18.100B
PROBLEM SET 8

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1. Problem 5.2

Theorem 1.1. *Suppose $f'(x) > 0$ in (a,b). Prove that f is strictly increasing in (a,b), and let g be its inverse function. Prove that g is differentiable and that $g'(f(x)) = \frac{1}{f'(x)}$ for $a < x < b$.*

Proof. First, we know f is continuous because of the existence of its derivative for all $x \in (a,b)$. Thus, we can use the specialized mean value theorem, which states that for some $x_1, x_2 \in (a,b)$, there exists an $x \in (a,b)$ such that $f(x_2) - f(x_1) = (x_2 - x_1)f'(x)$. Since $f'(x) > 0$ for all $x \in (a,b)$, we can see that:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \tag{1.2}$$

Without loss of generality, if $x_2 > x_1$, we see that $f(x_2) - f(x_1) > 0$, which shows that f is strictly increasing because $f(x_2) > f(x_1)$.

Now, we shall show that g is continuous and differentiable. First, we will show that g is strictly increasing by contradiction. If we assume not, then there exists some $z > w$, where $z, w \in (f(a), f(b)$ such that $g(z) \leq g(w)$. We know there must exist corresponding values $x, y \in (a,b)$ such that $x = g(z)$ and $y = g(w)$. Thus, we see that $x \leq y$ but that $z > w$ which implies that $f(x) > f(y)$ because $f(x) = f(g(z)) = z$ and $f(y) = f(g(w)) = w$. However, we have shown that f is strictly increasing which implies $f(x) < f(y)$, which is a contradiction because $f(x) > f(y)$ and $f(x) < f(y)$ cannot both be true. Thus, we see that g is stricly increasing.

To show that g is continuous, we assume the contrary. We now note that strictly increasing functions can only have jump discontinuities. This would mean that there exists some $z \in (f(a), f(b))$ such that $g(z-) < g(z+)$. Without loss of generality, assume that $g(z) = g(z-)$. Then we must have a corresponding value of $x \in (a,b)$ such that $f(x) = z$. This implies that $x = g(z) = g(z-) < g(z+)$. However, we must have the following:

$$g(z+) = \lim_{f(y) \to z^+} g(f(y)) = \lim_{y \to x^+} g(f(y)) = \lim_{y \to x^+} y = x \tag{1.3}$$

Thus, we have shown that $x = g(z) < g(z+) = x$, which is a contradiction. Therefore, g must be continuous. Since it is continuous, we obtain an expression for $g'(f(x))$ if it exists:

$$g'(f(x)) = \lim_{f(t) \to f(x)} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} = \lim_{t \to x} \frac{t - x}{f(t) - f(x)} = \lim_{t \to x} \frac{1}{f'(x) + u(t,x)} \tag{1.4}$$

Where $\lim_{t \to x} u(t,x) = 0$. Thus, since $f'(x) > 0$ for all $x \in (a,b)$, we can see that the limit exists, and that $g'(f(x)) = \frac{1}{f'(x)}$. □

2. Problem 5.5

Theorem 2.1. *Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.*

Proof. Since f is differentiable on $(0, \infty)$, it must also be continuous. Therefore, we can use the mean value theorem for points $x, x+1$ such that $x \in (0, \infty)$, which will ensure that $x + 1$ is also inside the domain. Therefore, by mean value theorem, we see that:

$$f(x+1) - f(x) = (x + 1 - x)f'(y) \tag{2.2}$$

For some $y \in (x, x+1)$. This shows that $g(x) = f'(y)$ for some $y \in (x, x+1)$. If we take the limit as $x \to +\infty$, we obtain:

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} f'(y) = \lim_{y\to\infty} f'(y) = 0 \tag{2.3}$$

This is because y has a lower bound of x, and as $x \to +\infty$, we also force $y \to +\infty$. Since $\lim_{y\to\infty} f'(y) = 0$, we can see that $g(x) \to 0$ and $x \to \infty$. □

3. Problem 5.14

Theorem 3.1. *Let f be a differentiable real function defined in (a, b). Prove that f is convex if and only if f' is monotonically increasing.*

Proof. Given a monotonically increasing function f', assume by contradiction that f is not convex. Then there exists some $x, y \in (a, b)$ such that for some $\lambda \in (0, 1)$, we have $f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda)f(y)$. Let $p = \lambda x + (1-\lambda)y$ so that $f(p) > \lambda f(x) + (1-\lambda)f(y)$. Moreover, we can assume without loss of generality that $y > x$ and we thus see that $p \in (x, y)$. We can use the mean value theorem, because f is differentiable by assumption and hence continuous. This shows that $f(y) - f(p) = (y-p)f'(z)$ for some $z \in (p, y)$. Using the mean value theorem again, we can see that $f(p) - f(x) = (p - x)f'(w)$ for some $w \in (x, p)$. Next, since $w \in (x, p)$ and $z \in (p, y)$, we can see that necessarily, $w < z$. Since f' is a monotonically increasing function, we must therefore have $f'(w) \le f'(z)$. Combining this, we find:

$$f'(w) = \frac{f(p) - f(x)}{p - x} \quad \le \quad \frac{f(y) - f(p)}{y - p} = f'(z) \tag{3.2}$$

$$(y - x)f(p) \quad \le \quad f(y)(p - x) + f(x)(y - p) \tag{3.3}$$

$$\lambda f(x) + (1-\lambda)f(y) \quad < \quad f(p) \le \frac{f(y)(p - x) + f(x)(y - p)}{y - x} \tag{3.4}$$

$$0 < \frac{f(y)((p - x) - (y - x)(1 - \lambda)) + f(x)((y - p) - (y - x)\lambda)}{y - x} \tag{3.5}$$

Since we have assumed $y > x$, we can divide by $y - x$ in equation 3.4. Next, we know that $p = \lambda x + (1-\lambda)y$, so substituting this into our expression and multiplying by the positive term $y - x$, we obtain:

$$0 < f(y)(\lambda x + (1-\lambda)y - x - (1-\lambda)y + (1-\lambda)x) + f(x)(y - \lambda x - (1-\lambda)y - y\lambda + x\lambda) \tag{3.6}$$

$$0 \quad < \quad f(y)(0) + f(x)(0) = 0 \tag{3.7}$$

$$0 \quad < \quad 0 \tag{3.8}$$

Since this is a strict inequality, this cannot be the case and we have shown a contradiction. Thus, we see that given a monotonically increasing function f', then f is convex. To show the converse, we will assume that f is convex. Then, we must show that f' is monotonically increasing.

Assume that $x, y \in (a, b)$. Without loss of generality, suppose that $y > x$. Then, since the derivative exists everywhere, we have the following two limits due to the definition of the derivative:

$$f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x} = \lim_{t\to x^-} \frac{f(t) - f(x)}{t - x} \tag{3.9}$$

$$f'(y) = \lim_{s\to y} \frac{f(s) - f(y)}{s - y} = \lim_{s\to y^+} \frac{f(s) - f(y)}{s - y} \tag{3.10}$$

Set $t < x < y < s$. We have shown in problem 5.23 of the last problem set that the following inequalities holds for convex functions, and hence for f:

$$\frac{f(x) - f(t)}{x - t} \le \frac{f(s) - f(t)}{s - t} \le \frac{f(s) - f(y)}{s - y} \tag{3.11}$$

Therefore, taking the left and right limits of x and y respectively, we obtain:

$$f'(x) = \lim_{t\to x^-} \frac{f(t) - f(x)}{t - x} leq \lim_{s\to y^+} \frac{f(y) - f(s)}{y - s} = f'(y) \tag{3.12}$$

Thus, we have shown that for $x < y$, we have $f'(x) \le f'(y)$ for all $x, y \in (a, b)$. Therefore, we have shown that f' is monotonically increasing.

□

Theorem 3.13. *Assume that $f''(x)$ exists for every $x \in (a,b)$ and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a,b)$.*

Proof. Since we have show that f is convex if and only if f' is monotonically increasing, we only must show that $f''(x) \geq 0$ if and only if f' is monotonically increasing. First, we will assume that $f''(x) \geq 0$. Then, since f' is differentiable everywhere on (a,b), we can use the mean value theorem since continuity is also required. Thus means that $f'(x_2)-f'(x_1) = (x_2-x_1)f''(x)$ for some $x_2, x_1 \in (a,b)$ and $x \in (x_2, x_1)$. Assume without loss of generality that $x_2 > x_1$. Then this implies that

$$\frac{f'(x_2) - f'(x_1)}{x_2 - x_1} \geq 0 \tag{3.14}$$

Which shows that for $x_2 \geq x_1$, we must have $f'(x_2) \geq f'(x_1)$. This shows that f' must be monotonically increasing. To prove the opposite way, assume that f' is monotonically increasing. Then for some $t > x$ where $t, x \in (a,b)$, we must have $f'(t) \geq f'(x)$. Alternatively, this means $f'(t) - f'(x) \geq 0$. Since $t > x$ implies that $t - x \neq 0$, we can divide by $t - x$ to obtain:

$$\phi^+(t) = \frac{f'(t) - f'(x)}{t - x} \geq 0 \tag{3.15}$$

We can also show for some $t < x$, where $t, x \in (a,b)$, we must have $f'(t) \leq f'(x)$. Using the same method as above, we have:

$$\phi^-(t) = \frac{f'(t) - f'(x)}{t - x} \geq 0 \tag{3.16}$$

Since f'' exists for every $x \in (a,b)$, we have:

$$\lim_{t \to x^+} \phi^+(t) = \lim_{t \to x^-} \phi^-(t) = f''(x) \geq 0 \tag{3.17}$$

Since this holds for arbitrary $x \in (a,b)$, we have proven that $f''(x) \geq 0$ if and only if f' is monotonically increasing. Since we have also shown that f is convex if and only if f' is monotonically increasing, we have proven that f is convex if and only if $f''(x) \geq 0$. □

4. Problem 5.15

Theorem 4.1. *Suppose $a \in \mathbb{R}^1$, f is a twice differentiable real function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively on (a, ∞). Then $M_1^2 \leq 4M_0M_2$.*

Proof. Since f is continuous on (a,∞) by its differentiability, and since both f' and f'' exist for (a,∞), we can use Taylor's Theorem, which states that, setting $\alpha = x$ and $\beta = x + 2h$, we obtain

$$f(x+2h) = f(x) + f'(x)(2h) + \frac{f''(\xi)}{2}(2h)^2 \tag{4.2}$$

This reduces down to the form:

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi) \tag{4.3}$$

For some $\xi \in (x, x+2h)$ and $h > 0$. Therefore, since $|f(x)|$ is bounded by M_0 and $|f''(x)|$ is bounded by M_2, we can obtain:

$$|f'(x)| \leq hM_2 + \frac{M_0}{h} \tag{4.4}$$

Since $\frac{M_0}{h}$ is obviously larger than $\frac{M_0}{2h}$. Next, we can rearrange the equation to obtain:

$$0 \leq h^2M_2 - h|f'(x)| + M_0 \tag{4.5}$$

Since this holds for any $h > 0$, we can take $h = \sqrt{\frac{M_0}{M_2}}$, using the fact that M_0 and M_2 are positive. If $M_2 = 0$, then $f'(x)$ is constant and $f(x)$ is a linear function by the mean value theorem. We cannot have $f'(x) = c \neq 0$, or else M_0 would be infinite, a contradiction to the hypothesis. Then, if $f'(x) = 0$, then

$M_1 = 0$, and the inequality is trivial. Moreover, if $M_0 = 0$, then the inequality is trivial. Therefore, we can take $M_0 > 0$ and $M_2 > 0$. Thus, substitute $h = \sqrt{\frac{M_0}{M_2}}$ into the expression:

$$0 \leq \frac{M_0}{M_2}M_2 - \sqrt{\frac{M_0}{M_2}}|f'(x)| + M_0 \tag{4.6}$$

Which leads to:

$$|f'(x)|^2\frac{M_0}{M_2} \leq 4M_0^2 \tag{4.7}$$

Since we have let $x \in (a, \infty)$ be any arbitrary value, we can see that $|f'(x)| \leq M_1$, which gives us:

$$M_1^2 \leq 4M_0M_2 \tag{4.8}$$

□

Theorem 4.9. *We will show that the strict equality $M_1^2 = 4M_0M_2$ can occur.*

Proof. Consider the following continuous function for $a = -1$ and $x \in (-1, \infty)$:

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0) \\ \frac{x^2-1}{x^2+1} & (0 \leq x < \infty) \end{cases} \tag{4.10}$$

Since we know $f(x)$ is differentiable everywhere, we can use the quotient and product rules (using right and left derivatives where appropriate) to obtain:

$$f'(x) = \begin{cases} 4x & (-1 < x < 0) \\ \frac{4x}{(x^2+1)^2} & (0 \leq x < \infty) \end{cases} \tag{4.11}$$

It is clear that for $x \in (-1, 0)$, we have $f'(x) < 0$ and for $x \in (0, \infty)$, we have $f'(x) > 0$. At $x = 0$, $f'(x) = 0$. Therefore, on $x \in (-1, 0)$, $f(x)$ is monotonically decreasing and on $x \in (0, \infty)$, $f(x)$ is monotonically increasing. Since we have:

$$\lim_{x\to -1^+} f(x) = 1, \qquad \lim_{x\to\infty} f(x) = 1, \qquad f(0) = -1 \tag{4.12}$$

Therefore, $M_0 = 1$. Now, we will use the same analysis to show that $M_1 = 4$. Differentiate $f'(x)$ using the appropriate right and left derivatives to obtain:

$$f''(x) = \begin{cases} 4 & (-1 < x < 0) \\ \frac{4(x^2-4x+1)}{(x^2+1)^3} & (0 \leq x < \infty) \end{cases} \tag{4.13}$$

On $x \in (-1, 0)$ we see that $f''(x) > 4$ so that $f'(x)$ is monotonically increasing. Since $\lim_{x\to 0^-} f'(x) = 0$ and $\lim_{x\to -1^+} f'(x) = -4$, we have $|f'(x)| < 4$ on $x \in (-1, 0)$. On $x \in [0, \infty)$ we see that

$$|f'(x)| = \frac{4x}{(x^2+1)^2} \leq 4\frac{x}{x^2+1}\frac{1}{x^2+1} \leq 4 \times \frac{1}{2} \times 1 = 2 \tag{4.14}$$

Therefore, since $f'(0) = 0$ as well, we can see that $M_1 = 4$. Next, for $x \in [0, \infty)$, we have

$$|f''(x)| = \frac{4}{(x^2+1)^2} - \frac{16x}{(x^2+1)^3} \leq \frac{4}{(x^2+1)^2} \leq 4 \tag{4.15}$$

For $x \in (-1, 0)$, we can see that $f''(x) = 4$ is a constant function. Therefore $M_2 = 4$. Now, we can see that $M_1^2 = 4^2 = 16$ and $4M_0M_2 = 4 \times 1 \times 4 = 16$. Therefore, we see that $M_1^2 = 4M_0M_2 = 16$. □

Theorem 4.16. *The same result holds for real vector valued functions f.*

Proof. Let $f = (f_1, \ldots, f_k)$ be a vector-valued function and fix

$$M_j = \sup_{x\in(a,\infty)} \left(\sum_{i=1}^{k} |f_i^{(j)}(x)|^2\right)^{\frac{1}{2}} \tag{4.17}$$

If $M_1 = 0$, we know that $M_1^2 \leq 4M_0M_2 = 0$. Otherwise, for any point $y \in (a, \infty)$, define $g(x) = f_1'(y)f_1(x) + \ldots + f_k'(y)f_k(x)$. Since $g(x)$ for $x \in (a, \infty)$ is a twice differentiable function, we can use the first part of the exercise to find:

$$|g'(x)|^2 \leq 4 \sup_{x\in(a,\infty)} |f_1'(y)f_1(x) + \ldots + f_k'(y)f_k(x)| \sup_{x\in(a,\infty)} |f_1'(y)f_1''(x) + \ldots f_k'(y)f_k''(x)| \tag{4.18}$$

Using the Cauchy-Swarchz inequality, we obtain

$$|g'(x)|^2 \leq 4\left(\sum_{i=1}^{k} |f_i'(y)|^2\right) M_0 M_2 \tag{4.19}$$

Since we have defined M_j^2 in a specific manner, we can see that $|g'(x)|^2 \leq 4M_1^2 M_0 M_2$. Moreover, we have let this inequality hold for arbitrary values of $x, y \in (a,\infty)$. Therefore, we can set $x = y$ and see that $|g'(x)| = |f_1'(x)|^2 + \ldots |f_k(x)|^2$. Thus, since this holds for any $x \in (a,\infty)$, we obtain:

$$\left(\sum_{i=1}^{k} |f_i'(x)|^2\right)^2 = M_1^4 \leq 4M_1^2 M_0 M_2 \tag{4.20}$$

This shows that $M_1^2 \leq 4M_0M_2$ by division because we know that $M_1 = 0$ is a trivial case. □

5. Problem 5.16

Theorem 5.1. *Suppose f is twice differentiable on $(0,\infty)$, f'' is bounded on $(0,\infty)$, and $f(x) \to 0$ as $x \to \infty$. Then $f'(x) \to 0$ as $x \to \infty$.*

Proof. Suppose that $a \in (0,\infty)$. Then since $f(x)$ for $x \in (a,\infty)$ is a twice differentiable function on (a,∞), we can use the result from the last exercise. This states that for least upper bounds M_0, M_1, M_2 of $|f(x)|, |f'(x)|, |f''(x)|$, respectively, the following holds true: $M_1^2 \leq 4M_0M_2$. Moreover, as we take the limit as $a \to \infty$, we can see that $M_0 \to 0$. We know this because $x \in (a,\infty)$, so as $a \to \infty$, we must have $x \to \infty$. Moreover, we know from assumption that $f(x) \to 0$ as $x \to \infty$. Therefore, we have discovered the following:

$$\lim_{a\to\infty} M_0 = \lim_{a\to\infty} \sup |f(x)| = \lim_{x\to\infty} \sup |f(x)| = 0 \tag{5.2}$$

Therefore, we take can our expression from the previous exercise and show that the right hand side converges to 0, because $f''(x)$ is bounded on $(0,\infty)$.

$$\lim_{a\to\infty} M_1^2 \leq \lim_{a\to\infty} 4M_0M_2 = 0 \tag{5.3}$$

This shows that $0 \leq \lim_{a\to\infty} M_1 \leq 0$, which by the squeeze law forces $\lim_{a\to\infty} M_1 = 0$. This means that:

$$0 = \lim_{a\to\infty} \sup |f'(x)| = \lim_{x\to\infty} \sup |f'(x)| \tag{5.4}$$

Since the supremum of the absolute value of $f'(x)$ is forced to equal zero in the limit as $x \to \infty$, we must therefore have $f'(x) \to 0$ as $x \to \infty$. This completes the proof.

□