18.100B
FINAL EXAM STUDY GUIDE

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1. Problem 1

Theorem 1.1. *Show that* $\sup A = \sqrt{2}$ *where* $A = \{x \in \mathbb{Q} : x > 0, x^2 < 2\}$.

Proof. First, we define $\sup A = s$ and note that $s^2 \geq 2$. Suppose not, then $s < 2$ and there exists an $\epsilon > 0$ such that $s + \epsilon \in A$, which contradicts the fact that s is the least upper bound. Now suppose that $s^2 > 2$. Then we can always find some $\delta > 0$ such that $s^2 - \delta > 2$, so that $s^2 - \delta \notin A$ and $s^2 - \delta > x$ for all $x \in A$, which is a contradiction of s being a supremum. Thus, we must have $s^2 = 2$, or by the uniqueness of radicals, $s = \sqrt{2}$. $\square$

2. Problem 2

Theorem 2.1. *Let* $\mathbf{A} \subset \mathbb{R}$ *be a nonempty set. Define* $-\mathbf{A} = \{x : -x \in \mathbf{A}\}$. *Show that* $\sup(-\mathbf{A}) = -\inf \mathbf{A}$ *and* $\inf(-\mathbf{A}) = -\sup \mathbf{A}$.

Proof. Suppose that A is bounded from below and let $a = \inf \mathbf{A}$. Then we must have $a \leq x$ for all $x \in \mathbf{A}$ and $-a \geq y$ for all $y \in -\mathbf{A}$. Therefore, we see that $y \leq -a \leq a \leq x$. Now, suppose that there exists some $\epsilon > 0$ such that $-a - \epsilon > y$ for all $y \in -\mathbf{A}$. Then, we must have $a + \epsilon < x$ for all $x \in \mathbf{A}$. Therefore, we see that a is not the infimum of $\mathbf{A}$, which is a contradiction. Therefore, no such epsilon exists, so that $-a$ is the supremum of $-\mathbf{A}$. If $\mathbf{A}$ is not bounded from below, then $\inf \mathbf{A} = -\infty$, so that $\sup(-\mathbf{A}) = \infty$. Therefore, we have shown that $\sup(-\mathbf{A}) = -\inf \mathbf{A}$. The proof that $\inf(-\mathbf{A}) = -\sup \mathbf{A}$ is similar. $\square$

3. Problem 3

Theorem 3.1. *Let* $A, B \subset \mathbb{R}$ *be nonempty. Define* $A + B = \{z = x + y : x \in A, y \in B\}$ *and* $A - B = \{z = x - y : x \in A, y \in B\}$. *Show that* $\sup(A + B) = \sup A + \sup B$ *and* $\sup(A - B) = \sup A - \inf B$.

Proof. Let A and B be bounded from above, and define $a = \sup A$ and $b = \sup B$, and $s = \sup(A + B)$. By the definition of supremum, we see that $a \geq x - \epsilon/2$ for all $x \in A$ and $b \geq y - \epsilon/2$ for all $y \in B$. Therefore, we see that $a + b \geq x + y - \epsilon$. Since $z = x + y \in A + B$, we have shown that $s = a + b$.

For the next part, we choose the set $C = -B = \{y : -y \in B\}$. We have previously shown that $\sup C = -\inf B$. Moreover, we see that $\sup(A - B) = \sup(A + C)$ because $A - B = \{z = x - y : x \in A, y \in B\} = \{z = x + y : x \in A, -y \in B\} = A + C$. We have just shown that $\sup(A + C) = \sup A + \sup C$, and since $\sup C = -\inf(B)$, we have $\sup(A - B) = \sup(A + C) = \sup A - \inf B$. $\square$

4. Problem 4

Theorem 4.1. *Let* A, B *be nonempty subsets of real numbers. Show that* $\sup(A \cup B) = \max\{\sup A, \sup B\}$ *and* $\inf(A \cup B) = \min\{\inf A, \inf B\}$.

Proof. First, suppose that A and B are bounded above so that $\sup A = a$ and $\sup B = b$. Then we can assume without loss of generality that $a \geq b$. Thus, we see that $a \geq x - \epsilon$ for all $x \in A$ and some $\epsilon > 0$. Moreover, we see that $a \geq b$ implies that $a \geq y - \epsilon$ for all $y \in B$ and some $\epsilon > 0$. This implies that $a = \sup(A \cup B)$. If either A or B is unbounded, then $\sup A = \infty$ or $\sup B = \infty$ and $\sup(A \cup B) = \infty$. This proves the theorem for $\sup(A \cup B) = \max\{\sup A, \sup B\}$, and the proof for $\inf(A \cup B) = \min\{\inf A, \inf B\}$ is similar. $\square$

5. Problem 5

Theorem 5.1. *Prove that if a sequence $\{a_n\}$ is monotonically increasing then* $\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$.

Proof. If the sequence $\{a_n\}$ is unbounded, we see that $\lim_{n\to\infty} a_n = \infty$ and $\sup\{a_n : n \in \mathbb{N}\} = \infty$. If the sequence $\{a_n\}$ is bounded, we can assume that $\sup\{a_n : n \in \mathbb{N}\} = a$. Since $\{a_n\}$ is monotonically increasing, we see that $a_{n+1} \geq a_n$. Moreover, we see that by definition of upper bound that $a_n \leq a$ for all $n \in \mathbb{N}$. This means that for each $\epsilon > 0$, we can find some a_{n_0} such that $a - \epsilon < a_{n_0}$ because $a - \epsilon$ is not an upper bound. Therefore, we see that $a > a_{n_0} > a - \epsilon$, which implies that $|a_{n_0} - a| < \epsilon$. Thus, we have shown that $\lim_{n\to\infty} a_n = a$. □

6. Problem 6

Theorem 6.1. *Let $a_1, a_2, \ldots, a_p$ be fixed positive numbers and consider the sequence $s_n = \frac{a_1^n + a_2^n + \ldots a_p^n}{p}$ and $x_n = \sqrt[n]{s_n}$. Prove that x_n is monotonically increasing.*

Proof. First we will show that $\frac{s_n}{s_{n+1}}$ is monotonically increasing. If each $a_1, \ldots a_p \leq 1$, then we see that $a_i^n \geq a_i^{n+1}$, which means that $s_n \geq s_{n+1}$. This means that $\frac{s_n}{s_{n+1}} \leq \frac{s_{n+1}}{s_{n+2}}$ for $n \geq 2$. The same is true when some a_i is greater than 1. Therefore, $\frac{s_n}{s_{n+1}}$ is monotonically increasing, and so $s_n^2 \leq s_{n+1}s_{n-1}$.

We know that $x_1 \leq x_2$ because:

$$\left(\sum_{i=1}^{p} a_i\right)^2 \leq \sum_{i=1}^{p} (a_i)^2 \tag{6.2}$$

Now assume $x_{n-1} \leq x_n$, so that $s_{n-1} \leq s_n^{\frac{n-1}{n}}$. We will show that $x_n \leq x_{n+1}$:

$$x_{n+1} = \sqrt[n+1]{s_{n+1}} \geq \sqrt[n+1]{\frac{s_n^2}{s_{n-1}}} \geq \sqrt[n+1]{\frac{s_n^2}{s_n^{\frac{n-1}{n}}}} = \sqrt[n+1]{s_n^{1+\frac{1}{n}}} = s_n^{\frac{1+\frac{1}{n}}{n+1}} = s_n^{\frac{1}{n}} = x_n \tag{6.3}$$

Since $x_{n+1} \geq x_n$, we have proven that $\{x_n\}$ is monotonically increasing by induction. □

7. Problem 7

Theorem 7.1. *Let $\{a_n\}$ be a bounded sequence which satisfies the condition $a_{n+1} \geq a_n - \frac{1}{2^n}$ for $n \in \mathbb{N}$. Show that the sequence $\{a_n\}$ is convergent.*

Proof. Since $\{a_n\}$ is bounded, we know that $|a_n| < M$ for some $M > 0$ for all $n \in \mathbb{N}$. Moreover, the assumption shows that $\frac{1}{2^n} \geq a_n - a_{n+1}$. This means that $|a_n - a_{n+1}| \leq \frac{1}{2^n}$. Particularly, we know that $\frac{1}{2^n} \to 0$ as $n \to \infty$. This tells us that $|a_n - a_{n+1}| \to 0$ as $n \to \infty$. By the Cauchy criterion, we see that $\{a_n\}$ converges. □

8. Problem 8

Theorem 8.1. *Establish the convergence and find the limit of the sequence defined by $a_1 = 0$ and $a_{n+1} = \sqrt{6 + a_n}$ for $n \geq 1$.*

Proof. First we note that $a_1 = 0$ and $a_2 = \sqrt{6}$. We know that since $3^2 = 9$ that $a_2 = \sqrt{6} \leq 3$, so that $0 < a_2 < 3$. This implies $6 < a_2 + 6 < 9$, and since square root preserves ordering, we have $\sqrt{6} < \sqrt{a_2 + 6} < 3$. Therefore, we see that $a_2 < a_3 < 3$. Continuing this process, we see that $\sqrt{6 + \sqrt{6}} < a_4 < 3$, which implies $a_3 < a_4 < 3$. We see that this process continues indefinitely, and that $a_n < a_{n+1} < 3$. This means that $\{a_n\}$ is monotonically increasing and bounded from above by 3. First, this establishes the convergence of $\{a_n\}$. Next, this shows that $\{a_n\} \to 3$ as $n \to \infty$. This is because for any $\epsilon > 0$, we can choose N large enough so that $|3 - a_N| < \epsilon$ because $\{a_n\}$ is monotonically increasing. □

9. Problem 9

Theorem 9.1. *Show that the sequence defined by $a_1 = 0$, $a_2 = \frac{1}{2}$, and $a_{n+1} = \frac{1}{3}(1 + a_n + a_{n-1}^3)$ for $n > 1$ converges and determine its limit.*

Proof. We see that $0 \leq a_n \leq 1$ because $1 + a_n + a_{n-1}^3$ is always less than 3 for our given starting values. First, we will show convergence by assuming $a_n \geq a_{n-1}$:

$$a_{n+1} - a_n = \frac{1}{3}(1 + a_n + a_{n-1}^3) - a_n = \frac{1}{3}\left(1 - \frac{2}{3}a_n - a_{n-1}^3\right) \tag{9.2}$$

Since $a_n \leq 1$ and $a_{n-1} \leq a_n$, we see that $a_{n+1} - a_n \geq \frac{1}{3}(1 - a_n(\frac{2}{3} + a_n^2)) \geq \frac{1}{3}(1 - 1(\frac{2}{3} + 1)) \geq 0$. Therefore, we see that $a_{n+1} \geq a_n$ if we assume that $a_n \geq a_{n+1}$. This shows that $\{a_n\}$ is monotonically increasing and bounded, which implies convergence. Moreover, we see that $\{a_n\} \to 1$ as $n \to \infty$ because for every $\epsilon > 0$ we can always choose an N such that $|1 - a_N| < \epsilon$. □

10. Problem 10

Theorem 10.1. *Let $\{a_n\}$ be defined recursively by $a_{n+1} = \frac{1}{4-3a_n}$ for $n \geq 1$. Determine for which a_1 the sequence converges and in the case of convergence find its limit.*

Proof. We will show by induction that the following is true:

$$a_n = \frac{(3^{n-1} - 1) - (3^{n-1} - 3)a_1}{(3^n - 1) - (3^n - 3)a_1} \tag{10.2}$$

First, it is clear that this follows for $a_2 = \frac{3-1-(3-3)a_1}{9-1-(9-3)a_1} = \frac{2}{8-6a_1} = \frac{1}{4-3a_1}$. Thus, we have established the base case. Now, we will show that this works for a_{n+1}:

$$a_{n+1} = \frac{1}{4 - 3a_n} = \frac{(3^n - 1) - (3^n - 3)a_1}{4((3^n - 1) - (3^n - 3)a_1) - 3((3^{n-1} - 1) - (3^{n-1} - 3)a_1)} \tag{10.3}$$

$$= \frac{(3^n - 1) - (3^n - 3)a_1}{(3^{n+1} - 1) - (3^{n+1} - 3)a_1} \tag{10.4}$$

Therefore, we see that the sequence does indeed converge as $n \to \infty$ for $a_1 \neq \frac{3^n-1}{3^n-3}$ for all $n \in \mathbb{N}$. If $a_1 = 1$, then $a_n = 1$. For all other allowable values of a_1, we have $a_n \to \frac{1}{3}$ as $n \to \infty$. □

11. Problem 11

Theorem 11.1.

Proof. □