MIT integration bee - 2010

Notación y Consideraciones Generales

A lo largo del documento vamos a utilizar la siguiente notación, en algunos problemas más complejos utilizaremos la notación usual:

$$\{\ f(x)\ \}_a^b := \int_a^b f(x)\ dx \ \ , \ \ [\ f(x)\ ] := \frac{d}{dx} f(x)$$

Recursos obtenidos de: https://math.mit.edu/~yyao1/integrationbee.html

Herramientas que te pueden ser útiles:

- Calculadora gráfica: https://www.desmos.com/calculator?lang=es
- Calculadora de integrales: https://mathdf.com/es/
- Calculadora de integrales: https://www.wolframalpha.com/
- Motor de busqueda para fórmulas en latex https://approach0.xyz/search/
- Te ayuda con ideas: https://chat.deepseek.com

Ejercicio 1

$$\int_0^{\pi/2} \sin(x)\sin(2x)\sin(3x)dx$$

$$---\text{ Demostración }---$$

$$\begin{aligned}
I &= \{\ \sin(x)\sin(2x)\sin(3x)\ \}_0^{\pi/2} \\
&= \frac{1}{2}\{\ \sin(x)(\cos(x)-\cos(5x))\ \}_0^{\pi/2} && (1) \\
&= \frac{1}{2}\left(\{\ \sin(x)\cos(x)\ \}_0^{\pi/2} - \{\ \sin(x)\cos(5x)\ \}_0^{\pi/2}\right) \\
&= \frac{1}{2}\left(\left(\frac{\sin^2(x)}{2}\right)_0^{\pi/2} - \{\ \sin(x)\cos(5x)\ \}_0^{\pi/2}\right) \\
&= \frac{1}{2}\left(\frac{1}{2} - \{\ \sin(x)\cos(5x)\ \}_0^{\pi/2}\right) \\
&= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\{\ \sin(6x)-\sin(4x)\ \}_0^{\pi/2}\right) && (2) \\
&= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\left(-\frac{\cos(6x)}{6}+\frac{\cos(4x)}{4}\right)_0^{\pi/2}\right) \\
&= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\left(\frac{2}{6}\right)\right) \\
&= \frac{1}{6}
\end{aligned}$$

- (1): $\sin(x)\sin(y) = \frac{1}{2}(\cos(x-y)-\cos(x+y))$
- (2): $\sin(x)\cos(y) = \frac{1}{2}(\sin(x-y)+\sin(x+y))$ y $\sin(-x) = -\sin(x)$

Ejercicio 2

$$\int_0^{\pi/2} \sin^3(2x)\cos(x)dx$$

$$- - - \text{Demostración} \; - - -$$

$$\begin{aligned}
I &= \{ \sin^3(2x)\cos(x) \}_0^{\pi/2} \\
&= \{ \sin^3(2x)\cos(x) \}_0^{\pi/2} \\
&= \{ (2\sin(x)\cos(x))^3\cos(x) \}_0^{\pi/2} && (1) \\
&= 8\{ \sin^3(x)\cos^4(x) \}_0^{\pi/2} && (1) \\
&= -8\{ (1-t^2)t^4 \}_1^0 && (2) \\
&= 8\{ (1-t^2)t^4 \}_0^1 && (3) \\
&= 8\left(\frac{t^5}{5} - \frac{t^7}{7}. \right)_0^1 \\
&= \frac{16}{35}
\end{aligned}$$

- (1): $\sin(2x) = 2\sin(x)\cos(x)$
- (2): $\left\{ \begin{array}{lcl} t &=& \cos(x) \\ dt &=& -\sin(x)dx \end{array} \right.$ y $\cos^2(x) + \sin^2(x) = 1$
- (3): $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Ejercicio 3

$$\int (x+1)^2(x-1)^{1/3}dx$$

$$- - - \text{Demostración} \; - - -$$

$$\begin{aligned}
I &= \{ (t+2)^2 \, t^{1/3} \} \\
&= \{ (t^2+4t+4) \, t^{1/3} \} \\
&= \frac{t^{10/3}}{10/3} + 4\frac{t^{7/3}}{7/3} + 4\frac{t^{4/3}}{4/3} \\
&= 3t^{4/3}\left(\frac{t^2}{10} + \frac{4t}{7} + 1 \right) + C
\end{aligned}$$

- (1): $\left\{ \begin{array}{lcl} t &=& x-1 \\ dt &=& dx \end{array} \right.$

Ejercicio 4

$$\int x \log\left(1 + \frac{1}{x}\right) dx$$

$$- - - \text{Demostración} \; - - -$$

$$\begin{aligned}
I &= \frac{x^2\log(1+1/x)}{2} + \frac{1}{2}\left\{ \frac{x}{x+1} \right\} && (1) \\
&= \frac{x^2\log(1+1/x)}{2} + \frac{1}{2}\,(\, x - \ln(x+1) \,) + C
\end{aligned}$$

- (1): $\left| \begin{array}{c|c} D & I \\ \log(1+1/x) & x \\ \frac{x}{1+x}(-\frac{1}{x^2}) & \frac{x^2}{2} \end{array} \right|$

Ejercicio 5

$$\int_0^1 \sin^2(\log x)dx$$

— — — Demostración — ——

$$
\begin{aligned}
I_1 &= (\ x\ \sin^2(\log(x))\ )_0^1 - \{\ \sin(2\log(x))\ \}_0^1 & (1) \\
&= -\{\ \sin(2\log(x))\ \}_0^1 \\
\\
I_2 &= \{\ \sin(2\log(x))\ \}_0^1 & (2) \\
&= (\ x\ \sin(2\log(x))\ )_0^1 - 2\{\ \cos(2\log(x))\ \}_0^1 \\
&= -2\{\ \cos(2\log(x))\ \}_0^1 \\
&= -2(\ x\ \cos(2\log(x))\ )_0^1 - 4I_2 & (3) \\
&= -2 - 4I_2 \\
I_2 &= -\tfrac{2}{5} \\
\\
I_1 &= \tfrac{2}{5}
\end{aligned}
$$

- (1): $\left|\begin{array}{c|c} D & I \\ \sin^2(\log(x)) & 1 \\ \frac{\sin(2\log(x))}{x} & x \end{array}\right|$
- (2): $\left|\begin{array}{c|c} D & I \\ \sin(2\log(x)) & 1 \\ \frac{2\ \cos(2\log(x))}{x} & x \end{array}\right|$
- (3): $\left|\begin{array}{c|c} D & I \\ \cos(2\log(x)) & 1 \\ -\frac{2\ \sin(2\log(x))}{x} & x \end{array}\right|$

Ejercicio 6

$$\int \frac{1}{1+3e^x}dx$$

— — — Demostración — ——

$$
\begin{aligned}
I &= \left\{ \frac{1}{1+3t} \cdot \frac{1}{t} \right\} & (1) \\
&= \left\{ -\frac{3}{1+3t} + \frac{1}{t} \right\} \\
&= -\ln(1+3t) + \ln(t) + C \\
&= \ln\left(\frac{t}{1+3t}\right) + C
\end{aligned}
$$

- (1): $\left\{\begin{array}{ccc} t &=& e^x \\ dt &=& e^x dx \end{array}\right.$

Ejercicio 7

$$\int_{\pi/4}^{\pi/3} \frac{1}{\sin^3 x \cos^5 x}dx$$

— — — Demostración — ——

$$
\begin{aligned}
I &= \left\{ \frac{\sin^2(x)+\cos^2(x)}{\sin^3(x)\cos^5(x)} \right\}_{\pi/4}^{\pi/3} && (1 \\
&= \left\{ \frac{1}{\sin(x)\cos^5(x)} + \frac{1}{\sin^3(x)\cos^3(x)} \right\}_{\pi/4}^{\pi/3} && (1 \\
&= \left\{ \frac{\sin(x)}{\cos^5(x)} + \frac{1}{\sin(x)\cos^3(x)} + \frac{1}{\sin^3(x)\cos^3(x)} \right\}_{\pi/4}^{\pi/3} && (1 \\
&= \left\{ \tan(x)\sec^4(x) + 2\frac{1}{\sin(x)\cos^3(x)} + \frac{1}{\sin^3(x)\cos(x)} \right\}_{\pi/4}^{\pi/3} && (1 \\
&= \left\{ \tan(x)\sec^4(x) + 2\left(\frac{\sin(x)}{\cos^3(x)} + \frac{1}{\sin(x)\cos(x)}\right) + \frac{1}{\sin(x)\cos(x)} + \frac{\cos(x)}{\sin^3(x)} \right\}_{\pi/4}^{\pi/3} && (1 \\
&= \left\{ \tan(x)\sec^4(x) + 2\tan(x)\sec^2(x) + 3\frac{1}{\sin(x)\cos(x)} + \cot(x)\csc^2(x) \right\}_{\pi/4}^{\pi/3} \\
&= \{ \tan(x)\sec^4(x) + 2\tan(x)\sec^2(x) + 3\left(\tan(x)+\cot(x)\right) + \cot(x)\csc^2(x) \}_{\pi/4}^{\pi/3} && (1 \\
&= \left(\frac{\tan^2(x)}{2} + \frac{\tan^4(x)}{4}\right)_{\pi/4}^{\pi/3} + 2\left(\frac{\tan^2(x)}{2}\right)_{\pi/4}^{\pi/3} + 3\left(-\ln(\cos(x)) + \ln(\sin(x))\right)_{\pi/4}^{\pi/3} - \left(\frac{\cot^2(x)}{2}\right)_{\pi/4}^{\pi/3} \\
&= \tfrac{16}{3} + \tfrac{3}{2}\ln(3)
\end{aligned}
$$

- (1): $\cos^2(x)+\sin^2(x)=1$

Nota:

1. Tenemos la siguiente fórmula de reducción de potencias:

$$\cos^n(\theta) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \cos(\ (2k-n)\theta\ )$$

2. Estudiar la integral $I(n,m) = \left\{ \frac{1}{\sin^n(\theta)\cos^m(\theta)} \right\}$ mediante la siguiente recursión:

$$I(n,m) = I(n-2,m) + I(n,m-2) \ \ , \ \ \text{para } n,m \geq 2$$

Utilizando el metodo de series formales, es decir, estudiando la serie formal en dos variables:

$$\sum_{n,m\geq 0} x^n y^m I(n,m)$$

Ejercicio 8

$$\int_1^\infty \frac{1}{x\sqrt{x^4-1}} dx$$

$$---\ \text{Demostración}\ ---$$

$$
\begin{aligned}
I &= \frac{1}{2}\left\{ \frac{\tan\theta}{|\tan\theta|} \right\}_0^{\pi/2} && (1) \\
&= \frac{\pi}{4} && (2)
\end{aligned}
$$

- (1): $\begin{cases} x^2 &= \sec\theta \\ 2x\ dx &= \sec\theta\ \tan\theta\ dx \end{cases}$, $1+\tan^2\theta = \sec^2\theta$
- (2): $\tan\theta \geq 0$ para todo $\theta \in [0,\pi/2)$

Nota:

1. Al hacer sustituciones debemos de asegurarnos que en el intervalo de integración la sustitución sea una biyección.

Ejercicio 9

$$\int \frac{1}{x(x^5+1)}dx$$

$$--- \text{Demostración} ---$$

$$\begin{aligned} I &= -\left\{ \frac{t^4}{t^5+1} \right\} \\ &= -\frac{\ln(t^5+1)}{5} + C \end{aligned}$$

- (1): $\left\{ \begin{array}{rcl} x &=& \frac{1}{t} \\ dx &=& -\frac{1}{t^2}\,dt \end{array} \right.$

Ejercicio 10

$$\int_0^{\pi/2} \sqrt{\tan(x)}\,dx$$

$$--- \text{Demostración} ---$$

$$\begin{aligned} I &= \left\{ \sqrt{\cot(x)} \right\}_0^{\pi/2} && (1) \\ 2I &= \left\{ \sqrt{\tan x} + \sqrt{\cot x} \right\}_0^{\pi/2} \\ &= \left\{ \frac{\cos x + \sin x}{\sqrt{\sin(x)\cos(x)}} \right\}_0^{\pi/2} \\ &= \sqrt{2}\left\{ \frac{\cos x + \sin x}{\sqrt{1-(\sin(x)-\cos(x))^2}} \right\}_0^{\pi/2} && (2) \\ &= -\sqrt{2}\left\{ \frac{1}{\sqrt{1-t^2}} \right\}_{-1}^{1} && (3) \\ &= -\sqrt{2}\left(\arcsin(t) \right)_{-1}^{1} && (4) \\ &= \sqrt{2}\pi && (4) \\ I &= \frac{\pi}{\sqrt{2}} \end{aligned}$$

- (1): $\left\{ \begin{array}{rcl} x &=& \pi/2-\theta \\ dx &=& -dt \end{array} \right.$, $\sin(\pi/2-x) = \cos(x)$ y $\cos(\pi/2-x) = \sin(x)$
- (2): $(\sin(x)-\cos(x))^2 = 1 - 2\sin(x)\cos(x)$
- (3): $\left\{ \begin{array}{rcl} t &=& \sin(x)-\cos(x) \\ dt &=& (\cos(x)+\sin(x))\,dx \end{array} \right.$
- (4): $[\arcsin(t)] = \frac{1}{\sqrt{1-t^2}}$

Nota:

1. Tener cuidado al usar integrales indefinidas pues debemos de determinar sobre que intervalo queremos encontrar la antiderivada.

   Supongamos que $x \in (0, \pi/4)$, entonces si queremos de terminar una antiderivada de:

   $$I = \left\{ \sqrt{\tan(x)} \right\} = -\left\{ \sqrt{\cot(\theta)} \right\}$$

   Sin embargo, al hacer esta sustitución el intervalo de definición de $\theta \in (\pi/2, \pi/4)$ y tenemos que:

$$2I \neq \left\{ \sqrt{\tan(x)} - \sqrt{\cot(\theta)} \right\}$$

2. Bibliografía interesante:

   - https://www.quora.com/What-is-the-integral-of-sqrt-tan-x-1/answer/Lai-Johnny?ch=17&oid=229121919&share=bbb0ceb7&srid=CCOmo&target_type=answer
   - https://math.stackexchange.com/questions/409332/calculus-indefinite-integration-find-int-sqrt-cot-x-sqrt-tan-x-dx?noredirect=1

Ejercicio 11

$$\int_0^{\pi/4} \sqrt{\tan(x)}\, dx$$

$$---\ \text{Demostración}\ ---$$

$$\begin{aligned}
I &= \left\{ \sqrt{\frac{\cos(\theta)-\sin(\theta)}{\sin(\theta)+\cos(\theta)}} \right\}_0^{\pi/4} && (1)\\
&= \left\{ \frac{\cos(\theta)-\sin(\theta)}{\sqrt{\cos^2(\theta)-\sin^2(\theta)}} \right\}_0^{\pi/4}\\
&= \left\{ \frac{\cos(\theta)}{\sqrt{\cos^2(\theta)-\sin^2(\theta)}} - \frac{\sin(\theta)}{\sqrt{\cos^2(\theta)-\sin^2(\theta)}} \right\}_0^{\pi/4}\\
&= \left\{ \frac{\cos(\theta)}{\sqrt{1-2\sin^2(\theta)}} - \frac{\sin(\theta)}{\sqrt{2\cos^2(\theta)-1}} \right\}_0^{\pi/4}
\end{aligned}$$

$$\begin{aligned}
I_1 &= \left\{ \frac{\cos(\theta)}{\sqrt{1-2\sin^2(\theta)}} \right\}_0^{\pi/4}\\
&= \left\{ \frac{1}{\sqrt{1-2t^2}} \right\}_0^{\sqrt{2}/2} && (2)\\
&= \left(\frac{\arcsin(\sqrt{2}t)}{\sqrt{2}} \right)_0^{\sqrt{2}/2}\\
&= \frac{\pi}{2\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
I_2 &= \left\{ \frac{-\sin(\theta)}{\sqrt{2\cos^2(\theta)-1}} \right\}_0^{\pi/4}\\
&= \left\{ \frac{1}{\sqrt{2t^2-1}} \right\}_{\sqrt{2}/2}^{1} && (3)\\
&= \frac{1}{\sqrt{2}} \left\{ \frac{\sec(u)\tan(u)}{|\tan(u)|} \right\}_0^{\pi/4} && (4)\\
&= \frac{1}{\sqrt{2}} \{ \sec(u) \}_0^{\pi/4} && (5)\\
&= \frac{1}{\sqrt{2}} (\ln(|\tan(u)+\sec(u)|))_0^{\pi/4} && (6)\\
&= \frac{\ln(1+\sqrt{2})}{\sqrt{2}}
\end{aligned}$$

$$I = \frac{\pi}{2\sqrt{2}} + \frac{\ln(1+\sqrt{2})}{\sqrt{2}}$$

- (1): $\left\{\begin{array}{rcl} x & = & \pi/4-\theta \\ dx & = & -d\theta \end{array}\right.$, $\sin(\pi/4-\theta)=\frac{\sqrt{2}}{2}(\cos(\theta)-\sin(\theta))$ y $\cos(\pi/4-\theta)=\frac{\sqrt{2}}{2}(\cos(\theta)+\sin(\theta))$
- (2): $\left\{\begin{array}{rcl} t & = & \sin(\theta) \\ dt & = & \cos(\theta)\ d\theta \end{array}\right.$
- (3): $\left\{\begin{array}{rcl} t & = & \cos(\theta) \\ dt & = & -\sin(\theta)\ d\theta \end{array}\right.$
- (4): $\left\{\begin{array}{rcl} \sqrt{2}t & = & \sec(u) \\ \sqrt{2}\ dt & = & \sec(u)\tan(u)\ du \end{array}\right.$
- (5): $\tan(u)\geq 0$ para todo $u\in(0,\pi/2)$
- (6): $\sec(u)=\frac{\sec(u)(\tan(u)+\sec(u))}{\tan(u)+\sec(u)}$ y $[\tan(u)+\sec(u)]=\sec(u)(\tan(u)+\sec(u))$

Nota:

1. Para diferentes intervalos de integración tenemos diferentes métodos para integrar la misma función.

Ejercicio 12

$$\int_0^1 \frac{\log(1+x)}{1+x^2}dx$$

$$---\text{ Demostración }---$$

$$\begin{aligned} I &= \{\ln(1+\tan(\theta))\}_0^{\pi/4} && (1)\\ &= \left\{\ln\left(\frac{2}{1+\tan(u)}\right)\right\}_0^{\pi/4} && (2)\\ &= \frac{\pi}{4}\ln(2)-I \end{aligned}$$

$$I = \frac{\pi}{8}\ln(2)$$

- (1): $\left\{\begin{array}{rcl} x & = & \tan(\theta) \\ dx & = & \sec^2(\theta)d\theta \end{array}\right.$ y $1+\tan^2(\theta)=\sec^2(\theta)$
- (2): $\left\{\begin{array}{rcl} \theta & = & \pi/4-u \\ d\theta & = & -du \end{array}\right.$ y $\tan(\pi/4-u)=\frac{1-\tan(u)}{1+\tan(u)}$

Ejercicio 13

$$\int_{64}^{729} \frac{x^{1/2}}{x^{1/2}-x^{1/3}}dx$$

$$---\text{ Demostración }---$$

$$
\begin{aligned}
I &= 6\left\{\frac{t^6}{t-1}\right\}_2^3 && (1)\\
&= 6\left\{\frac{(u+1)^6}{u}\right\}_1^2 && (2)\\
&= 6\left(\ln(|u|)+\sum_{k=1}^{6}\binom{6}{k}\frac{1}{k}\cdot u^k\right)_1^2 && (3)\\
&= 6\left(\ln(2)+\sum_{k=1}^{6}\binom{6}{k}\frac{1}{k}\cdot(2^k-1)\right)\\
&= 6\ln(2)+\frac{10747}{10} && (4)
\end{aligned}
$$

- (1): $\begin{cases} t &= x^{1/6} \\ dt &= \frac{1}{6}t^{-5}dx \end{cases}$
- (2): $\begin{cases} u &= t-1 \\ du &= dt \end{cases}$
- (3): $(x+1)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$
- (4): Usar el triángulo de pascal para este tipo de cuentas.

Ejercicio 14

$$\int x^x(1+\ln x)dx$$

$$---\text{ Demostración }---$$

$$I = x^x + C \quad (1)$$

- (1): $[x^x] = [e^{x\ln(x)}] = x^x(1+\ln(x))$

Ejercicio 15

$$\int_0^1 x^{13/2}\sqrt{1+x^{5/2}}dx$$

$$---\text{ Demostración }---$$

$$
\begin{aligned}
I &= 2\left\{t^4(t^5)^2\sqrt{1+t^5}\right\}_0^1 && (1)\\
&= \frac{2}{5}\left\{(u-1)^2\sqrt{u}\right\}_1^2 && (2)\\
&= \frac{2}{5}\left(\frac{2}{7}u^{7/2}-\frac{4}{5}u^{5/2}+\frac{2}{3}u^{3/2}\right)_1^2\\
&= \frac{2}{5}\left(\frac{2}{7}u^{7/2}-\frac{4}{5}u^{5/2}+\frac{2}{3}u^{3/2}\right)_1^2\\
&\approx 0.1761
\end{aligned}
$$

- (1): $\begin{cases} x &= t^2 \\ dx &= 2t\,dt \end{cases}$
- (2): $\begin{cases} u &= t^5+1 \\ du &= 5t^4\,dt \end{cases}$

Ejercicio 16

$$\int_1^{\infty} \frac{1}{(x^2+1)^2} dx$$

$$\color{green}{---\text{ Demostración }---}$$

$$\begin{aligned} I &= \left\{ \cos^2(\theta) \right\}_{\pi/4}^{\pi/2} & (1) \\ &= \frac{1}{2} \left\{ \cos(2\theta) + 1 \right\}_{\pi/4}^{\pi/2} & (2) \\ &= \frac{1}{2} \left(\frac{\sin(2\theta)}{2} + x \right)_{\pi/4}^{\pi/2} \\ &= \frac{\pi}{8} - \frac{1}{4} \end{aligned}$$

- (1): $\left\{ \begin{array}{rcl} x &=& \tan(\theta) \\ dx &=& \sec^2(\theta)\, d\theta \end{array} \right.$
- (2): $\cos(2\theta) = 2\cos(\theta) - 1$

Ejercicio 17

$$\int_0^1 \frac{1}{x^4 - 13x^2 + 36} dx$$

$$\color{green}{---\text{ Demostración }---}$$

$$\begin{aligned} I &= -\frac{1}{5} \left\{ \frac{1}{x^2-4} - \frac{1}{x^2-9} \right\}_0^1 \\ &= -\frac{1}{5} \left\{ -\frac{1}{4} \left(\frac{1}{x+2} - \frac{1}{x-2} \right) + \frac{1}{6} \left(\frac{1}{x+3} - \frac{1}{x-3} \right) \right\}_0^1 \\ &= -\frac{1}{5} \left(-\frac{1}{4} \ln \left(\left| \frac{x+2}{x-2} \right| \right) + \frac{1}{6} \ln \left(\left| \frac{x+3}{x-3} \right| \right) \right)_0^1 \\ &= \frac{\ln(3)}{20} - \frac{\ln(2)}{30} \end{aligned}$$

Nota:

1. Hacer fracciones parciales con dos términos es más sencillo que hacer con todos los términos.
2. $\{1/x\} = \ln(|x|)$, importante poner el valor absoluto.

Ejercicio 18

$$\int \frac{\log(\log x)}{x} dx$$

$$\color{green}{---\text{ Demostración }---}$$

$$\begin{aligned} I &= \left\{ \log(t) \right\} & (1) \\ &= t\ln(t) - t + C & (2) \end{aligned}$$

- (1): $\left\{ \begin{array}{rcl} t &=& \ln(x) \\ dt &=& \frac{1}{x}\, dx \end{array} \right.$
- (2): integral importante $\{\ln(x)\} = x\ln(x) - x + C$, usar integración por partes.

Ejercicio 19

$$\int \frac{1 + \cot x}{1 - \cot x} dx$$

$$--- \text{Demostración} ---$$

$$\begin{aligned} I &= -\{\cot(\pi/4 - x)\} & (1)\\ &= \ln(\,|\sin(\pi/4-x)|\,) + C & (2) \end{aligned}$$

- (1): $\tan(\pi/4 - x) = \frac{\cos(x)-\sin(x)}{\cos(x)+\sin(x)} = \frac{1-\tan(x)}{1+\tan(x)} = \frac{\cot(x)-1}{\cot(x)+1}$
- (2): $\{\cot(x)\} = \ln(|\sin(x)|)$

Ejercicio 20

$$\int \frac{\cos x + x\sin x}{x(x+\cos x)}dx$$

$$--- \text{Demostración} ---$$

$$\begin{aligned} I &= \left\{ \frac{1 + x\tan(x)}{x(x\sec(x)+1)} \right\} \\ &= \left\{ \frac{\sec(x) + x\sec(x)\tan(x)}{x\sec(x)(x\sec(x)+1)} \right\} \\ &= \left\{ \frac{1}{t(t+1)} \right\} & (1)\\ &= \left\{ \frac{1}{t} - \frac{1}{t+1} \right\} \\ &= \ln|t| - \ln|t+1| + C \end{aligned}$$

- (1): $\left\{ \begin{array}{rcl} t &=& x\sec(x) \\ dt &=& (\sec(x) + x\sec(x)\tan(x))\,dx \end{array} \right.$

Ejercicio 21

$$\int_0^{\pi/2} \frac{1}{\sin x + \sec x}dx$$

$$--- \text{Demostración} ---$$

$$\begin{aligned} I &= \left\{ \frac{1}{\cos(\theta)+\csc(\theta)} \right\}_0^{\pi/2} & (1)\\ 2I &= \left\{ \frac{1}{\sin x + \sec x} + \frac{1}{\cos(\theta)+\csc(\theta)} \right\}_0^{\pi/2} \\ &= \left\{ \frac{\sin(x)+\cos(x)}{\sin(x)\cos(x)+1} \right\}_0^{\pi/2} \\ &= 2\left\{ \frac{\sin(x)+\cos(x)}{3-(\sin(x)-\cos(x))^2} \right\}_0^{\pi/2} & (2)\\ &= 2\left\{ \frac{1}{3-t^2} \right\}_{-1}^{1} & (3)\\ &= \frac{1}{\sqrt{3}}\left\{ \frac{1}{\sqrt{3}-t} + \frac{1}{\sqrt{3}+t} \right\}_{-1}^{1} \\ &= \frac{1}{\sqrt{3}}\left(\ln|\sqrt{3}+t| - \ln|\sqrt{3}-t| \right)_{-1}^{1} \\ &= \frac{1}{\sqrt{3}}\ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right) \end{aligned}$$

- (1): $\left\{ \begin{array}{rcl} x &=& \pi/2-\theta \\ dx &=& -d\theta \end{array} \right.$, $\sin(\pi/2-\theta) = \cos(\theta)$ y $\cos(\pi/2-\theta) = \sin(\theta)$
- (2): $(\sin(x) - \cos(x))^2 = 1 - 2\sin(x)\cos(x)$

- (3): $\begin{cases} t &= \sin(x) - \cos(x) \\ dt &= (\sin(x) + \cos(x))\, dx \end{cases}$

Ejercicio 22

$$\int_0^{\infty} \frac{1}{\sqrt{1+e^x+e^{2x}}} dx$$

$$--- \text{Demostración} ---$$

$$\begin{aligned} I &= \left\{ \frac{e^t}{\sqrt{1+e^t+e^{2t}}} \right\}_{-\infty}^{0} && (1) \\ &= \left\{ \frac{1}{\sqrt{1+u+u^2}} \right\}_{0}^{1} && (2) \\ &= \left\{ \frac{1}{\sqrt{(u+1/2)^2+3/4}} \right\}_{0}^{1} \\ &= \{ \sec(\theta) \}_{\pi/6}^{\pi/3} && (3) \\ &= (\ln|\tan(\theta)+\sec(\theta)|)_{\pi/6}^{\pi/3} \\ &= \left(\ln(2+\sqrt{3}) - \frac{1}{2}\ln(3) \right)_{\pi/6}^{\pi/3} \end{aligned}$$

- (1): $\begin{cases} x &= -t \\ dx &= -dt \end{cases}$
- (2): $\begin{cases} u &= e^t \\ du &= e^t\, dt \end{cases}$
- (3): $\begin{cases} u+1/2 &= \frac{\sqrt{3}}{2}\tan(\theta) \\ du &= \frac{\sqrt{3}}{2}\sec^2(\theta) dt \end{cases}$

Ejercicio 23

$$\int_0^1 x^3 e^{x^2} dx$$

$$--- \text{Demostración} ---$$

$$\begin{aligned} I &= \left\{ x x^2 e^{x^2} \right\}_0^1 \\ &= \frac{1}{2}\{ u e^u \}_0^1 \\ &= \frac{1}{2}(u e^u - e^u)_0^1 && (2) \\ &= \frac{1}{2} \end{aligned}$$

- (1): $\begin{cases} u &= x^2 \\ du &= 2x\, dx \end{cases}$
- (2): $\left| \begin{array}{c|c} D & I \\ u & e^u \\ 1 & e^u \\ 0 & e^u \end{array} \right|$

Ejercicio 24

$$\int_0^1 \sqrt{1+x\sqrt{1+x\sqrt{1+x\sqrt{\cdots}}}}\,dx$$

$$\color{red}{---\ \text{Demostración}\ ---}$$

Considere la siguiente sucesión:

$$a_1 = \sqrt{1+x} \quad , \quad a_{n+1} = \sqrt{1+xa_n} \quad \text{para } x \in [0,1]$$

Veamos que $\{a_n\}_{n\geq 1}$ converge para todo $x \in [0,1]$.

(Monótona Creciente) Usando induccion matemática tenemos que:

$$\begin{array}{rcl} a_1 & = & \sqrt{1+x} \\ & \geq & 1 \\ \\ a_1 - 1 & \geq & 0 \\ x(a_1-1) & \geq & 0 \\ xa_1 & \geq & x \\ 1+xa_1 & \geq & 1+x \\ a_2^2 & \geq & a_1^2 \\ a_2 & \geq & a_1 \end{array}$$

Suponiendo que para $n \in \mathbb{N}$ se tiene que $a_n \geq a_{n-1}$, entonces:

$$\begin{array}{rcl} a_n & \geq & a_{n-1} \\ xa_n & \geq & xa_{n-1} \\ 1+xa_n & \geq & 1+xa_{n-1} \\ a_{n+1}^2 & \geq & a_n^2 \\ a_{n+1} & \geq & a_n \end{array}$$

Así pues, la sucesión $\{a_n\}_{n\geq 1}$ es monótona creciente.

(Acotada Superiormente) veamos que la sucesión $\{a_n\}_{n\geq 1}$ es acotada superiormente por 2. Razonemos por inducción matemática.

Claramente tenemos que $a_1 \leq \sqrt{2} \leq 2$ pues $x \in [0,1]$, ahora suponiendo que para $n \in \mathbb{N}$ se tiene que $a_n \leq 2$, entonces:

$$\begin{array}{rcl} a_n & \leq & 2 \\ xa_n & \leq & 2 \\ 1+xa_n & \leq & 3 \\ a_{n+1}^2 & \leq & 3 \\ a_{n+1} & \leq & \sqrt{3} \\ & \leq & 2 \end{array}$$

Así pues, la sucesión $\{a_n\}_{n\geq 1}$ es acotada superiormente por 2.

Considere la función $f(x) = \lim_{n\to\infty} a_n(x)$ para todo $x \in [0,1]$, esta función esta bien definida pues ya vimos que la sucesión $\{a_n\}_{n\geq 1}$ converge para todo $x \in [0,1]$.

$$\begin{array}{rcl} a^2 & = & \lim\limits_{n\to\infty} a_n^2 \\ & = & \lim\limits_{n\to\infty} 1+xa_{n-1} \\ & = & 1+xa \\ a & = & \dfrac{x+\sqrt{x^2+4}}{2} \end{array}$$

Así pues, tenemos que:

$$f(x) = \lim_{n\to\infty} a_n(x) = \frac{x+\sqrt{x^2+4}}{2}$$

$$
\begin{aligned}
I &= \left\{ \frac{x+\sqrt{x^2+4}}{2} \right\}_0^1 \\
&= \frac{1}{2}\left\{ x+\sqrt{x^2+4} \right\}_0^1
\end{aligned}
$$

$$
\begin{aligned}
I_1 &= \left\{ \sqrt{x^2+4} \right\}_0^1 \\
&= 4\left\{ \sec^3(\theta) \right\}_0^{\arctan(1/2)} && (1) \\
&= 2\left(\sec(\theta)\tan(\theta) + \ln|\tan(\theta)+\sec(\theta)| \right)_0^{\arctan(1/2)} && (2) \\
&= 2\left(\frac{x\sqrt{x^2+4}}{4} + \ln\left| \frac{x+\sqrt{x^2+4}}{2} \right| \right)_0^1 && (3) \\
&= \frac{\sqrt{5}}{2} + 2\ln\left(\frac{1+\sqrt{5}}{2} \right)
\end{aligned}
$$

$$
I = \frac{1}{4} + \frac{\sqrt{5}}{4} + \ln\left(\frac{1+\sqrt{5}}{2} \right)
$$

- (1): $\left\{ \begin{aligned} x &= 2\tan(\theta) \\ dx &= 2\sec^2(\theta)\, d\theta \end{aligned} \right.$
- (2):

$$
\begin{aligned}
I(n) &= \{\sec^n(\theta)\} \\
&= \sec^{n-2}(\theta)\tan(\theta) - (n-2)\{\sec^{n-2}(\theta)\tan^2(\theta)\} + C \\
&= \sec^{n-2}(\theta)\tan(\theta) - (n-2)\{\sec^n(\theta) - \sec^{n-2}(\theta)\} + C \\
I(n)(n-1) &= \sec^{n-2}(\theta)\tan(\theta) + (n-2)I(n-2) + C \\
I(n) &= \tfrac{\sec^{n-2}(\theta)\tan(\theta)}{n-1} + \tfrac{n-2}{n-1}I(n-2) + C
\end{aligned}
$$

- (3): Resolver para θ la siguinete ecuación $\tan(\theta) = \frac{1}{2}$

Nota:

1. Toda sucesión monótona (creciente / decreciente) acotada (superiormente / inferiormente) es convergente.

Ejercicio 25

$$
\int \left(\frac{1}{\log x} - \frac{1}{(\log x)^2} \right) dx
$$

$$
--- \text{ Demostración } ---
$$

$$
\begin{aligned}
I &= \left\{ \frac{\ln(x)-1}{(\ln x)^2} \right\} \\
&= \frac{x}{\ln(x)} + C && (1)
\end{aligned}
$$

- (1): $\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x)-g'(x)f(x)}{g^2(x)}$

Ejercicio 26

$$
\int_1^2 \sqrt{(x-1)(2-x)}dx
$$

$$
--- \text{ Demostración } ---
$$

$$
\begin{aligned}
I &= \left\{ \sqrt{1/4 - (x-3/2)^2} \right\}_1^2 \\
&= \frac{1}{4} \left\{ |\cos(\theta)| \cos(\theta) \right\}_{3\pi/2}^{\pi/2} && (1) \\
&= -\frac{1}{4} \left\{ \cos^2(\theta) \right\}_{3\pi/2}^{\pi/2} \\
&= -\frac{1}{8} \left\{ \cos(2\theta) + 1 \right\}_{3\pi/2}^{\pi/2} && (2) \\
&= -\frac{1}{8} \left(\frac{\sin(2\theta)}{2} + \theta \right)_{3\pi/2}^{\pi/2} \\
&= \frac{\pi}{8}
\end{aligned}
$$

- (1): $\left\{ \begin{array}{rcl} x - 3/2 &=& 1/2 \sin(\theta) \\ dx &=& 1/2 \cos(\theta)\, d\theta \end{array} \right.$
- (2): $\cos(2\theta) = 2\cos^2(\theta) - 1$