RUDIN CHAPTER 8
SOLUTIONS

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1. Problem 8.1

Theorem 1.1. *Define $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(x) = 0$ if $x = 0$. Prove that f has derivatives of all orders at $x = 0$ and that $f^{(n)}(0) = 0$ for $n \in \mathbb{N}$.*

Proof. Define $g(x) = e^{-1/x^2}$. We know that e^x is differentiable by a theorem in Rudin. Moreover, we know that $-\frac{1}{x^2}$ is differentiable at every point except $x = 0$ because x^2 is differentiable. Thus, we see by the chain rule that $g(x) = e^{-1/x^2}$ is differentiable everywhere but at $x = 0$. Moreover, the chain rule tells us that the derivative is given by $g'(x) = -\frac{2}{x^3}e^{-1/x^2} = -\frac{2}{x^3}g(x)$. We will use induction to find $g^(n)(x)$. First, we have already established the base case of $g'(x) = -\frac{2}{x^3}g(x)$. Assume that $g^{(n)}(x) = \sum_{k=1}^{n} c_k x^{-(2k+n)} g(x)$ where c_k are constants. Then we have:

$$
\begin{aligned}
g^{(n+1)}(x) &= \frac{d}{dx}\sum_{k=1}^{n} c_k x^{-(2k+n)} g(x) && (1.2)\\
&= \sum_{k=1}^{n} -c_k(2k+n)x^{-(2k+n)-1}g(x) + \sum_{k=1}^{n} c_k x^{-(2k+n)} g'(x) && (1.3)\\
&= \sum_{k=1}^{n} c'_k x^{-(2k+(n+1))} g(x) + \sum_{k=1}^{n} c''_k x^{-(2k+n)-3} g(x) && (1.4)\\
&= \sum_{k=1}^{n} c'_k x^{-(2k+(n+1))} g(x) + \sum_{k=1}^{n} c''_k x^{-(2(k+1)+(n+1))} g(x) && (1.5)\\
&= \sum_{k=1}^{n} c'_k x^{-(2k+(n+1))} g(x) + \sum_{k=2}^{n+1} c''_k x^{-(2k+(n+1))} g(x) && (1.6)\\
&= \sum_{k=1}^{n+1} \bar{c}_k x^{-(2k+(n+1))} g(x) && (1.7)
\end{aligned}
$$

The last step comes from the fact that for indices $k = 2$ through $k = n$, we have $(c'_k + c''_k)x^{-(2k+(n+1))}g(x)$, where $c'_k + c''_k$ is just another constant. Therefore, we have shown that $g^{(n)}(x) = \sum_{k=1}^{n} c_k x^{-(2k+n)} g(x)$ using mathematical induction. Therefore, since $f(x) = g(x)$ if $x \neq 0$, we see that $f(x)$ has derivatives of all order if $x \neq 0$, since we have already shown that $g(x)$ has derivatives of all orders. Now we are left to show that this still works at $x = 0$.

To do so, we will show that for any $r > 0$, we have $\lim_{x\to 0} x^{-r} g(x) = 0$. First, we note that for any $r \in \mathbb{R}$, we have $\lim_{h\to\infty} h^{r/2}e^{-h} = 0$ for $h > 1$ by a theorem in Rudin. Therefore, if we substitute $h = \frac{1}{x^2}$, we obtain:

$$
\begin{aligned}
0 &= \lim_{h\to\infty} h^{r/2} e^{-h} && (1.8)\\
&= \lim_{\frac{1}{x^2}\to\infty} \frac{1}{x^r} e^{-\frac{1}{x^2}} && (1.9)\\
&= \lim_{x\to 0} x^{-r} g(x) && (1.10)
\end{aligned}
$$

Thus, for every $n \in \mathbb{N}$, we have the following limit for the derivative of $g^{(n)}(0)$:

$$\lim_{x\to 0} g^{(n)}(x) = \lim_{x\to 0} \sum_{k=1}^{n} c_k x^{-(2k+n)} g(x) \tag{1.11}$$

$$= \sum_{k=1}^{n} c_k \lim_{x\to 0} x^{-(d_k)} g(x) \tag{1.12}$$

$$= 0 \tag{1.13}$$

Where we have replaced $(2k+n) = d_k$ as a positive constant. Since we have shown $\lim_{x\to 0} x^{-d_k} g(x) = 0$ for all positive constants d_k, we discover that $f^{(n)}(0) = 0 = \lim_{x\to 0} g^{(n)}(x)$. Therefore, for all $n \in \mathbb{N}$, we have shown that $f^{(n)}(0) = 0$ exists. This completes the proof. □

2. Problem 8.2

Theorem 2.1. *Let a_{ij} be defined so that*

$$a_{ij} = \begin{cases} 0 & i < j \\ -1 & i = j \\ 2^{j-i} & i > j \end{cases} \tag{2.2}$$

Prove that $\sum_i \sum_j a_{ij} = -2$ and that $\sum_j \sum_i a_{ij} = 0$.

Proof. First we will show that $\sum_i \sum_j a_{ij} = -2$. First pick $i \in \mathbb{N}$. The we have the following:

$$\sum_j a_{ij} = \sum_{j=1}^{i} a_{ij} = -1 + \sum_{j=1}^{i-1} 2^{j-i} = -1 + \sum_{n=1}^{i-1} 2^{-n} = \frac{-1}{2^{i-1}} \tag{2.3}$$

Where we have made a substitution of $n = i - j$ and used the formula for computing geometric series. Thus, we find that when we sum over all $i \in \mathbb{N}$, we obtain:

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \frac{-1}{2^{i-1}} = \frac{1}{2} - 4 = -2 \tag{2.4}$$

Next, we shall pick some $j \in \mathbb{N}$. Then we have the following:

$$\sum_i a_{ij} = \sum_{i=j}^{\infty} a_{ij} = -1 + \sum_{i=j+1}^{\infty} 2^{j-i} = -1 + \sum_{n=1}^{\infty} \frac{1}{2^n} = -1 + 1 = 0 \tag{2.5}$$

Therefore, summing over all $j \in \mathbb{N}$, we find:

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} 0 = 0 \tag{2.6}$$

This completes the proof. □

3. Problem 8.4

Theorem 3.1. *Prove that $\lim_{x\to 0} \frac{b^x - 1}{x} = \log b$ for $b > 0$.*

Proof. First assume that the derivative of b^x for $b > 1$ is given by $b^x \ln b$. If this is the case, then we can use L'Hospital's Theorem to obtain the following: $\lim_{x\to 0} \frac{b^x - 1}{x} = \lim_{x\to 0} b^x \ln b = \ln b$. Thus, we only need to prove that $\frac{d}{dx} b^x = b^x \ln b$.

First, we note that for $b > 0$, we have $b^x = e^{\ln b^x} = e^{x \ln b}$. Now, we can use the chain rule to obtain the following:

$$\frac{d}{dx} b^x = \frac{d}{dx} e^{x \ln b} \tag{3.2}$$

$$= e^{x \ln b} \ln b \tag{3.3}$$

$$= b^x \ln b \tag{3.4}$$

Thus, we have completed the proof. □

Theorem 3.5. *Prove that $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$.*

Proof. We can use L'Hospital's Theorem to obtain the following:

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = \lim_{x\to 0}\frac{1}{1+x} = 1 \tag{3.6}$$

This completes the proof. ☐

Theorem 3.7. *Prove that* $\lim_{x\to 0}(1+x)^{1/x} = e$.

Proof. We will prove that $\lim_{x\to 0}(1+x)^{1/x} = e^{\lim_{x\to 0}\frac{1}{x}\ln(1+x)}$. Suppose that this is the case, then we have:

$$\lim_{x\to 0}(1+x)^{1/x} = e^{\lim_{x\to 0}\frac{1}{x}\ln(1+x)} = e^{\lim_{x\to 0}\frac{1}{1+x}} = e^1 = e \tag{3.8}$$

Thus, we only need to show that $\lim_{x\to 0}(1+x)^{1/x} = e^{\lim_{x\to 0}\frac{1}{x}\ln(1+x)}$. Well, a theorem in Rudin shows that that $y^\alpha = E(\alpha L(y)) = e^{\alpha \ln y}$ for any $\alpha\in\mathbb{Q}$ and $y>0$. Therefore, substituting $y=(1+x)$ and $\alpha = 1/x$, we obtain $(1+x)^{1/x} = e^{\frac{1}{x}\ln(1+x)}$. Next, we will show that the limit commutes. We have:

$$\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1+x)} \tag{3.9}$$

A theorem in Rudin says that if f and g are both continuous functions at 0 and $f(0)$ respectively, then $h(x) = g(f(x))$ is continuous at 0. Let $f(x) = \frac{1}{x}\ln(1+x)$ and $g(x) = e^x$. A theorem in Rudin says that e^x is continuous for every $x\in\mathbb{R}$. Moreover, we can show that $f(x)$ is continuous at 0. This is because the limit is well defined:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{1}{1+x} = 1 \tag{3.10}$$

Therefore, all the conditions are satisfied, and we know that $h(x) = g(f(x)) = (1+x)^{1/x}$ is continuous at 0. Therefore, we can commute the limit by a theorem in Rudin so that $\lim_{x\to 0} h(x) = g(\lim_{x\to 0} f(x))$, which is what we wanted to show. Note that for later problems, it is possible to generalize this, and we will prove a lemma to which we will refer later. ☐

Lemma 3.11. *If* $p\in\bar{\mathbb{R}}$ *and* $\lim_{x\to p} g(x) = P$ *is defined, then* $\lim_{x\to p} g(x) = e^{\lim_{x\to p}\ln g(x)}$.

Proof. We know that $g(x)$ is continuous at p. Now define $g(x) = e^x$. We know that $g(x)$ is continuous at all $x\in\mathbb{R}$ by a theorem in Rudin. Therefore, we can define $h(x) = g(f(x))$ and see that $h(x)$ is continuous at p by a theorem in Rudin. Therefore, it is possible to commute the limit and have $\lim_{x\to p} h(x) = \lim_{x\to p} g(f(x)) = g(\lim x \to p f(x))$. Moreover, since we know that $e^{\ln f(x)} = f(x)$ is also continuous at p by the fact that $\ln f(x)$ is continuous, we have the shown that $\lim_{x\to p} f(x) = \lim_{x\to p} e^{\ln f(x)} = e^{\lim_{x\to p}\ln f(x)}$, which is what we wanted. ☐

Theorem 3.12. *Prove that* $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$.

Proof. We will use L'Hospital's Theorem and a change of variable for $t=\frac{1}{n}$ to obtain:

$$\lim_{n\to\infty}\ln\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty} n\ln\left(1+\frac{x}{n}\right) \tag{3.13}$$

$$= \lim_{t\to 0}\frac{\ln(1+tx)}{t} \tag{3.14}$$

$$= \lim_{t\to 0}\frac{x}{1+tx} \tag{3.15}$$

$$= x \tag{3.16}$$

By lemma 5.8, we see that $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^{\lim_{n\to\infty}\ln(1+\frac{x}{n})^n} = e^{-x}$, which is what we wanted. ☐

4. Problem 8.5

Theorem 4.1. *Find* $\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{x}$.

Proof. Using L'Hospital's Theorem and lemma 5.8 we obtain:

$$\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{x} = \lim_{x\to 0}\frac{e-e^{\frac{1}{x}\ln(1+x)}}{x} \tag{4.2}$$

$$= \lim_{x\to 0} -e^{\frac{1}{x}\ln(1+x)}\left(\frac{1}{x(1+x)}-\frac{1}{x^2}\ln(1+x)\right) \tag{4.3}$$

$$= \lim_{x\to 0} -(1+x)^{1/x}\left(\frac{x-(1+x)\ln(1+x)}{x^2(1+x)}\right) \tag{4.4}$$

$$= -e\lim_{x\to 0}\left(\frac{x-(1+x)\ln(1+x)}{x^2(1+x)}\right) \tag{4.5}$$

$$= -e\lim_{x\to 0}\frac{1-\frac{1}{1+x}-\ln(1+x)-\frac{x}{1+x}}{3x^2+2x} \tag{4.6}$$

$$= -e\lim_{x\to 0}\frac{-\ln(1+x)}{3x^2+2x} \tag{4.7}$$

$$= e\lim_{x\to 0}\frac{1}{(1+x)(6x+2)} \tag{4.8}$$

$$= \frac{e}{2} \tag{4.9}$$

□

Theorem 4.10. *Find* $\lim_{n\to\infty}\frac{n}{\ln n}(n^{1/n}-1)$.

Proof. Using L'Hospital's Theorem, lemma 5.8, and the theorem in Rudin stating $\lim_{n\to\infty} n^{1/n}=1$, we find:

$$\lim_{n\to\infty}\frac{n}{\ln n}(n^{1/n}-1) = \lim_{n\to\infty}\frac{n}{\ln n}(e^{\frac{1}{n}\ln n}-1) \tag{4.11}$$

$$= \lim_{n\to\infty}\frac{e^{\frac{1}{n}\ln n}-1}{\frac{1}{n}\ln n} \tag{4.12}$$

$$= \lim_{n\to\infty}\frac{e^{\frac{1}{n}\ln n}\left(\frac{\ln n-1}{n^2}\right)}{\frac{\ln n-1}{n^2}} \tag{4.13}$$

$$= \lim_{n\to\infty} n^{1/n} \tag{4.14}$$

$$= 1 \tag{4.15}$$

□

Theorem 4.16. *Find* $\lim_{x\to 0}\frac{\tan x - x}{x(1-\cos x)}$.

Proof. Using L'Hospital's Theorem and the trigonometric identities $\sec^2 x - 1 = \tan^2 x$ and the fact that $\frac{d}{dx}\tan x = \sec^2 x$, we obtain:

$$\lim_{x\to 0}\frac{\tan x - x}{x(1-\cos x)} = \lim_{x\to 0}\frac{\sec^2 x - 1}{1 + x\sin x - \cos x} \tag{4.17}$$

$$= \lim_{x\to 0}\frac{\tan^2 x}{1-\cos x + x\sin x} \tag{4.18}$$

$$= \lim_{x\to 0}\frac{2\tan x\sec^2 x}{2\sin x + x\cos x} \tag{4.19}$$

$$= \lim_{x\to 0}\frac{2\tan x}{2\cos^2 x\sin x + x\cos^3 x} \tag{4.20}$$

$$= \lim_{x\to 0}\frac{-4\sec^2 x}{\cos x(3x\sin 2x - 7\cos 2x + 1)} \tag{4.21}$$

$$= \frac{-4}{\lim_{x\to 0}\cos^3 x(3x\sin 2x - 7\cos 2x + 1)} \tag{4.22}$$

$$= \frac{-4}{-7+1} \tag{4.23}$$

$$= \frac{2}{3} \tag{4.24}$$

□

Theorem 4.25. *Find* $\lim_{x\to 0} \frac{x-\sin x}{\tan x - x}$.

Proof. Using some common trigonometric identities, such as $\sin 2x = 2\cos x \sin x$, $\sec^2 x - 1 = \tan^2 x$, and $\tan x = \frac{\sin x}{\cos x}$, as well as L'Hospital's Theorem, we obtain the following expression:

$$\lim_{x\to 0} \frac{x-\sin x}{\tan x - x} = \lim_{x\to 0} \frac{1-\cos x}{\sec^2 x - 1} \tag{4.26}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{\tan^2 x} \tag{4.27}$$

$$= \lim_{x\to 0} \frac{(1-\cos x)\cos^2 x}{\sin^2 x} \tag{4.28}$$

$$= \lim_{x\to 0} \frac{\sin x \cos x(3\cos x - 2)}{\sin 2x} \tag{4.29}$$

$$= \lim_{x\to 0} \frac{1}{2}\frac{\sin 2x}{\sin 2x}(3\cos x - 3) \tag{4.30}$$

$$= \frac{1}{2} \tag{4.31}$$

□

5. Problem 8.9

Theorem 5.1. *Put* $s_N = 1 + \frac{1}{2} + \ldots + \frac{1}{N}$. *Prove that* $\lim_{N\to\infty}(s_N - \ln N)$ *exists.*

Proof. We let $\gamma = \lim_{N\to\infty} s_N - \ln N$ and note that we have the following telescoping sum:

$$g(x) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - \ln(n+1) + \ln n = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - \ln(N+1) \tag{5.2}$$

If we can show that $\lim_{N\to\infty} g_N(x)$ converges, then we can show that γ converges as well. This is because $\ln(N+1)$ and $\ln(N)$ converge to the same thing. Formally, we have $\lim_{N\to\infty} \ln(N+1) - \ln N = \lim_{N\to\infty} \ln(1+\frac{1}{N}) = \ln 1 = 0$. Therefore, we only must show that $g(x)$ converges to show that γ converges. By the properties of logarithms, we have:

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{n} - \ln(n+1) + \ln n \tag{5.3}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} + \ln\left(1 + \frac{1}{n}\right) \tag{5.4}$$

(5.5)

Now, we will show that the summand is bounded by x^2. First we take the function $f(x) = x^2 - x - \ln(1-x)$. We must show that for $x = \frac{1}{n}$, we always have $f(x) > 0$. First, we show that $f'(x) = 2x - 1 - \frac{1}{1-x}$ and $f''(x) = 2 + \frac{1}{(1-x)^2}$. We see that $f''(x) > 0$ for all $x \in \mathbb{R}$, and that $f'(0) = 0$. Therefore, $f'(x) > 0$ for all $x > 0$. Next, we see that $f(0) = 0$, which shows that $f(x) > 0$ for all $x > 0$. Thus, substituting $x = \frac{1}{n}$, we have discovered that $f(1/n) = \frac{1}{n^2} - \frac{1}{n} - \ln(1 - \frac{1}{n}) > 0$. Rearranging, we have the following inequalities:

$$0 < \frac{1}{n} + \ln\left(1 - \frac{1}{n}\right) < \frac{1}{n^2} \tag{5.6}$$

The lower bound of 0 comes from the fact that $n \in \mathbb{N}$ must be positive and so $\ln(1 + \frac{1}{n}) > 0$. Therefore, taking sums and limits, we have:

$$\lim_{N\to\infty} \sum_{n=1}^{N} g(x) < \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^2} \quad \Rightarrow \quad \sum_{n=1}^{\infty} g(x) < \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{5.7}$$

We know that $\sum \frac{1}{n^2}$ converges because it is a geometric series with $p = 2$, so by the comparison test, we know that $\sum g(x)$ also converges. Since we have shown above that the convergence of $g(x)$ implies the convergence of γ, we have completed our proof. □

Theorem 5.8. *Roughly how large must* m *be so that* $N = 10^m$ *satisfies* $s_N > 100$*?*

Proof. Since we know that $0 \leq s_N - \ln N \leq \sum \frac{1}{n^2}$ for all $N \in \mathbb{N}$, we have the following inequality for s_N:

$$\ln N \leq s_N \leq \ln N + \frac{1}{n^2} \tag{5.9}$$

Therefore, in order for $s_N > 100$, we must have $\ln N > 100$. This implies that we need $\ln 10^m > 100$, which is the same as $e^{100} < 10^m$. Therefore, we want $m = \log_{10}(e^{100})$ in order for $s_N > 100$. □