18.100B
PROBLEM SET 4

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1. Problem 2.17

Theorem 1.1. *Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Then E is not countable.*

Proof. By contradiction, assume that E is countable. Then one can list all of the elements of E as follows:

$$\begin{gathered} 0.s_{11}s_{12}s_{13}\dots \\ 0.s_{21}s_{22}s_{23}\dots \\ 0.s_{31}s_{32}s_{33}\dots \\ \vdots \end{gathered}$$

Then one can construct a sequence $s = 0.s_1s_2s_3\dots$ such that s_k is the opposite digit of s_{kk}, where s_{kk} is the kth digit of the kth element of E. For example, if $s_{11} = 4$, then $s_1 = 7$. Thus, the sequence s is different than every sequence on the list by at least one digit. Moreover, it is a decimal expansion composed of only the digits 4 and 7, so the list does not contain all the elements of E. This is a contradiction, so E is not countable. □

Theorem 1.2. *The set E is not dense in $[0,1]$.*

Proof. We must show that $\bar{E} \neq [0,1]$. To do this, note that $0.\bar{4} = \inf E$. This is because for any $x \in E$, $0.\bar{4} \leq x$ which means $0.\bar{4}$ is a lower bound and $0.\bar{4} \in E$, so $\forall h > 0$ one can see that $0.\bar{4} + h$ is not a lower bound of E. Thus any number $p < 0.\bar{4}$ is not contained in E. Let $P = [0, 0.\bar{4})$, then for any $p \in P$, not every neighborhood $N_r(p)$ contains a $q \in E$. This is because there exists an $r = \frac{0.\bar{4}-p}{2}$ such that $N_r(p) \cap E = \emptyset$. Thus, these p are not limit points of E or elements of E. Moreover, since $p \in [0,1]$, we can see that E is not dense. □

Theorem 1.3. *The set E is compact.*

Proof. We will show that E is closed and bounded, and because $E \subset \mathbb{R}$, the Heine-Borel Theorem shows that E is compact. First, it is easy to show that E is bounded because $E \subset [0,1]$. By the archimedean principle, $\exists q, N, r \in \mathbb{R}$ such that $Nd(p,q) < r$ for all $p \in [0,1]$ because $|p - q| \in \mathbb{R}$. Thus, $d(p,q) < \frac{r}{N}$ and $\frac{r}{N} \in \mathbb{R}$. Since there always exists a number $\frac{r}{N}$ such that $d(p,q) < \frac{r}{N}$ for $p \in [0,1]$ and $q \in \mathbb{Q}$, we know that $[0,1]$ is bounded. Since $E \subset [0,1]$, we know that E is also bounded.

To show that E is closed, we must show that $E' \subset E$. Let us assume the contrary, namely that there exists an $x \in E'$ such that $x \notin E$. Since x is not in E, there is at least one digit that is not a 4 or 7. Let us denote the decimal expansion of x by $x = x_1x_2\dots x_k\dots$ where x_k is the digit which is not a 4 or 7. Now we can see that it is not true that $\forall r > 0$, $N_r(x)$ contains a $q \neq x$ such that $q \in E$. This is because the neighborhood with radius $r = \left(\frac{1}{10}\right)^{k+1}$ will never contain a $q \in E$. The closest $q \in E$ to the neighborhood with this radius would have a decimal expansion of $q_{min} = x_1x_2\dots\bar{q_k}$, where q_k is given by $q_k = 7$ if $|7 - x_k| < |4 - x_k|$ or $q_k = 4$ if $|4 - x_k| < |7 - x_k|$. Every other $q \in E$ has a further distance from any element $x \in E$, namely $d(x, q_{min}) \leq d(x,q)$. Thus, if $q_{min} \notin N_r(x)$, then no $q \in E$ belongs to $N_r(x)$.

But it is easy to see that $q_{min} \notin E$ because $|q_{min} - x| > \left(\frac{1}{10}\right)^k (|q_k - x_k|) > \left(\frac{1}{10}\right)^{k+1}$. This is because $|q_k - x_k| > 1$, as $q_k \in \{0 < p < 9 : p \in \mathbb{N}, p \neq 4, p \neq 7\}$ and $x_k \in \{4, 7\}$. Thus, E is closed. Since we have already shown E is bounded, by the Heine-Borel Theorem, we can conclude that E is compact. □

Theorem 1.4. *The set E is perfect.*

Proof. Since we know E is closed by the above theorem, we must show that any $x \in E$ is a limit point of E. First, take the decimal expansion $x = 0.x_1x_2x_3 \ldots x_n \ldots$ where each $x_1, x_2, x_3, \ldots \in \{4, 7\}$. Then let $z_k = 0.q_1q_2q_3 \ldots q_n \ldots$ such that

$$q_n = \begin{cases} x_k & \text{if } k \neq n \\ 4 & \text{if } x_k = 7 \\ 7 & \text{if } x_k = 4 \end{cases}$$

□

Thus, x is the same as z_k up to the kth digit. After the kth digit, x and z_k are completely different. It follows that for every $r > 0$, there is a neighborhood $N_r(x)$ that contains a $q \neq x$ such that $q \in E$. This is because one can always find a z_k such that $d(x, z_k) < r$. If the decimal approximation of r begins with a significant digit at the ith position, then one can always take z_{i+1}, which will be an element of E inside the neighborhood $N_r(x)$. Thus, each point x is a limit point of E, making it a perfect set.

2. Problem 2.18

Theorem 2.1. *There is a nonempty perfect set in $\mathbb{R}^1$ which contains no rational number.*

Proof. Consider the set constructed in the following way. Take the closed interval $[a, b]$ between two irrational numbers $a, b \in \mathbb{I}$. Since the rationals are countable, it is possible to order the rational numbers in the interval into a list. So assume $E = q_1, q_2, q_3, \ldots, q_n, \ldots$ is a list of all the rational numbers in the interval $[a, b]$. Now take an open interval (a_1, b_1) where $a_1, b_1 \in \mathbb{I}$ around q_1 such that $a < a_1 < q_1 < b_1 < b$. Define the set $A_1 = [a, b] \setminus (a_1, b_1)$. Keep producing these intervals around q_n and define $A_n = A_{n-1} \setminus (a_n, b_n)$. If a point q_n has already been removed, then instead move to the next A_n with the following $A_n = A_{n-1}$. In order to make sure that no endpoints are removed from a different interval being removed, we must have the conditions $q_n - a_n < \max_{i=1}^{\infty}\{|q_n - b_i|\}$ and $q_n - b_n < \max_{i=1}^{\infty}\{|q_n - a_i|\}$. Now take

$$A = \bigcap_{n=1}^{\infty} A_n \tag{2.2}$$

We know that each A_n is closed because the intervals removed were open, and their complements are closed. Moreover, we made sure that no endpoints were accidentally removed by other intervals. Moreover, each A_n is bounded because $A_n \subset [a, b]$ and $[a, b]$ is bounded by the archimedean principle (see Problem 2.17, Theorem 1.3, paragraph 2 for the identical proof which works for any interval in $\mathbb{R}^1$). Thus, we can see that A_n is compact by the Heine-Borel Theorem. Moreover, each A_n is nonempty because there will always exist the irrational endpoints a, b. Since $A_n \supset A_{n+1}$ for $n = 1, 2, 3, \ldots$ and $\{A_n\}$ is a sequence of nonempty compact sets, then $\bigcap_1^{\infty} A_n$ is not empty.

Now we must show that A is a perfect set, and it is sufficient to show that A contains no isolated point. Let $x \in A$ and let S be any segment containing x. Let I_n be the interval of A_n which contains x. Choose n large enough so that $I_n \subset S$. Then let x_n be an endpoint of I_n such that $x_n \neq x$. It follows from the construction of A that $x_n \in A$. Hence, x is a limit point of A and A is perfect. □

3. Problem 2.19

Theorem 3.1. *If A and B are disjoint closed sets in some metric space X, then they are separated.*

Proof. We must show that $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$ if $A \cap B = \emptyset$ with A and B closed. Thus, if A and B are closed, then $A = \bar{A}$ and $B = \bar{B}$. Since $A \cap B = \emptyset$, then we can also see that $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$, which shows the two sets are separated. □

Theorem 3.2. *If A and B are disjoint open sets, then they are separated.*

Proof. If A and B are disjoint, then $A \cap B = \emptyset$. Thus, we must show that $A' \cap B = \emptyset$ and $A \cap B' = \emptyset$. By contradiction, assume $A' \cap B \neq \emptyset$ without loss of generality. Then there exists some $x \in A'$ which is also an element of B. Since $x \in A'$, x is a limit point of A and for every $r > 0$, there exists an $a \in N_r(x)$ such that $a \neq x$ and $a \in A$. Since $x \in B$ and B is open, there also exists a neighborhood $N(x) \subset B$. Thus, there must be some $a \in N(x)$ such that $a \neq x$ and $a \in A$. This is a contradiction because then a would be an element of $N(x) \subset B$ and A, implying that A and B are not disjoint. □

Theorem 3.3. *Fix $p \in X$, $\delta > 0$. Define A to be the set of all $q \in X$ for which $d(p,q) < \delta$. Define B similarly, with $>$ in place of $<$. Then A and B are separated.*

Proof. We must show that $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$. Assume the contrary of the first relation, namely that $\bar{A} \cap B \neq \emptyset$. Then there exists an $x \in A \cup A'$ such that $x \in B$. If $x \in A$, then $d(p,q) < \delta$ and $d(p,q) > \delta$ because $x \in B$ as well. This is a contradiction, so $x \notin A$.

If $x \in A'$, then for all $r > 0$, there exists an $s \in N_r(x)$ such that $s \neq x$ and $s \in A$. Thus, $d(x,s) < r$ by the definition of neighborhood and $d(p,s) < \delta$ because $s \in A$. Therefore, by the triangle inequality, we know $d(p,x) \leq d(x,s) + d(p,s) < \delta + r$. Since $x \in B$, we know $d(p,x) > \delta$ so that $\delta < d(p,x) < \delta + r$ for all $r > 0$. As we take smaller and smaller values of r, then this expression becomes $\delta < d(p,x) \leq \delta$. This cannot be true because if $d(p,x) = \delta$, then $\delta \not< d(p,x)$ and $d(p,x)$ cannot be greater or less than δ either, as one part of the inequality would not hold for both cases. This is a contradiction and shows that $\bar{A} \cap B = \emptyset$.

To show that $A \cap \bar{B} = \emptyset$, we proceed along the same lines and assume the contrary: $A \cap \bar{B} \neq \emptyset$. Then there exists some $x \in B \cup B'$ such that $x \in A$. From the same argument as above, $x \notin B$. Now, if $x \in B'$, then for every $r > 0$, there exists an $s \in N_r(x)$ such that $s \neq x$ and $s \in B$. Thus, $d(x,s) < r$ by the definition of neighborhood and $d(p,x) < \delta$ since $x \in A$. By the triangle inequality, we have $d(p,s) < d(x,s) + d(p,x) < \delta + r$. Moreover, since $s \in B$, we know that $d(p,s) < \delta$. This shows that $\delta < d(p,s) < \delta + r$ for all $r > 0$. By the identical argument as above, this is a contradiction and $A \cap \bar{B} = \emptyset$. Thus, A and B are separated. □

Theorem 3.4. *Every connected metric space with at least two points is uncountable.*

Proof. If a metric space has at least two elements $a, b \in X$, then $d(a,b) > 0$. Choose some $\delta_n = nd(a,b)$ where $n \in (0,1)$. Then let $A_n = \{p \in X, d(p,a) < \delta_n\}$ and $B_n = \{p \in X, d(p,a) > \delta_n\}$. Thus, A_n and B_n are nonempty because $0 < \delta_n < d(a,b)$, so $a \in A_n$ and $b \in B_n$ for all $n \in (0,1)$. By the last theorem, we know that A_n and B_n are separated. This means that $A_n \cup B_n \neq X$ because X is connected. Since $A_n \subset X$ and $B_n \subset X$, there must be some $E_n \subset X$ such that $E_n \cap (A_n \cup B_n) = \emptyset$. The only possible set E_n is allowed to be by construction is $E_n = \{p \in X, d(p,a) = \delta_n\}$. Now let

$$E = \bigcap_{n \in (0,1)} E_n. \tag{3.5}$$

Thus it is clear that $E \subset X$. Moreover, E is uncountable because $(0,1)$ is uncountable. Hence, there does not exist a bijective mapping between E and $\mathbb{N}$. Since $E \subset X$ is uncountable, X itself is also uncountable. □

4. Problem 2.20

Theorem 4.1. *The closure of a connected set E is always connected.*

Proof. Assume the contrary, namely that $\bar{E} = A \cup B$ such that $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$. We will show that either A or B must be empty. We can express E as $E = (E \cap A) \cup (E \cap B)$ because $E \subset A \cup B$. Since A and B are separated, we know that A and B are disjoint, so that $(E \cap A)$ and $(E \cap B)$ are disjoint as well. Since E is connected, we must have that $E \cap A = \emptyset$ or $E \cap B = \emptyset$. Assume without loss of generality that $E \cap A = \emptyset$, then $E \subset B$ because $E = E \cap B$. This implies that $\bar{E} \subset \bar{B}$ and further that $\bar{E} = A \cup B \subset \bar{B}$. However, since $A \cap \bar{B} = \emptyset$, we can find that

$$A = A \cap (A \cup B) \subset A \cap \bar{B} = \emptyset \tag{4.2}$$

which shows that $\bar{E}$ is in fact connected. This completes the proof. □

Theorem 4.3. *Not all interiors of connected sets are connected.*

Proof. Consider the two closed balls $A := \{(x, y) \in \mathbb{R}^2 : d((x, y), (-1, 0)) \leq 1\}$ and $B := \{(x, y) \in \mathbb{R}^2 : d((x, y), (1, 0)) \leq 1\}$. Then $E = A \cup B$ is a connected set, but its interior will be given by the union of the two open balls $A^\circ \cup B^\circ$. This is not connected because $A \cap B^\circ = \emptyset$ and $A^\circ \cap B = \emptyset$. □

5. Problem 2.29

Theorem 5.1. *Every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments.*

Proof. We know that $\mathbb{R}^1$ is separable and contains a countable dense subset E by the theorem proven in exercise 2.22. Then if $x \in \mathbb{R}^1$, x must be either a limit point of E or an element of E. Now take an open interval $(a, b) \subset \mathbb{R}^1$. For every $x \in (a, b)$, we must have either $x \in E$ and/or $x \in E'$. Define $A := \{x \in (a, b), x \in E\}$ and $B := \{x \in (a, b), x \in E', x \notin E\}$, then $A \cup B = (a, b)$. It is clear that A and B are disjoint by their construction. Moreover, they are at most countable because $A \subset E$ and $B \subset E$, and E is countable. If A and B are infinite, then they are an infinite subset of a countable set, which makes them countable. If A and B are finite, then they are at most countable by the definition.

Now it remains to show that we have either 1) $A \neq \emptyset$ and $B \neq \emptyset$ or 2) $B = \emptyset$ and A is a union of disjoint segments (which is still at most countable from the above argument). If we can show that only these two cases occur, then the proof will be completed. Assume by contradiction that $A = \emptyset$, then $E \cap (a, b) = \emptyset$. Thus, we must have $(a, b) = B$ and so $(a, b) \subset E'$. Then for every $x \in (a, b)$ and every $r > 0$, there exists a neighborhood $N_r(x)$ containing a $q \neq x$ such that $q \in E$. Moreover, since (a, b) is open, there exists an $N(x) \subset (a, b)$. Thus, some $q \in N(x)$ is also $q \in (a, b)$. This implies that $E \cap (a, b) \neq \emptyset$ because $q \in E$. This is a contradiction because we assumed $A = \emptyset$ and thus that $E \cap (a, b) = \emptyset$. Thus, we must have $A \neq \emptyset$.

Now we will show that if $B = \emptyset$, then A is a union of disjoint segments. If $B = \emptyset$, then $(a, b) = A$. Moreover, for every $x \in (a, b)$, we have $x \notin E'$ even though $x \in E$, implying that x is an isolated point of E. Thus, for some neighborhood $N_\delta(x)$, there does not exist a $q \in N_\delta(x)$ such that $q \neq x$ and $q \in E$. In other words, for some $\delta > 0$, we have $d(x, q) > \delta$ for every $q \neq x$ and $q \in E$. Thus, one can make nonempty partitions $C := \{q \in A : d(x, q) < \delta\}$ and $D := \{q \in A : d(x, q) > \delta\}$. By the theorem proven in problem 2.19(c), we know that C and D are separated and hence disjoint. Moreover, $(C \cup D) \cup \{x\} = A$ is a union of disjoint segments. Thus, we have shown that if B is empty, the open set (a, b) is still the union of an at most countable collection of disjoint segments. □