18.100B
PROBLEM SET 2

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1. Problem 2.11

Theorem 1.1. *The distance $d_1(x, y) = (x - y)^2$ is not a metric.*

Proof. Here, the third requirement for a metric does not hold, namely that $d(x, y) \leq d(x, r) + d(r, y)$. This is because $d_1(x, y) = (x - y)^2 = x^2 - 2xy + y^2$ and $d_1(x, r) + d_1(r, y) = x^2 - 2xr + r^2 + r^2 - 2ry + y^2 = x^2 + y^2 + 2r^2 - 2xr - 2ry$. Thus, one must have $-2xy \leq 2r^2 - 2xr - 2ry$ for all $r \in \mathbb{R}^1$ for d_1 to be a metric. This is the same as $xy \geq r(x + y - r)$. However, if one sets $x = 2$ and $y = 0$, this inequality does not hold for all values of r. For instance, $0 \ngeq 1(2 - 1) = 1$ which shows that d_1 is not a metric. $\square$

Theorem 1.2. *The distance $d_2(x, y) = \sqrt{|x - y|}$ is a metric.*

Proof. The first two properties of a metric are easy to prove. We know $d_2(x, y) > 0$ holds for all $x \neq y$ and $d_2(x, x) = 0$ because square roots of positive numbers are always positive. Next, $d_2(x, y) = d_2(y, x)$ because $|x - y| = |y - x|$. Finally, we have $d_2(x, y) \leq d_2(x, r) + d_2(r, y)$ for all $r \in \mathbb{R}^1$. This is because the triangle inequality for absolute values states that $|x - y| \leq |x - r| + |r - y|$, which means $d_2(x, y) \leq \sqrt{|x - r| + |r - y|} = \sqrt{d_2(x, r)^2 + d_2(r, y)^2}$. However, by the triangle inequality, we know that $\sqrt{d_2(x, r)^2 + d_2(r, y)^2} \leq d_2(x, r) + d_2(r, y)$ because $\sqrt{|x - r| + |r - y|} \leq \sqrt{|x - y|} + \sqrt{|r - y|}$. This means that $d_2(x, y) \leq d_2(x, r) + d_2(r, y)$ for all $r \in \mathbb{R}^1$. Thus, d_2 is a metric. $\square$

Theorem 1.3. *The distance $d_3(x, y) = |x^2 - y^2|$ is not a metric.*

Proof. The first property of metrics does not hold. For instance, if $x = 1$ and $y = -1$, then $x \neq y$, but $d_3(x, y) = 0$, which means d_3 is not a metric. $\square$

Theorem 1.4. *The distance $d_4(x, y) = |x - 2y|$ is not a metric.*

Proof. We know that a metric must have the property $d_4(x, y) > 0$ if $x \neq y$. However, this property does not hold for $x = 2$ and $y = 1$, where $d_4(x, y) = 0$ and $x \neq y$. Thus, d_4 is not a metric. $\square$

Theorem 1.5. *The distance $d_5(x, y) = \frac{|x-y|}{1+|x-y|}$ is a metric.*

Proof. We see that $d_5(x, y) = 0 \iff x = y$, and that $d_5(x, y) > 0$ for all $x, y \in \mathbb{R}^1$. Also, we see that since $|x - y| = |y - x|$, that $d_5(x, y) = d_5(y, x)$. Now, we must prove that $d_5(x, y) \leq d_5(x, r) + d_5(r, y)$. This can be done by looking at the quantity

$$d_5(x, r) + d_5(r, y) - d_5(x, y) = \frac{|x - r|}{1 + |x - r|} + \frac{|r - y|}{1 + |r - y|} - \frac{|x - y|}{1 + |x - y|} \tag{1.6}$$

By expanding out the denominator, one finds that this expression is equal to the following

$$\begin{aligned} |x-r|(1+|x-y|)(1+|r-y|) &+ |r-y|(1+|x-y|)(1+|x-r|) \\ &- |x-y|(1+|x-r|)(1+|r-y|) \end{aligned}$$

$$= |x-r||r-y||x-y| + 2|r-y||x-r| + |r-y| + |x-r| - |x-y| \tag{1.7}$$

The first two terms are non-negative because they are products of absolute values. The last term is also non-negative because the triangle inequality states that $0 \leq |x-r| + |r-y| - |x-y|$, which means that the entire expression is non-negative. This shows that the final property of metrics is true, namely that $0 \leq d_5(x,r) + d_5(r,y) - d_5(x,y)$. This shows that d_5 is a metric. ☐

2. Problem 2.12

Theorem 2.1. *If $K \subset \mathbb{R}^1$ consist of 0 and the numbers $\frac{1}{n}$ for $n = 1, 2, 3, \ldots$, then K is compact (directly from the definition).*

Proof. We must show that for every open cover of K, there exists a finite subcover. To do this, let $\{G_\alpha\}_{\alpha \in A}$ be an open cover of K. There must exist an index α_0 such that $0 \in G_{\alpha_0}$. Since G_{α_0} is open, we know that there exists an $r > 0$ such that $N_r(0) \subset G_{\alpha_0}$. Moreover, by the archimedean principle, we know there exists some n such that $\frac{1}{n} < r$. Thus, we have that $N_{\frac{1}{n}}(0) \subset N_r(0) \subset G_{\alpha_0}$, which means that $N_{\frac{1}{n}}(0) \subset G_{\alpha_0}$. Correspondingly, we know that $\frac{1}{n} \in G_{\alpha_0}$. We can apply the above argument to each member of K such that $n = 1, 2, 3, \ldots$ and obtain $\frac{1}{n} \in G_{\alpha_n}$. Thus, for all $N > n$, we have $\frac{1}{N} < \frac{1}{n} < r$, meaning that we also have $\frac{1}{N} \in G_{\alpha_0}$. Because of this, for all $N \leq n$, there exist indices $\alpha_N \in A$ such that $\frac{1}{N} \in G_{\alpha_N}$. Thus, we know that $K \subset G_{\alpha_0} \cup \cdots \cup G_{\alpha_N}$ for some finite N. This shows that every open cover $\{G_\alpha\}_{\alpha \in A}$ of K has a finite subcover, and that K is compact. ☐

3. Problem 2.13

Theorem 3.1. *It is possible to construct a compact set of real numbers whose limit points form a countable set.*

Proof. Consider the set E_i for $0 \leq i \leq 1$ and $i \in \mathbb{Q}$ that is defined as $E_i := \{i + \frac{1}{p} : p \in \mathbb{N}\} \cup \{i\}$. In the case of $i = 0$, E_i is the set consisting of $\frac{1}{n}$ for all $n = 1, 2, \ldots$ joined with 0. Now, we can create a set E that is compact and whose limit points form a countable set by defining for all $i \in \mathbb{Q}$ the following

$$E := \bigcup_{0 \leq i \leq 1} E_i \tag{3.2}$$

This means that the set is closed, because it contains all its limit points. One can see that in this case, all the limit points of E are given by the union of all the limit points of E_i. There is only one limit point of E_i by construction, which is i. Thus, all the limit points of E are the set of all $0 \leq i \leq 1$ for $i \in \mathbb{Q}$, which is definitely a subset of E. Moreover, we can show that the set is bounded. To do this, we must show that there is a real number M and a point $q \in \mathbb{R}$ such that $d(p,q) < M$ for all $p \in E$. This is clearly the case because all elements of E are confined to the closed interval $[0,2]$. By the archimedean principle, there exists an $M \in \mathbb{R}$ such that $2 * q < M$ for all $q \in \mathbb{R}$. Since $|p - q| < M$ for all $p \in [0,2]$, we know that $d(p,q) < M$ for all $p \in E$ and $q \in \mathbb{R}$, as $E \subset [0,2]$. This shows that E is both closed and bounded, and since $E \subset \mathbb{R}$, we know by Heine-Borel that E is compact.

Now, it is easy to show that the limit points of E form a countable set. We have already shown that E' is the set $\{i : 0 \leq i \leq 1, i \in \mathbb{Q}\}$. We also know that the rationals are countable and that E' is an infinite subset of $\mathbb{Q}$. Since every infinite subset of a countable set is countable, we know that E' is countable. $\square$

4. Problem 2.14

Theorem 4.1. *There is an open cover of the segment* $(0,1)$ *which has no finite subcover.*

Proof. All we must do is show that there exists a single open cover without any finite subcovers. To do this, consider the open cover $\{G_\alpha\}$, where $\alpha \in \mathbb{N}$. Now define each interval as follows $G_\alpha := (\frac{1}{\alpha}, 1)$. Therefore, we have $\bigcup_{\alpha\in\mathbb{N}} G_\alpha = (0,1)$. Thus, $(0,1) \subset \bigcup_{\alpha\in\mathbb{N}} G_\alpha$, and we have an open cover of the interval $(0,1)$. However, this open cover has no finite subcover. We will show this by contradiction. If there does exist a finite subcover of $\{G_\alpha\}$, then there is some largest number $N \in \mathbb{N}$ such that $(0,1) \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_N}$. This means that $(0,1) \subset (1,1) \cup \cdots \cup (\frac{1}{N}, 1)$ and that $(0,1) \subset (\frac{1}{N}, 1)$ for $N \in \mathbb{N}$. This is a contradiction, which means there does not exist a finite subcover of $\{G_\alpha\}$. Thus, we have provided an example of an open cover of the segment $(0,1)$ which has no finite subcover. $\square$

5. Problem 2.16

Theorem 5.1. *Regard* $\mathbb{Q}$*, the set of all rational numbers, as a metric space, with* $d(p,q) = |p-q|$*. Let* E *be the set of all* $p \in \mathbb{Q}$ *such that* $2 < p^2 < 3$*. Show that* E *is closed and bounded in* $\mathbb{Q}$*, but that* E *is not compact. In addition,* E *is open in* $\mathbb{Q}$*.*

Proof. To show that E is closed in $\mathbb{Q}$, it is sufficient to show that $E = \mathbb{Q} \cap G$ for some $G \subset \mathbb{R}$ such that G is closed in $\mathbb{R}$. Let $G = \{x : 2 \le x^2 \le 3, x \in \mathbb{R}\}$, then it is easy to see that $E = \mathbb{Q} \cap G$. Now we must show that G is closed in $\mathbb{R}$. To do this, note that $\sqrt{3}$ is an upper bound of G because $p \le \sqrt{3}$ for every $p \in G$. Thus, $\sqrt{3} = \sup G$ because for every $h > 0$, $\sqrt{3} - h \in G$. Moreover, we can see that $\sqrt{3} \in G$, which means $\sup G \in G$ and that G is closed. Since we have shown that G is closed in $\mathbb{R}$, we now know that $E = \mathbb{Q} \cap G$ is closed in $\mathbb{Q}$.

To show that E is bounded in $\mathbb{Q}$, we must show that there exists an $M \in \mathbb{R}$ and a $q \in \mathbb{Q}$ such that $d(p,q) < M$ for all $p \in E$. For $p > 0$, pick any $q \in \mathbb{Q}$ such that $0 < q < p$, and it is clear that $|p - q| < p$. For $p < 0$, pick any $q \in \mathbb{Q}$ such that $p < q < 0$, and we have $|p-q| < -p$. Thus, there exists an $M \in \mathbb{R}$ and a $q \in \mathbb{Q}$ such that $d(p,q) < M$ for all $p \in E$, showing that E is bounded in $\mathbb{Q}$.

We have now shown that E is closed and bounded in $\mathbb{Q}$, but have yet to prove that E is not compact. To do this, we will use the Heine-Borel theorem, and show that E is not closed and bounded in $\mathbb{R}$. To do this, we will show that E is not closed in $\mathbb{R}$, which can be seen if one examines $y = \sup E$. One can see that $y = \sqrt{3} = \sup E$, because for each $p \in E$, $p \le \sqrt{3}$, and for every $h > 0$, $\sqrt{3} - h \in E$. However, since $\sqrt{3} \notin E$, we can see that $\sup E \notin E$, showing that E does not contain all of its limit points. Thus, E is open in $\mathbb{R}$. This implies that E is not compact by Heine-Borel.

Now we want to know whether E is open in $\mathbb{Q}$. To do this, we must know whether every point $p \in E$ is an interior point. In other words, does there exist an $r > 0$ such that $N_r(p) \subset E$? If one picks $r = \min\{|-\sqrt{3}-p|, |-\sqrt{2}-p, |\sqrt{2}-p|, |\sqrt{3}-p|\}$, then one will always be able to find a distance $\frac{r}{2}$ such that $N_{\frac{r}{2}}(p) \subset E$ for every $p \in E$. Thus, E is open in $\mathbb{Q}$. $\square$

6. Problem 2.22

Theorem 6.1. *A metric space is called separable if it contains a countable dense subset. We shall show that* $\mathbb{R}^k$ *is separable.*

Proof. First, consider the set $\{P\}$ of points $p \in \mathbb{R}^k$ such that $p = (p_1, p_2, \ldots, p_k)$ and $p_n \in \mathbb{Q}$ for all $n = 1, \ldots, k$. In other words, $\{P\}$ is the set of points with only rational coordinates, $\mathbb{Q}^k$. We know that $\{P\}$ is countable because $\mathbb{Q}$ is countable, and $\{P\}$ is a finite grouping of rational coordinates.

Now, we must show that $\{P\}$ is dense in $\mathbb{R}^k$. In other words, each $x \in \mathbb{R}^k$ must have every neighborhood contain a point $q \neq x$ such that $q \in \mathbb{Q}^k$, or x must be an element of $\mathbb{Q}^k$. Given $r > 0$, there exists some $q_n \in \mathbb{Q}$ such that $d(x_n, q_n) < \frac{r}{k}$ for $k \in \mathbb{N}$ by the archimedean property. Thus, we know that

$$\begin{aligned} d(x,q) &= \sqrt{d(x_1,q_1)^2 + d(x_2,q_2)^2 + \ldots + d(x_k,q_k)^2} && (6.2) \\ &< \underbrace{\sqrt{\left(\frac{r}{k}\right)^2 + \ldots + \left(\frac{r}{k}\right)^2}}_{k} && (6.3) \end{aligned}$$

Since the expression in (6.3) is equal to $\sqrt{\frac{kr^2}{k^2}} = \frac{r}{\sqrt{k}} \leq r$, where the last inequality comes from the fact that $k \in \mathbb{N}$, we have that $d(x,q) < r$ for each $x \in \mathbb{R}^k$. This shows for all $x \in \mathbb{R}^k$, each neighborhood of x contains a $q \neq x$ such that $q \in \{P\}$. This means that $\{P\}$ is dense in $\mathbb{R}^k$, and since we have already shown countability, $\{P\}$ is also separable. $\square$