18.100B
PROBLEM SET 2

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1. Problem 2.11

Theorem 1.1. *The distance $d_1(x, y) = (x - y)^2$ is not a metric.*

Proof. Here, the third requirement for a metric does not hold, namely that $d(x, y) \leq d(x, r) + d(r, y)$. This is because $d_1(x, y) = (x - y)^2 = x^2 - 2xy + y^2$ and $d_1(x, r) + d_1(r, y) = x^2 - 2xr + r^2 + r^2 - 2ry + y^2 = x^2 + y^2 + 2r^2 - 2xr - 2ry$. Thus, one must have $-2xy \leq 2r^2 - 2xr - 2ry$ for all $r \in \mathbb{R}^1$ for d_1 to be a metric. This is the same as $xy \geq r(x + y - r)$. However, if one sets $x = 2$ and $y = 0$, this inequality does not hold for all values of r. For instance, $0 \not\leq 1(2 - 1) = 1$ which shows that d_1 is not a metric. $\square$

Theorem 1.2. *The distance $d_2(x, y) = \sqrt{|x - y|}$ is a metric.*

Proof. The first two properties of a metric are easy to prove. We know $d_2(x, y) > 0$ holds for all $x \neq y$ and $d_2(x, x) = 0$ because square roots of positive numbers are always positive. Next, $d_2(x, y) = d_2(y, x)$ because $|x - y| = |y - x|$. Finally, we have $d_2(x, y) \leq d_2(x, r) + d_2(r, y)$ for all $r \in \mathbb{R}^1$. This is because the triangle inequality for absolute values states that $|x - y| \leq |x - r| + |r - y|$, which means $d_2(x, y) \leq \sqrt{|x - r| + |r - y|} = \sqrt{d_2(x, r)^2 + d_2(r, y)^2}$. However, by the triangle equality, we know that $\sqrt{d_2(x, r)^2 + d_2(r, y)^2} \leq d_2(x, r) + d_2(r, y)$, and so that $d_2(x, y) \leq d_2(x, r) + d_2(r, y)$ for all $r \in \mathbb{R}^1$. Thus, d_2 is a metric. $\square$

Theorem 1.3. *The distance $d_3(x, y) = |x^2 - y^2|$ is not a metric.*

Proof. The first property of metrics does not hold. For instance, if $x = 1$ and $y = -1$, then $x \neq y$, but $d_3(x, y) = 0$, which means d_3 is not a metric. $\square$

Theorem 1.4. *The distance $d_4(x, y) = |x - 2y|$ is not a metric.*

Proof. We know that a metric must have the property $d_4(x, y) > 0$ if $x \neq y$. However, this property does not hold for $x = 2$ and $y = 1$, where $d_4(x, y) = 0$ and $x \neq y$. Thus, d_4 is not a metric. $\square$