

Final Project Parameter Values

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1 Parameter Interpretations

It is evident that K_0 represents the carrying capacity of the scallops, and R_0 represents the logistic growth rate of the population. The conversion factors C_1 and C_2 must have dimensions [scallops/rays] and [sharks/rays] in order for them to nondimensionalize y and z . These parameters represent the carrying capacity of rays and sharks in terms of K_0 . One can also see that D_1 is the death rate of a rays and D_2 is the death rate of sharks.

Analyzing the functions F_i show that they must have dimensions of [1/time]. This means that A_i has units of [1/time] and B_i has units of population. A_i represents the frequency of predation in a unit of time, while B_i represents a close to average population size.

2 Parameter Values

In light of these observations, we can attempt to fit these parameter values to our problem. First, we will only consider the region where scallops and cownose rays are prominent, which is on the eastern coast of the United States. We will pick the area where scallops usually inhabit, which roughly corresponds to the coastline between North Carolina and Cape Cod. This is about 800 miles of coastline, and scallops inhabit up to about 3 miles offshore [3]. Scallop densities in the late 1970s and early 1980s were about 20 scallops per square meter, which is historically close to the highest density reached [3]. This means that we have a carrying capacity of roughly $K_0 = 1 \times 10^{11}$ scallops in our region. The average density of scallops now has dropped dramatically and is probably closer to 5 scallops per square meter, which leads to roughly $B_1 = 2 \times 10^{10}$.

Next, we note that cownose rays consume about 1.5% of their body weight each day according to [4] who determines consumption based on VO_2 respiration. The average cownose ray weighs about 10kg while the average scallop weighs about 0.02kg. Thus, if we assume the diet of a cownose rays consists of about 50% scallops, the average ray consumes about 1300 scallops per year. We have parameter values of roughly $A_1 = 0.5$ and $C_1 = 1300$.

The average density of cownose rays in Chesapeake Bay is roughly 0.001 rays per square meter [1]. We can estimate C_1 from this data as well. Assuming we have roughly a density of 2.5 scallops per square meter (since scallop densities dropped off dramatically since [3] performed their measurements), we see that there are about $C_1 = 2.5/0.001 = 2500$ scallops per ray. Taking the average of the two C_1 values we obtained, we will use $C_1 = 1900$. We can also obtain the parameter for the average population of rays $B_2 = 5 \times 10^6$ using this estimate.

Hammerhead sharks consume about 2% of their body weight each day [2]. Since hammerheads weigh about 150kg, and cownose rays are about 10kg, we see that hammerheads consume about 50 cownose rays per year if we assume cownose rays account for about 50% of the hammerhead diet. Thus, we find the parameter values of $A_2 = 0.5$ and $C_2 = 50$.

Death rates can be obtained by looking at average lifetimes and assuming uniform distributions of age. Cownose rays live for about 15 years, so we obtain $D_1 = 0.07$, while hammerheads live about 25 years, so $D_2 = 0.04$. We assume a growth rate of $R_0 = 20$.

Table 1: Parameter Value Estimates

Parameter	Dimension	Estimated Value	References
Dimensionless Conversion Parameters			
C_1	[scallops/rays]	1900	[4] and [1]
C_2	[sharks/rays]	50	[2]
F_1 and F_2 Function Parameters			
A_1	[1/time]	0.5	
A_2	[1/time]	0.5	
B_1	[scallops]	2×10^{10}	[3]
B_2	[rays]	5×10^6	[1]
Scallop Population Parameters			
K_0	[scallops]	1×10^{11}	[3]
R_0	[1/time]	20	
Death Rates			
D_1	[1/time]	0.07	
D_2	[1/time]	0.04	

3 Dimensionless Parameters

Since we have estimates for the parameters in our original equations, we can find the nondimensionalized parameters. We can express the nondimensionalized equations in terms of the dimensionalized coefficients and find:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{dX}{dT} \frac{1}{K_0 R_0} = \frac{1}{K_0 R_0} \left(R_0 X \left(1 - \frac{X}{K_0} \right) - C_1 \frac{A_1 X}{B_1 + X} Y \right) \\
&= x(1 - x) - \frac{\frac{A_1 K_0}{R_0 B_1} x}{1 + \frac{K_0}{B_1} x} y
\end{aligned} \tag{1}$$

$$\begin{aligned}
\frac{dy}{dt} &= \frac{dY}{dT} \frac{C_1}{R_0 K_0} = \frac{C_1}{R_0 K_0} \left(\frac{A_1 X}{B_1 + X} Y - \frac{A_2 Y}{B_2 + Y} Z - D_1 Y \right) \\
&= \frac{\frac{A_1 K_0}{R_0 B_1} x}{1 + \frac{K_0}{B_1} x} y - \frac{\frac{A_2 K_0 C_2}{R_0 B_2 C_1} y}{1 + \frac{K_0}{B_2 C_1} y} z - \frac{D_1}{R_0} y
\end{aligned} \tag{2}$$

$$\begin{aligned}
\frac{dz}{dt} &= \frac{dZ}{dT} \frac{C_1}{R_0 C_2 K_0} = \frac{C_1}{R_0 C_2 K_0} \left(C_2 \frac{A_2 Y}{B_2 + Y} Z - D_2 Z \right) \\
&= \frac{\frac{C_2 A_2 K_0}{R_0 B_2 C_1} y}{1 + \frac{K_0}{B_2 C_1} y} z - \frac{D_2}{R_0} z
\end{aligned} \tag{3}$$

Using these equations, we can obtain the expressions for the dimensionless parameters in terms of the parameters from the original equation. Using these expressions, we can obtain estimated values for a_i, b_i , and d_i based on the estimated values for the parameters from the original equation. Substituting these values into the dimensionless equations, we have:

$$\dot{x} = x(1 - x) - \frac{0.13x}{1 + 5x} y \tag{4}$$

$$\dot{y} = \frac{0.13x}{1+5x}y - \frac{13.16y}{1+10.53y}z - d_1y \quad (5)$$

$$\dot{z} = \frac{13.16y}{1+10.53y}z - d_2z \quad (6)$$

Table 2: Dimensionless Parameters

Parameter	Expression	Estimated Value
F_1 and F_2 Function Parameters		
a_1	$\frac{A_1 K_0}{R_0 B_1}$	0.13
a_2	$\frac{A_2 K_0 C_2}{R_0 B_2 C_1}$	13.16
b_1	$\frac{K_0}{B_1}$	5.00
b_2	$\frac{K_0}{B_2 C_1}$	10.53
Death Rates		
d_1	$\frac{D_1}{R_0}$	3.50×10^{-3}
d_2	$\frac{D_2}{R_0}$	2.00×10^{-3}

References

- [1] BLAYLOCK, R. Distribution and abundance of the cownose ray, *Rhinoptera bonasus*, in lower chesapeake bay. *Estuaries* 16 (1993), 255–263.
- [2] BUSH, A., AND HOLLAND, K. Food limitation in a nursery area: Estimates of daily ration in juvenile scalloped hammerheads, *Sphyrna lewini*. *Journal of Exploratory Marine Biology and Ecology* 278 (2002), 157–178.
- [3] FAY, C., NEVES, R., AND PARDUE, G. Species profiles: Life histories and environmental requirements of coastal fishes and invertebrates (mid-atlantic) – bay scallop. *U.S. Fish and Wildlife Service Division of Biological Services* 82 (1983).
- [4] NEER, J. Aspects of the life history, ecophysiology, bioenergetics, and population dynamics of the cownose ray, *rhinoptera bonasus*, in the northern gulf of mexico. *Population* (2005), 1–124.