RUDIN CHAPTER 3
SOLUTIONS

JOHN WANG

1. Problem 3.1

Theorem 1.1. *The convergence of* $\{s_n\}$ *implies the convergence of* $\{|s_n|\}$. *The converse is not true, however.*

Proof. We will assume that $\{s_n\}$ is a convergent set in $\mathbb{R}$ because the norm is not defined in Rudin for any other metric space. Since $\{s_n\}$ is a convergent sequence, it must be Cauchy. Therefore, there exists and N such that $m, n \geq N$ implies that $d(s_m, s_n) < \epsilon$ for all $\epsilon > 0$. We can use the traingle inequality, then, to show the following:

$$\begin{aligned} |s_n| &= |(s_n + s_m) - s_m| \\ &\leq |(s_n + s_m) - s_n| + |s_n - s_m| \\ &= |s_m| + d(s_n, s_m) \\ &< |s_m| + \epsilon \end{aligned} \tag{1.2}$$

This shows that $|s_n| - |s_m| < \epsilon$, which further implies that $d(|s_n|, |s_m|) < \epsilon$ for $m, n \geq N$. Thus, we see that the sequence $\{|s_n|\}$ is Cauchy, and since it is in $\mathbb{R}$, it converges.

It is easy to show that the converse is not true, namely that the convergence of $\{|s_n|\}$ does not imply the convergence of $\{s_n\}$. Take the sequence defined by $s_n = (-1)^n$. Thus, we can see that $|s_n| = |(-1)^n| = 1$ is a constant sequence, and thus definitely converges. However, s_n switches between -1 and 1 and thus will never converge for any n.

□

2. Problem 3.2

Theorem 2.1. *We show that* $\lim_{n\to\infty} \sqrt{n^2+n} - n = \frac{1}{2}$.

Proof. We multiply through by the conjugate to obtain the following:

$$\begin{aligned} \lim_{n\to\infty} \sqrt{n^2+n} - n &= \lim_{n\to\infty} \sqrt{n^2+n} - n \frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} \\ &= \lim_{n\to\infty} \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} \\ &= \lim_{n\to\infty} \frac{n}{n\sqrt{\frac{n^2+n}{n^2}}+n} \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} \\ &= \frac{1}{\sqrt{1}+1} \\ &= \frac{1}{2} \end{aligned} \tag{2.2}$$

□

3. Problem 3.3

Theorem 3.1. *If* $s_1 = \sqrt{2}$ *and* $s_{n+1} = \sqrt{2+\sqrt{s_n}}$ *for* $n = 1, 2, 3, \ldots$ *then* $\{s_n\}$ *converges and* $s_n < 2$ *for all* $n \in \mathbb{N}$.

Proof. We know that $0 < \sqrt{2} < 2$, so we have $0 < s_1 < 2$. Moreover, if we add 2 to each side of the inequality, we obtain $2 < 2+\sqrt{2} < 4$. Taking the square roots, we obtain $\sqrt{2} < \sqrt{2+\sqrt{2}} < 2$. Adding 2 to each side of this, we obtain $2+\sqrt{2} < 2+\sqrt{2+\sqrt{2}} < 4$. Taking square roots, we have $\sqrt{2} < \sqrt{2+\sqrt{2+\sqrt{2}}} < 2$. We can repeat the process infinitely many times and notice that $s_1 = \sqrt{2}, s_2 = \sqrt{2+\sqrt{2}}, s_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$. Thus, we see that $s_1 < s_2 < s_3 < \ldots < 2$. We have therefore shown that $\{s_n\}$ is a monotonically increasing sequence that is bounded because $s_n < 2 \; \forall n \in \mathbb{N}$. By a theorem in Rudin, we know that $\{s_n\}$ must converge. □

4. Problem 3.4

Theorem 4.1. *Consider the sequence $\{s_n\}$ defined by $s_1 = 0$ with $s_{2m} = \frac{s_{2m-1}}{2}$ and $s_{2m+1} = \frac{1}{2} + s_{2m}$. The lower limit of the sequence is $\frac{1}{2}$ and the upper limit is 1.*

Proof. If we compute the sequence, then we see the following, starting at $s_1 = 0$:

$$\{s_n\} = 0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}, \ldots \tag{4.2}$$

We can use induction to derive the following formulas:

$$\text{Even n} \rightarrow s_n = \frac{2^{\frac{n}{2}-1} - 1}{2^{\frac{n}{2}-1}} = \frac{1}{2} - \frac{1}{2^{\frac{n}{2}-1}} \qquad \text{Odd n} \rightarrow s_n = \frac{2^{\frac{n-1}{2}} - 1}{2^{\frac{n-1}{2}}} = 1 - \frac{1}{2^n}$$

We can see that for even n, we have $s_n < \frac{1}{2}$ and for odd n, we have $s_n < 1$. It is clear by inspection that they are subsequential limits. Moreover, we can show that these are the only two subsequential limits of $\{s_n\}$. This is because each subsequence $\{s_{n_k}\}$, in order to converge, must contain either a finite number of even terms or a finite number of odd terms. There must exist some N such that $n_k \geq N$ implies that each subsequent s_{n_k} is either all odd or all even. If this is not the case, and for $n_k \geq N$ we have s_{n_k} odd (even) and $s_{n_{k+1}}$ even (odd), then $d(s_{n_k}, s_{n_{k+1}}) = \frac{1}{2}$ because $s_{2m+1} - s_{2m} = \frac{1}{2}$. This would imply that the subsequence does not converge. Thus, it is clear that $\{\frac{1}{2}, 1\}$ are the only two subsequential limits. Thus, we have that the upper bound $s^* = \sup\{\frac{1}{2}, 1\} = 1$ and the lower bound $s_* = \inf\{\frac{1}{2}, 1\} = \frac{1}{2}$.

□

5. Problem 3.20

Theorem 5.1. *Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Then the full sequence $\{p_n\}$ converges to p.*

Proof. Since we know that $\{p_{n_i}\}$ converges, we know that there exits an N_0 such that $n_i \geq N_0$ implies $d(p_{n_i}, p) < \epsilon$ for all $\epsilon > 0$. Since the sequence $\{p_n\}$ is Cauchy, there exists an M such that $n \geq M$ and $m \geq M$ implies $d(p_n, p_m) < \epsilon$ for all $\epsilon > 0$. By the triangle inequality, we obtain for $n, n_i \geq \max\{N_0, M\}$ that

$$\begin{aligned} d(p_n, p) &\leq d(p_n, p_{n_i}) + d(p_{n_i}, p) \\ &< \epsilon + \epsilon = 2\epsilon \end{aligned} \tag{5.2}$$

We know that $d(p_n, p_{n_i}) < \epsilon$ by the fact that $\{p_n\}$ is Cauchy, because $p_{n_i} \in \{p_n\}$ as it p_{n_i} part of a subsequence of $\{p_n\}$. We obtain $d(p_{n_i}, p) < \epsilon$ because the subsequence $\{p_{n_i}\}$ converges to p. Thus, since ϵ is arbitrary, we obtain that the full sequence $\{p_n\}$ converges to p. □

6. Problem 3.21

Theorem 6.1. *If $\{E_n\}$ is a sequence of closed, nonempty, and bounded sets in a complete metric space X, if $E_n \supset E_{n+1}$, and if $\lim_{n\to\infty} diam E_n = 0$, then $\bigcap_1^\infty E_n$ consists of exactly one point.*

Proof. Suppose $\{p_n\}$ is any sequence with $p_n \in E_n$. Then the assumption that $\lim_{n\to\infty} \text{diam} E_n = 0$ means that there exists an N such that for all $\epsilon > 0$, we have $d(\text{diam} E_n, 0) < \epsilon$ for $n \geq N$. Thus, we have $\text{diam} E_n < \epsilon$. Moreover, since we have $E_m \supset E_n$ for $m \geq n \geq N$, we know that $d(p_n, p_m) < \text{diam} E_n < \epsilon$ for $p_n \in E_n$ and $p_m \in E_m$. This implies that $\{p_n\}$ is Cauchy, and since we have assumed that X is complete, $\{p_n\}$ must converge to some limit p in X. Moreover, since p is the limit of $\{p_n\}$, it is also a limit point of E_n. This is because for $n > N$, $d(p_n, p) < \epsilon$ which implies that there exists a point $p_n \in E_n$ for every neighborhood of p. Since we assumed each E_n is closed, we know that p must be contained in each E_n. Thus, we know that $p \in \bigcap_1^\infty E_n$.

It is clear that there are no more elements in $\bigcap_1^\infty E_n$. Assume by contradiction that there exists a $q \neq p$ such that $q \in \bigcap_1^\infty E_n$. Since $q \neq p$, we know that $d(p,q) > 0$ by definition of a metric space. Since both p and q belong to $E_1, E_2, \ldots$ we see that $\sup d(p_n, q_n) > 0$ for $p_n, q_n \in E_n$. Thus, the diameter $\operatorname{diam} E_n > 0$ for all n. Thus shows that $\lim_{n\to\infty} \operatorname{diam} E_n \neq 0$, which is a contradiction of our assumption. Thus, the set $\bigcap_1^\infty E_n$ contains exactly one element. ∎

7. Problem 3.23

Theorem 7.1. *Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges.*

Proof. Since $\{p_n\}$ is a Cauchy sequence, we know that there exists some N such that $d(p_n, p_m) < \epsilon$ for all $\epsilon > 0$ and $m, n \geq N$. Since $\{q_n\}$ is Cauchy, we know there exists some M such that $d(q_n, q_m) < \epsilon$ for all $\epsilon > 0$ and $m, n \geq N$. Using the triangle inequality, we find that

$$\begin{aligned} d(p_n, q_n) &\leq d(p_n, p_m) + d(p_m, q_n) && (7.2) \\ &\leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) \\ &< 2\epsilon + d(p_m, q_m) \\ d(p_n, q_n) - d(p_m, q_m) &< 2\epsilon && (7.3) \end{aligned}$$

This implies that $d(d(p_n, q_n), d(p_m, q_m)) = |d(p_n, q_n) - d(p_m, q_m)| < 2\epsilon$ for all $\epsilon > 0$ and $m, n \geq N$. Thus we know that $\{d(p_n, q_n)\}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, we know that $\{d(p_n, q_n)\}$ converges. ∎

8. Problem 3.6

Theorem 8.1. *The series $\sum a_n$ diverges if $a_n = \sqrt{n+1} - \sqrt{n}$.*

Proof. If we multiply a_n by its conjugate, then we obtain the following:

$$\begin{aligned} a_n &= (\sqrt{n+1} - \sqrt{n}) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} && (8.2) \\ &= \frac{1}{\sqrt{n+1} + \sqrt{n}} && (8.3) \end{aligned}$$

Moreover, since we know that $\sqrt{n} \leq n$ for all $n \in \mathbb{N}$, we know that $\sqrt{n+1} + \sqrt{n} \leq (n+1) + n = 2n+1$. Thus, we can see that $a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} \geq \frac{1}{2n+1}$. Finally, the series $\sum \frac{1}{2n+1}$ diverges because it has $p = 1$. Using the comparison test, we can see that $\sum a_n$ also diverges because $a_n \geq \frac{1}{2n+1}$. ∎

Theorem 8.4. *The series $\sum a_n$ converges if $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$.*

Proof. If we multiply a_n by its conjugate, we will obtain:

$$\begin{aligned} a_n &= \frac{\sqrt{n+1} - \sqrt{n}}{n} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} && (8.5) \\ &= \frac{1}{n(\sqrt{n+1} + \sqrt{n})} && (8.6) \\ &< \frac{1}{n\sqrt{n}} && (8.7) \end{aligned}$$

The last inequality comes from the fact that $\sqrt{n+1} + \sqrt{n} > \sqrt{n}$. Moreover, we knowt that $\sum \frac{1}{n\sqrt{n}}$ converges because it is has $p = \frac{3}{2}$. By the comparison test, we know that $\sum a_n$ also converges. ∎

Theorem 8.8. *The series $\sum a_n$ converges if $a_n = (\sqrt[n]{n} - 1)^n$.*

Proof. If we perform the ratio test, then we obtain the following:

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{n} - 1 && (8.9) \\ &= 1 - 1 = 0 && (8.10) \end{aligned}$$

This is because $\lim_{n\to\infty} \sqrt[n]{n} = 1$ for $n > 0$, as proved in Rudin. Thus, since $0 < 1$, the ratio test shows that $\sum a_n$ converges. $\square$

Theorem 8.11. *If $|z| \leq 1$ for $a_n = \frac{1}{1+z^n}$, then $\sum a_n$ diverges. If $|z| > 1$, then $\sum a_n$ converges.*

Proof. We will prove that if $|z| \leq 1$, then $\sum a_n$ diverges. This is because $|1 + z^n| \leq 1 + |z|^n \leq 2$. It is obvious that $\sum \frac{1}{2}$ diverges because its sequences do not converge to zero. Thus, since $\sum \frac{1}{1+z^n} \geq \sum \frac{1}{2}$, we know that $\sum a_n$ diverges.

Now, if $|z| > 1$, then we know that $\sum \frac{1}{z^n}$ converges by a theorem in Rudin. Since $1 + z^n > z^n$, we have $\frac{1}{1+z^n} < \frac{1}{z^n}$. By the comparison test, we know that $\sum \frac{1}{1+z^n}$ converges as well. $\square$

9. Problem 3.7

Theorem 9.1. *The convergence of $\sum a_n$ implies the convergence of $\sum \frac{\sqrt{a_n}}{n}$ if $a_n \geq 0$.*

Proof. Using the Cauchy Swarchz Inequality, we have:

$$\left|\sum \frac{\sqrt{a_n}}{n}\right|^2 \leq \sum |\sqrt{a_n}|^2 \sum \left|\frac{1}{n}\right|^2 \tag{9.2}$$

$$= \sum a_n \sum \frac{1}{n^2} \tag{9.3}$$

For two given series $\sum b_n$ and $\sum c_n$ converging to B and C respectively, the Cauchy product is defined as $d_n = \sum b_n z^n \sum c_n z^n$ if one sets $z = 1$. A theorem in Rudin has shown that $\sum d_n = BC$ if $\sum b_n$ is absolutely convergent and $\sum c_n$ is convergent, which means that $\sum d_n$ and all the partial sums of d_n are bounded. Since $\sum a_n$ is convergent by assumption and $\sum \frac{1}{n^2}$ is absolutely convergent by a theorem in Rudin, we know that $\sum a_n \sum \frac{1}{n^2}$ is bounded. Thus, we know that $\left|\sum \frac{\sqrt{a_n}}{n}\right|^2$ is bounded, and hence, so is $\sum \frac{\sqrt{a_n}}{n}$. Since we assumed $a_n \geq 0$, we know that $\sum \frac{\sqrt{a_n}}{n}$ is monotonically increasing so it converges because it is also bounded. $\square$

10. Problem 3.9

Theorem 10.1. *The radius of convergence is $R = 1$ for the power series $\sum n^3 z^n$.*

Proof. We know that $R = \frac{1}{\alpha}$, where $\alpha = \lim_{n\to\infty} \sup \sqrt[n]{|n^3|} = 1$ because if we let $x_n = \sqrt[n]{n^3} - 1$, then we have for $k > 0$ and $n > 2k$:

$$n^3 = (x_n + 1)^n > \binom{n}{k} x_n^k = \frac{n(n-1)\dots(n-k+1)}{k!} x_n^k > \frac{n^k x_n^k}{2^k k!} \tag{10.2}$$

We obtain (3.2) by the binomial theorem and since x_n cannot be negative for $n \in \mathbb{N}$, we have:

$$0 \leq x_n^k < \frac{n^3 2^k k!}{n^k} \tag{10.3}$$

$$0 \leq x_n < 2(k!)^{\frac{1}{k}} n^{\frac{3}{k}-1} \tag{10.4}$$

Since $\frac{3}{k} - 1 < 0$ for all $k > 3$, we have can take the limit of both sides of the final inequality and show that $\lim_{n\to\infty} x_n = 0$ by the squeeze law. Thus, we have shown that $\alpha = \lim_{n\to\infty} \sup \sqrt[n]{|n^3|} = 1$ and that $R = 1$.

$\square$

Theorem 10.5. *The radius of convergence is $R = +\infty$ for the power series $\sum \frac{2^n}{n!} z^n$.*

Proof. We must find α in order to find $R = \frac{1}{\alpha}$:

$$\alpha = \lim_{n\to\infty} \sup \sqrt[n]{\frac{2^n}{n!}} \tag{10.6}$$

$$= \lim_{n\to\infty} \sup \frac{2}{\sqrt[n]{n!}} \tag{10.7}$$

$$= \lim_{n\to\infty} \sup \frac{2}{n^{\frac{1}{n}}(n-1)^{\frac{1}{n}}\dots(1)^{\frac{1}{n}}} \tag{10.8}$$

$$< \lim_{n\to\infty} \frac{2}{n(1)^{\frac{1}{n}}} \tag{10.9}$$

Since we know that $\lim_{n\to\infty}(1)^{\frac{1}{n}} = 1$, we also know that $\lim_{n\to\infty} \frac{2}{n(1)^{\frac{1}{n}}} = 0$. Since we must have $\sqrt[n]{\frac{2^n}{n!}} > 0$ for all $n \in \mathbb{N}$, we have the following squeeze law:

$$0 \leq \lim_{n\to\infty} \sup \sqrt[n]{\frac{2^n}{n!}} < \lim_{n\to\infty} \frac{2}{n(1)^{\frac{1}{n}}} < 0 \tag{10.10}$$

This shows that $\alpha = 0$, and that $R = +\infty$ because all terms are positive. □

Theorem 10.11. *The radius of convergence is $R = \frac{1}{2}$ for the power series $\sum \frac{2^n}{n^2} z^n$.*

Proof. We must find α, so we have the following

$$\alpha = \lim_{n\to\infty} \sup \sqrt[n]{\frac{2^n}{n^2}} \tag{10.12}$$

$$= \lim_{n\to\infty} \sup \frac{2}{n^{\frac{2}{n}}} \tag{10.13}$$

$$= \lim_{n\to\infty} \sup \frac{2}{(\sqrt[n]{n})^2} \tag{10.14}$$

$$= 2 \tag{10.15}$$

The final step comes from the fact that $\lim_{n\to\infty} \sqrt[n]{n} = 1$ according to a theorem in Rudin. Thus, we find that $R = \frac{1}{\alpha} = \frac{1}{2}$. □

Theorem 10.16. *The radius of convergence is $R = 3$ for the power series $\sum \frac{n^3}{3^n} z^n$.*

Proof. Using the same procedure as before, we shall calculate α.

$$\alpha = \lim_{n\to\infty} \sup \sqrt[n]{\frac{n^3}{3^n}} \tag{10.17}$$

$$= \lim_{n\to\infty} \sup \frac{n^{\frac{3}{n}}}{3} \tag{10.18}$$

$$= \lim_{n\to\infty} \sup \frac{(\sqrt[n]{n})^3}{3} \tag{10.19}$$

$$= \frac{1}{3} \tag{10.20}$$

The final step comes from the same fact as before, namely that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. Thus, we see that $R = \frac{1}{\alpha} = \frac{1}{3}$. □

11. Problem 3.13

Theorem 11.1. *The Cauchy product of two absolutely convergent series converges absolutely.*

Proof. Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series. This means that $\sum_{n=0}^{\infty} |a_n| = A$ and $\sum_{n=0}^{\infty} |b_n| = B$ are convergent series as well. Now define the Cauchy product as $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}$. We will give the following definitions:

$$A_n = \sum_{k=0}^{n} |a_k|, \quad B_n = \sum_{k=0}^{n} |b_k|, \quad C_n = \sum_{k=0}^{n} |c_n| \tag{11.2}$$

Thus, if we expand out the terms of C_n, we obtain the following:

$$\begin{aligned}
C_n &= |a_0b_0| + |a_0b_1 + a_1b_0| + \ldots + |a_0b_n + \ldots + a_nb_0| && (11.3)\\
&\leq |a_0|B_n + |a_1|B_{n-1} + \ldots + |a_n|B_0 && (11.4)\\
&\leq |a_0|B_n + |a_1|B_n + \ldots + |a_n|B_n && (11.5)\\
&= (|a_0| + \ldots + |a_n|)B_n && (11.6)\\
&= A_nB_n && (11.7)
\end{aligned}$$

Thus we see that $C_n \leq A_nB_n$ and if we take the limits of both sides, then we have:

$$\begin{aligned}
\sum_{n=0}^{\infty} |c_n| &= \lim_{n\to\infty} C_n && (11.8)\\
&\leq \lim_{n\to\infty} A_nB_n && (11.9)\\
&= AB && (11.10)
\end{aligned}$$

Thus, we can see that $\sum_{n=0}^{\infty} |c_n|$ is bounded. Moreover, since $|c_j| \geq 0$ for all $j \in \mathbb{Z}_{j\geq 0}$, we know that $\sum_{n=0}^{\infty} |c_n|$ is monotonically increasing. Thus, we know that $\sum_{n=0}^{\infty} |c_n|$ is both bounded and monotonically increasing, which shows that it is convergent. This means that the Cauchy product $\sum_{n=0}^{\infty} c_n$ is absolutely convergent.

□

12. Problem 3.16

Theorem 12.1. *Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$ and define $x_2, x_3, \ldots$ by the recursion formula $x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$. Then $\{x_n\}$ is decreasing monotonically and has $\lim x_n = \sqrt{\alpha}$.*

Proof. First, we will show that the sequence is decreasing monotonically. Note that $x_1 > 0$ because we fixed an $\alpha > 0$. Next, take $x_{n+1} - x_n$:

$$\begin{aligned}
x_{n+1} - x_n &= \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - x_n && (12.2)\\
&= \frac{1}{2}\left(\frac{\alpha}{x_n} - x_n\right) && (12.3)\\
&= \frac{1}{2}\left(\frac{\alpha - x_n^2}{x_n}\right) && (12.4)
\end{aligned}$$

Thus, we must determine the sign of $\alpha - x_n^2$. To begin, we have assumed $\alpha - x_1^2 < 0$ because we set $x_1 > \sqrt{\alpha}$. Moreover, we can see that $x_n^2 \geq \alpha$ for all $n \in \mathbb{N}$ because as $\{x_n\}$ approaches α, we have $\alpha - x_n^2 << x_n$. If for some n, $x_n^2 = \alpha$, then $x_{n+1} - x_n = 0$ and the sequence will become constant. Thus because $\alpha - x_n^2 \leq 0$, we see that the sequence is monotonically decreasing.

To show the second part of the problem, we know that the series is decreasing monotonically and is bounded by 0. This is because x_1 starts out positive, so we have $x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) > 0$ for all $n \in \mathbb{N}$. This show that $\{x_n\}$ converges. Let us define x as its limit: $\{x_n\} \to x$. Then we have

$$\begin{aligned}
\lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) && (12.5)\\
x &= \frac{1}{2}\left(x + \frac{\alpha}{x}\right) && (12.6)\\
x^2 &= \alpha && (12.7)
\end{aligned}$$

Since $\alpha > 0$ and also $x_n > 0$, we know that $x = \sqrt{\alpha}$ and thus we have shown that $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.

□

Theorem 12.8. *Put $\epsilon_n = x_n - \sqrt{\alpha}$ and show that $\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$ so that setting $\beta = 2\sqrt{\alpha}$, we obtain $\epsilon_{n+1} < \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n}$ for $n = 1, 2, \ldots$.*

Proof. We know that $\epsilon_n^2 = (x_n - \sqrt{\alpha})^2 = x_n^2 + \alpha - 2x_n\sqrt{\alpha}$, so that $x_n^2 + \alpha = \epsilon_n^2 + 2x_n\sqrt{\alpha}$. Then, we can use our recursive formulas to determine ϵ_{n+1}:

$$\begin{aligned}
\epsilon_{n+1} &= \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} && (12.9)\\
&= \frac{1}{2}\left(\frac{x_n^2 + \alpha - 2x_n\sqrt{\alpha}}{x_n}\right) && (12.10)\\
&= \frac{1}{2}\frac{\epsilon_n^2}{x_n} && (12.11)
\end{aligned}$$

We have shown in the first part of the exercise that $x_n^2 > \alpha$ for all $n \in \mathbb{N}$, so we know that $\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$.

Next, we will show that $\epsilon_{n+1} < \frac{\epsilon_n^2}{\beta} = \beta\left(\frac{\epsilon_n}{\beta}\right)^2$. For ϵ_2 we have $\epsilon_2 < \frac{\epsilon_1^2}{\beta}$. Thus, we have the base case for our inductive argument. Next, we have

$$\begin{aligned}
\epsilon_{n+1} &= \frac{\epsilon_n^2}{\beta} && (12.12)\\
&< \frac{1}{\beta}\left(\frac{\epsilon_{n-1}^2}{\beta}\right)^2 && (12.13)\\
&< \frac{1}{\beta}\frac{1}{\beta^2}\left(\frac{\epsilon_{n-2}^2}{\beta}\right)^{2^2} && (12.14)\\
&\vdots && (12.15)\\
&< \frac{1}{\beta}\frac{{\epsilon_1}^{2^n}}{\beta^{2^{n-1}}} && (12.16)\\
&= \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n} && (12.17)
\end{aligned}$$

This completes the proof, since we have shown $\epsilon_{n+1} < \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n}$.

□

Theorem 12.18. *If $\alpha = 3$ and $x_1 = 2$, then $\frac{\epsilon_1}{\beta} < \frac{1}{10}$ and therefore that $\epsilon_5 < 4 \times 10^{-16}$ and $\epsilon_6 < 4 \times 10^{-32}$.*

Proof. We know that $\epsilon_n = x_n - \sqrt{\alpha}$ so that $\epsilon_1 = x_1 - \sqrt{\alpha} = 2 - \sqrt{3}$. We also know that $\beta = 2\sqrt{\alpha} = 2\sqrt{3}$. Thus, we can multiply by the conjugate to obtain

$$\begin{aligned}
\frac{\epsilon_1}{\beta} &= \frac{2-\sqrt{3}}{2\sqrt{3}}\frac{2+\sqrt{3}}{2+\sqrt{3}} && (12.19)\\
&= \frac{1}{2\sqrt{3}(2+\sqrt{3})} && (12.20)\\
&= \frac{1}{4\sqrt{3}+6} && (12.21)
\end{aligned}$$

We know that $1 < \sqrt{3}$ so that $4 < 4\sqrt{3}$. This means that $\frac{\epsilon_1}{\beta} < \frac{1}{10}$. Next, we can approximate $\beta = 2\sqrt{3} < 4$ because we know that $\sqrt{3} < 2$, so we can obtain the following bounds for ϵ_5 and ϵ_6:

$$\begin{aligned}
\epsilon_5 &= \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^4} && (12.22)\\
\epsilon_5 &< 4 \times 10^{-16} && (12.23)\\
\epsilon_6 &= \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^5} && (12.24)\\
\epsilon_6 &< 4 \times 10^{-32} && (12.25)
\end{aligned}$$

Because we have $2^4 = 16$ and $2^5 = 32$. This completes the proof. □

13. Problem 3.18

Theorem 13.1. *Fix a positive integer p and a positive number α and define $x_{n+1} = \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1}$. Then if $x_1 > \alpha^{\frac{1}{p}}$, the sequence decreases monotonically and has $\lim_{n\to\infty} x_n = \alpha^{\frac{1}{p}}$.*

Proof. First, we must show that the sequence is monotonically decreasing if $x_1 > \alpha^{\frac{1}{p}}$. To do this, we observe the following:

$$x_{n+1} - x_n = \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1} - x_n \tag{13.2}$$

$$= \frac{1}{p}\left((p-1)x_n + \alpha x_n^{-p+1} - px_n\right) \tag{13.3}$$

$$= \frac{1}{p}\left(-x_n + \frac{\alpha}{x_n^{p-1}}\right) \tag{13.4}$$

$$= \frac{1}{p}\left(\frac{-x_n x_n^{p-1} + \alpha}{x_n^{p-1}}\right) \tag{13.5}$$

$$= \frac{1}{p}\left(\frac{\alpha - x_n^p}{x_n^{p-1}}\right) \tag{13.6}$$

Since we have defined $x_1 > \alpha^{\frac{1}{p}}$, we know that $x_1^p > \alpha$ which gives $\alpha - x_1^p < 0$. Thus, we see that $x_2 - x_1 < 0$. Using inductive reasoning, we can see that $x_n > \alpha^{\frac{1}{p}}$ for all $n \in \mathbb{N}$. The sequence is decreasing from x_1, but the sequence does not go below $\alpha^{\frac{1}{p}}$ because if for some n we have $x_n = \alpha^{\frac{1}{p}}$, then $x_{n+1} - x_n = 0$. Thus, the sequence is monotonically decreasing since $x_{n+1} - x_n < 0$ for all $n \in \mathbb{N}$. Moreover, the sequence is bounded by zero because p is a positive integer, so $x_{n+1} > \frac{\alpha}{p}x_n^{-p+1}$, and since we have defined $x_1 > 0$, we have that x_{n+1} can never be negative.

Thus we have shown the sequence is bounded and monotonically decreasing, showing that it is convergent. Let $\lim_{n\to\infty} x_n = x$, then we have:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1} \tag{13.7}$$

$$x = \frac{p-1}{p}x + \frac{\alpha}{p}x^{-p+1} \tag{13.8}$$

$$x\left(1 - \frac{p-1}{p}\right) = \frac{\alpha}{p}x^{-p}x \tag{13.9}$$

$$\frac{p}{\alpha}\frac{1}{p} = x^{-p} \tag{13.10}$$

$$x^p = \alpha \tag{13.11}$$

$$x = \alpha^{\frac{1}{p}} \tag{13.12}$$

We have thus shown that $\lim_{n\to\infty} x_n = \alpha^{\frac{1}{p}}$, which is what we wanted. □

Theorem 13.13. *Let $\epsilon_n = x_n - \alpha^{\frac{1}{p}}$, then we have $\epsilon_{n+1} < \beta^{1-p}(\beta\epsilon_1)^{p^n}$.*

Proof. First, we note that $\epsilon_n^p = (x_n - \alpha^{\frac{1}{p}})^p > \binom{p}{1}x_n(\alpha^{\frac{1}{p}})^{p-1} > x_n\alpha^{\frac{p-1}{p}}$ by the binomial theorem and because $p \geq 1$. We do not have to worry about the negative sign in front of α because if p is an odd integer, then $p-1$ is even so $\alpha^{\frac{p-1}{p}} > 0$ and if p is an even integer then p is even so that $\alpha^{\frac{p-1}{p}} > 0$ as well. The binomial theorem also tells us that $\epsilon_n^{-p} = (x_n - \alpha^{\frac{1}{p}})^{-p} > x_n^{-p}$. Since p is a positive integer, we can see that that $x_n < \epsilon_n^p \leq p\epsilon_n^p$. This gives us the following relationships:

$$x_n^{-p} < p\epsilon_n^{-p}, \qquad x_n < \frac{\epsilon_n^p}{\alpha^{\frac{p-1}{p}}} \tag{13.14}$$

Since these are true, we can then find a bound for ϵ_{n+1}:

$$\begin{aligned}
\epsilon_{n+1} &= x_{n+1} - \alpha^{\frac{1}{p}} && (13.15)\\
&= \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p}x_n - \alpha^{\frac{1}{p}} && (13.16)\\
&< \frac{p-1}{p}\frac{\epsilon_n^p}{\alpha^{\frac{p-1}{p}}} + \frac{\alpha}{p}p\epsilon_n^{-p}\frac{\epsilon_n^p}{\alpha^{\frac{p-1}{p}}} - \alpha^{\frac{1}{p}} && (13.17)\\
&= \frac{p-1}{p}\frac{\epsilon_n^p}{\alpha^{\frac{p-1}{p}}} + \frac{\alpha p}{p\alpha^{\frac{p-1}{p}}} - \alpha^{\frac{1}{p}} && (13.18)\\
&= \frac{1}{p\alpha^{\frac{p-1}{p}}}\left((p-1)\epsilon_n^p + \alpha p - \alpha^{\frac{1}{p}}p\alpha^{\frac{p-1}{p}}\right) && (13.19)\\
&= \frac{1}{p\alpha^{\frac{p-1}{p}}}\left((p-1)\epsilon_n^p + \alpha p - \alpha^{\frac{1}{p}}p\alpha\alpha^{-\frac{1}{p}}\right) && (13.20)\\
&= \left(\frac{p-1}{p\alpha^{\frac{p-1}{p}}}\right)\epsilon_n^p && (13.21)
\end{aligned}$$

If we define the following constant:

$$\beta = \frac{p-1}{p\alpha^{\frac{p-1}{p}}} \tag{13.22}$$

Then we have found a relationship between ϵ_{n+1} and ϵ_n.

$$\epsilon_{n+1} < \beta\epsilon_n^p \tag{13.23}$$

Thus, we can use induction on the sequence $\{\epsilon_n\}$.

$$\begin{aligned}
\epsilon_1 &< \beta\epsilon_1^p && (13.24)\\
\epsilon_2 &< \beta(\beta\epsilon_1^p)^p && (13.25)\\
&\vdots && (13.26)\\
\epsilon_{n+1} &< \beta(\beta^p)^{n-1}(\epsilon_1^p)^n && (13.27)
\end{aligned}$$

If we collect terms, then we have completed the proof:

$$\epsilon_{n+1} < \beta^{1-p}(\beta\epsilon_1)^{pn} \tag{13.28}$$

□

Theorem 13.29. *This is a good algorithm for computing nth roots, especially for large values of α. For example, if $\alpha = 9$, $p = 10$, and $x_1 = 3$, then we have $\beta\epsilon_1 < \frac{1}{10}$ with $\epsilon_5 < 10^{-9,991}$ and $\epsilon_6 < 10^{-99,991}$.*

Proof. First, we can compute $\beta = \frac{p-1}{p\alpha^{\frac{p-1}{p}}} = \frac{9}{10*9^{\frac{9}{10}}}$. We know that $9^{\frac{9}{10}} < 9^1 < 9$ so we have $\beta < \frac{1}{10}$. We also have $\beta\epsilon_1 = \beta(x_1 - \alpha^{\frac{1}{p}})$. Since we know that $9 < 2^{10} = 1024$, we have $9^{\frac{1}{10}} < 2$. This gives:

$$\begin{aligned}
\beta\epsilon_1 &= \beta(3 - 9^{\frac{1}{10}}) && (13.30)\\
&< \frac{1}{10}(3-2) && (13.31)\\
&< \frac{1}{10} && (13.32)
\end{aligned}$$

Therefore, we can use the bounds of $\beta < \frac{1}{10}$ and $\beta\epsilon < \frac{1}{10}$ to find bounds for ϵ_5 and ϵ_6 by using the formula $\epsilon_{n+1} < \beta^{1-p}(\beta\epsilon_1)^{pn}$:

$$\epsilon_5 < \left(\frac{1}{10}\right)^{-9}\left(\frac{1}{10}\right)^{10^4} \tag{13.33}$$

$$\epsilon_5 < 10^{-9,991} \tag{13.34}$$

$$\epsilon_6 < \left(\frac{1}{10}\right)^{-9}\left(\frac{1}{10}\right)^{10^5} \tag{13.35}$$

$$\epsilon_6 < 10^{-99,991} \tag{13.36}$$

It is clear that this algorithm converges extremely quickly for large values of α and p. $\square$