

MIT integration bee - 2011

Notación y Consideraciones Generales

A lo largo del documento vamos a utilizar la siguiente notación, en algunos problemas más complejos utilizaremos la notación usual:

$$\{ f(x) \}_a^b := \int_a^b f(x) dx, \quad [f(x)] := \frac{d}{dx} f(x)$$

Recursos obtenidos de: <https://math.mit.edu/~yyao1/integrationbee.html>

Herramientas que te pueden ser útiles:

- Calculadora gráfica: <https://www.desmos.com/calculator?lang=es>
- Calculadora de integrales: <https://mathdf.com/es/>
- Calculadora de integrales: <https://www.wolframalpha.com/>
- Motor de búsqueda para fórmulas en latex <https://approach0.xyz/search/>
- Te ayuda con ideas: <https://chat.deepseek.com>

Ejercicio 1

$$\int \frac{x^6 - 1}{x^4 + x^3 - x - 1} dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \{ x^2 - x + 1 \} \\ &= \frac{x^3}{3} - \frac{x^2}{2} + x + C \end{aligned}$$

Nota:

1. Si tenemos una función racional $\frac{p(x)}{q(x)}$ donde $\deg p(x) \geq \deg q(x)$, entonces por el algoritmo de la división para polinomios tenemos que $p(x) = s(x)q(x) + r(x)$ donde $\deg r(x) < \deg q(x)$.

Ejercicio 2

$$\int (2 \ln x + (\ln x)^2) dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \{ 2te^t + t^2e^t \} \quad (1) \\ &= \{ 2te^t + t^2e^t \} \quad (1) \\ &= t^2e^t + C \quad (2) \end{aligned}$$

$$\bullet (1): \begin{cases} x &= e^t \\ dx &= e^t dt \end{cases}$$

$$\bullet (2): [e^t f(t)] = e^t(f(t) + f'(t))$$

Ejercicio 3

$$\int \frac{2x}{\sqrt{1-x^4}} dx$$

--- Demostración ---

$$\begin{aligned} I &= \left\{ \frac{1}{\sqrt{1-t^2}} \right\} \quad (1) \\ &= \arcsin(t) + C \end{aligned}$$

$$\bullet (1): \begin{cases} t &= x^2 \\ dt &= 2x dx \end{cases}$$

Ejercicio 4

$$\int \frac{x^2 + 1}{x + 1} dx$$

--- Demostración ---

$$\begin{aligned} I &= \left\{ x - 1 + \frac{2}{x + 1} \right\} \\ &= \frac{x^2}{2} - x + 2 \ln |x + 1| + C \end{aligned}$$

Ejercicio 5

$$\int \frac{\sin(x)^3 + \sin(x)^2 - 2 \sin(x) - 2}{\sin(x)^2 + 2 \sin(x) + 1} dx$$

--- Demostración ---

$$\begin{aligned} I &= \left\{ \sin(x) - 1 - \frac{\sin(x) + 1}{\sin^2(x) + 2 \sin(x) + 1} \right\} \quad (1) \\ &= \left\{ \sin(x) - 1 - \frac{1}{\sin(x) + 1} \right\} \end{aligned}$$

$$\begin{aligned} I_1 &= \{ \sin(x) - 1 \} \\ &= -\cos(x) - x \end{aligned}$$

$$\begin{aligned} I_2 &= \left\{ \frac{1}{\sin(x) + 1} \right\} \\ &= \left\{ \frac{\sec(x)(\tan(x) + \sec(x))}{(\tan(x) + \sec(x))^2} \right\} \\ &= \left\{ \frac{1}{t^2} \right\} \quad (2) \\ &= -\frac{1}{t} \end{aligned}$$

$$I = \frac{1}{\tan(x) + \sec(x)} - \cos(x) - x + C$$

• (1): Hacer una división de polinomios con la variable $t = \sin(x)$

$$\bullet (2): \begin{cases} t &= \tan(x) + \sec(x) \\ dt &= \sec(x)(\tan(x) + \sec(x)) dx \end{cases}$$

Ejercicio 6

$$\int \sinh(x)^{-2} dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \{ \operatorname{csch}^2(x) \} \\ &= -\operatorname{coth}(x) + C \end{aligned}$$

Nota:

1. Propiedades básicas de las funciones trigonométricas hiperbólicas:

$$\left\{ \begin{array}{ll} \cosh^2(x) - \sinh^2(x) &= 1 \\ 2 \sinh^2(x) &= \cosh(2x) - 1 \\ 2 \cosh^2(x) &= \cosh(2x) + 1 \\ \sinh^2(x) + \cosh^2(x) &= \cosh(2x) \\ 2 \sinh(x) \cosh(x) &= \sinh(2x) \\ (\sinh(x) + \cosh(x))^2 &= \cosh(2x) + \sinh(2x) \end{array} \right.$$

2. Derivadas de las funciones trigonométricas hiperbólicas:

$$\left\{ \begin{array}{ll} [\sinh(x)] &= \cosh(x) \\ [\cosh(x)] &= \sinh(x) \\ [\tanh(x)] &= \operatorname{sech}^2(x) \\ [\coth(x)] &= -\operatorname{csch}^2(x) \\ [\operatorname{sech}(x)] &= -\operatorname{sech}(x)\tanh(x) \\ [\operatorname{csch}(x)] &= -\operatorname{csch}(x)\coth(x) \end{array} \right.$$

Ejercicio 7

$$\int \sec^4 x \tan^2 x dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \{ \tan^2(x)(1 + \tan^2(x)) \sec^2(x) \} \\ &= \{ u^2(1 + u^2) \} \\ &= \frac{u^3}{3} + \frac{u^5}{5} + C \end{aligned} \quad (1)$$

$$\bullet (1): \left\{ \begin{array}{ll} u &= \tan^2(x) \\ du &= \sec^2(x) dx \end{array} \right.$$

Ejercicio 8

$$\int \sqrt{\csc x - \sin x} dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \left\{ \frac{|\cos(x)|}{\sqrt{\sin(x)}} \right\} \\ &= \pm \left\{ \frac{\cos(x)}{\sqrt{\sin(x)}} \right\} \\ &= \pm 2\sqrt{\sin(x)} + C \end{aligned} \quad (2)$$

- (1): $\begin{cases} u &= \tan^2(x) \\ du &= \sec^2(x)dx \end{cases}$

- (2): $[\sqrt{f(x)}] = \frac{1}{2} \frac{f'(x)}{\sqrt{f(x)}}$

Ejercicio 9

$$\int \cos^6 x dx$$

— — — Demostración — — —

$$I = \frac{1}{8} \{ (\cos(2x) + 1)^3 \} \quad (1)$$

$$= \frac{1}{8} \{ \cos^3(2x) + 3 \cos^2(2x) + 3 \cos(2x) + 1 \}$$

$$= \frac{1}{8} \{ -\cos(2x) \sin^2(2x) + 3 \cos^2(2x) + 4 \cos(2x) + 1 \} \quad (2)$$

$$= \frac{1}{8} \left(-\frac{\sin^3(2x)}{6} + \frac{3}{2} \left(\frac{\sin(4x)}{4} + x \right) + 2 \sin(2x) + x \right) + C \quad (1) \text{ y } (3)$$

- (1): $\cos(2x) = 2 \cos^2(x) - 1$

- (2): $\sin^2(x) + \cos^2(x) = 1$

- (3): $[f^n(x)] = n f^{n-1}(x) f'(x)$

Ejercicio 10

$$\int \frac{1}{1 + 2x^2 + x^4} dx$$

— — — Demostración — — —

$$I = \left\{ \frac{1}{(x^2 + 1)^2} \right\}$$

$$= \left\{ \cos^2(\theta) \right\} \quad (1)$$

$$= \frac{1}{2} \{ \cos(2\theta) + 1 \} \quad (2)$$

$$= \frac{1}{2} \left(\frac{\sin(2\theta)}{2} + \theta \right)$$

$$= \frac{1}{2} \left(\frac{x}{x^2 + 1} + \arctan(x) \right) + C$$

- (1): $\begin{cases} x &= \tan(\theta) \\ dx &= \sec^2(\theta) d\theta \end{cases}$

- (2): $\cos(2x) = 2 \cos^2(x) - 1$

Ejercicio 11

$$\int \cos(\log x) dx$$

— — — Demostración — — —

$$I = x \cos(\ln x) + \{ \sin(\ln x) \} \quad (1)$$

$$= x \cos(\ln x) + x \sin(\ln x) - I \quad (2)$$

$$I = \frac{x}{2} (\cos(\ln x) + \sin(\ln x)) + C$$

$$\bullet (1): \left| \begin{array}{c} D \\ \cos(\ln x) \\ -\frac{\sin(\ln x)}{x} \end{array} \right| \left| \begin{array}{c} I \\ dx \\ x \end{array} \right|$$

$$\bullet (2): \left| \begin{array}{c} D \\ \sin(\ln x) \\ \frac{\cos(\ln x)}{x} \end{array} \right| \left| \begin{array}{c} I \\ dx \\ x \end{array} \right|$$

Ejercicio 12

$$\int \frac{1}{\cos x} dx$$

--- Demostración ---

$$\begin{aligned} I &= \{ \sec(x) \} \\ &= \ln |\tan(x) + \sec(x)| + C \quad (1) \end{aligned}$$

$$\bullet (1): [\tan(x) + \sec(x)] = \sec(x)(\tan(x) + \sec(x))$$

Ejercicio 13

$$\int \frac{1}{9 \cos^2 x + 4 \sin^2 x} dx$$

--- Demostración ---

$$\begin{aligned} I &= \left\{ \frac{\sec^2(x)}{9 + 4 \tan^2(x)} \right\} \\ &= \left\{ \frac{1}{9 + 4u^2} \right\} \quad (1) \\ &= \frac{1}{9} \left\{ \frac{1}{1 + \frac{4}{9}u^2} \right\} \\ &= \frac{1}{6} \arctan \left(\frac{2}{3}u \right) + C \end{aligned}$$

$$\bullet (1): \begin{cases} u = \tan(x) \\ du = \sec^2(x) dx \end{cases}$$

Ejercicio 14

$$\int \frac{1}{x^2(x^4 + 1)^{3/4}} dx$$

--- Demostración ---

$$\begin{aligned} I &= - \left\{ \frac{t^3}{(t^4 + 1)^{3/4}} \right\} \quad (1) \\ &= - \frac{1}{4} \left\{ \frac{1}{u^{3/4}} \right\} \quad (2) \\ &= -u^{1/4} + C \end{aligned}$$

- (1): $\begin{cases} x &= \frac{1}{t} \\ dx &= -\frac{1}{t^2} dt \end{cases}$
- (2): $\begin{cases} u &= t^4 + 1 \\ du &= 4t^3 dt \end{cases}$

Ejercicio 15

$$\int_0^\pi \cos x \cos 3x \cos 5x dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \frac{1}{2} \{ \cos(x)(\cos(8x) + \cos(2x)) \}_0^\pi & (1) \\ &= \frac{1}{4} \{ \cos(9x) + \cos(7x) + \cos(3x) + \cos(x) \}_0^\pi & (1) \\ &= \frac{1}{4} \left(\frac{\sin(9x)}{9} + \frac{\sin(7x)}{7} + \frac{\sin(3x)}{3} + \sin(x) \right)_0^\pi \\ &= 0 \end{aligned}$$

- (1): $2 \cos(a) \cos(b) = \cos(a+b) + \cos(a-b)$

Nota:

1. Una forma mas facil de resolver la integral es ver que si f es una función impar, entonces:

$$\{f(x)\}_{-a}^a = 0$$

2. Resolver el límite-integral:

$$\lim_{n \rightarrow \infty} \left\{ \prod_{i=1}^n \cos(nx) \right\}_0^\pi$$

Ejercicio 16

$$\int \left(\frac{1}{\log x} + \log(\log x) \right) dx$$

— — — Demostración — — —

$$\begin{aligned} I &= x \left(\frac{1}{\ln x} + \ln(\ln x) \right) - \left\{ \frac{\ln(x) - 1}{\ln^2(x)} \right\} & (1) \\ &= x \left(\frac{1}{\ln x} + \ln(\ln x) \right) - \frac{x}{\ln(x)} + C & (2) \\ &= x \ln(\ln x) + C \end{aligned}$$

$$\bullet (1): \left| \begin{array}{c} D \\ \frac{1}{\log x} + \log(\log x) \\ \frac{1}{x} \left(\frac{\ln(x)-1}{\ln^2(x)} \right) \end{array} \right| \left| \begin{array}{c} I \\ dx \\ x \end{array} \right|$$

$$\bullet (2): \left[\frac{f(x)}{\ln(x)} \right] = \frac{f'(x) \ln(x) - f(x) \frac{1}{x}}{\ln^2(x)}$$

Ejercicio 17

$$\int \frac{1}{2 + e^x} dx$$

— — — Demostración — — —

$$I = - \left\{ \frac{e^t}{2e^t + 1} \right\} \quad (1)$$

$$= - \left\{ \frac{1}{2u + 1} \right\} \quad (2)$$

$$= - \frac{\ln(2u + 1)}{2} + C$$

$$\bullet (1): \begin{cases} x &= -t \\ dx &= -dt \end{cases}$$

$$\bullet (2): \begin{cases} u &= e^t \\ du &= e^t dt \end{cases}$$

Ejercicio 18

$$\int \sqrt{\frac{x}{1 - x^3}} dx$$

— — — Demostración — — —

$$I = 2 \left\{ \frac{t^2}{\sqrt{1 - t^6}} \right\} \quad (1)$$

$$= \frac{2}{3} \left\{ \frac{1}{\sqrt{1 - u^2}} \right\} \quad (2)$$

$$= \frac{2}{3} \arcsin(u) + C$$

$$\bullet (1): \begin{cases} x &= t^2 \\ dx &= 2t dt \end{cases}$$

$$\bullet (2): \begin{cases} u &= t^3 \\ du &= 3t^2 dt \end{cases}$$

Ejercicio 19

$$\int \frac{4x}{1 - x^4} dx$$

— — — Demostración — — —

$$I = 2 \left\{ \frac{1}{1 - t^2} \right\} \quad (1)$$

$$= \left\{ \frac{1}{1 - t} + \frac{1}{1 + t} \right\}$$

$$= \ln(1 + t) - \ln(1 - t) + C$$

$$\bullet (1): \begin{cases} t &= x^2 \\ dt &= 2x dx \end{cases}$$

Ejercicio 20

$$\int x^x (1 + \log x) dx$$

— — — Demostración — — —

$$I = x^x + C \quad (1)$$

- (1): $[x^x] = [e^{x \ln(x)}] = x^x(1 + \ln(x))$

Ejercicio 21

$$\int_0^6 \sqrt{6x - x^2} dx$$

— — — Demostración — — —

Sea $y = \sqrt{6x - x^2}$, entonces:

$$(x - 3)^2 + y^2 = 9, \quad \text{para } y \geq 0$$

Así pues, la integral esta calculando el área de media circunferencia de radio 3 centrada en el punto (3, 0).

$$I = \frac{9}{2}\pi$$

Ejercicio 22

$$\int \sin(101x) \sin^{99}(x) dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \left\{ \sin((\alpha + 1)x) \sin^{\alpha-1}(x) \right\} \\ &= \frac{\sin^{\alpha} \sin(\alpha x)}{\alpha} + C \end{aligned}$$

- (1):

$$\begin{aligned} [\sin^{\alpha}(x) \sin(\alpha x)] &= \alpha \sin^{\alpha-1}(x) \cos(x) \sin(\alpha x) + \alpha \cos(\alpha) \sin^{\alpha}(x) \\ &= \alpha \sin^{\alpha-1}(x) (\cos(x) \sin(\alpha x) + \cos(\alpha) \sin(x)) \\ &= \alpha \sin^{\alpha-1}(x) \sin((\alpha + 1)x) \end{aligned}$$

Ejercicio 23

$$\int x e^{e^{x^2} + x^2} dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \left\{ x e^{e^{x^2}} e^{x^2} \right\} \\ &= \left\{ x e^{e^{x^2}} e^{x^2} \right\} \\ &= \frac{e^{e^{x^2}}}{2} + C \end{aligned}$$

- (1): $[e^{e^{x^2}}] = 2x e^{e^{x^2}} e^{x^2}$

Ejercicio 24

$$\int_0^1 \frac{x^3 - 3x^2 + 3x - 1}{x^4 + 4x^3 + 6x^2 + 4x + 1} dx$$

— — — Demostración — — —

$$I = \left\{ \frac{(x-1)^3}{(x+1)^4} \right\}_0^1 \quad (1)$$

$$= \left\{ \frac{(t-2)^3}{t^4} \right\}_1^2 \quad (2)$$

$$= \left\{ t^{-1} - 6t^{-2} + 12t^{-3} - 8t^{-4} \right\}_1^2$$

$$= \left(\ln|t| + \frac{6}{t} - \frac{6}{t^2} + \frac{8}{3} \frac{1}{t^3} \right)_1^2$$

$$= \ln(2) - \frac{5}{6}$$

- (1): Triángulo de Pascal

- (2): $\begin{cases} t = x+1 \\ dt = dx \end{cases}$

Ejercicio 25

$$\int \sqrt{\frac{1-x}{1+x}} dx$$

— — — Demostración — — —

$$\begin{aligned} I &= \left\{ \frac{1-x}{\sqrt{1-x^2}} \right\} \\ &= \left\{ \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \right\} \end{aligned}$$

$$\begin{aligned} I_1 &= \left\{ \frac{1}{\sqrt{1-x^2}} \right\} \\ &= \arcsin(x) \end{aligned}$$

$$\begin{aligned} I_2 &= \left\{ \frac{x}{\sqrt{1-x^2}} \right\} \\ &= \frac{1}{2} \left\{ \frac{1}{\sqrt{1-t}} \right\} \\ &= -\sqrt{1-t} \\ &= -\sqrt{1-x^2} \end{aligned}$$

$$I = \arcsin(x) + \sqrt{1-x^2} + C$$

- (1): $\begin{cases} t = x^2 \\ dt = 2x dx \end{cases}$