18.781
PROBLEM SET 9

JOHN WANG

1. Problem 1

Problem 1.1. *Recall that* $x = [a_0, \ldots, a_{n-1}, x_n] = \frac{x_n p_{n-1} + p_{n-2}}{x_n q_{n-1} + q_{n-2}}$. *Show that* $x - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_{n-1}(x_n q_{n-1} + q_{n-2})}$. *Use this to show that the convergents* p_n/q_n *indeed converge to* x, *and even and odd-numbered convergents lie on opposite sides of* x.

Solution We know that the following is true, using the fact that $x = [a_0, \ldots, a_{n-1}, x_n]$:

$$x - \frac{p_{n-1}}{q_{n-1}} = \frac{x_n p_{n-1} + p_{n-2}}{x_n q_{n-1} + q_{n-2}} - \frac{x_n p_{n-1} + p_{n-2}}{x_n q_{n-1} + q_{n-2}} \tag{1.1}$$

$$= \frac{x_n q_{n-1} p_{n-1} + p_{n-2} q_{n-1} - x_n q_{n-1} p_{n-1} - p_{n-1} q_{n-2}}{(x_n q_{n-1} q_{n-2}) q_{n-1}} \tag{1.2}$$

$$= \frac{p_{n-2} q_{n-1} - p_{n-1} q_{n-2}}{(x_n q_{n-1} + q_{n-2}) q_{n-1}} \tag{1.3}$$

Also, we know that from a theorem proven in class, that $\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_{k-1} q_k}$. This implies that $p_{k-1} q_k - p_k q_{k-1} = (-1)^k$. Equivalently, setting $k = n-1$, we see that $p_{n-2} q_{n-1} - p_{n-1} q_{n-2} = (-1)^{n-1}$. This shows

$$x - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_{n-1}(x_n q_{n-1} + q_{n-2})} \tag{1.4}$$

Now recall $q_n = a_n q_{n-1} + q_{n-2}$, and that $a_n > 0$ for all $n \in \mathrm{N}$. Moreover, since q_{-1} and q_{-2} are both non-negative, we see that q_n increases monotonically as $n \to \infty$. This means that the RHS of the equation goes to zero as $n \to \infty$ because q_{n-1} and q_{n-2} both converge to ∞ ans $n \to \infty$. Moreover, we see that p_{n-1}/q_{n-1} changes signs because $q_n > 0$ and $x_n > 0$ for all $n \in \mathrm{N}$, but $(-1)^{n-1}$ changes sign with odd and even numbered convergents. □

2. Problem 2

Problem 2.1. *It follows from the above problem that* $|x - p_n/q_n| < 1/(q_n q_{n+1})$ *and that* $|xq_n - p_n| < 1/q_{n+1}$. *Show that* $|xq_n - p_n| > 1/q_{n+2}$ *and therefore that* $|x - \frac{p_{n+1}}{q_{n+1}}| < |x - \frac{p_n}{q_n}|$.

Solution First we note that based on the fact that $x = [a_0, \ldots, a_{n-1}, x_n]$ and $x_{n+1} < a_{n+1} + 1$, we can obtain the following inequalities:

$$x - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(x_{n+1} q_n + q_{n-1})} > \frac{(-1)^n}{q_n((a_{n+1} + 1) q_n + q_{n-1})} \tag{2.1}$$

$$x q_n - p_n = \frac{(-1)^n}{q_n + q_{n+1}} \tag{2.2}$$

The last line uses the fact that $q_{n+1} = a_{n+1} q_n + q_{n-1}$ using the recursive definition of q_n. Moreover, we know that $q_n + q_{n+1} < q_{n+2}$ because $a_{n+2} > 0$. This implies the following:

$$|x q_n - p_n| > \frac{1}{q_{n+2}} \tag{2.3}$$

Which is the first part of what we wanted to prove. We know that $|x - \frac{p_{n+1}}{q_{n+1}}| < \frac{1}{q_{n+1} q_{n+2}}$ by the first inequality mentioned in the beginning of the problem. This implies, since $q_n < q_{n+1}$ that $|x - \frac{p_{n+1}}{q_{n+1}}| < \frac{1}{q_n q_{n+2}}$. Moreover, we have already shown that $|xq_n - p_n| > 1/q_{n+2}$. This implies that

$$\left| x - \frac{p_{n+1}}{q_{n+1}} \right| < \frac{1}{q_{n+1} q_{n+2}} < \frac{1}{q_n q_{n+2}} < \left| x - \frac{p_n}{q_n} \right| \tag{2.4}$$

Which is what we wanted to show. □

3. Problem 3

Problem 3.1. *Let $n \geq 1$. Show that if a/b is a rational number, with a, b integers and b positive, such that $|bx - a| < |q_n x - p_n|$, then $b \geq q_{n+1}$.*

Solution First, we note that we can write the vector (a, b) as an integer linear combination of (p_n, q_n) and (p_{n+1}, q_{n+1}). Thus, we need to find integer coefficients α and β such that $a = \alpha p_n + \beta p_{n+1}$ and $b = \alpha q_n + \beta q_{n+1}$. In other words, we must solve the following matrix equations:

$$\begin{bmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \tag{3.1}$$

We know from a theorem proven in class that the determinant can be given by:

$$\begin{vmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{vmatrix} = (-1)^{n+1} \neq 0 \tag{3.2}$$

This shows that we have solutions $\alpha = (-1)^{n+1}(aq_{n+1} - bp_{n+1})$ and $\beta = (-1)^{n+1}(bp_n - aq_n)$ to the equations. Next, we shall show that α and β have differing signs. First suppose that $\beta < 0$. Then we know that $b = \alpha q_n + \beta q_{n+1}$ so that $\alpha q_n = b - \beta q_{n+1} > 0$. Since $q_n > 0$, we know that $\alpha > 0$. Now suppose that $\beta \geq 0$. Then we know that $b < q_{n+1}$ so that $\alpha q_n = b - \beta q_{n+1} < 0$. This implies that $\alpha < 0$. Therefore, we must have opposite signs for α and β.

Moreover, we know that $x - \frac{p_n}{q_n}$ and $x - \frac{p_{n+1}}{q_{n+1}}$ must have different signs from a theorem in class, which implies that $q_n x - p_n$ and $q_{n+1}x - p_{n+1}$ have different signs as well. This means, due to the opposite signs, that we can write:

$$|bx - a| = |(\alpha q_n + \beta q_{n+1})x - (\alpha p_n + \beta p_{n+1})| \tag{3.3}$$

$$= |\alpha||q_n x - p_n| + |\beta||q_{n+1}x - p_{n+1}| \tag{3.4}$$

Since we know by hypothesis that $|bx - a| < |q_n x - p_n|$, and we know that α and β are integers, we must have $\alpha = 0$. This implies that $0 = (-1)^{n+1}(aq_{n+1} - bp_{n+1})$ which further implies that $aq_{n+1} = bp_{n+1}$. Since we know that $gcd(q_{n+1}, p_{n+1}) = 1$ by a theorem shown in class, we must have $q_{n+1}|b$. This implies that $b \geq q_{n+1}$. This completes the proof. □

Problem 3.2. *Check that problem 3.1 implies that $|x - a/b| \geq |x - p_n/q_n|$ for every $1 \leq b \leq q_n$, i.e. p_n/q_n is a best approximation to x among rational numbers with denominators less than or equal to q_n.*

Solution We know from the above that if $b \leq q_n$, then we must have $|bx - a| \geq |q_n x - p_n|$. This follows because otherwise, we would have $|bx - a| < |q_n x - p_n|$, which would imply $b \geq q_{n+1}$. This would contradiction $b < q_n$ because $q_n < q_{n+1}$. This shows the following:

$$b\left|x - \frac{a}{b}\right| \geq q_n \left|x - \frac{p_n}{q_n}\right| \tag{3.5}$$

$$\left|x - \frac{a}{b}\right| \geq \frac{q_n}{b}\left|x - \frac{p_n}{q_n}\right| \tag{3.6}$$

$$\left|x - \frac{a}{b}\right| \geq \left|x - \frac{p_n}{q_n}\right| \tag{3.7}$$

Where the last line follows because $b \leq q_n$ implies that $q_n/b \geq 1$. This completes the proof. □

4. Problem 4

Problem 4.1. *If a/b is a rational approximation to x as above (a, b integers, b positive), such that $|x - \frac{a}{b}| < \frac{1}{2b^2}$, then show that a/b must be a convergent of the simple continued fraction of x.*

Solution We can suppose $g = gcd(a, b) = 1$, otherwise we could reqrite the problem so that $|x - \frac{a/g}{b/g}| < \frac{1}{2(b/g)^2}$. Now, let $\frac{p_n}{q_n}$ be the convergents of x and suppose by contradiction that a/b is not a convergent of x. Then we know that there exists an n such that $q_n \leq b < q_{n+1}$ since the sequence $\{q_n\}$ increases monotonically. Thus, we see from the previous theorem that $|xb - a| \geq |xq_n - p_n|$. Therefore, we find:

$$|xq_n - p_n| \leq |xb - a| < \frac{1}{2b} \tag{4.1}$$

$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{2bq_n} \tag{4.2}$$

Since we know that $\frac{a}{b} \neq \frac{p_n}{q_n}$ and that $bp_n - aq_n$ is an integer by virtue of a, b, p_n, q_n all being integers, we find that the following is true:

$$\begin{aligned}
\frac{1}{bq_n} &\leq \frac{|bp_n - aq_n|}{bq_n} && (4.3)\\
&= \left|\frac{p_n}{q_n} - \frac{a}{b}\right| && (4.4)\\
&\leq \left|x - \frac{a}{b}\right| + \left|x - \frac{p_n}{q_n}\right| && (4.5)\\
&< \frac{1}{2bq_n} + \frac{1}{2b^2} && (4.6)\\
&= \frac{1}{2b}\left(\frac{b+q_n}{q_n b}\right) && (4.7)
\end{aligned}$$

This implies that $2b < b + q_n$ or that $b < q_n$. However, this is a contradiction due to the previous problem which showed that $b \geq q_{n+1}$. □

5. Problem 5

Problem 5.1. *Let $\phi = (1+\sqrt{5})/2$ be the golden ratio, and let $\kappa > \sqrt{5}$. Show that there are only finitely many rational numbers p/q such that $|\phi - \frac{p}{q}| < \frac{1}{\kappa q^2}$.*

Solution We showed in class that the continued fraction expansion of $\phi = (1+\sqrt{5})/2$ is given by $[1, 1, 1, \ldots]$. We see then that $p_n = a_n p_{n-1} + p_{n-2} = p_{n-1} + p_{n-2}$ and that $q_n = a_n q_{n-1} + q_{n-2} = q_{n-1} + q_{n-2}$ are the numerator and denominator of the convergents. We can show inductively that $q_n = p_{n-1}$. Clearly it holds for $n = 1$ because $q_1 = 1$ and $p_0 = 1$. Now, suppose this for all integers less than or equal to n. We see that $q_{n+1} = q_n + q_{n-1}$ by the recurrence. Using the induction hypothesis, we find $q_{n+1} = p_{n-1} + p_{n-2} = p_n$, which completes the induction step. Therefore, we find:

$$\lim_{n\to\infty} \frac{q_{n-1}}{q_n} = \lim_{n\to\infty} \frac{q_{n-1}}{p_{n-1}} = \frac{1}{\phi} = \frac{2}{1+\sqrt{5}} \tag{5.1}$$

We can simplify the above expression by multiplying by the conjugate to obtain $\frac{2}{1+\sqrt{5}} \frac{\sqrt{5}-1}{\sqrt{5}-1} = \frac{\sqrt{5}-1}{2}$. We can then obtain the expression:

$$\lim_{n\to\infty} \phi_n + \frac{q_{n-1}}{q_n} = \frac{\sqrt{5}+1}{2} + \frac{\sqrt{5}-1}{2} = \sqrt{5} \tag{5.2}$$

This shows us that for $\kappa > \sqrt{5}$, we can only have $\phi_{n+1} + q_{n+1}/q_n > \kappa$ for a finite number of values of n. We know the following as well:

$$\left|\phi - \frac{p_n}{q_n}\right| = \frac{1}{q_n(\phi_{n+1}q_n + q_{n-1})} = \frac{1}{q_n^2(\phi_{n+1} + q_{n-1}/q_n)} \tag{5.3}$$

Which uses a fact proven in problem 1 on this problem set. Thus, we see that there are only a finite number of values n such that $|\phi - \frac{p_n}{q_n}| < \frac{1}{\kappa q_n^2}$. Using the previous problem, we see that all rational numbers p/q satisfying the inequality $|\phi - \frac{p}{q}| < \frac{1}{\kappa q^2}$ must be a convergent of p/q. However, we have just shown that there are finitely many convergents that satisfy this property, so therefore, there are a finite number of rational numbers that satisfy the property. □

6. Problem 6

Let p be a prime congruent to $1 \pmod 4$ and suppose u is an integer such that $u^2 \equiv -1 \pmod p$.

Problem 6.1. *Write the rational number $u/p = [a_0, a_1, \ldots, a_n]$ and let i be the largest integer such that $q_i \leq \sqrt{p}$. Show that $|p_i/q_i - u/p| < 1/(q_i\sqrt{p})$ and therefore that $|p_i p - uq_i| < \sqrt{p}$.*

Solution First notice that if $|p_i/q_i - u/p| < 1/(q_i\sqrt{p})$, then we can multiply both sides by q_i and p to obtain $|p_i p - uq_i| < \sqrt{p}$. Thus, we only need to show that $|p_i/q_i - u/p| < 1/(q_i\sqrt{p})$, and the proof will be complete.

Now, let us first make some observations. We know that the sequence $\{q_i\}$ is increasing as $i \to \infty$. This means that $q_i < q_k$ whenever $i < k$. Moreover, we know from problem 1 in this problem set, that the following holds where p_i/q_i are convergents to u/p:

$$\left|\frac{u}{p} - \frac{p_i}{q_i}\right| = \frac{1}{q_i(a_{i+1}q_i + q_{i-1})} \tag{6.1}$$

Thus, in order for us to show that $|p_i/q_i - u/p| < 1/(q_i\sqrt{p})$, we need to show that $a_{i+1}q_i + q_{i-1} > \sqrt{p}$. However, we know that i is the largest integer such that $q_i \leq \sqrt{p}$. Since q_i increases as i increases, this implies that for all $k > 0$, we know that $q_{i+k} > \sqrt{p}$. Therefore, we know that $q_{i+1} = a_{i+1}q_i + q_{i-1} > \sqrt{p}$. This completes what we wanted to show, and finishes the proof. □

Problem 6.2. *Letting $x = q_i$ and $y = p_ip - uq_i$, show that $0 < x^2 + y^2 < 2p$ and that $x^2 + y^2 \equiv 0 \pmod{p}$. Conclude that $p = x^2 + y^2$.*

Solution First, we know that $x^2 + y^2 = q_i^2 + (p_ip - uq_i)^2 = q_i^2 + p_i^2p^2 - 2p_ipuq_i + u^2q_i^2$. Regrouping terms, we find $x^2 + y^2 = q_i^2(u_i^2 + 1) + p_i^2p^2 - 2p_ipuq_i$. Since we know that $u^2 \equiv -1 \pmod{p}$, we find that $q_i^2(u_i^2 + 1) \equiv 0 \pmod{p}$. Moreover, we know that $p(pp_i^2 - 2p_iuq_i) \equiv 0 \pmod{p}$. This implies that $x^2 + y^2 \equiv 0 \pmod{p}$. Next, we want to show that $0 < x^2 + y^2 < 2p$. This follows because $x = q_i$ so that $x^2 = q_i^2 \leq (\sqrt{p})^2 = p$ by the hypothesis that $q_i \leq \sqrt{p}$. Next, we know from the previous problem that $|y| = |p_ip - uq_i| < \sqrt{p}$, which implies that $y^2 < (\sqrt{p})^2 = p$. This shows that $x^2 + y^2 < p + p = 2p$. Moreover, we know that $x, y > 0$ because $q_i \in \mathrm{Z}$ so that $x^2 > 0$. Thus, we see that $0 < x^2 + y^2 < 2p$.

The first fact, that $x^2 + y^2 \equiv 0 \pmod{p}$ implies that $x^2 + y^2 = kp$ for some $k \geq 1$. However, since we know that $0 < x^2 + y^2 < 2p$, we know that $k < 2$ as well. This forces $k = 1$, which shows that $p = x^2 + y^2$, which is what we wanted. □

7. Problem 7

Problem 7.1. *Let d be a positive non-square integer. For which positive integers c does the quadratic irrational $([\sqrt{d}] + \sqrt{d})/c$ have a purely periodic expansion?*

Solution By a theorem proven in class, an irrational number x has a purely periodic expansion if and only if $x \geq 1$ and $-1 < \bar{x} < 0$. For $x = ([\sqrt{d}] + \sqrt{d})/c$, these two conditions imply that $[\sqrt{d}] + \sqrt{d} \geq c$ and that $-1 < \bar{x} < 0$. This corresponds to:

$$-1 < \bar{x} = \frac{[\sqrt{d}] - \sqrt{d}}{c} < 0 \tag{7.1}$$

Where $\bar{x} = ([\sqrt{d}] - \sqrt{d})/c$ since $\sqrt{d}/c$ is the only irrational part of x. This implies:

$$-c < -\sqrt{d} + [\sqrt{d}] < 0 \tag{7.2}$$

However, we know that $[\sqrt{d}] \leq \sqrt{d} < [\sqrt{d}] + 1$ which implies that $-1 < [\sqrt{d}] - \sqrt{d} < 0$. This, the second condition shows that $c \geq 1$ must hold. Putting the two conditions together, we find that all positive integers c such that:

$$1 \leq c \leq [\sqrt{d}] + \sqrt{d} \tag{7.3}$$

will allow x to have a purely periodic expansion. □

8. Problem 8

Let x be an irrational real number.

Problem 8.1. *Given any positive integer N, show that there is a rational number p/q with $p, q \in \mathrm{Z}$ and $1 \leq q \leq N$ such that $|x - \frac{p}{q}| < \frac{1}{q(N+1)}$.*

Solution Let us examine the fractional parts of the rational numbers ix as i ranges from 0 through N. Let $\{0, \{1x\}, \{2x\}, \ldots, \{Nx\}, 1\}$ be the set of fractional parts of numbers, where $\{kx\} = kx - [kx]$ denotes the fractional part of kx. Note we have also included 0 and 1 in the above set.

Now, we see that at least one of the $N+1$ intervals $[k/(N+1), (k+1)/(N+1))$ in $[0, 1]$ has two elements from the above set using the pigeonhole principle, since there are $N+2$ elements for $N+1$ intervals. If 1 or 0 is one of these two elements, then $\{kx\}$ is the other element and we can choose $p \leq N$ and $q = k$. This would give $|kx - p| \leq \frac{1}{n+1}$.

Otherwise, we must have $0 \leq |\{kx\} - \{lx\}| \leq \frac{1}{N+1}$. Writing this out, we obtain:

$$\frac{1}{N+1} \geq |kx - [kx] - lx + [lx]| \tag{8.1}$$

$$= |x(k-l) + [lx] - [kx]| \tag{8.2}$$

Now set $k - l = q$ and $[lx] - [kx] = p$ and we obtain $|xq - p| < \frac{1}{N+1}$ which implies that $|x - \frac{p}{q}| < \frac{1}{q(N+1)}$, which is what we wanted to show. □

Problem 8.2. *Use part (a) to show that there are infinitely many rational numbers p/q such that $|x-p/q| < 1/q^2$.*

Solution Suppose not by contradiction and let $S = \{p_1/q_1, p_2/q_2, \ldots, p_r/q_r\}$ be the set of all rational numbers that satisfy $|x - p_i/q_i| < 1/q_i^2$. Since the set is finite, we can choose N large enough so that $\frac{1}{q_i(N+1)} < |x - p_i/q_i|$ for all $i \in \{1, 2, \ldots r\}$. We know from the previous part that there exists a rational number p/q with $p, q \in \mathrm{Z}$ and $1 \leq q \leq N$ such that $|x - \frac{p}{q}| < \frac{1}{q(N+1)}$. This shows that $|x - \frac{p}{q}| < \frac{1}{q(N+1)} < \frac{1}{q^2}$ because $q < N+1$. Yet, we know that $p/q \notin S$ because we have shown that $\frac{1}{q_i(N+1)} < |x - p_i/q_i|$ for all $p_i/q_i \in S$. This is a contradiction, so there must be infinitely many rational numbers that satisfy $|x - p/q| < 1/q^2$. □

9. Problem 9

Let m be a positive integer and let x have continued fraction $[m, m, m, \ldots]$.

Problem 9.1. *Compute the value of x.*

Solution We know that x can be expanded out to $m + \frac{1}{x_1}$ where $x_1 = [m, m, m, \ldots]$. Thus, we see that $x_1 = x$. This means that we can write the expression $x = m + \frac{1}{x}$. We obtain the quadratic equation $x^2 - mx - 1 = 0$. Solving for x, we find:

$$x = \frac{m \pm \sqrt{m^2+4}}{2} \tag{9.1}$$

However, since $m - \sqrt{m^2+4} < 0$ for all $m \geq 1$, we know that $x = (m - \sqrt{m^2+4})/2$ will be negative. If we restrict ourselves to positive values of x, we find that we must have $x = \frac{m+\sqrt{m^2+4}}{2}$. □

Problem 9.2. *Let p_n/q_n be the nth convergent to x. Write down and solve a linear reucrrence with constant coefficients for p_n and q_n, and thereby calculate an explicit formula for p_n/q_n.*

Solution We know from the recurrences for the convergents that $p_n = mp_{n-1} + p_{n-2}$. This means we have the recurrence $p_n - mp_{n-1} - p_{n-2} = 0$ with the characteristic polynomial of $\lambda^2 - m\lambda - 1 = 0$. The roots of the polynomial are given by:

$$\lambda = \frac{m \pm \sqrt{m^2+4}}{2} \tag{9.2}$$

The recurrence for q_n is the same, so we find that we have the following expressions for p_n and q_n:

$$p_n = c_1 \left(\frac{m + \sqrt{m^2+4}}{2}\right)^n + c_2 \left(\frac{m - \sqrt{m^2+4}}{2}\right)^2 \tag{9.3}$$

$$q_n = c_3 \left(\frac{m + \sqrt{m^2+4}}{2}\right)^n + c_4 \left(\frac{m - \sqrt{m^2+4}}{2}\right)^2 \tag{9.4}$$

Since we know the starting conditions for the recurrences are $p_0 = m, p_1 = m^2 + 1$ and $q_0 = 1, q_1 = m$, we can solve for c_1, c_2, c_3,and c_4 by substituting for $n = 0$ and $n = 1$. We find that the constants are given by:

$$c_1, c_2 = \frac{m}{2} \pm \frac{1}{2}\sqrt{\frac{m^4 + 4m^2 + 4}{m^2+4}} \tag{9.5}$$

$$c_3, c_4 = \frac{m^2 \pm m\sqrt{m^2+4} + 4}{2(m^2+4)} \tag{9.6}$$

To obtain a closed form expression for the convergents, simply take p_n/q_n where the formulas for p_n and q_n are given as above. □

10. Problem 10

Recall the AM-GM inequality: $\frac{r_1+r_2+\ldots+r_n}{n} \geq \sqrt[n]{r_1 \ldots r_n}$ for positive real numbers $r_1, \ldots, r_n$. We proved it for $n = 2$.

Problem 10.1. *Prove the inequality for $n = 2^k$ any power of 2.*

Solution We will proceed by induction. First we have shown that the AM-GM inequality holds when $n = 2$ by a proof given in class. Now suppose that it holds for all powers of 2 such up to $n = 2^k$. We shall show it holds for $n = 2^{k+1}$. We shall group the elements $r_1, \ldots, r_n$ into two groups:

$$\frac{r_1 + r_2 + \ldots + r_{2^{k+1}}}{2^{k+1}} = \frac{1}{2}\left(\frac{r_1 + \ldots + r_{2^k}}{2^k} + \frac{r_{2^k+1} + \ldots + r_{2^{k+1}}}{2^k}\right) \tag{10.1}$$

$$\geq \frac{1}{2}\left(\sqrt[2^k]{r_1 \ldots r_{2^k}} + \sqrt[2^k]{r_{2^k+1} \ldots r_{2^{k+1}}}\right) \tag{10.2}$$

We can again use the AM-GM inequality, this time for the case of $n = 2$ and we find:

$$\frac{r_1 + r_2 + \ldots + r_{2^{k+1}}}{2^{k+1}} \geq \sqrt{\sqrt[2^k]{r_1 \ldots r_{2^k}} \sqrt[2^k]{r_{2^k+1} \ldots r_{2^{k+1}}}} \tag{10.3}$$

$$= \sqrt[2^{k+1}]{r_1 r_2 \ldots r_{2^{k+1}}} \tag{10.4}$$

This completes the induction step and finishes the proof. □

Problem 10.2. *Prove the inequality for any n, by choosing a k such that $2^{k-1} < n \leq 2^k$ and applying the inequality from part (a) to the 2^k numbers $r_1, \ldots, r_n, r, r, \ldots, r$ where r is chosen appropriately.*

Solution Let us set $r = \frac{r_1+\ldots+r_n}{n}$ and $2^k = m$ where $2^{k-1} < n \leq 2^k$. Then know the following is true:

$$r = \frac{r_1 + \ldots + r_n}{n} = \frac{(r_1 + \ldots + r_n)\frac{m}{n}}{m} \tag{10.5}$$

$$= \frac{(r_1 + \ldots + r_n)\frac{m-n}{n} + (r_1 + \ldots + r_n)}{m} \tag{10.6}$$

$$= \frac{(m-n)r + (r_1 + \ldots + r_n)}{m} \tag{10.7}$$

$$= \frac{r_1 + \ldots + r_n + r \ldots + r}{m} \tag{10.8}$$

$$\geq \sqrt[m]{r^{m-n} r_1 \ldots r_n} \tag{10.9}$$

Where we have used the AM-GM inequality for $m = 2^k$ in the last line. Now, we see that $r \geq \sqrt[m]{r_1 \ldots r_n} r^{1-n/m}$. This expression simplifies to $r^{n/m} \geq \sqrt[m]{r_1 \ldots r_n}$. Exponentiating both sides to the mth power, then taking the nth root of both sides, we find that $r \geq \sqrt[n]{r_1 \ldots r_n}$ which is what we wanted to show. □