin(x)^3 + \sin(x)^2 - 2\sin(x) - 2}{\sin(x)^2 + 2\sin(x) + 1}dx$$

$$--- \text{ Demostración } ---$$

$$\begin{aligned} I &= \left\{ \sin(x) - 1 - \frac{\sin(x)+1}{\sin^2(x) + 2\sin(x) + 1} \right\} && (1) \\ &= \left\{ \sin(x) - 1 - \frac{1}{\sin(x)+1} \right\} \end{aligned}$$

$$\begin{aligned} I_1 &= \{ \sin(x) - 1 \} \\ &= -\cos(x) - x \end{aligned}$$

$$\begin{aligned} I_2 &= \left\{ \frac{1}{\sin(x)+1} \right\} \\ &= \left\{ \frac{\sec(x)(\tan(x) + \sec(x))}{(\tan(x) + \sec(x))^2} \right\} \\ &= \left\{ \frac{1}{t^2} \right\} && (2) \\ &= -\frac{1}{t} \end{aligned}$$

$$I = \frac{1}{\tan(x) + \sec(x)} - \cos(x) - x + C$$

- (1): Hacer una división de polinomios con la variable $t = \sin(x)$
- (2): $\begin{cases} t &= \tan(x) + \sec(x) \\ dt &= \sec(x)(\tan(x) + \sec(x))dx \end{cases}$

Ejercicio 6

$$\int \sinh(x)^{-2} dx$$

$$--- \text{ Demostración } ---$$

$$\begin{aligned} I &= \{ \operatorname{csch}^2(x) \} \\ &= -\coth(x) + C \end{aligned}$$

Nota:

1. Propiedades básicas de las funciones trigonometricas hiperbólicas:

$$\begin{cases} \cosh^2(x) - \sinh^2(x) & = & 1 \\ 2\sinh^2(x) & = & \cosh(2x) - 1 \\ 2\cosh^2(x) & = & \cosh(2x) + 1 \\ \sinh^2(x) + \cosh^2(x) & = & \cosh(2x) \\ 2\sinh(x)\cosh(x) & = & \sinh(2x) \\ (\sinh(x) + \cosh(x))^2 & = & \cosh(2x) + \sinh(2x) \end{cases}$$

2. Derivadas de las funciones trigonométricas hiperbólicas:

$$\begin{cases} [\sinh(x)] & = & \cosh(x) \\ [\cosh(x)] & = & \sinh(x) \\ [\tanh(x)] & = & \operatorname{sech}^2(x) \\ [\coth(x)] & = & -\operatorname{csch}^2(x) \\ [\operatorname{sech}(x)] & = & -\operatorname{sech}(x)\tanh(x) \\ [\operatorname{csch}(x)] & = & -\operatorname{csch}(x)\coth(x) \end{cases}$$

Ejercicio 7

$$\int \sec^4 x \tan^2 x dx$$

$$--- \text{ Demostración } ---$$

$$\begin{aligned} I &= \{ \tan^2(x)(1 + \tan^2(x)) \sec^2(x) \} \\ &= \{ u^2(1 + u^2) \} && (1) \\ &= \frac{u^3}{3} + \frac{u^5}{5} + C \end{aligned}$$

- (1): $\begin{cases} u & = & \tan^2(x) \\ du & = & \sec^2(x)dx \end{cases}$

Ejercicio 8

$$\int \sqrt{\csc x - \sin x} dx$$

$$--- \text{ Demostración } ---$$

$$\begin{aligned} I &= \left\{ \frac{|\cos(x)|}{\sqrt{\sin(x)}} \right\} \\ &= \pm \left\{ \frac{\cos(x)}{\sqrt{\sin(x)}} \right\} \\ &= \pm 2\sqrt{\sin(x)} + C \quad (2) \end{aligned}$$

- (1): $\left\{ \begin{array}{rcl} u & = & \tan^2(x) \\ du & = & \sec^2(x)dx \end{array} \right.$

- (2): $[\sqrt{f(x)}] = \frac{1}{2}\frac{f'(x)}{\sqrt{f(x)}}$

Ejercicio 9

$$\int \cos^6 x dx$$

$$--- \text{Demostración} ---$$

$$\begin{aligned} I &= \frac{1}{8}\{ (\cos(2x)+1)^3 \} && (1) \\ &= \frac{1}{8}\{ \cos^3(2x) + 3\cos^2(2x) + 3\cos(2x) + 1 \} \\ &= \frac{1}{8}\{ -\cos(2x)\sin^2(2x) + 3\cos^2(2x) + 4\cos(2x) + 1 \} && (2) \\ &= \frac{1}{8}\left(-\frac{\sin^3(2x)}{6} + \frac{3}{2}\left(\frac{\sin(4x)}{4} + x\right) + 2\sin(2x) + x \right) + C && (1) \text{ y } (3) \end{aligned}$$

- (1): $\cos(2x) = 2\cos^2(x) - 1$

- (2): $\sin^2(x) + \cos^2(x) = 1$

- (3): $[f^n(x)] = nf^{n-1}(x)f'(x)$

Ejercicio 10

$$\int \frac{1}{1+2x^2+x^4}dx$$

$$--- \text{Demostración} ---$$

$$\begin{aligned} I &= \left\{ \frac{1}{(x^2+1)^2} \right\} \\ &= \{ \cos^2(\theta) \} && (1) \\ &= \frac{1}{2}\{ \cos(2\theta) + 1 \} && (2) \\ &= \frac{1}{2}\left(\frac{\sin(2\theta)}{2} + \theta \right) \\ &= \frac{1}{2}\left(\frac{x}{x^2+1} + \arctan(x) \right) + C \end{aligned}$$

- (1): $\left\{ \begin{array}{rcl} x & = & \tan(\theta) \\ dx & = & \sec^2(\theta)d\theta \end{array} \right.$

- (2): $\cos(2x) = 2\cos^2(x) - 1$

Ejercicio 11

$$\int \cos(\log x)dx$$

$$--- \text{Demostración} ---$$

$$
\begin{aligned}
I &= x\cos(\ln x) + \{\, \sin(\ln x)\, \} & (1)\\
&= x\cos(\ln x) + x\sin(\ln x) - I & (2)
\end{aligned}
$$

$$
I = \frac{x}{2}\left(\cos(\ln x) + \sin(\ln x)\right) + C
$$

- (1): $\left|\begin{array}{c|c} D & I \\ \cos(\ln x) & dx \\ -\frac{\sin(\ln x)}{x} & x \end{array}\right|$
- (2): $\left|\begin{array}{c|c} D & I \\ \sin(\ln x) & dx \\ \frac{\cos(\ln x)}{x} & x \end{array}\right|$

Ejercicio 12

$$
\int \frac{1}{\cos x} dx
$$

--- Demostración ---

$$
\begin{aligned}
I &= \{\, \sec(x)\, \} \\
&= \ln|\tan(x) + \sec(x)| + C & (1)
\end{aligned}
$$

- (1): $[\tan(x) + \sec(x)] = \sec(x)(\tan(x) + \sec(x))$

Ejercicio 13

$$
\int \frac{1}{9\cos^2 x + 4\sin^2 x} dx
$$

--- Demostración ---

$$
\begin{aligned}
I &= \left\{ \frac{\sec^2(x)}{9 + 4\tan^2(x)} \right\} \\
&= \left\{ \frac{1}{9 + 4u^2} \right\} & (1)\\
&= \frac{1}{9}\left\{ \frac{1}{1 + \frac{4}{9}u^2} \right\} \\
&= \frac{1}{6}\arctan\left(\frac{2}{3}u\right) + C
\end{aligned}
$$

- (1): $\left\{ \begin{array}{rcl} u &=& \tan(x) \\ du &=& \sec^2(x)dx \end{array} \right.$

Ejercicio 14

$$
\int \frac{1}{x^2(x^4+1)^{3/4}} dx
$$

--- Demostración ---

$$
\begin{aligned}
I &= -\left\{ \frac{t^3}{(t^4+1)^{3/4}} \right\} & (1)\\
&= -\frac{1}{4}\left\{ \frac{1}{u^{3/4}} \right\} & (2)\\
&= -u^{1/4} + C
\end{aligned}
$$

- (1): $\begin{cases} x &= \frac{1}{t} \\ dx &= -\frac{1}{t^2}dt \end{cases}$
- (2): $\begin{cases} u &= t^4+1 \\ du &= 4t^3dt \end{cases}$

Ejercicio 15

$$\int_0^{\pi} \cos x \cos 3x \cos 5x dx$$

$$--- \text{Demostración} ---$$

$$\begin{aligned} I &= \frac{1}{2}\{\cos(x)(\cos(8x)+\cos(2x))\}_0^{\pi} && (1) \\ &= \frac{1}{4}\{\cos(9x)+\cos(7x)+\cos(3x)+\cos(x)\}_0^{\pi} && (1) \\ &= \frac{1}{4}\left(\frac{\sin(9x)}{9}+\frac{\sin(7x)}{7}+\frac{\sin(3x)}{3}+\sin(x)\right)_0^{\pi} \\ &= 0 \end{aligned}$$

- (1): $2\cos(a)\cos(b) = \cos(a+b)+\cos(a-b)$

Nota:

1. Una forma mas facil de resolver la integral es ver que si f es una función impar, entonces:

$$\{f(x)\}_{-a}^{a} = 0$$

2. Resolver el límite-integral:

$$\lim_{n\to\infty}\left\{\prod_{i=1}^{n}\cos(nx)\right\}_0^{\pi}$$

Ejercicio 16

$$\int\left(\frac{1}{\log x}+\log(\log x)\right)dx$$

$$--- \text{Demostración} ---$$

$$\begin{aligned} I &= x\left(\frac{1}{\ln x}+\ln(\ln x)\right)-\left\{\frac{\ln(x)-1}{\ln^2(x)}\right\} && (1) \\ &= x\left(\frac{1}{\ln x}+\ln(\ln x)\right)-\frac{x}{\ln(x)}+C && (2) \\ &= x\ln(\ln x)+C \end{aligned}$$

- (1): $\left|\begin{array}{c|c} D & I \\ \frac{1}{\log x}+\log(\log x) & dx \\ \frac{1}{x}\left(\frac{\ln(x)-1}{\ln^2(x)}\right) & x \end{array}\right|$
- (2): $\left[\frac{f(x)}{\ln(x)}\right] = \frac{f'(x)\ln(x)-f(x)\frac{1}{x}}{\ln^2(x)}$

Ejercicio 17

$$\int \frac{1}{2+e^x}dx$$

$$\color{green}{---\text{ Demostración }---}$$

$$\begin{aligned}
I &= -\left\{\frac{e^t}{2e^t+1}\right\} && (1)\\
&= -\left\{\frac{1}{2u+1}\right\} && (2)\\
&= -\frac{\ln(2u+1)}{2}+C
\end{aligned}$$

- (1): $\begin{cases} x &= -t \\ dx &= -dt \end{cases}$
- (2): $\begin{cases} u &= e^t \\ du &= e^t dt \end{cases}$

Ejercicio 18

$$\int \sqrt{\frac{x}{1-x^3}}dx$$

$$\color{green}{---\text{ Demostración }---}$$

$$\begin{aligned}
I &= 2\left\{\frac{t^2}{\sqrt{1-t^6}}\right\} && (1)\\
&= \frac{2}{3}\left\{\frac{1}{\sqrt{1-u^2}}\right\} && (2)\\
&= \frac{2}{3}\arcsin(u)+C
\end{aligned}$$

- (1): $\begin{cases} x &= t^2 \\ dx &= 2tdt \end{cases}$
- (2): $\begin{cases} u &= t^3 \\ du &= 3t^2dt \end{cases}$

Ejercicio 19

$$\int \frac{4x}{1-x^4}dx$$

$$\color{green}{---\text{ Demostración }---}$$

$$\begin{aligned}
I &= 2\left\{\frac{1}{1-t^2}\right\} && (1)\\
&= \left\{\frac{1}{1-t}+\frac{1}{1+t}\right\}\\
&= \ln(1+t)-\ln(1-t)+C
\end{aligned}$$

- (1): $\begin{cases} t &= x^2 \\ dt &= 2xdx \end{cases}$

Ejercicio 20

$$\int x^x(1+\log x)dx$$

$$- - - \text{ Demostración } - - -$$

$$I = x^x + C \quad (1)$$

- (1): $[x^x] = [e^{x\ln(x)}] = x^x(1+\ln(x))$

Ejercicio 21

$$\int_0^6 \sqrt{6x - x^2}dx$$

$$- - - \text{ Demostración } - - -$$

Sea $y = \sqrt{6x - x^2}$, entonces:

$$(x-3)^2 + y^2 = 9 \ , \quad \text{para } y \geq 0$$

Así pues, la integral esta calculando el área de media circunferencia de radio 3 centrada en el punto $(3, 0)$.

$$I = \frac{9}{2}\pi$$

Ejercicio 22

$$\int \sin(101x)\sin^{99}(x)dx$$

$$- - - \text{ Demostración } - - -$$

$$\begin{aligned} I &= \{ \sin((\alpha+1)x) \ \sin^{\alpha-1}(x) \ \} \\ &= \frac{\sin^{\alpha}\sin(\alpha x)}{\alpha} + C \end{aligned}$$

- (1):

$$\begin{aligned} [\sin^{\alpha}(x)\sin(\alpha x)] &= \alpha\sin^{\alpha-1}(x)\cos(x)\sin(\alpha x) + \alpha\cos(\alpha)\sin^{\alpha}(x) \\ &= \alpha\sin^{\alpha-1}(x)(\ \cos(x)\sin(\alpha x) + \cos(\alpha)\sin(x)\ ) \\ &= \alpha\sin^{\alpha-1}(x)\sin((\alpha+1)x) \end{aligned}$$

Ejercicio 23

$$\int xe^{e^{x^2}+x^2}dx$$

$$- - - \text{ Demostración } - - -$$

$$\begin{aligned} I &= \left\{ xe^{e^{x^2}} e^{x^2} \right\} \\ &= \left\{ xe^{e^{x^2}} e^{x^2} \right\} \\ &= \frac{e^{e^{x^2}}}{2} + C \end{aligned}$$

- (1): $\left[e^{e^{x^2}}\right] = 2xe^{e^{x^2}}e^{x^2}$

Ejercicio 24

$$\int_0^1 \frac{x^3-3x^2+3x-1}{x^4+4x^3+6x^2+4x+1}dx$$

$$--- \text{Demostración} ---$$

$$\begin{aligned}
I &= \left\{ \frac{(x-1)^3}{(x+1)^4} \right\}_0^1 && (1)\\
&= \left\{ \frac{(t-2)^3}{t^4} \right\}_1^2 && (2)\\
&= \left\{ t^{-1} - 6t^{-2} + 12t^{-3} - 8t^{-4} \right\}_1^2 \\
&= \left(\ln|t| + \frac{6}{t} - \frac{6}{t^2} + \frac{8}{3}\frac{1}{t^3} \right)_1^2 \\
&= \ln(2) - \frac{5}{6}
\end{aligned}$$

- (1): Triángulo de Pascal
- (2): $\left\{ \begin{array}{rcl} t &=& x+1 \\ dt &=& dx \end{array} \right.$

Ejercicio 25

$$\int \sqrt{\frac{1-x}{1+x}}dx$$

$$--- \text{Demostración} ---$$

$$\begin{aligned}
I &= \left\{ \frac{1-x}{\sqrt{1-x^2}} \right\} \\
&= \left\{ \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \right\}
\end{aligned}$$

$$\begin{aligned}
I_1 &= \left\{ \frac{1}{\sqrt{1-x^2}} \right\} \\
&= \arcsin(x)
\end{aligned}$$

$$\begin{aligned}
I_2 &= \left\{ \frac{x}{\sqrt{1-x^2}} \right\} \\
&= \frac{1}{2}\left\{ \frac{1}{\sqrt{1-t}} \right\} \\
&= -\sqrt{1-t} \\
&= -\sqrt{1-x^2}
\end{aligned}$$

$$I = \arcsin(x) + \sqrt{1-x^2} + C$$

- (1): $\left\{ \begin{array}{rcl} t &=& x^2 \\ dt &=& 2x\,dx \end{array} \right.$