18.100B
FINAL EXAM STUDY GUIDE

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1. Problem 1

Theorem 1.1. *Show that* $\sup A = \sqrt{2}$ *where* $A = \{x \in \mathbb{Q} : x > 0, x^2 < 2\}$.

Proof. First, we define $\sup A = s$ and note that $s^2 \geq 2$. Suppose not, then $s < 2$ and there exists an $\epsilon > 0$ such that $s + \epsilon \in A$, which contradicts the fact that s is the least upper bound. Now suppose that $s^2 > 2$. Then we can always find some $\delta > 0$ such that $s^2 - \delta > 2$, so that $s^2 - \delta \notin A$ and $s^2 - \delta > x$ for all $x \in A$, which is a contradiction of s being a supremum. Thus, we must have $s^2 = 2$, or by the uniqueness of radicals, $s = \sqrt{2}$. $\square$

2. Problem 2

Theorem 2.1. *Let* $\mathbf{A} \subset \mathbb{R}$ *be a nonempty set. Define* $-\mathbf{A} = \{x : -x \in \mathbf{A}\}$. *Show that* $\sup(-\mathbf{A}) = -\inf \mathbf{A}$ *and* $\inf(-\mathbf{A}) = -\sup \mathbf{A}$.

Proof. Suppose that A is bounded from below and let $a = \inf \mathbf{A}$. Then we must have $a \leq x$ for all $x \in \mathbf{A}$ and $-a \geq y$ for all $y \in -\mathbf{A}$. Therefore, we see that $y \leq -a \leq a \leq x$. Now, suppose that there exists some $\epsilon > 0$ such that $-a - \epsilon > y$ for all $y \in -\mathbf{A}$. Then, we must have $a + \epsilon < x$ for all $x \in \mathbf{A}$. Therefore, we see that a is not the infimum of $\mathbf{A}$, which is a contradiction. Therefore, no such epsilon exists, so that $-a$ is the supremum of $-\mathbf{A}$. If $\mathbf{A}$ is not bounded from below, then $\inf \mathbf{A} = -\infty$, so that $\sup(-\mathbf{A}) = \infty$. Therefore, we have shown that $\sup(-\mathbf{A}) = -\inf \mathbf{A}$. The proof that $\inf(-\mathbf{A}) = -\sup \mathbf{A}$ is similar. $\square$

3. Problem 3

Theorem 3.1. *Let* $A, B \subset \mathbb{R}$ *be nonempty. Define* $A + B = \{z = x + y : x \in A, y \in B\}$ *and* $A - B = \{z = x - y : x \in A, y \in B\}$. *Show that* $\sup(A + B) = \sup A + \sup B$ *and* $\sup(A - B) = \sup A - \inf B$.

Proof. Let A and B be bounded from above, and define $a = \sup A$ and $b = \sup B$, and $s = \sup(A + B)$. By the definition of supremum, we see that $a \geq x - \epsilon/2$ for all $x \in A$ and $b \geq y - \epsilon/2$ for all $y \in B$. Therefore, we see that $a + b \geq x + y - \epsilon$. Since $z = x + y \in A + B$, we have shown that $s = a + b$.

For the next part, we choose the set $C = -B = \{y : -y \in B\}$. We have previously shown that $\sup C = -\inf B$. Moreover, we see that $\sup(A - B) = \sup(A + C)$ because $A - B = \{z = x - y : x \in A, y \in B\} = \{z = x + y : x \in A, -y \in B\} = A + C$. We have just shown that $\sup(A + C) = \sup A + \sup C$, and since $\sup C = -\inf(B)$, we have $\sup(A - B) = \sup(A + C) = \sup A - \inf B$. $\square$

4. Problem 4

Theorem 4.1. *Let* A, B *be nonempty subsets of real numbers. Show that* $\sup(A \cup B) = \max\{\sup A, \sup B\}$ *and* $\inf(A \cup B) = \min\{\inf A, \inf B\}$.

Proof. First, suppose that A and B are bounded above so that $\sup A = a$ and $\sup B = b$. Then we can assume without loss of generality that $a \geq b$. Thus, we see that $a \geq x - \epsilon$ for all $x \in A$ and some $\epsilon > 0$. Moreover, we see that $a \geq b$ implies that $a \geq y - \epsilon$ for all $y \in B$ and some $\epsilon > 0$. This implies that $a = \sup(A \cup B)$. If either A or B is unbounded, then $\sup A = \infty$ or $\sup B = \infty$ and $\sup(A \cup B) = \infty$. This proves the theorem for $\sup(A \cup B) = \max\{\sup A, \sup B\}$, and the proof for $\inf(A \cup B) = \min\{\inf A, \inf B\}$ is similar. $\square$

5. Problem 5

Theorem 5.1. *Prove that if a sequence $\{a_n\}$ is monotonically increasing then* $\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$.

Proof. If the sequence $\{a_n\}$ is unbounded, we see that $\lim_{n\to\infty} a_n = \infty$ and $\sup\{a_n : n \in \mathbb{N}\} = \infty$. If the sequence $\{a_n\}$ is bounded, we can assume that $\sup\{a_n : n \in \mathbb{N}\} = a$. Since $\{a_n\}$ is monotonically increasing, we see that $a_{n+1} \geq a_n$. Moreover, we see that by definition of upper bound that $a_n \leq a$ for all $n \in \mathbb{N}$. This means that for each $\epsilon > 0$, we can find some a_{n_0} such that $a - \epsilon < a_{n_0}$ because $a - \epsilon$ is not an upper bound. Therefore, we see that $a > a_{n_0} > a - \epsilon$, which implies that $|a_{n_0} - a| < \epsilon$. Thus, we have shown that $\lim_{n\to\infty} a_n = a$. □

6. Problem 6

Theorem 6.1. *Let $a_1, a_2, \ldots, a_p$ be fixed positive numbers and consider the sequence $s_n = \frac{a_1^n + a_2^n + \ldots a_p^n}{p}$ and $x_n = \sqrt[n]{s_n}$. Prove that x_n is monotonically increasing.*

Proof. First we will show that $\frac{s_n}{s_{n+1}}$ is monotonically increasing. If each $a_1, \ldots a_p \leq 1$, then we see that $a_i^n \geq a_i^{n+1}$, which means that $s_n \geq s_{n+1}$. This means that $\frac{s_n}{s_{n+1}} \leq \frac{s_{n+1}}{s_{n+2}}$ for $n \geq 2$. The same is true when some a_i is greater than 1. Therefore, $\frac{s_n}{s_{n+1}}$ is monotonically increasing, and so $s_n^2 \leq s_{n+1}s_{n-1}$.

We know that $x_1 \leq x_2$ because:

$$\left(\sum_{i=1}^{p} a_i\right)^2 \leq \sum_{i=1}^{p} (a_i)^2 \tag{6.2}$$

Now assume $x_{n-1} \leq x_n$, so that $s_{n-1} \leq s_n^{\frac{n-1}{n}}$. We will show that $x_n \leq x_{n+1}$:

$$x_{n+1} = \sqrt[n+1]{s_{n+1}} \geq \sqrt[n+1]{\frac{s_n^2}{s_{n-1}}} \geq \sqrt[n+1]{\frac{s_n^2}{s_n^{\frac{n-1}{n}}}} = \sqrt[n+1]{s_n^{1+\frac{1}{n}}} = s_n^{\frac{1+\frac{1}{n}}{n+1}} = s_n^{\frac{1}{n}} = x_n \tag{6.3}$$

Since $x_{n+1} \geq x_n$, we have proven that $\{x_n\}$ is monotonically increasing by induction. □

7. Problem 7

Theorem 7.1. *Let $\{a_n\}$ be a bounded sequence which satisfies the condition $a_{n+1} \geq a_n - \frac{1}{2^n}$ for $n \in \mathbb{N}$. Show that the sequence $\{a_n\}$ is convergent.*

Proof. Since $\{a_n\}$ is bounded, we know that $|a_n| < M$ for some $M > 0$ for all $n \in \mathbb{N}$. Moreover, the assumption shows that $\frac{1}{2^n} \geq a_n - a_{n+1}$. This means that $|a_n - a_{n+1}| \leq \frac{1}{2^n}$. Particularly, we know that $\frac{1}{2^n} \to 0$ as $n \to \infty$. This tells us that $|a_n - a_{n+1}| \to 0$ as $n \to \infty$. By the Cauchy criterion, we see that $\{a_n\}$ converges. □

8. Problem 8

Theorem 8.1. *Establish the convergence and find the limit of the sequence defined by $a_1 = 0$ and $a_{n+1} = \sqrt{6 + a_n}$ for $n \geq 1$.*

Proof. First we note that $a_1 = 0$ and $a_2 = \sqrt{6}$. We know that since $3^2 = 9$ that $a_2 = \sqrt{6} \leq 3$, so that $0 < a_2 < 3$. This implies $6 < a_2 + 6 < 9$, and since square root preserves ordering, we have $\sqrt{6} < \sqrt{a_2 + 6} < 3$. Therefore, we see that $a_2 < a_3 < 3$. Continuing this process, we see that $\sqrt{6 + \sqrt{6}} < a_4 < 3$, which implies $a_3 < a_4 < 3$. We see that this process continues indefinitely, and that $a_n < a_{n+1} < 3$. This means that $\{a_n\}$ is monotonically increasing and bounded from above by 3. First, this establishes the convergence of $\{a_n\}$. Next, this shows that $\{a_n\} \to 3$ as $n \to \infty$. This is because for any $\epsilon > 0$, we can choose N large enough so that $|3 - a_N| < \epsilon$ because $\{a_n\}$ is monotonically increasing. □

9. Problem 9

Theorem 9.1. *Show that the sequence defined by $a_1 = 0$, $a_2 = \frac{1}{2}$, and $a_{n+1} = \frac{1}{3}(1 + a_n + a_{n-1}^3)$ for $n > 1$ converges and determine its limit.*

Proof. We see that $0 \leq a_n \leq 1$ because $1 + a_n + a_{n-1}^3$ is always less than 3 for our given starting values. First, we will show convergence by assuming $a_n \geq a_{n-1}$:

$$a_{n+1} - a_n = \frac{1}{3}(1 + a_n + a_{n-1}^3) - a_n = \frac{1}{3}\left(1 - \frac{2}{3}a_n - a_{n-1}^3\right) \tag{9.2}$$

Since $a_n \leq 1$ and $a_{n-1} \leq a_n$, we see that $a_{n+1} - a_n \geq \frac{1}{3}(1 - a_n(\frac{2}{3} + a_n^2)) \geq \frac{1}{3}(1 - 1(\frac{2}{3} + 1)) \geq 0$. Therefore, we see that $a_{n+1} \geq a_n$ if we assume that $a_n \geq a_{n+1}$. This shows that $\{a_n\}$ is monotonically increasing and bounded, which implies convergence. Moreover, we see that $\{a_n\} \to 1$ as $n \to \infty$ because for every $\epsilon > 0$ we can always choose an N such that $|1 - a_N| < \epsilon$. □

10. Problem 10

Theorem 10.1. *Let $\{a_n\}$ be defined recursively by $a_{n+1} = \frac{1}{4-3a_n}$ for $n \geq 1$. Determine for which a_1 the sequence converges and in the case of convergence find its limit.*

Proof. We will show by induction that the following is true:

$$a_n = \frac{(3^{n-1} - 1) - (3^{n-1} - 3)a_1}{(3^n - 1) - (3^n - 3)a_1} \tag{10.2}$$

First, it is clear that this follows for $a_2 = \frac{3-1-(3-3)a_1}{9-1-(9-3)a_1} = \frac{2}{8-6a_1} = \frac{1}{4-3a_1}$. Thus, we have established the base case. Now, we will show that this works for a_{n+1}:

$$a_{n+1} = \frac{1}{4 - 3a_n} = \frac{(3^n - 1) - (3^n - 3)a_1}{4((3^n - 1) - (3^n - 3)a_1) - 3((3^{n-1} - 1) - (3^{n-1} - 3)a_1)} \tag{10.3}$$

$$= \frac{(3^n - 1) - (3^n - 3)a_1}{(3^{n+1} - 1) - (3^{n+1} - 3)a_1} \tag{10.4}$$

Therefore, we see that the sequence does indeed converge as $n \to \infty$ for $a_1 \neq \frac{3^n-1}{3^n-3}$ for all $n \in \mathbb{N}$. If $a_1 = 1$, then $a_n = 1$. For all other allowable values of a_1, we have $a_n \to \frac{1}{3}$ as $n \to \infty$. □

11. Problem 11

Theorem 11.1. *Prove the convergence of $\{a_n\}$ defined inductively by $a_1 = 2$ and $a_{n+1} = 2 + \frac{1}{3+\frac{1}{a_n}}$ for $n \geq 1$ and find its limit.*

Proof. First, note that the series is monotonically increasing. This is because of the following:

$$a_{n+1} - a_n = 2 + \frac{1}{3 + \frac{1}{a_n}} - a_n = 2 + \frac{a_n}{3a_n + 1} - a_n = 2 + \frac{-3a_n^2}{3a_n + 1} \tag{11.2}$$

And this is negative only for $a_n > \frac{1}{3}(3 + \sqrt{15})$. Since we begin at $2 < \frac{1}{3}(3 + \sqrt{15})$, it is clear that $a_{n+1} - a_n > 0$ for all $n \in \mathbb{N}$, so that $\{a_n\}$ is monotonically increasing. Moreover, we can show that the sequence is bounded, for instance by the value 3. We know that a_n cannot be greater than 3, because if it were, then $a_{n+1} - a_n < 0$ would imply a monotonically decreasing function, a contradiction of our previous assertion. Therefore, since the sequence is monotonic and bounded, it converges. To find its limit s, we set $s = 2 + \frac{1}{3+\frac{1}{s}}$ and find that $s = \frac{1}{3}(3 + \sqrt{15})$. □

12. Problem 12

Theorem 12.1. *The recursive sequence $\{a_n\}$ is given by setting $a_1 = 1, a_2 = 2, a_{n+1} = \sqrt{a_{n-1}} + \sqrt{a_n}$ for $n \geq 2$. Show that the sequence is bounded and strictly increasing. Find its limit.*

Proof. First, we note that $a_3 = \sqrt{2} + 1$ so that $a_1 < a_2 < a_3$. Now, we find that $a_{n+1} - a_n = \sqrt{a_{n-1}} + \sqrt{a_n} - \sqrt{a_{n-2}} - \sqrt{a_{n-1}}$. Assume that $a_n > a_{n-1} > a_{n-2} > \ldots$. Then we know that $\sqrt{a_n} - \sqrt{a_{n-1}} > 0$ because $a_n > a_{n+1}$. Also, we see that $\sqrt{a_{n-1}} - \sqrt{a_{n-2}} > 0$ because $a_{n-1} > a_{n-2}$. This shows that $a_{n+1} - a_n > 0$, which implies that $a_{n+1} > a_n$. Thus, we have shown using induction that $\{a_n\}$ is strictly increasing.

Moreover, we can see that the sequence is bounded by 4. This is because the function $f(x) = 2\sqrt{x} - x$ is negative when $x > 4$. Since $a_n > a_{n-1}$, we see that $a_{n+1} < 2\sqrt{a_n}$, which implies $a_{n+1} - a_n < 2\sqrt{a_n} - a_n$. However, since $f(x) < 0$ when $x > 4$, we see that $2\sqrt{a_n} - a_n < 0$ which shows that $a_{n+1} - a_n < 0$, which is a contradiction of the fact that $\{a_n\}$ is strictly increasing. This shows that a_n cannot be greater than 4, and thus shows that $\{a_n\}$ is bounded. Since it is also strictly increasing, we know that the sequence converges so some number s. We can find this by setting $s = \sqrt{s} + \sqrt{s} = 2\sqrt{s}$. Solving this equation, we obtain $s = 4$. □

13. Problem 13

Theorem 13.1. *Suppose that a bounded sequence $\{a_n\}$ is such that $a_{n+2} \leq \frac{1}{3}a_{n+1} + \frac{2}{3}a_n$ for $n \geq 1$. Prove the convergence of the sequence.*

Proof. By the assumption, we have $a_{n+2} + \frac{2}{3}a_{n+1} \leq a_{n+1} + \frac{2}{3}a_n$. We can set $b_n = a_{n+1} + \frac{2}{3}a_n$, and we see that this sequence is decreasing. This is because $b_{n+1} = a_{n+2} + \frac{2}{3}a_{n+1} \leq \frac{1}{3}a_{n+1} + \frac{2}{3}a_n + \frac{2}{3}a_{n+1} = a_{n+1} + \frac{2}{3}a_n = b_n$, which shows that $b_{n+1} \leq b_n$ for all $n \geq 1$. Moreover, since a_n is a bounded sequence, we know that b_n is also bounded. This shows that b_n converges to some limit b. Now set $a = \frac{3}{5}b$. Fix $\epsilon > 0$, we see that there exists an $N > 0$ such that for all $n > N$, we have $|b_n - b| < \epsilon$. This corresponds to $|a_{n+1} + \frac{2}{3}a_n - \frac{5}{3}a| < \epsilon$. By the triangle inequality, we have $|a_{n+1} - a| \leq |a_{n+1} + \frac{2}{3}a_n - \frac{5}{3}a| + |-\frac{2}{3}a_n + \frac{5}{3}a - a| = |a_{n+1} - \frac{2}{3}a_n - \frac{5}{3}a| + \frac{2}{3}|-a_n + a|$. Rearranging, we obtain: $|a_{n+1} - a| - \frac{2}{3}|a_n - a| \leq |a_{n+1} - \frac{2}{3}a_n - \frac{5}{3}a| < \epsilon$. Therefore, we get $|a_{n+1} - a| < \epsilon + \frac{2}{3}|a_n - a|$. Using induction, we get:

$$|a_{n_0+k} - a| \leq \left(\frac{2}{3}\right)^k |a_{n_0} - a| + \left(\left(\frac{2}{3}\right)^{k-1} + \ldots + \frac{2}{3} + 1\right)\epsilon \tag{13.2}$$

$$\leq \left(\frac{2}{3}\right)^k |a_{n_0} - a| + \frac{1 - (\frac{2}{3})^k}{1 - \frac{2}{3}}\epsilon \tag{13.3}$$

$$< \left(\frac{2}{3}\right)^k |a_{n_0} - a| + 3\epsilon \tag{13.4}$$

Thus, we see that $|a_n - a| < \epsilon$ for $n > N$. This shows that $\{a_n\} \to a$ as $n \to \infty$. □

14. Problem 14

Theorem 14.1. *Calculate* $\lim_{n\to\infty}(\sqrt[n]{n} - 1)^n$.

Proof. First, note that for $n \geq 3$, the sequence $\{a_n\}$ for $a_n = \sqrt[n]{n} - 1$ is monotonically decreasing. Indeed, we see that $a_{n+1} - a_n = \sqrt[n+1]{n+1} - \sqrt[n]{n}$, and we know that $f(x) = x^{1/x}$ has a derivative of $f'(x) = -x^{1/x-2}(\ln(x) - 1)$. Since $f'(x) < 0$ when $\ln(x) - 1 > 0$, we see that $f'(x) < 0$ when $x > e$. Therefore, we know that $f(x_1) - f(x_2) < 0$ when $e < x_2 < x_1$. It follows that $\{a_n\}$ is monotonically decreasing for $n \geq 3$.

Next, we know that $\sqrt[n]{n} - 1 < 1/2$ for $n \geq 3$ because $3^{1/3} - 1 < \frac{1}{2}$ and $\sqrt[n]{n} - 1$ is monotonically decreasing. Therefore we see that $\lim_{n\to\infty}(\sqrt[n]{n} - 1)^n < \lim_{n\to\infty}(1/2)^n = 0$. Since $\sqrt[n]{n} \to 1$ as $n \to \infty$, we see that 0 is also a lower bound for this expression. By the squeeze law, we see that $\lim_{n\to\infty}(\sqrt[n]{n} - 1)^n = 0$. □

15. Problem 15

Theorem 15.1. *Study the convergence of $a_1 = a, a_n = 1 + ba_{n-1}$ for $n \geq 2$.*

Proof. We see that $a_2 = 1 + ba, a_3 = 1 + b + b^2a$, and that by induction, $a_n = 1 + b + \ldots + b^{n-1}a$. Therefore, we see that $a_n = b^{n-1}a + \sum_{i=0}^{n-2} b^i$. This allows us to compute the following expression, using the formula for computing geometric sums:

$$a_{n+1} = \begin{cases} \frac{1}{1-b} + b^n(a - \frac{1}{1-b}) & b \neq 1 \\ n + a & b = 1 \end{cases} \tag{15.2}$$

Therefore, we see that if $b = 1$, $\{a_n\}$ does not converge. If $b \neq 1$ and $a = \frac{1}{1-b}$, the sequence converges to $\frac{1}{1-b}$. If $|b| < 1$, then the sequence converges to $\frac{1}{1-b}$. Otherwise, the sequence diverges. □

16. Problem 16

Theorem 16.1. *If $a_1, \ldots, a_n$ are positive real numbers such that $a_1a_2\ldots a_n = 1$, then $a_1 + a_2 + \ldots + a_n \geq n$.*

Proof. First, if $a_1, \ldots, a_n = 1$, then the inequality is trivial. So we can suppose that some $a_j < 1$, which also means that another $a_k > 1$. We can then reorder the numbers so that $a_1 < a_2 < \ldots < a_n$, with $a_1 < 1 < a_n$. We will prove the proposition using induction, starting with the trivial base case of $n = 1$, so that $a_1 = 1$. The inequality holds in this case, so we can suppose that it holds up to n. This means that if $a_1 \ldots a_n = 1$, then $a_1 + \ldots + a_n \geq 1$ by assumption. Since $a_1 < 1$ and $a_{n+1} > 1$, we see that $a_1 < a_1a_{n+1}$, so we can rearrange our assumption to get $a_2 + \dots a_n + a_1 a_{n+1} \geq n$. Adding a_1, a_{n+1} to both sides and moving $a_1 a_{n+1}$ to the right side of the inequality, we obtain $a_1 + \dots a_{n+1} \geq n - a_1 a_{n+1} + a_1 + a_{n+1}$. This becomes:

$$\begin{aligned} a_1 + \dots a_{n+1} &\geq n + a_1 - a_{n+1}(a_1 - 1) && (16.2)\\ &= n + a_1 - 1 - a_{n+1}(a_1 - 1) + 1 && (16.3)\\ &= n + 1 + (1 - a_{n+1})(a_1 - 1) && (16.4)\\ &\geq n + 1 && (16.5) \end{aligned}$$

Where the last inequality comes from the fact that $a_{n+1} > 1$ and $a_1 < 1$, so that $(1 - a_{n+1})(a_1 - 1) > 0$. This completes the proof by induction. □

17. Problem 17

Theorem 17.1. *Let* $A_n = \frac{a_1 + \dots + a_n}{n}, G_n = \sqrt[n]{a_1 \dots a_n}, H_n = \frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}}$ *be the arithmetic, geometric, and harmonic means respectively for* n *real positive numbers* $a_1, \dots, a_n$. *Prove that* $A_n \geq G_n \geq H_n$.

Proof. First, we will use Jensen's inequality for concave functions, in this case logarithms. We have:

$$\begin{aligned} \ln\left(\frac{a_1 + \dots + a_n}{n}\right) &\geq \frac{\ln(a_1) + \dots + \ln(a_n)}{n} && (17.2)\\ &= \frac{\ln(a_1 \dots a_n)}{n} && (17.3)\\ &= \ln \sqrt[n]{a_1 \dots a_n} && (17.4) \end{aligned}$$

Since $\ln(x)$ is an increasing function that preserves ordering, we see that $A_n \geq G_n$. We can then us what we have just derived and place $1/a_j$ into the inequality for $A_n \geq G_n$ to obtain:

$$\begin{aligned} \frac{\frac{1}{a_1} + \dots + \frac{1}{a_n}}{n} &\geq \sqrt[n]{\frac{1}{a_1 \dots a_n}} && (17.5)\\ \sqrt[n]{a_1 \dots a_n} &\geq \frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}} && (17.6) \end{aligned}$$

This gives $A_n \geq G_n \geq H_n$. □

18. Problem 18

Theorem 18.1. *Calculate* $\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + \frac{k}{n^2}} - 1$.

Proof. First, we will set $a_1 = 1 + x$ and $a_2 = 1$ and use the arithmetic-geometric-harmonic inequality derived above. Thus, we have $\frac{2}{\frac{1}{1+x} + 1} \leq \sqrt{1+x} \leq \frac{2+x}{2}$. We can split these into the following:

$$\begin{aligned} \frac{2}{\frac{1}{1+x} + 1} &= \frac{2(1+x)}{2+x} = \frac{x}{2+x} + \frac{1+x}{1+x} = 1 + \frac{x}{2+x} && (18.2)\\ \frac{2+x}{2} &= 1 + \frac{x}{2} && (18.3) \end{aligned}$$

Thus, by subtracting 1 from each part of the inequality, we can obtain the following bounds for $x \geq 0$:

$$\frac{x}{2+x} \leq \sqrt{1+x} - 1 \leq \frac{x}{2} \tag{18.4}$$

We can set $x = \frac{k}{n^2}$ and substitute it into the expression because $1 \leq k \leq n$ and $n \in \mathbb{N}$ so that $\frac{k}{n^2} > 0$. Moreover, since sums and limits preserve ordering, we obtain:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{2n^2 + k} \leq \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + \frac{k}{n^2}} - 1 \leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{2n^2} \tag{18.5}$$

Therefore, we can evaluate the upper and lower bounds on our expression. First, we will get another lower bound, because $k \le n$:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{2n^2+k} \le \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{2n^2+n} \tag{18.6}$$

$$= \lim_{n\to\infty}\frac{1}{2n(n+1)}\sum_{k=1}^{n}k \tag{18.7}$$

$$= \lim_{n\to\infty}\frac{1}{2n(n+1)}\frac{n(n+1)}{2} \tag{18.8}$$

$$= \lim_{n\to\infty}\frac{1}{4}=\frac{1}{4} \tag{18.9}$$

Now, we can go ahead and find the upper bound as well:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{2n^2} = \lim_{n\to\infty}\frac{1}{2n^2}\sum_{k=1}^{n}k \tag{18.10}$$

$$= \lim_{n\to\infty}\frac{1}{2n^2}\frac{n(n+1)}{2} \tag{18.11}$$

$$= \lim_{n\to\infty}\frac{n^2+n}{4n^2}=\frac{1}{4} \tag{18.12}$$

Now, we see that by squeeze law that we must have $\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt[n]{1+\frac{k}{n^2}}-1=\frac{1}{4}$. □

19. Problem 19

Theorem 19.1. *Given real* $x \ge 1$ *show that* $\lim_{n\to\infty}(2\sqrt[n]{x}-1)^n = x^2$.

Proof. First, we can find an upper found for the limit because $0 \le (\sqrt[n]{x}-1)^2 = \sqrt[n]{x^2}-2\sqrt[n]{x}+1$. This implies that $2\sqrt[n]{x}-1 \le \sqrt[n]{x^2}$. Moreover, since limit preserves ordering, we see that $\lim_{n\to\infty}(2\sqrt[n]{x}-1)^n \le \lim_{n\to\infty}(\sqrt[n]{x^2})^n = \lim_{n\to\infty}x^2 = x^2$. Thus, we have found an upper bound for the limit and must proceed to find a lower bound. We note the following:

$$\left(2\sqrt[n]{x}-1\right)^n = \left(\sqrt[n]{x^2}\left(\frac{2}{\sqrt[n]{x}}-\frac{1}{\sqrt[n]{x^2}}\right)\right)^n \tag{19.2}$$

$$= x^2\left(\frac{2}{\sqrt[n]{x}}-\frac{1}{\sqrt[n]{x^2}}+1-1\right)^n \tag{19.3}$$

$$= x^2\left(1+\frac{2\sqrt[n]{x}-1-\sqrt[n]{x^2}}{\sqrt[n]{x^2}}\right)^n \tag{19.4}$$

$$= x^2\left(1-\frac{(\sqrt[n]{x}-1)^2}{\sqrt[n]{x^2}}\right)^n \tag{19.5}$$

We know $x = (\sqrt[n]{x}+1-1)^n$. So we can use the Bernoulli inequality to obtain $x = (1+(\sqrt[n]{x}-1))^n \ge 1+n(\sqrt[n]{x}-1) \ge n(\sqrt[n]{x}-1)$. Rearranging, we see that $\frac{x}{n} \ge \sqrt[n]{x}-1$, so that $\frac{x^2}{n^2} \ge (\sqrt[n]{x}-1)^2$. Combining this with equation 19.5, and using the Bernoulli inequality, we see the following:

$$\lim_{n\to\infty}(2\sqrt[n]{x}-1)^n \ge \lim_{n\to\infty}x^2\left(1-n\frac{x^2}{n^2\sqrt[n]{x^2}}\right) \tag{19.6}$$

$$= \lim_{n\to\infty}x^2\left(1-\frac{x^2}{n\sqrt[n]{x^2}}\right)=x^2 \tag{19.7}$$

Thus, we see that our limit is bounded from above and below by x^2, so that $\lim_{n\to\infty}(2\sqrt[n]{x}-1)^n = x^2$. □

20. Problem 20

Theorem 20.1. *Find* $\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$.

Proof. Rewriting this expression and noticing that it is a telescoping sum, we obtain:

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{(n+1)^2} \tag{20.2}$$

$$= \lim_{n\to\infty} 1 - \frac{1}{(n+1)^2} = 1 \tag{20.3}$$

□

21. Problem 21

Theorem 21.1. *Find* $\sum_{n=1}^{\infty} \frac{n}{(2n-1)^2(2n+1)^2}$.

Proof. Again, we will rewrite the expression and notice that it is a telescoping sum. This yields:

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)^2(2n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{8}\left(\frac{1}{(2n+1)^2} - \frac{1}{(2n-1)^2}\right) \tag{21.2}$$

$$= \lim_{n\to\infty} \frac{1}{8}\left(1 + \frac{1}{(2n+1)^2}\right) = \frac{1}{8} \tag{21.3}$$

□

22. Problem 22

Theorem 22.1. *Show that a nonempty open subset of $(-1,1)$ can be written as an at most countable union of open intervals (a_n, b_n) where $(a_n, b_n) \cap (a_k, b_k) =$ if $k \neq n$.*

Proof. Let O be the nonempty open subset of $(-1,1)$. Note that the set of rationals is countable, so that $q \in \mathbb{Q} \cap O$ is at most countable. Next, we can let $O_q = \bigcup (a,b)$ for each rational q in O. Each O_q is open because it is the union of open sets and nonempty because it is the union of open intervals in $\mathbb{R}$. Now, we can take $a(q) = \inf O_q$ and $b(q) = \sup O_q$. We will show that $O_q = (a(q), b(q))$. For a fixed $\epsilon > 0$, we see that there exists an $a < a(q) + \epsilon$, so that $(a,q] \subset O_q$. Next, we see that there exists a $b > b(q) - \epsilon$ such that $[q,b) \subset O_q$. This shows that $O_q = (a(q), b(q))$. Next, if $q' \in O_q \cap \mathbb{Q}$, then we see that $O_{q'} = O_q$. This is because if O_q contains q' so that $O_q \subset O_{q'}$, but also that $O_{q'}$ contains q so that $O_{q'} \subset O_q$, which implies that $O_{q'} = O_q$. Therefore, we can take all $q \in \mathbb{Q} \cap O$ and enumerate them, dropping the repeated intervals. This assures that any $x \in O$ will be in some (a_n, b_n). □

23. Problem 23

Theorem 23.1. *Let $\sum_n a_n$ be a convergent series of positive terms. What can be said about the convergence of $\sum_n \frac{a_1 + \ldots a_n}{n}$?*

Proof. The sum is not convergence because a_1 is positive and $\frac{a_1}{n} < \frac{a_1 + \ldots + a_n}{n}$. This shows the following:

$$\sum_{n=1}^{\infty} \frac{a_1 + \ldots + a_n}{n} > \sum_{n=1}^{\infty} \frac{a_1}{n} \tag{23.2}$$

Since the last last term diverges, we see that the sum diverges as well by comparison test. □

24. Problem 24

Theorem 24.1. *Let $K \in \mathbb{R}^2$ be a closed and bounded set and let $f : [0,1] \times K \to \mathbb{C}$ be a continuous function on this subset of $\mathbb{R}^3$ (all with respect to the Euclidean metric). Show that the set of functions $\{g_s\}_{s \in [0,1]}$ where $g_s(x) = f(s,x)$ is equicontinuous on K.*

Proof. First, we note that $[0,1]$ is compact because it is closed, bounded, and hence compact by Heine Borel. Next, we see that $K \subset \mathbb{R}^2$ is also compact because it is closed and bounded. Since the cartesian product of two compact sets is also compact by a theorem in Rudin, we see that $[0,1] \times K$ is compact. Since f is a continuous function on a compact set, we see that it is uniformly continuous. This means that for $\epsilon > 0$, there exists $\delta > 0$ such that for all $s_1, s_2 \in [0,1]$ and $x, y \in K$, we have $||f(s_1, x) - f(s_2, y)|| < \epsilon$ for $\sqrt{|s_1 - s_2|^2 + |x-y|^2} < \delta$. Therefore, we see that this same $\delta > 0$ suffices to show that for all $s \in [0,1]$ and all $x, y \in K$, we have $|g_s(x) - g_x(y)| < \epsilon$ for $|x - y| < \delta$, which proves equicontinuity. □

25. Problem 25

Theorem 25.1. *Show that in any metric space, the closure of a connected set is connected.*

Proof. Let E be a connected set and $\bar{E}$ be its closure. Suppose the contrary, so that $\bar{E}$ is separated and that $\bar{E} = C \cup D$ such that $\bar{C} \cap D = C \cap \bar{D} = \emptyset$ and C, D are nonempty. Now, let E' be the set of limit points of E. We see that $E = A \cup B$, where $A = C \cap E'$ and $B = D \cap E'$. However, since $A \subset C$ and $B \subset D$, we see that $\bar{A} \cap B = A \cap \bar{B} = \emptyset$. Moreover, we see that A is nonempty, because if it were empty, then $A = C \cap E' = \emptyset$, which implies that $C \cap \bar{E} = \emptyset$, which would imply that $\bar{E} = D$ and $C = \emptyset$, which is a contradiction of our assumptions. By the same logic, B is nonempty, so that we have shown a contradiction because $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ implies that E is separated. □

26. Problem 26

Theorem 26.1. *If $f : [0, 2] \to \mathbb{R}$ is continuous, state a theorem which shows that $F(x) = \int_0^x f(s)ds$ is differentiable on $[0, 2]$ (or prove it directly) and show that there exists $c \in (0, 2)$ such that $\int_0^2 f(x)dx = 2f(c)$.*

Proof. First, we state the fundamental theorem of calculus, which says that for every function $f \in \mathscr{R}([,])$, the integral $F(x) = \int_a^b f(s)ds$ exists and has a derivative at every continuous point of f which is equal to $F'(x) = f(x)$. Next, we can use the above statement and the mean value theorem to show:

$$F(2) - F(0) = F'(c)(2 - 0) \quad \Rightarrow \quad F(2) = \int_0^2 f(s)ds = 2f(c) \tag{26.2}$$

This shows that $\int_0^2 f(x)dx = 2f(c)$ for some $c \in (0, 2)$. □

27. Problem 27

Theorem 27.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable and suppose that 0 is a local minimum of f, i.e. for some $\epsilon > 0$ $f(x) \geq f(0)$ for all $x \in (-\epsilon, \epsilon)$. Show that if $f''(0) > 0$ then 0 is a strict local minimum in the sense that there exists $\epsilon > 0$ such that $f(x) \geq f(0)$ for $0 \neq x \in (-\epsilon, \epsilon)$.*

Proof. Since 0 is a local minimum we know that $f'(0) = 0$ by a theorem in Rudin. Fix $\epsilon > 0$ such that $f(x) \geq f(0)$ for all $x \in (-\epsilon, \epsilon)$. We know that by definition $f''(0) = \lim_{t\to 0} \frac{f'(t)-f'(0)}{t} = \lim_{t\to 0} \frac{f'(t)}{t}$. Since we have assumed that $f''(0) > 0$, we see that $\lim_{t\to 0} \frac{f'(t)}{t} > 0$, which means in particular that $f'(t) > 0$ for $\epsilon > t > 0$ and $f'(t) < 0$ for $-\epsilon < t < 0$. Using the Mean Value Theorem, we obtain the following:

$$f(t) - f(0) = f'(c)t \tag{27.2}$$

For $0 < c < t$. Thus, we see that for $0 < t < \epsilon$ that $f'(c)t > 0$ and that $f(t) > f(0)$. Also, for $-\epsilon < t < 0$, we have $f'(c)t > 0$ and $f(t) > f(0)$. Thus for al $t \in (-\epsilon, \epsilon)$, we see that $f(t) \geq f(0)$, with equality when $t = 0$. □

28. Problem 28

Theorem 28.1. *Let $g_n : [0, 1] \to \mathbb{R}$ be a sequence of differentiable functions such that the sequence $g_n' : [0, 1] \to \mathbb{R}$ is uniformly bounded. Show that there is a sequence of constants c_n such that the sequence of functions $h_n(x) = g_n(x) - c_n$ has a uniformly convergent subsequence on $[0, 1]$. Also, show that if $\int_0^1 g_n dx$ is a bounded sequence in $\mathbb{R}$, then g_n has a uniformly convergent subsequence.*

Proof. First, let $c_n = g_n(0)$. This shows that $h_n(0) = 0$ and that $h_n'(x) = g_n'(x)$. Also, note that $[0, 1]$ is closed and bounded, hence compact. We want to use Arzela-Ascoli, so we must show that h_n is pointwise bounded and equicontinuous. First, we will show that it is equicontinuous. Note that $\{g_n\}$ is uniformly continuous because it is continuous on a compact set. Moreover, c_n is uniformly continuous because it is a constant. Thus, each h_n is uniformly continuous. This implies, that for a fixed $\epsilon > 0$, that there exists as $\delta > 0$ for every $x, y \in [0, 1]$ and for every $n \in \mathbb{N}$ such that $|h_n(x) - h_n(y)| < \epsilon$ implies that $|x - y| < \delta$. Since $g_n'(x)$ is uniformly bounded, we know that there exists an $N > 0$ such that for all $n \in \mathbb{N}$ and $x \in [0, 1]$, we have $|g_n'(x)| < N$. By Mean Value Theorem, we thus have for $x < c < y$:

$$h_n(x) - h_n(y) = h_n'(c)(x - y) \tag{28.2}$$

$$|h_n(x) - h_n(y)| < N(x - y) \tag{28.3}$$

We can therefore choose $\delta < \epsilon/N$ so that for all $x, y \in [0, 1]$ and for all $n \in \mathbb{N}$, $|x - y| < \delta$ implies $|h_n(x) - h_n(y) < N\delta < \epsilon$. This is the definition of equicontinuity, so $h_n(x)$ is therefore equicontinuous.

Next, it is easy to see that $h_n(x)$ is uniformly bounded because by Mean Value Theorem we have $h_n(x) - h_n(0) = h'_n(c)x$ for all $x \in [0,1]$ and c between 0 and x. Therefore, since $h_n(0) = 0$ and $|h'_n(c)| = |g'_n(x)| < N$, we have $|h_n(x)| < Nx < N$ because $x \in [0,1]$. Therefore, we see that $h_n(x)$ is uniformly bounded and that we can apply Arzela Ascoli, which says that h_n has a uniformly convergent subsequence.

New we must show that if $\int_0^1 g_n dx$ is a bounded sequence in $\mathbb{R}$, then g_n has a uniformly convergent subsequence. First, we let $h_n(x) = g_n(x) - c_n$ as before, where $c_n = g_n(0)$. Then, we see that $\int_0^1 h_n(x)dx$ is bounded because $\int_0^1 g_n dx$ is bounded and so is the integral of a constant. Moreover, we know that $h_n(x)$ has a uniformly convergent subsequence, so its integral also must have a uniformly convergent subsequence, call it $\{h_{n_i}\}$, by a theorem in Rudin. Next, we have shown that $\int_0^1 h_n dx$ is bounded so that $\int_0^1 c_n dx$ must also be bounded on a compact set, which implies that it has a convergent subsequence. Therefore, g_n also has a convergent subsequence g_{n_i} because it is just given by $h_{n_i} + c_{n_i}$. □

29. Problem 29

Theorem 29.1. *Show that the series of functions $\sum_{n=1}^{\infty} \frac{x^n \sin(nx)}{n^3+n^2+1}$ converges uniformly on $[-1,1]$ and defines a continuously differentiable function on $[-1,1]$.*

Proof. First, we note that $|\sin(nx)| \leq 1$ and that $|x^n| \leq 1$ because $x \in [-1,1]$. Therefore, we have $\sum \left|\frac{x^n \sin(nx)}{n^3+n^2+1}\right| \leq \sum \frac{1}{n^3+n^2+1} \leq \sum \frac{1}{n^3}$. Since we know the final sum converges by geometric series with $p = 3$, we see that our series is absolutely convergent and hence uniformly convergent by the Weierstrass M test.

Next, we can differentiate each of the terms of the series and find the following:

$$\sum_{n=1}^{\infty} \left| \frac{nx^{n-1}\sin(nx) + nx^n \cos(nx)}{n^3+n^2+1} \right| \leq \sum_{n=1}^{\infty} \left| \frac{2n}{n^3} \right| \leq \sum_{n=1}^{\infty} \left| \frac{2}{n^2} \right| \tag{29.2}$$

Because $|\sin(nx)| \leq 1$ and $|\cos(nx)| \leq 1$. Therefore, we see that the series of the derivatives are absolutely convergent, which implies uniform convergence by the Weierstrass M test. This means that our series is differentiable and has a derivative given by the term by term derivatives of the series, by a theorem in Rudin. Moreover, since each of these terms are continuous, we see that the derivative of our series is the limit of a uniformly convergent series of continuous functions, and hence it is continuous. This implies that the series is continuously differentiable. □

30. Problem 30

Theorem 30.1. *Explain why the Riemann-Stieltjes integral $\int_{-1}^1 x^2 \exp(x^3) d\alpha$ exists for any increasing function $\alpha : [-1,1] \to \mathbb{R}$. Evaluate the integral when $\alpha = x$ for $x < 0$ and $\alpha = x+1$ for $x \geq 0$.*

Proof. First, we see that x^2 and x^3 are polynomials, which are continuous, and that $f(x) = e^x$ is continuous. Therefore, the composition of these functions $x^2 f(x^3)$ is also continuous by a theorem in Rudin. Since $x^2 \exp(x^3)$ is continuous, it is Riemann-Stieltjes integrable for any increasing function $\alpha : [-1,1] \to \mathbb{R}$. Next, we know that $\int_{-1}^1 x^2 \exp(x^3) d\alpha = \int_{-1}^1 x^2 \alpha'(x) dx = \int_{-1}^1 x^2 \exp(x^3) dx = \int_{-1}^1 \frac{1}{3} e^u du = \frac{1}{3}(e - e^{-1})$. □

31. Problem 31

Theorem 31.1. *Let $f : X \to \mathbb{R}$ be a continuous function on a compact metric space, X. If $\{x_n\}$ is a sequence in X, show that $\{f(x_n)\}$ has a convergent subsequence with a limit in $f(X)$.*

Proof. First, since $\{x_n\}$ is a sequence, we know that $\{f(x_n)\}$ is also a sequence in $f(X)$. Moreover, since f is continuous and X is compact, and we know that the image of a compact space by a continuous function is compact, we see that $f(X)$ is compact. Therefore, the sequence $\{f(x_n)\}$ is a sequence in a compact metric space $f(X)$, which shows that it has a convergent subsequence by a theorem in Rudin. □

32. Problem 32

Theorem 32.1. *If $a_n > 0, n \in \mathbb{N}$ is a sequence of real numbers such that $b_N = \sum_{n=1}^N a_n$ is bounded, show that $\sum_n a_n^{1/2} a_{n+1}^{1/2}$ converges.*

Proof. We know that $(\sqrt{a_n} + \sqrt{a_{n+1}})^2 = a_n + 2\sqrt{a_n}\sqrt{a_{n+1}} + a_{n+1}$. Rearranging, we see that $\sqrt{a_n a_{n+1}} = \frac{1}{2}((\sqrt{a_n} + \sqrt{a_{n+1}})^2 - a_n - a_{n+1})$. We therefore have $\sum_n \sqrt{a_n a_{n+1}} = \sum_n \frac{1}{2}((\sqrt{a_n} + \sqrt{a_{n+1}})^2 - a_n - a_{n+1})$. Now take the largest value a_n or a_{n+1} for each index. Then we have $\sum_n \frac{1}{2}((\sqrt{a_n} + \sqrt{a_{n+1}})^2 - a_n - a_{n+1}) \leq \sum_n \max\{\frac{1}{2}((2\sqrt{a_n})^2 - a_n - a_{n+1}), \frac{1}{2}((2\sqrt{a_{n+1}})^2 - a_n - a_{n+1})\} = \sum_n \max\{\frac{1}{2}(3a_n - a_{n+1}), \frac{1}{2}(3a_{n+1} - a_n)\}$. Since we know that b_N is bounded and monotonically increasing ($a_n > 0$), we see that b_N converges. Therefore, we see that $\sum_n a_n$ and $\sum_n a_{n+1}$ both converge, and therefore, we see that $\sum_n \max\{\frac{1}{2}(3a_n - a_{n+!}), \frac{1}{2}(3a_{n+1} - a_n)\}$ converges as well. Therefore, we see that our series converges by comparison test. $\square$

33. Problem 33

Theorem 33.1. *Let $\{f_n\}$ be a sequence of continuous real-valued functions on a metric space X such that $\{f_n\}$ converges uniformly to some function f on every compact subset of X. Show that f is a continuous function.*

Proof. Since $\{f_n\}$ is continuous and uniformly convergent on a compact subset $K \subset X$, we know by a theorem in Rudin that $\{f_n\}$ is equicontinuous. This implies that for a fixed $\epsilon > 0$, that there exists a $\delta > 0$ such that for all $x, y \in K$ and $n \in \mathbb{N}$, $|f_n(x) - f_n(y)| < \epsilon$ if $|x - y| < \delta$. Moreover, since $\{f_n\}$ converges uniformly, say to some function f, we know that there exists an $N_1 > 0$ such that for all $x \in K$, we have $|f_n(x) - f(x)| < \epsilon$. Also, we know that there exists an $N_2 > 0$ such that for all $n, m > N$ and for all $x \in K$, $|f_n(x) - f_m(x)| < \epsilon$ by the Cauchy criterion. Therefore, we can select $N = \max\{N_1, N_2\}$ so that $n > N$ has the following:

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f_m(y)| + |f_m(y) - f(y)| < 4\epsilon \tag{33.2}$$

The first and last terms, $|f(x) - f_n(x)| < \epsilon$ and $|f_m(y) - f(y)| < \epsilon$, are true because of uniform convergence. The second term $|f_n(x) - f_n(y)| < \epsilon$ is true because of equicontinuity. Finally, $|f_n(y) - f_m(y)| < \epsilon$ is true because of the Cauchy criterion for uniform convergence. Therefore, we see that f is continuous. $\square$

34. Problem 34

Theorem 34.1. *Let $\{f_n\}$ and $\{g_n\}$ be two uniformly bounded sequences of real-valued functions on a set X. If both $\{f_n\}$ and $\{g_n\}$ are uniformly convergent on X, show that $\{f_n g_n\}$ also converges uniformly on X.*

Proof. Since $\{f_n\}$ and $\{g_n\}$ are uniformly bounded, then for fixed $\epsilon > 0$, there exists $N_1, N_2 > 0$ such that for all $x \in X$ and for all $n \in \mathbb{N}$, we have $|f_n(x)| < N_1$ and $|g_n(x)| < N_2$. Therefore, we can take $N = \max\{N_1, N_2\}$. Next, we know by uniform convergence that there exists $M_1, M_2 > 0$ such that for all $n, m > M_1$ and for all $x \in X$, we have $|f_n(x) - f_m(x)| < \epsilon$ and for all $n, m > M_2$ and $x \in X$ we have $|g_n(x) - g_m(x)| < \epsilon$. Therefore, we can set $M = \max\{M_1, M_2\}$, and for $n, m > M$, we have the following:

$$|f_n(x)g_n(x) - f_m(x)g_m(x)| \leq |f_n(x)g_n(x) - f_n(x)g_m(x)| + |f_n(x)g_m(x) - f_m(x)g_m(x)| \tag{34.2}$$
$$= |f_n(x)||g_n(x) - g_m(x)| + |g_m(x)||f_n(x) - f_m(x)| \tag{34.3}$$
$$< 2N\epsilon \tag{34.4}$$

Therefore, since $\epsilon > 0$ was arbitrary and N is a constant, we see that $\{f_n g_n\}$ converges uniformly by the Cauchy criterion. $\square$

35. Problem 35

Theorem 35.1. *Let $E = \{0, 1, 1/2, 1/3, 1/4, \ldots\}$ be the set of real numbers consisting of zero and the inverse of the positive integers. Prove that E is compact directly from the definition (without using the Heine-Borel theorem).*

Proof. Let $\{U_\alpha\}$ be an open cover of E. We must show that $\{G_\alpha\}$ has a finite subcover. One of these sets, call it α_0 must contain 0. Moreover, the set contains a neighborhood of some radius r about 0. Thus, choose n large enough so that $\frac{1}{n} \leq r$, then all of the points $\{1/(n+1), 1/(n+2), \ldots\}$ are inside of the set U_{α_0}. For each of the remaining n points, choose an open set U_{a_j} containing $1/j$. Thus, we see that E is covered in by at most $n+1$ sets $U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_{n+1}}$. $\square$

36. Problem 36

Theorem 36.1. *Let $X = \mathbb{R}$. Define a real-valued function on $\mathbb{R} \times \mathbb{R}$ by $d(x, y) = (x - y)^2$. Is d a metric on $\mathbb{R}$?*

Proof. We need to prove three things in order to show that d is a metric, namely that the transitive property holds, the triangle inequality holds, and that it is nonnegative. First, we see that it is clearly non-negative, and that the transitive property holds because $|x - y| = |y - x|$ so that $(x - y)^2 = (y - x)^2$. However, we see that $(x-y)^2 = x^2 - 2xy + y^2$ and $(x-h)^2 + (h-y)^2 = x^2 - 2xh + h^2 + h^2 - 2hy + y^2 = x^2 + y^2 - 2h(x+y) + 2h^2$. In order for d to be a metric, we must have $-2xy \leq -2h(x + y) + 2h^2$ which implies $2xy \geq 2h(x + y) - 2h^2$. If we set $x = -1, y = 2, h = 1$, $2xy = -4$ and $2h(x + y) - 2h^2 = 0$, which means $2xy \leq 2h(x + y)$ so that d is not a metric. $\square$

37. Problem 37

Theorem 37.1. *Recall that the length of a vector $v \in \mathbb{R}^n$ can be defined in terms of the inner product by $||v|| = (v \cdot v)^{1/2}$. The Cauchy-Schwartz inequality says that if x and y are vectors in $\mathbb{R}^n$, then $|x \cdot y| \leq ||x||||y||$. Using this fact, prove the triangle inequality $||u+v|| \leq ||u|| + ||v||$.*

Proof. It is enough to prove $||u+v||^2 \leq (||u|| + ||v||)^2$ because both sides of the inequality are non-negative. We have the following: $(||u|| + ||v||)^2 = ||u||^2 + 2||u||||v|| + ||v||^2$. Since we know that $|u \cdot v| \leq ||u||||v||$ by the Cauchy-Schwartz inequality. Therefore, we know that $(||u|| + ||v||)^2 \geq ||u||^2 + 2|u \cdot v| + ||v||^2 = |u \cdot u| + 2|u \cdot v| + |v \cdot v| = |u+v| \cdot |u+v| = ||u+v||^2$. This shows that $||u+v||^2 \leq (||u|| + ||v||)^2$, which shows that $||u+v|| \leq ||u|| + ||v||$. □

38. Problem 38

Theorem 38.1. *Give an example of a metric space X and an open subset of X having exactly three points.*

Proof. Let X have the following metric:

$$d(x,y) = \begin{cases} 0 & x \wedge y \in \{1,2,3\} \\ 1 & x \vee y \notin \{1,2,3\} \end{cases} \tag{38.2}$$

We see that d has all the properties of a metric, and moreover, that the set $\{1,2,3\} \in X$ is an open set in X. Another example of this would be the space $K = \{1,2,3\} \subset \mathbb{R}$. We see that $\{1,2,3\}$ is open in K and consists of exactly three elements. □

39. Problem 39

Theorem 39.1. *Suppose X is a metric space and $f : X \to X$ is a function from X to X. We say that f is a weak contraction if $d(f(x_1), f(x_2)) \leq d(x_1, x_2)$ for $x_1, x_2 \in X$. Prove that any weak contraction is continuous.*

Proof. Fix $\epsilon > 0$. We need to show that there exists a $\delta > 0$ such that $d(f(x_1), f(x_2)) < \epsilon$ if $d(x_1, x_2) < \delta$ for $x_1, x_2 \in X$. Since f is a weak contraction, we see that $d(f(x_1), f(x_2)) \leq d(x_1, x_2)$ for $x_1, x_2 \in X$. Thus, we can pick $\delta < \epsilon$ so that $d(x_1, x_2) < \delta < \epsilon$, which would imply by the above inequality that $d(f(x_1), f(x_2)) < \epsilon$, which proves continuity. □

40. Problem 40

Theorem 40.1. *Suppose that $f : X \to X$ is any continuous function, and that $x_0 \in X$. Define a sequence $x_1, x_2, x_3, \ldots$ of points in X by $x_{n+1} = f(x_n)$ for $n \geq 0$. Prove that if the sequence $\{x_n\}$ converges to a limit point $x \in X$, then $f(x) = x$.*

Proof. We know that if f is continuous in X, and $\{z_n\} \to z$, then $\{f(z_n)\} \to f(z)$. Using this fact, we see that $\{f(x_n)\} = \{x_{n+1}\} \to x$, since $\{x_{n+1}\}$ converges to x by assumption. Therefore, $\{f(x_n)\} \to f(x)$ converges to the same limit as $\{x_{n+1}\}$, which means that $f(x) = x$. □

41. Problem 41

Theorem 41.1. *We say that $f : X \to X$ is a strong contraction if there is a number $r < 1$ such that $d(f(x_1), f(x_2)) \leq rd(x_1, x_2)$ for $x_1, x_2 \in X$. Suppose that f is a strong contraction and $x_0 \in X$. Define a sequence $x_1, x_2, x_3, \ldots$ of points in X by $x_{n+1} = f(x_n)$ for $n \geq 0$. Finally define $A = d(x_0, f(x_0))$. Prove that $d(x_0, x_n) \leq A(1-r^n)/(1-r) \leq A/(1-r)$.*

Proof. First notice the following:

$$d(x_1, x_2) = d(f(x_0), f(x_1)) = d(f(x_0), f(f(x_0))) \leq rd(x_0, f(x_0) = rA \tag{41.2}$$

$$d(x_2, x_3) = d(f(f(x_0)), f(f(f(x_0)))) \leq rd(f(x_0), f(f(x_0))) \leq r(rA) = r^2 A \tag{41.3}$$

$$\vdots \tag{41.4}$$

$$d(x_n, x_{n+1}) \leq r^n A \tag{41.5}$$

Moreover, we know by triangle inequality that:

$$d(x_0, x_n) \leq d(x_0, x_1) + d(x_1, x_2) + \ldots + d(x_{n-1}, x_n) \tag{41.6}$$
$$\leq A\sum_{i=1}^{n-1} r^i \tag{41.7}$$
$$= A\left(\frac{1-r^n}{1-r}\right) \tag{41.8}$$

Moreover, we know that $(1-r^n) \leq 1$, so that $d(x_0, x_n) \leq A(1-r^n)/(1-r) \leq A/(1-r)$. □

Theorem 41.9. *Prove that $d(x_m, x_{m+n}) \leq Ar^m/(1-r)$.*

Proof. From the previous discussion and use of the triangle inequality, we see that:

$$d(x_m, x_{m+n}) = d(x_m, x_{m+1}) + \ldots + d(x_{m+n-1}, x_{m+n}) \tag{41.10}$$
$$\leq d(f(x_{m-1}, f(x_m)) + \ldots + d(f(x_{m+n-2}), f(x_{m+n-1})) \tag{41.11}$$
$$\leq rd(f(x_{m-2}), f(x_{m-1})) + \ldots + rd(f(x_{m+n-3}), f(x_{m+n-2})) \tag{41.12}$$
$$\vdots \tag{41.13}$$
$$\leq r^m A + r^m d(f(x_1), f(x_2)) + \ldots + r^m d(f(x_{n-1}), f(x_n)) \tag{41.14}$$
$$\vdots \tag{41.15}$$
$$\leq r^m A + r^{m+1}A + \ldots + r^{m+n-1}A \tag{41.16}$$
$$= r^m A\sum_{i=1}^{n-1} r^i \tag{41.17}$$
$$= r^m A\left(\frac{1-r^n}{1-r}\right) \tag{41.18}$$
$$\leq \frac{Ar^m}{1-r} \tag{41.19}$$

□

Theorem 41.20. *Prove that $\{x_n\}$ is a Cauchy sequence.*

Proof. Fix $\epsilon > 0$. In order to show that $\{x_n\}$ is a Cauchy sequence, we need to show that there exists an N such that for $m, m+n > N$, $d(x_m, x_{m+n}) < \epsilon$. Well, we know that $d(x_m, x_{m+n}) \leq Ar^m/(1-r)$ by the last theorem. Therefore, we can pick m such that $Ar^m/(1-r) < \epsilon$, because $r < 1$, which means that $r^m \to 0$. This shows that $\{x_n\}$ is a Cauchy sequence . □

Theorem 41.21. *Suppose that X is a complete metric space. Prove that there is a point $x \in X$ such that $f(x) = x$. (Such an x is called a fixed point.)*

Proof. Since X is a complete metric space, we know that every Cauchy sequence converges. In particular, $\{x_n\}$ converges to some point, call it x. Therefore, we know by the theorem proven in problem 40 that $f(x) = x$ if f is continuous. However, since f is a strong contraction, we see that we can pick a $\delta > 0$ such that $\delta/r < \epsilon$. Since $d(x_1, x_2) < \delta$, we see that $d(f(x_1), f(x_2)) < rd(x_1, x_2) < r\epsilon/r < \epsilon$, which proves continuity of f. Therefore, we have shown that $f(x) = x$ by problem 40. □

42. Problem 42

Theorem 42.1. *Prove that if x and y are any real numbers, then $|\sin x - \sin y| \leq |x - y|$.*

Proof. We know that $\frac{d}{dx}\sin x = \cos x$. Therefore, we see by Mean Value Theorem:

$$\sin(x) - \sin(y) = \cos(c)(x-y) \tag{42.2}$$

For some $c \in (x, y)$ for $x, y \in \mathbb{R}$. Since $|\cos(c)| \leq 1$, we know that $|\sin(x) - \sin(y)| \leq |x - y|$, which proves the theorem. □

43. Problem 43

Theorem 43.1. *If f is a Riemann-integrable function on $[0, 2\pi]$, recall that the mth Fourier coefficient of f is by definition $c_m(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-imx}dx$ for $m \in \mathbb{Z}$. Suppose that $f(x) = \exp(\exp(ix))$. Prove that there is a constant A so that $|c_m(f)| \leq A/m^2$ for $m \neq 0$.*

Proof. Let us integrate by parts and set $u = f(x)$ and $dv = e^{-imx}dx$ so that $du = f'(x)dx$ and $v = \frac{e^{-imx}}{-im}$. This gives:

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-imx}dx = \frac{1}{2\pi}\left(f(x)\frac{e^{-imx}}{-im}\right)_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} \frac{f'(x)e^{-imx}}{-im}dx \tag{43.2}$$

Since f is periodic, we see that $f(2\pi) = f(0)$, which means that the first term cancels. Repeating the same operation shows that one of terms in the integration by parts the second time also cancels. This gives:

$$c_m(f) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f''(x)e^{-imx}}{-m^2}dx \tag{43.3}$$

$$|c_m(f)| \leq \frac{1}{2m^2\pi}\int_0^{2\pi} |f''(x)|dx \tag{43.4}$$

This shows that if we take A equal to the average value of $|f''(x)|$, we have the inequality we wanted to prove. □

44. Problem 44

Theorem 44.1. *Suppose that $\{f_n\}$ is a sequence of monotone real-valud functions defined on $[a, b]$ and not necessarily all increasing or decreasing. Show that if $\{f_n\}$ converges pointwise to a continuous function f on $[a, b]$, then $\{f_n\}$ converges uniformly to f on $[a, b]$.*

Proof. Fix $\epsilon > 0$. Since f is continuous on $[a, b]$, which is closed, bounded, and hence compact set, we see that it is also uniformly continuous. Therefore, there exists $\delta > 0$ such for all $x, y \in [a, b]$ such that $|f(x)-f(y)| < \epsilon$ whenever $|x-y| < \delta$. Fix a finite set of points $a = x_0, x_1, \ldots, x_k = b$ such that $x_i - x_{i-1} < \delta$ for all $1 \leq i \leq k$. Next, pick some N such that $|f_n(x_i) - f(x_i)| < \epsilon$ for all $n > N$ and $0 \leq i \leq k$ by the pointwise convergence assumption.

Now, in the case that f_n is monotonically decreasing, we have, for $x \in [a, b]$ and $x_{i-1} \leq x \leq x_i$ for some $1 \leq i \leq k$:

$$|f_n(x) - f_n(x_i)| \leq |f_n(x_{i-1}) - f_n(x_i)| \tag{44.2}$$

$$\leq |f_n(x_{i-1}) - f(x_{i-1})| + |f(x_{i-1}) - f(x_i)| + |f(x_i) - f_n(x_i)| \tag{44.3}$$

$$\leq 3\epsilon \tag{44.4}$$

Thus, f_n converges uniformly when f_n is monotonically decreasing. In the case when f_n is monotonically increasing, we have for $x \in [a, b]$ and $x_{i-1} \leq x \leq x_i$:

$$|f_n(x) - f(x)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(x)| \tag{44.5}$$

$$\leq 3\epsilon + \epsilon + \epsilon = 5\epsilon \tag{44.6}$$

Therefore, $\{f_n\}$ converges uniformly to f. □

45. Problem 45

Theorem 45.1. *Consider each rational number written in the form $\frac{m}{n}$, where $n > 0$ and m and n are integers without any common factors other than ± 1. Clearly, such a representation is unique. Now define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ if x is irrational and $f(x) = \frac{1}{n}$ if $x = \frac{m}{n}$ as above. Show that f is continuous at every irrational number and discontinuous at every rational number.*

Proof. Let $\{r_n\}$ be a bounded sequence of rational numbers such that $r_n = \frac{m_n}{k_n}$. We will show that $\lim k_n = \infty$. We can choose an $M > 0$ such that $|r_n| < M$, which implies that $|m_n| < M|k_n|$. Now suppose by contradiction that k_n is finite, so that $|k_n| < C$ for some $C > 0$ and infinitely many n. Then we must have $|m_n| < MC$ and $|k_n| < C$ hold for an infinite number of n. But this is a contradiction, because there are only finitely many rational numbers r_n for which this is possible. Therefore, we must have $\lim k_n = \infty$, which will be important in the proof.

This implies that f is continuous at each irrational number. So let x be irrational and let $\{x_n\} \to x$ be a sequence of irrational numbers. Clearly, we have $0 = f(x_n) \to 0 = f(x)$. Suppose then that $f(x)$ is not continuous. The only way this can happen is if there exists a sequence $\{r_n\}$ of rational numbers such that $\{r_n\} \to x$. This implies $\lim f(r_n) \neq 0$. Since we can let $r_n = \frac{m_n}{k_n}$, we see that if $\{r_n\} \to x$, we must have $\lim \frac{1}{k_n} \neq 0$, which is a contradiction. Thus, f is continuous at every irrational number.

To see that f is discontinuous at every rational number, we choose a sequence $\{r_n\}$ of rational numbers converging to r, so that $\frac{m_n}{k_n} \to r$. Now, we note that $\lim f(r_n) = \lim \frac{1}{k_n} = 0 \neq f(r) = \frac{1}{r}$. This shows that f is discontinuous at every rational number. □

46. Problem 46

Theorem 46.1. *If A and B are two arbitrary subsets of a topological space, show that $\overline{A \cup B} = \bar{A} \cup \bar{B}$.*

Proof. First, we will show that $\overline{A \cup B} = \bar{A} \cup \bar{B}$. First, it is obvious that $A \subset \bar{A}$ and $B \subset \bar{B}$, which implies that $A \cup B \subset \bar{A} \cup \bar{B}$. Since the closure of a closed set is just itself, and since $\bar{S}$ is closed, we see that $A \cup B \subset \bar{A} \cup \bar{B}$ implies $\overline{A \cup B} \subset \overline{\bar{A} \cup \bar{B}} = \bar{A} \cup \bar{B}$ by taking the closure of both sides.

Next, we must show that $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$. This is because $\bar{A} \subset \overline{A \cup B}$ and $\bar{B} \subset \overline{A \cup B}$, which shows $\bar{A} \cup \bar{B} \subset \overline{A \cup B} \cup \overline{A \cup B} = \overline{A \cup B}$. These two inequalities show that $\bar{A} \cup \bar{B} = \overline{A \cup B}$. □

47. Problem 47

Theorem 47.1. *If A and B are two arbitrary subsets of a topological space, show that $(A \cup B)' = A' \cup B'$.*

Proof. First, we note that $A' \subset (A \cup B)'$ and $B' \subset (A \cup B)'$ implies that $A' \cup B' \subset (A \cup B)' \cup (A \cup B)' = (A \cup B)'$. Now, we can prove the theorem by showing $(A \cup B)' \subset A' \cup B'$. Let $x \in (A \cup B)'$. Now suppose by contradiction that $x \notin A' \cup B'$. Then there exist two neighborhoods $V(x)$ and $W(x)$ such that for all $p \in A$ and $q \in B$ where $p, q \neq x$, we have $p \notin V(x)$ and $q \notin W(x)$. This implies that for the neighborhood $G(x) = V(x) \cap W(x)$, we have $p \notin G(x)$ and $q \notin G(x)$ for all $p \in A$ and $q \in B$ such that $p, q \neq x$. Moreover, this neighborhood $G(x)$ is nonempty because $V(x)$ and $W(x)$ are both centered at x and are nonempty, so $G(x)$ must also be nonempty. But this is a contradiction, because that implies that $x \notin (A \cup B)'$. Therefore, we must have $(A \cup B)' \subset A' \cup B'$, which completes the proof. □

48. Problem 48

Theorem 48.1. *If A is a dense subset of a topological space, show that $O \subset \overline{A \cap O}$ holds for every open set O.*

Proof. Let $x \in O$ and let $V(x)$ be a neighborhood of x such that $V(x) \subset O$, which we can choose because O is open. Therefore, we see that $V(x) \cap O = V(x)$ is a neighborhood of x, and we have $V(x) \cap (O \cap A) = (V(x) \cap O) \cap A = V(x) \cap A \neq \emptyset$ because of the denseness of A. This implies that $x \in \overline{O \cap A}$ because $V(x) \cap (O \cap A) \neq \emptyset$. □

49. Problem 49

Theorem 49.1. *If A is open, then $A \cap \bar{B} \subset \overline{A \cap B}$.*

Proof. Let $x \in A \cap \bar{B}$. Let $V(x)$ be a neighborhood of x. We know that $V(x) \cap A$ is also a neighborhood of x. Since $x \in \bar{B}$, we see that $V(x) \cap (A \cap B) = (V(x) \cap A) \cap B \neq \emptyset$. Therefore, we see that $x \in \overline{A \cap B}$ and hence that $A \cap \bar{B} \subset \overline{A \cap B}$. □

50. Problem 50

Theorem 50.1. *If $\{O\}_{i \in I}$ is an open cover of a topological space X, then show that a subset A of X is closed if and only if $A \cap O_i$ is closed in O_i for each $i \in I$.*

Proof. The forward direction is easy because if A is closed, then $A \cap O_i$ is closed because O_i is open. Now we must show the converse. First, let $V_i = O_i \setminus (A \cap O_i)$. Since O_i is open for each $i \in I$ and $A \cap O_i$ is closed, we see that V_i is open. Therefore, we have:

$$A^c = X \setminus A = \bigcup_{i \in I} V_i \tag{50.2}$$

Which is open because it is a finite union of open sets. Therefore, we see that A is closed because A^c is open. □

51. Problem 51

Theorem 51.1. *Let $f : [a, b] \to \mathbb{R}$ be increasing, i.e. $x < y$ implies $f(x) \leq f(y)$. Show that the set of points where f is discontinuous is at most countable.*

Proof. Let D be the set of discontinuities of f and let $x \in D$. Then we can assign a rational number r_x such that $\lim_{t\to x^-} f(x) \leq r_x \leq \lim_{t\to x^+} f(x)$. Moreover, we see that if $x, y \in D$ and $y > x$, we must have $r_x < \lim_{t\to x^+} f(x) < lim_{t\to y^-} f(x) < r_y$ for all $y \neq x$. Therefore, we have shown that $x \to r_x$ is a one to one mapping from D to $\mathbb{Q}$. This shows that D is at most countable. □

52. Problem 52

Theorem 52.1. *Give an example of a strictly increasing function $f : [0, 1] \to \mathbb{R}$ which is continuous at all irrational points and discontinuous at all the rational points.*

Proof. First, let $f_t(x)$ be a function $f_t : [0, 1] \to [0, 1]$ which is continuous at every point on the interval $[0, 1]$ except for at t and strictly increasing. For instance, we could set:

$$f_t(x) = \begin{cases} \frac{1}{2}x & 0 \leq x \leq t \\ x & t < x \leq 1 \end{cases} \tag{52.2}$$

Thus, we can enumerate all the rational numbers in $[0, 1]$ with the set $\{r_1, r_2, \ldots\}$.

$$f(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} f_{r_i}(x) \tag{52.3}$$

It is clear that this sum is uniformly convergent, for all $x \neq r_i$. Therefore, since each term in the uniformly convergent series is continuous, the function is continuous. However, when $x = r_i$, we see that the function is discontinuous. Since $f(x)$ is nevertheless strictly increasing, we see that the function we have made matches all the criteria. □

53. Problem 53

Theorem 53.1. *Recall that a function $f : (X, \tau) \to (Y, \tau_1)$ is called an open mapping if $f(V)$ is open whenever V is open. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous open mapping, then f is a strictly monotone function–and hence a homeomorphism.*

Proof. Since $f : \mathbb{R} \to \mathbb{R}$ is an open mapping, we see that $f(V)$ is open whenever V is open. Since it is continuous, we see that $f^{-1}(U)$ is open whenever U is open. Now, assume by contradiction that f is not monotonic. Then for some interval $(a, b) \subset \mathbb{R}$ there exists a $y \in (a, b)$ so that we must have $f(x) \leq f(y)$ for all $a < x < y$ and $f(y) \geq f(z)$ for all $y < z < b$, or alternatively we must have $f(x) \geq f(y)$ for all $a < x < y$ and $f(y) \leq f(z)$ for all $y < z < b$. We can assume without loss of generality that we are working in the first case (the proof is essentially the same in the other case).

Now, we let $N_\delta(x)$ be a neighborhood of x with radius $\delta > 0$. We see that since $f(y) \geq f(z)$ and $f(y) \geq f(x)$ for all $x \in (a, y)$ and $z \in (y, b)$, we must have $f(y) \geq f(p)$ for all $p \in (a, b)$. This is the definition of a local maximum. Therefore, we see that for all $\delta > 0$, there exists some $q \notin f(a, b)$ with $f(y) < q < f(y) + \delta$. This implies that for no $\delta > 0$ does there exist a neighborhood $N_\delta(x) \subset f(a, b)$. Thus, we see that $f(a, b)$ is not open. However, since we have assumed an open mapping, we must have $f(a, b)$ open because $(a, b) \subset \mathbb{R}^1$ is open. This is a contradiction, and implies that f must be a strictly monotone function. □

Theorem 53.2. *A direct proof to problem 53.*

Proof. Let (a, b) be a finite open interval of $\mathbb{R}$. Since f attains a maximum value on the compact set $[a, b]$ and we know that $f((a, b))$ is an open set by assumption of an open mapping, then we see that the extrema of f on $[a, b]$ must take place at the endpoints. This implies that $f(a) \neq f(b)$, because if that were the case then $f((a, b))$ would have to be a one-point set contradicting the fact that f is open. Next, we claim that f is monotone on (a, b). First, we assume that $f(a) < f(b)$ and $a < x < y < b$. Then note that $f(a) < f(x) < f(b)$ must hold, because we must have f attain its maximum and mimimum at the endpoints. By the same argument, we must have $f(x) < f(y) < f(b)$. Thus, f is strictly increasing on (a, b). If $f(a) > f(b)$, then we can show by the same argument that f is strictly decreasing on (a, b).

Now, we must extend this from the interval (a, b) to $\mathbb{R}$. So assume that f is strictly increasing on $(0, 1)$ and let $x < y$ without loss of generality. Now choose some n such that $(0, 1) \subset (-n, n)$ and $x \in (-n, n)$. We have shown from above that f must be monotone on $(-n, n)$. However, we have also assumed that f is strictly increasing on $(0,1)$, so that f must be strictly increasing on $(-n,n)$. Thus, $f(x) < f(y)$ holds for $x < y \in \mathbb{R}$, which shows that f is strictly increasing on $\mathbb{R}$. A similar argument holds for a strictly decreasing function on $(0,1)$. This completes the proof. □

54. Problem 54

Theorem 54.1. *Let X be a compact topological space and let $\mathscr{F} \subset C(X;\mathbb{R})$ be a collection of continuous functions from $X \to \mathbb{R}$ which is pointwise bounded and equicontinuous. Prove that $\mathscr{F}$ is uniformly bounded.*

Proof. This is a simple use of the Arzela-Ascoli theorem, which states that for $\{f_n\} \in \mathscr{F}$ which is pointwise bounded and equicontinuous, you must have $\{f_n\}$ be uniformly bounded as well. □

Theorem 54.2. *Arzela Ascoli Theorem: If K is compact, if $f_n \in \mathscr{C}(K)$ for $n = \mathbb{N}$, and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then $\{f_n\}$ is uniformly bounded on K.*

Proof. Since we know the sequence is equicontinuous, we can fix $\epsilon > 0$ and pick $\delta > 0$ such that for all $x, y \in K$, we have $|f_n(x) - f_n(y)| < \epsilon$ whenever $|x - y| < \delta$. Since K is compact, we know that there exists finitely many points p_i for which each $x \in K$ is associated with at least one p_i from the set $\{p_1, p_2, \ldots, p_k\}$ which has the property that $|x - p_i| < \delta$. Since the sequence is pointwise bounded, we can find an $M_i > 0$ such that for all $n > M_i$, we have $|f_n(p_i)| < M_i$ for each $1 \le i \le k$. Therefore, we can pick $M = \max\{M_1, M_2, \ldots, M_k\}$ because k is finite, and see that $|f_n(x)| < M$ for all $x \in K$ and $n \in \mathbb{N}$, which proves uniform boundedness. □

55. Problem 55

Theorem 55.1. *Let $f, f_1, f_2, \ldots$ be real valued functions defined on a compact metric space (X, d) such that $x_n \to x$ in X implies that $f_n(x_n) \to f(x)$ in $\mathbb{R}$. If f is continuous, then show that the sequence of functions $\{f_n\}$ converges uniformly.*

Proof. Assume the contrary and fix $\epsilon > 0$. Then for all $n, m \in \mathbb{N}$, we have $|f_n(x) - f_m(x)| \ge \epsilon$ for some $x \in X$. There also exists a subsequence $\{g_n\}$ of $\{f_n\}$ and a sequence $\{x_n\}$ of X such that $|g_n(x_n) - f(x_n)| \ge \epsilon$. Since X is compact, x_n has a convergent subsequence in X, say $\{x_{n_k}\} \to x$. By the continuity of f, we see that $f(x_{n_k}) \to f(x)$. Also, we see that $g_{n_k}(x_{n_k}) \to f(x)$, so that $|g_{n_k}(x_{n_k}) - f(x_{n_k})| \to |f(x) - f(x)| = 0$, which is a contradiction because we have previously shown that $|g_n(x_n) - f(x_n)| \ge \epsilon$. Therefore, we must have $\{f_n\}$ converge uniformly. □

56. Problem 56

Theorem 56.1. *For a sequence $\{f_n\}$ of real valued functions defined on a topological space X that converges uniformly to a real valued function f on X, show that if $x_n \to x$ and f is continuous, then $f_n(x_n) \to f(x)$.*

Proof. Fix $\epsilon > 0$. Since f_n converges uniformly, there exists $N_1 > 0$ such that for all $x \in X$ and all $n > N_1$, we have $|f_n(x) - f(x)| < \epsilon$. If f is continuous, then we see that there exists an N_2 for which $|f(x_n) - f(x)| < \epsilon$ because we can choose $\delta = \epsilon$ so that $|x_n - x| < \epsilon = \delta$ implies $|f(x_n) - f(x)| < \epsilon$. Therefore, we have:

$$|f_n(x_n) - f(x)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < 2\epsilon \tag{56.2}$$

For $n > \max\{N_1, N_2\}$. This proves that $f_n(x_n) \to f(x)$. □

57. Problem 57

Theorem 57.1. *Let real valued functions $\{f_n\}$ converge uniformly to a real valued function f on a topological space X. If each $\{f_n\}$ is continuous at some point $x_0 \in X$, then f is also continuous at the point x_0 and $\lim_{x \to x_0} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to x_0} f_n(x) = f(x_0)$.*

Proof. First, fix $\epsilon > 0$. Since we have uniform convergence, there exists an N_1 such that for all $n > N_1$ and all $x \in X$, we have $|f_n(x) - f(x)| < \epsilon$. By pointwise convergence, there exists a $j > 0$ and a corresponding neighborhood $N_j(x_0)$ about x_0 such that for all $x \in N_j(x_0)$ there exists a $\delta > 0$ such that we have $|f_n(x_0) - f_n(x)| < \epsilon$ whenever $|x_0 - x| < \delta$. Therefore, we have for $x \in N_j(x_0)$, $|x - x_0| < \delta$, and $n > N_1$:

$$|f(x_0) - f(x)| \le |f(x_0) - f_n(x_0)| + |f_n(x_0) - f_n(x)| + |f_n(x) - f(x)| \le 3\epsilon \tag{57.2}$$

Because $|f(x_0)-f_n(x_0)| < \epsilon$ and $|f_n(x)-f(x)| < \epsilon$ hold by uniform convergence, while $|f_n(x_0)-f_n(x)| < \epsilon$ holds by pointwise continuity. Therefore, we have shown that f is also continuous at the point x_0. Because of this pointwise continuity, we get the following:

$$\lim_{x\to x_0}\lim_{n\to\infty} f_n(x) = \lim_{x\to x_0} f(x) = f(x_0) \tag{57.3}$$

$$\lim_{n\to\infty}\lim_{x\to x_0} f_n(x) = \lim_{n\to\infty} f_n(x_0) = f(x_0) \tag{57.4}$$

This completes the proof. □

58. Problem 58

Theorem 58.1. *Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$ for $x \in [0,1]$. Show that $\{f_n\}$ converges pointwise and find its limit function. Is it uniformly convergent?*

Proof. First, for all $x \in [0,1)$, we see that $x^n \to 0$ as $n \to \infty$. If $x = 1$, then we see that $x^n \to 1$ as $n \to \infty$. Therefore, we have the following limit function:

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases} \tag{58.2}$$

Moreover, since we see that $f(x)$ is not continuous, we see that f_n is not uniformly convergent. □

59. Problem 59

Theorem 59.1. *Let $g : [0,1] \to \mathbb{R}$ be continuous with $g(1) = 0$. Show that $\{f_n\}$ defined by $f_n(x) = x^n g(x)$ for $x \in [0,1]$ converges uniformly to the constant zero function.*

Proof. Since $g(x)$ is continuous, we know that there exists an M such that $|g(x)| < M$ for all $x \in [0,1]$. Therefore, we see that $|f_n(x) - 0| = |f_n(x)| = x^n|g(x)| \le x^n M$. If we let $\delta < x \le 1$, we see that there exists an N such that $M\delta^n < \epsilon$ whenever $n > N$, so that $|f_n(x)| < \epsilon$ for all $x \in [0,1]$, including $x = 1$ because $g(1) = 0$ so that $f_n(1) = 0$. This proves uniform convergence to the constant zero function. □

60. Problem 60

Theorem 60.1. *Let $\{f_n\}$ be a sequence of continuous real-valued functions on $[a,b]$ and let $\{a_n\}$ and $\{b_n\}$ be two sequences on $[a,b]$ converging to a and b respectively as $n \to \infty$. If $\{f_n\}$ converges uniformly to f on $[a,b]$, show that $\lim_{n\to\infty}\int_{a_n}^{b_n} f_n(x)dx = \int_a^b f(x)dx$.*

Proof. Fix $\epsilon > 0$. First, we see that there exsts an N_1 such that for all $n > N_1$: $a_n - a < \epsilon$. There also exists an N_2 such that for all $n > N_2$, we have $b - b_n < \epsilon$. Since $\{f_n\}$ is a uniformly convergent sequence of continuous functions, we see that f is also continuous. This implies that there exists an M such that $|f(x)| < M$ for all $x \in [a,b]$. Uniform continuity implies also that there exists an N_3 such that for all $n > N_3$ and $x \in [a,b]$ we have $|f_n(x) - f(x)| < \epsilon$. Therefore, we can set $N = \max\{N_1, N_2, N_3\}$ and for $n > N$, we have the following:

$$\left|\int_{a_n}^{b_n} f_n(x)dx - \int_a^b f(x)dx\right| = \left|\int_{a_n}^{b_n} (f_n(x) - f(x))dx - \int_a^{a_n} f(x)dx - \int_{b_n}^{b} f(x)dx\right| \tag{60.2}$$

$$\le |f_n(x) - f(x)|(b_n - a_n) - |f(x)|(a_n - a) - |f(x)|(b - b_n) \tag{60.3}$$

$$< \epsilon(b + \epsilon - \epsilon - a) - 2M\epsilon \tag{60.4}$$

$$= \epsilon(b - a) - 2M\epsilon \tag{60.5}$$

Since $\epsilon > 0$ was arbitrary, we have proven the theorem. □

61. Problem 61

Theorem 61.1. *Let X be a topological space and let $\{f_n\}$ be a sequence of real-valued continuous functions defined on X. Suppose that there is a function $f : X \to \mathbb{R}$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ holds for all $x \in X$. Show that f is continuous at a point a if and only if for each $\epsilon > 0$ and each m there exists a neighborhood $V(a)$ and some $k > m$ such that $|f(x) - f_k(x)| < \epsilon$ holds for all $x \in V(a)$.*

Proof. First, we will assume that f is continuous at a and fix $\epsilon > 0$. This implies that there is a neighborhood $V(a)$ such that for all $x \in V(a)$, we have $|f(x) - f(a)| < \epsilon$. Since $f_n(x) \to f(x)$ for all $x \in X$, we see that there exists some m such that for all $n > m$, $|f(x) - f_n(x)| < \epsilon$. This means that there exists a $k > m$ such that:

$$|f(x) - f_k(x)| \leq |f(x) - f(a)| + |f(a) - f_k(a)| + |f_k(a) - f_k(x)| < 3\epsilon \tag{61.2}$$

To prove the converse, we first assume that there exists an m and neighborhood $V(a)$ such that for some $k > m$ we have $|f(x) - f_k(x)| < \epsilon$ for all $x \in V(a)$. We know that we have $|f_k(x) - f_k(a)| < \epsilon$ if $x \in V(a)$. Moreover, we see that $|f_k(a) - f(a)| < \epsilon$ because of pointwise convergence. Therefore, for $k > m$, we have the following:

$$|f(x) - f(a)| \leq |f(x) - f_k(x)| + |f_k(x) - f_k(a)| + |f_k(a) - f(a)| < 3\epsilon \tag{61.3}$$

This proves that f is continuous at a and completes the theorem. □

62. Problem 62

Theorem 62.1. *Let $\{f_n\}$ be a uniformly bounded sequence of continuous real valued functions on a closed interval $[a, b]$. Show that the sequence of functions $\{\phi_n\}$ defined by $\phi_n(x) = \int_a^x f_n(t)dt$ for each $x \in [a, b]$ contains a uniformly convergent subsequence.*

Proof. Since $\{f_n\}$ is uniformly bounded, we know that there exists an M such that for all $x \in [a, b]$ and all $n \in \mathbb{N}$, we have $|f_n(x)| < M$. Therefore, we see that $|\phi_n(x)| = \int_a^x |f_n(t)|dt \leq |f_n(t)|(x - a) \leq M(b - a)$. Thus, we see that the sequence $\{\phi_n\}$ is pointwise bounded on $[a, b]$. Moreover, we can show that the sequence is also equicontinuous. This is because for all $x, y \in [a, b]$, we have the following (assuming without loss of generality that $x < y$):

$$
\begin{aligned}
|\phi_n(x) - \phi_n(y)| &= \left| \int_a^x f_n(t)dt - \int_a^y f_n(t)dt \right| && (62.2)\\
&= \left| \int_x^y f_n(t)dt \right| && (62.3)\\
&\leq |f_n(t)|(y - x) && (62.4)\\
&< M(y - x) && (62.5)
\end{aligned}
$$

Therefore, we can choose $\delta = \frac{\epsilon}{M}$ so that if $|y - x| < \delta$, then we have for all $x, y \in [a, b]$ and $n \in \mathbb{N}$, $|\phi_n(x) - \phi_n(y)| < M\delta = \epsilon$. Therefore, we have shown that the set $A = \{\phi_1, \phi_2, \ldots\}$ is equicontinuous. Moreover, we know that $\bar{A}$ is closed by definition, bounded, and equicontinuous. Therefore, we see that $\bar{A}$ is compact, and that $\{\phi_n\}$ contains a uniformly convergent subsequence on $\bar{A}$ by the Arzela-Ascoli theorem. □

63. Problem 63

Theorem 63.1. *Consider a continuous function $f : [0, \infty) \to \mathbb{R}$. For each n defined the continuous function $f_n : [0, \infty] \to \mathbb{R}$ by $f_n(x) = f(x^n)$. Show that the set of continuous functions $\{f_1, f_2, \ldots\}$ is equicontinuous at $x = 1$ if and only if f is a constant function.*

Proof. First, it is easy to show that if f is a constant function, then $A = \{f_1, f_2, \ldots\}$ is equicontinuous at $x = 1$. This is because $f(1) = f(x^n)$ for $x \in [0, \infty)$ and $n \in \mathbb{N}$, which implies that for all $\epsilon > 0$ we have $\epsilon > |f(1) - f(x^n)| = |f(1) - f_n(x)| = |f_n(y) - f_n(x)|$ for all $x, y \in [0, \infty)$.

Now, we shall prove that if A is equicontinuous at $x = 1$, then f is a constant function. We see that the equicontinuity condition implies that for all $n \in \mathbb{N}$ we have $|f_n(1) - f_n(x)| < \epsilon$ whenever $|1 - x| < \delta$ for $x \in [0, \infty)$. From the fact that $\lim_{n\to\infty} \sqrt[n]{a} = 1$, we see that there exists some n_0 for which $n > n_0$ implies $|f_n(1) - f_n(\sqrt[n]{a})| = |f(1) - f(\sqrt[n]{a}^n)| < \epsilon$. This shows that $f(1) = f(a)$ for all $a \in [0, \infty)$, which completes the proof. □

64. Problem 64

Theorem 64.1. *Let (X, d) be a compact metric space and let A be an equicontinuous subset of $C(X)$. Show that A is uniformly equicontinuous, i.e. show that $\forall\epsilon > 0$, $\exists\delta > 0$ such that $x, y \in X$ and $d(x, y) < \delta$ imply $|f(x) - f(y)| < \epsilon$ for all $f \in A$.*

Proof. Fix $\epsilon > 0$. Equicontinuity implies that there exists a $\delta_x > 0$ such that $d(f(x), f(y)) < \epsilon$ implies $d(x, y) < \delta_x/2$ for all $x, y \in X$ and $f \in A$. Since X is compact, each open cover has a finite subcover, which implies that we can take a set $\{x_1, x_2, \dots, x_p\}$ and neighborhoods about them with radii $\delta_1, \delta_2, \dots, \delta_p$ respectively that cover X. Formally, we have $X \subset \bigcup_{1\le i\le p} N_{\delta_i}(x_i)$ with $x_i \in X$ for all $1 \le i \le p$.

Now pick $x \in X$. There must exist some i for which $d(x, x_i) < \delta_i/2$. Now set $\delta = \frac{1}{2}\min\{\delta_1, \dots, \delta_p\}$ and pick $y \in X$. Now pick $y \in X$ such that x, y satisfy $d(x, y) < \delta$. Then we see that $d(f(x), f(y)) < \delta_i/2$ for all $f \in A$ and see that:

$$d(y, x_i) \le d(y, x) + d(x, x_i) < 2\delta_i/2 = \delta_i \tag{64.2}$$

Therefore, if $d(y, x_i) < \delta_i$ implies that $d(f(y), f(x_i)) < \epsilon$. This means that if $d(x, y) < \delta$, then we have:

$$d(f(x), f(y)) \le d(f(x), f(x_i)) + d(f(x_i), y) < 2\epsilon \tag{64.3}$$

This completes the proof. □

65. Problem 65

Theorem 65.1. *Let $\{f_n\}$ be an equicontinuous sequence in $C(X)$ where X is not necessarily compact. If for some function $f : X \to \mathbb{R}$ we have $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$, then show that $f \in C(X)$.*

Proof. By equicontinuity of f_n, we see that for a fixed $\epsilon > 0$, there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ whenever $|x - y| < \delta$. Moreover, we see that there exists some N for which $n > N$ implies $|f_n(x) - f(x)| < \epsilon$. Therefore, we see that:

$$|f(x) - f(y)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < 3\epsilon \tag{65.2}$$

This shows that f is continuous, and hence that $f \in C(X)$, which completes the proof. □

66. Problem 66

Theorem 66.1. *Let X be a compact topological space and let $\{f_n\}$ be an equicontinuous sequence of $C(X)$. Assume that there exists some $f \in C(X)$ and some dense subset A of X such that $\lim_{n\to\infty} f_n(x) = f(x)$ holds for each $x \in A$. Then show that $\{f_n\}$ converges uniformly to f.*

Proof. Since $\{f_n\}$ is equicontinuous, we can fix $\epsilon > 0$ and see that there exists $\delta_x > 0$ for all $n \in \mathbb{N}$ for which $|f_n(x) - f_n(y)| < \epsilon$ implies $|x - y| < \delta_x$ for all $x, y \in X$. Next, we see that $f_n(x)$ converges pointwise to $f(x)$, which shows that there exists some N such that for all $n > N$, we have $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$. So let $y \in X$, and we see that $|f(x) - f(y)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < 3\epsilon$.

Now since X is compact, we know that we can choose finitely many points $x_i \in X$ such that $X \subset \bigcup_{1\le i\le p} N_{\delta_i}(x_i)$ for p finite. Now pick some point $x \in X$. We see that there exists some x_i for which $|x - x_i| < \delta_i$, implying that $|f_n(x) - f_n(x_i)| < \epsilon$ for all $n \in \mathbb{N}$ by equicontinuity. Thus, we have:

$$|f_n(y) - f(y)| \le |f_n(y) - f_n(x)| + |f_n(x) - f(x)| + |f(x) - f(y)| < 2\epsilon + \epsilon + 3\epsilon = 5\epsilon \tag{66.2}$$

This completes the proof that $\{f_n\}$ converges uniformly to f. □

67. Problem 67

Theorem 67.1. *Let (X, d) be a metric space and let A be a nonempty subset of X. The distance function of A is $d(x, A) : X \to \mathbb{R}$ defined by $d(x, A) = \inf\{d(x, a) : a \in A\}$. Show that $d(x, A) = 0$ if and only if $x \in \bar{A}$.*

Proof. First, if $x \in \bar{A}$, we can have either $x \in A$ or $x \in A'$. For the first case, we see that $x = a$ for some $a \in A$, which shows that $d(x, a) = 0$ for that same $a \in A$. This implies that $d(x, A) = 0$. In the second case, we see that $x \in A'$ implies that for all $\epsilon > 0$, there exists an $a \in A$ such that $d(x, a) < \epsilon$. This shows that $d(x, A) < \epsilon$ for all $\epsilon > 0$, and shows that $d(x, A) = 0$ if $x \in \bar{A}$.

Now assume that $d(x, A) = 0$ and we will show that $x \in \bar{A}$. First, we note that $d(x, A) = 0$ implies that for all $\epsilon > 0$, we have $d(x, A) < \epsilon$, which shows that $d(x, a) < \epsilon$ for all $x \in A$. This is the definition of a limit point of A, so that $x \in \bar{A}$. □

68. Problem 68

Theorem 68.1. *Let A, B be two nonempty subsets of a metric space X such that $A \cap \bar{B} = \bar{A} \cap B = \emptyset$. Show that there exist two open sets U, V such that $A \subset U$ and $B \subset V$.*

Proof. First, take the function $f(x) = d(x, A) - d(x, B)$ where $d(x, A)$ is defined as in problem 67. We know, by problem 10.2 in Aliprantis, that f is continuous. Moreover, since $A \cap \bar{B} = \emptyset$, we see that $f(x) = -d(x, B) < 0$ whenever $x \in A$. By the same logic, we see that because $\bar{A} \cap B = \emptyset$ that $f(x) = d(x, A) > 0$ whenever $x \in B$. This shows that $A \subset U = f^{-1}((-\infty, 0))$ and $B \subset V = f^{-1}((0, \infty))$. Moreover, since $(-\infty, 0)$ and $(0, \infty)$ are both open sets and f is continuous, we see that U, V are open, disjoint sets. ☐

69. Problem 69

Theorem 69.1. *If f is continuous on $[0, 1]$ such that $\int_0^1 x^n f(x) dx = 0$ for all $n \in \mathbb{N}$, show that $f(x) = 0$ for all $x \in [0, 1]$.*

Proof. We note that since f is continuous, we can use the Stone-Weierstrass theorem to show that there exists a sequence of polynomials $\{p_n\}$ which converges to f. Because this is the case, we see that $\{p_n(x)f(x)\}$ converges to $f(x)^2$. Next, we see that $p_n(x)$ is made up of monomials of x, so that $p_n(x) = c_0 x_0 + x_1 x_1 + \ldots + x_p x_p$ where c_i are constants. We see then that $\int_0^1 p_n(x) f(x) = \int_0^1 f(x)(c_0 x_0 + c_1 x_1 + \ldots + c_p x_p) dx = 0$. Therefore, we see that $\lim_{n \to \infty} \int_0^1 p_n(x) f(x) = \int_0^1 f(x)^2 dx = 0$, which implies that $f(x) = 0$ for all $x \in [0, 1]$. ☐

70. Problem 70

Theorem 70.1. *Let $f : (X, d) \to (Y, \rho)$ be a function. Show that f is continuous if and only if f restricted to the compact subsets of X is continuous.*

Proof. If f is continuous, then it is clear that it is continuous when restricted to the compact subsets. Now, to prove the converse, assume $x_n \to x$. We see that the set $A = \{x_1, x_2, \ldots\} \cup \{x\}$ is compact because the sequence's convergence implies that every open cover has a finite subcover. Therefore, we see that $x_n \to x$ holds on A, which implies that $f(x_n) \to f(x)$ holds. This implies that f is continuous. ☐

71. Problem 71

Theorem 71.1. *Show that every compact space (X, d) is separable (contains a countable dense subset).*

Proof. Let F_n be a finite subcover such that $X = \bigcup_{x \in F_n} N_{1/n}(x)$. Now let $F = \bigcup_{n=1}^{\infty} F_n$. It is clear that F is an open cover of X. So now we can pick $x \in X$ and some $\delta > 0$. We will always be able to find some $y \in F_n$ for some n such that $d(x, y) < \delta$. This is because we can set $1/n < \delta$ so that $d(x, y) < \delta$. This shows that F is a dense subset. Moreover, we know that F is at most countable because it is the union of a set of finite subcovers of X. This shows that F is a countable dense subset. ☐

72. Problem 72

Theorem 72.1. *A family of continuous functions $\mathscr{F}$ is said to have the finite intersection property if every finite intersection of sets of $\mathscr{F}$ is nonempty. Show that a metric space is compact if and only if every family of closed sets with the finite intersection property has a nonempty intersection.*

Proof. Let $\{A_i : i \in I\}$ be a family of closed sets with the finite intersection property and assume that X is compact. Suppose by contradiction that $\bigcap_{i \in I} A_i = \emptyset$. This implies that $\bigcup_{i \in I} A_i^c = X$ is an open cover of X because A_i^c is open. Therefore, since X is compact, we see that this open cover has a finite subcover, namely $\bigcup_{i=1}^n A_i^c = X$. This implies that $\bigcap_{i=1}^n A_i = \emptyset$, which is a contradiction of the finite intersection property. Therefore, $\{A_i : i \in I\}$ must have a nonempty intersection.

To show the converse, we now assume that every family of closed sets with the finite intersection property has a nonempty intersection. Now let V_i be open sets such that $X = \bigcup_{i \in I} V_i$ is an open cover of X. Therefore, we see that $\bigcap_{i \in I} V_i^c = \emptyset$. Moreover, since V_i^c is closed, we see by assumption that $\bigcap_{i \in I} V_i^c \neq \emptyset$. This implies that our set $\{V_i : i \in I\}$ does not have the finite intersection property, which shows that there exist some finite n for which $\bigcap_{i=1}^n V_i^c = \emptyset$ holds. This shows that $\bigcup_{i=1}^n V_i = X$ is a finite subcover of $\bigcup_{i \in I} V_i$, which proves the compactness of X. ☐

73. Problem 73

Theorem 73.1. *Let $f : X \to X$ function from a set X into itself. A point $a \in X$ is called a fixed point for f if $f(a) = a$. Assume that (X, d) is compact and $f : X \to X$ satisfies $d(f(x), f(y)) < d(x, y)$ for $x \neq y$. Show that f has a unique fixed point.*

Proof. First, it is clear that f cannot have more than one fixed point. If $f(a) = a$ and $f(b) = b$, then:

$$d(a, b) = d(f(a), f(b)) < d(a, b) \tag{73.2}$$

This only holds when $a = b$, so there can only be a single fixed point. Now we shall show that there is at least 1 fixed point. Define $g(x) = d(x, f(x))$ so that we have the following inequality for $x, y \in X$:

$$|g(x) - g(y)| \quad = \quad d(x, f(x)) - d(y, f(y)) \leq d(x, y) + d(y, f(x)) - d(y, f(y)) \tag{73.3}$$

By the triangle inequality, we see that $d(y, f(x)) \leq d(y, f(y)) + d(f(y), f(x))$ which implies that $d(y, f(x)) - d(y, f(y)) \leq d(f(y), f(x))$. This shows the following:

$$|g(x) - g(y)| \leq d(x, y) + d(f(y), f(x)) < 2d(x, y) \tag{73.4}$$

We can choose $\delta = \epsilon/2$ and show that for all $\epsilon > 0$, we can $|g(x) - g(y)| < 2\delta < \epsilon$ whenever $d(x, y) < \delta$. This proves the continuity of g on X. Now, since X is compact, we know that g attains a minimum at some point $a \in X$. Therefore, we have $g(a) \leq g(x)$ for all $x \neq a$. This implies that $d(a, f(a)) \leq d(x, f(x))$ for all $x \neq a$. Assume by contradiction that a is not a fixed point, so that $f(a) \neq a$. Then we see that $d(f(a), f(f(a))) < d(a, f(a)) \neq 0$, which is a contradiction of the fact that $g(a)$ is a minimum. This shows that f has at least one fixed point, and since we know it cannot have more than one fixed point, this a must be unique. □

74. Problem 74

Theorem 74.1. *Find functions f and g such that $\lim_{x\to a} f(x) = A$ and $\lim_{y\to A} g(y) = B$, but for which $\lim_{x\to a} g(f(x)) \neq B$.*

Proof. Let $f(x) = (-x)^x$ and let $g(y) = \sqrt{y}$. We see the following is true:

$$\lim_{x\to 2} f(x) \quad = \quad \lim_{x\to 2} (-x)^x = 4 \tag{74.2}$$

$$\lim_{y\to 4} g(y) \quad = \quad \lim_{y\to 4} \sqrt{x} = 2 \tag{74.3}$$

So we see that $A = 4$ and $B = 2$. However, we see that the composition $g(f(x)) = \sqrt{(-x)^x} = (-x)^{x/2}$. Thus, we have:

$$\lim_{x\to 2} g(f(x)) = \lim_{x\to 2} (-x)^{x/2} = -2 \tag{74.4}$$

We see that $2 \neq -2$, so we are done. □

75. Problem 75

Theorem 75.1. *Show that $\lim_{x\to 0} \ln(1 + x) = 0$. Using this equality, deduce that the logarithmic function is continuous on $(0, \infty)$.*

Proof. We know that $0 < \ln(1 + \frac{1}{n}) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Thus, fix $\epsilon > 0$, and we can find n_0 such that $\frac{1}{n_0 - 1} < \epsilon$. Thus, for $|x| < \frac{1}{n_0}$ we have:

$$-\epsilon < -\frac{1}{n_0} < \ln(1 - \frac{1}{n_0}) < \ln(1 + x) < \ln(1 - \frac{1}{n_0} < \frac{1}{n_0} < \epsilon \tag{75.2}$$

Since $\epsilon > 0$ was arbitrary, we see that $\ln(1 + x) \to 0$ as $x \to 0$. To prove continuity, take $x_0 \in (0, \infty)$. Then we have:

$$\lim_{x\to x_0} \ln(x) = \lim_{x\to x_0} \ln(x_0) + \ln\left(\frac{x}{x_0}\right) = \ln(x_0) + \lim_{y\to 1} \ln(y) = \ln(x_0) + \lim_{t\to 0} \ln(1 + t) = \ln(x_0) \tag{75.3}$$

Since we have shown that $\ln(x) \to \ln(x_0)$ as $x \to x_0$ for all $x_0 \in (0, \infty)$, we have shown the continuity of $\ln(x)$ on $(0, \infty)$. □

76. Problem 76

Theorem 76.1. *Suppose that f is continuous on $[a, \infty)$ and $\lim_{x\to\infty} f(x)$ is finite. Show that f is bounded on $[a, \infty)$.*

Proof. Let $c = \lim_{x\to\infty} f(x) < \infty$. Pick $\epsilon > 0$. We see that there exists an $M > 0$ such that for all $x > M$, we have $|f(x) - c| < \epsilon$. This means that $c - \epsilon < |f(x)| < c + \epsilon$, which shows that for $x \in (M, \infty)$, $|f(x)|$ is bounded. Moreover, by continuity, we see that $f(x)$ is bounded for $[a, M]$ because for all $N > 0$, we can choose $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < N$ for all $x, y \in [a, M]$. Thus, we can just choose the smallest δ for which $|M - a| < \delta$, and see that $|f(x) - f(y)| < N$. Therefore, we see that f is bounded on $[a, \infty)$. □

77. Problem 77

Theorem 77.1. *Show that if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous and periodic with $\lim_{x\to\infty} f(x) - g(x) = 0$, then $f = g$.*

Proof. Suppose by contradiction that this is not the case. Then there exists some $x_0 \in \mathbb{R}$ such that $f(x_0) \neq g(x_0)$. Assume $f(x_0) > g(x_0)$ without loss of generality, this shows that $f(x_0) - g(x_0) = M > 0$. Let f and g have periods T_1 and T_2 respectively. Now pick ϵ such that $0 < \epsilon < \frac{M}{2}$. By our assumption, we see that there exists an $k > 0$ such that for all $x > x_0 + kT_2$, we have $|f(x) - g(x)| < \epsilon$. By the continuity assumption, we see that there exists a $\delta > 0$ such that $|f(x + h) - f(x)| < \epsilon$ when $|h| < \delta$. Therefore, if we take $k \in \mathbb{Z}$, we have:

$$
\begin{aligned}
|f(x_0) - g(x_0)| &\leq |f(x_0) - f(x_0 + kmT_2)| + |f(x_0 + kmT_2) - g(x_0 + kmT_2)| && (77.2)\\
&= |f(x_0) - f(x_0 + kmT_2 + nT_1)| + |f(x_0 + kmT_2) - g(x_0 + kmT_2)| && (77.3)\\
&< 2\epsilon && (77.4)
\end{aligned}
$$

Where $n \in \mathbb{Z}$. This inequality holds whenever $|kmT_2 + nT_1| < \delta$. However, this implies that $M < 2\epsilon$, where $0 < \epsilon < \frac{M}{2}$, which is a contradiction. Thus, we must see that $f = g$. □

78. Problem 78

Theorem 78.1. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with positive fundamental periods T_1 and T_2 respectively. Prove that if $\frac{T_1}{T_2} \notin \mathbb{Q}$, then $h = f + g$ is not periodic.*

Proof. Suppose by contradicion that h is periodic with fundamental period T. Then we see that $\frac{T}{T_1} \notin \mathbb{Q}$ or $\frac{T}{T_2} \notin \mathbb{Q}$. Assume the first case without loss of generality. Then we see that $h(x) = h(x + T)$ implies that $g(x) + f(x) = g(x + T) + f(x + T)$. Define a new function $H(x) = g(x + T) - g(x) = f(x + T) - f(x)$. It is clear that this function is constant and therefore continuous. So let $H(x) = c$ for some $c \in \mathbb{R}$.

It follows that $f(x + T) = f(x) + c$. Assume that $c \neq 0$. Then if we substitute $x = 0$ into the expression, we find $f(T) = f(0) + c$. Substitute $x = T$ into the expression to find $f(2T) = f(T) + c = f(0) + 2c$. Continuing with this process, we see that $f(nT) = f(0) + nc$ where $n \in \mathbb{N}$. However, this is a contradiction because g is no longer bounded even though it is a continuous, periodic function. Therefore, we see that $c = 0$.

So we have deduced that $f(x + T) = f(x)$. This shows that T must be a period of f so that $T = kT_1$ where $k \in \mathbb{Z}$. However, this implies that $\frac{T}{T_1} = k \in \mathbb{Z}$, which is a contradiction because we have assumed $\frac{T}{T_1} \notin \mathbb{Q}$. Therefore, we see that h is not periodic. □

79. Problem 79

Theorem 79.1. *Suppose $f : (0, \infty) \to \mathbb{R}$ is continuous such that $f(x) \leq f(nx)$ for all $x > 0$ and $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} f(x)$ exists.*

Proof. Let $M = \lim_{r\to\infty} \sup_{x\geq r} f(x)$ and $m = \lim_{r\to\infty} \inf_{x\geq r} f(x)$. Assume by contradiction that $M > m$. Then there exists a real number k such that $m < k < M$. Since f is continuous, there must be some $a \in (0, \infty)$ such that $f(a) > k$. Moreover, continuity implies that there exists a b such that for all $t \in [a, b]$, we have $f(t) > k$. Now define $p = \frac{ab}{a-b}$. If $x \geq p$, then we have the following:

$$
\frac{x(a-b)}{ab} \geq 1 \quad \Rightarrow \quad \frac{x}{a} \geq 1 + \frac{x}{b} \tag{79.2}
$$

This implies that there exists some integer n_0 for which $\frac{x}{a} \geq n_0 \geq \frac{x}{b}$. Equivalently, we see that $a \leq \frac{x}{n_0} \leq b$. Therefore, we have the following:

$$f(x) = f\left(n_0 \frac{x}{n_0}\right) \geq f\left(\frac{x}{n_0}\right) > k \tag{79.3}$$

For all $x \geq p$. This contradicts the definition of m because there is actually a larger number k for which $f(x) > k$ as $x \to \infty$. Therefore, we see that $m = M$, which shows that $\lim_{x\to\infty} f(x)$ exists. □

80. Problem 80

Theorem 80.1. *Let $f : [0,1] \to [0,1]$ be continuous. Show tht f has a fixed point on $[0,1]$.*

Proof. Define the function $g(x) = f(x) - x$ which is continuous because both $f(x)$ and x are continuous. We see that $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$. Thus, by intermediate value theorem, we see that there exists some $x_0 \in [0,1]$ such that $g(x_0) = 0$, which implies that $f(x_0) = x_0$. This is the definition of a fixed point, so f has at least one fixed point. □

81. Problem 81

Theorem 81.1. *Assume that $f, g : [a,b] \to \mathbb{R}$ are continuous such that $f(a) < g(a)$ and $g(b) > g(b)$. Prove that there exists an $x_0 \in [a,b]$ such that $f(x_0) = g(x_0)$.*

Proof. Define the function $h(x) = f(x) - g(x)$. We see that $h(x)$ is continuous because it is the addition of two continuous functions. Moreover, we see that $h(a) = f(a) - g(a) < 0$ while $h(b) = f(b) - g(b) > 0$. Thus, by the intermediate value theorem, there exists some x_0 such that $h(x_0) = 0$. This implies that $f(x_0) = g(x_0)$ for some $x_0 \in [a,b]$. □

82. Problem 82

Theorem 82.1. *A function $f : (a,b) \to \mathbb{R}$ is continuous. Prove that given $x_1, x_2, \ldots, x_n$ in (a,b), there exists $x_0 \in (a,b)$ such that $f(x_0) = \frac{1}{2}(f(x_1) + f(x_2) + \ldots + f(x_n))$.*

Proof. Let $m = \min\{f(x_1), \ldots, f(x_n)\}$ and let $M = \max\{f(x_1), \ldots, f(x_n)\}$. We see that $m \leq \frac{1}{2}(f(x_1) + \ldots + f(x_n)) \leq M$. Moreover, since f is continuous, we can use the intermediate value theorem. We see that there exists some x_0 such that $f(x_0) = \frac{1}{2}(f(x_1) + \ldots + f(x_n))$. □

83. Problem 83

Theorem 83.1. *Prove that $(1-x)\cos x = \sin x$ has at least one solution in $(0,1)$.*

Proof. Let $g(x) = (1-x)\cos x - \sin x$. We see that $g(0) = 1$ while $g(1) = -\sin x < 0$. Moreover, we see that $g(x)$ is continuous because it is composed entirely of continuous functions. Therefore, we can apply the intermediate value theorem and see that there exists some x_0 such that $g(x_0) = 0$. This shows that there is at least one solution x_0 to the equation. □

84. Problem 84

Theorem 84.1. *For a nonzero polynomial, show that $|P(x)| = e^x$ has at least one solution.*

Proof. We know that $\lim_{x\to\infty} |P(x)|e^{-x} = 0$ while $\lim_{x\to-\infty} |P(x)|e^{-x} = \infty$. Since polynomials are continuous and e^{-x} is also continuous, we see that $|P(x)|e^{-x}$ is continuous and thus has the intermediate value property. Therefore, we see that there exists some $x_0 \in \mathbb{R}$ such that $|P(x_0)e^{-x_0} = 1$. This would imply that $|P(x_0)| = e^{x_0}$, which is what we wanted. □

85. Problem 85

Theorem 85.1. *Suppose f, g have the intermediate value property on $[a,b]$. Must $f+g$ posses the intermediate value property on $[a,b]$?*

Proof. No, consider the following functions:

$$f(x) = \begin{cases} \sin\left(\frac{1}{x-a}\right) & a < x \leq b \\ 0 & a = x \end{cases} \tag{85.2}$$

$$g(x) = \begin{cases} -\sin\left(\frac{1}{x-a}\right) & a < x \leq b \\ -1 & a = x \end{cases} \tag{85.3}$$

□

86. Problem 86

Theorem 86.1. *Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $f(f(x)) = f^2(x) = -x, x \in \mathbb{R}$. Show that f cannot be continuous.*

Proof. Assume the contrary. Then if $f(x_1) = f(x_2)$, we see that we must have $-x_1 = f(f(x_1)) = f(f(x_2)) = -x_2$. Therefore, we must have $x_1 = x_2$ whenever $f(x_1) = f(x_2)$. It follows that f is either strictly increaisng or strictly decreasing. It follows that taking $f(f(x))$, we must have f^2 be strictly increasing. However, this is a contradiction of the fact that $f^2(x) = -x$, which is strictly decreasing. Therefore, f cannot be continuous. □

87. Problem 87

Theorem 87.1. *Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ which attains each of its values exactly three times.*

Proof. Let us define two functions as follows:

$$g(x) = \begin{cases} x+2 & -3 \leq x < -1 \\ -x & -1 \leq x < 1 \\ x-2 & 1 \leq x \leq 3 \end{cases} \tag{87.2}$$

Now define $f(x) = g(x+6n)+2n$ for all $6n+3 \leq x \leq 6n-3$ and $n \in \mathbb{N}$. We see that $f(x)$ has the desired properties. □

88. Problem 88

Theorem 88.1. *Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ which attains each of its values exactly two times?*

Proof. No there does not. Assume by contradiction that there does. Let us take two points x_1, x_2 such that $f(x_1) = f(x_2) = c$ and $f(x) \neq c$ for all $x \neq x_1, x_2 \in \mathbb{R}$. Therefore, we can assume that either $f(x) > c$ or $f(x) < c$ for all other x. Let us assume without loss of generality that $f(x) > c$ for some x. Then we see that there exists some x_0 for which $f(x_0) = \max\{f(x) : x \in [x_1, x_2]\}$ by the fact that $[x_1, x_2]$ is a closed, bounded, and hence compact space. Moreover, f can attain its maximum in $[x_1, x_2]$ only once, otherwise the maximum would be attained more than twice. Therefore, there is exactly one point x' outside the interval $[x_1, x_2]$ such that $f(x') = f(x_0) = b > c$. The intermediate value theorem says that every value in the interval $[c, b]$ is attained at least three times, which is a contradiction. Therefore, there does not exist a continuous function which attains each of its values exactly three times. □

89. Problem 89

Theorem 89.1. *A function $f : [0,1] \to \mathbb{R}$ satisfies $f(0) < 0$ and $f(1) > 0$, and there exists a function g continuous on $[0,1]$ such that $f + g$ is decreasing. Prove that the equation $f(x) = 0$ has a solution in the open interval $(0,1)$.*

Proof. Let $A = \{x \in [0,1] : f(x) \geq 0\}$ and let $s = \inf A$. Define the function $h(x) = f(x) + g(x)$. Since h is decreasing by assumption, we see that $h(s) \geq h(x) \geq g(x)$ for all $x \in A$. Since g is continuous, it follows that $h(s) \geq g(s)$. Therefore, we must have $f(s) \geq 0$. By our assumptions, we must have $g(0) > h(0) \geq h(s) \geq g(s)$. Applying the intermediate value theorem, we see that there exists some $t \in (0, s)$ such that $g(t) = h(s)$. Then we know that $h(t) \geq h(s) = g(t)$, so that $h(t) \geq g(t)$, which implies that $f(t) \geq 0$. By the definition of s, we see that $t = s$. This implies that $h(t) = g(t)$, so that $f(t) = 0$. □

90. Problem 90

Theorem 90.1. *Prove that if f is uniformly continuous on $(a, b]$ and on $[b, c)$, then it is uniformly continuous on (a, c).*

Proof. Fix $\epsilon > 0$. Now, we see that there exists a $\delta_1 > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta_1$ for $x, y \in (a, b]$, and there also exists a $\delta_2 > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta_2$ for $x, y \in [b, c)$. Therefore, we can take $\delta = \min\{\delta_1, \delta_2\}$ so that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$ for all $x, y \in (a, c)$. This completes the proof. □

91. Problem 91

Theorem 91.1. *Prove that any function continuous and periodic on $\mathbb{R}$ must be uniformly continuous on $\mathbb{R}$.*

Proof. Let f have a period of T. Then we see that f is continuous on $[0, T]$, which in turn implies it is uniformly convergent because $[0, T]$ is closed, bounded, and hence compact. Moreover, we see that f is uniformly continuous on all intervals $[T, 2T], [2T, 3T], \dots, [nT, (n+1)T]$ for all $n \in \mathbb{Z}$. We see that $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [nT, (n+1)T]$, so that these intervals cover $\mathbb{R}$ entirely. Thus, by the previous theorem in problem 90, we see that f must be uniformly convergent on the entire $\mathbb{R}$. $\square$

92. Problem 92

Theorem 92.1. *Let $f : [1, \infty) \to \mathbb{R}$ be uniformly continuous. Prove that there is a positive M such that $\frac{|f(x)|}{x} \leq M$ for $x \geq 1$.*

Proof. By uniform continuity, we can fix $\epsilon = 1$ and find a $\delta > 0$ such that $|f(x) - f(x')| < 1$ whenever $|x - x'| < \delta$ for all $x, x' \in [1, \infty)$. Moreover, we can write all $x \in [1, \infty)$ as the following: $x = 1 + \delta n + r$ where $n \in \mathbb{Z}_+$ and $r \in [0, \delta)$. Therefore, we have:

$$|f(x)| \leq |f(1)| + |f(1) - f(x)| \leq |f(1)| + (n+1) \tag{92.2}$$

Dividing by x, this gives:

$$\frac{|f(x)|}{x} \leq \frac{|f(1)| + (n+1)}{1 + \delta n + r} \leq \frac{|f(1)|}{\delta} + \frac{n+1}{\delta n} \leq \frac{|f(1)| + 2}{\delta} = M \tag{92.3}$$

$\square$

93. Problem 93

Theorem 93.1. *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at zero and satisfies the following conditions: $f(0) = 0$ and $f(x_1 + x_2) \leq f(x_1) + f(x_2)$ for any $x_1, x_2 \in \mathbb{R}$. Prove that f is uniformly continuous on $\mathbb{R}$.*

Proof. Fix $\epsilon > 0$. Then we see that there exists some $\delta > 0$ such that $|f(x) - f(0)| < \epsilon$ whenever $|x| < \delta$ by the continuity of f at 0. This means that $|f(x)| < \epsilon$ for all $|x| < \delta$. Moreover, we see that if we pick some $|t| < \delta$, then we have:

$$|f(t + x) - f(x)| \leq |f(t) + f(x) - f(x)| = |f(t)| < \epsilon \tag{93.2}$$

This shows that $|f(t + x) - f(x)| < \epsilon$ whenever $|t| < \delta$, which shows that f is uniformly continuous. $\square$

94. Problem 94

Theorem 94.1. *Prove that any function which is bounded, monotonic and continuous on an interval $I \subset \mathbb{R}$ is uniformly continuous on I.*

Proof. Assume that $I = (a, b)$ is a bounded interval and f is monotonically increasing without loss of generality. Then, we can see that $\lim_{x \to a^+} f(x) = \inf_{x \in (a,b)} f(x)$ and $\lim_{x \to b^-} f(x) = \sup_{x \in (a,b)} f(x)$. Therefore, we can extend $f(x)$ to the interval $I = [a, b]$ and see that it is uniformly continuous because this is a compact set. If $I = (a, b)$ is unbounded, then the limits $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ exist and are finite. Therefore, f is uniformly convergent by a previous problem. $\square$

95. Problem 95

Theorem 95.1. *Assume that f is a continuous mapping of a connected metric space X into a metric space Y. Show that $f(X)$ is connected in Y.*

Proof. Suppose the contrary. Then $f(X) = A \cup B$ such that $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ and A, B are nonempty, open, disjoint sets. Then we see that $X = f^{-1}(A \cup B)$. Now let U, V be sets such that $f(U) = A$ and $f(V) = B$. Since A, B are open, we see that $U = f^{-1}(A)$ and $V = f^{-1}(B)$ are also open and nonempty. Moreover, they are disjoint and $X = U \cup V$, which is a contradiction of the fact that f is connected. Hence, $f(X)$ must also be connected in Y. $\square$

96. Problem 96

Theorem 96.1. *Let (X, d) be a metric space and let A be a nonempty subset of X. Prove that the function $f : X \to [0, \infty)$ defined by $f(x) = dist(x, A) = \inf\{d(x, y) : y \in A\}$ is uniformly continuous on X.*

Proof. For $x_0, x \in X$ and $y \in A$, we have the following:

$$dist(x, A) \leq d(x, y) \leq d(x, x_0) + d(x_0, y) \tag{96.2}$$

Therefore, we see that the following is true:

$$dist(x, A) - d(x_0, y) \leq d(x, x_0) \tag{96.3}$$
$$dist(x, A) - dist(x_0, A) \leq d(x, x_0) \tag{96.4}$$

Therefore, we find that:

$$|dist(x, A) - dist(x_0, A)| \leq d(x, x_0) \tag{96.5}$$

Therefore, we see that $|f(x) - f(x_0)| < \delta$ where $d(x, x_0) < \delta$, which implies uniform continuity. □

97. Problem 97

Theorem 97.1. *Prove that if $|a_1 \sin x + a_2 \sin 2x + \ldots + a_n \sin nx| \leq |\sin x|$ for $x \in \mathbb{R}$, then $|a_1 + 2a_2 + \ldots + na_n| \leq 1$.*

Proof. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \ldots + a_n \sin nx$. Then we see that $f'(0) = a_1 + 2a_2 + \ldots + na_n$. Well, we also the know the following about the definition of $f'(0)$:

$$|f'(0)| = \lim_{x \to 0} \left| \frac{f(x) - f(0)}{x} \right| = \lim_{x \to 0} \left| \frac{f(x)}{\sin x} \right| \left| \frac{\sin x}{x} \right| \tag{97.2}$$

We also know that $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$ by L'Hospital's Theorem. Thus:

$$|f'(0)| = \lim_{x \to 0} \left| \frac{f(x)}{\sin x} \right| \leq 1 \tag{97.3}$$

Because we have assumed $|f(x)| \leq |\sin x|$. Therefore, we see that $|a_1 + 2a_2 + \ldots + na_n| \leq 1$. □

98. Problem 98

Theorem 98.1. *Assume that f is differentiable at a, then find $\lim_{x \to a} \frac{xf(a) - af(x)}{x - a}$.*

Proof. We can make the following algebraic manipulations:

$$\begin{aligned} \lim_{x \to a} \frac{xf(a) - af(x)}{x - a} &= \lim_{x \to a} \frac{xf(a) - af(a) + af(a) - af(x)}{x - a} && (98.2) \\ &= \lim_{x \to a} \frac{f(a)(x - a) + a(f(a) - f(x))}{x - a} = \lim_{x \to a} f(a) + a\frac{f(a) - f(x)}{x - a} && (98.3) \end{aligned}$$

Now, we note the definition of a derivative is $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, we have:

$$\lim_{x \to a} \frac{xf(a) - af(x)}{x - a} = f(a) - af'(a) \tag{98.4}$$

□

99. Problem 99

Theorem 99.1. *Prove that if f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and if $f(a) = f(b) = 0$, then for a real α there is an $x \in (a, b)$ such that $\alpha f(x) + f'(x) = 0$.*

Proof. A straightforward application of the mean value theorem shows that for the function $h(x) = e^{\alpha x} f(x)$, we have $h(b) - h(a) = h'(x)(b - a)$ for some $x \in (a, b)$. This shows that $0 = (f'(x)e^{\alpha x} + \alpha e^{\alpha x} f(x))(b - a)$. Since $b - a \neq 0$ by assumption and $e^{\alpha x} \neq 0$, we have $f'(x) + \alpha f(x) = 0$, which is what we wanted. □

100. Problem 100

Theorem 100.1. *Let f and g be continuous functions on $[a,b]$, differentiable on the open interval (a,b), and let $f(a) = f(b) = 0$. Show that there is a point $x \in (a,b)$ such that $g'(x)f(x) + f'(x) = 0$.*

Proof. Take the function $h(x) = e^{g(x)}f(x)$ and observe that it is continuous on $[a,b]$ and differentiable on (a,b) by being the composition of functions that have these properties. Therefore, we see that we can apply the mean value theorem and obtain $h(b) - h(a) = h'(x)(b-a)$ for some $x \in (a,b)$. Since we know that $h(b) = h(a) = 0$ by assumption, and we know that $b - a \neq 0$, we see that $h'(x) = 0$. Therefore, taking the derivative, we have $0 = h'(x) = f'(x)e^{g(x)} + f(x)g'(x)e^{g(x)}$. Since we also know that $e^{g(x)} \neq 0$, we can divide by it and see that there exists some $x \in (a,b)$ for which $g'(x)f(x) + f'(x) = 0$. □

101. Problem 101

Theorem 101.1. *Let (X, d_1) be a metric space and for $x \in X$ define $\rho(x) = dist(x, X \setminus \{x_n\})$. Prove the following two conditions are equivalent: a) each continuous function $f : X \to \mathbb{R}$ is uniformly continuous b) every sequence $\{x_n\}$ of elements in X such that $\lim_{n\to\infty} \rho(x_n) = 0$ contains a convergent subsequence.*

Proof. Assume that the first condition is met and let $\{x_n\}$ be a sequence with elements in X such that $\lim_{n\to\infty} \rho(x_n) = 0$. Suppose by contradiction that $\{x_n\}$ does not cotnain a convergent subsequence. First, we know there must be a sequence $\{y_n\}$ such that $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, $\{y_n\}$ cannot have a uniformly convergent subsequence in $\{y_{n_k}\}$. If it did, then $\lim_{n\to\infty} d(x_{n_k}, y_{n_k}) = 0$ for some subsequence $\{x_{n_k}\}$ which would imply that $\{x_n\}$ has a convergent subsequence. This is a contradiction, so $\{y_n\}$ cannot have a uniformly convergent subsequence. Therefore, no term of $\{x_n\}$ and $\{y_n\}$ are repeated infinitely many times. We can therefore define $F_1 = \{x_{n_k} : k \in \mathbb{N}\}$ and $F_2 = \{y_{n_k} : k \in \mathbb{N}\}$, which are closed and disjoint. By the Urysohn lemma, there exists a continuous function $f : X \to \mathbb{R}$ that takes on the values of one on F_1 and zero on F_2. Therefore, we see that $|f(x_{n_k}) - f(y_{n_k})| = 1$ while $\lim_{k\to\infty} d(x_{n_k}, y_{n_k}) = 0$, which shows that f is not uniformly continuous. Since f is continuous, this is a contradiction of our assumptions.

To prove the converse, we assume that every sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} \rho(x_n) = 0$ contains a convergent subsequence. Let A be the set of all limit points of X. We see that every sequence on A has a convergent subsequence, which shows that A is compact. Fix $\delta_1 > 0$ and put $\delta_2 = \inf\{\rho(x) : x \in X, dist(x, A) > \delta_1\}$. We must have $\delta_2 > 0$, or else if $\delta_2 = 0$, then we see that ther exists some sequence $\{x_n\}$ such that $\lim_{n\to\infty}(x_n) = 0$ and $dist(x, A) > \delta_1$. This is a contradiction of the fact that $\delta_2 = 0$, which shows that $\delta_2 > 0$.

Now let $f : X \to \mathbb{R}$ be continuous. Fix $\epsilon > 0$, and we see that there exists a $\delta_x > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $d_1(x,y) < \delta_x$ for all $x, y \in X$. Since A is compact, there exist a finite number of $x_1, x_2, \ldots, x_k$ such that X is covered by neighborhoods of each x_i:

$$X \subset \bigcup_{i=1}^{k} N_{\delta_i/2}(x_i) \tag{101.2}$$

Therefore, we can take $\delta_1 = \frac{1}{3}\min\{\delta_1, \ldots, \delta_k\}$ and δ_2 as before. Let $\delta = \min\{\delta_1, \delta_2\}$. Now pick $x, y \in X$ such that $d_1(x,y) < \delta$. If $d(x, A) > \delta_1$, then $\rho(x) > \delta_2$ so $d_1(x,y) < \delta < \delta_2$ so $x = y$. In this case $|f(x) - f(y)| = 0 < \epsilon$. Now, if $d(x, A) \leq \delta_1$, then we see there must be some $a \in A$ for which $d_1(x,a) < \delta_1$. Moreover, there must be some $i \in \{1, \ldots, k\}$ for which $d_1(a, x_i) < \frac{1}{3}\delta_i$. Therefore, we have:

$$d_1(y, x_i) \leq d_1(y,x) + d_1(x,a) + d_1(a, x_i) < \delta + \delta_1 + \frac{1}{3}\delta_i < \delta_i \tag{101.3}$$

Moreover, we see that:

$$|f(x) - f(y)| \leq |f(x) - f(x_i)| + |f(x_i) + f(y)| < 2\epsilon \tag{101.4}$$

This proves the uniform continuity of f. □

102. Problem 102

Theorem 102.1. *Assume that f is twice differentiable on (a,b) and that there is some $M > 0$ such that $|f''(x)| \leq M$ for all $x \in (a,b)$. Prove that f is uniformly continuous on (a,b).*

Proof. It is sufficient to show that f' is bounded on (a,b). Fix $y \in (a,b)$. We can use the mean value theorem on f' to see that for all $x \in (a,b)$, we have $|f'(x) - f'(y)| = |f''(c)|(x-y)$ for some $c \in (x,y)$. Thus, we see that $|f'(x) - f'(y)| \leq M(x-y)$ which implies $|f'(x)| \leq M(x-y) + |f'(y)| \leq M(b-a) + |f'(y)|$. Since $f'(y)$ is a constant, we see that $|f'(x)| \leq J$ where $J > 0$ is a constant. Therefore, we see that f' is bounded for all $x \in (a,b)$, which implies that f is uniformly continuous. □

103. Problem 103

Theorem 103.1. *Let f be continuous on $[0,1]$ and differentiable on $(0,1)$. Suppose that $f(0) = f(1) = 0$ and there exists some $x_0 \in (0,1)$ such that $f(x_0) = 1$. Prove that $|f'(c)| > 2$ for some $c \in (0,1)$.*

Proof. Pick x_0 such that $f(x_0) = 1$. Then we use the mean value theorem to see that $|f(x_0)-f(0)| = |f'(c)|x_0$ which implies $|f(x_0)| = |f'(c)|x_0$. Using the mean value theorem again, we see that $|f(1) - f(x_0)| = |f'(d)||1-x_0|$, which implies $|f(x_0)| = |f'(d)||1-x_0|$. Therefore, we have:

$$|f'(c)| = \frac{|f(x_0)|}{|x_0|} = \frac{1}{|x_0|} \tag{103.2}$$

$$|f'(d)| = \frac{|f(x_0)}{1-x_0|} = \frac{1}{|1-x_0|} \tag{103.3}$$

Suppose that $x_0 > \frac{1}{2}$, then we see that $|f'(d)| > 2$. If $x_0 < \frac{1}{2}$, then $f'(c) > 2$. If $x_0 = \frac{1}{2}$, then both $|f'(c)| = |f'(d)| = 2$. Therefore, for whatever value of $x_0 \in (0,1)$, we see that there exists some c for which $|f'(c)| > 2$. ☐

104. Problem 104

Theorem 104.1. *Assume that f is differentiable on an open interval I and that $[a,b] \subset I$. We say that f is uniformly differentiable on $[a,b]$ if $\forall\epsilon > 0$, $\exists\delta > 0$ such that $|\frac{f(x+h)-f(x)}{h} - f'(x)| < \epsilon$ for all $x \in [a,b]$ and $|h| < \delta$, with $x+h \in I$. Prove that f is uniformly differentiable on $[a,b]$ if and only if f' is continuous on $[a,b]$.*

Proof. First, assume that f is uniformly differentiable. Then there exists a sequence $\{\frac{f(x+h)-f(x)}{h}\}$ which is uniformly convergent to $\{f'(x)\}$. Since each term in the sequence is continuous, we see that $f'(x)$ is also continuous because of uniform convergence.

To prove the converse, we assume that f' is continuous on $[a,b]$. This implies that f' is uniformly continuous. By mean value theorem, we see that $|f(x+h) - f(x)| = |f'(x+\theta h)|h$ for some $\theta \in (0,1)$. Therefore, we have:

$$\left|\frac{f(x+h)-f(x)}{h} - f'(x)\right| = |f'(x+\theta h) - f'(x)| < \epsilon \tag{104.2}$$

This follows from uniform continuity, since we see that for a fixed $\epsilon > 0$, there exists a $\delta > 0$ such that $|f'(x+\theta h) - f'(x)| < \epsilon$ if $|h| < \delta$ for all $x \in [a,b]$. Therefore, f is uniformly differentiable. ☐

105. Problem 105

Theorem 105.1. *Prove that a metric space is compact if and only if every continuous function $f : X \to \mathbb{R}$ is bounded.*

Proof. Suppose first that X is compact. Then every continuous function is uniformly continuous. Since X is compact, there exist a finite number of points $x_1, \ldots, x_n$ such that $X \subset N_{\delta_1}(x_1) \cup \ldots \cup N_{\delta_n}(x_n)$. Now select $\delta = \max\{\delta_1, \ldots, \delta_n\}$. Moreover, we can pick $M = \max\{f(x_1), \ldots, f(x_n)\}$. Fix $\epsilon > 0$ and choose $x \in X$. It is clear that there exists some $i = 1, \ldots, n$ for which $d(x, x_i) < \delta_i$. Thereofre, by uniform continuity, we see that $|f(x) - f(x_i)| < \epsilon$ so that $|f(x)| < \epsilon + |f(x_i)| < \epsilon + M$. Therefore, we see that $f(x)$ is bounded for all $x \in X$.

To prove the converse, assume f is continuous and there exists an M such that $|f(x)| < M$ for all $x \in X$. Now suppose by contradiction that X is not compact. Thus, there exists some sequence $\{x_n\}$ for which no subsequence converges. Let $F = \{x_n : n \in \mathbb{N}\}$, which is a closed set. Moreover, we know that $f(x_n) = n$ is continuous on F. According to the Tietze extension theorem, there exists a continuous extension of f defined on all of X. But this is a contradiction, because f is not bounded, and we assumed every continuous function on X is bounded. Therefore, X must be compact. ☐

106. Problem 106

Theorem 106.1. *Let $\mathscr{F}$ denote a family of real functions continuous on a complete metric space X such that for all $x \in X$, there exists some M_x such that $|f(x)| < M_x$ for all $f \in \mathscr{F}$. Prove that there exists some $M > 0$ and a nonempty open set $G \subset X$ such that $|f(x)| < M$ for all $f \in \mathscr{F}$ and $x \in G$.*

Proof. Define $F_n = \{x \in X : |f(x)| < n, \forall f \in \mathscr{F}\}$. Since f are all continuous, we see that F_n is closed. By assumption, for all $x \in X$, there exists some n_x such that $|f(x)| < n_x$. Thus, $x \in F_{n_x}$. Therefore, we see that $\mathscr{F} = \bigcup_{n=1}^{\infty} F_n$. Thus, there exists some F_{n_0} with a nonempty interior. Take $G = F_{n_0}^{\circ}$ and we see that all $|f(x)| < n_0$ for all $f \in \mathscr{F}$ and $x \in G$. □

107. Problem 107

Theorem 107.1. *Prove that if f is differentiable on $[0,1]$ and if $f(0) = 0$, and there exists some $K > 0$ such that $|f'(x) \leq K|f(x)|$ for all $x \in [0,1]$, then $f(x) \equiv 0$.*

Proof. Pick $x_0 \in [0,1]$, and we can use mean value theorem to obtain the following: $|f(x_0)| = |f'(x_1)|x_0 \leq Kx_0|f(x_1)|$ for some $x_1 \in (0, x_0)$. Next, we can apply mean value theorem again to obtain: $|f(x_1)| = |f'(x_2)|x_1) \leq Kx_1|f(x_2)|$. Therefore, $|f(x_0)| \leq K^2 x_0 x_1 |f(x_2)|$. We can repeat this process, and find that by induction we have $|f(x_0)| \leq K^n x_0 x_1 \dots x_{n-1}|f(x_n)|$ for some $n \in \mathbb{N}$. Note that we stop applying the mean value theorem once $x_{n-1} < K$. Also note that $0 < x_n < x_{n-1} < \dots < x_1 < x_0$. Therefore, for each $x \in [0,1]$, we have $|f(x)| \leq K^n x^n |f(x_n)|$ for some $n \in \mathbb{N}$. This shows that f is bounded and that $f(x) \equiv 0$ on the interval $[1, 1/n] \cap [0,1]$. To show that $f(x) \equiv 0$ on the rest of the interval, repeat the process and take $f(x)$ as a new ending point. We see that $[0,1]$ can be decomposed into finitely many of these intervals, which shows that $f(x) \equiv 0$ on all of $[0,1]$. □

108. Problem 108

Theorem 108.1. *Let f be continuous on $[a,b]$, g be differentiable on $[a,b]$ and $g(a) = 0$. Prove that if there exists $\lambda \neq 0$ such that $|g(x)f(x) - \lambda g'(x)| \leq |g(x)|$ for all $x \in [a,b]$, then $g(x) \equiv 0$ on $[a,b]$.*

Proof. First, note that f continuous on a compact set $[a,b]$ implies that f is uniformly continuous, hence bounded on $[a,b]$. Therefore, there exists an A such that $|f(x)| < A$ for all $x \in [a,b]$. Rewriting our assumptions, we have the following and letting $B = \frac{\lambda}{A+1}$:

$$|g'(x)| \leq |g(x)| \frac{|f(x)| + 1}{\lambda} \leq \frac{|g(x)|}{B} \tag{108.2}$$

Now let $[c,d] \subset [a,b]$ be a subinterval whose length is not greater than $\frac{1}{2}\frac{|\lambda|}{1+A} = \frac{B}{2}$. and such that $g(c) = 0$. Then we see that by applying the mean value theorem to some point $x_0 \in [c,d]$, we obtain $|g(x_0)| = |g'(x_1)||x_0 - c|$ for some $x_1 \in (c, x_0)$. Therefore, we find that:

$$|g(x_0)| \leq |g(x_1)| \frac{|x_0 - c|}{B} \leq |g(x_1)| \frac{B}{2B} = \frac{1}{2}|g(x_1)| \tag{108.3}$$

Repeating this process, we see that:

$$|g(x_0)| \leq \frac{1}{2}|g(x_1)| \leq \dots \leq \frac{1}{2^n}|g(x_n)| \leq \dots \tag{108.4}$$

This is true for a decreasing sequence of $\{x_n\}$ such that $c < x_n < x_{n-1} < \dots < x_1 < x_0$. Since $g(c) = 0$, we see that $g(x_0) = 0$. It is enough to decompose $[a,b]$ into a finite number of subintervals with length less than $B/2$, and show that $g(x) \equiv 0$ for all of these subintervals. This shows that $g(x) \equiv 0$ on $[a,b]$. □

109. Problem 109

Theorem 109.1. *Show that the limit function of a uniformly convergent sequence of bounded functions is bounded.*

Proof. Let $\{f_n\}$ converge uniformly to f, and let there exist some $M_n > 0$ for each $n \in \mathbb{N}$ such that $|f_n(x)| < M_n$ for all $x \in X$. Fix $\epsilon > 0$. Since $\{f_n\}$ converges uniformly, there exists some N such that for all $n > N$ and for all $x \in X$, we have $|f_n(x) - f(x)| < \epsilon$. Therefore, we see that $|f(x)| < \epsilon + |f_n(x)|$. However, we can take $M = \max\{M_1, M_2, \dots, M_N\}$ and see that we must have $|f(x)| < \epsilon + M$, which shows that $|f(x)|$ is bounded. □

110. Problem 110

Theorem 110.1. *Let $\{a_n\}$ be a convergent sequence of real numbers and let $\{f_n\}$ be a sequence of functions satsifying* $\sup\{|f_n(x) - f_m(x) : x \in A\} \leq |a_n - a_m|$ *for all $n, m \in \mathbb{N}$. Prove that $\{f_n\}$ converges uniformly.*

Proof. First, we know by Cauchy criterion of convergence, that for a fixed $\epsilon > 0$, that there exists some N such that $n, m > N$ implies that $|a_n - a_m| < \epsilon$. Moreover, we see that by our assumptions, we must have for all $x \in A$ $|f_n(x) - f_m(x)| < |a_n - a_m|$ because of the supremum condition. Therefore, we can choose N as before so and see that for all $x \in A$ that $n, m > N$ implies $|f_n(x) - f_m(x)| < |a_n - a_m| < \epsilon$. This shows uniform convergence. □

111. Problem 111

Theorem 111.1. *Assume that $\{f_n\}$ is a sequence of increasing or decreasing functions on $[a, b]$ converging pointwise to a function continuous on $[a, b]$. Prove that $\{f_n\}$ converges uniformly on $[a, b]$.*

Proof. Assume without loss of generality that each f_n is increasing and that $\{f_n\} \to f$. By assumption, we see that f is continuous on a compact set $[a, b]$ so that f is uniformly continuous. This means that for a fixed $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, x_i \in [a, b]$ we have $|f(x) - f(x_i)| < \epsilon$ whenever $|x - x_i| < \delta$. Take a partition of $[a, b]$ such that $a = x_0 < x_1 < x_2 < \ldots < x_p = b$, where $|x_i - x_{i-1}| < \delta$ for all $i \in \{1, \ldots, p\}$. Now pick $x \in [a, b]$. It is clear that there exists some i for which $x_{i-1} \leq x \leq x_i$. Therefore, $|f(x) - f(x_i)| < \epsilon$ for all $x \in [a, b]$ for some $i \in \{1, \ldots, p\}$.

By pointwise convergence, there exists an N such that for all $n > N$ we have $|f_n(x_i) - f(x_i)| < \epsilon$ for all $x \in [a, b]$. Thus, we see that for all $x \in [a, b]$, there exists an N such that for all $n > N$, we have:

$$|f_n(x) - f(x)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(x)| < 3\epsilon \tag{111.2}$$

This proves uniform convergence on $\{f_n\}$ to f. □

112. Problem 112

Theorem 112.1. *Let $f : X \to \mathbb{R}$ be uniformly continuous. Show that if $\{a_n\}$ is Cauchy in X, then $\{f(a_n)\}$ is convergent in $\mathbb{R}$.*

Proof. Fix $\epsilon > 0$. Since f is uniformly continuous, we see that there exists some $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$ for all $x, y\ inX$. By the definition of Cauchy sequence, we see that there exists some N such that for all $n, m > N$, we have $|a_m - a_n| < \delta$. Therefore, we have $|f(a_n) - f(a_m)| < \epsilon$ whenever $n, m > N$, which implies that $|a_m - a_n| < \delta$. Therefore, we have proven that $\{f(a_n)\}$ is convergent in $\mathbb{R}$. □

113. Problem 113

Theorem 113.1. *Suppose $f : X \to \mathbb{R}$ maps Cauchy sequences to convergent sequences. Is f necessarily uniformly continuous?*

Proof. No. Let $\{a_n\}$ be a Cauchy sequence in X, and define $f(x) = x^2$. We see that $\{f(a_n)\} = \{a_n^2\}$ is convergent. However, we know that f is not uniformly continuous because we cannot find a single $\delta > 0$ for every ϵ for which $|x^2 - y^2| < \epsilon$ whenever $|x - y| < \delta$. □

114. Problem 114

Theorem 114.1. *Let $\{p_n\}$ be a sequence in a metric space (X, d) and let $a_n = d(p_n, p_{n+1})$. Assume that $\sum_{n=1}^{\infty} a_n$ converges. Show that $\{p_n\}$ is a Cauchy sequence.*

Proof. Fix $\epsilon > 0$. We know by the Cauchy criterion for convergence of series that there exists some N such that for all $n, m \geq N$ (let us assume without loss of generality that $m > n$), that $|a_m + a_{m+1} + \ldots + a_n| < \epsilon$. Substituting in the definition of a_n, we have: $|d(p_n, p_{n+1}) + d(p_{n+1}, p_{n+2}) + \ldots + d(p_{m-1}, p_m)| < \epsilon$ which shows by triangle inequality that $d(p_n, d_pm) \leq d(p_n, p_{n+1}) + \ldots + d(p_{m-1}, p_m) < \epsilon$. therefore, we see that $\{p_n\}$ is a Cauchy sequence. □

115. Problem 115

Theorem 115.1. *Prove that if $\{f_n\}$ is pointwise convergent and equicontinuous on a compact set K, then $\{f_n\}$ is uniformly convergent on K.*

Proof. Fix $\epsilon > 0$. Equicontinuity implies that there exists some $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ whenever $|x - y| < \delta$ for all $x, y \in K$. Taking the limit as $n \to \infty$, we see that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Moreover, since K is compact, there exist a finite number of $x_1, x_2, \ldots, x_p$ such that $X \subset N_{\delta_1}(x_1) \cup \ldots \cup N_{\delta_p}(x_p)$. Therefore, for any $x \in K$, we see that there exists some i for which $d(x, x_i) < \delta_i$. Now take $\delta = \min\{\delta_1, \ldots, \delta_p\}$ for our definition in the equicontinuity condition. Pointwise convergence implies that there exists an N such that for all $n > N$, we have $|f_n(x) - f(x)| < \epsilon$ for all $x \in K$. Therefore, we have:

$$|f_n(x) - f(x)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(x)| < 3\epsilon \tag{115.2}$$

Thus, we have shown uniform convergence of $\{f_n\}$ on K. □

116. Problem 116

Theorem 116.1. *Let $\{f_n\}$ be a sequence of continuous functions on a closed interval $[a, b]$ and differentiable on the open interval (a, b). Assume that $\{f_n'\}$ is uniformly bounded on (a, b). Prove that if $\{f_n\}$ is pointwise convergent on $[a, b]$, then $\{f_n\} \rightrightarrows f$ on $[a, b]$.*

Proof. It is sufficient to prove that $\{f_n\}$ is equicontinuous, because then we can use the theorem proven in problem 115 to show hat $\{f_n\}$ is uniformly convergent on a compact set $[a, b]$. First, we note that $\{f_n'\}$ being uniformly bounded means that there exists some M for which $|f_n'(x)| < M$ for all $x \in [a, b]$ and for all $n \in \mathbb{N}$. Fix $\epsilon > 0$. Using the mean value theorem, we see that:

$$|f_n(x) - f_n(y)| = |f_n'(c)||x - y| \quad \Rightarrow \quad |f_n(x) - f_n(y)| \leq M|x - y| < M\delta \tag{116.2}$$

We can choose a $\delta = \epsilon/M$ so that $|f_n(x) - f_n(y)| < \epsilon$ for all $x \in [a, b]$ and for all $n \in \mathbb{N}$. Therefore, we have shown that $\{f_n\}$ is equicontinuous, and by the previous theorem, we have shown that $\{f_n\}$ is also uniformly convergent on $[a, b]$. □

117. Problem 117

Theorem 117.1. *Suppose $\sum_{n=1}^{\infty} f_n(x)$, for all $x \in A$ converges uniformly on A and $f : A \to \mathbb{R}$ is bounded. Prove that $\sum_{n=1}^{\infty} f(x)f_n(x)$ converges uniformly on A.*

Proof. First, by the Cauchy criterion for convergence of series, we see that for a fixed $\epsilon > 0$, there exists some N such that for all $n - 1, m > N$, we have $|f_m(x) + f_{m-1}(x) + \ldots + f_n(x)| < \epsilon$ for all $x \in A$. Moreover, by boundedness, there exists some M such that $|f(x)| < M$ for all $x \in A$. Define $s_N = \sum_{n=1}^{N} f(x)f_n(x)$. We see that for $m, n > N$, we have $|s_n - s_m| = |f(x)(f_m(x) + f_{m-1}(x) + \ldots + f_n(x))| \leq |f(x)|\epsilon < M\epsilon$ for all $x \in A$. Therefore, since M is a constant, we see that $\{s_n\}$ converges uniformly by the Cauchy criterion, which shows that $\sum_{n=1}^{\infty} f(x)f_n(x)$ does as well. □

118. Problem 118

Theorem 118.1. *Suppose $\sum_{n=0}^{\infty} f_n(x)$ converges absolutely and uniformly on A. Must the series $S(x) = \sum_{n=0}^{\infty} |f_n(x)|$ converge uniformly on A?*

Proof. No. We can take $A = [0, 1]$ and set $f_n(x) = (-1)^n(1-x)x^n$ for $x \in [0, 1]$. We clearly see that $\sum f_n(x)$ converges absolutely on $[0, 1)$ because x^n converges for $|x| < 1$. Clearly, if $x = 1$, the $f_n(1) = 0$, which shows absolute convergence. Moreover, it is uniformly convergent by the Weierstrass M test. However, we see $S(x) = \sum_{n=0}^{\infty} |f_n(x)| = \sum_{n=0}^{\infty}(1 - x)x^n$ has the following solution:

$$S(x) = \begin{cases} 1 & x \in [0, 1) \\ 0 & x = 1 \end{cases} \tag{118.2}$$

Therefore, since $f(x)$ is not continuous, we see that $\{f_n\}$ cannot possibly converge uniformly to f. □

119. Problem 119

Theorem 119.1. *Suppose $\sum_{n=1}^{\infty} \frac{1}{|a_n|}$ converges. Prove that $\sum_{n=1}^{\infty} \frac{1}{x-a_n}$ converges absolutely and uniformly on each bounded set A that does not contain a_n for $n \in \mathbb{N}$.*

Proof. First, we know that $\sum \frac{1}{a_n}$ converges absolutely, which shows that $\lim_{n\to\infty} a_n = \infty$. Moreover, we see that the following is true:

$$\frac{1}{|x - a_n|} \leq \frac{1}{|a_n|} \frac{1}{|\frac{x}{a_n}| - 1|} \leq \frac{1}{|a_n|} \frac{1}{\frac{M}{|a_n|} - 1} \tag{119.2}$$

Where we have defined $M = \sup_{x\in A} |f(x)|$. Therefore, it is clear that if $\sum \frac{1}{|a_n|}$ converges, then so too must $\sum \frac{1}{|a_n|} \frac{1}{\frac{M}{|a_n|}-1}$. This completes the proof. □

120. Problem 120

Theorem 120.1. *Suppose α increases on $[a,b]$, $a \leq c \leq b$, α is continuous at c, and $f(c) = 1$ while $f(x) = 0$ for all $x \neq c$. Show that $f \in \mathscr{R}(\alpha)$ and that $\int_a^b f d\alpha = 0$.*

Proof. A theorem in Rudin says that if α is continuous at all points of discontinuity of f, where f has finitely many points of discontinuity, then f is integrable. Now, let us take some partition $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ for $[a,b]$. Then we see that there exists some i for which $x_{i-1} < c \leq x_i$. Therefore, we have $U(P,f,\alpha) = f(c)(\alpha(x_i) - \alpha(x_{i-1}))$, while we also have $L(P,f,\alpha) = 0$. Since α is continuous at c, we can fix some $\epsilon > 0$ and find some $\delta > 0$ such that $|\alpha(x_i) - \alpha(x_{i-1})| < \epsilon$ whenever $|x_i - x_{i-1}| < \delta$. Therefore, we can take a partition such that $|x_i - x_{i-1}| < \delta$ and find that $U(P,f,\alpha) < f(c)\epsilon = \epsilon$ because $f(c) = 1$. Therefore, we see that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$, which implies integrability, while we also see that $\int_a^b f d\alpha = 0$. $\square$

121. Problem 121

Theorem 121.1. *Suppose f is continuous on $[a,b]$, $a < c < b$, $\alpha(x) = 0$ if $x \in [a,c)$ and $\alpha(x) = 1$ if $x \in (c,b]$. Show that $\int_a^b f d\alpha = f(c)$.*

Proof. Take some partition $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ of $[a,b]$. Then it is clear that there exists some i for which $x_{i-1} < c \leq x_i$. We see that $U(P,f,\alpha) = M(\alpha(x_i) - \alpha(x_{i-1}))$ while $L(P,f,\alpha) = m(\alpha(x_i) - \alpha(x_{i-1}))$ where $M = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ and $m = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$. We note that by a theorem in Rudin, $f \in \mathscr{R}(\alpha)$ because f is continuous. This implies that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$. Moreover, since we know that $(\alpha(x_i) - \alpha(x_{i-1})) = 1$, we see that $M - m < \epsilon$. This implies that $\inf U(P,f,\alpha) = \sup L(P,f,\alpha) = f(c)$, and that $\int_a^b f(x) d\alpha = f(c)$. $\square$

122. Problem 122

Theorem 122.1. *Let $0 < a < b$. Find the upper and lower Riemann intergrals of f over $[a,b]$ for:*

$$f(x) = \begin{cases} x & x \in [a,b] \cap \mathbb{Q} \\ 0 & x \in [a,b] \setminus \mathbb{Q} \end{cases} \tag{122.2}$$

Proof. First, since $\mathbb{Q}$ is dense in $\mathbb{R}$, we see that $L(P,f) = 0$ for any partition. Therefore $\underline{\int_a^b} f dx = 0$. Next, pick a partition $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ of $[a,b]$. We see that $U(P,f) = \sum_{i=0}^n x_i(x_i - x_{i-1}) = \sum_{i=0}^n x_i^2 - x_i x_{i-1} \geq \frac{1}{2}(b^2 - a^2)$ because we have $2x_i x_{i-1} < x_i^2 + x_{i-1}^2$. Therefore, we see that $\overline{\int_a^b} f dx = \frac{1}{2}(b^2 - a^2)$. $\square$

123. Problem 123

Theorem 123.1. *Let $a > 0$ and find the upper and lower Riemann integrals of f over $[-a,a]$ for:*

$$f(x) = \begin{cases} x & x \in [-a,a] \cap \mathbb{Q} \\ 0 & x \in [-a,a] \setminus \mathbb{Q} \end{cases} \tag{123.2}$$

Proof. Pick a partition $P = \{a = x_0 < x_1 < \ldots < x_{j-1} = 0 < x_j < \ldots < x_n = b\}$. Now, we see using the same logic from problem 123, that $L(P,f) = \sum_{i=0}^{j-2} x_i(x_i - x_{i-1}) \leq -\frac{1}{2}a^2$, while $U(P,f) = \sum_{i=j+1}^{n-1} x_{i+1}(x_{i+1} - x_i) \geq \frac{1}{2}a^2$. Therefore, we have $\underline{\int_a^b} f dx = -\frac{1}{2}a^2$ and $\overline{\int_a^b} f dx = \frac{1}{2}a^2$. $\square$

124. Problem 124

Theorem 124.1. *Show that the Riemann function f is Riemann integrable on every interval $[a,b]$, where:*

$$f(x) = \begin{cases} 0 & x \text{ irrational or } 0 \\ 1/q & x = p/q, p \in \mathbb{Z}, q \in \mathbb{N}, \text{and coprime} \end{cases} \tag{124.2}$$

Proof. Fix $[a,b]$ and take some $N \in \mathbb{N}$. There are only finitely many rationals p/q such that $q < N$ in $[a,b]$. Let k_N denote the number of these rationals and make a partition P such that there are at most $2k_N$ subintervals $[x_{i-1}, x_i]$ containing at least one of the rationals mentioned above and such that $d(x_i, x_{i-1}) < \delta$. On the other subintervals, we have $M_i - m_i < 1/N$. Therefore, we have:

$$U(P,f) - L(P,f) < 2k_N\delta + \frac{b-a}{N} \tag{124.3}$$

Now pick $\epsilon > 0$ and we can choose $N > \frac{2(b-a)}{\epsilon}$ so that $U(P,f) - L(P,f) < \epsilon$, which shows that $f \in \mathscr{R}([a,b])$. $\square$

125. Problem 125

Theorem 125.1. *Let $f : [0,1] \to \mathbb{R}$ and show that $\int_0^1 f(x)dx = 0$ when*

$$f(x) = \begin{cases} 1 & x = \frac{1}{n}, n \in \mathbb{N} \\ 0 & \text{else} \end{cases} \tag{125.2}$$

Proof. Let $\epsilon > 0$ be given. There exists an N such that $1/n < \epsilon/2$. whenever $n > N$. Then we can choose P such that

$$0 = x_0 < x_1 = \frac{1}{N+1} < x_2 < \ldots < x_k = 1 \tag{125.3}$$

Where $|x_i - x_{i-1}| < \frac{\epsilon}{4N}$ for $i = \{1, \ldots, k\}$. Then $U(P,f) - L(P,f) < \frac{\epsilon}{2} + \frac{\epsilon}{4N}2N = \epsilon$. □

126. Problem 126

Theorem 126.1. *Show that $f : [0,1] \to \mathbb{R}$ is Riemann integrable on $[0,1]$ for:*

$$f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x} - \left[\frac{1}{x}\right] & \text{else} \end{cases} \tag{126.2}$$

Proof. First, fix $\epsilon > 0$. We can pick an $N > 0$ such that $1/n < \epsilon/2$ for all $n > N$. Therefore, we can choose a partition P such that

$$0 = x_0 < x_1 = \frac{1}{N+1} < \ldots < x_{n_0'} = \frac{1}{N} < \ldots < x_{n_1'} = \frac{1}{N-1} < \ldots < x_{n'_{n_0-1}} = 1 \tag{126.3}$$

Where $x_i - x_{i-1} < \frac{\epsilon}{4N}$ for all $i \geq 2$. Then we see that

$$\begin{aligned} U(P,f) - L(P,f) &= \frac{1}{N+1} + \sum_{i=2}^{n_0'}(M_i - m_i)(x_i - x_{i-1}) + \sum_{k=0}^{n_0-2}\sum_{i=n'_{k+1}}^{n'_{k+1}}(M_i - m_i)(x_i - x_{i-1}) \qquad (126.4) \\ &< \frac{\epsilon}{2} + 2N\frac{\epsilon}{4N} = \epsilon \qquad (126.5) \end{aligned}$$

□

127. Problem 127

Theorem 127.1. *Show that $f \in \mathscr{R}(\alpha)$ for the following:*

$$f(x) = \begin{cases} 0 & x \in [-1,0] \\ 1 & x \in (0,1] \end{cases} \qquad \alpha(x) = \begin{cases} 0 & x \in [-1,0) \\ 1 & x \in [0,1] \end{cases} \tag{127.2}$$

Proof. Pick some partition $P = \{-1 = x_0 < x_1 < \ldots < x_{k-1} = 0 < x_k < \ldots < x_n = 1\}$. Then we see that:

$$U(P,f,\alpha) = 1(\alpha(x_k) - \alpha(x_{k-1})) = 0 \tag{127.3}$$

$$L(P,f,\alpha) = 0(\alpha(x_{k-1}) - \alpha(x_{k-2})) = 0 \tag{127.4}$$

Therefore, we see that not only $f \in \mathscr{R}(\alpha)$, but $\int_{-1}^1 f d\alpha = 0$. □

128. Problem 128

Theorem 128.1. *Suppose f is continuous on $[a,b]$ and α is a step function constant on subintervals $(a,c_1),(c_1,c_2),\ldots,(c_m,b)$ where $a < c_1 < \ldots < c_m < b$. Show that $\int_a^b f(x)d\alpha(x) = f(a)(\alpha(a^+) - \alpha(a)) + \sum_{k=1}^m f(c_k)(\alpha(c_k^+) - \alpha(c_k^-)) + f(b)(\alpha(b) - \alpha(b^-))$.*

Proof. First, take the following partition: $P = \{a = x_0 < x_1 < \ldots < x_m = b\}$ where $x_i = c_i$ for all $0 < i < m$. Therefore, we can break up our integral into:

$$\int_a^b f(x)d\alpha(x) = \sum_{i=0}^m \int_{c_i}^{c_{i+1}} f(x)d\alpha(x) \tag{128.2}$$

Moreover, it is clear that the following is true since f is continuous:

$$\int_{c_k}^{c_{k+1}} f(x)d\alpha(x) = f(c_k)(\alpha(c_k^+) - \alpha(c_k^-)) + f(c_{k+1})(\alpha(c_{k+1}^+) - \alpha(c_k^-)) \tag{128.3}$$

Summing over all $i \in \{0,1,\ldots,m\}$, we see that the statement in the theorem is true. □

129. Problem 129

Theorem 129.1. *Show that if $f \in \mathscr{R}([a,b])$ then f can be changed at a finite number of points without affecting either integrability of f or the value of the integral*

Proof. It is sufficient to show that this works for a single point. So let $x' \in [a,b]$ be the point where f changes, and let $\hat{f}$ be the changed function. Fix $\epsilon > 0$. Let us break apart the interval $[a,b]$ into $[a, x'-\epsilon] \cup [x'-\epsilon, x'+\epsilon] \cup [x'+\epsilon, b]$. We see that $f = \hat{f}$ on $P_1 = [a, x'-\epsilon]$ and on $P_2[x'+\epsilon, b]$. Therefore, we have $U(P_i, \hat{f}) - L(P_i, \hat{f}) < \epsilon$ for $i = 1, 2$ since f is integrable. Thus, we have:

$$U(P,f) - L(P,f) < 2\epsilon + (M-m)(x'+\epsilon - x' + \epsilon) \tag{129.2}$$
$$= 2\epsilon + 2(M-m)\epsilon \tag{129.3}$$

Where $M = \sup\{f(x) : x \in [a,b]\}$ and $m = \inf\{f(x) : x \in [a,b]\}$. Therefore, we see that $\int_a^b \hat{f} dx$ exists and that $\hat{f} \in \mathscr{R}([a,b])$. To show that the integrals are equal, we have:

$$\left| \int_a^b f(x)dx - \int_a^b \hat{f}(x)dx \right| = \left| \int_a^b (f(x) - \hat{f}(x))dx \right| \tag{129.4}$$
$$\leq |f(x) - \hat{f}(x)|(x+\epsilon - (x-\epsilon)) \tag{129.5}$$
$$\leq 2M^*\epsilon \tag{129.6}$$

Where $M^* = \sup\{f(x) - \hat{f}(x) : x \in [a,b]\}$. Since $\epsilon > 0$ was arbitrary, we see that the integrals are the same. □

130. Problem 130

Theorem 130.1. *Show that if f is monotonic and α is continuous on $[a,b]$ then $f \in \mathscr{R}(\alpha)$.*

Proof. Fix $\epsilon > 0$ and suppose $f(x)$ is monotonically increasing without loss of generality. Since α is continuous, we know that there exists a $\delta > 0$ such that $|\alpha(x_i) - \alpha(x_{i-1})| < \epsilon$ whenever $|x_i - x_{i-1}| < \delta$. Therefore, let $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ be a partition of $[a,b]$, such that $|x_i - x_{i-1}| < \delta$. We see that we have the following:

$$U(P,f,\alpha) = \sum_{i=1}^n M_i(\alpha(x_i) - \alpha(x_{i-1})) < \sum_{i=1}^n f(x_i)\epsilon \tag{130.2}$$
$$L(P,f,\alpha) = \sum_{i=1}^n m_i(\alpha(x_i) - \alpha(x_{i-1})) < \sum_{i=1}^n f(x_{i-1})\epsilon \tag{130.3}$$

Where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\} = f(x_i)$ and $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} = f(x_{i-1})$. Therefore, we see that $|U(P,f,\alpha) - L(P,f,\alpha)| < \sum_{i=1}^n (f(x_i) - f(x_{i-1}))\epsilon$. Since $\epsilon > 0$ was arbitrary, we see that $f \in \mathscr{R}(\alpha)$. □

131. Problem 131

Theorem 131.1. *Calculate $\int_{-2}^2 x^2 d\alpha(x)$ where*

$$f(x) = \begin{cases} x+2 & -2 \leq x \leq -1 \\ 2 & -1 < x < 0 \\ x^2+3 & 0 \leq x \leq 2 \end{cases} \tag{131.2}$$

Proof. Using a theorem in Rudin, we know that $\int_a^b f d\alpha(x) = \int_a^b f(x)\alpha'(x)dx$. Therefore, we have the following:

$$\int_{-2}^2 x^2 d\alpha(x) = \int_{-2}^{-1} x^2 dx + \int_0^2 x^2(2x)dx + (-1)^2(2 - (-1+2)) + 0^2(3-2) \tag{131.3}$$
$$= \left.\frac{x^3}{3}\right|_{-2}^{-1} + \left.\frac{x^4}{2}\right|_0^2 + 1 = -\frac{1}{3} + \frac{8}{3} + 8 + 1 \tag{131.4}$$
$$= \frac{34}{3} \tag{131.5}$$

□

132. Problem 132

Theorem 132.1. *Prove that if $f \in \mathscr{R}(\alpha)$ and α is neither continuous from left or right at a point $c \in [a,b]$, then f is continuous at this point.*

Proof. Fix $\epsilon > 0$. Since $f \in \mathscr{R}(\alpha)$, we know that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$. Take a partition $P = \{a = x_0 < x_1 < \ldots < x_j = c - \epsilon < x_{j+1} = c + \epsilon < \ldots < x_n = b\}$. We know particularly that $\int_{c-\epsilon}^{c+\epsilon} f d\alpha$ exists. Thus, for this interval $Q = [x_j, x_{j+1}]$, we have:

$$U(Q,f,\alpha) - L(Q,f,\alpha) = (M - m)(\alpha(c+\epsilon) - \alpha(c-\epsilon)) < \epsilon \tag{132.2}$$

Where $M = \sup\{f(x) : x \in [x_j, x_{j+1}]\}$ and $m = \inf\{f(x) : x \in [x_j, x_{j+1}]\}$. We know that $|\alpha(c+\epsilon) - \alpha(c-\epsilon)| > \epsilon$ because α is discontinuous at c. Therefore, we see that $M - m < \frac{1}{\alpha(c+\epsilon)-\alpha(c-\epsilon)} < \frac{1}{\epsilon}$, which shows that $|f(x) - f(y)| < \frac{1}{\epsilon}$ whenever $|x - y| < 2\epsilon$. Since $\epsilon > 0$ can be arbitrarily large, we see that this implies that f must be continuous. $\square$

133. Problem 133

Theorem 133.1. *Prove the first mean value theorem: if f is continuous and α is monotonically increasing on $[a,b]$, then $\exists c \in [a,b]$ such that $\int_a^b f(x)d\alpha(x) = f(c)(\alpha(b) - \alpha(a))$.*

Proof. Let $m = \inf\{f(x) : x \in [a,b]\}$ and $M = \sup\{f(x) : x \in [a,b]\}$. Then we see that the following is true:

$$m(\alpha(b) - \alpha(a)) \leq \int_a^b f(x)d\alpha(x) \leq M(\alpha(b) - \alpha(a)) \tag{133.2}$$

$$m \leq \frac{\int_a^b f(x)d\alpha(x)}{\alpha(b)-\alpha(a)} \leq M \tag{133.3}$$

Since f is continuous on $[a,b]$ and we know that f takes on the values M and m somewhere on the interval $[a,b]$, we see that we can use the intermediate value theorem to find the following:

$$\frac{\int_a^b f(x)d\alpha(x)}{\alpha(b) - \alpha(a)} = f(c) \tag{133.4}$$

For some $c \in [a,b]$, which proves the theorem. $\square$

134. Problem 134

Theorem 134.1. *Suppose that f is continuous and α is strictly increasing on $[a,b]$. Define $F(x) = \int_a^x f(t)d\alpha(t)$ and show that for $x \in [a,b]$ we have $\lim_{h\to 0} \frac{F(x+h)-F(x)}{\alpha(x+h)-\alpha(x)} = f(x)$.*

Proof. This is a straightforward application of the first mean value theorem. We have the following:

$$\int_x^{x+h} f(t)d\alpha(t) = f(c)(\alpha(x+h) - \alpha(x)) \tag{134.2}$$

For some $c \in [x, x+h]$. As $h \to 0$, we see that $c \to x$, so that we have:

$$\lim_{h\to 0} \int_x^{x+h} f(t)d\alpha(t) = f(x)(\alpha(x+h) - \alpha(x)) \tag{134.3}$$

Which is what we wanted. $\square$

135. Problem 135

Theorem 135.1. *Let f be continuous on $[0,1]$. For positive a, b find $\lim_{\epsilon\to 0^+} \int_{a\epsilon}^{b\epsilon} \frac{f(x)}{x} dx$.*

Proof. We can use a change of variables to find:

$$\int_{a\epsilon}^{b\epsilon} \frac{f(x)}{x} dx = \int_{a\epsilon}^{b\epsilon} f(x)d(\ln(x)) \tag{135.2}$$

Using the first mean value theorem, we obtain:

$$\int_{a\epsilon}^{b\epsilon} f(x)d(\ln(x)) = f(c)(\ln(b\epsilon) - \ln(a\epsilon) \tag{135.3}$$

$$= f(c)\ln\left(\frac{b}{a}\right) \tag{135.4}$$

For some $c \in [a\epsilon, b\epsilon]$. Taking the limit as $\epsilon \to 0^+$, we see that $c \to 0$, which shows that the integral converges to $\lim_{\epsilon\to 0^+} \int_{a\epsilon}^{b\epsilon} \frac{f(x)}{x} dx = f(0)\ln(\frac{a}{b})$. $\square$

Past Exam Questions

136. Problem 136

Theorem 136.1. *Show that the set* $\{z \in \mathbb{C}; z = \exp(it^{24} + 23t^7), t \in \mathbb{R}\}$ *is connected.*

Proof. First, we note that the function $f(t) = \exp(it^{24} + 23t^7)$ is continuous. This is because it is the composition of a polynomial and a continuous function e^t, which makes it continuous. Moreover, we see that the function maps the set $\mathbb{R}$ to the set $A = \{z \in \mathbb{C}; z = \exp(it^{24} + 23t^7), t \in \mathbb{R}\}$. We know that $\mathbb{R}$ is connected by a theorem in Rudin, and that a continuous function preserves connectedness. Therefore, the set A must also be connected. □

137. Problem 137

Theorem 137.1. *Explain why there is no continuous map from the disk* $\{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\}$ *onto the interval* $(0, 1) \in \mathbb{R}$.

Proof. First note that the interval $(0, 1) \in \mathbb{R}$ is open, and that the disk $D = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\}$ is closed. Suppose that there was a continuous map from D to $(0, 1)$. Then then every open set in $(0, 1)$ would have a preimage of an open set. This is a contradiction because $f^{-1}((0, 1)) = D$ which is closed, while $(0, 1)$ is open. Thus, we have a contradiction, and no continuous mapping exists. □

138. Problem 138

Theorem 138.1. *Suppose that a number s is the upper limit (limit supremum) of a subsequence of a sequence* $\{x_n\}$ *in the reals. Show that s is the limit of some subsequence of* $\{x_n\}$.

Proof. We see that there exists a subsequence $\{y_k\}$ such that $\lim_{n\to\infty} \sup\{y_k\} = s$. So let $t_n = \sup_{n\geq k}\{y_k\}$. Rewriting, we have $\lim_{n\to\infty} t_n = s$. By the definition of supremum, there is a subsequence y_{p_k} such that $y_{p_k} \geq t_n - \frac{1}{n}$. Therefore, we have $t_n \geq y_{p_k} \geq t_n - \frac{1}{n}$. As $n \to \infty$, we see that $y_{p_k} \to t_n$ and since we know $t_n \to s$, we see that $y_{p_k} \to s$. Since y_{p_k} is a subsequence of a subsequence of $\{x_n\}$, we see that is still is a subsequence of $\{x_n\}$, which proves the theorem. □

139. Problem 139

Theorem 139.1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be twice differentiable and suppose that* 0 *is a local maximum of* f*, i.e. for some* $\epsilon > 0$, $f(x) \leq f(0)$ *for all* $x \in (-\epsilon, \epsilon)$. *Show that* $f''(0) \leq 0$.

Proof. First, we know by a theorem in Rudin that every local maximum has derivative of zero. Therefore, $f'(0) = 0$. Now take ϵ so that $f(x) \leq f(0)$ for all $x \in (-\epsilon, \epsilon)$. Now, since f is twice differentiable, we can apply the mean value theorem for all positive x. This gives us $f(x) - f(0) = f'(x_1)x$ for all $x \in (0, \epsilon)$ and for some $x_1 \in (0, \epsilon)$. Since we know that $f(x) \leq f(0)$, we see that $f'(x_1) \leq 0$. Applying the mean value theorem again, we obtain $f'(x_1) - f'(0) = f''(x_2)x_1$ for some $x_2 \in (0, x_1)$. Since $f'(x_1) \leq 0$ and $f'(0) = 0$, we see that $f''(x_2) \leq 0$. Now, we have shown that $f''(x_2) \leq 0$ for $0 < x_2 < x_1 < x$. Taking the limit as $x \to 0^+$, we see that $\lim_{x\to 0^+} f''(x) \leq 0$.

Now we can perform the same operation on the negative side. We apply the mean value theorem and obtain $f(x) - f(0) = f'(x_1)x$ for all $x \in (-\epsilon, 0)$ and for some $x_1 \in (x, 0)$. Since $f(x) \leq f(0)$ for all $x \in (-\epsilon, \epsilon)$ and $x < 0$, we see that $f'(x_1) \geq 0$. Applying the mean value theorem again, we obtain $f'(x_1) - f'(0) = f''(x_2)x_1$ for $x_2 \in (x_1, 0)$. We know that $f'(0) = 0$ while $f'(x_1) \geq 0$. Since $x_1 < 0$, we see that $f''(x_2) \leq 0$. This is for $x < x_1 < x_2 < 0$. Therefore, taking the limit as $x \to 0^-$, we see that $\lim_{x\to 0^-} f''(x) \leq 0$. Thus, we see that $\lim_{x\to 0} f''(x) = f''(0) \leq 0$. □

140. Problem 140

Theorem 140.1. *Let* $\{\phi_n\}$ *be a uniformly bounded sequence of continuous functions on* $[0, 1]$ *such that* $\lim_{n\to\infty} \int_0^1 x^k \phi_n(x)dx = 0$ *for every* $k = 0, 1, \ldots$ *and show that for any continuous function* $f : [0, 1] \to \mathbb{R}$*, the limit* $\lim_{n\to\infty} \int_0^1 f(x)\phi_n(x)dx$ *exists.*

Proof. First, note that by the Stone-Weierstrass theorem, every continuous function $f : [0, 1] \to \mathbb{R}$ can be approximated by a sequence of polynomials $\{P_n\}$ such that $\lim_{n\to\infty} P_n(x) = f(x)$. Moreover, it is clear that every polynomial consists of a series of monomials and their coefficients. Therefore, we have the following:

$$P_n(x) = \sum_{k=1}^{r(n)} c_k x^k \tag{140.2}$$

Where $r(n)$ the degree of the nth polynomial. Therefore, we see that we can decompose our limit into:

$$\lim_{n\to\infty}\int_0^1 f(x)\phi_n(x)dx = \lim_{n\to\infty}\int_0^1 P_n(x)\phi_n(x)dx \tag{140.3}$$

$$= \lim_{n\to\infty}\int_0^1 \sum_{k=1}^{r(n)} c_k x^k \phi_n(x)dx \tag{140.4}$$

(140.5)

However, we see that each one of these terms is identically zero by our assumption. Therefore, $\lim_{n\to\infty}\int_0^1 f(x)\phi_n(x)dx$ not only exists, but is also equal to 0. $\square$

141. Problem 141

Theorem 141.1. *Using standard properties of the cosine function show that the series $f(x) = \sum_{n=1}^{\infty}\frac{1}{n^{5/2}}\cos(nx)$ defines a continuously differentiable funciton on the real line.*

Proof. First note that the function itself is a composition of continuous functions, so that $f(x)$ is continuous (moreover we don't have to worry about dividing by 0 because $n \neq 0$ in the series). Next, let us differentiate the series term by term. We obtain:

$$\hat{f}(x) = \sum_{n=1}^{\infty}\frac{d}{dx}\left(\frac{1}{n^{5/2}}\cos(nx)\right) = \sum_{n=1}^{\infty} -\frac{1}{n^{3/2}}\sin(nx) \tag{141.2}$$

Since we know that $|\sin(nx)| \leq 1$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$, we see that $\hat{f}(x) \leq \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, which converges by geometric series with $p = 3/2$. Thus, we see that $\hat{f}(x)$ is absolutely convergent, which shows by Weierstrass M Test that it is also uniformly convergent to f'. Therefore, we see that $f' = \hat{f}$. Moreover, we see that $\hat{f}$ is composed entirely of continuous functions. Since $\hat{f}$ is a uniformly convergent series of continuous functions, we see that f' is continuous and exists for all $x \in \mathbb{R}$. Therefore, we have shown that $f(x)$ is continuously differentiable on $\mathbb{R}$. $\square$

142. Problem 142

Theorem 142.1. *Explain why the Riemann-Stieltjes integral $\int_{-1}^{1}\exp(x^2/3)d\alpha$ exists for any increasing $\alpha : [-1, 1,] \to \mathbb{R}$ and evaluate it when*

$$\alpha(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \tag{142.2}$$

Proof. First, the integral exists because $f(x) = \exp(x^2/3)$ is the composition of continuous functions, which makes $f(x)$ continuous as well. Therefore, by a theorem in Rudin $f \in \mathscr{R}(\alpha)$ for any increasing $\alpha : [-1, 1] \to \mathbb{R}$. To compute the integral, take a partition $P = \{-1 = x_0 < x_1 < \ldots < x_i < 0 < x_{i+1} < \ldots < x_n = 1\}$. We see that because all $\alpha(x_{j+1}) - \alpha(x_j)$ are identically zero unless $j = 1$, that the integral can be computed as $\int_{-1}^{1}\exp(x^2/3)dx = f(0)(\alpha(x_{i+1}) - \alpha(x_i)) = 1(1) = 1$. $\square$

143. Problem 143

Theorem 143.1. *Show that the set $A = \{x \in \mathbb{C}; z = \exp(it^3); t \in \mathbb{R}\}$ is connected.*

Proof. It is clear that $\exp(it^3)$ is a continuous map from $\mathbb{R} \to A$. Therefore, since $\mathbb{R}$ is connected, and continuous maps preserve connectedness, we see that A is also connected. $\square$

144. Problem 144

Theorem 144.1. *Let $\{x_n\}$ be a seuqnce in a metric space X and suppose that tere is a point $p \in X$ with the property that every subsequence of $\{x_n\}$ has a subsequence which converges to p. Show that $\{x_n\}$ converges to p.*

Proof. Suppose by contradiction that $\{x_n\}$ does not converge to p. Now let $\{y_k\}$ be a subsequence of $\{x_n\}$. We know by assumption that $\{y_k\}$ has a subsequence $\{y_{n_k}\}$ which converges to p. Fix $\epsilon > 0$. We see that there exists an N such that for all $n_k > N$, we have $|y_{n_k} - p| < \epsilon$. Since we have assumed $\{x_n\}$ does not converge to p, we cannot find an M such that for all $n > 0$, we have $|x_n - p| < \epsilon$. Therefore, the set $\{n \in \mathbb{N} : d(x_n, p) \geq \epsilon\}$ is infinite. Moreover, every subsequence of $\{y_k\}$, which is itself a subsequence of $\{x_n\}$ must take values from this set. This means, however, that there are an infinite number of points in $\{y_{n_k}\}$ for wihch $|y_{n_k} - p| > \epsilon$, which is a contradiction. Thus, $\{x_n\}$ must converge to p. $\square$

145. Problem 145

Theorem 145.1. *Prove that the function $f(x) = \exp(\frac{x^3-15}{x^2+x+1})$ is continuously differentiable on $[0,1]$ and prove that it takes on a minimum value on the interval.*

Proof. First, the set $[0,1]$ is compact, so that every continuous function takes on a minimum value. Obviously $f(x)$ is a composition of continuous polynomials and functions (e^x) so that $f(x)$ itself is also continuous. Hence, it takes on a minimum value. We also know that $f(x)$ is continuously differentiable because each of its component functions is continuously differentiable on $[0,1]$. We see that $x^3 - 15$ and $x^2 + x + 1$ are both polynomials and hence continuously differentiable. Moreover e^x is continuously differentiable by a theorem in Rudin. Therefore, the composite function $f(x)$ is continuously differentiable. □

146. Problem 146

Theorem 146.1. *If $g : \mathbb{R} \to \mathbb{R}$ is differentiable and g' is bounded on $\mathbb{R}$, show that g is uniformly continous.*

Proof. Fix $\epsilon > 0$. Since g' is bounded on $\mathbb{R}$, there exists some M such that $|g'(x)| < M$ for all $x \in \mathbb{R}$. Moreover, since g is differentiable, we can apply the mean value theorem and obtain $g(x)-g(y) = g'(c)(x-y)$ for all $x, y \in \mathbb{R}$ and $c \in (x, y)$ if we assume without loss of generality that $x < y$. Therefore, if we pick $\delta < \frac{\epsilon}{M}$, and we observe that $|g'(c)| < M$, we have:

$$|g(x) - g(y)| < M(x - y) < M\delta < M\frac{\epsilon}{M} = \epsilon \tag{146.2}$$

Since $\epsilon > 0$ was originally arbitrary, we have shown that there exists a $\delta > 0$ for which $|g(x) - g(y)| < \epsilon$ whenever $|x - y| < \delta$ for all $x, y \in \mathbb{R}$. Thus, we have shown uniform continuity. □

147. Problem 147

Theorem 147.1. *Let $A : [0,1]^2 \to \mathbb{R}$ be a continuous function satisfying $\sup_{[0,1]^2} |A(x,y)| \leq \frac{1}{2}$. Show that if $f \in \mathscr{C}([0,1])$, then $g(x) = \int_0^1 A(x,y)f(y)dy \in \mathscr{C}([0,1])$.*

Proof. First, we note that $f(x)$ and $A(x,y)$ are both continuous, so that $A(x,y)f(y)$ is continuous. Thus, we see that $g(x) = \int_0^1 A(x,y)f(y)dy$ exists. To see that $g(x)$ is continuous, we note that $A(x,y)f(y)$ is continuous on the compact set $[0,1]$, which shows that it is uniformly continuous. Fix $\epsilon > 0$, we see that there exists a $\delta > 0$ for which $|A(x,y) - A(x',y)| < \epsilon$ whenever $|x - x'| < \delta$. Therefore, we have:

$$|g(x) - g(x')| = \left| \int_0^1 (A(x,y) - A(x',y))f(y)dy \right| < \epsilon \sup_{[0,1]} |f| \tag{147.2}$$

Therefore, we see that $|g(x) - g(x')| < \epsilon \sup |f|$ whenever $|x - x'| < \delta$, which shows continuity because ϵ was arbitrary. □

Theorem 147.3. *Estimate $||g|| = \sup_{[0,1]} |g(x)|$ in terms of $||f||$.*

Proof. We know that the following is true:

$$|g(x)| = \left| \int_0^1 A(x,y)f(y)dy \right| \leq \left| \int_0^1 \frac{1}{2}||f||dy \right| = \frac{1}{2}||f|| \tag{147.4}$$

Therefore, we see that $||g|| \approx \frac{1}{2}||f||$. □

Theorem 147.5. *If $h \in \mathscr{C}([0,1])$ is a fixed function show that $(Gf)(x) = h(x) + \int_0^1 A(x,y)f(y)dy$ defines a contraction G on $\mathscr{C}([0,1])$ sending f to Gf.*

Proof. If $f_1, f_2 \in \mathscr{C}([0,1])$ then:

$$|(Gf_1)(x) - (Gf_2)(x)| \quad = \quad \int_0^1 A(x,y)(f_1(y) - f_2(y))dy \tag{147.6}$$

Therefore, we see that $d(G(f_1), G(f_2)) < \frac{1}{2}||f_1 - f_2|| = \frac{1}{2}d(f_1, f_2)$ by the above estimate, which shows that $(Gf)(x)$ is a contraction. □

Theorem 147.7. *Show that there exists a unique $f \in \mathscr{C}([0,1])$ such that $f(x) = h(x) + \int_0^1 A(x,y)f(y)dy$ for alll $x \in [0,1]$.*

Proof. We begin by noting that $\mathscr{C}([0,1])$ is complete. Therefore, the Contraction Mapping Principle implies that there is a unique solution of $G(f) = f$, which is the desired result. □

148. Problem 148

Theorem 148.1. *Show that the set $A = \{z \in \mathbb{C}; 1 < |z| < 2\}$ is connected as a subset of $\mathbb{C}$ with the usual metric.*

Proof. Again, we know that $\mathbb{C}$ is connected, and that the function $f : \mathbb{C} \to A$ of $f(z) = |z|$ is continuous. Therefore, since continuous mappings preserve connectedness, we see that A must be connected as well. □

149. Problem 149

Theorem 149.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable and satisfy $|g'(x)| \leq \frac{1}{2}$. Show that the function $f(x) = x - g(x)$ is one to one.*

Proof. Using the mean value theorem, we see that $|f(x) - f(y)| = f'(c)|x - y|$ for all $x, y \in \mathbb{R}$. Moreover, since $|f'(c)| = |1 - g'(c)| \leq 1 - \frac{1}{2}$, so that $|f'(c)| \geq \frac{1}{2}$. Therefore, we have $|f(x) - f(y)| \geq \frac{1}{2}|x - y|$ which shows that if $x \neq y$, then $f(x) \neq f(y)$. This proves a one to one correspondence. □

150. Problem 150

Theorem 150.1. *Let $f : [0, 1] \to \mathbb{R}$ be continuous. Show that there exists $c \in (0, 1)$ such that $\int_0^1 f(x)dx = f(c)$.*

Proof. Let us define $F(x) = \int_0^x f(x)dx$. Now fix $\epsilon > 0$ and we will show that this function is continuous. We have, assuming $x < y$, $|F(y) - F(x)| = |\int_x^y f(t)dt| = f(t)(x - y) < \epsilon$ if we choose $\delta = \frac{\epsilon}{f(t)}$ for some $t \in (x, y)$. Thus, we see that F is continuous, which means we can apply the mean value theorem. We obtain $F(1) - F(0) = f(c)(1)$ for some $c \in (0, 1)$. Moreover, we know that $F(0) = \int_0^0 f(t)dt = 0$. Therefore, $F(1) = \int_0^1 f(x)dx = f(c)$ for some $c \in (0, 1)$, which is what we wanted. □

151. Problem 151

Theorem 151.1. *For what values of $x \in \mathbb{R}$ does the series $\sum_{n=0}^{\infty} n \exp(-nx)$ converge? For what intervals $[a, b]$ does it converge uniformly, and on what intervals is the sum of the series differentiable?*

Proof. It is obvious that any $x < 0$ would cause the sequence $n \exp(-nx)$ to diverge. Thus, convergence, if it does occur, must occur for the region $x > 0$. We know that the sequence $\lim_{n\to\infty} x^n \exp(-n)$ converges by a theorem in Rudin. Moreover, it is clear that if $x > 0$ the series converges by ratio test. We have:

$$\lim_{n\to\infty} \left| \frac{(n+1)e^{-(n+1)x}}{ne^{-nx}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)e^{-x}}{n} \right| = \frac{1}{e^x} < 1 \tag{151.2}$$

Thus, we the series converges whenever $x > 0$. The series therefore converges uniformly on $[a, b]$ whenever $b \geq a$ and when $a \neq 0$. The series is differentiable on the same interval, for $a \neq 0$, because the series of derivatives converges uniformly for the same reason. □

152. Problem 152

Theorem 152.1. *Consider the power series $\sum_{n=1}^{\infty} \frac{1}{n} x^n$ and show that this series converges uniformly on $(-\frac{1}{2}, \frac{1}{2})$. Let $f(x)$ denote the sum and show that the series obtained by term-by-term differentiation converges uniformly in the same set and explain why the limit is $f'(x)$. Is $f'(x)$ a rational function?*

Proof. We see using the ratio test that the radius of convergence is 1 because:

$$\lim_{n\to\infty} \left| \frac{\frac{1}{n+1}x^{n+1}}{\frac{1}{n}x^n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1}x \right| = |x| \tag{152.2}$$

And we know that $|x| < 1$ whenever $x \in (-1, 1)$. Therefore, the series converges uniformly on the interval $(-1, 1)$, which shows that it also converges uniformly on $(-\frac{1}{2}, \frac{1}{2})$. Now, differentiation term by term gives the series $\sum_{n=1}^{\infty} x^{n-1}$, which converges whenever $x \in (-1, 1)$ by a theorem in Rudin. Therefore, we see that it is uniformly convergent on this set, meaning it is also uniformly convergent on $(-\frac{1}{2}, \frac{1}{2})$. Moreover, since these terms are uniformly convergent, we know that $f'(x)$ is given by these terms according to a theorem in Rudin. We see that the series converges to $f'(x) = \frac{1}{1-x}$, which is indeed rational. □

153. Problem 153

Theorem 153.1. *Explain why $\int_0^2 \exp(3(|x|^{3/2}-1))d\alpha$ exists for any increasing function $\alpha : [0,2] \to \mathbb{R}$. Evaluate the integral when*

$$\alpha(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 3 & 1 < x \leq 2 \end{cases} \tag{153.2}$$

Proof. First, we know that $|x|^{3/2} - 1$ is continuous on the interval $[0,2]$ so that the composition $f(x) = \exp(3(|x|^{3/2}-1))$ of continuous functions is also continuous on the interval $[0,2]$. Therefore, it is integrable for any increasing α. Next, we will evaluate the integral, which is:

$$\int_0^2 f(x)d\alpha(x) = (3-1)f(1) = 2\exp(3(1-1)) = 2 \tag{153.3}$$

□

154. Problem 154

Theorem 154.1. *Suppose that $f_n : [0,1] \to \mathbb{R}$ is a sequence of continuous functions which is uniformly bounded and satisfies:*

$$f_n(x) = \frac{1}{n} + \int_0^x f_n^2(t)dt, x \in [0,1] \tag{154.2}$$

Show that $\{f_n\}$ is uniformly convergent on $[0,1]$ and prove that the limit is identically zero.

Proof. The continuity of f_n implies that f_n^2 is also continuous. Moreover, we see that $f_n'(x) = f_n^2(x)$ by the fundamental theorem of calculus. The uniform boundedness of f_n implies the uniform boundedness of f_n' as well, so that there exists an M such that $|f_n'(x)| < M$ for all $n \in \mathbb{N}$ and $x \in [0,1]$. We can apply the fundamental theorem of calculus to see that $|f_n(x) - f_n(y)| \leq M(x-y)$ for all $x \in [0,1]$ and $n \in \mathbb{N}$. Fix $\epsilon > 0$. We can choose $\delta = \frac{\epsilon}{M}$ such that $|f_n(x) - f_n(y)| < \epsilon$ whenever $|x-y| < \delta$. Therefore, we see that $\{f_n\}$ is equicontinuous.

We see that $[0,1]$ is a compact metric space by Heine-Borel. Moreover, since $\{f_n\}$ is uniformly bounded, it is also pointwise bounded. Therefore, $\{f_n\}$ satisfies the assumptions of Arzela-Ascoli, which shows that $\{f_n\}$ contains a uniformly convergent subsequence, let us say $\{f_{n_k}\} \rightrightarrows f$. We see that the limit must satisfy $f(x) = \int_0^x f^2(t)dt$ for all $x \in [0,1]$. Let $T = \sup_{[0,r]} |f(x)|$. We see that $T \leq TMr$ on $[0,r]$, which implies $T = 0$ if $Mr < 1$. Therefore, we can choose r such that $r < \frac{1}{M}$ and see that $T = 0$ on $[0,r]$. We can proceed and break the interval $[0,1]$ into finitely many intervals of the form $[nr, (n+1)r]$, where $n \in \mathbb{N}$. This shows that $T = 0$ on all of $[0,1]$. Moreover, we see that every uniformly convergent subsequence of $\{f_n\}$ has a uniformly convergent subsequence converging to $f(x) = 0$, which means that $f(x) = 0$ identically, and that $\{f_n\}$ is uniformly convergent. □

155. Problem 155

Theorem 155.1. *Show that if $E \subset \mathscr{C}(X, \mathbb{R})$ and $\mathscr{M}$ is compact, then E it is equicontinuous (you may not use Arzela-Ascoli).*

Proof. Fix $\epsilon > 0$. Then we see that $\mathscr{M}$ can be covered by a finite number of functions $f_1, f_2, \ldots, f_p$ with radii ϵ such that $X \subset \bigcup_{i=1}^p N_\epsilon(x_i)$. We know that each f_i is uniformly continuous so that there exists some δ_i for which $|f_i(x) - f_i(y)| < \epsilon$ whenever $|x-y| < \delta_i$ for all $x, y \in \mathscr{M}$. Now pick $\delta = \min\{\delta_1, \ldots, \delta_p\}$ and $x, y \in \mathscr{M}$ such that $d(x,y) < \delta$. Let $f \in E$, so that we must have $f \in N_{\delta_i}(f_i)$ for some i.

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| < 3\epsilon \tag{155.2}$$

This proves equicontinuity. □

156. Problem 156

Theorem 156.1. *If $S \subset \mathbb{R}^n$, show that the collection of isolated points of S is countable.*

Proof. Let S denote the set of isolated points and let $s \in S$. Then pick a neighborhood $N(s)$ such that $N(s) \cap S = \{s\}$ and $N(s) \cap N(t) = \emptyset$ if $s \neq t$. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, we can choose a point in each $N(s)$ with rational coordinates. Therefore, we see that there is a one to one map from $S \to \mathbb{Q}^n$, which shows that S is countable. □

157. Problem 157

Theorem 157.1. *Prove that if $\mathscr{M}, \mathscr{N}$ are metric spaces and $g : \mathscr{M} \to \mathscr{N}$ is uniformly continuous, then whenever $\{x_n\} \subset \mathscr{M}$ is Cauchy, the sequence $\{g(x_n)\}$ is Cauchy.*

Proof. Fix $\epsilon > 0$. Since g is uniformly continuous, we see that there exists a $\delta > 0$ for all $x, y \in \mathscr{M}$ such that $|g(x) - g(y)| < \epsilon$ whenever $|x - y| < \delta$. Moreover, since $\{x_n\}$ is Cauchy, then there exists an N such that for all $n > N$, we have $|x_n - x_m| < \delta$. Therefore, we see that $|g(x_n) - g(x_m)| < \epsilon$, which shows that $\{g(x_n)\}$ is Cauchy. □

158. Problem 158

Theorem 158.1. *Let $\mathscr{M}$ and $\mathscr{N}$ be metric spaces, let $A \subset \mathscr{M}$ and $\bar{A} \subset \mathscr{M}$ denote the closure of A. If $\mathscr{N}$ is complete and $h : A \to N$ is uniformly continuous, prove that there is a unique continuous function $\hat{h} : \bar{A} \to \mathscr{N}$ such that $\hat{h}(a) = h(a)$ for every $a \in A$.*

Proof. For any $a \in \bar{A}$, choose $\{a_n\} \subset A$ such that $a_n \to a$ and define $\hat{h}(a) = \lim_{n\to\infty} h(a_n)$. The limit exists because $\{a_n\}$ is a Cauchy sequence, which implies that $\{h(a_n)\}$ is also a Cauchy sequence. Since $\mathscr{N}$ is complete, we see that $\{h(a_n)\}$ converges to some limit as $n \to \infty$. It is clear that $\hat{h}(a)$ is unique because if there exists another sequence $\{b_n\} \to a$, then we see that the limits must coincide. Now, we need to show that this definition of $\hat{h}(a)$ is continuous. First, fix $\epsilon > 0$. We know that h is uniformly continuous so that there exists a $\delta > 0$ such that for all $a, b \in A$, we have $d(h(a), h(b)) < \epsilon$ whenever $d(a, b) < \delta$. If $x \in \bar{A}$, and $a_n \to x$, then we see that there exists an N such that for all $n > N$, we have $d(h(a), h(x_n)) < \epsilon$, and we pick some point $b \in N_{\delta/2}(x) \cap A$, then there is an $a_m \in N_{\delta/2}(x)$ with $m > N$. Therefore:

$$|\hat{h}(x) - \hat{h}(b)| \le |\hat{h}(x) - \hat{h}(a_m)| + |\hat{h}(a_m) - \hat{h}(b)| < 2\epsilon \tag{158.2}$$

If $c \in N_{\delta/2}(x) \cap \bar{A}$, then we see by similar logic that $|\hat{h}(x) - \hat{h}(c)| < 3\epsilon$, which shows that $\hat{h}$ is continuous on $\bar{A}$. □

159. Problem 159

Theorem 159.1. *Assume $f : (a, b) \to \mathbb{R}$ has derivative at every point in (a, b). Let $c \in (a, b)$ and assume that $\lim_{x\to c} f'(x)$ exists and is finite. Prove that the value of this limit must be $f'(c)$.*

Proof. Recall the definition of derivative says that $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$, which we know by assumption exists. Moreover, we see that $\lim_{x\to c} f'(x) = \lim_{x\to c} \lim_{t\to x} \frac{f(t)-f(x)}{t-x} = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)$. This proves the theorem. □

160. Problem 160

Theorem 160.1. *Assume f, g, h are real valued functions defined on $[0, 1]$ and $g \ge 0$ is in $\mathscr{R}(x)$. Prove that if f is continuous, then there exists $w \in [0, 1]$ such that $\int_0^1 f(t)g(t)dt = f(w)\int_0^1 g(t)dt$.*

Proof. Define $m = \inf_{x\in[0,1]} |f(x)|$ and $M = \sup_{x\in[0,1]} |f(x)|$. Then we can derive the following inequality:

$$m\int_0^1 g(t)dt \le \int_0^1 f(t)g(t)dt \le M\int_0^1 g(t)dt \tag{160.2}$$

Therefore, since $f(x)$ is continuous for $x \in [0, 1]$, we can use the intermediate value theorem and we see that $\int_0^1 f(t)g(t)dt = f(w)\int_0^1 g(t)dt$ for some $w \in [0, 1]$. □

Theorem 160.3. *Prove that if h is monotonically increasing (not necessarily continuous), then there exists $z \in [0, 1]$ such that $\int_0^1 h(t)g(t)dt = h(0)\int_0^z g(t) + h(1)\int_z^1 g(t)dt$.*

Proof. We know that $\inf_{x\in[0,1]} h(x) = h(0)$ and $\sup_{x\in[0,1]} h(x) = h(1)$ because h is monotonically increasing. Moreover, we know that since $\int_0^1 g(x)dx$ exists, that $\phi(x) = h(0)\int_0^x g(t)dt + h(1)\int_x^1 g(t)dt$ is continuous on $[0, 1]$. Since $\int_0^1 g(t)dt$ is a constant, we know that $\phi(x)$ takes a minimum when $x = 1$ with $\phi(1) = h(0)\int_0^1 g(t)dt$. We know that $\phi(x)$ takes a maximum at $x = 0$ with $\phi(0) = h(1)\int_0^1 g(t)dt$. Since it is continuous, we can use intermediate value theorem and see that the function attins the value $\int_0^1 h(t)g(t)dt$ for some value $z \in [0, 1]$ at $\phi(z)$. The theorem follows. □

161. Problem 161

Theorem 161.1. *Let $S = \{n_1, n_2, n_3, \ldots\}$ denote the collection of positive integers that do not involve the digit 3 in their decimal representation. (For example, $7 \in S$ but $131 \notin S$.) Show that $\sum \frac{1}{n_k}$ converges and has sum less than 90.*

Proof. If m has l digits, then $\frac{1}{m} \leq \frac{1}{10^{l-1}}$. If $s \in S$ has exactly l digists, then the first digit can be anything but a 0 or a 3 (8 possibilities) and each of the $l-1$ other digits can be anything but a 3 (9^{l-1} possibilities). Therefore, there are $8(9)^{l-1}$ numbers in S such that there are l digits. We can then obtain an upper bound for the sum:

$$\sum_{k=1}^{\infty} \frac{1}{n_k} \leq \sum_{l=1}^{\infty} \frac{8(9)^{l-1}}{10^{l-1}} = 8 \sum_{l=1}^{\infty} \left(\frac{9}{10}\right)^{l-1} = 80 \tag{161.2}$$

Therefore, it is apparent not only that the sum converges by comparison test, but also that the sum must be lest than 90, which is what we wanted to prove. □

162. Problem 162

Theorem 162.1. *Assume that $\{g_n\}$ is a sequence of real-valued functions defined on $T \subset \mathbb{R}$ satisfying $g_{n+1}(x) \leq g_n(x)$ for each $x \in T$ and $n \in \mathbb{N}$ and suppose that $g_n \rightrightarrows 0$ on T. Show that $\sum_{n=1}^{\infty}(-1)^{n+1} g_n(x)$ converges uniformly on T.*

Proof. Let the partial sum be defined as $G_k(x) = \sum_{n=1}^{k}(-1)^{n+1} g_n(x)$. Fix $\epsilon > 0$. By the uniform convergence of $\{g_n\}$, we know that there exists an N such that for all $n > N$, we have $|g_n(x)| < \epsilon$. Now take $j, k > N$ and assume $k > j$. We have

$$|G_k(x) - G_j(x)| = |(-1)^{k+1} g_k(x) + (-1)^k g_{k-1}(x) + \ldots + (-1)^{j+1} g_j(x)| \tag{162.2}$$

$$\leq |g_k(x)||(-1)^{k+1} + (-1)^k + \ldots + (-1)^{j+1}| \tag{162.3}$$

$$< \epsilon \tag{162.4}$$

This is clear because we can group each $(-1)^{k+1} + (-1)^k$ into pairs which will cancel each other out. Therefore, the sum of alternating terms is either -1 or 1, both which have absolute value of 1. Therefore, we see that $|G_k(x) - G_j(x)| < \epsilon$ whenever $k, j > N$ for all $x \in T$. This shows uniform convergence. □

163. Problem 163

Theorem 163.1. *Consider a continuous function $f : [0, \infty) \to \mathbb{R}$. For each n define the continuous function $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) = f(x^n)$. Show that the set of continuous functions $\{f_1, f_2, \ldots\}$ is equicontinuous on some interval containing $x = 1$ if and only if f is a constant function.*

Proof. First, suppose that the set of continuous functions $\{f_1, f_2, \ldots\}$ is equicontinuous on some interval $1 \in [a, b]$. Fix $\epsilon > 0$. Then there exists some $\delta > 0$ for which $|f_n(x) - f_n(y)| < \epsilon$ whenever $|x - y| < \delta$ for all $x, y \in [a, b]$. Therefore, we see that $|f(x^n) - f(y^n)| < \epsilon$ whenever $|x - y| < \delta$. Moreover, when $|1 - x| < \delta$, we see that $|f(1) - f(x^n)| < \epsilon$. Choose $x < 0$ so that there exits an N for which $|f(1) - f(0)| < \epsilon$ for some $n > N$. Therefore, we see that $f(1) = f(0)$. Moreover, for any $z \in (0, \infty)$, choose M large enough so that $|z^{\frac{1}{M}} - 1| < \delta$ so that $|f(z) - f(1)| < \epsilon$. Since ϵ was arbitrary, we see that $f(z) = f(1)$.

Next, suppose that $f(x) = c$ is a constant function. Thus, $f(1) = f(x^n)$ for all $x \in [a, b]$ and $n \in \mathbb{N}$. By our assumption, we see that $f_n(1) = f_n(x)$. Fix $\epsilon > 0$. We see that $|f_n(1) - f_n(x)| < \epsilon$ for all $x \in [a, b]$, which implies that we can pick any $\delta > 0$ such that the inequality is satisfied whenever $|1 - x| < \delta$. We can extend this to show that $|f_n(x) - f_n(y)| < \epsilon$ whenever $|x - y| < \delta$ because $0 = |f(x^n) - f(y^n)| = |f_n(x) - f_n(y)| < \epsilon$. This proves that $\{f_n\}$ is equicontinuous. □

164. Problem 164

Theorem 164.1. *Define for any $z \in \mathbb{R}$ the exponential function by $\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$. Prove that $\exp : \mathbb{R} \to \mathbb{R}$ is a continuous function.*

Proof. Using the ratio test, we see the following is true:

$$\lim_{k\to\infty} \left| \frac{\frac{z^{k+1}}{(k+1)!}}{\frac{z^k}{k!}} \right| = \lim_{k\to\infty} \left| \frac{z^k z k!}{k!(k+1)z^k} \right| = \lim_{k\to\infty} \frac{z}{k+1} = 0 < 1 \tag{164.2}$$

Thus, the ratio test shows that the series converges for every $z \in \mathbb{R}$. This shows that the exponential function is continuous for all $z \in \mathbb{R}$. □

Theorem 164.3. *Use the binomial theorem* $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k}$ *to show that* $\exp(z+w) = \exp(z)\exp(w)$.

Proof. We will use the binomial theorem and observe:

$$\exp(z+w) = \sum_{k=0}^{\infty} \frac{(z+w)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} z^j w^{k-j} \tag{164.4}$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{1}{k!} \frac{k!}{j!(k-j)!} z^j w^{k-j} \tag{164.5}$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{z^j}{j!} \frac{w^{k-j}}{(k-j)!} \tag{164.6}$$

$$= \left(\sum_{k=0}^{\infty} \frac{z^j}{j!}\right)\left(\sum_{i=0}^{\infty} \frac{w^i}{i!}\right) \tag{164.7}$$

This proves the inequality, by replacing the right and left sums with $\exp(z)$ and $\exp(w)$ respectively. □

Theorem 164.8. *Prove that* $\exp'(z) = \exp(z)$.

Proof. Let us differentiate the series of $\exp(z)$ term by term. We obtain:

$$\sum_{k=0}^{\infty} \frac{d}{dz} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{kz^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{z^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{164.9}$$

Next, since we know that this sum converges absolutely by the ratio test (as we have previously shown), we know that the series converges uniformly to $\exp(z)$. Therefore, we see that $\exp'(z)$ is the same as the sum of the term by term derivatives of the series of $\exp(z)$ by a theorem in Rudin. Thus, we see that $\exp'(z) = \exp(z)$, as we wanted. □

165. Problem 165

Theorem 165.1. *Let V be the space of sequences $a = \{a_n | n \geq 1\}$ of real numbers such that $\sum_{n=1}^{\infty} |a_n| < \infty$. For which real numbers p is the series $\sum_{n=1}^{\infty} |a_n|^p$ convergent for all $a \in V$?*

Proof. It is clear that $\sum_{n=1}^{\infty} |a_n|^p$ is convergent when $p > 1$. This is because in order for $\sum |a_n|$ to converge, it is a necessary condition for $|a_n| \to 0$ as $n \to \infty$. Therefore, $|a_n|^p$ when $p < 0$ will diverge. When $0 < p < 1$, we could have convergence for some a_n, but not for all. Therefore, we must have $p > 1$ to gaurantee convergence, which is easily seen by comparison test. □

Theorem 165.2. *Define the function $d_p(a,b) = (\sum_{n=1}^{\infty} |a_n - b_n|^p)^{1/p}$ on $V \times V$. Show that d_p is a metric on V for $p = 1, 2$.*

Proof. First, it is clear that the first two conditions of a metric are satisfied for both $p = 1, 2$. This is because $d_p(a,b) \geq 0$ unless $a = b$, in which case $|a_n - b_n| = 0 \forall n \in \mathbb{N}$, which implies $d_p(a,b) = 0$. Next, we see that $d_p(a,b) = d_p(b,a)$ because $|a_n - b_n| = |b_n - a_n|$. Therefore, the only thing left to prove is the triangle inequality. In the case of $p = 1$, this is just the regular triangle inequality, since we have $|a_n - b_n| \leq |a_n - c_n| + |c_n - b_n|$, and summing both sides term by term, we get that $\sum_{n=1}^{\infty} |a_n - b_n| \leq \sum_{n=1}^{\infty} |a_n - c_n| + \sum_{n=1}^{\infty} |c_n - b_n|$, which is what we wanted. This shows that $p = 1$ is a metric.

To show that the triangle inequality holds for $p = 2$, we will let $||.||$ be the function $(\sum_{n=1}^{\infty} |.|^2)^{1/2}$. Then we have to show that $||a+b|| < ||a|| + ||b||$. First, we note that $||a+b||^2 = ||a||^2 + 2|ab| + ||b||^2$. By Cauchy Schwartz, we see that $2|ab| \leq 2||a|| \cdot ||b||$, so we have $||a+b||^2 \leq ||a||^2 + 2||u|| \cdot ||b|| + ||b||^2 = (||a|| + ||b||)^2$. Taking square roots, we have $||a+b|| \leq ||a|| + ||b||$, which is what we wanted. □

Theorem 165.3. *Is V complete with respect to the metric d_1 and d_2?*

Proof. We see that V is complete with respect to d_1 because each sequence in V is such that $\sum_{n=1}^{\infty} |a_n|$ converges, which means every Cauchy sequence in d_1 converges by definition. Recall that $\sum_{n=1}^{\infty} |a_n|^2 \geq (\sum_{n=1}^{\infty} |a_n|)^2$ by Cauchy-Schwartz inequality. Taking the square root of both sides does not affect ordering, so we see that $(\sum_{n=1}^{\infty} |a_n|^2)^{1/p} \geq \sum_{n=1}^{\infty} |a_n|$. Therefore, each Cauchy sequence with metric d_2 does not necessarily converge. □

166. Problem 166

Theorem 166.1. *Let $s_n = 1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}}$. Show that there exists a limit $z = \lim_{n\to\infty}(4\sqrt{n} - 2s_n)$ and find the integer part of z.*

Proof. First, we note that $\int_1^n \frac{dx}{\sqrt{x}}$ can be approximated with a lower integral sum with a partition $P = \{x_1 = 1 < x_2 = 2 < \ldots < x_n = n$. Thus, we have $L(P, f) \leq \int_1^n \frac{dx}{\sqrt{x}} \leq U(P, f)$. Now, using the above partition, we have $L(P, f) = \sum_{i=1}^{n-1} \frac{1}{\sqrt{i+1}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} = s_n - 1$. The upper integral sum with the partition is $U(P, f) = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} = 1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} = s_n$. Thus we have:

$$s_n - 1 \leq \int_1^n \frac{dx}{\sqrt{x}} \leq s_n \tag{166.2}$$

Taking smaller and smaller partitions is the same as taking $n \to \infty$. Therefore, we can compute the integral and obtain $\int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{n} - 2$. Thus, rearranging our inequality we obtain:

$$2 \leq 4\sqrt{n} - 2s_n \leq 4 \tag{166.3}$$

Since the sequence is monotonically increasing and bounded, we see that the limit z exists, whose integer part is 2. □

167. Problem 167

Theorem 167.1. *Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(x)dx = 0$. Show that there exists $c \in [0, 1]$ such that $f(c) = 0$.*

Proof. Let $F(x) = \int_0^x f(x)dx$. We see that F exists and is continuous and differentiable for all $x \in [0, 1]$ because f is continuous on this interval, and by the fundamental theorem of calculus. Therefore, we can apply the mean value theorem and see that $F(1) - F(0) = F'(c)$ for some $c \in [0, 1]$. Since $F'(c) = f(c)$ and $F(1) = \int_0^1 f(x)dx = 0$ while $F(0) = \int_0^0 f(x)dx = 0$, we have $f(c) = 0$ for some $c \in [0, 1]$. □

Theorem 167.2. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with period 1. Show that for each integer $n \geq 1$ there exists a point c such that $f(c) + f(c + \frac{1}{n}) + \ldots + f(c + \frac{n-1}{n}) = n\int_0^1 f(x)dx$.*

Proof. □

168. Problem 168

Theorem 168.1. *Find $\lim_{x\to 0}(1 + x - \frac{x^2}{2} - \log(1 + x))^{1/x^3}$.*

Proof. Take the limit of the log and put e to that power. Eventually, after a lot of math, you get $e^{-1/3}$. □

169. Problem 169

Theorem 169.1. *Show that if a real function $f(x)$ has positive second derivative on $[a, b]$ then f is convex on $[a, b]$.*

Proof. If $f''(x) > 0$ for $x \in [a, b]$, then we see that $f'(x_1) < f(x_2)$ for all $x_1 < x_2$ in that interval. So let $x_1 < x < x_2$ and use the mean value theorem for some $c \in [x_1, x]$ and $d \in [x, x_2]$. we obtain:

$$\frac{f(x) - f(x_1)}{x - x_1} = f'(c) < f'(d) = \frac{f(x_2) - f(x)}{x_2 - x} \tag{169.2}$$

This shows us that the following is true:

$$(x_2 - x)(f(x) - f(x_1)) < (x - x_1)(f(x_2) - f(x)) \tag{169.3}$$

$$f(x_1)(x - x_2) < (x - x_1)f(x_2) + (x_1 - x_2)f(x) \tag{169.4}$$

$$f(x) < f(x_1)\frac{x_2 - x}{x_2 - x_1} + f(x_2)\frac{x - x_1}{x_2 - x_1} \tag{169.5}$$

Now, letting $x = \lambda x_1 + (1 - \lambda)x_2$ be an interpolation of the two points x_1, x_2 where $\lambda \in [0, 1]$, we see that:

$$f(\lambda x_1 + (1 - \lambda)x_2) < \frac{f(x_1)(-\lambda x_1 + \lambda x_2) + f(x_2)((\lambda - 1)x_1 + (1 - \lambda)x_2)}{x_2 - x_1} \tag{169.6}$$

$$f(\lambda x_1 + (1 - \lambda)x_2) < \lambda f(x_1) + (1 - \lambda)f(x_2) \tag{169.7}$$

Thus, we see that f must be convex. □

170. Problem 170

Theorem 170.1. *Suppose that $x \in \mathbb{R}$ satisfies $0 \leq x \leq \epsilon$ for all $\epsilon > 0$. Show that $x = 0$, using only axioms of $\mathbb{R}$ as an ordered field. State the axioms you are using.*

Proof. Suppose by contradiction that $x > 0$. Then $\frac{1}{2}x > 0$ because the product of two positive quantities is positive. Moreover, $\frac{x}{2} + 0 < \frac{x}{2} + \frac{x}{2}$ because $y < z$ implies $y + x < z + x$ for all x. Therefore, we see that $\frac{x}{2} < x$ by addition. Also, we can set $\epsilon = \frac{x}{2}$ so that by assumption $0 \leq x \leq \frac{x}{2}$. But only one of the following: $\frac{x}{2} < x$ and $x \leq \frac{x}{2}$ can be true. Therefore, we arrive at a contradiction, so that $x = 0$. $\square$

171. Problem 171

Theorem 171.1. *Let $\{a_n\}$ be a sequence of positive real numbers. Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ also converges.*

Proof. We know that $0 < (x - y)^2$ so that $2xy < x^2 + y^2$. Setting $x = \sqrt{a_n}$ and $y = \sqrt{a_{n+1}}$, we see that $\sqrt{a_n a_{n+1}} < \frac{1}{2}a_n + \frac{1}{2}a_{n+1}$. Next, we see that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} a_{n+1}$ converges. Therefore, we see that $\sum_{n=1}^{\infty} \frac{1}{2}a_n + \frac{1}{2}a_{n+1}$ also converges by grouping the series together. We see that $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ converges by comparison test. $\square$

Theorem 171.2. *Prove the converse is true if $\{a_n\}$ is monotonically decreasing.*

Proof. Since $\{a_n\}$ is monotonically decreasing, we see that $a_n > a_{n+1}$ for all $n \in \mathbb{N}$. Therefore, $\sqrt{a_n a_{n+1}} > \sqrt{a_{n+1} a_{n+1}} = a_n$. Thus, we see that if $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$, then we see by comparison test that $\sum_{n=1}^{\infty} a_{n+1}$ as converges. Since $\sum_{n=1}^{\infty} a_{n+1}$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges, we see that the latter also converges. $\square$

172. Problem 172

Theorem 172.1. *For each of the following examples, either give an example of a continuous function f on S such that $f(S) = T$, or explain why there can be no such continuous function. (a) $S = (0, 1), T = (0, 1]$.*

Proof. Let $f(x) = 1 - |2x - 1|$. $\square$

Theorem 172.2. $S = (0, 1), T = (0, 1) \cup (1, 2)$.

Proof. There does not exist such a continuous function because $(0, 1)$ is connected, while $(0, 1) \cup (1, 2)$ is separated, and continuous functions preserve connectedness. $\square$

Theorem 172.3. $S = [0, 1] \cup [2, 3], T = \{0, 1\}$.

Proof. Let $f(x)$ be defined as follows:

$$f(x) = \begin{cases} 0 & x \in [0, 1] \\ 1 & x \in [2, 3] \end{cases} \tag{172.4}$$

We see that the function is continuous on T. $\square$

Theorem 172.5. $S = \mathbb{R}, T = \mathbb{Q}$.

Proof. No function exists because $\mathbb{R}$ is connected, while $\mathbb{Q}$ is not connected. Continuous mappings preserve connectedness, so that no possible function can exist. $\square$

Theorem 172.6. $S = [0, 1] \times [0, 1], T = (0, 1) \times (0, 1)$.

Proof. No function exists. The pre-image of an open set under a continuous map is an open set. We see that T is open, but that S is closed. Therefore, it is impossible to have an open map such that $S = f^{-1}(T)$. $\square$

173. Problem 173

Theorem 173.1. *Assume $f_n : E \to \mathbb{R}$, $E \subset \mathbb{R}$ are uniformly continuous functions. Assume f_n converges uniformly to f. Prove that f is also uniformly continuous.*

Proof. Fix $\epsilon > 0$. By uniform convergence, we see that there exists an $N > 0$ such that for all $n > N$, we have $|f_n(x) - f(x)| < \epsilon$ for all $x \in E$. Moreover, by uniform continuity, there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ whenever $|x - y| < \delta$ for all $n \in \mathbb{N}$ and all $x, y \in E$. Therefore, we have:

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < 3\epsilon \tag{173.2}$$

Whenever $n > N$ and $|x - y| < \delta$, and for all $x, y \in E$. This proves uniform continuity of f. $\square$

174. Problem 174

Theorem 174.1. *Let $f : X \to Y$ be a continuous map between metric spaces and let $K \subset X$ be compact. Prove that $f(K) \subset Y$ is compact using the definition of compactness through open covers.*

Proof. Let $\{G_\alpha\}$ be an open cover of Y. Since f is continuous, we see that for any open set $V \in Y$, the inverse image $f^{-1}(V)$ must also be open. Therefore, take the inverse images of all the sets $\{G_\alpha\}$ that make up the open cover. We see that $\{f^{-1}(G_\alpha)\}$ for all α in $\{G_\alpha\}$ forms an open cover of X, since if $x \in K$, then $f(x) \subset f(K)$ and $x \in G_\alpha$ for some $\alpha in A$, hence $x \in f^{-1}(G_\alpha)$. Since every open cover permits a finite subcover, we see that there exist $i \in \{1, 2, \dots, n\}$ such that $K \subset \bigcup_{i=1}^n f^{-1}(U_{\alpha_i})$. Moreover, we see that $f(K) \subset \bigcup_{i=1}^n U_{\alpha_i}$, since if $y \in f(K)$ then $y = f(x)$ for some $x \in K$, and hence $x \in f^{-1}(U_{\alpha_i})$ for some $i \in \{1, 2, \dots, n\}$. Hence, $\{G_\alpha\}$ permits a finite subcover, which shows that $f(K) \subset Y$ is compact. □

175. Problem 175

Theorem 175.1. *Let $\alpha : [0, 1] \to \mathbb{R}$ be given by:*

$$\alpha(x) = \begin{cases} x - 1 & 0 \le x < \frac{1}{2} \\ x + 1 & \frac{1}{2} \le x \le 1 \end{cases} \tag{175.2}$$

Let $f(x) = 2x$. Show that the integral $\int_0^1 f d\alpha$ exists and compute its value.

Proof. We know that α is monotonically increasing, and that $f(x)$ is continuous. Therefore, by a theorem in Rudin, we see that the integral $\int_0^1 f d\alpha$ exists. Moreover, computing its integral, we see that $\int_0^1 f d\alpha = \int_0^{\frac{1}{2}} 2x dx + 1(\frac{3}{2} - (-\frac{1}{2})) + \int_{\frac{1}{2}}^1 2x dx = \int_0^1 2x dx + 2 = 1 + 2 = 3$. □

176. Problem 176

Theorem 176.1. *Let $\mathscr{F} = \{f_1, \dots, f_n\}$ be a finite collection of uniformly continuous functions. Prove that $\mathscr{F}$ is equicontinuous.*

Proof. Fix $\epsilon > 0$. There exists a δ_i for each $i \in \{1, \dots, n\}$ such that $|f_i(x) - f_i(y)| < \epsilon$ whenever $|x - y| < \delta_i$. Thus, we can pick $\delta = \min\{\delta_1, \dots, \delta_n\}$. Now choose two points x, y such that $|x - y| < \delta$. We see that for all $i \in \{1, \dots, n\}$, we have $|f_i(x) - f_i(y)| < \epsilon$. Therefore, we have shown that $\mathscr{F}$ is equicontinuous. □

177. Problem 177

Theorem 177.1. *Consider the infinite sequence of functions $f_n(x) = \frac{x}{x + \frac{1}{n}}$ for $x \in [0, 1]$ and $n \in \mathbb{N}$. Show that each function f_n is uniformly continuous.*

Proof. Fix $\epsilon > 0$. With a bit of algebra, we obtain:

$$|f_n(x) - f_n(y)| = \frac{\frac{|x-y|}{n}}{|x - \frac{1}{n}||y - \frac{1}{n}|} \le \frac{\frac{|x-y|}{n}}{\frac{1}{n^2}} = n|x + y| \tag{177.2}$$

Now choose $\delta < \epsilon/n$ so that $|f_n(x) - f_n(y)| < \epsilon$ whenever $|x - y| < \delta$. This shows uniform continuity. □

Theorem 177.3. *Show that the sequence of functions f_n from above has no uniformly convergent subsequence. Conclude that it is not equicontinuous.*

Proof. We see that $f_n(x) = \frac{nx}{nx+1}$. If we take $n \to \infty$, we see that $f_n(x) \to 1$ if $x \neq 0$, and $f_n(0) \to 0$ as $n \to \infty$. Therefore, we see that the sequence of functions converges pointwise to the following:

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x \in (0, 1] \end{cases} \tag{177.4}$$

However, this function is not continuous, so it has no hope of being uniformly convergent. If the function were equicontinuous, then we would know by Arzela-Ascoli that the funciton contains a uniformly convergent subsequence (since it is clearly also pointwise bounded). Since it does not, we know that the function is not equicontinuous. □