18.100B
PROBLEM SET 9

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1. Problem 5.19

Theorem 1.1. *Suppose f is defined in $(-1,1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $a_n \to 0$, and $b_n \to 0$ as $n \to \infty$. Define the difference quotients $D_n = \frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}$. Then if $\alpha_n < 0 < \beta_n$, $\lim D_n = f'(0)$.*

Proof. Because the derivative exists at $x = 0$, we know the following to be true by the definition of derivative:

$$f'(0) = \lim_{n\to\infty} \frac{f(\alpha_n) - f(0)}{\alpha_n} - u(n) \tag{1.2}$$

$$f'(0) = \lim_{n\to\infty} \frac{f(\beta_n) - f(0)}{\beta_n} - v(n) \tag{1.3}$$

Here, the functions $u(t) \to 0$ and $v(t) \to 0$ as $n \to \infty$. Therefore, rearranging these, we can obtain:

$$\lim_{n\to\infty} f(\alpha_n) = \lim_{n\to\infty} f(0) + (f'(0) + u(n))\alpha_n \tag{1.4}$$

$$\lim_{n\to\infty} f(\beta_n) = \lim_{n\to\infty} f(0) + (f'(0) + v(n))\beta_n \tag{1.5}$$

Thus, since $\alpha < 0 < \beta$, we can determine the difference quotient by substituting values of $f(\beta_n)$ and $f(\alpha_n)$ that we have just derived.

$$D_n = \frac{f(0) + (f'(0) + v(n))\beta_n - f(0) - (f'(0) + u(n))\alpha_n}{\beta_n - \alpha_n} \tag{1.6}$$

$$= f'(0) + \frac{v(n)\beta_n - u(n)\alpha_n}{\beta_n - \alpha_n} \tag{1.7}$$

Since we have $\alpha_n < 0 < \beta_n$, we see that $|\alpha_n| \le \beta_n - \alpha_n$ and $\beta_n \le \beta_n - \alpha_n$. This allows us to use the triangle inequality and show:

$$|D_n - f'(0)| = v(n)\frac{|\beta_n|}{|\beta_n - \alpha_n|} - u(n)\frac{|\alpha_n|}{|\beta_n - \alpha_n|} \tag{1.8}$$

$$\le v(n) - u(n) \tag{1.9}$$

Taking this limit as $n \to \infty$, we see that $D_n - f'(0) \to 0$, which shows that $D_n \to f'(0)$ as $n \to \infty$. □

Theorem 1.10. *If $0 < \alpha_n < \beta_n$ and $\{\beta_n/(\beta_n - \alpha_n)\}$ is bounded, then $\lim D_n = f'(0)$.*

Proof. Since we have previously derived $D_n - f'(0)$, we can just use the expression from above to prove this theorem. First, we know that since $0 < \alpha_n < \beta_n$, we can say that $\alpha_n < \beta_n$. Therefore, we have:

$$D_n - f'(0) = v(n)\frac{\beta_n}{\beta_n - \alpha_n} - u(n)\frac{\alpha_n}{\beta_n - \alpha_n} \tag{1.11}$$

$$\le (v(n) - u(n))\frac{\beta_n}{\beta_n - \alpha_n} \tag{1.12}$$

Since we know that $\{\beta_n/(\beta_n - \alpha_n)\}$ is bounded, we can see that as we take $n \to \infty$, we see that the right hand side goes to zero because $v(n) \to 0$ and $u(n) \to 0$ individually.

$$\lim_{n\to\infty} |D_n - f'(0)| \le \lim_{n\to\infty} |v(n) - u(n)| \left|\frac{\beta_n}{\beta_n - \alpha_n}\right| = 0 \tag{1.13}$$

Thus, we see that $\lim D_n = f'(0)$. □

Theorem 1.14. *If f' is continuous in $(-1,1)$, then $\lim D_n = f'(0)$.*

Proof. We can apply the mean value theorem to the function f since it is both continuous and differentiable on $(-1,1)$. Thus, for each $n \in \mathbb{N}$, there exists a t_n with $\alpha_n \le t_n \le \beta_n$ such that:

$$f'(t_n) = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = D_n \tag{1.15}$$

Therefore, we see that $\lim \alpha_n \le \lim t_n \le \lim \beta_n$. Since both $\alpha_n \to 0$ and $\beta_n \to 0$, we see that $t_n \to 0$ as $n \to \infty$. Therefore, taking the limit as $n \to \infty$ in the above expression, we see that $\lim D_n = f'(0)$. □

Theorem 1.16. *There exists a function f which is differentiable in $(-1,1)$ and in which α_n, β_n tend to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.*

Proof. Consider the following function defined for $x \in (-1,1)$:

$$f = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{1.17}$$

We can pick $\beta_n = \frac{2}{\pi(4n-1)}$ and $\alpha_n = \frac{1}{2\pi n}$. We see that both $\beta_n \to 0$ and $\alpha_n \to 0$ as $n \to \infty$. However, we also see that $f(\alpha_n) = 0$ for all $n \in \mathbb{N}$ and that $f(\beta_n) = -\beta_n^2$. Therefore, we have:

$$\lim_{n\to\infty} D_n = \lim_{nto\infty} \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} \tag{1.18}$$

$$= \lim_{n\to\infty} -\frac{\beta_n^2}{\beta_n - \alpha_n} \tag{1.19}$$

$$= \lim_{n\to\infty} -\frac{4}{\pi^2(4n-1)^2} \frac{2\pi n(4n-1)}{1} \tag{1.20}$$

$$= -\frac{2}{\pi} \tag{1.21}$$

Thus, since $f'(0) = 0$, and we can see that $0 \neq -\frac{2}{\pi}$, we have given an example for the theorem. □

2. Problem 5.25

Theorem 2.1. *Suppose f is twice differentiable on $[a,b]$, $f(a) < 0, f(b) > 0, f'(x) \ge \delta > 0$, and $0 \le f''(x) \le M$ for all $x \in [a,b]$. Let ξ be the unique point in (a,b) at which $f(\xi) = 0$. Choose $x_1 \in (\xi, b)$ and define x_n by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Interpret this goemetrically in terms of a tangent to the graph of f.*

Proof. We see that the formula for x_{n+1} computes the intercept of the tangent line of the function at point x_n with the x axis. This will then be the next point, and the process will continue until x_n converges to the root of the function (when $f = 0$). □

Theorem 2.2. *Prove that $x_{n+1} < x_n$ and that $\lim_{n\to\infty} x_n = \xi$.*

Proof. We will use induction to show that $\xi < x_{n+1} < x$. We can use the mean value theorem to show that for some $c_n \in (\xi, x_n)$, we have: $(x_n - \xi)f'(c_n) = f(x_n) - f(\xi) = f(x_n)$ because $f(\xi) = 0$. Moreover, we know that f' is increasing on $[a,b]$, which means that $f'(c_n) < f'(x_n)$ because $c_n < x_n$. Thus,

$$f'(c_n) = \frac{f(x_n)}{(x_n - \xi)} < f'(x_n) = \frac{f(n)}{x_n - x_n + \frac{f(x_n)}{f'(x_n)}} = \frac{f(x_n)}{x_n - x_{n+1}} \tag{2.3}$$

Therefore, we can rearrange the inequality and see that $x_n - x_{n+1} < x_n - \xi$. This completes the first part of the inequality, because now we see that $\xi < x_{n+1}$. Next, we know that since $f(x) > 0$ and $f'(x) > 0$ for all $x \in [a,b]$, we see that $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} < x_n$. Thus, we have shown that $\xi < x_{n+1} < x_n$.

Next, we must show that $\lim x_n = \xi$. First, we know that $\{x_n\}$ is a bounded, strictly decreasing sequence. This means that its limit λ exists. Therefore, we have the following:

$$\lambda = \lim_{n\to\infty} x_{n+1} \tag{2.4}$$

$$\lambda = \lim_{n\to\infty} x_n - \frac{f(x_n)}{f'(x_n)} \tag{2.5}$$

$$\lambda = \lambda - \frac{f(\lambda)}{f'(\lambda)} \tag{2.6}$$

$$0 = f(\lambda) \tag{2.7}$$

Since $f(\xi) = 0$ is the unique point in (a,b) for which $f(\xi) = 0$, we must have $\lambda = \xi$. Therefore, $\lim x_n = \xi$. □

Theorem 2.8. *Use Taylor's theorem to show that* $x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$ *for some* $t_n \in (\xi, x_n)$.

Proof. Using Taylor's theorem for some $t_n \in (\xi, x_n)$, we can obtain:

$$f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2 \tag{2.9}$$

$$0 = \frac{f(x_n)}{f'(x_n)} + (\xi - x_n) + \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \tag{2.10}$$

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \tag{2.11}$$

We can divide by $f'(x_n)$ because we know that $f'(x) > 0$ for all $x \in (a,b)$. We also know that $(x_n - \xi)^2 = (\xi - x_n)^2$, so we can substitute one for the other. □

Theorem 2.12. *If* $A = M/2\delta$, *deduce that* $0 \le x_{n+1} - \xi \le \frac{1}{A}[A(x_1 - \xi)]^{2^n}$.

Proof. First, since we have shown that $\xi < x_{n+1}$, we see that $0 \le x_{n+1} - \xi$. Also, since $f''(x) < M$ and $f'(x) \ge \delta$ for all $x \in (a,b)$, we see that $\frac{f''(t_n)}{2f'(x_n)} \le \frac{M}{2\delta} = A$ for $t_n \in (\xi, x_n)$. We have found that $x_{n+1} - \xi \le A(x_n - \xi)^2$. Then we can use mathematical induction. For the base case, we have $x_2 - \xi \le A(x_1 - \xi)^2 = \frac{1}{A}[A(x_1 - \xi)]^2$. Now assume that the inequality has been proven for all cases up to x_n. We shall prove that it works for x_{n+1}:

$$x_{n+1} - \xi \le A(x_n - \xi)^2 \tag{2.13}$$

$$= A\left(\frac{1}{A}[A(x_1 - \xi)]^{2^{n-1}}\right)^2 \tag{2.14}$$

$$= \frac{1}{A}[A(x_1 - \xi)]^{2^n} \tag{2.15}$$

This proves the inequality. □

Theorem 2.16. *Show that Newton's method amounts to finding a fixed point of the function* g *defined by* $g(x) = x - \frac{f(x)}{f'(x)}$.

Proof. We want to show that Newton's method finds x_0 such that $g(x_0) = x_0$, or that $x_0 - \frac{f(x_0)}{f'(x_0)} = x_0$ which implies $f(x_0) = 0$. Therefore, we only must show that Newton's method finds $f(x_0) = 0$, because $f'(x_0) > 0$ for all $x \in (a,b)$.

Since we have previously shown that $\lim x_n = \xi$, we know that $\lim f(x_n) = f(\xi) = 0$. Thus, Newton's method finds an approximation to x_0, where $f(x_0) = 0$ as we take larger and larger $n \in \mathbb{N}$ for $\{x_n\}$. This is what we wanted to show.

As x approaches ξ, we see that $g'(x) = \frac{f(x)f''(x)}{f'(x)^2}$, so that $0 \le g'(x) \le f(x)\frac{M}{\delta^2}$. Thus, we see that as x approaches ξ, we have $g'(x)$ approaching 0. □

Theorem 2.17. *Put* $f(x) = x^{1/3}$ *on* $(-\infty, \infty)$ *and try Newton's method.*

Proof. We see that $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x^{1/3}}{x^{-2/3}} = x_n - 3x_n = -2x_n$. Thus, we see that $x_2 = -2x_1$. Using induction, we can assume that $x_n = (-2)^{n-1}x_1$ has been proven up to x_n. Then, we can show that

$$x_{n+1} = -2x_n = -2(-2)^{n-1}x_1 = (-2)^n x_1 \tag{2.18}$$

With mathematical induction, we have shown that $x_n = (-2)^{n-1}x_1$. Therefore, we see that for any choice of x_1, x_n does not converge. □

3. Problem 5.26

Theorem 3.1. *Suppose* f *is differentiable on* $[a,b]$, $f(a) = 0$, *and there is a real number* A *such that* $|f'(x)| \le A|f(x)|$ *on* $[a,b]$. *Prove that* $f(x) = 0$ *for all* $x \in [a,b]$.

Proof. If $A = 0$, then we can see that $f'(x) = 0$, which implies that $f(x) = f(a) = 0$ for all $x \in [a,b]$. Moreover, A cannot be negative because $|.|$ cannot be negative. Thus, we can assume $A > 0$. Next, fix $x_0 \in [a,b]$ and let $M_0 = \sup |f(x)|$ and $M_1 = \sup |f'(x)|$ for $a \le x \le x_0$. Next, we can use the mean value theorem, because f is differentiable and hence continuous, to obtain:

$$f'(x) = \frac{f(x_0) - f(a)}{x_0 - a} \tag{3.2}$$

$$f'(x)(x_0 - a) = f(x_0) \tag{3.3}$$

Therefore, since $|f'(x)| \leq \sup|f'(x)| = M_1$, we see that $f(x_0) \leq M(x_0 - a)$. Next, since we have $|f'(x)| \leq A|f(x)|$, we find that

$$|f(x)| \leq M_1(x_0 - a) \leq AM_0(x_0 - a) \tag{3.4}$$

Since we can pick any value for x_0, we can choose $x_0 - a < \frac{1}{A}$ such that $A(x_0 - a) < 1$. Then we see that $|f(x)| < A(x_0 - a)M_0$ for all $x \in [a, x_0]$. However, we can only have $M_0 = 0$ because otherwise a number stricly smaller than the supremum would be an upper bound, which shows that $f = 0$ on $[a, x_0]$. To show that $f = 0$ on $[x_0, b]$, we note that we can fix $x_0^1 \in [x_0, b]$ such that $|f(x)| \leq AM_0(x_0^1 - x_0)$. Repeating the same argument, we see that $f = 0$ on $[a, x_0] \cup [x_0, x_0^1]$. Since $[x_0, x_0^1]$ is a fixed interval, we can see that using the Archimedean principle, we will eventually cover $[a, b]$ with enough intervals $[x_0^n, x_0^{n+1}]$. Thus, we see that $f(x) = 0$ for all $x \in [a, b]$. □

4. Problem 5.27

Theorem 4.1. *Let ϕ be a real function defined on a rectangle R in the plane, given by $a \leq x \leq b$, $\alpha \leq y \leq \beta$. A solution of the initial value problem $y' = \phi(x, y), y(a) = c, (\alpha \leq c \leq \beta)$ is by definition a differentiable function f on $[a, b]$ such that $f(a) = c, \alpha \leq f(x) \leq \beta$, and $f'(x) = \phi(x, f(x))$ for $(a \leq x \leq b)$. Prove that such a problem has at most one solution if there is a constant A such that $|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$ whenever $(x, y_1) \in \mathbb{R}$ and $(x, y_2) \in \mathbb{R}$.*

Proof. Assume we have two solutions $f_1(x)$ and $f_2(x)$. We will show that they are equal by defining the function $g(x) = f_2(x) - f_1(x)$. Then since both of the solutions are such that $f_2(a) = f_1(a) = c$, we know that $g(a) = f_2(a) - f_1(a) = 0$. Next, since we have $f_1'(x) = \phi(x, f_1(x))$ and $f_2'(x) = \phi(x, f_2(x))$, we know that by the assumed condition, we have:

$$|g'(x)| = |\phi(x, f_2(x)) - \phi(x, f_1(x))| = |f_2'(x) - f_1'(x)| \leq A|f_2(x) - f_1(x)| \tag{4.2}$$

Thus, we see that $|g'(x)| \leq A|g(x)|$, so that g satisfies the conditions of problem 5.26 above. This means that we have $g(x) = 0$ for all $x \in [a, b]$. Thus, we see that $f_2(x) = f_1(x)$ for all $x \in [a, b]$, and that the two solutions are actually the same. Therefore, the problem has at most one solution. □

5. Problem 6.1

Theorem 5.1. *Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathbb{R}(\alpha)$.*

Proof. Fix $\epsilon > 0$. Since we know that α is continuous at x_0, we know that $|\alpha(x) - \alpha(x_0)| < \epsilon$ if $|x - x_0| < \delta$ for all $x \in [a, b]$. Thus, choose some partition $P = \{a = x_0 < \ldots < x_{i-1} < x_i < \ldots < x_n < b\}$ for $[a, b]$ and let $x_0 \in [x_{i-1}, x_i]$ the be interval in which x_0 lies. We can choose a particular interval $[x_{i-1}, x_i]$ such that $|x_{i-1} - x_0| < \delta/2$ and $|x_i - x_0| < \delta/2$. Moreover, we see that for this interval, we have:

$$\sup_{x \in [x_{i-1}, x_i]} f(x) = 1 \qquad \inf_{x \in [x_{i-1}, x_i]} f(x) = 0 \tag{5.2}$$

Because we know that $f(x_0) = 1$ but at all other points in the interval $f(x) = 0$. Next, we see that for the other intervals, we have:

$$\sup_{x \in [x_{j-1}, x_j]} f(x) = \inf_{x \in [x_{j-1}, x_j]} f(x) = 0 \qquad (0 \leq j \neq i \leq n) \tag{5.3}$$

Therefore, we see that $m_j = \inf_{x \in [x_{j-1}, x_j]} f(x) = M_j = \sup_{x \in [x_{j-1}, x_j]} f(x)$. This means that $M_j - m_j = 0$ for all $j \neq i$ and $0 \leq j \leq n$. Thus, we have the following:

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{j=0}^{n} (M_j - m_j)\Delta\alpha_j \tag{5.4}$$

$$= \Delta\alpha_i \tag{5.5}$$

$$= \alpha(x_i) - \alpha(x_{i-1}) \tag{5.6}$$

By the triangle inequality, we know that $|\alpha(x_i) - \alpha(x_{i-1})| \leq |\alpha(x_i) - \alpha(x_0)| + |\alpha(x_0) - \alpha(x_{i-1})|$. Since we have chosen $|x_{i-1} - x_0| < \delta/2$ and $|x_i - x_0| < \delta/2$, we have by continuity that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon/2 + \epsilon/2 = \epsilon$, which shows that $f \in \mathbb{R}(\alpha, [a, b])$. □

Theorem 5.7. *Prove that $\int f d\alpha = 0$.*

Proof. We know that we must have:

$$\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha \tag{5.8}$$

Since we know the definition of each of these upper and lower integrals, we can write out the following:

$$\underline{\int_a^b} f d\alpha = \inf U(P, f, \alpha) = 0 \tag{5.9}$$

$$\overline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0 \tag{5.10}$$

Therefore, since we know it exists, we see that $\int f d\alpha = 0$. □

6. Problem 6.2

Theorem 6.1. *Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$.*

Proof. Assume the contrary and fix $\epsilon > 0$. Then for some $x_0 \in [a, b]$, we have $f(x_0) > 0$ (since we have assumed $f(x) > 0$ as well). We know that f is continuous, so that $|f(x) - f(x_0)| < \epsilon$ if $0 < |x - x_0| < \delta$. Since $\int_a^b f(x)dx$ exists, we can choose any partition $P = \{a = x_0 < \ldots < x_{i-1} < x_j < \ldots < x_n = b\}$ such that $0 < |x_{i-1} - x_0| < \delta/2$ and $0 < |x_i - x_0| < \delta/2$. Moreover, we know the following must be true:

$$0 = \int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx \tag{6.2}$$

This means that $0 = \sup L(P, f) = \inf U(P, f)$ over all the possible partitions P of $[a, b]$. Thus, for every possible partition, we must have:

$$0 = \sum_{j=1}^{n} M_j \Delta x_j = \sum_{j=1}^{n} m_j \Delta x_j \tag{6.3}$$

Particularly, since $M_j = \sup f(x)$ for $x_{j-1} \leq x \leq x_j$, and we know that $\sum_{j=1}^n \Delta x_j = a - b \neq 0$, we must have $M_j = 0$ for all j such that $|x_{j-1} - x_j| \neq 0$. However, we have constructed a partition P such that $0 < |x_{i-1} - x_i| < \delta$ and where $f(x_0) > 0$ for some $x_0 \in [x_{i-1}, x_i]$, which means that $M_i = f(x_0) > 0$. This is a contradiction because we have shown all $M_j = 0$. Therefore, we must have $f(x) = 0$ for all $x \in [a, b]$. □

7. Problem 6.3

Theorem 7.1. *Define three functions $\beta_1, \beta_2, \beta_3$ as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta_1(0) = 0, \beta_2(0) = 1, \beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1, 1]$. Prove that $f \in \mathbb{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then $\int f d\beta_1 = f(0)$.*

Proof. Consider the partition $P = \{x_0, x_1, x_2, x_3\}$ where $x_0 = -1$ and $x_1 = 0 < x_2 < x_3 = 1$. Then $U(P, f, \alpha) = M_2$ and $L(P, f, \alpha) = m_2$. Here, we denote $M_2 = \sup_{x \in [0, x_2]} f(x)$ and $m_2 = \inf_{x \in [0, x_2]} f(x)$. Thus, we only need to have knowledge of the interval $[0, x_2]$, which approaches 0 from the right. If $f(0+) = f(0)$, then we see that $M_2, m_2 \to f(0)$ as $x_2 \to 0$. Therefore $f \in \mathbb{R}(\beta_1)$.

To prove the converse, assume the contrary. If $f(0+) \neq f(0)$, then either M_2 or m_2 does not converge to $f(0)$ as $x_2 \to 0$, which is a contradiction of the assumption that $f \in \mathbb{R}(\beta_1)$. Therefore, we must have $f(0+) = f(0)$. Finally, note that in the course of this proof, we have shown that $\int_{-1}^1 f d\beta_1 = f(0)$ because $M_2 = m_2 = f(0)$ as $x_2 \to 0$. □

Theorem 7.2. *Prove that $f \in \mathbb{R}(\beta_2)$ if and only if $f(0-) = f(0)$ and that then $\int f d\beta_2 = f(0)$.*

Proof. Take the partition $P = \{x_0, x_1, x_2, x_3\}$ where $-1 = x_0 < x_1 < x_2 = 0$ and $x_3 = 1$. Thus, it is clear that $\Delta\beta_{2,i} = 0$ for all i except $i = 2$. For $i = 2$, we see that $\Delta\beta_{2,2} = \beta_2(x_2) - \beta_2(x_1) = 1$. Therefore $U(P, f, \beta_2) = M_2$ and $L(P, f, \beta_2) = m_2$. If $f(0-) = f(0)$, then $M_2, m_2 \to f(0)$ as $x_1 \to 0-$, which shows that $f \in \mathbb{R}(\beta_2)$.

To show the converse, we can assume the contrary, and we see that if $f(0-) \neq f(0)$, then either M_2 or m_2 does not converge to $f(0)$ as $x_1 \to 0-$. This is a contradiction because we assumed $f \in \mathbb{R}(\beta_2)$, so we must have $f(0-) = f(0)$. Like the above theorem, we have shown that $\int_{-1}^1 f d\beta_2 = f(0)$ because $M_2, m_2 \to f(0)$ as $x_2 \to 0$. □

Theorem 7.3. *Prove that $f \in \mathbb{R}(\beta_3)$ if and only if f is continuous at 0.*

Proof. Fix $\epsilon > 0$. If f is continuous at 0, then we have $|f(x) - f(0)| < \epsilon$ if $|x| < \delta$ for all $x \in [-1, 1]$. Now take the partition $P = \{x_0, x_1, x_2, x_3, x_4\}$ such that $-1 = x_0 < x_1 < x_2 = 0 < x_3 < x_4 = 1$. Thus, we see that the only two indices for which $\Delta\beta_{3,i} \neq 0$ are $i = 2, 3$. We have $\Delta\beta_{3,2} = \beta_3(x_2) - \beta_3(x_1) = \Delta\beta_{3,3} = \beta_3(x_3) - \beta_3(x_2) = 1/2$. Therefore, we can see that $L(P, f, \beta_3) = (m_2 + m_3)/2$ and $U(P, f, \beta_3) = (M_2 + M_3)/2$. Since f is continuous, we know that as $x_1 \to 0-$ and $x_3 \to 0+$, we have:

$$U(P, f, \beta_3) - L(P, f, \beta_3) = \frac{1}{2}(M_2 - m_2) + \frac{1}{2}(M_3 - m_3) \tag{7.4}$$

$$= \frac{1}{2}\left(\sup_{x \in [x_1, 0]} f(x) - \inf_{x \in [x_1, 0]} f(x)\right) + \frac{1}{2}\left(\sup_{x \in [0, x_3]} f(x) - \inf_{x \in [0, x_3]} f(x)\right) \tag{7.5}$$

$$< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon \tag{7.6}$$

Therefore, we see that $f \in \mathbb{R}(\beta_3)$.

Next, if we assume $f \in \mathbb{R}(\beta_3)$, we know that $U(P, f, \beta_3) - L(P, f, \beta_3) < \epsilon$. Considering the same partition as before, it is clear that must have $f(x) \to f(0)$ as $x \to 0$ as $x_1 \to 0-$ and $x_3 \to 0+$. This is because we can assume the contrary and say that either $f(0-) \neq f(0)$ or $f(0+) \neq f(0)$. Then we would see that either $M_2 - m_2$ or $M_3 - m_3$ does not converge to zero, so that $U(P, f, \beta_3) - L(P, f, \beta_3)$ does not converge to 0, which is a contradiction. ☐

Theorem 7.7. *If f is continuous at 0 prove that $\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0)$.*

Proof. If f is continuous at 0, then $f(0) = f(0-) = f(0+) = \lim_{x \to 0} f(x)$. This means that $f \in \mathbb{R}(\beta_1)$ by part 1 and $f \in \mathbb{R}(\beta_2)$ by part 2. The third part shows that $f \in \mathbb{R}(\beta_3)$ by continuity of f at 0. Thus, all the above integrals exist. Moreover, parts 1 and 2 show that $\int f d\beta_1 = \int f d\beta_2 = f(0)$. We have seen that part 3 implies $f(0-) = f(0+) = f(0)$, which also shows that $U(P, f, \beta_3) = \frac{1}{2}(M_2 + M_3) \to f(0)$ and $U(P, f, \beta_3) = \frac{1}{2}(m_2 + m_3) \to f(0)$ as $x_1 \to 0-$ and $x_3 \to 0+$. Since we have:

$$L(P, f, \beta_3) \leq \underline{\int_{-1}^{1}} f d\beta_3 \leq \overline{\int_{-1}^{1}} f d\beta_3 \leq U(P, f, \beta_3) \tag{7.8}$$

We know that as $x_1 \to 0-$ and $x_3 \to 0+$, we must have $\int_{-1}^{1} f d\beta_3 = f(0)$. ☐