18.100B
PROBLEM SET 6

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1. Problem 3.6

Theorem 1.1. *The series* $\sum a_n$ *diverges if* $a_n = \sqrt{n+1} - \sqrt{n}$.

Proof. If we multiply a_n by its conjugate, then we obtain the following:

$$a_n = (\sqrt{n+1} - \sqrt{n})\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \tag{1.2}$$

$$= \frac{1}{\sqrt{n+1}+\sqrt{n}} \tag{1.3}$$

Moreover, since we know that $\sqrt{n} \le n$ for all $n \in \mathbb{N}$, we know that $\sqrt{n+1}+\sqrt{n} \le (n+1)+n = 2n+1$. Thus, we can see that $a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} \ge \frac{1}{2n+1}$. Finally, the series $\sum \frac{1}{2n+1}$ diverges because it has $p = 1$. Using the comparison test, we can see that $\sum a_n$ also diverges because $a_n \ge \frac{1}{2n+1}$. □

Theorem 1.4. *The series* $\sum a_n$ *converges if* $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$.

Proof. If we multiply a_n by its conjugate, we will obtain:

$$a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \tag{1.5}$$

$$= \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \tag{1.6}$$

$$< \frac{1}{n\sqrt{n}} \tag{1.7}$$

The last inequality comes from the fact that $\sqrt{n+1}+\sqrt{n} > \sqrt{n}$. Moreover, we knowt that $\sum \frac{1}{n\sqrt{n}}$ converges because it is has $p = \frac{3}{2}$. By the comparison test, we know that $\sum a_n$ also converges. □

Theorem 1.8. *The series* $\sum a_n$ *converges if* $a_n = (\sqrt[n]{n} - 1)^n$.

Proof. If we perform the ratio test, then we obtain the following:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{n} - 1 \tag{1.9}$$

$$= 1 - 1 = 0 \tag{1.10}$$

This is because $\lim_{n\to\infty} \sqrt[n]{n} = 1$ for $n > 0$, as proved in Rudin. Thus, since $0 < 1$, the ratio test shows that $\sum a_n$ converges. □

Theorem 1.11. *If* $|z| \le 1$ *for* $a_n = \frac{1}{1+z^n}$, *then* $\sum a_n$ *diverges. If* $|z| > 1$, *then* $\sum a_n$ *converges.*

Proof. We will prove that if $|z| \le 1$, then $\sum a_n$ diverges. This is because $|1 + z^n| \le 1 + |z|^n \le 2$. It is obvious that $\sum \frac{1}{2}$ diverges because its sequences do not converge to zero. Thus, since $\sum \frac{1}{1+z^n} \ge \sum \frac{1}{2}$, we know that $\sum a_n$ diverges.

Now, if $|z| > 1$, then we know that $\sum \frac{1}{z^n}$ converges by a theorem in Rudin. Since $1 + z^n > z^n$, we have $\frac{1}{1+z^n} < \frac{1}{z^n}$. By the comparison test, we know that $\sum \frac{1}{1+z^n}$ converges as well. □

2. Problem 3.7

Theorem 2.1. *The convergence of $\sum a_n$ implies the convergence of $\sum \frac{\sqrt{a_n}}{n}$ if $a_n \geq 0$.*

Proof. Using the Cauchy Swarchz Inequality, we have:

$$\left|\sum \frac{\sqrt{a_n}}{n}\right|^2 \leq \sum |\sqrt{a_n}|^2 \sum \left|\frac{1}{n}\right|^2 \tag{2.2}$$

$$= \sum a_n \sum \frac{1}{n^2} \tag{2.3}$$

For two given series $\sum b_n$ and $\sum c_n$ converging to B and C respectively, the Cauchy product is defined as $d_n = \sum b_n z^n \sum c_n z^n$ if one sets $z = 1$. A theorem in Rudin has shown that $\sum d_n = BC$ if $\sum b_n$ is absolutely convergent and $\sum c_n$ is convergent, which means that $\sum d_n$ and all the partial sums of d_n are bounded. Since $\sum a_n$ is convergent by assumption and $\sum \frac{1}{n^2}$ is absolutely convergent by a theorem in Rudin, we know that $\sum a_n \sum \frac{1}{n^2}$ is bounded. Thus, we know that $\left|\sum \frac{\sqrt{a_n}}{n}\right|^2$ is bounded, and hence, so is $\sum \frac{\sqrt{a_n}}{n}$. Since we assumed $a_n \geq 0$, we know that $\sum \frac{\sqrt{a_n}}{n}$ is monotonically increasing so it converges because it is also bounded. □

3. Problem 3.9

Theorem 3.1. *The radius of convergence is $R = 1$ for the power series $\sum n^3 z^n$.*

Proof. We know that $R = \frac{1}{\alpha}$, where $\alpha = \lim_{n\to\infty} \sup \sqrt[n]{|n^3|} = 1$ because if we let $x_n = \sqrt[n]{n^3} - 1$, then we have for $k > 0$ and $n > 2k$:

$$n^3 = (x_n + 1)^n > \binom{n}{k} x_n^k = \frac{n(n-1)\dots(n-k+1)}{k!} x_n^k > \frac{n^k x_n^k}{2^k k!} \tag{3.2}$$

We obtain (3.2) by the binomial theorem and since x_n cannot be negative for $n \in \mathbb{N}$, we have:

$$0 \leq x_n^k < \frac{n^3 2^k k!}{n^k} \tag{3.3}$$

$$0 \leq x_n < 2(k!)^{\frac{1}{k}} n^{\frac{3}{k}-1} \tag{3.4}$$

Since $\frac{3}{k} - 1 < 0$ for all $k > 3$, we have can take the limit of both sides of the final inequality and show that $\lim_{n\to\infty} x_n = 0$ by the squeeze law. Thus, we have shown that $\alpha = \lim_{n\to\infty} \sup \sqrt[n]{|n^3|} = 1$ and that $R = 1$.

□

Theorem 3.5. *The radius of convergence is $R = +\infty$ for the power series $\sum \frac{2^n}{n!} z^n$.*

Proof. We must find α in order to find $R = \frac{1}{\alpha}$:

$$\alpha = \lim_{n\to\infty} \sup \sqrt[n]{\frac{2^n}{n!}} \tag{3.6}$$

$$= \lim_{n\to\infty} \sup \frac{2}{\sqrt[n]{n!}} \tag{3.7}$$

$$= \lim_{n\to\infty} \sup \frac{2}{n^{\frac{1}{n}} (n-1)^{\frac{1}{n}} \dots (1)^{\frac{1}{n}}} \tag{3.8}$$

$$< \lim_{n\to\infty} \frac{2}{n(1)^{\frac{1}{n}}} \tag{3.9}$$

Since we know that $\lim_{n\to\infty} (1)^{\frac{1}{n}} = 1$, we also know that $\lim_{n\to\infty} \frac{2}{n(1)^{\frac{1}{n}}} = 0$. Since we must have $\sqrt[n]{\frac{2^n}{n!}} > 0$ for all $n \in \mathbb{N}$, we have the following squeeze law:

$$0 \le \lim_{n\to\infty} \sup \sqrt[n]{\frac{2^n}{n!}} < \lim_{n\to\infty} \frac{2}{n(1)^{\frac{1}{n}}} < 0 \tag{3.10}$$

This shows that $\alpha = 0$, and that $R = +\infty$ because all terms are positive. $\square$

Theorem 3.11. *The radius of convergence is $R = \frac{1}{2}$ for the power series $\sum \frac{2^n}{n^2} z^n$.*

Proof. We must find α, so we have the following

$$\begin{aligned}
\alpha &= \lim_{n\to\infty} \sup \sqrt[n]{\frac{2^n}{n^2}} && (3.12)\\
&= \lim_{n\to\infty} \sup \frac{2}{n^{\frac{2}{n}}} && (3.13)\\
&= \lim_{n\to\infty} \sup \frac{2}{(\sqrt[n]{n})^2} && (3.14)\\
&= 2 && (3.15)
\end{aligned}$$

The final step comes from the fact that $\lim_{n\to\infty} \sqrt[n]{n} = 1$ according to a theorem in Rudin. Thus, we find that $R = \frac{1}{\alpha} = \frac{1}{2}$. $\square$

Theorem 3.16. *The radius of convergence is $R = 3$ for the power series $\sum \frac{n^3}{3^n} z^n$.*

Proof. Using the same procedure as before, we shall calculate α.

$$\begin{aligned}
\alpha &= \lim_{n\to\infty} \sup \sqrt[n]{\frac{n^3}{3^n}} && (3.17)\\
&= \lim_{n\to\infty} \sup \frac{n^{\frac{3}{n}}}{3} && (3.18)\\
&= \lim_{n\to\infty} \sup \frac{(\sqrt[n]{n})^3}{3} && (3.19)\\
&= \frac{1}{3} && (3.20)
\end{aligned}$$

The final step comes from the same fact as before, namely that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. Thus, we see that $R = \frac{1}{\alpha} = \frac{1}{3}$. $\square$

4. Problem 3.13

Theorem 4.1. *The Cauchy product of two absolutely convergent series converges absolutely.*

Proof. Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series. This means that $\sum_{n=0}^{\infty} |a_n| = A$ and $\sum_{n=0}^{\infty} |b_n| = B$ are convergent series as well. Now define the Cauchy product as $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}$. We will give the following definitions:

$$A_n = \sum_{k=0}^{n} |a_k|, \quad B_n = \sum_{k=0}^{n} |b_k|, \quad C_n = \sum_{k=0}^{n} |c_n| \tag{4.2}$$

Thus, if we expand out the terms of C_n, we obtain the following:

$$\begin{aligned}
C_n &= |a_0 b_0| + |a_0 b_1 + a_1 b_0| + \ldots + |a_0 b_n + \ldots + a_n b_0| && (4.3)\\
&\le |a_0| B_n + |a_1| B_{n-1} + \ldots + |a_n| B_0 && (4.4)\\
&\le |a_0| B_n + |a_1| B_n + \ldots + |a_n| B_n && (4.5)\\
&= (|a_0| + \ldots + |a_n|) B_n && (4.6)\\
&= A_n B_n && (4.7)
\end{aligned}$$

Thus we see that $C_n \le A_n B_n$ and if we take the limits of both sides, then we have:

$$\sum_{n=0}^{\infty} |c_n| = \lim_{n\to\infty} C_n \tag{4.8}$$
$$\leq \lim_{n\to\infty} A_n B_n \tag{4.9}$$
$$= AB \tag{4.10}$$

Thus, we can see that $\sum_{n=0}^{\infty} |c_n|$ is bounded. Moreover, since $|c_j| \geq 0$ for all $j \in \mathbb{Z}_{j\geq 0}$, we know that $\sum_{n=0}^{\infty} |c_n|$ is monotonically increasing. Thus, we know that $\sum_{n=0}^{\infty} |c_n|$ is both bounded and monotonically increasing, which shows that it is convergent. This means that the Cauchy product $\sum_{n=0}^{\infty} c_n$ is absolutely convergent. □

5. Problem 3.16

Theorem 5.1. *Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$ and define $x_2, x_3, \ldots$ by the recursion formula $x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$. Then $\{x_n\}$ is decreasing monotonically and has $\lim x_n = \sqrt{\alpha}$.*

Proof. First, we will show that the sequence is decreasing monotonically. Note that $x_1 > 0$ because we fixed an $\alpha > 0$. Next, take $x_{n+1} - x_n$:

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - x_n \tag{5.2}$$
$$= \frac{1}{2}\left(\frac{\alpha}{x_n} - x_n\right) \tag{5.3}$$
$$= \frac{1}{2}\left(\frac{\alpha - x_n^2}{x_n}\right) \tag{5.4}$$

Thus, we must determine the sign of $\alpha - x_n^2$. To begin, we have assumed $\alpha - x_1^2 < 0$ because we set $x_1 > \sqrt{\alpha}$. Moreover, we can see that $x_n^2 \geq \alpha$ for all $n \in \mathbb{N}$ because as $\{x_n\}$ approaches α, we have $\alpha - x_n^2 << x_n$. If for some n, $x_n^2 = \alpha$, then $x_{n+1} - x_n = 0$ and the sequence will become constant. Thus because $\alpha - x_n^2 \leq 0$, we see that the sequence is monotonically decreasing.

To show the second part of the problem, we know that the series is decreasing monotonically and is bounded by 0. This is because x_1 starts out positive, so we have $x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) > 0$ for all $n \in \mathbb{N}$. This show that $\{x_n\}$ converges. Let us define x as its limit: $\{x_n\} \to x$. Then we have

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) \tag{5.5}$$
$$x = \frac{1}{2}\left(x + \frac{\alpha}{x}\right) \tag{5.6}$$
$$x^2 = \alpha \tag{5.7}$$

Since $\alpha > 0$ and also $x_n > 0$, we know that $x = \sqrt{\alpha}$ and thus we have shown that $\lim_{n\to\infty} x_n = \sqrt{\alpha}$. □

Theorem 5.8. *Put $\epsilon_n = x_n - \sqrt{\alpha}$ and show that $\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$ so that setting $\beta = 2\sqrt{\alpha}$, we obtain $\epsilon_{n+1} < \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n}$ for $n = 1, 2, \ldots$.*

Proof. We know that $\epsilon_n^2 = (x_n - \sqrt{\alpha})^2 = x_n^2 + \alpha - 2x_n\sqrt{\alpha}$, so that $x_n^2 + \alpha = \epsilon_n^2 + 2x_n\sqrt{\alpha}$. Then, we can use our recursive formulas to determine ϵ_{n+1}:

$$\epsilon_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} \tag{5.9}$$

$$= \frac{1}{2}\left(\frac{x_n^2 + \alpha - 2x_n\sqrt{\alpha}}{x_n}\right) \tag{5.10}$$

$$= \frac{1}{2}\frac{\epsilon_n^2}{x_n} \tag{5.11}$$

We have shown in the first part of the exercise that $x_n^2 > \alpha$ for all $n \in \mathbb{N}$, so we know that $\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$.

Next, we will show that $\epsilon_{n+1} < \frac{\epsilon_n^2}{\beta} = \beta\left(\frac{\epsilon_n}{\beta}\right)^2$. For ϵ_2 we have $\epsilon_2 < \frac{\epsilon_1^2}{\beta}$. Thus, we have the base case for our inductive argument. Next, we have

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{\beta} \tag{5.12}$$

$$< \frac{1}{\beta}\left(\frac{\epsilon_{n-1}^2}{\beta}\right)^2 \tag{5.13}$$

$$< \frac{1}{\beta}\frac{1}{\beta^2}\left(\frac{\epsilon_{n-2}^2}{\beta}\right)^{2^2} \tag{5.14}$$

$$\vdots \tag{5.15}$$

$$< \frac{1}{\beta}\frac{{\epsilon_1}^{2^n}}{\beta^{2^{n-1}}} \tag{5.16}$$

$$= \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n} \tag{5.17}$$

This completes the proof, since we have shown $\epsilon_{n+1} < \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n}$.

□

Theorem 5.18. *If $\alpha = 3$ and $x_1 = 2$, then $\frac{\epsilon_1}{\beta} < \frac{1}{10}$ and therefore that $\epsilon_5 < 4\times 10^{-16}$ and $\epsilon_6 < 4 \times 10^{-32}$.*

Proof. We know that $\epsilon_n = x_n - \sqrt{\alpha}$ so that $\epsilon_1 = x_1 - \sqrt{\alpha} = 2 - \sqrt{3}$. We also know that $\beta = 2\sqrt{\alpha} = 2\sqrt{3}$. Thus, we can multiply by the conjugate to obtain

$$\frac{\epsilon_1}{\beta} = \frac{2-\sqrt{3}}{2\sqrt{3}}\frac{2+\sqrt{3}}{2+\sqrt{3}} \tag{5.19}$$

$$= \frac{1}{2\sqrt{3}(2+\sqrt{3})} \tag{5.20}$$

$$= \frac{1}{4\sqrt{3}+6} \tag{5.21}$$

We know that $1 < \sqrt{3}$ so that $4 < 4\sqrt{3}$. This means that $\frac{\epsilon_1}{\beta} < \frac{1}{10}$. Next, we can approximate $\beta = 2\sqrt{3} < 4$ because we know that $\sqrt{3} < 2$, so we can obtain the following bounds for ϵ_5 and ϵ_6:

$$\epsilon_5 = \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^4} \tag{5.22}$$

$$\epsilon_5 < 4 \times 10^{-16} \tag{5.23}$$

$$\epsilon_6 = \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^5} \tag{5.24}$$

$$\epsilon_6 < 4 \times 10^{-32} \tag{5.25}$$

Because we have $2^4 = 16$ and $2^5 = 32$. This completes the proof. □

6. Problem 3.18

Theorem 6.1. *Fix a positive integer p and a positive number α and define $x_{n+1} = \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1}$. Then if $x_1 > \alpha^{\frac{1}{p}}$, the sequence decreases monotonically and has $\lim_{n\to\infty} x_n = \alpha^{\frac{1}{p}}$.*

Proof. First, we must show that the sequence is monotonically decreasing if $x_1 > \alpha^{\frac{1}{p}}$. To do this, we observe the following:

$$x_{n+1} - x_n = \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1} - x_n \tag{6.2}$$

$$= \frac{1}{p}\left((p-1)x_n + \alpha x_n^{-p+1} - px_n\right) \tag{6.3}$$

$$= \frac{1}{p}\left(-x_n + \frac{\alpha}{x_n^{p-1}}\right) \tag{6.4}$$

$$= \frac{1}{p}\left(\frac{-x_n x_n^{p-1} + \alpha}{x_n^{p-1}}\right) \tag{6.5}$$

$$= \frac{1}{p}\left(\frac{\alpha - x_n^p}{x_n^{p-1}}\right) \tag{6.6}$$

Since we have defined $x_1 > \alpha^{\frac{1}{p}}$, we know that $x_1^p > \alpha$ which gives $\alpha - x_1^p < 0$. Thus, we see that $x_2 - x_1 < 0$. Using inductive reasoning, we can see that $x_n > \alpha^{\frac{1}{p}}$ for all $n \in \mathbb{N}$. The sequence is decreasing from x_1, but the sequence does not go below $\alpha^{\frac{1}{p}}$ because if for some n we have $x_n = \alpha^{\frac{1}{p}}$, then $x_{n+1} - x_n = 0$. Thus, the sequence is monotonically decreasing since $x_{n+1} - x_n < 0$ for all $n \in \mathbb{N}$. Moreover, the sequence is bounded by zero because p is a positive integer, so $x_{n+1} > \frac{\alpha}{p}x_n^{-p+1}$, and since we have defined $x_1 > 0$, we have that x_{n+1} can never be negative.

Thus we have shown the sequence is bounded and monotonically decreasing, showing that it is convergent. Let $\lim_{n\to\infty} x_n = x$, then we have:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1} \tag{6.7}$$

$$x = \frac{p-1}{p}x + \frac{\alpha}{p}x^{-p+1} \tag{6.8}$$

$$x\left(1 - \frac{p-1}{p}\right) = \frac{\alpha}{p}x^{-p}x \tag{6.9}$$

$$\frac{p}{\alpha}\frac{1}{p} = x^{-p} \tag{6.10}$$

$$x^p = \alpha \tag{6.11}$$

$$x = \alpha^{\frac{1}{p}} \tag{6.12}$$

We have thus shown that $\lim_{n\to\infty} x_n = \alpha^{\frac{1}{p}}$, which is what we wanted. □

Theorem 6.13. *Let $\epsilon_n = x_n - \alpha^{\frac{1}{p}}$, then we have $\epsilon_{n+1} < \beta^{1-p}(\beta\epsilon_1)^{p^n}$.*

Proof. First, we note that $\epsilon_n^p = (x_n - \alpha^{\frac{1}{p}})^p > \binom{p}{1}x_n(\alpha^{\frac{1}{p}})^{p-1} > x_n\alpha^{\frac{p-1}{p}}$ by the binomial theorem and because $p \geq 1$. We do not have to worry about the negative sign in front of α because if p is an odd integer, then $p-1$ is even so $\alpha^{\frac{p-1}{p}} > 0$ and if p is an even integer then p is even so that $\alpha^{\frac{p-1}{p}} > 0$ as well. The binomial theorem also tells us that $\epsilon_n^{-p} = (x_n - \alpha^{\frac{1}{p}})^{-p} > x_n^{-p}$. Since p is a positive integer, we can see that that $x_n < \epsilon_n^p \leq p\epsilon_n^p$. This gives us the following relationships:

$$x_n^{-p} < p\epsilon_n^{-p}, \qquad x_n < \frac{\epsilon_n^p}{\alpha^{\frac{p-1}{p}}} \tag{6.14}$$

Since these are true, we can then find a bound for ϵ_{n+1}:

$$\begin{aligned}
\epsilon_{n+1} &= x_{n+1} - \alpha^{\frac{1}{p}} && (6.15)\\
&= \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p}x_n - \alpha^{\frac{1}{p}} && (6.16)\\
&< \frac{p-1}{p}\frac{\epsilon_n^p}{\alpha^{\frac{p-1}{p}}} + \frac{\alpha}{p}p\epsilon_n^{-p}\frac{\epsilon_n^p}{\alpha^{\frac{p-1}{p}}} - \alpha^{\frac{1}{p}} && (6.17)\\
&= \frac{p-1}{p}\frac{\epsilon_n^p}{\alpha^{\frac{p-1}{p}}} + \frac{\alpha p}{p\alpha^{\frac{p-1}{p}}} - \alpha^{\frac{1}{p}} && (6.18)\\
&= \frac{1}{p\alpha^{\frac{p-1}{p}}}\left((p-1)\epsilon_n^p + \alpha p - \alpha^{\frac{1}{p}}p\alpha^{\frac{p-1}{p}}\right) && (6.19)\\
&= \frac{1}{p\alpha^{\frac{p-1}{p}}}\left((p-1)\epsilon_n^p + \alpha p - \alpha^{\frac{1}{p}}p\alpha\alpha^{-\frac{1}{p}}\right) && (6.20)\\
&= \left(\frac{p-1}{p\alpha^{\frac{p-1}{p}}}\right)\epsilon_n^p && (6.21)
\end{aligned}$$

If we define the following constant:

$$\beta = \frac{p-1}{p\alpha^{\frac{p-1}{p}}} \tag{6.22}$$

Then we have found a relationship between ϵ_{n+1} and ϵ_n.

$$\epsilon_{n+1} < \beta\epsilon_n^p \tag{6.23}$$

Thus, we can use induction on the sequence $\{\epsilon_n\}$.

$$\begin{aligned}
\epsilon_1 &< \beta\epsilon_1^p && (6.24)\\
\epsilon_2 &< \beta(\beta\epsilon_1^p)^p && (6.25)\\
&\vdots && (6.26)\\
\epsilon_{n+1} &< \beta(\beta^p)^{n-1}(\epsilon_1^p)^n && (6.27)
\end{aligned}$$

If we collect terms, then we have completed the proof:

$$\epsilon_{n+1} < \beta^{1-p}(\beta\epsilon_1)^{pn} \tag{6.28}$$

□

Theorem 6.29. *This is a good algorithm for computing nth roots, especially for large values of α. For example, if $\alpha = 9$, $p = 10$, and $x_1 = 3$, then we have $\beta\epsilon_1 < \frac{1}{10}$ with $\epsilon_5 < 10^{-9,991}$ and $\epsilon_6 < 10^{-99,991}$.*

Proof. First, we can compute $\beta = \frac{p-1}{p\alpha^{\frac{p-1}{p}}} = \frac{9}{10*9^{\frac{9}{10}}}$. We know that $9^{\frac{9}{10}} < 9^1 < 9$ so we have $\beta < \frac{1}{10}$. We also have $\beta\epsilon_1 = \beta(x_1 - \alpha^{\frac{1}{p}})$. Since we know that $9 < 2^{10} = 1024$, we have $9^{\frac{1}{10}} < 2$. This gives:

$$\begin{aligned}
\beta\epsilon_1 &= \beta(3 - 9^{\frac{1}{10}}) && (6.30)\\
&< \frac{1}{10}(3-2) && (6.31)\\
&< \frac{1}{10} && (6.32)
\end{aligned}$$

Therefore, we can use the bounds of $\beta < \frac{1}{10}$ and $\beta\epsilon < \frac{1}{10}$ to find bounds for ϵ_5 and ϵ_6 by using the formula $\epsilon_{n+1} < \beta^{1-p}(\beta\epsilon_1)^{p^n}$:

$$\epsilon_5 < \left(\frac{1}{10}\right)^{-9}\left(\frac{1}{10}\right)^{10^4} \tag{6.33}$$

$$\epsilon_5 < 10^{-9,991} \tag{6.34}$$

$$\epsilon_6 < \left(\frac{1}{10}\right)^{-9}\left(\frac{1}{10}\right)^{10^5} \tag{6.35}$$

$$\epsilon_6 < 10^{-99,991} \tag{6.36}$$

It is clear that this algorithm converges extremely quickly for large values of α and p. $\square$