

MIT integration bee - 2010

Notación y Consideraciones Generales

A lo largo del documento vamos a utilizar la siguiente notación, en algunos problemas más complejos utilizaremos la notación usual:

$$\{ f(x) \}_a^b := \int_a^b f(x) dx , \quad [f(x)] := \frac{d}{dx} f(x)$$

Recursos obtenidos de: <https://math.mit.edu/~yyao1/integrationbee.html>

Herramientas que te pueden ser útiles:

- Calculadora gráfica: <https://www.desmos.com/calculator?lang=es>
- Calculadora de integrales: <https://mathdf.com/es/>
- Calculadora de integrales: <https://www.wolframalpha.com/>
- Motor de búsqueda para fórmulas en latex <https://approach0.xyz/search/>
- Te ayuda con ideas: <https://chat.deepseek.com>

Ejercicio 1

$$\int_0^{\pi/2} \sin(x) \sin(2x) \sin(3x) dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \{ \sin(x) \sin(2x) \sin(3x) \}_0^{\pi/2} \\ &= \frac{1}{2} \{ \sin(x)(\cos(x) - \cos(5x)) \}_0^{\pi/2} \tag{1} \\ &= \frac{1}{2} \left(\{ \sin(x) \cos(x) \}_0^{\pi/2} - \{ \sin(x) \cos(5x) \}_0^{\pi/2} \right) \\ &= \frac{1}{2} \left(\left(\frac{\sin^2(x)}{2} \right)_0^{\pi/2} - \{ \sin(x) \cos(5x) \}_0^{\pi/2} \right) \\ &= \frac{1}{2} \left(\frac{1}{2} - \{ \sin(x) \cos(5x) \}_0^{\pi/2} \right) \\ &= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{2} \{ \sin(6x) - \sin(4x) \}_0^{\pi/2} \right) \tag{2} \\ &= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{2} \left(-\frac{\cos(6x)}{6} + \frac{\cos(4x)}{4} \right)_0^{\pi/2} \right) \\ &= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{2} \left(\frac{2}{6} \right) \right) \\ &= \frac{1}{6} \end{aligned}$$

- (1): $\sin(x) \sin(y) = \frac{1}{2}(\cos(x - y) - \cos(x + y))$
- (2): $\sin(x) \cos(y) = \frac{1}{2}(\sin(x - y) + \sin(x + y))$ y $\sin(-x) = -\sin(x)$

Ejercicio 2

$$\int_0^{\pi/2} \sin^3(2x) \cos(x) dx$$

— — — Demostración — — —

$$\begin{aligned}
 I &= \{ \sin^3(2x) \cos(x) \}_0^{\pi/2} \\
 &= \{ \sin^3(2x) \cos(x) \}_0^{\pi/2} \\
 &= \{ (2 \sin(x) \cos(x))^3 \cos(x) \}_0^{\pi/2} \quad (1) \\
 &= 8 \{ \sin^3(x) \cos^4(x) \}_0^{\pi/2} \quad (1) \\
 &= -8 \{ (1-t^2)t^4 \}_1^0 \quad (2) \\
 &= 8 \{ (1-t^2)t^4 \}_0^1 \quad (3) \\
 &= 8 \left(\frac{t^5}{5} - \frac{t^7}{7} \right)_0^1 \\
 &= \frac{16}{35}
 \end{aligned}$$

- (1): $\sin(2x) = 2 \sin(x) \cos(x)$
- (2): $\begin{cases} t &= \cos(x) \\ dt &= -\sin(x)dx \end{cases} \Rightarrow \cos^2(x) + \sin^2(x) = 1$
- (3): $\int_a^b f(x)dx = - \int_b^a f(x)dx$

Ejercicio 3

$$\int (x+1)^2(x-1)^{1/3} dx$$

— — — Demostración — — —

$$\begin{aligned}
 I &= \{ (t+2)^2 t^{1/3} \} \\
 &= \{ (t^2 + 4t + 4) t^{1/3} \} \\
 &= \frac{t^{10/3}}{10/3} + 4 \frac{t^{7/3}}{7/3} + 4 \frac{t^{4/3}}{4/3} \\
 &= 3t^{4/3} \left(\frac{t^2}{10} + \frac{4t}{7} + 1 \right) + C
 \end{aligned}$$

- (1): $\begin{cases} t &= x-1 \\ dt &= dx \end{cases}$

Ejercicio 4

$$\int x \log \left(1 + \frac{1}{x} \right) dx$$

— — — Demostración — — —

$$\begin{aligned}
 I &= \frac{x^2 \log(1+1/x)}{2} + \frac{1}{2} \left\{ \frac{x}{x+1} \right\} \quad (1) \\
 &= \frac{x^2 \log(1+1/x)}{2} + \frac{1}{2} (x - \ln(x+1)) + C
 \end{aligned}$$

- (1): $\begin{vmatrix} D & I \\ \log(1+1/x) & x \\ \frac{x}{1+x} (-\frac{1}{x^2}) & \frac{x^2}{2} \end{vmatrix}$

Ejercicio 5

$$\int_0^1 \sin^2(\log x) dx$$

— — — Demostración — — —

$$\begin{aligned} I_1 &= (\ x \ \sin^2(\log(x)) \)_0^1 - \{ \sin(2\log(x)) \ }_0^1 & (1) \\ &= -\{ \sin(2\log(x)) \ }_0^1 \end{aligned}$$

$$\begin{aligned} I_2 &= \{ \sin(2\log(x)) \ }_0^1 & (2) \\ &= (\ x \ \sin(2\log(x)) \)_0^1 - 2 \{ \cos(2\log(x)) \ }_0^1 \\ &= -2 \{ \cos(2\log(x)) \ }_0^1 \\ &= -2(\ x \ \cos(2\log(x)) \)_0^1 - 4I_2 & (3) \\ &= -2 - 4I_2 \\ I_2 &= -\frac{2}{5} \end{aligned}$$

$$I_1 = \frac{2}{5}$$

$$\bullet \ (1): \left| \begin{array}{c|c} D & I \\ \sin^2(\log(x)) & 1 \\ \hline \frac{\sin(2\log(x))}{x} & x \end{array} \right|$$

$$\bullet \ (2): \left| \begin{array}{c|c} D & I \\ \sin(2\log(x)) & 1 \\ \hline \frac{2\cos(2\log(x))}{x} & x \end{array} \right|$$

$$\bullet \ (3): \left| \begin{array}{c|c} D & I \\ \cos(2\log(x)) & 1 \\ \hline -\frac{2\sin(2\log(x))}{x} & x \end{array} \right|$$

Ejercicio 6

$$\int \frac{1}{1+3e^x} dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \left\{ \frac{1}{1+3t} \cdot \frac{1}{t} \right\} & (1) \\ &= \left\{ -\frac{3}{1+3t} + \frac{1}{t} \right\} \\ &= -\ln(1+3t) + \ln(t) + C \\ &= \ln\left(\frac{t}{1+3t}\right) + C \end{aligned}$$

$$\bullet \ (1): \left\{ \begin{array}{lcl} t & = & e^x \\ dt & = & e^x dx \end{array} \right.$$

Ejercicio 7

$$\int_{\pi/4}^{\pi/3} \frac{1}{\sin^3 x \cos^5 x} dx$$

— — — Demostración — — —

$$\begin{aligned}
I &= \left\{ \frac{\sin^2(x) + \cos^2(x)}{\sin^3(x) \cos^5(x)} \right\}_{\pi/4}^{\pi/3} \\
&= \left\{ \frac{1}{\sin(x) \cos^5(x)} + \frac{1}{\sin^3(x) \cos^3(x)} \right\}_{\pi/4}^{\pi/3} \\
&= \left\{ \frac{\sin(x)}{\cos^5(x)} + \frac{1}{\sin(x) \cos^3(x)} + \frac{1}{\sin^3(x) \cos^3(x)} \right\}_{\pi/4}^{\pi/3} \\
&= \left\{ \tan(x) \sec^4(x) + 2 \frac{1}{\sin(x) \cos^3(x)} + \frac{1}{\sin^3(x) \cos(x)} \right\}_{\pi/4}^{\pi/3} \\
&= \left\{ \tan(x) \sec^4(x) + 2 \left(\frac{\sin(x)}{\cos^3(x)} + \frac{1}{\sin(x) \cos(x)} \right) + \frac{1}{\sin(x) \cos(x)} + \frac{\cos(x)}{\sin^3(x)} \right\}_{\pi/4}^{\pi/3} \\
&= \left\{ \tan(x) \sec^4(x) + 2 \tan(x) \sec^2(x) + 3 \frac{1}{\sin(x) \cos(x)} + \cot(x) \csc^2(x) \right\}_{\pi/4}^{\pi/3} \\
&= \left\{ \tan(x) \sec^4(x) + 2 \tan(x) \sec^2(x) + 3 (\tan(x) + \cot(x)) + \cot(x) \csc^2(x) \right\}_{\pi/4}^{\pi/3} \\
&= \left(\frac{\tan^2(x)}{2} + \frac{\tan^4(x)}{4} \right)_{\pi/4}^{\pi/3} + 2 \left(\frac{\tan^2(x)}{2} \right)_{\pi/4}^{\pi/3} + 3 (-\ln(\cos(x)) + \ln(\sin(x)))_{\pi/4}^{\pi/3} - \left(\frac{\cot^2(x)}{2} \right)_{\pi/4}^{\pi/3} \\
&= \frac{16}{3} + \frac{3}{2} \ln(3)
\end{aligned} \tag{1}$$

- (1): $\cos^2(x) + \sin^2(x) = 1$

Nota:

1. Tenemos la siguiente fórmula de reducción de potencias:

$$\cos^n(\theta) = 2^{-n} \sum_{k=0}^n \binom{n}{k} \cos((2k-n)\theta)$$

2. Estudiar la integral $I(n, m) = \left\{ \frac{1}{\sin^n(\theta) \cos^m(\theta)} \right\}$ mediante la siguiente recursión:

$$I(n, m) = I(n-2, m) + I(n, m-2), \text{ para } n, m \geq 2$$

Utilizando el metodo de series formales, es decir, estudiando la serie formal en dos variables:

$$\sum_{n,m \geq 0} x^n y^m I(n, m)$$

Ejercicio 8

$$\int_1^\infty \frac{1}{x\sqrt{x^4-1}} dx$$

— — — Demostración — — —

$$\begin{aligned}
I &= \frac{1}{2} \left\{ \frac{\tan \theta}{|\tan \theta|} \right\}_0^{\pi/2} \quad (1) \\
&= \frac{\pi}{4} \quad (2)
\end{aligned}$$

- (1): $\left\{ \frac{x^2}{2x \, dx} \right. = \frac{\sec \theta}{\sec \theta \, \tan \theta \, dx}, 1 + \tan^2 \theta = \sec^2 \theta \right.$

- (2): $\tan \theta \geq 0$ para todo $\theta \in [0, \pi/2)$

Nota:

1. Al hacer sustituciones debemos de asegurarnos que en el intervalo de integración la sustitución sea una biyección.

Ejercicio 9

$$\int \frac{1}{x(x^5+1)} dx$$

— — — Demostración — — —

$$\begin{aligned} I &= -\left\{ \frac{t^4}{t^5+1} \right\} \\ &= -\frac{\ln(t^5+1)}{5} + C \end{aligned}$$

$$\bullet (1): \left\{ \begin{array}{l} x = \frac{1}{t} \\ dx = -\frac{1}{t^2} dt \end{array} \right.$$

Ejercicio 10

$$\int_0^{\pi/2} \sqrt{\tan(x)} dx$$

— — — Demostración — — —

$$I = \left\{ \sqrt{\cot(x)} \right\}_0^{\pi/2} \quad (1)$$

$$\begin{aligned} 2I &= \left\{ \sqrt{\tan x} + \sqrt{\cot x} \right\}_0^{\pi/2} \\ &= \left\{ \frac{\cos x + \sin x}{\sqrt{\sin(x) \cos(x)}} \right\}_0^{\pi/2} \\ &= \sqrt{2} \left\{ \frac{\cos x + \sin x}{\sqrt{1 - (\sin(x) - \cos(x))^2}} \right\}_0^{\pi/2} \quad (2) \end{aligned}$$

$$= -\sqrt{2} \left\{ \frac{1}{\sqrt{1-t^2}} \right\}_{-1}^1 \quad (3)$$

$$= -\sqrt{2} (\arcsin(t))_{-1}^1 \quad (4)$$

$$= \sqrt{2}\pi \quad (4)$$

$$I = \frac{\pi}{\sqrt{2}}$$

$$\bullet (1): \left\{ \begin{array}{l} x = \pi/2 - \theta \\ dx = -dt \end{array} \right., \sin(\pi/2 - x) = \cos(x) \text{ y } \cos(\pi/2 - x) = \sin(x)$$

$$\bullet (2): (\sin(x) - \cos(x))^2 = 1 - 2 \sin(x) \cos(x)$$

$$\bullet (3): \left\{ \begin{array}{l} t = \sin(x) - \cos(x) \\ dt = (\cos(x) + \sin(x)) dx \end{array} \right.$$

$$\bullet (4): [\arcsin(t)] = \frac{1}{\sqrt{1-t^2}}$$

Nota:

1. Tener cuidado al usar integrales indefinidas pues debemos de determinar sobre que intervalo queremos encontrar la antiderivada.

Supongamos que $x \in (0, \pi/4)$, entonces si queremos de terminar una antiderivada de:

$$I = \left\{ \sqrt{\tan(x)} \right\} = -\left\{ \sqrt{\cot(\theta)} \right\}$$

Sin embargo, al hacer esta sustitución el intervalo de definición de $\theta \in (\pi/2, \pi/4)$ y tenemos que:

$$2I \neq \left\{ \sqrt{\tan(x)} - \sqrt{\cot(\theta)} \right\}$$

2. Bibliografía interesante:

- https://www.quora.com/What-is-the-integral-of-sqrt-tan-x-1/answer/Lai-Johnny?ch=17&oid=229121919&share=bbb0ceb7&srid=CCOMo&target_type=answer
- <https://math.stackexchange.com/questions/409332/calculus-indefinite-integration-find-int-sqrt-cot-x-sqrt-tan-x-dx?noredirect=1>

Ejercicio 11

$$\int_0^{\pi/4} \sqrt{\tan(x)} dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \left\{ \sqrt{\frac{\cos(\theta) - \sin(\theta)}{\sin(\theta) + \cos(\theta)}} \right\}_0^{\pi/4} \\ &= \left\{ \frac{\cos(\theta) - \sin(\theta)}{\sqrt{\cos^2(\theta) - \sin^2(\theta)}} \right\}_0^{\pi/4} \\ &= \left\{ \frac{\cos(\theta)}{\sqrt{\cos^2(\theta) - \sin^2(\theta)}} - \frac{\sin(\theta)}{\sqrt{\cos^2(\theta) - \sin^2(\theta)}} \right\}_0^{\pi/4} \\ &= \left\{ \frac{\cos(\theta)}{\sqrt{1 - 2\sin^2(\theta)}} - \frac{\sin(\theta)}{\sqrt{2\cos^2(\theta) - 1}} \right\}_0^{\pi/4} \end{aligned} \tag{1}$$

$$\begin{aligned} I_1 &= \left\{ \frac{\cos(\theta)}{\sqrt{1 - 2\sin^2(\theta)}} \right\}_0^{\pi/4} \\ &= \left\{ \frac{1}{\sqrt{1 - 2t^2}} \right\}_0^{\sqrt{2}/2} \\ &= \left(\frac{\arcsin(\sqrt{2}t)}{\sqrt{2}} \right)_0^{\sqrt{2}/2} \\ &= \frac{\pi}{2\sqrt{2}} \end{aligned} \tag{2}$$

$$\begin{aligned} I_2 &= \left\{ \frac{-\sin(\theta)}{\sqrt{2\cos^2(\theta) - 1}} \right\}_0^{\pi/4} \\ &= \left\{ \frac{1}{\sqrt{2t^2 - 1}} \right\}_0^1 \end{aligned} \tag{3}$$

$$= \frac{1}{\sqrt{2}} \left\{ \frac{\sec(u)\tan(u)}{|\tan(u)|} \right\}_0^{\pi/4} \tag{4}$$

$$= \frac{1}{\sqrt{2}} \{ \sec(u) \}_{0}^{\pi/4} \tag{5}$$

$$= \frac{1}{\sqrt{2}} (\ln(|\tan(u) + \sec(u)|))_0^{\pi/4} \tag{6}$$

$$= \frac{\ln(1 + \sqrt{2})}{\sqrt{2}}$$

$$I = \frac{\pi}{2\sqrt{2}} + \frac{\ln(1 + \sqrt{2})}{\sqrt{2}}$$

- (1): $\begin{cases} x &= \pi/4 - \theta \\ dx &= -d\theta \end{cases}$, $\sin(\pi/4 - \theta) = \frac{\sqrt{2}}{2}(\cos(\theta) - \sin(\theta))$ y $\cos(\pi/4 - \theta) = \frac{\sqrt{2}}{2}(\cos(\theta) + \sin(\theta))$
- (2): $\begin{cases} t &= \sin(\theta) \\ dt &= \cos(\theta) d\theta \end{cases}$
- (3): $\begin{cases} t &= \cos(\theta) \\ dt &= -\sin(\theta) d\theta \end{cases}$
- (4): $\begin{cases} \sqrt{2}t &= \sec(u) \\ \sqrt{2} dt &= \sec(u) \tan(u) du \end{cases}$
- (5): $\tan(u) \geq 0$ para todo $u \in (0, \pi/2)$
- (6): $\sec(u) = \frac{\sec(u)(\tan(u)+\sec(u))}{\tan(u)+\sec(u)}$ y $[\tan(u) + \sec(u)] = \sec(u)(\tan(u) + \sec(u))$

Nota:

1. Para diferentes intervalos de integración tenemos diferentes métodos para integrar la misma función.

Ejercicio 12

$$\int_0^1 \frac{\log(1+x)}{1+x^2} dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \{ \ln(1 + \tan(\theta)) \}_{0}^{\pi/4} && (1) \\ &= \left\{ \ln \left(\frac{2}{1 + \tan(u)} \right) \right\}_{0}^{\pi/4} && (2) \\ &= \frac{\pi}{4} \ln(2) - I \end{aligned}$$

$$I = \frac{\pi}{8} \ln(2)$$

- (1): $\begin{cases} x &= \tan(\theta) \\ dx &= \sec^2(\theta)d\theta \end{cases}$ y $1 + \tan^2(\theta) = \sec^2(\theta)$
- (2): $\begin{cases} \theta &= \pi/4 - u \\ d\theta &= -du \end{cases}$ y $\tan(\pi/4 - u) = \frac{1 - \tan(u)}{1 + \tan(u)}$

Ejercicio 13

$$\int_{64}^{729} \frac{x^{1/2}}{x^{1/2} - x^{1/3}} dx$$

— — — Demostración — — —

$$I = 6 \left\{ \frac{t^6}{t-1} \right\}_2^3 \quad (1)$$

$$= 6 \left\{ \frac{(u+1)^6}{u} \right\}_1^2 \quad (2)$$

$$= 6 \left(\ln(|u|) + \sum_{k=1}^6 \binom{6}{k} \frac{1}{k} \cdot u^k \right)_1^2 \quad (3)$$

$$= 6 \left(\ln(2) + \sum_{k=1}^6 \binom{6}{k} \frac{1}{k} \cdot (2^k - 1) \right) \quad (4)$$

- (1): $\begin{cases} t &= x^{1/6} \\ dt &= \frac{1}{6}t^{-5}dx \end{cases}$

- (2): $\begin{cases} u &= t-1 \\ du &= dt \end{cases}$

- (3): $(x+1)^n = \sum_{k=0}^n \binom{n}{k} x^k$

- (4): Usar el triángulo de pascal para este tipo de cuentas.

Ejercicio 14

$$\int x^x (1 + \ln x) dx$$

— — — Demostración — — —

$$I = x^x + C \quad (1)$$

- (1): $[x^x] = [e^{x \ln(x)}] = x^x (1 + \ln(x))$

Ejercicio 15

$$\int_0^1 x^{13/2} \sqrt{1+x^{5/2}} dx$$

— — — Demostración — — —

$$I = 2 \left\{ t^4 (t^5)^2 \sqrt{1+t^5} \right\}_0^1 \quad (1)$$

$$= \frac{2}{5} \left\{ (u-1)^2 \sqrt{u} \right\}_1^2 \quad (2)$$

$$= \frac{2}{5} \left(\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} \right)_1^2$$

$$= \frac{2}{5} \left(\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} \right)_1$$

$$\approx 0.1761$$

- (1): $\begin{cases} x &= t^2 \\ dx &= 2t dt \end{cases}$

- (2): $\begin{cases} u &= t^5 + 1 \\ du &= 5t^4 dt \end{cases}$

Ejercicio 16

$$\int_1^\infty \frac{1}{(x^2+1)^2} dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \left\{ \cos^2(\theta) \right\}_{\pi/4}^{\pi/2} & (1) \\ &= \frac{1}{2} \left\{ \cos(2\theta) + 1 \right\}_{\pi/4}^{\pi/2} & (2) \\ &= \frac{1}{2} \left(\frac{\sin(2\theta)}{2} + x \right)_{\pi/4}^{\pi/2} \\ &= \frac{\pi}{8} - \frac{1}{4} \end{aligned}$$

- (1): $\left\{ \begin{array}{l} x = \tan(\theta) \\ dx = \sec^2(\theta) d\theta \end{array} \right.$

- (2): $\cos(2\theta) = 2\cos(\theta) - 1$

Ejercicio 17

$$\int_0^1 \frac{1}{x^4 - 13x^2 + 36} dx$$

— — — Demostración — — —

$$\begin{aligned} I &= -\frac{1}{5} \left\{ \frac{1}{x^2 - 4} - \frac{1}{x^2 - 9} \right\}_0^1 \\ &= -\frac{1}{5} \left\{ -\frac{1}{4} \left(\frac{1}{x+2} - \frac{1}{x-2} \right) + \frac{1}{6} \left(\frac{1}{x+3} - \frac{1}{x-3} \right) \right\}_0^1 \\ &= -\frac{1}{5} \left(-\frac{1}{4} \ln \left(\left| \frac{x+2}{x-2} \right| \right) + \frac{1}{6} \ln \left(\left| \frac{x+3}{x-3} \right| \right) \right)_0^1 \\ &= \frac{\ln(3)}{20} - \frac{\ln(2)}{30} \end{aligned}$$

Nota:

1. Hacer fracciones parciales con dos términos es más sencillo que hacer con todos los términos.
2. $\{1/x\} = \ln(|x|)$, importante poner el valor absoluto.

Ejercicio 18

$$\int \frac{\log(\log x)}{x} dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \{ \log(t) \} & (1) \\ &= t \ln(t) - t + C & (2) \end{aligned}$$

- (1): $\left\{ \begin{array}{l} t = \ln(x) \\ dt = \frac{1}{x} dx \end{array} \right.$

- (2): integral importante $\{\ln(x)\} = x \ln(x) - x + C$, usar integración por partes.

Ejercicio 19

$$\int \frac{1 + \cot x}{1 - \cot x} dx$$

— — — Demostración — — —

$$I = -\{\cot(\pi/4 - x)\} \quad (1)$$

$$= \ln(|\sin(\pi/4 - x)|) + C \quad (2)$$

- (1): $\tan(\pi/4 - x) = \frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)} = \frac{1 - \tan(x)}{1 + \tan(x)} = \frac{\cot(x) - 1}{\cot(x) + 1}$

- (2): $\{\cot(x)\} = \ln(|\sin(x)|)$

Ejercicio 20

$$\int \frac{\cos x + x \sin x}{x(x + \cos x)} dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \left\{ \frac{1 + x \tan(x)}{x(x \sec(x) + 1)} \right\} \\ &= \left\{ \frac{\sec(x) + x \sec(x) \tan(x)}{x \sec(x)(x \sec(x) + 1)} \right\} \\ &= \left\{ \frac{1}{t(t+1)} \right\} \\ &= \left\{ \frac{1}{t} - \frac{1}{t+1} \right\} \\ &= \ln|t| - \ln|t+1| + C \end{aligned} \quad (1)$$

- (1): $\begin{cases} t &= x \sec(x) \\ \frac{dt}{dx} &= (\sec(x) + x \sec(x) \tan(x)) \end{cases}$

Ejercicio 21

$$\int_0^{\pi/2} \frac{1}{\sin x + \sec x} dx$$

— — — Demostración — — —

$$I = \left\{ \frac{1}{\cos(\theta) + \csc(\theta)} \right\}_0^{\pi/2} \quad (1)$$

$$\begin{aligned} 2I &= \left\{ \frac{1}{\sin x + \sec x} + \frac{1}{\cos(\theta) + \csc(\theta)} \right\}_0^{\pi/2} \\ &= \left\{ \frac{\sin(x) + \cos(x)}{\sin(x) \cos(x) + 1} \right\}_0^{\pi/2} \\ &= 2 \left\{ \frac{\sin(x) + \cos(x)}{3 - (\sin(x) - \cos(x))^2} \right\}_0^{\pi/2} \end{aligned} \quad (2)$$

$$= 2 \left\{ \frac{1}{3 - t^2} \right\}_{-1}^1 \quad (3)$$

$$= \frac{1}{\sqrt{3}} \left\{ \frac{1}{\sqrt{3} - t} + \frac{1}{\sqrt{3} + t} \right\}_{-1}^1$$

$$= \frac{1}{\sqrt{3}} \left(\ln|\sqrt{3} + t| - \ln|\sqrt{3} - t| \right)_{-1}^1$$

$$= \frac{1}{\sqrt{3}} \ln \left(\frac{2 + \sqrt{3}}{2 - \sqrt{3}} \right)$$

- (1): $\begin{cases} x &= \pi/2 - \theta \\ \frac{dx}{d\theta} &= -1 \end{cases}, \sin(\pi/2 - \theta) = \cos(\theta) \text{ y } \cos(\pi/2 - \theta) = \sin(\theta)$

- (2): $(\sin(x) - \cos(x))^2 = 1 - 2 \sin(x) \cos(x)$

- (3):
$$\begin{cases} t &= \sin(x) - \cos(x) \\ dt &= (\sin(x) + \cos(x)) dx \end{cases}$$

Ejercicio 22

$$\int_0^\infty \frac{1}{\sqrt{1+e^x+e^{2x}}} dx$$

— — — Demostración — — —

$$I = \left\{ \frac{e^t}{\sqrt{1+e^t+e^{2t}}} \right\}_{-\infty}^0 \quad (1)$$

$$= \left\{ \frac{1}{\sqrt{1+u+u^2}} \right\}_0^1 \quad (2)$$

$$= \left\{ \frac{1}{\sqrt{(u+1/2)^2 + 3/4}} \right\}_0^1$$

$$= \{ \sec(\theta) \}_{\pi/6}^{\pi/3} \quad (3)$$

$$= (\ln |\tan(\theta) + \sec(\theta)|)_{\pi/6}^{\pi/3}$$

$$= \left(\ln(2 + \sqrt{3}) - \frac{1}{2} \ln(3) \right)_{\pi/6}^{\pi/3}$$

- (1):
$$\begin{cases} x &= -t \\ dx &= -dt \end{cases}$$

- (2):
$$\begin{cases} u &= e^t \\ du &= e^t dt \end{cases}$$

- (3):
$$\begin{cases} u + 1/2 &= \frac{\sqrt{3}}{2} \tan(\theta) \\ du &= \frac{\sqrt{3}}{2} \sec^2(\theta) dt \end{cases}$$

Ejercicio 23

$$\int_0^1 x^3 e^{x^2} dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \left\{ x x^2 e^{x^2} \right\}_0^1 \\ &= \frac{1}{2} \{ u e^u \}_0^1 \\ &= \frac{1}{2} (u e^u - e^u)_0^1 \quad (2) \\ &= \frac{1}{2} \end{aligned}$$

- (1):
$$\begin{cases} u &= x^2 \\ du &= 2x dx \end{cases}$$

- (2):
$$\begin{array}{c|cc} D & I \\ u & e^u \\ 1 & e^u \\ 0 & e^u \end{array}$$

Ejercicio 24

$$\int_0^1 \sqrt{1 + x\sqrt{1 + x\sqrt{1 + x\sqrt{\dots}}}} dx$$

— — — Demostración — — —

Considere la siguiente sucesión:

$$a_1 = \sqrt{1+x} , \quad a_{n+1} = \sqrt{1+xa_n} \quad \text{para } x \in [0, 1]$$

Veamos que $\{a_n\}_{n \geq 1}$ converge para todo $x \in [0, 1]$.

(Monótona Creciente) Usando inducción matemática tenemos que:

$$\begin{aligned} a_1 &= \sqrt{1+x} \\ &\geq 1 \\ \\ a_1 - 1 &\geq 0 \\ x(a_1 - 1) &\geq 0 \\ xa_1 &\geq x \\ 1 + xa_1 &\geq 1 + x \\ a_2^2 &\geq a_1^2 \\ a_2 &\geq a_1 \end{aligned}$$

Suponiendo que para $n \in \mathbb{N}$ se tiene que $a_n \geq a_{n-1}$, entonces:

$$\begin{aligned} a_n &\geq a_{n-1} \\ xa_n &\geq xa_{n-1} \\ 1 + xa_n &\geq 1 + xa_{n-1} \\ a_{n+1}^2 &\geq a_n^2 \\ a_{n+1} &\geq a_n \end{aligned}$$

Así pues, la sucesión $\{a_n\}_{n \geq 1}$ es monótona creciente.

(Acotada Superiormente) veamos que la sucesión $\{a_n\}_{n \geq 1}$ es acotada superiormente por 2. Razonemos por inducción matemática.

Claramente tenemos que $a_1 \leq \sqrt{2} \leq 2$ pues $x \in [0, 1]$, ahora suponiendo que para $n \in \mathbb{N}$ se tiene que $a_n \leq 2$, entonces:

$$\begin{aligned} a_n &\leq 2 \\ xa_n &\leq 2 \\ 1 + xa_n &\leq 3 \\ a_{n+1}^2 &\leq 3 \\ a_{n+1} &\leq \sqrt{3} \\ &\leq 2 \end{aligned}$$

Así pues, la sucesión $\{a_n\}_{n \geq 1}$ es acotada superiormente por 2.

Considere la función $f(x) = \lim_{n \rightarrow \infty} a_n(x)$ para todo $x \in [0, 1]$, esta función está bien definida pues ya vimos que la sucesión $\{a_n\}_{n \geq 1}$ converge para todo $x \in [0, 1]$.

$$\begin{aligned} a^2 &= \lim_{n \rightarrow \infty} a_n^2 \\ &= \lim_{n \rightarrow \infty} 1 + xa_{n-1} \\ &= 1 + xa \\ a &= \frac{x + \sqrt{x^2 + 4}}{2} \end{aligned}$$

Así pues, tenemos que:

$$f(x) = \lim_{n \rightarrow \infty} a_n(x) = \frac{x + \sqrt{x^2 + 4}}{2}$$

$$I = \left\{ \frac{x + \sqrt{x^2 + 4}}{2} \right\}_0^1$$

$$= \frac{1}{2} \left\{ x + \sqrt{x^2 + 4} \right\}_0^1$$

$$I_1 = \left\{ \sqrt{x^2 + 4} \right\}_0^1$$

$$= 4 \left\{ \sec^3(\theta) \right\}_0^{\arctan(1/2)} \quad (1)$$

$$= 2 (\sec(\theta) \tan(\theta) + \ln |\tan(\theta) + \sec(\theta)|)_0^{\arctan(1/2)} \quad (2)$$

$$= 2 \left(\frac{x\sqrt{x^2 + 4}}{4} + \ln \left| \frac{x + \sqrt{x^2 + 4}}{2} \right| \right)_0^1 \quad (3)$$

$$= \frac{\sqrt{5}}{2} + 2 \ln \left(\frac{1 + \sqrt{5}}{2} \right)$$

$$I = \frac{1}{4} + \frac{\sqrt{5}}{4} + \ln \left(\frac{1 + \sqrt{5}}{2} \right)$$

- (1): $\begin{cases} x &= 2 \tan(\theta) \\ dx &= 2 \sec^2(\theta) d\theta \end{cases}$

- (2):

$$\begin{aligned} I(n) &= \{\sec^n(\theta)\} \\ &= \sec^{n-2}(\theta) \tan(\theta) - (n-2) \{\sec^{n-2}(\theta) \tan^2(\theta)\} + C \\ &= \sec^{n-2}(\theta) \tan(\theta) - (n-2) \{\sec^n(\theta) - \sec^{n-2}(\theta)\} + C \\ I(n)(n-1) &= \sec^{n-2}(\theta) \tan(\theta) + (n-2)I(n-2) + C \\ I(n) &= \frac{\sec^{n-2}(\theta) \tan(\theta)}{n-1} + \frac{n-2}{n-1} I(n-2) + C \end{aligned}$$

- (3): Resolver para θ la siguiente ecuación $\tan(\theta) = \frac{1}{2}$

Nota:

1. Toda sucesión monótona (creciente / decreciente) acotada (superiormente / inferiormente) es convergente.

Ejercicio 25

$$\int \left(\frac{1}{\log x} - \frac{1}{(\log x)^2} \right) dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \left\{ \frac{\ln(x) - 1}{(\ln x)^2} \right\} \\ &= \frac{x}{\ln(x)} + C \quad (1) \end{aligned}$$

- (1): $\left[\frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$

Ejercicio 26

$$\int_1^2 \sqrt{(x-1)(2-x)} dx$$

— — — Demostración — — —

$$\begin{aligned}
I &= \left\{ \sqrt{1/4 - (x - 3/2)^2} \right\}_1^2 \\
&= \frac{1}{4} \left\{ |\cos(\theta)| \cos(\theta) \right\}_{3\pi/2}^{\pi/2} \quad (1) \\
&= -\frac{1}{4} \left\{ \cos^2(\theta) \right\}_{3\pi/2}^{\pi/2} \\
&= -\frac{1}{8} \left\{ \cos(2\theta) + 1 \right\}_{3\pi/2}^{\pi/2} \quad (2) \\
&= -\frac{1}{8} \left(\frac{\sin(2\theta)}{2} + \theta \right)_{3\pi/2}^{\pi/2} \\
&= \frac{\pi}{8}
\end{aligned}$$

- (1): $\begin{cases} x - 3/2 &= 1/2 \sin(\theta) \\ dx &= 1/2 \cos(\theta) d\theta \end{cases}$

- (2): $\cos(2\theta) = 2 \cos^2(\theta) - 1$