18.100B
PROBLEM SET 5

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1. Problem 3.1

Theorem 1.1. *The convergence of* $\{s_n\}$ *implies the convergence of* $\{|s_n|\}$. *The converse is not true, however.*

Proof. We will assume that $\{s_n\}$ is a convergent set in $\mathbb{R}$ because the norm is not defined in Rudin for any other metric space. Since $\{s_n\}$ is a convergent sequence, it must be Cauchy. Therefore, there exists and N such that $m, n \geq N$ implies that $d(s_m, s_n) < \epsilon$ for all $\epsilon > 0$. We can use the traingle inequality, then, to show the following:

$$\begin{aligned} |s_n| &= |(s_n + s_m) - s_m| \\ &\leq |(s_n + s_m) - s_n| + |s_n - s_m| \\ &= |s_m| + d(s_n, s_m) \\ &< |s_m| + \epsilon \end{aligned} \tag{1.2}$$

This shows that $|s_n| - |s_m| < \epsilon$, which further implies that $d(|s_n|, |s_m|) < \epsilon$ for $m, n \geq N$. Thus, we see that the sequence $\{|s_n|\}$ is Cauchy, and since it is in $\mathbb{R}$, it converges.

It is easy to show that the converse is not true, namely that the convergence of $\{|s_n|\}$ does not imply the convergence of $\{s_n\}$. Take the sequence defined by $s_n = (-1)^n$. Thus, we can see that $|s_n| = |(-1)^n| = 1$ is a constant sequence, and thus definitely converges. However, s_n switches between -1 and 1 and thus will never converge for any n.

□

2. Problem 3.2

Theorem 2.1. *We show that* $\lim_{n\to\infty} \sqrt{n^2+n} - n = \frac{1}{2}$.

Proof. We multiply through by the conjugate to obtain the following:

$$\begin{aligned} \lim_{n\to\infty} \sqrt{n^2+n} - n &= \lim_{n\to\infty} \sqrt{n^2+n} - n \frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} \\ &= \lim_{n\to\infty} \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} \\ &= \lim_{n\to\infty} \frac{n}{n\sqrt{\frac{n^2+n}{n^2}}+n} \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} \\ &= \frac{1}{\sqrt{1}+1} \\ &= \frac{1}{2} \end{aligned} \tag{2.2}$$

□

3. Problem 3.3

Theorem 3.1. *If* $s_1 = \sqrt{2}$ *and* $s_{n+1} = \sqrt{2+\sqrt{s_n}}$ *for* $n = 1, 2, 3, \ldots$ *then* $\{s_n\}$ *converges and* $s_n < 2$ *for all* $n \in \mathbb{N}$.

Proof. We know that $0 < \sqrt{2} < 2$, so we have $0 < s_1 < 2$. Moreover, if we add 2 to each side of the inequality, we obtain $2 < 2 + \sqrt{2} < 4$. Taking the square roots, we obtain $\sqrt{2} < \sqrt{2+\sqrt{2}} < 2$. Adding 2 to each side of this, we obtain $2+\sqrt{2} < 2+\sqrt{2+\sqrt{2}} < 4$. Taking square roots, we have $\sqrt{2} < \sqrt{2+\sqrt{2+\sqrt{2}}} < 2$. We can repeat the process infinitely many times and notice that $s_1 = \sqrt{2}, s_2 = \sqrt{2+\sqrt{2}}, s_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$. Thus, we see that $s_1 < s_2 < s_3 < \ldots < 2$. We have therefore shown that $\{s_n\}$ is a monotonically increasing sequence that is bounded because $s_n < 2$ $\forall n \in \mathbb{N}$. By a theorem in Rudin, we know that $\{s_n\}$ must converge. □

4. Problem 3.4

Theorem 4.1. *Consider the sequence* $\{s_n\}$ *defined by* $s_1 = 0$ *with* $s_{2m} = \frac{s_{2m}-1}{2}$ *and* $s_{2m+1} = \frac{1}{2} + s_{2m}$. *The lower limit of the sequence is* $\frac{1}{2}$ *and the upper limit is* 1.

Proof. If we compute the sequence, then we see the following, starting at $s_1 = 0$:

$$\{s_n\} = 0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}, \ldots \tag{4.2}$$

We can use induction to derive the following formulas:

$$\text{Even n} \rightarrow s_n = \frac{2^{\frac{n}{2}-1}-1}{2^{\frac{n}{2}-1}} = \frac{1}{2} - \frac{1}{2^{\frac{n}{2}-1}} \qquad \text{Odd n} \rightarrow s_n = \frac{2^{\frac{n-1}{2}}-1}{2^{\frac{n-1}{2}}} = 1 - \frac{1}{2^n}$$

We can see that for even n, we have $s_n < \frac{1}{2}$ and for odd n, we have $s_n < 1$. It is clear by inspection that they are subsequential limits. Moreover, we can show that these are the only two subsequential limits of $\{s_n\}$. This is because each subsequence $\{s_{n_k}\}$, in order to converge, must contain either a finite number of even terms or a finite number of odd terms. There must exist some N such that $n_k \geq N$ implies that each subsequent s_{n_k} is either all odd or all even. If this is not the case, and for $n_k \geq N$ we have s_{n_k} odd (even) and $s_{n_{k+1}}$ even (odd), then $d(s_{n_k}, s_{n_{k+1}}) = \frac{1}{2}$ because $s_{2m+1} - s_{2m} = \frac{1}{2}$. This would imply that the subsequence does not converge. Thus, it is clear that $\{\frac{1}{2}, 1\}$ are the only two subsequential limits. Thus, we have that the upper bound $s^* = \sup\{\frac{1}{2}, 1\} = 1$ and the lower bound $s_* = \inf\{\frac{1}{2}, 1\} = \frac{1}{2}$.

□

5. Problem 3.20

Theorem 5.1. *Suppose* $\{p_n\}$ *is a Cauchy sequence in a metric space* X, *and some subsequence* $\{p_{n_i}\}$ *converges to a point* $p \in X$. *Then the full sequence* $\{p_n\}$ *converges to* p.

Proof. Since we know that $\{p_{n_i}\}$ converges, we know that there exits an N_0 such that $n_i \geq N_0$ implies $d(p_{n_i}, p) < \epsilon$ for all $\epsilon > 0$. Since the sequence $\{p_n\}$ is Cauchy, there exists an M such that $n \geq M$ and $m \geq M$ implies $d(p_n, p_m) < \epsilon$ for all $\epsilon > 0$. By the triangle inequality, we obtain for $n, n_i \geq \max\{N_0, M\}$ that

$$\begin{aligned} d(p_n, p) &\leq d(p_n, p_{n_i}) + d(p_{n_i}, p) \\ &< \epsilon + \epsilon = 2\epsilon \end{aligned} \tag{5.2}$$

We know that $d(p_n, p_{n_i}) < \epsilon$ by the fact that $\{p_n\}$ is Cauchy, because $p_{n_i} \in \{p_n\}$ as it p_{n_i} part of a subsequence of $\{p_n\}$. We obtain $d(p_{n_i}, p) < \epsilon$ because the subsequence $\{p_{n_i}\}$ converges to p. Thus, since ϵ is arbitrary, we obtain that the full sequence $\{p_n\}$ converges to p. □

6. Problem 3.21

Theorem 6.1. *If $\{E_n\}$ is a sequence of closed, nonempty, and bounded sets in a complete metric space X, if $E_n \supset E_{n+1}$, and if $\lim_{n\to\infty} diam E_n = 0$, then $\bigcap_1^\infty E_n$ consists of exactly one point.*

Proof. Suppose $\{p_n\}$ is any sequence with $p_n \in E_n$. Then the assumption that $\lim_{n\to\infty} \operatorname{diam} E_n = 0$ means that there exists an N such that for all $\epsilon > 0$, we have $d(\operatorname{diam} E_n, 0) < \epsilon$ for $n \geq N$. Thus, we have $\operatorname{diam} E_n < \epsilon$. Moreover, since we have $E_m \supset E_n$ for $m \geq n \geq N$, we know that $d(p_n, p_m) < \operatorname{diam} E_n < \epsilon$ for $p_n \in E_n$ and $p_m \in E_m$. This implies that $\{p_n\}$ is Cauchy, and since we have assumed that X is complete, $\{p_n\}$ must converge to some limit p in X. Moreover, since p is the limit of $\{p_n\}$, it is also a limit point of E_n. This is because for $n > N$, $d(p_n, p) < \epsilon$ which implies that there exists a point $p_n \in E_n$ for every neighborhood of p. Since we assumed each E_n is closed, we know that p must be contained in each E_n. Thus, we know that $p \in \bigcap_1^\infty E_n$.

It is clear that there are no more elements in $\bigcap_1^\infty E_n$. Assume by contradiction that there exists a $q \neq p$ such that $q \in \bigcap_1^\infty E_n$. Since $q \neq p$, we know that $d(p, q) > 0$ by definition of a metric space. Since both p and q belong to $E_1, E_2, \ldots$ we see that $\sup d(p_n, q_n) > 0$ for $p_n, q_n \in E_n$. Thus, the diameter $\operatorname{diam} E_n > 0$ for all n. Thus shows that $\lim_{n\to\infty} \operatorname{diam} E_n \neq 0$, which is a contradiction of our assumption. Thus, the set $\bigcap_1^\infty E_n$ contains exactly one element. □

7. Problem 3.23

Theorem 7.1. *Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges.*

Proof. Since $\{p_n\}$ is a Cauchy sequence, we know that there exists some N such that $d(p_n, p_m) < \epsilon$ for all $\epsilon > 0$ and $m, n \geq N$. Since $\{q_n\}$ is Cauchy, we know there exists some M such that $d(q_n, q_m) < \epsilon$ for all $\epsilon > 0$ and $m, n \geq N$. Using the triangle inequality, we find that

$$\begin{aligned} d(p_n, q_n) &\leq d(p_n, p_m) + d(p_m, q_n) \qquad (7.2)\\ &\leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) \\ &< 2\epsilon + d(p_m, q_m) \\ d(p_n, q_n) - d(p_m, q_m) &< 2\epsilon \qquad (7.3) \end{aligned}$$

This implies that $d(d(p_n, q_n), d(p_m, q_m)) = |d(p_n, q_n) - d(p_m, q_m)| < 2\epsilon$ for all $\epsilon > 0$ and $m, n \geq N$. Thus we know that $\{d(p_n, q_n)\}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, we know that $\{d(p_n, q_n)\}$ converges. □