RUDIN CHAPTER 5
SOLUTIONS

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1. Problem 5.2

Theorem 1.1. *Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable and that $g'(f(x)) = \frac{1}{f'(x)}$ for $a < x < b$.*

Proof. First, we know f is continuous because of the existence of its derivative for all $x \in (a, b)$. Thus, we can use the specialized mean value theorem, which states that for some $x_1, x_2 \in (a, b)$, there exists an $x \in (a, b)$ such that $f(x_2) - f(x_1) = (x_2 - x_1)f'(x)$. Since $f'(x) > 0$ for all $x \in (a, b)$, we can see that:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \tag{1.2}$$

Without loss of generality, if $x_2 > x_1$, we see that $f(x_2) - f(x_1) > 0$, which shows that f is strictly increasing because $f(x_2) > f(x_1)$.

Now, we shall show that g is continuous and differentiable. First, we will show that g is strictly increasing by contradiction. If we assume not, then there exists some $z > w$, where $z, w \in (f(a), f(b)$ such that $g(z) \leq g(w)$. We know there must exist corresponding values $x, y \in (a, b)$ such that $x = g(z)$ and $y = g(w)$. Thus, we see that $x \leq y$ but that $z > w$ which implies that $f(x) > f(y)$ because $f(x) = f(g(z)) = z$ and $f(y) = f(g(w)) = w$. However, we have shown that f is strictly increasing which implies $f(x) < f(y)$, which is a contradiction because $f(x) > f(y)$ and $f(x) < f(y)$ cannot both be true. Thus, we see that g is stricly increasing.

To show that g is continuous, we assume the contrary. We now note that strictly increasing functions can only have jump discontinuities. This would mean that there exists some $z \in (f(a), f(b))$ such that $g(z-) < g(z+)$. Without loss of generality, assume that $g(z) = g(z-)$. Then we must have a corresponding value of $x \in (a, b)$ such that $f(x) = z$. This implies that $x = g(z) = g(z-) < g(z+)$. However, we must have the following:

$$g(z+) = \lim_{f(y)\to z^+} g(f(y)) = \lim_{y\to x^+} g(f(y)) = \lim_{y\to x^+} y = x \tag{1.3}$$

Thus, we have shown that $x = g(z) < g(z+) = x$, which is a contradiction. Therefore, g must be continuous. Since it is continuous, we obtain an expression for $g'(f(x))$ if it exists:

$$g'(f(x)) = \lim_{f(t)\to f(x)} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} = \lim_{t\to x} \frac{t - x}{f(t) - f(x)} = \lim_{t\to x} \frac{1}{f'(x) + u(t, x)} \tag{1.4}$$

Where $\lim_{t\to x} u(t, x) = 0$. Thus, since $f'(x) > 0$ for all $x \in (a, b)$, we can see that the limit exists, and that $g'(f(x)) = \frac{1}{f'(x)}$. □

2. Problem 5.5

Theorem 2.1. *Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x + 1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.*

Proof. Since f is differentiable on $(0, \infty)$, it must also be continuous. Therefore, we can use the mean value theorem for points $x, x + 1$ such that $x \in (0, \infty)$, which will ensure that $x + 1$ is also inside the domain. Therefore, by mean value theorem, we see that:

$$f(x + 1) - f(x) = (x + 1 - x)f'(y) \tag{2.2}$$

For some $y \in (x, x + 1)$. This shows that $g(x) = f'(y)$ for some $y \in (x, x + 1)$. If we take the limit as $x \to +\infty$, we obtain:

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} f'(y) = \lim_{y\to\infty} f'(y) = 0 \tag{2.3}$$

This is because y has a lower bound of x, and as $x \to +\infty$, we also force $y \to +\infty$. Since $\lim_{y\to\infty} f'(y) = 0$, we can see that $g(x) \to 0$ and $x \to \infty$. $\square$

3. Problem 5.14

Theorem 3.1. *Let f be a differentiable real function defined in (a, b). Prove that f is convex if and only if f' is monotonically increasing.*

Proof. Given a monotonically increasing function f', assume by contradiction that f is not convex. Then there exists some $x, y \in (a, b)$ such that for some $\lambda \in (0, 1)$, we have $f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda)f(y)$. Let $p = \lambda x + (1-\lambda)y$ so that $f(p) > \lambda f(x) + (1-\lambda)f(y)$. Moreover, we can assume without loss of generality that $y > x$ and we thus see that $p \in (x, y)$. We can use the mean value theorem, because f is differentiable by assumption and hence continuous. This shows that $f(y) - f(p) = (y - p)f'(z)$ for some $z \in (p, y)$. Using the mean value theorem again, we can see that $f(p) - f(x) = (p - x)f'(w)$ for some $w \in (x, p)$. Next, since $w \in (x, p)$ and $z \in (p, y)$, we can see that necessarily, $w < z$. Since f' is a monotonically increasing function, we must therefore have $f'(w) \le f'(z)$. Combining this, we find:

$$f'(w) = \frac{f(p) - f(x)}{p - x} \quad \le \quad \frac{f(y) - f(p)}{y - p} = f'(z) \tag{3.2}$$

$$(y - x)f(p) \quad \le \quad f(y)(p - x) + f(x)(y - p) \tag{3.3}$$

$$\lambda f(x) + (1-\lambda)f(y) \quad < \quad f(p) \le \frac{f(y)(p - x) + f(x)(y - p)}{y - x} \tag{3.4}$$

$$0 < \frac{f(y)((p - x) - (y - x)(1 - \lambda)) + f(x)((y - p) - (y - x)\lambda)}{y - x} \tag{3.5}$$

Since we have assumed $y > x$, we can divide by $y - x$ in equation 3.4. Next, we know that $p = \lambda x + (1-\lambda)y$, so substituting this into our expression and multiplying by the positive term $y - x$, we obtain:

$$0 < f(y)(\lambda x + (1-\lambda)y - x - (1-\lambda)y + (1-\lambda)x) + f(x)(y - \lambda x - (1-\lambda)y - y\lambda + x\lambda) \tag{3.6}$$

$$0 \quad < \quad f(y)(0) + f(x)(0) = 0 \tag{3.7}$$

$$0 \quad < \quad 0 \tag{3.8}$$

Since this is a strict inequality, this cannot be the case and we have shown a contradiction. Thus, we see that given a monotonically increasing function f', then f is convex. To show the converse, we will assume that f is convex. Then, we must show that f' is monotonically increasing.

Assume that $x, y \in (a, b)$. Without loss of generality, suppose that $y > x$. Then, since the derivative exists everywhere, we have the following two limits due to the definition of the derivative:

$$f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x} = \lim_{t\to x^-} \frac{f(t) - f(x)}{t - x} \tag{3.9}$$

$$f'(y) = \lim_{s\to y} \frac{f(s) - f(y)}{s - y} = \lim_{s\to y^+} \frac{f(s) - f(y)}{s - y} \tag{3.10}$$

Set $t < x < y < s$. We have shown in problem 5.23 of the last problem set that the following inequalities holds for convex functions, and hence for f:

$$\frac{f(x) - f(t)}{x - t} \le \frac{f(s) - f(t)}{s - t} \le \frac{f(s) - f(y)}{s - y} \tag{3.11}$$

Therefore, taking the left and right limits of x and y respectively, we obtain:

$$f'(x) = \lim_{t\to x^-} \frac{f(t) - f(x)}{t - x} leq \lim_{s\to y^+} \frac{f(y) - f(s)}{y - s} = f'(y) \tag{3.12}$$

Thus, we have shown that for $x < y$, we have $f'(x) \le f'(y)$ for all $x, y \in (a, b)$. Therefore, we have shown that f' is monotonically increasing.

□

Theorem 3.13. *Assume that $f''(x)$ exists for every $x \in (a,b)$ and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a,b)$.*

Proof. Since we have show that f is convex if and only if f' is monotonically increasing, we only must show that $f''(x) \geq 0$ if and only if f' is monotonically increasing. First, we will assume that $f''(x) \geq 0$. Then, since f' is differentiable everywhere on (a,b), we can use the mean value theorem since continuity is also required. Thus means that $f'(x_2) - f'(x_1) = (x_2 - x_1)f''(x)$ for some $x_2, x_1 \in (a,b)$ and $x \in (x_2, x_1)$. Assume without loss of generality that $x_2 > x_1$. Then this implies that

$$\frac{f'(x_2) - f'(x_1)}{x_2 - x_1} \geq 0 \tag{3.14}$$

Which shows that for $x_2 \geq x_1$, we must have $f'(x_2) \geq f'(x_1)$. This shows that f' must be monotonically increasing. To prove the opposite way, assume that f' is monotonically increasing. Then for some $t > x$ where $t, x \in (a,b)$, we must have $f'(t) \geq f'(x)$. Alternatively, this means $f'(t) - f'(x) \geq 0$. Since $t > x$ implies that $t - x \neq 0$, we can divide by $t - x$ to obtain:

$$\phi^+(t) = \frac{f'(t) - f'(x)}{t - x} \geq 0 \tag{3.15}$$

We can also show for some $t < x$, where $t, x \in (a,b)$, we must have $f'(t) \leq f'(x)$. Using the same method as above, we have:

$$\phi^-(t) = \frac{f'(t) - f'(x)}{t - x} \geq 0 \tag{3.16}$$

Since f'' exists for every $x \in (a,b)$, we have:

$$\lim_{t \to x^+} \phi^+(t) = \lim_{t \to x^-} \phi^-(t) = f''(x) \geq 0 \tag{3.17}$$

Since this holds for arbitrary $x \in (a,b)$, we have proven that $f''(x) \geq 0$ if and only if f' is monotonically increasing. Since we have also shown that f is convex if and only if f' is monotonically increasing, we have proven that f is convex if and only if $f''(x) \geq 0$. □

4. Problem 5.15

Theorem 4.1. *Suppose $a \in \mathbb{R}^1$, f is a twice differentiable real function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively on (a,∞). Then $M_1^2 \leq 4M_0M_2$.*

Proof. Since f is continuous on (a,∞) by its differentiability, and since both f' and f'' exist for (a,∞), we can use Taylor's Theorem, which states that, setting $\alpha = x$ and $\beta = x + 2h$, we obtain

$$f(x+2h) = f(x) + f'(x)(2h) + \frac{f''(\xi)}{2}(2h)^2 \tag{4.2}$$

This reduces down to the form:

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi) \tag{4.3}$$

For some $\xi \in (x, x+2h)$ and $h > 0$. Therefore, since $|f(x)|$ is bounded by M_0 and $|f''(x)|$ is bounded by M_2, we can obtain:

$$|f'(x)| \leq hM_2 + \frac{M_0}{h} \tag{4.4}$$

Since $\frac{M_0}{h}$ is obviously larger than $\frac{M_0}{2h}$. Next, we can rearrange the equation to obtain:

$$0 \leq h^2 M_2 - h|f'(x)| + M_0 \tag{4.5}$$

Since this holds for any $h > 0$, we can take $h = \sqrt{\frac{M_0}{M_2}}$, using the fact that M_0 and M_2 are positive. If $M_2 = 0$, then $f'(x)$ is constant and $f(x)$ is a linear function by the mean value theorem. We cannot have $f'(x) = c \neq 0$, or else M_0 would be infinite, a contradiction to the hypothesis. Then, if $f'(x) = 0$, then

$M_1 = 0$, and the inequality is trivial. Moreover, if $M_0 = 0$, then the inequality is trivial. Therefore, we can take $M_0 > 0$ and $M_2 > 0$. Thus, substitute $h = \sqrt{\frac{M_0}{M_2}}$ into the expression:

$$0 \leq \frac{M_0}{M_2} M_2 - \sqrt{\frac{M_0}{M_2}} |f'(x)| + M_0 \tag{4.6}$$

Which leads to:

$$|f'(x)|^2 \frac{M_0}{M_2} \leq 4M_0^2 \tag{4.7}$$

Since we have let $x \in (a, \infty)$ be any arbitrary value, we can see that $|f'(x)| \leq M_1$, which gives us:

$$M_1^2 \leq 4M_0M_2 \tag{4.8}$$

□

Theorem 4.9. *We will show that the strict equality $M_1^2 = 4M_0M_2$ can occur.*

Proof. Consider the following continuous function for $a = -1$ and $x \in (-1, \infty)$:

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0) \\ \frac{x^2-1}{x^2+1} & (0 \leq x < \infty) \end{cases} \tag{4.10}$$

Since we know $f(x)$ is differentiable everywhere, we can use the quotient and product rules (using right and left derivatives where appropriate) to obtain:

$$f'(x) = \begin{cases} 4x & (-1 < x < 0) \\ \frac{4x}{(x^2+1)^2} & (0 \leq x < \infty) \end{cases} \tag{4.11}$$

It is clear that for $x \in (-1, 0)$, we have $f'(x) < 0$ and for $x \in (0, \infty)$, we have $f'(x) > 0$. At $x = 0$, $f'(x) = 0$. Therefore, on $x \in (-1, 0)$, $f(x)$ is monotonically decreasing and on $x \in (0, \infty)$, $f(x)$ is monotonically increasing. Since we have:

$$\lim_{x \to -1^+} f(x) = 1, \qquad \lim_{x \to \infty} f(x) = 1, \qquad f(0) = -1 \tag{4.12}$$

Therefore, $M_0 = 1$. Now, we will use the same analysis to show that $M_1 = 4$. Differentiate $f'(x)$ using the appropriate right and left derivatives to obtain:

$$f''(x) = \begin{cases} 4 & (-1 < x < 0) \\ \frac{4(x^2-4x+1)}{(x^2+1)^3} & (0 \leq x < \infty) \end{cases} \tag{4.13}$$

On $x \in (-1, 0)$ we see that $f''(x) > 4$ so that $f'(x)$ is monotonically increasing. Since $\lim_{x \to 0^-} f'(x) = 0$ and $\lim_{x \to -1^+} f'(x) = -4$, we have $|f'(x)| < 4$ on $x \in (-1, 0)$. On $x \in [0, \infty)$ we see that

$$|f'(x)| = \frac{4x}{(x^2+1)^2} \leq 4 \frac{x}{x^2+1} \frac{1}{x^2+1} \leq 4 \times \frac{1}{2} \times 1 = 2 \tag{4.14}$$

Therefore, since $f'(0) = 0$ as well, we can see that $M_1 = 4$. Next, for $x \in [0, \infty)$, we have

$$|f''(x)| = \frac{4}{(x^2+1)^2} - \frac{16x}{(x^2+1)^3} \leq \frac{4}{(x^2+1)^2} \leq 4 \tag{4.15}$$

For $x \in (-1, 0)$, we can see that $f''(x) = 4$ is a constant function. Therefore $M_2 = 4$. Now, we can see that $M_1^2 = 4^2 = 16$ and $4M_0M_2 = 4 \times 1 \times 4 = 16$. Therefore, we see that $M_1^2 = 4M_0M_2 = 16$. □

Theorem 4.16. *The same result holds for real vector valued functions f.*

Proof. Let $f = (f_1, \ldots, f_k)$ be a vector-valued function and fix

$$M_j = \sup_{x \in (a, \infty)} \left(\sum_{i=1}^{k} |f_i^{(j)}(x)|^2 \right)^{\frac{1}{2}} \tag{4.17}$$

If $M_1 = 0$, we know that $M_1^2 \leq 4M_0M_2 = 0$. Otherwise, for any point $y \in (a, \infty)$, define $g(x) = f_1'(y)f_1(x) + \ldots + f_k'(y)f_k(x)$. Since $g(x)$ for $x \in (a, \infty)$ is a twice differentiable function, we can use the first part of the exercise to find:

$$|g'(x)|^2 \leq 4 \sup_{x\in(a,\infty)} |f_1'(y)f_1(x) + \ldots + f_k'(y)f_k(x)| \sup_{x\in(a,\infty)} |f_1'(y)f_1''(x) + \ldots f_k'(y)f_k''(x)| \tag{4.18}$$

Using the Cauchy-Swarchz inequality, we obtain

$$|g'(x)|^2 \leq 4 \left(\sum_{i=1}^{k} |f_i'(y)|^2\right) M_0 M_2 \tag{4.19}$$

Since we have defined M_j^2 in a specific manner, we can see that $|g'(x)|^2 \leq 4M_1^2 M_0 M_2$. Moreover, we have let this inequality hold for arbitrary values of $x, y \in (a,\infty)$. Therefore, we can set $x = y$ and see that $|g'(x)| = |f_1'(x)|^2 + \ldots |f_k(x)|^2$. Thus, since this holds for any $x \in (a,\infty)$, we obtain:

$$\left(\sum_{i=1}^{k} |f_i'(x)|^2\right)^2 = M_1^4 \leq 4M_1^2 M_0 M_2 \tag{4.20}$$

This shows that $M_1^2 \leq 4M_0M_2$ by division because we know that $M_1 = 0$ is a trivial case. □

5. Problem 5.16

Theorem 5.1. *Suppose f is twice differentiable on $(0,\infty)$, f'' is bounded on $(0,\infty)$, and $f(x) \to 0$ as $x \to \infty$. Then $f'(x) \to 0$ as $x \to \infty$.*

Proof. Suppose that $a \in (0,\infty)$. Then since $f(x)$ for $x \in (a,\infty)$ is a twice differentiable function on (a,∞), we can use the result from the last exercise. This states that for least upper bounds M_0, M_1, M_2 of $|f(x)|, |f'(x)|, |f''(x)|$, respectively, the following holds true: $M_1^2 \leq 4M_0M_2$. Moreover, as we take the limit as $a \to \infty$, we can see that $M_0 \to 0$. We know this because $x \in (a,\infty)$, so as $a \to \infty$, we must have $x \to \infty$. Moreover, we know from assumption that $f(x) \to 0$ as $x \to \infty$. Therefore, we have discovered the following:

$$\lim_{a\to\infty} M_0 = \lim_{a\to\infty} \sup |f(x)| = \lim_{x\to\infty} \sup |f(x)| = 0 \tag{5.2}$$

Therefore, we take can our expression from the previous exercise and show that the right hand side converges to 0, because $f''(x)$ is bounded on $(0,\infty)$.

$$\lim_{a\to\infty} M_1^2 \leq \lim_{a\to\infty} 4M_0M_2 = 0 \tag{5.3}$$

This shows that $0 \leq \lim_{a\to\infty} M_1 \leq 0$, which by the squeeze law forces $\lim_{a\to\infty} M_1 = 0$. This means that:

$$0 = \lim_{a\to\infty} \sup |f'(x)| = \lim_{x\to\infty} \sup |f'(x)| \tag{5.4}$$

Since the supremum of the absolute value of $f'(x)$ is forced to equal zero in the limit as $x \to \infty$, we must therefore have $f'(x) \to 0$ as $x \to \infty$. This completes the proof. □

6. Problem 5.19

Theorem 6.1. *Suppose f is defined in $(-1,1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $a_n \to 0$, and $b_n \to 0$ as $n \to \infty$. Define the difference quotients $D_n = \frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}$. Then if $\alpha_n < 0 < \beta_n$, $\lim D_n = f'(0)$.*

Proof. Because the derivative exists at $x = 0$, we know the following to be true by the definition of derivative:

$$f'(0) = \lim_{n\to\infty} \frac{f(\alpha_n) - f(0)}{\alpha_n} - u(n) \tag{6.2}$$

$$f'(0) = \lim_{n\to\infty} \frac{f(\beta_n) - f(0)}{\beta_n} - v(n) \tag{6.3}$$

Here, the functions $u(t) \to 0$ and $v(t) \to 0$ as $n \to \infty$. Therefore, rearranging these, we can obtain:

$$\lim_{n\to\infty} f(\alpha_n) = \lim_{n\to\infty} f(0) + (f'(0) + u(n))\alpha_n \tag{6.4}$$

$$\lim_{n\to\infty} f(\beta_n) = \lim_{n\to\infty} f(0) + (f'(0) + v(n))\beta_n \tag{6.5}$$

Thus, since $\alpha < 0 < \beta$, we can determine the difference quotient by substituting values of $f(\beta_n)$ and $f(\alpha_n)$ that we have just derived.

$$D_n = \frac{f(0) + (f'(0) + v(n))\beta_n - f(0) - (f'(0) + u(n))\alpha_n}{\beta_n - \alpha_n} \tag{6.6}$$

$$= f'(0) + \frac{v(n)\beta_n - u(n)\alpha_n}{\beta_n - \alpha_n} \tag{6.7}$$

Since we have $\alpha_n < 0 < \beta_n$, we see that $|\alpha_n| \leq \beta_n - \alpha_n$ and $\beta_n \leq \beta_n - \alpha_n$. This allows us to use the triangle inequality and show:

$$|D_n - f'(0)| = v(n)\frac{|\beta_n|}{|\beta_n - \alpha_n|} - u(n)\frac{|\alpha_n|}{|\beta_n - \alpha_n|} \tag{6.8}$$

$$\leq v(n) - u(n) \tag{6.9}$$

Taking this limit as $n \to \infty$, we see that $D_n - f'(0) \to 0$, which shows that $D_n \to f'(0)$ as $n \to \infty$. □

Theorem 6.10. *If $0 < \alpha_n < \beta_n$ and $\{\beta_n/(\beta_n - \alpha_n)\}$ is bounded, then $\lim D_n = f'(0)$.*

Proof. Since we have previously derived $D_n - f'(0)$, we can just use the expression from above to prove this theorem. First, we know that since $0 < \alpha_n < \beta_n$, we can say that $\alpha_n < \beta_n$. Therefore, we have:

$$D_n - f'(0) = v(n)\frac{\beta_n}{\beta_n - \alpha_n} - u(n)\frac{\alpha_n}{\beta_n - \alpha_n} \tag{6.11}$$

$$\leq (v(n) - u(n))\frac{\beta_n}{\beta_n - \alpha_n} \tag{6.12}$$

Since we know that $\{\beta_n/(\beta_n - \alpha_n)\}$ is bounded, we can see that as we take $n \to \infty$, we see that the right hand side goes to zero because $v(n) \to 0$ and $u(n) \to 0$ individually.

$$\lim_{n\to\infty} |D_n - f'(0)| \leq \lim_{n\to\infty} |v(n) - u(n)| \left|\frac{\beta_n}{\beta_n - \alpha_n}\right| = 0 \tag{6.13}$$

Thus, we see that $\lim D_n = f'(0)$. □

Theorem 6.14. *If f' is continuous in $(-1, 1)$, then $\lim D_n = f'(0)$.*

Proof. We can apply the mean value theorem to the function f since it is both continuous and differentiable on $(-1, 1)$. Thus, for each $n \in \mathbb{N}$, there exists a t_n with $\alpha_n \leq t_n \leq \beta_n$ such that:

$$f'(t_n) = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = D_n \tag{6.15}$$

Therefore, we see that $\lim \alpha_n \leq \lim t_n \leq \lim \beta_n$. Since both $\alpha_n \to 0$ and $\beta_n \to 0$, we see that $t_n \to 0$ as $n \to \infty$. Therefore, taking the limit as $n \to \infty$ in the above expression, we see that $\lim D_n = f'(0)$. □

Theorem 6.16. *There exists a function f which is differentiable in $(-1, 1)$ and in which α_n, β_n tend to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.*

Proof. Consider the following function defined for $x \in (-1, 1)$:

$$f = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{6.17}$$

We can pick $\beta_n = \frac{2}{\pi(4n-1)}$ and $\alpha_n = \frac{1}{2\pi n}$. We see that both $\beta_n \to 0$ and $\alpha_n \to 0$ as $n \to \infty$. However, we also see that $f(\alpha_n) = 0$ for all $n \in \mathbb{N}$ and that $f(\beta_n) = -\beta_n^2$. Therefore, we have:

$$\lim_{n\to\infty} D_n = \lim_{nto\infty} \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} \tag{6.18}$$

$$= \lim_{n\to\infty} -\frac{\beta_n^2}{\beta_n - \alpha_n} \tag{6.19}$$

$$= \lim_{n\to\infty} -\frac{4}{\pi^2(4n-1)^2}\frac{2\pi n(4n-1)}{1} \tag{6.20}$$

$$= -\frac{2}{\pi} \tag{6.21}$$

Thus, since $f'(0) = 0$, and we can see that $0 \neq -\frac{2}{\pi}$, we have given an example for the theorem. □

7. Problem 5.25

Theorem 7.1. *Suppose f is twice differentiable on $[a,b]$, $f(a)<0, f(b)>0, f'(x)\geq\delta>0$, and $0\leq f''(x)\leq M$ for all $x\in[a,b]$. Let ξ be the unique point in (a,b) at which $f(\xi)=0$. Choose $x_1\in(\xi,b)$ and define x_n by $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$. Interpret this goemetrically in terms of a tangent to the graph of f.*

Proof. We see that the formula for x_{n+1} computes the intercept of the tangent line of the function at point x_n with the x axis. This will then be the next point, and the process will continue until x_n converges to the root of the function (when $f=0$). □

Theorem 7.2. *Prove that $x_{n+1}<x_n$ and that $\lim_{n\to\infty}x_n=\xi$.*

Proof. We will use induction to show that $\xi<x_{n+1}<x$. We can use the mean value theorem to show that for some $c_n\in(\xi,x_n)$, we have: $(x_n-\xi)f'(c_n)=f(x_n)-f(\xi)=f(x_n)$ because $f(\xi)=0$. Moreover, we know that f' is increasing on $[a,b]$, which means that $f'(c_n)<f'(x_n)$ because $c_n<x_n$. Thus,

$$f'(c_n)=\frac{f(x_n)}{(x_n-\xi)}<f'(x_n)=\frac{f(n)}{x_n-x_n+\frac{f(x_n)}{f'(x_n)}}=\frac{f(x_n)}{x_n-x_{n+1}} \tag{7.3}$$

Therefore, we can rearrange the inequality and see that $x_n-x_{n+1}<x_n-\xi$. This completes the first part of the inequality, because now we see that $\xi<x_{n+1}$. Next, we know that since $f(x)>0$ and $f'(x)>0$ for all $x\in[a,b]$, we see that $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}<x_n$. Thus, we have shown that $\xi<x_{n+1}<x_n$.

Next, we must show that $\lim x_n=\xi$. First, we know that $\{x_n\}$ is a bounded, strictly decreasing sequence. This means that its limit λ exists. Therefore, we have the following:

$$\lambda = \lim_{n\to\infty} x_{n+1} \tag{7.4}$$

$$\lambda = \lim_{n\to\infty} x_n-\frac{f(x_n)}{f'(x_n)} \tag{7.5}$$

$$\lambda = \lambda-\frac{f(\lambda)}{f'(\lambda)} \tag{7.6}$$

$$0 = f(\lambda) \tag{7.7}$$

Since $f(\xi)=0$ is the unique point in (a,b) for which $f(\xi)=0$, we must have $\lambda=\xi$. Therefore, $\lim x_n=\xi$. □

Theorem 7.8. *Use Taylor's theorem to show that $x_{n+1}-\xi=\frac{f''(t_n)}{2f'(x_n)}(x_n-\xi)^2$ for some $t_n\in(\xi,x_n)$.*

Proof. Using Taylor's theorem for some $t_n\in(\xi,x_n)$, we can obtain:

$$f(\xi) = f(x_n)+f'(x_n)(\xi-x_n)+\frac{f''(t_n)}{2}(\xi-x_n)^2 \tag{7.9}$$

$$0 = \frac{f(x_n)}{f'(x_n)}+(\xi-x_n)+\frac{f''(t_n)}{2f'(x_n)}(x_n-\xi)^2 \tag{7.10}$$

$$x_{n+1}-\xi = \frac{f''(t_n)}{2f'(x_n)}(x_n-\xi)^2 \tag{7.11}$$

We can divide by $f'(x_n)$ because we know that $f'(x)>0$ for all $x\in(a,b)$. We also know that $(x_n-\xi)^2=(\xi-x_n)^2$, so we can substitute one for the other. □

Theorem 7.12. *If $A=M/2\delta$, deduce that $0\leq x_{n+1}-\xi\leq\frac{1}{A}[A(x_1-\xi)]^{2^n}$.*

Proof. First, since we have shown that $\xi<x_{n+1}$, we see that $0\leq x_{n+1}-\xi$. Also, since $f''(x)<M$ and $f'(x)\geq\delta$ for all $x\in(a,b)$, we see that $\frac{f''(t_n)}{2f'(x_n)}\leq\frac{M}{2\delta}=A$ for $t_n\in(\xi,x_n)$. We have found that $x_{n+1}-\xi\leq A(x_n-\xi)^2$. Then we can use mathematical induction. For the base case, we have $x_2-\xi\leq A(x_1-\xi)^2=\frac{1}{A}[A(x_1-\xi)]^2$. Now assume that the inequality has been proven for all cases up to x_n. We shall prove that it works for x_{n+1}:

$$x_{n+1}-\xi \leq A(x_n-\xi)^2 \tag{7.13}$$

$$= A\left(\frac{1}{A}[A(x_1-\xi)]^{2^{n-1}}\right)^2 \tag{7.14}$$

$$= \frac{1}{A}[A(x_1-\xi)]^{2^n} \tag{7.15}$$

This proves the inequality. □

Theorem 7.16. *Show that Newton's method amounts to finding a fixed point of the function g defined by $g(x) = x - \frac{f(x)}{f'(x)}$.*

Proof. We want to show that Newton's method finds x_0 such that $g(x_0) = x_0$, or that $x_0 - \frac{f(x_0)}{f'(x_0)} = x_0$ which implies $f(x_0) = 0$. Therefore, we only must show that Newton's method finds $f(x_0) = 0$, because $f'(x_0) > 0$ for all $x \in (a, b)$.

Since we have previously shown that $\lim x_n = \xi$, we know that $\lim f(x_n) = f(\xi) = 0$. Thus, Newton's method finds an approximation to x_0, where $f(x_0) = 0$ as we take larger and larger $n \in \mathbb{N}$ for $\{x_n\}$. This is what we wanted to show.

As x approaches ξ, we see that $g'(x) = \frac{f(x)f''(x)}{f'(x)^2}$, so that $0 \leq g'(x) \leq f(x)\frac{M}{\delta^2}$. Thus, we see that as x approaches ξ, we have $g'(x)$ approaching 0. □

Theorem 7.17. *Put $f(x) = x^{1/3}$ on $(-\infty, \infty)$ and try Newton's method.*

Proof. We see that $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x^{1/3}}{x^{-2/3}} = x_n - 3x_n = -2x_n$. Thus, we see that $x_2 = -2x_1$. Using induction, we can assume that $x_n = (-2)^{n-1}x_1$ has been proven up to x_n. Then, we can show that

$$x_{n+1} = -2x_n = -2(-2)^{n-1}x_1 = (-2)^n x_1 \tag{7.18}$$

With mathematical induction, we have shown that $x_n = (-2)^{n-1}x_1$. Therefore, we see that for any choice of x_1, x_n does not converge. □

8. Problem 5.26

Theorem 8.1. *Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.*

Proof. If $A = 0$, then we can see that $f'(x) = 0$, which implies that $f(x) = f(a) = 0$ for all $x \in [a, b]$. Moreover, A cannot be negative because $|.|$ cannot be negative. Thus, we can assume $A > 0$. Next, fix $x_0 \in [a, b]$ and let $M_0 = \sup|f(x)|$ and $M_1 = \sup|f'(x)|$ for $a \leq x \leq x_0$. Next, we can use the mean value theorem, because f is differentiable and hence continuous, to obtain:

$$f'(x) = \frac{f(x_0) - f(a)}{x_0 - a} \tag{8.2}$$

$$f'(x)(x_0 - a) = f(x_0) \tag{8.3}$$

Therefore, since $|f'(x)| \leq \sup|f'(x)| = M_1$, we see that $f(x_0) \leq M(x_0 - a)$. Next, since we have $|f'(x)| \leq A|f(x)|$, we find that

$$|f(x)| \leq M_1(x_0 - a) \leq AM_0(x_0 - a) \tag{8.4}$$

Since we can pick any value for x_0, we can choose $x_0 - a < \frac{1}{A}$ such that $A(x_0 - a) < 1$. Then we see that $|f(x)| < A(x_0 - a)M_0$ for all $x \in [a, x_0]$. However, we can only have $M_0 = 0$ because otherwise a number stricly smaller than the supremum would be an upper bound, which shows that $f = 0$ on $[a, x_0]$. To show that $f = 0$ on $[x_0, b]$, we note that we can fix $x_0^1 \in [x_0, b]$ such that $|f(x)| \leq AM_0(x_0^1 - x_0)$. Repeating the same argument, we see that $f = 0$ on $[a, x_0] \cup [x_0, x_0^1]$. Since $[x_0, x_0^1]$ is a fixed interval, we can see that using the Archimedean principle, we will eventually cover $[a, b]$ with enough intervals $[x_0^n, x_0^{n+1}]$. Thus, we see that $f(x) = 0$ for all $x \in [a, b]$. □

9. Problem 5.27

Theorem 9.1. *Let ϕ be a real function defined on a rectangle R in the plane, given by $a \leq x \leq b$, $\alpha \leq y \leq \beta$. A solution of the initial value problem $y' = \phi(x, y), y(a) = c, (\alpha \leq c \leq \beta)$ is by definition a differentiable function f on $[a, b]$ such that $f(a) = c, \alpha \leq f(x) \leq \beta$, and $f'(x) = \phi(x, f(x))$ for $(a \leq x \leq b)$. Prove that such a problem has at most one solution if there is a constant A such that $|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$ whenever $(x, y_1) \in \mathbb{R}$ and $(x, y_2) \in \mathbb{R}$.*

Proof. Assume we have two solutions $f_1(x)$ and $f_2(x)$. We will show that they are equal by defining the function $g(x) = f_2(x) - f_1(x)$. Then since both of the solutions are such that $f_2(a) = f_1(a) = c$, we know that $g(a) = f_2(a) - f_1(a) = 0$. Next, since we have $f_1'(x) = \phi(x, f_1(x))$ and $f_2'(x) = \phi(x, f_2(x))$, we know that by the assumed condition, we have:

$$|g'(x)| = |\phi(x, f_2(x)) - \phi(x, f_1(x))| = |f_2'(x) - f_1'(x)| \leq A|f_2(x) - f_1(x)| \tag{9.2}$$

Thus, we see that $|g'(x)| \leq A|g(x)|$, so that g satisfies the conditions of problem 5.26 above. This means that we have $g(x) = 0$ for all $x \in [a, b]$. Thus, we see that $f_2(x) = f_1(x)$ for all $x \in [a, b]$, and that the two solutions are actually the same. Therefore, the problem has at most one solution. $\square$