Smile Dynamics

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Abstract

Traditionally smile models have been assessed according to how well they fit market option prices across strikes and maturities. However, the pricing of most of the recent exotic structures, such as reverse cliquets or Napoleons, is more dependent on the assumptions made for the future dynamics of implied vols than on today's vanilla option prices. In this article we study examples of some popular classes of models, such as stochastic volatility and Jump / Lévy models, to highlight structural features of their dynamic properties.

1. Introduction

This article focuses on dynamic properties of smile models. In the Black-Scholes model, by construction, (a) implied volatilities for different strikes are equal, (b) they are also frozen. Over the years several alternate models, starting with local volatility, have appeared with the aim of fitting market implied volatilities across strikes and maturities.

This capability is a desirable feature of any smile model: the model price then incorporates by construction the cost of trading vanilla options to hedge the exotic option's vega risk - at least for the initial trade. Otherwise, the price has to be manually adjusted to reflect hedging costs, i.e. the difference between market and model prices of vanilla options used for the hedge. This may be sufficient if the vega hedge is stable, which is usually the case for barrier options.

However, most of the recent exotic structures, such as Napoleons and reverse cliquets[1], require rebalancing of the vega hedge when the underlier or its implied volatilities move substantially. To ensure that future hedging costs are priced-in correctly, the model has to be designed so that it incorporates from the start a dynamics for implied volatilities which is consistent with the historically experienced one.

Stated differently, for this type of options, $\frac{\partial^2 P}{\partial \hat{\sigma}^2}$ and $\frac{\partial^2 P}{\partial S \partial \hat{\sigma}}$ are sizeable and a suitable model needs to price in a Theta to match these Gammas. In our view this issue is far more important than the model's ability to exactly reproduce today's smile surface.

1.0.1. An example

As an illustration let us consider the following example of a Napoleon option of maturity 6 years. The client initially invests 100, then gets a 6% coupon for the first 2 years and at the end of years 3, 4, 5, 6, an annual coupon of 8% augmented by the worst of the 12 monthly performances of the Eurostoxx 50 index observed each year, with the coupon floored at zero. At maturity he also gets his 100 back. The payoff for the last four coupons is designed so that their value at inception is very small, thereby financing the "large" fixed initial coupons[2] which we remove from the option in what follows.

The graph 1.1 shows on the left the Black-Scholes value of the option at time $t = 0$, as a function of volatility. As we can see the Napoleon is in substance a Put option on long (1 year) forward volatility, for which no time value has been appropriated for in the Black-Scholes price (no Theta matching $\frac{\partial^2 P}{\partial \hat{\sigma}^2}$) .

[1] See review article by C. Jeffery (Jeffery, 2004).
[2] As well as the distributor's fee, typically 1% per year.

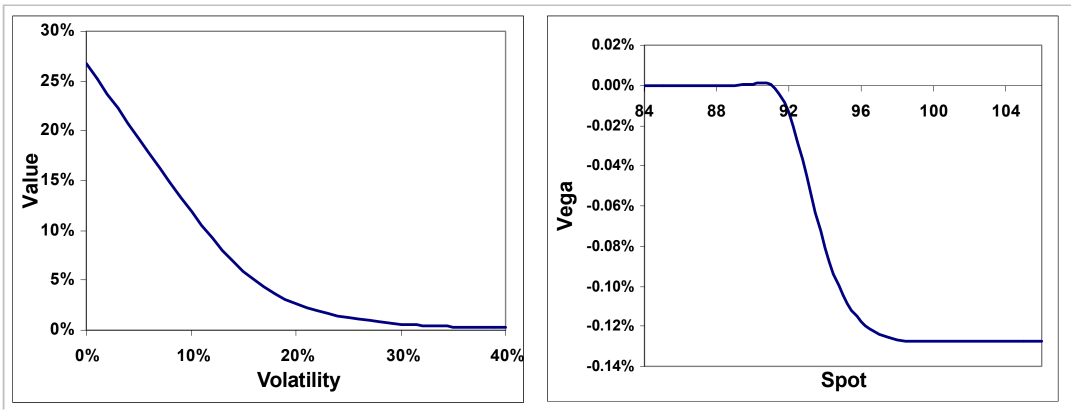

Figure 1.1: Left: initial value of coupons of years 3, 4, 5, 6, as a function of volatility. Right: Vega of a coupon at the end of the first month, as a function of the spot price.

Now let us move to the end of the first month of year 3. The graph on the right pictures the Vega of the coupon of year 3 at 20% volatility, as a function of the spot price, assuming the spot value at the beginning of the year was 100: it is a decreasing function of the spot, and goes to zero for low spot values, as the coupon becomes worthless. Now, as the spot decreases, the options' seller will need to buy back Vega; however moves in spot prices are historically negatively correlated with moves in implied vols, resulting in a negative P&L to the seller, not accounted for in the Black-Scholes price (no Theta matching $\frac{\partial^2 P}{\partial S \partial \hat{\sigma}}$).

The Black-Scholes price should thus be adjusted for the effect of the two cross-gammas mentioned, as well as for the one-month forward skew contribution.

Local vol models (Dupire, 1994), whose *raison d'etre* is their ability to exactly fit observed market smiles, have historically been used to price skew-sensitive options. Even though implied vols do move in these models, their motion is purely driven by the spot and is dictated by the shape of the market smile used for calibration. This also materializes in the fact that forward smiles depend susbtantially on the forward date and the spot value at the forward date.

It would be desirable to be able to independently (a) calibrate today's market smile, (b) specify its future dynamics. One can attempt to directly specify an *ab initio* joint process for implied vols and the spot. This approach has been explored (Schönbucher, 1999) and is hampered by the difficulty to ensure no-arbitrage in future smiles.

In this article we choose to focus on models based on a specification of the spot process. We consider some of the most popular models and characterize the dynamics of implied vols that they generate.

Our purpose here is not to be exhaustive; rather we select examples of models and products to point out specific properties of the models at hand and, more importantly, structural features which are shared by classes of models. We comment on the pricing of specific products.

This article is organized as follows. In section 2 we set up a simple pricing and hedging framework for models in incomplete markets, specializing to the case of stochastic volatility and jump and Lévy processes. We discuss pricing equations and deltas. Section 3 deals with the Heston model, typical of one-factor stochastic volatility models. Jump and Lévy processes and one of their stochastic volatility extensions are covered in Section 4. The concluding section then draws a summary and presents our views on future work.

2. Pricing and hedging

Pricing and hedging is in essence a stochastic control problem: once a measure of the replication risk has been specified, what is the optimal hedging strategy, and what price should be quoted?

In the usual Black-Scholes and local volatility framework, the only source of randomness is the spot process, which is diffusive. It turns out that the delta strategy not only minimizes the replication risk, it makes it vanish. This peculiar feature is typical of this framework and is not generic. Actually, the variance of the hedger's final P&L will be finite only because trading in the underlier does not occur continuously in time.

In more general settings, the variance of the final P&L will be finite even though trading occurs constinuously, either because the spot process is not continuous - this is typical of jump and Lévy processes - or because additional sources of randomness are present - this is the case of stochastic volatility models - or both.

In this paper, we derive pricing equations assuming that we only trade in the underlier. Our criterion is to minimize the variance of the hedger's discounted final P&L[3], which, for a European option reads:

$$P\&L \;=\; -e^{-r(T-t)}f(S_T) + \int_t^T e^{-r(\tau-t)}\Delta(\tau, S, \ldots)\left(dS_\tau - (r-q)Sd\tau\right) \tag{2.1}$$

Here f denotes the payoff function, r is the interest rate, T is the maturity, q incorporates both repo cost and dividend yield. Δ is a function of S, the spot, t, and may depend on other variables. Δ is determined by requiring that it minimizes the variance of the P&L. We then define the price of the option as $-E[P\&L]$.

In contrast with approaches based on utility functions, we do not adjust the price for the residual risk. One reason is that, in practice, the option will be added to an existing book: the marginal variation in the risk upon adding an extra option depends on the exisiting book. The other reason is that, for the sake of simplicity, we want pricing to remain a linear operation: the price of a book is the sum of the prices of each option in the book. In the two following sections, we carry out this analysis, first for the Heston model, typical of one-factor stochastic volatility models, then for a jump model.

2.1. Stochastic volatility - the Heston model

In the Heston model (Heston, 1993), the historical dynamics for the spot process is:

$$\begin{aligned} dS &= \mu S dt + \sqrt{V} S dZ_t \\ dV &= -k(V - V_0)dt + \sigma\sqrt{V}dW_t \end{aligned} \tag{2.2}$$

where W, Z are Brownian motions with correlation ρ.

Let $m_\Delta(t,S,V)$ be the expectation and $v_\Delta(t,S,V)$ the variance of the hedger's discounted final P&L assuming zero initial wealth at time t. The subscript Δ indicates that m and v depend on the - as yet unknown - function $\Delta(t,S,V)$. In the Hamilton-Jacobi-Bellman (HJB) stochastic control formalism one derives a partial differential equation for the "value function" J. Here the role of J is played by v_Δ, the control being Δ. In contrast to the usual HJB setting, the equation for v_Δ is not autonomous; it has to be supplemented with an equation for m_Δ. In the sequel we will drop the Δ subscripts for notational economy. From the dynamics 2.2 we derive the following coupled equations for m and v.

$$\begin{aligned} \frac{\partial m}{\partial t} + \mathcal{L}m - rm &= -(\mu - r + q)S\Delta \\ \frac{\partial v}{\partial t} + \mathcal{L}v - 2rv &= -VS^2\left(\Delta + \frac{\partial m}{\partial S} + \frac{\rho\sigma}{S}\frac{\partial m}{\partial V}\right)^2 - (1-\rho^2)\sigma^2 V\left(\frac{\partial m}{\partial V}\right)^2 \end{aligned}$$

where the differential operator $\mathcal{L}$ reads:

$$\mathcal{L} = \mu S\frac{\partial}{\partial S} - k(V - V_0)\frac{\partial}{\partial V} + \frac{1}{2}VS^2\frac{\partial}{\partial S^2} + \frac{1}{2}\sigma^2 V^2\frac{\partial}{\partial V^2} + \rho\sigma SV\frac{\partial^2}{\partial S\partial V}$$

At maturity, the P&L has no uncertainty anymore, hence the boundary conditions for m and v:

$$\begin{aligned} m(T,S,V) &= -f(S) \\ v(T,S,V) &= 0 \end{aligned}$$

As expected the source term in the equation for m only involves the difference between the historical drift μ and $r-q$, which is the cost of trading in the underlying. The source term for v is the sum of two positive contributions: one generated by the spot, the other one generated by the portion of volatility degrees of freedom which cannot be hedged by the spot. By variationally

[3]See Bouchaud *et al.*, (Bouchaud, 1994), for a treatment in discrete time

differentiating v with respect to Δ and requiring that v be minimal, we get the following expression for Δ:

$$\Delta = -\frac{\partial m}{\partial S} - \frac{\rho\sigma}{S}\frac{\partial m}{\partial V}$$

This expression of Δ makes the first source term in the equation for v vanish. The second term remains: the variance of the final P&L does not vanish and there is no risk-neutral price for the option. We define the price P as $P = -m$.

By plugging the expression of Δ in the equation for m, we get the following equation for P:

$$\frac{\partial P}{\partial t} + (r-q)S\frac{\partial P}{\partial S} - k(V - \overline{V_0})\frac{\partial P}{\partial V} + \frac{1}{2}VS^2\frac{\partial^2 P}{\partial S^2} + \frac{1}{2}\sigma^2 V\frac{\partial^2 P}{\partial V^2} + \rho\sigma SV\frac{\partial^2 P}{\partial S\partial V} = rP \tag{2.3}$$

where

$$\overline{V_0} = V_0 - \frac{(\mu - r + q)\rho\sigma}{k}$$

Δ is given by:

$$\Delta = \frac{\partial P}{\partial S} + \frac{\rho\sigma}{S}\frac{\partial P}{\partial V} \tag{2.4}$$

A few observations are in order:

- As expected, the pricing drift for the spot is its financing cost $r - q$.
- The Black-Scholes delta and price are recovered when σ vanishes
- The second portion of the delta is the ratio of the covariance of V and S increments to the variance of S increments.
- V_0 is renormalized; this is due to the fact the volatility degree of freedom is partially hedged by trading in the underlying. Note that $\overline{V_0}$ keeps the same functional form as V_0 (here a constant) so that the pricing equation keeps its usual form. In other stochastic volatility models, the functional form for the pricing drift of V as a function of S and V will be different, unless $\mu = r - q$.

We will use the above pricing equation in the sequel and replace $\overline{V_0}$ with V_0 for notational economy. As in the Black-Scholes framework, the pricing equation generalizes to path-dependent options.

2.2. Jump models

Now we apply the same ideas to jump models (Merton, 1976). Let the process for the spot be a jump-diffusion process where σ is the volatility, λ the intensity of the Poisson process, J the size of the jumps, itself a random variable:

$$dS = \mu S dt + \sigma S dZ_t + JS dq_t$$

Now Δ will depend solely on S and t. The equations for m and v are:

$$\begin{aligned}
\frac{\partial m}{\partial t} + \mathcal{L}m - rm &= -(\mu - r + q + \lambda\overline{J})S\Delta \\
\frac{\partial v}{\partial t} + \mathcal{L}v - 2rv &= -\sigma^2 S^2\left(\frac{\partial m}{\partial S} + \Delta\right)^2 - \lambda\left(\overline{\delta m^2} + \overline{J^2}(\Delta S)^2 + 2\overline{\delta m J}\Delta S\right)
\end{aligned}$$

with the same boundary conditions as in the previous section. We have used the following notation: $\delta m = m(S(1+J), t) - m(S,t)$ and $\overline{f} = E[f]$, where the expectation is taken over J, the amplitude of the jump. The integro-differential operator $\mathcal{L}$ is defined as:

$$\mathcal{L}f = \mu S\frac{\partial f}{\partial S} + \lambda\overline{\delta f} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2}$$

Differentiating with respect to Δ yields the following expression:

$$\Delta = -\frac{\sigma^2\frac{\partial m}{\partial S} + \lambda\overline{J^2}\frac{\overline{\delta m J}}{S\overline{J^2}}}{\sigma^2 + \lambda\overline{J^2}} \tag{2.5}$$

which is readily interpreted as the ratio of the covariance of the price and spot increments – either generated by diffusion or by jumps – to the variance of the spot increments. Let us now set $P = -m$ and use the above expression for Δ.

A power expansion in the size of jumps yields at the lowest non-trivial order:

$$\Delta \;\approx\; \frac{\partial P}{\partial S} \;+\; \frac{1}{2}\frac{\lambda\overline{J^3}}{\sigma^2+\lambda\overline{J^2}}S\frac{\partial^2 P}{\partial S^2} \tag{2.6}$$

Because Δ is different than $\frac{\partial P}{\partial S}$, one can see in the equation for m that μ remains in the pricing equation. Thus by using the delta in eq. 2.5 we are making a bet on the realized historical drift. This materializes in the fact that our P&L between two rehedges comprises a linear term of the form $(\Delta - \frac{\partial P}{\partial S})\delta S$, where δS is the variation of the spot. This is the price we pay for having an "optimal" delta that takes jumps into account: in exchange for reducing the contribution of jumps to the variance of the P&L, we increase the contribution of the "normal" diffusive behaviour.

Thus it will be sensible to use the delta in eq. 2.5 only if the specification of the jump model is in agreement with the historical dynamics of the underlying. As this is not guaranteed, it may be wiser to choose $\Delta = -\frac{\partial m}{\partial S}$ so as to remove the linear contribution in the P&L. This is the choice we make here[4]. The pricing equation then reads:

$$\frac{\partial P}{\partial t} + (r-q)S\frac{\partial P}{\partial S} \;+\; \lambda(\overline{\delta P} - \overline{J}S\frac{\partial P}{\partial S}) \;+\; \frac{\sigma^2 S^2}{2}\frac{\partial^2 P}{\partial S^2} \;=\; rP \tag{2.7}$$

and the associated Delta is:

$$\Delta = \frac{\partial P}{\partial S} \tag{2.8}$$

This can be generalized to Lévy processes and path-dependent options.

3. Dynamic properties - Stochastic volatility - The Heston model

In this section we examine the Heston model, a typical example within the class of one-factor stochastic volatility models. First, we characterize its static properties. Next we compare the model-generated dynamics of implied vols with their historical dynamics. Then we comment on the pricing of forward-start options and end with a discussion of the Delta and a comparison with local vol models.

3.1. The Heston model

The Heston model has 5 parameters V, V_0, ρ, σ, k among which k plays a special role: $\tau = 1/k$ is a cutoff that separates short and long maturities. The Heston model is homogeneous: implied vols are a function of V and moneyness: $\hat{\sigma} = f(\frac{K}{F}, V)$, where F is the forward. Perturbation of the pricing equation at 1st order in σ yields the following expressions for the skew and at-the-money-forward (ATMF) volatility:

- $T \ll \tau$, at order zero in T:

$$\hat{\sigma}_F \;=\; \sqrt{V}, \qquad \left.\frac{d\hat{\sigma}}{d\ln K}\right|_F \;=\; \frac{\rho\sigma}{4\sqrt{V}} \tag{3.1}$$

- $T \gg \tau$, at order 1 in $\frac{1}{T}$:

$$\hat{\sigma}_F \;=\; \sqrt{V_0}(1+\frac{\rho\sigma}{4k}) \;+\; \frac{\sqrt{V_0}}{2kT}\left(\frac{V-V_0}{V_0} \;+\; \frac{\rho\sigma}{4k}\frac{V-3V_0}{V_0}\right), \qquad \left.\frac{d\hat{\sigma}}{d\ln K}\right|_F \;=\; \frac{\rho\sigma}{2kT\sqrt{V_0}} \tag{3.2}$$

The long-term behavior of the skew is what we expect: in a stochastic vol model with mean reversion, increments of $\ln(S)$ become stationary and independent at long times. Thus the *skewness* of $\ln(S)$ decays like $1/\sqrt{T}$; consequently [5] the *skew* decays like $1/T$.

Let us write the expression of the Variance Swap volatility $\hat{\sigma}_{VS}(T)$, defined such that $T\hat{\sigma}^2_{VS}(T)$ is the expectation of the realized variance for maturity T:

$$\hat{\sigma}^2_{VS}(T) \;=\; V_0 + (V-V_0)\frac{1-e^{-kT}}{kT} \tag{3.3}$$

[4]In a context where jumps are used to model rare and extreme events with the purpose of reducing the size of the P&L upon a jump, the delta in eq. 2.5 would be used.

[5]See Backus *et al.*, (Backus, 1997)

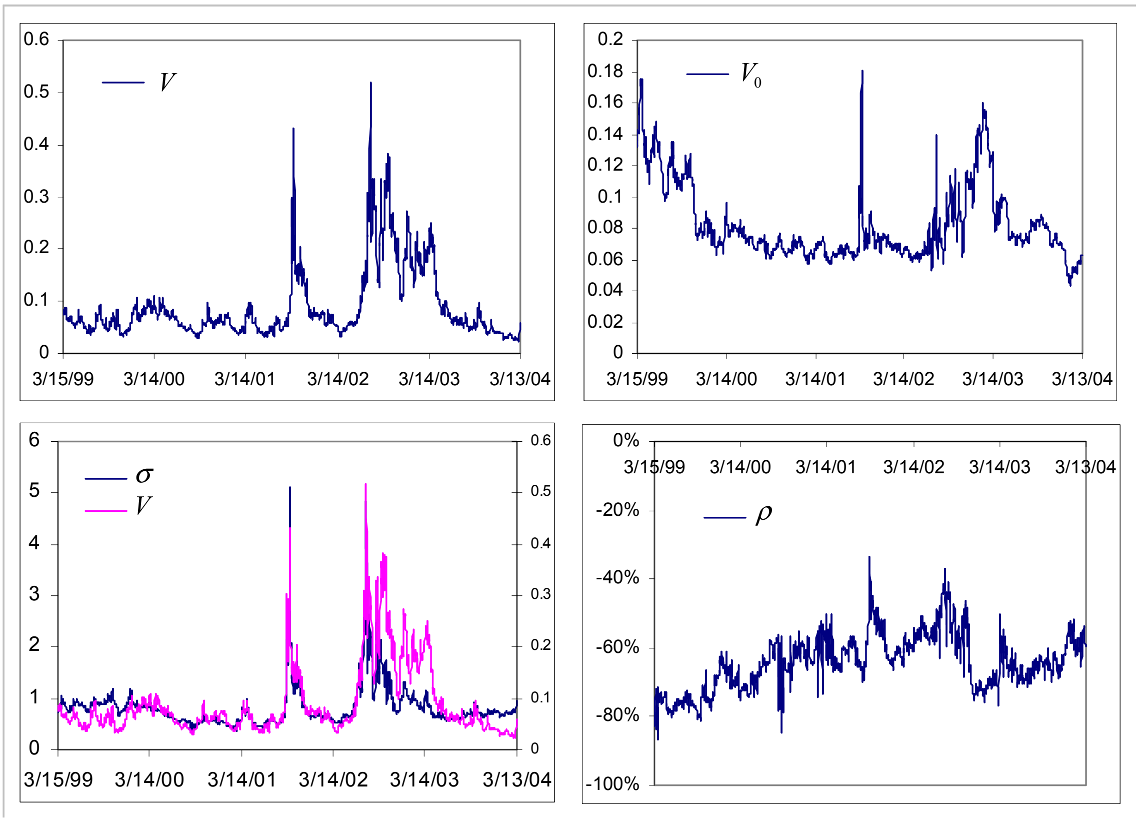

Figure 3.1: Fitted values of V, V_0, σ, ρ.

3.2. Dynamics of implied volatilities

We have calibrated market implied vols of the Eurostoxx 50 index from March 12 1999 to March 12 2004 for options of maturities 1 month, 3 months, 6 months and 1 year.

Although the dynamics of both short and long implied vols in the model is driven by V, eq. 3.3 shows that the dynamics of V is mostly reflected in that of short vols. We thus choose $k = 2$ and fit all other parameters. The daily historical values for V, V_0, σ, ρ are shown on figure 3.1.

We can see surges in volatility on September 11th 2001, then again in the summer of 2002, following the Worldcom collapse, and in the spring of 2003 at the beginning of the 2nd Gulf war.

Figure 3.2 illustrates how well levels of short and long implied vols are tracked. The graph on the left shows the at-the-money (ATM) 1-month implied vol and $\sqrt{V}$: $\sqrt{V}$ is a good proxy for the 1-month ATM vol.

The right-hand graph in figure 3.2 pictures the 1-year ATM vol as well as the 1-year Variance Swap volatility, computed from V and V_0 using eq. (3.3). We see that, as we would expect for equity smiles, the variance swap vol lies higher than the ATM vol. Here too the calibration is satisfactory.

3.2.1. Discussion

In the Heston model while S and V are dynamic, V_0, ρ, σ are supposed to be constant: their dynamics is not priced-in by the model. Figure 3.1 shows that:

- V_0 moves, but this is expected as we are asking the model to fit both short and long implied vols.
- ρ is fairly stable, and does not seem correlated with other parameters.
- σ is the most interesting parameter: we have superimposed the graph of V with a scale 10 times larger. We see that σ varies substantially and seems very correlated with V.

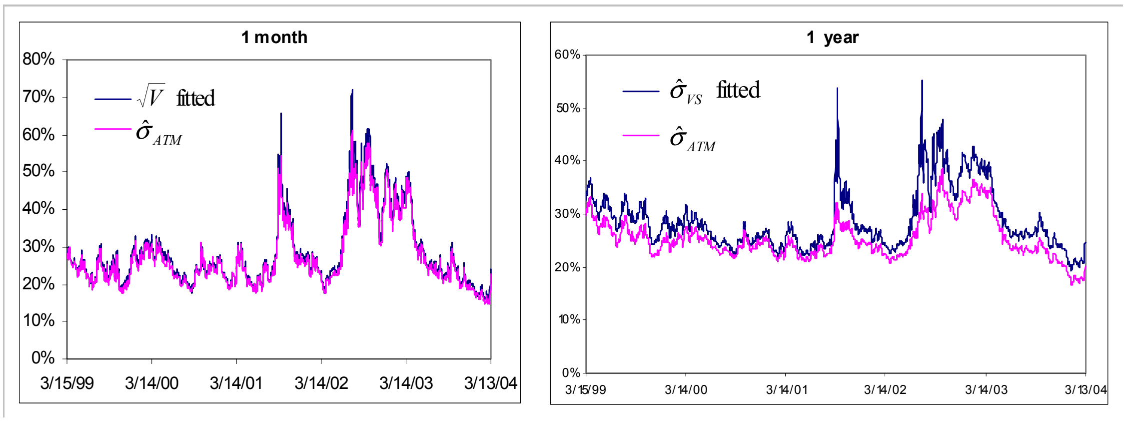

Figure 3.2: left: $\sqrt{V}$ and 1-month ATM vol, right: $\hat{\sigma}_{VS}$ and 1-year ATM vol

The last observation can be accounted for by looking at the approximate expression for the short-term skew. Equation 3.1 shows that in the Heston model it is inversely proportional to $\sqrt{V}$, which is approximately equal to the ATM vol. The fact that fitted values for σ are roughly proportional to V suggests that market skews are proportional to ATM vols, rather than inversely proportional.

In this respect the model is misspecified, since it is not pricing in the observed correlation between V and σ. This correlation is very visible in graphs for V and σ, mostly for extreme events. However it is high even in more normal regimes. For example daily *variations* of V and σ measured from March 15 1999 to September 10 2001 have a correlation of 59%.

The recent past shows a different behaviour: starting in the summer of 2003, while ATM vols decreased, skews steepened sharply, an effect that the Heston model naturally generates. Figure 3.1 indeed shows that during that period σ remains stable while V decreases.

Let us now turn to the dynamics of implied volatilities generated by the model, as compared to the historical one. In the Heston model, the implied vol dynamics is determined, by construction, by that of S and V.

We can use daily values for the couple (S, V) to check whether their dynamics is consistent with the model specification 2.2. Let us compute the following averages, which in theory should all be equal to 1:

$$
\begin{aligned}
R_S &= \left\langle \frac{\delta S^2}{S^2 V \delta t} \right\rangle &= 0.75 \\
R_V &= \left\langle \frac{\delta V^2}{\sigma^2 V} \right\rangle &= 0.4 \\
R_{SV} &= \left\langle \frac{\delta S \delta V}{\rho \sigma S V \delta t} \right\rangle &= 0.6
\end{aligned}
$$

where brackets denote historical averages using daily variations.

From these numbers we estimate that

$$
\begin{aligned}
\frac{\sigma_{realized}}{\sigma_{implied}} &= \sqrt{R_V} = 0.63 \\
\frac{\rho_{realized}}{\rho_{implied}} &= \frac{R_{SV}}{\sqrt{R_V R_S}} = 1.1
\end{aligned}
$$

suggesting that calibration on market smiles overestimates the "vol of vol" σ by 40%, while the value of the "spot/vol correlation" ρ is captured with an acceptable accuracy.

Surprisingly, R_S is notably different than 1, showing that short implied vols overrestimated historical volatility by 13% on our historical sample, possibly accounting for the enduring popularity of dispersion trades.

It is possible that these global averages are excessively impacted by extreme events. Let us then look at running monthly averages. Figure 3.3 shows the results for the six following quantities:

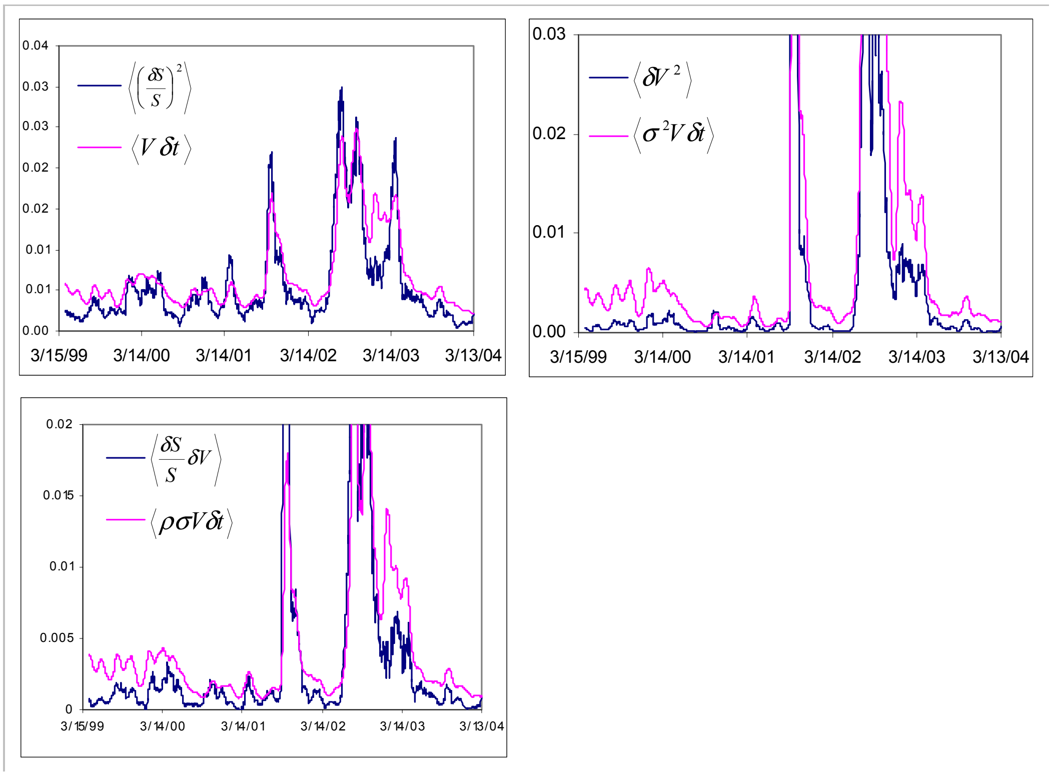

Figure 3.3: $V^{real}_{\frac{\delta S}{S}}$ and $V^{imp}_{\frac{\delta S}{S}}$, $V^{real}_{\delta V}$ and $V^{impl}_{\delta V}$, $C^{real}_{\frac{\delta S}{S}\delta V}$ and $C^{impl}_{\frac{\delta S}{S}\delta V}$

$$\begin{aligned} V^{real}_{\frac{\delta S}{S}} &= \left\langle \frac{\delta S^2}{S^2} \right\rangle \quad \text{and} \quad V^{impl}_{\frac{\delta S}{S}} = \langle V\delta t\rangle \\ V^{real}_{\delta V} &= \left\langle \delta V^2 \right\rangle \quad \text{and} \quad V^{impl}_{\delta V} = \left\langle \sigma^2 V\delta t\right\rangle \\ C^{real}_{\frac{\delta S}{S}\delta V} &= \left\langle \frac{\delta S}{S}\delta V \right\rangle \quad \text{and} \quad C^{impl}_{\frac{\delta S}{S}\delta V} = \langle \rho\sigma V\delta t\rangle \end{aligned}$$

where brackets now denote running monthly averages. The sign of $C^{real}_{\frac{\delta S}{S}\delta V}$ and $C^{impl}_{\frac{\delta S}{S}\delta V}$ has been changed.

We see that even during "normal" market conditions, the difference between "realized" and "implied" quantities is substantial. For example, using monthly running averages estimated on data from March 15 1999 to September 10 2001 gives the following numbers:

$$R_S \ = \ 0.73, \quad R_V \ = \ 0.30, \quad R_{SV} \ = \ 0.44$$

corresponding to the following ratios:

$$\frac{\sigma_{realized}}{\sigma_{implied}} = 0.54, \qquad \frac{\rho_{realized}}{\rho_{implied}} = 0.95$$

again showing that, while the "spot/vol correlation" ρ is well captured by market smiles, the "vol of vol" σ is overestimated by roughly a factor of 2.

Concretely this means that the model is pricing in a "volatility of volatility" for 1-month ATM vols which is twice larger than its historical value: future Vega rehedging costs are not properly priced-in. It also implies that the delta:

$$\Delta \ = \ \frac{\partial P}{\partial S} + \frac{\rho\sigma}{S}\frac{\partial P}{\partial V}$$

is not efficient, as it overhedges the systematic impact of spot on volatility.

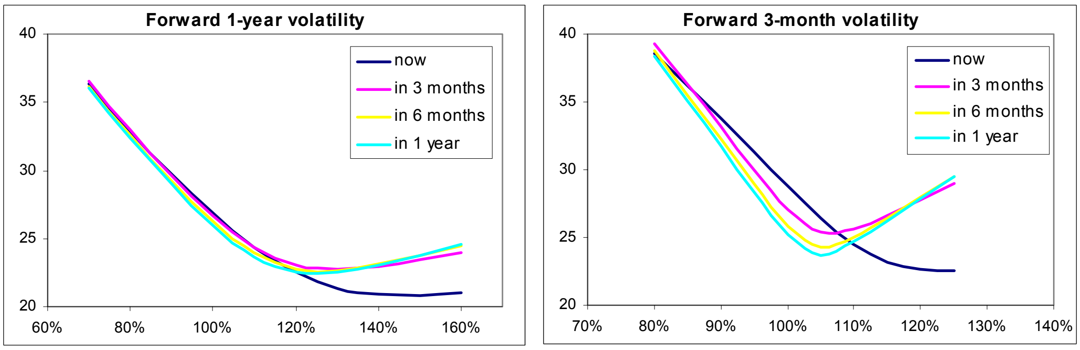

Figure 3.4: $\widehat{\sigma}(\xi)$ for a 1-year maturity (left) and a 3-month maturity (right)

The main results of our historical analysis are : (a) σ and V are very correlated, (b) the value of σ determined from calibration on market smiles is a factor of 2 larger than its historical value.

While (a) could be solved by altering the model's specification, (b) is structural. Indeed, we have only one device in the model – namely the "vol of vol" σ – to achieve two different objectives, one static, the other dynamic: (a) create skewness in the distribution of $\ln(S)$ so as to match market smiles, (b) drive the dynamics of implied vols in a way which is consistent with their historical behaviour. It is natural that we are unable to fulfill both objectives. We view this as a structural limitation of any one-factor stochastic volatility model.

We have concentrated here on the dynamics of short-term vols. Space prevents us from examining the crucial issue of the term-structure of the vol of vols, which is controlled by the correlation function of V.

3.3. Forward start options

Here we consider a one-period forward call option which pays $\left(\frac{S_{T_1+\theta}}{S_{T_1}} - \xi\right)^+$ at date $T_1 + \theta$, for different values of moneyness ξ. From the model-generated price of the forward start option we imply Black-Scholes volatilities to get what is generally termed the "forward smile" $\hat{\sigma}(\xi)$.

Figure 3.4 shows the forward smile computed using the following typical values: $V = V_0 = 0.1$, $\sigma = 1$, $\rho = -0.7$, $k = 2$, for two values of θ: 0.25 (3 months) and 1 (1 year). Todays' smile ($T_1 = 0$) is also plotted for reference.

Note that forward smiles are more convex than today's smile: since the price of a Call option is an increasing and convex function of its implied volatility, uncertainty in the value of future implied volatility increases the option price.

As T_1 is more distant, the distribution for V becomes stationary in the Heston model. Thus forward smiles collapse onto a single curve for $T_1 \gg 6$ months, in our example. This is manifest in figure 3.4.

The graphs also show that the increased convexity with respect to today's smile is larger for strikes $\xi > 100\%$ than for strikes $\xi < 100\%$. This can be traced to the dependence of the skew to the level of ATM volatility. Since the short-term skew is inversely proportional to the ATM volatility, implied volatilities for strikes lower than 100% will move more than those for symmetrical strikes in the money. This is specific to the Heston model.

While the forward smile is a global measure of the distribution of implied vols at a forward date it is instructive to look at the distribution itself. Let $T_1 \gg \frac{1}{k}$. The density of V has the following stationary form:

$$\rho(V) \propto V^{\left(\frac{2kV_0}{\sigma^2}-1\right)} e^{-\frac{2k}{\sigma^2}V}$$

Using the parameter values listed above, we find that $\frac{2kV_0}{\sigma^2} - 1 \;=\; -0.6$ i.e. the density for V *diverges* for small values of V.

Thus even simple cliquets are substantially affected by the model specification: the practical conclusion for pricing is that, for short-term forward-start options, the Heston model is likely to overemphasize low ATM vol/high skew scenarios.

3.4. Local dynamics and Delta

We here study the local dynamics of the Heston model: how do implied vols move when the spot moves? This sheds light on the model's Delta since its deviation from the Black-Scholes value is related to the model's expected shift in implied vols when the spot moves.

In local vol models, the motion of implied volatilities is driven by the spot. From the expression of the local volatility (Dupire, 1994), in the limit of short maturity and weak skew, one can derive the following well-known relationship linking the skew to the dynamics of the at-the-money volatility as a function of the spot:

$$\frac{d\hat{\sigma}_{K=S}}{d\ln S} \;=\; 2\,\frac{d\hat{\sigma}}{d\ln K}\bigg|_{K=S}$$

showing that $\hat{\sigma}_{K=S}$ moves "twice as fast" as the skew.

In stochastic volatility models, while implied volatilities are not a function of S, they are correlated with S: this is what the second piece of the Delta in eq. 2.4 hedges against. Conditional on a small move of the spot δS , V moves on average by $\delta V = \frac{\rho\sigma}{S}\delta S$.

Let us compute the expected variation in $\hat{\sigma}_F$, for short and long maturities:

- For $T \ll \tau$ we use expressions 3.1, correct at order 0 in T. At this order, F and S can be identified. The expression for σ_F gives: $E[\delta\hat{\sigma}_{K=S}] \;=\; \frac{\rho\sigma}{2\sqrt{V}}\frac{\delta S}{S}$. Looking at the expression for the skew, we notice that:

$$\frac{E[\delta\hat{\sigma}_{K=S}]}{\delta\ln S} \;=\; 2\,\frac{d\hat{\sigma}}{d\ln K}\bigg|_{K=S} \tag{3.4}$$

  This shows that, locally, the shift in implied vols expected by the Heston model when the spot moves is identical to that of a local vol model; thus the Deltas of vanilla options for strikes near the money will be the same for both models - at order 1 in σ. This result is generic and holds for all stochastic volatility models.

- For $T \gg \tau$ we use expressions 3.2, correct at order 1 in $\frac{1}{T}$. We get, keeping only terms linear in σ:

$$E[\delta\hat{\sigma}_{K=F}] \;=\; \frac{\rho\sigma}{2kT\sqrt{V_0}}\frac{\delta S}{S}$$

  Comparing with the expression of the skew in eq. 3.2 we see that:

$$\frac{E[\delta\hat{\sigma}_{K=F}]}{\delta\ln S} \;=\; \frac{d\hat{\sigma}}{d\ln K}\bigg|_{K=F}$$

  The ATMF volatility slides on the smile, the Heston model behaves like a *sticky-strike* model: implied volatilities for fixed strikes do not move as the spot moves. Thus the Deltas of vanilla options for strikes near the forward will be equal to their Black-Scholes Deltas - again at order 1 in σ. The extension to other stochastic volatility models is left for future work.

These results are obtained for the Heston model at first order in σ and are relevant for equity smiles. If ρ is small, as is the case for currency smiles, the contribution from terms of order σ^2 dominates, altering the conclusions: for example the similarity to local vol models for short maturities will be lost.

4. Dynamic properties - Jump / Lévy models

4.1. Jump / Lévy models

In this section we consider jump models for which the size of the relative jump experienced by the spot does not depend on the spot level. Such models are homogeneous: implied volatilities are a function of moneyness $\hat{\sigma}(K,S) \;=\; \hat{\sigma}\left(\frac{K}{S}\right)$.

The spot is the only degree of freedom in the model. As it moves the smile experiences a translation along with it: for a fixed moneyness, implied vols are frozen. This has two main consequences:

- Forward smiles do not depend on the forward date and are the same as today's smile: a graph similar to Figure 3.4 would show all smiles collapsing onto a single curve. When pricing a cliquet this is equivalent to impacting all forward-start options by the same smile cost.
- The Deltas for vanilla options are model-independent and can be read off the smile directly. The Delta for strike K is given by:

$$\Delta_K \;=\; \Delta_K^{BS} \;-\; \frac{1}{S}\mathrm{Vega}_K^{BS}\frac{d\hat{\sigma}_K}{d\ln K}$$

  where Δ_K^{BS} and Vega_K^{BS} are the Black-Scholes Delta and Vega of the vanilla option of strike K computed with its implied vol $\hat{\sigma}_K$.

In Jump / Lévy models, increments of $\ln(S)$ are independent, thus the skewness of the distribution of $\ln(S_T)$ decays as $\frac{1}{\sqrt{T}}$, and, at 1st order in the skewness, that the *skew* decays as $\frac{1}{T}$, too fast in comparison with market smiles.

Stochastic volatility models generate a smile by starting with a process for $\ln(S)$ which is Gaussian at short time scales and making volatility stochastic and correlated with the spot process. In contrast, Jump / Lévy models generate a skew without additional degrees of freedom by starting with a process for $\ln(S)$ at short time scales with sufficient embedded skewness and kurtosis so that both are still large enough at longer time scales to generate a smile, even though they decay as $\frac{1}{\sqrt{T}}$ and $\frac{1}{T}$, respectively.

In the next section we use the example of Variance Swaps to illustrate how the behaviour of Jump / Levy models on short time scales impacts the price of very path-dependent options.

4.2. Variance Swaps

A Variance Swap (VS) is a now standard option which pays at maturity the realized variance of the spot, measured as the sum of squared returns observed at discrete dates - usually daily.

If the observations are frequent enough, its price P_{VS} is just the discounted expected variance by construction:

$$P_{VS} \;=\; e^{-rT}\,\hat{\sigma}_{VS}^2$$

We now introduce the Log Swap volatility $\hat{\sigma}_{LS}(T)$. $\hat{\sigma}_{LS}(T)$ is the implied volatility of the LogSwap, which is the European payoff $-2\ln(S)$. This profile, when delta-hedged, generates a Gamma P&L which is exactly equal[6] to the squared return of the spot between two rehedging dates. Because this statically replicates the payout of a VS, VSs are usually priced using $\hat{\sigma}_{LS}(T)$. In the Black-Scholes model, in the limit of very frequent observations, $\hat{\sigma}_{LS} \;=\; \hat{\sigma}_{VS} \;=\; \sigma$.

The value of $\hat{\sigma}_{LS}(T)$ is the implied vol of a European payoff; it is thus model-independent and is derived from the market smile. For equity smiles, $\hat{\sigma}_{LS}(T)$ usually lies higher than the ATM vol. For example, in early March of this year, because of the high skew / low vol context, the 1-year $\hat{\sigma}_{VS}$ for the Eurostoxx 50 index was about 4 points higher than the ATM vol, which was around 18%.

In the Heston model, direct computation yields $\hat{\sigma}_{VS}(T) = \hat{\sigma}_{LS}(T)$. This self-consistence can be shown to hold for all diffusive models.

In Jump / Lévy models, however, $\hat{\sigma}_{VS}$ is usually *lower* than $\hat{\sigma}_{LS}$ and even *lower* than $\hat{\sigma}_{ATM}$. For example, in the limit of frequent jumps of small amplitude, the following relationship can be derived, at 1st order in the skewness:

$$\hat{\sigma}_{K=F} - \hat{\sigma}_{VS} \;=\; 3(\hat{\sigma}_{LS} - \hat{\sigma}_{K=F})$$

where $\hat{\sigma}_{K=F}$ is the volatility for a strike equal to the forward.

The question then is: to price VSs, should we use $\hat{\sigma}_{VS}$, or $\hat{\sigma}_{LS}$ or yet another vol?

To understand the difference, imagine hedging the profile $-2\ln(S)$ with the Black-Scholes delta computed with an implied volatility $\hat{\sigma}$. If there are no dividends, the delta is independent on the volatility, equal to $\frac{-2}{S}$. The Gamma portion of the Gamma/Theta P&L realized during Δt, stopping at 3d order terms in ΔS reads:

$$\left(\frac{\Delta S}{S}\right)^2 \;-\; \frac{2}{3}\left(\frac{\Delta S}{S}\right)^3$$

Introducing the volatility σ, given by $\sigma^2\Delta t = E[\left(\frac{\Delta S}{S}\right)^2]$, and the skewness $\mathcal{S}_{\Delta t}$ of $\frac{\Delta S}{S}$, we can write the expectation of this P&L as:

$$\sigma^2\Delta t\left(1 \;-\; \frac{2\mathcal{S}_{\Delta t}}{3}\sigma\sqrt{\Delta t}\right)$$

Let us take the limite $\Delta t \to 0$.

[6] Except if dividends are modelled as discrete cash amounts

- In stochastic volatility models, as $\Delta t \to 0$, returns become Gaussian and $\mathcal{S}_{\Delta t} \to 0$. Thus the P&L generated by delta-hedging the LogSwap profile is exactly the realized variance. This explains why $\hat{\sigma}_{LS}$ and $\hat{\sigma}_{VS}$ are the same.
- In Jump / Lévy models, because $\mathcal{S}_{\Delta t} \propto \frac{1}{\sqrt{\Delta t}}$, the 3d order term contribution tends to a finite constant as $\Delta t \to 0$: delta-hedging the LogSwap profile generates an additional contribution from 3d order terms[7].

  For equity smiles $\mathcal{S}$ is negative. Delta-hedging the Log Swap profile then generates in addition to the realized variance a spurious *positive* P&L. Thus, the VS should be priced using a vol lower than $\hat{\sigma}_{LS}$: $\hat{\sigma}_{VS} < \hat{\sigma}_{LS}$.

If real underliers behaved according to the Jump / Lévy model specification, we should then price VSs using $\hat{\sigma}_{VS}$. Inspection of daily returns of the Eurostoxx 50 index shows however that for daily returns $\mathcal{S}_{\Delta t}$ is a number of order 1. Using a daily volatility of 2% gives an estimation of the contribution of the 3d order term ≈ 50 times smaller than that of the 2nd order term, in sharp contrast with the model's estimation.

The practical conclusion for the pricing of VSs is that it will be more appropriate to use $\hat{\sigma}_{LS}$.

More generally, we have to be aware of the fact that, once their parameters are calibrated to market smiles, Jump / Lévy models will predict excessive skews at shorter time scales; this behaviour is structural.

4.3. Stochastic volatility extensions to Jump / Lévy models

A simple way of adding dynamics to implied vols in a Jump / Lévy model is to make the flow of time stochastic: replace t with a non-decreasing process τ_t and evaluate the Lévy process L at τ. This is a particular case of a subordinated process. If the characteristic functions of both L_t and τ_t are known, then the characteristic function of L_τ is also known and an inverse Laplace transform yields European option prices. P. Carr *et al.* (Carr, 2003) choose τ_t as the integral of a CIR process:

$$\begin{aligned} \tau_t &= \int_0^t \lambda_u du \\ d\lambda &= -k(\lambda - \lambda_0)dt + \sigma\sqrt{\lambda}dZ_t \end{aligned}$$

What is the dynamics of implied vols in such a model? Here we look at short-term options. The shape of the smile for maturity T is determined by the distribution of $\ln(S_T)$. Given the variance $\mathcal{V}$ and the skewness $\mathcal{S}$ of a distribution for $\ln(S_T)$, perturbation at 1st order in $\mathcal{S}$ gives (Backus, 1997):

$$\hat{\sigma}_{K=F} = \sqrt{\frac{\mathcal{V}}{T}} \tag{4.1}$$

$$K\left.\frac{d\hat{\sigma}}{dK}\right|_{K=F} = \frac{\mathcal{S}}{6\sqrt{T}} \tag{4.2}$$

where F is the forward of maturity T.

Because λ_t is a continuous process, for short maturities: $L_{\tau_T} \approx L_{\lambda T}$; in other words, λ acts as a pure scale factor on time. Since the cumulants of L all scale linearly with time, we have

$$\begin{aligned} \mathcal{V} &\propto \lambda T \\ \mathcal{S} &\propto \frac{1}{\sqrt{\lambda T}} \end{aligned}$$

Plugging these expressions in equations (4.1), (4.2), we get the following form for the ATMF vol and skew, for short maturities:

$$\begin{aligned} \hat{\sigma}_{K=F} &\propto \sqrt{\lambda} \\ K\left.\frac{d\hat{\sigma}}{dK}\right|_{K=F} &\propto \frac{1}{\sqrt{\lambda}T} \end{aligned}$$

[7] Higher order terms would also yield at each order a non-vanishing contribution. For example the contribution from the 4th order term would have included a factor $\Delta t(3 + \kappa_{\Delta t})$, where $\kappa_{\Delta t}$ is the kurtosis of returns. Since, as mentioned in the previous section, $\kappa_{\Delta t} \propto \frac{1}{T}$, the contribution from the 4th order is finite as $\Delta_t \to 0$.

Let us examine the scaling behaviour of these expressions. The dependence of vol and skew on T is what we would expect; more interesting is the dependence on λ: combining both equations yields the following result:

$$K \left.\frac{d\hat{\sigma}}{dK}\right|_{K=F} \propto \frac{1}{\hat{\sigma}_{K=F}}$$

Thus, for short maturities, the skew is approximately inversely proportional to the ATMF vol.

This result is interesting in that it is general for the class of models considered: it depends neither on the choice of Lévy process nor the process for λ. Thus, impacting time with a stochastic scale factor allows implied vols to move but with a fixed dependence of the short-term skew on the level of ATMF vol. As noted in Section 3 of Part I this feature is also shared by the Heston model, for very different reasons. To get a different behaviour, we would need to make the parameters of the Lévy process λ-dependent, probably loosing the analytical tractability of the model.

5. Conclusion

As mentioned in the general introduction, we believe that analyzing and controlling the dynamics of implied volatilities is a central issue in the construction of models for the pricing of the recent breed of exotic structures as well as general path-dependent options which cannot be hedged statically.

In this article we have studied some aspects of the dynamic properties of implied vols for two of the most popular classes of models: stochastic volatility and Jump / Lévy models. We have also pointed out some of the structural implications of choosing a particular type of driving process for the spot.

It is our assessment that, in addition to the spot process, at least another driving process is needed to model the implied vol dynamics, and presumably more than one if our objective is to correctly match the term-structure of the "vol of vols", while accurately fitting market smiles. In order to avoid the inconsistencies noted in the analysis of the dynamical properties of the Heston model, such a model would have "state variables", calibrated to market smiles, whose dynamics is priced-in, and "structural parameters", either calibrated to the historical dynamics of implied vols, or chosen by the user. Within the set of model parameters, it would then be very useful to be able to precisely identify those governing static features and those governing dynamic features of the model-generated smiles. In this respect, associating stochastic volatility with Lévy processes seems a very promising avenue of research.

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