18.100B
PROBLEM SET 11

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1. Problem 7.2

Theorem 1.1. *If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E.*

Proof. Since $\{f_n\}$ converges uniformly to a limit, say f, then we see that for every $\epsilon > 0$, there exists an N_1 such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N_1$ and $x \in E$. The same is true for $\{g_n\}$, namely that for every $\epsilon > 0$, there exists an N_2 such that $|g_n(x) - f(x)| < \epsilon$ for all $n \geq N_2$ and $x \in E$. Thus, there exists an $N = \max\{N_1, N_2\}$ such that for every $n \geq N$ and every $\epsilon > 0$, we obtain:

$$|f_n(x) - f(x) + g_n(x) - g(x)| = |(f_n(x) + g_n(x)) - (f(x) + g(x))| < 2\epsilon \tag{1.2}$$

Therefore, since ϵ was arbitrary, we see that $\{f_n + g_n\}$ converges uniformly on E. □

Theorem 1.3. *If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.*

Proof. Fix $\epsilon > 0$. Again, we know that there exists an N_1 such that for $n > N_1$, we obtain $|f_n(x) - f(x)| < \epsilon$ for all $x \in E$. There also exists an N_2 such that for $n > N_2$ we obtain $|g_n(x) - g(x)| < \epsilon$. Both these statements are true because $\{f_n\} \to f$ and $\{g_n\} \to g$. Next, we will prove a lemma:

Lemma 1.4. *Every uniformly convergent sequence of bounded functions $\{h_n\}$ is uniformly bounded.*

Proof. A sequence of bounded functions means that for every n, we have $|h_n(x)| < M_n$. Now pick N so that for all $n > N$, $|h_n(x) - h(x)| < 1$ for all $x \in E$. Therefore, we see:

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x) - f_N(x)| + |f_N(x)| < 2 + M_N \tag{1.5}$$

Now pick $M = \max\{M_1, \ldots, M_{N-1}, 2 + M_N\}$. It is clear that $|f_n(x)| < M$ for all $n \in \mathbb{N}$. Therefore, we have shown that $\{h_n\}$ is uniformly bounded. □

Now, we can use the above lemma and say that $|f_n(x)| < M$ and $|g_n(x)| < L$. Therefore, we see that for $n > N_2$, we have

$$|g(x)| \leq |g(x) - g_n(x)| + |g_n(x)| < \epsilon + L \tag{1.6}$$

Thus, using the triangle inequality on $|f_n(x)g_n(x) - f(x)g(x)|$, we can obtain for $n > \max\{N_1, N_2\}$:

$$\begin{aligned}
|f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)| && (1.7)\\
&= |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| && (1.8)\\
&< M\epsilon + (\epsilon + L)\epsilon && (1.9)
\end{aligned}$$

Since ϵ was arbitrary, we see that $\{f_n g_n\}$ converges uniformly. □

2. Problem 7.3

Theorem 2.1. *Construct sequences $\{f_n\}, \{g_n\}$ which converge uniformly in some set E but such that $\{f_n g_n\}$ does not converge uniformly on E.*

Proof. Consider the two sequences $f_n(x) = x + \frac{1}{n}$ and $g_n(x) = x + \frac{1}{n}$ for $x \in \mathbb{R}$. Then we see that $f_n(x) \to x$ and $g_n(x) \to x$ as $n \to \infty$. It is easy to see that $x + \frac{1}{n}$ converges uniformly because $|x + \frac{1}{n} - x| = |\frac{1}{n}|$. Therefore, we use the Archimedean principle to pick an N such that for all $n > N$, $\frac{1}{n} < \epsilon$. Therefore, both $\{f_n\}$ and $\{g_n\}$ converge uniformly.

However, we see that $\{f_n g_n\}$ does not converge uniformly, even though it converges pointwise to x^2. We have $f_n(x)g_n(x) = (x + \frac{1}{n})^2$ for $x \in \mathbb{R}$. Next, we can see the following:

$$\left|\left(x+\frac{1}{n}\right)^2 - x^2\right| = \left|x^2 + \frac{2x}{n} + \frac{1}{n^2} - x^2\right| \tag{2.2}$$

$$= \left|\frac{2nx+1}{n^2}\right| \tag{2.3}$$

However, we can pick $n = N$ and set $\epsilon = 1$. Since we have $x \in \mathbb{R}$, we can choose $x = N$, which gives $|\frac{2N^2+1}{N^2}| = |2 + \frac{1}{N^2}| > \epsilon = 1$. Therefore, we see that $\{f_n g_n\}$ does not converge uniformly. $\square$

3. Problem 7.4

Theorem 3.1. *Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$. For what values x does the series converge absolutely?*

Proof. The series diverges for $x = 0$, simply because the sequence of partial sums of 1 does not converge to 0. Also, the series is not defined for $x = -\frac{1}{n^2}$, so it does not converge absolutely. However, for all $x \in \mathbb{R}$ other than the ones mentioned above, the series converges absolutely. We can use comparison test to show the following:

$$\sum_{n=1}^{\infty}\left|\frac{1}{1+n^2x}\right| \le \sum_{n=1}^{\infty}\left|\frac{1}{n^2x}\right| = |x|\sum_{n=1}^{\infty}\frac{1}{n^2} \tag{3.2}$$

And since $\sum \frac{1}{n^2}$ converges by being a geometric series with $p = 2$, we see that the series on the left converges by comparison test. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ converges absolutely for all x other than $x = 0$ and $x = -\frac{1}{n^2}$. $\square$

Theorem 3.3. *The series converges uniformly for all intervals $[a, b] \in E$ such that a, b are the same sign and there does not exist a number $-\frac{1}{n^2}$ in the interval.*

Proof. If the assumptions are satisfied, then we see that $\frac{1}{1+n^2x}$ is either monotonically increasing or monotonically decreasing, depending on the sign of x. Therefore, we have either $|\frac{1}{1+n^2x}| \le |\frac{1}{1+n^2a}|$ or we have $\frac{1}{1+n^2x}| \le |\frac{1}{1+n^2b}|$. Since all of the terms well defined (by our assumption that there do not exist terms of the form $-\frac{1}{n^2}$), we see that $|f_n(x)| \le M_n$ for all $x \in E$, where $M_n = |\frac{1}{1+n^2a}|$ or $M_n = |\frac{1}{1+n^2b}|$. Since we know that $\sum M_n$ converges for both M_n, we know that $\sum f_n$ also converges by a theorem in Rudin. $\square$

Theorem 3.4. *f is continuous wherever the series converges.*

Proof. We see that $f_n(x) = \frac{1}{1+n^2x}$ is continuous wherever $f(x)$ is defined. Since this corresponds to the intervals where $f(x)$ is uniformly convergent, we see that $f_n(x)$ is continuous on E, where E is the set of x for which $f(x)$ is uniformly convergent. Therefore, by a theorem in Rudin, since $\{f_n\}$ is a sequence of continuous functions on E, and $f_n \to f$ uniformly on E, then we know that f is continuous on E. $\square$

Theorem 3.5. *f is not bounded.*

Proof. Suppose by contradiction that f is bounded by some number M so that $|f(x)| < \frac{M}{2}$ for all $x \in E$. Then we can choose $x = \frac{1}{M^2}$ and see that

$$f\left(\frac{1}{M^2}\right) = \sum_{n=1}^{\infty}\frac{1}{1+\frac{n^2}{M^2}} \tag{3.6}$$

$$\ge \frac{1}{1+\frac{1}{M^2}} + \frac{1}{1+\frac{2^2}{M^2}} + \ldots + \frac{1}{1+\frac{M^2}{M^2}} \tag{3.7}$$

$$\ge \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} \tag{3.8}$$

$$= \frac{M}{2} \tag{3.9}$$

Thus, we have found a number x for which $|f(x)| \ge \frac{M}{2}$ which is a contradiction. Therefore, f is not bounded. $\square$

4. Problem 7.6

Theorem 4.1. *Prove that the series $\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$ converges uniformly in every bounded interval.*

Proof. We need to show that the sequence $\{s_i\}$ of partial sums converges uniformly on every closed interval $x \in [a,b]$. So let $s_i = \sum_{n=1}^{i}(-1)^n \frac{x^2+n}{n^2}$ and fix $\epsilon > 0$. Now we want to show that there exists some N such that for $i,j > N$ we have $|s_i(x) - s_j(x)| < \epsilon$ for all $x \in [a,b]$. Indeed, expanding this out, and assuming without loss of generality that $i > j > N$, we obtain the following:

$$|s_i(x) - s_j(x)| = \left|\sum_{n=1}^{i}(-1)^n \frac{x^2+n}{n^2} - \sum_{n=1}^{j}(-1)^n \frac{x^2+n}{n^2}\right| \tag{4.2}$$

$$= \left|\sum_{n=j}^{i}(-1)^n \frac{x^2+n}{n^2}\right| \tag{4.3}$$

$$= \left|\sum_{n=j}^{i}(-1)^n \frac{x^2}{n^2}\right| + \left|\sum_{n=j}^{i}(-1)^n \frac{1}{n}\right| \tag{4.4}$$

Clearly, the series on the right converges uniformly on x because it does not depend on x and it also is an alternating series that converges. The series on the left converges uniformly on some interval $[a,b]$ because we can let $M = \max\{a,b\}$ and get:

$$\left|(-1)^n \frac{x^2}{n^2}\right| \leq \frac{M^2}{n^2} \tag{4.5}$$

$$\left|\sum_{n=j}^{i}(-1)^n \frac{x^2}{n^2}\right| \leq \sum_{n=j}^{i}\left|(-1)^n \frac{x^2}{n^2}\right| \leq \sum_{n=j}^{i}\frac{M^2}{n^2} \tag{4.6}$$

Since the series $\sum \frac{M^2}{n^2}$ converges by begin a geometric series with $p = 2$, we see that $\sum(-1)^n \frac{x^2}{n^2}$ also converges by a theorem in Rudin. Therefore, $\{s_i\}$ is the sum of two convergent series which, by problem 7.2, shows that s_i converges uniformly and thus that the series $\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$ converges uniformly on every bounded interval $[a,b]$. ☐

Theorem 4.7. *The series $\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$ does not converge absolutely for any value of x.*

Proof. We must show that $\sum_{n=1}^{\infty}\left|(-1)^n \frac{x^2+n}{n^2}\right| = \sum_{n=1}^{\infty}\left|\frac{x^2+n}{n^2}\right|$ does not converge for any x. Indeed, we see that the following is true:

$$\sum_{n=1}^{\infty}\left|(-1)^n \frac{x^2+n}{n^2}\right| = \sum_{n=1}^{\infty}\left|\frac{x^2+n}{n^2}\right| \tag{4.8}$$

$$= \sum_{n=1}^{\infty}\left|\frac{x^2}{n^2}\right| + \sum_{n=1}^{\infty}\frac{1}{n} \tag{4.9}$$

Since the series on the right diverges by begin a geometric series with $p = 1$, we can only hope for convergence if the series on the left is negative. However, we see that it will never be negative, so that the entire series diverges. Therefore, $\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$ does not converge absolutely. ☐

5. Problem 7.7

Theorem 5.1. *For $n \in \mathbb{N}$ and $x \in \mathbb{R}$, put $f_n(x) = \frac{x}{1+nx^2}$. Then $\{f_n\}$ converges uniformly to a function f.*

Proof. Fix $\epsilon > 0$ and $x \in \mathbb{R}$. Using the Cauchy criterion, all we must do is show that there exists an N such that $|f_n(x) - f_m(x)| < \epsilon$ for $n,m > N$. Suppose $n > m$ without loss of generality. Now, we see that the following is true:

$$|f_n(x) - f_m(x)| = \left| \frac{x}{1+nx^2} - \frac{x}{1+mx^2} \right| \tag{5.2}$$

$$= \frac{x(1+mx^2) - x(1+nx^2)}{(1+nx^2)(1+mx^2)} \tag{5.3}$$

$$= \frac{x^3(m-n)}{1+mx^2+nx^2+nmx^4} \tag{5.4}$$

$$\leq \frac{x^3(m-n)}{nmx^4} \tag{5.5}$$

$$= \frac{1}{n} - \frac{1}{m} \tag{5.6}$$

Using the Archimedean principle, it is clear that we can select N large enough with $n, m > N$ such that $\frac{1}{n} - \frac{1}{m} < \epsilon$. Therefore, we see that the series converges uniformly by the Cauchy criterion for all $x \in \mathbb{R}$. ☐

Theorem 5.7. *The equation $f'(x) = \lim_{n\to\infty} f'_n(x)$ is correct if $x \neq 0$ but false for $x = 0$.*

Proof. We must show that $\{f'_n\}$ converges uniformly for all $x \neq 0$. Then we can apply a theorem in Rudin because we know that $\{f\}$ converges uniformly, and thus it converges pointwise at all $x_0 \in \mathbb{R} \setminus 0$. Therefore, let us examine $\{f'_n\}$ using the quotient rule:

$$f'_n(x) = \frac{d}{dx} \frac{x}{1+nx^2} \tag{5.8}$$

$$= \frac{1+nx^2 - x(2nx)}{(1+nx)^2} \tag{5.9}$$

$$= \frac{1-nx^2}{(1+nx)^2} \tag{5.10}$$

Now we must show that $\{f'_n\}$ converges uniformly for all $x \neq 0$. So, pick $x \in \mathbb{R} \setminus 0$ and fix $\epsilon > 0$. We can use Cauchy criterion and obtain for $n > m$:

$$|f'_n(x) - f'_m(x)| = \left| \frac{1-nx^2}{(1+nx)^2} - \frac{1-mx^2}{(1+mx)^2} \right| \tag{5.11}$$

$$\leq \left| \frac{(1-nx^2)(1+mx)^2 - (1-mx^2)(1+nx)^2}{m^2n^2x^4} \right| \tag{5.12}$$

$$= \left| \frac{2x(m-n) + x^2(m^2-n^2)}{m^2n^2x^4} \right| \tag{5.13}$$

$$= \left| \frac{2}{mn^2x^3} - \frac{2}{m^2nx^3} \right| + \left| \frac{1}{n^2x^2} - \frac{1}{m^2x^2} \right| \tag{5.14}$$

Thus, we can choose an N with $n, m > N$ so that $|f'_n(x) - f'_m(x)| < \epsilon$. To see this, we note that the term on the right can be made arbitrarily small using the archimedean principle, say to less than $\epsilon/2$. Next, the term on the left can be made arbitrarily small as well. We see that $\frac{2}{x^3}$ is divided either mn^2 or m^2n, and even if x is negative, we can choose m, n large enough so that the term becomes arbirarily small using the archimedean principle. Therefore, $|f'_n(x) - f'_m(x)| < \epsilon$ for $x \in \mathbb{R} \setminus 0$ implying uniform convergence on the same set.

This means we can apply the theorem in Rudin which states that for $\{f_n\}$ differentiable on $[a, b]$, if $f_n(x_0)$ converges for some point $x_0 \in [a, b]$ and if $\{f'_n\}$ converges unfiormly on $[a, b]$, then $f'(x) = \lim_{n\to\infty} f'_n(x)$. Thus, the first part of the problem is completed. We are left to show that this is false for $x = 0$.

This can be easily seen because $f'_n(0) = 1$ for all $n \in \mathbb{N}$. However, first we will show that $f(0) = 0$. It is clear that $f_n(0) = 0$. Therefore, we have $|f_n(0) - f(0)| = |0 - 0| = 0$. Moreover, $0 < \epsilon$ for all $\epsilon > 0$. Therefore, we see that $f(0) = 0$ by the uniform convergence we proved earlier. Moreover, $f'(0) = 0$. However, as we have seen, $f'_n(0) = 1$ for all $n \in \mathbb{N}$, which shows that $\lim_{n\to\infty} f'_n(0) = 1 \neq 0 = f'(0)$. ☐

6. Problem 7.10

Theorem 6.1. *Letting (x) denote the fractional part of the real number x, consider the function $f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}$ for $x \in \mathbb{R}$. Find all the discontinuities of f and show that they form a countable dense set.*

Proof. First, we will show that f converges uniformly on $\mathbb{R}$. Let $f_k(x) = \sum_{n=1}^{k} \frac{(nx)}{n^2}$ be the partial sum of $f(x)$. We must show that the sequence $\{f_k\}$ converges uniformly for all $x \in \mathbb{R}$. We observe that $(nx) < 1$ for all numbers $nx \in \mathbb{R}$. Therefore, we have the following:

$$\left|\frac{(nx)}{n^2}\right| \leq \frac{1}{n^2} \tag{6.2}$$

Since we know that $\sum \frac{1}{n^2}$ converges by being geometric with $p = 2$, we see that by a theorem in Rudin, $f(x)$ converges uniformly for all $x \in \mathbb{R}$.

Next, we will note that $g(x) = (x)$ is discontinuous for all $x \in \mathbb{Z}$. Now, let $g_n(x) = (nx)$. We see that $g_n(x)$ is discontinuous for all $nx \in \mathbb{Z}$. In other words, $g_n(x)$ is discontinuous for $x = \frac{m}{n}$ where $m, n \in \mathbb{Z}$. This means that $g_n(x)$ is discontinuous for all $x \in \mathbb{Q}$. Now, we will show that $f(x)$ is discontinuous for $x \in \mathbb{Q}$. If $x \in \mathbb{Q}$, we see the following:

$$f_k(x) = \sum_{n=1}^{k} \frac{(nx)}{n^2} = \sum_{n=1}^{k} \frac{g_n(x)}{n^2} \tag{6.3}$$

Moreover, we see that $\lim_{t\to x^-} g_n(t) = 1$ and $\lim_{t\to x^+} g_n(t) = 0$. This holds for all x and n, so that:

$$\lim_{t\to x^-} g_n(t) \geq \lim_{t\to x^+} g_n(t) \tag{6.4}$$

Thus, we can take the limit that of $f_k(t)$ as $t \to x^-$ and $t \to x^+$:

$$\lim_{t\to x^+} f_k(t) = \lim_{t\to x^+} \sum_{n=1}^{k} \frac{g_n(t)}{n^2} = \sum_{n=1}^{k} \lim_{t\to x^+} g_n(t) \frac{1}{n^2} = 0 \tag{6.5}$$

$$\lim_{t\to x^-} f_k(t) = \lim_{t\to x^-} \sum_{n=1}^{k} \frac{g_n(t)}{n^2} = \sum_{n=1}^{k} \lim_{t\to x^-} g_n(t) \frac{1}{n^2} = \sum_{n=1}^{k} \frac{1}{n^2} \tag{6.6}$$

Since we know that $f_k(x) \to f(x)$ uniformly, a theorem in Rudin says that we can swap limits in the following way:

$$\lim_{k\to\infty} \lim_{t\to x^+} f_k(t) = \lim_{t\to x^+} \lim_{k\to\infty} f_k(t) = \lim_{t\to x^+} f(t) \tag{6.7}$$

$$\lim_{k\to\infty} \lim_{t\to x^-} f_k(t) = \lim_{t\to x^-} \lim_{k\to\infty} f_k(t) = \lim_{t\to x^-} f(t) \tag{6.8}$$

Moreover, we already know the expressions for the term on the left:

$$\lim_{k\to\infty} \lim_{t\to x^+} f_k(t) = \lim_{k\to\infty} 0 = 0 \tag{6.9}$$

$$\lim_{k\to\infty} \lim_{t\to x^-} f_k(t) = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{6.10}$$

Thus, we have obtained expressions for the left and right limits of the function f at $x \in \mathbb{Q}$:

$$\lim_{t\to x^+} f(t) = 0 \qquad \lim_{t\to x^-} f(t) = \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{6.11}$$

Therefore, we see that the left and right limits of $f(x)$ are not equal, so that the function is discontinuous at $x \in \mathbb{Q}$. Now we need only show that $f(x)$ is continuous at all $x \notin \mathbb{Q}$. Well, we know by problem 4.16 in a previous problem set that (x) is continuous at all $x \notin \mathbb{N}$. Therefore, we see that $f_k(x)$ is continuous at all $x \notin \mathbb{Q}$. Since $f_k(x) \to f(x)$ uniformly, we see that $f(x)$ is continuous for all $x \notin \mathbb{Q}$ by a theorem in Rudin. Therefore, we have shown that the only points of discontinuity are $x \in \mathbb{Q}$.

We know that $\mathbb{Q} \subset \mathbb{R}$ is a countable dense subset of $\mathbb{R}$. Therefore, we have shown that the points of discontinuities of $f(x)$ are a countable dense set, which completes the proof. □

Theorem 6.12. *Show that f is nevertheless Riemann-integrable on every bounded interval $[a, b]$.*

Proof. We know that on any bounded interval $[a, b]$, we have only finitely many discontinuity points. In fact, we will have $n(b - a) + 1$ number of discontinuity points. Since $\alpha = x$ is continuous at every point in $[a, b]$, we see that $\alpha = x$ is continuous at every point for which $f_k(x) = \sum_{n=1}^{k} \frac{(nx)}{n^2}$ is discontinuous. Therefore, we can apply a theorem in Rudin and see that $f_k \in \mathscr{R}$.

Next, since we know that $f_k \to f$ uniformly and that $f_k \in \mathscr{R}$ on $[a, b]$, we also know that $f \in \mathscr{R}$ on $[a, b]$ by a theorem in Rudin. This completes the proof. □

7. Problem 7.12

Theorem 7.1. *Suppose g and f_n for $n \in \mathbb{N}$ are defined on $(0,\infty)$ and are Riemann-integrable on $[t,T]$ whenever $0 < t < T < \infty$, $|f_n| \leq g_n$, $f_n \to f$ uniformly on every compact subset of $(0,\infty)$, and $\int_0^\infty g(x)dx < \infty$. Prove that $\lim_{n\to\infty} \int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx$.*

Proof. First, we will show that f is integrable on $[0,\infty)$. We do this by noting that $f_n \to f$ uniformly and each $f_n \in \mathscr{R}$, which implies by a theorem in Rudin that $f \in \mathscr{R}$. Moreover, we can show that $\int_0^\infty$ is finite because we know that each $|f_n| \leq g$. Therefore, since f_n is uniformly convergent, which implies pointwise convergence, we see that $|f(x)| \leq g(x)$ for all $x \in [0,\infty)$. Thus, for $n > m \in [0,\infty)$, we must have $|\int_m^n f(x)dx| \leq \int_n^m |f(x)|dx \leq \int_n^m g(x)dx$.

Since we have $\int_0^\infty g(x)dx < \infty$, we know that there exists an J such that for all $j > J$, we have $\int_j^\infty g(x)dx < \epsilon$. To see why this is the case, we can assume the contrary. Then $\int_c^\infty g(x) > \epsilon$ for all $c \in [0,\infty)$. Thus, we would have:

$$\lim_{d\to\infty}\int_0^d g(x)dx = \int_0^c g(x)dx + \lim_{d\to\infty}\int_c^d g(x)dx \tag{7.2}$$

$$\leq \int_0^c g(x)dx + \lim_{d\to\infty}(d-c)\epsilon \tag{7.3}$$

Since the integral term on the left is finite for a finite c, and the term on the right diverges, this would imply that $\int_0^\infty g(x)dx \not< \infty$, which is a contradiction of our assumption. Hence, there must exist a J such that for $j > J$, we have $\int_j^\infty g(x)dx < \epsilon$.

Moreover, since $f_n \to f$ uniformly, we can choose an N such that for all $n > N$ and all $x \in [0,\infty)$, we have $|f_n(x) - f(x)| < \epsilon$. Therefore, we obtain the following:

$$\left|\int_0^\infty f_n(x) - \int_0^\infty f(x)\right| = \int_0^j |f_n(x) - f(x)|dx + \int_j^\infty |f_n(x) - f(x)|dx \tag{7.4}$$

$$\leq \int_0^j |f_n(x) - f(x)|dx + \int_j^\infty 2g(x)dx \tag{7.5}$$

$$\leq \epsilon(j-0) + 2\epsilon \tag{7.6}$$

$$\leq \epsilon(j+2) \tag{7.7}$$

Since $\epsilon > 0$ was arbitrary and j is a constant, we see that $\int_0^\infty f_n(x) \to \int_0^\infty f(x)$ as $n \to \infty$. □

8. Problem 7.14

Theorem 8.1. *Let f be a continuous real function on $\mathbb{R}^1$ with the following properties: $0 \leq f(t) \leq 1$, $f(t+2) = f(t)$ for every t, and*

$$f(t) = \begin{cases} 0 & (0 \leq t \leq \frac{1}{3}) \\ 1 & (\frac{2}{3} \leq t \leq 1) \end{cases} \tag{8.2}$$

Put $\Phi(t) = (x(t), y(t))$ where $x(t) = \sum_{n=1}^\infty 2^{-n} f(3^{2n-1}t)$ and $y(t) = \sum_{n=1}^\infty 2^{-n} f(3^{2n}t)$. Prove that Φ is continuous and that Φ maps $I = [0,1]$ onto the unit square $I^2 \subset \mathbb{R}^2$. In fact, show that Φ maps the Cantor set onto I^2.

Proof. First, we will show that Φ is continuous. To do this, it is enough to show that $x(t)$ and $y(t)$ are continuous. First, we know that f is a continuous function by assumption of the real line. Moreover, we see that $x_i(t)$ and $y_i(t)$ are bounded:

$$x_i(t) = \sum_{n=1}^{i} 2^{-n} f(3^{2n-1}t) \leq \sum_{n=1}^{i} |2^{-n} f(3^{2n-1}t)| \leq \sum_{n=1}^{i} 2^{-n} \tag{8.3}$$

$$y_i(t) = \sum_{n=1}^{i} 2^{-n} f(3^{2n}t) \leq \sum_{n=1}^{i} |2^{-n} f(3^{2n}t)| \leq \sum_{n=1}^{i} 2^{-n} \tag{8.4}$$

Since $\sum 2^{-n}$ converges, we see that $x_i \to x$ and $y_i \to y$ uniformly. Moreover, since each x_i and y_i is continuous, as it is a sum of multiples of a continuous function f, we see that x and y are continuous due to uniform convergence. Thus, we have shown that Φ is also continuous.

Now we must show that Φ maps the Cantor set onto I^2. It is clear that we must have each $(x_0, y_0) \in I^2$ of the form

$$x_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1}, \qquad y_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n} \tag{8.5}$$

Where each a_i is either 0 or 1. It is clear then that $t_0 = \sum_{i=1}^{\infty} 3^{-i-1}(2a_i)$ converges, since it is a geometric series, and moreover, it converges to a number in the range $[0,1]$ since a_i can be either 0 or 1. Therefore, we can compute $3^k t_0$ in the following manner:

$$\begin{aligned} 3^k t_0 &= \sum_{i=1}^{\infty} 3^{k-i-1}(2a_i) && (8.6)\\ &= 2\sum_{i=1}^{k-1} 3^{k-1-i}(a_i) + \sum_{j=0}^{\infty} 3^{-j-1}(2a_{k+j}) && (8.7)\\ &= 2N + \sum_{j=0}^{\infty} 3^{-j-1}(2a_{k+j}) && (8.8)\end{aligned}$$

Here, N is an integer. Since $f(x+2) = f(x)$, we see that $f(2N+x) = f(x)$ so that we obtain the following expression:

$$f(3^k t_0) = \sum_{j=0}^{\infty} 3^{-j-1}(2a_{k+j}) \tag{8.9}$$

Now there are two options for a_k. We can either have $a_k = 0$, in which case we see that the first term with $j = 0$ is 0, so we get:

$$\sum_{j=0}^{\infty} 3^{-j-1}(2a_{j+k}) = \sum_{j=1}^{\infty} 3^{-j-1} 2a_{j+k} \tag{8.10}$$

We can obtain a lower bound by assuming $a_i = 0$ for all $i > k$ and an upper bound by assuming $a_i = 1$ for all $i > k$. We see that the first series converges to 0. The second series converges as follows:

$$\begin{aligned} \sum_{j=1}^{\infty} 3^{-j-1}(2a_{j+k}) &= \frac{2}{3}\sum_{j=1}^{\infty} \frac{1}{3^j} && (8.11)\\ &= \frac{2}{3}\frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{3} && (8.12)\end{aligned}$$

Therefore, when $a_k = 0$, we see:

$$0 \le \sum_{j=0}^{\infty} 3^{-j-1}(2a_{j+k}) \le \frac{1}{3} \qquad \Rightarrow \qquad f(3^k t_0) = 0 = a_k \tag{8.13}$$

We can perform similar bounds for when $a_k = 1$. There, we see that the first term when $j = 0$ is equal to $\frac{2}{3}$. Since we have already found the bounds for $\sum_{j=1}^{\infty} 3^{-j-1} 2a_{j+k}$, we can just add them to $\frac{2}{3}$. Thus, we see that when $a_k = 1$, we have:

$$\frac{2}{3} \le \sum_{j=0}^{\infty} 3^{-j-1}(2a_{j+k}) \le 1 \qquad \Rightarrow \qquad f(3^k t_0) = 1 = a_k \tag{8.14}$$

By the definition of $x(t)$ and $y(t)$, we see that $\Phi(t_0) = (x_0, y_0)$, which implies that Φ is surjective. Moreover, the points t_0 are clearly the points appearing in the Cantor set. Thus, we have completed the proof. $\square$