Profit-and-Loss of Option Strategies under Quadratic Skew Parametrization

Artur Sepp

Bank of America Merrill Lynch, 2 King Edward Street, London, EC1 1HQ

E-mail: Artur.Sepp@baml.com

September 5, 2010

Abstract

We analyse the profit-and-loss (P&L) of delta-hedging strategies for vanilla options in the presence of the implied volatility skew and derive an approximation for the P&L under the quadratic parametrization of the implied volatility. We apply this approximation to study the P&L of a straddle, a risk-reversal, and a butterfly. Using our results, we derive the break-even realized skew and convexity that equate the average realized P&L of the risk-reversal and the butterfly, respectively, to zero. Furthermore, we analyse the impact of the volatility skew on the delta-hedging of these option strategies. We present some empirical results using implied volatilities of options on the S&P 500 index.

1 Introduction

The quadratic parametrization of the implied volatility skew is widely used in practice in one way or another for its simplicity. While pricing of vanilla options under this parametrization is considered in a number of papers (Dumas *et al* (1998), Demeterfi *et al* (1999), Backus-Foresi-Wu (2004), Daglish *et al* (2007), Bergomi (2009)), we have found no study that analyses the P&L of vanilla options under the quadratic parametrization. We derive an approximation for the P&L of a delta-hedged vanilla option predicted given changes in the time variable, the underlying price and the at-the-money (ATM) volatility. We apply this approximation to study the P&L of the straddle, the risk-reversal, and the butterfly.

As the implied volatility skew is inferred from option prices quoted in the market, it provides a static snapshot of the risk-neutral distribution of the underlying price implied by these option prices, by using the generic result of Breeden-Litzenberger (1978). More generally, Derman-Kani (1994), Rubinstein (1994), Dupire (1994) (also see Chriss *et al* (1996) and Jackwerth (1999)) develop the link between the implied volatilities and the risk-neutral distribution introducing the concept of the local volatility. One of the important conclusions of their analysis is that, given the martingale measure consistent with option prices, if an European option is valued and hedged using the local volatility and the hedging is continuous in time with no transaction costs, then the P&L, which is realized from the delta-hedging over infinitesimal time periods, is zero. Under the objective probability measure (note that the realized variance under the objective measure does not necessarily equal to the local variance), the realized P&L is equal to the difference between realized and local variances multiplied by the option gamma (see El Karoui *et al* (1998), Lipton (2001), Carr (2005)).

While the continuous time analysis simplifies theoretical developments, in practice the delta-hedging is implemented over discrete time intervals. Therefore, the realized P&L will be sensitive to realizations of the state variables such as the underlying price and the ATM volatility (see Sepp (2010)). In addition, the P&L realized from a delta-hedged option position is affected by the parameters of the implied volatility skew. The quadratic parametrization includes three parameters for the ATM volatility, the skew, and the convexity. An interesting question is what values of these parameters should be so that the average realized P&L is zero. In case of the straddle delta-hedged under the Black-Scholes-Merton (1973) model, this leads to the conclusion that the realized P&L is approximately the difference between the implied volatility and the realized volatility multiplied by the option vega (see Derman (1999a) and Sinclair (2008)). We derive similar expressions for the P&L of the risk-reversal and the butterfly. As a result, within the quadratic parametrization, we propose a definition of the break-even skew and convexity that equate the average realized P&L of the risk-reversal and the butterfly, respectively, to zero.

We also consider a single factor model with the dynamics of the ATM volatility being implied by the dynamics of the underlying price. We show that this model adequately describes the empirical data and apply it to simplify the approximate P&L of the straddle, the risk-reversal, and the butterfly, which provides useful insight. In addition, we study the delta-hedging error arising from the differences between an assumed and a realized change in the implied volatility given a change in the underlying price.

This article is organized as follows. In Section 2, we discuss the quadratic parametrization. In Section 3, we derive approximation for the P&L of the delta-hedged vanilla option. In Section 4, we apply this approximation to study the P&L of the straddle, the risk-reversal, and the butterfly. In Section 5, we introduce the single factor dynamics and discuss some implications. In Section 6, we present some empirical analysis.

Throughout this article we will provide some illustrations and discuss empirical findings. In the empirical analysis for this article, we use the time series of implied volatilities with maturity of one month for options on the S&P 500 index with strikes corresponding to 95%, 100%, 105% of one month forward price. The time series, obtained from the Bloomberg terminal, is based on the end-of-day observables for the period from January 3, 2007, to September 3, 2010 with the total of 626 days. In the analysis we assume the business calendar, with maturity of one month specified by $T = 1/12$ and daily interval specified by $\delta t = 1/252$.

2 Parametrization

We assume that the implied Black-Scholes-Merton (BSM) volatility, $\sigma_{BSM}(T, K)$, of vanilla options corresponding to strike K and maturity T is parametrized by function $\boldsymbol{\sigma}(x)$ defined as follows:

$$\boldsymbol{\sigma}(x) = \sigma_0 + \mathcal{S}x + \frac{\mathcal{C}}{2\sigma_0}x^2 \tag{2.1}$$

where x is the measure of moneyness:

$$x = \ln\left(\frac{K}{S}\right)$$

with S being the forward for maturity T. Here, σ_0 is the at-the-money-forward (ATM) volatility level, $\mathcal{S}$ is the skew parameter and $\mathcal{C}$ is the convexity (smile) parameter. The convexity parameter is normalized by σ_0 in line with equation (2.4). While it is straightforward to consider the case when the skew and convexity depend on the ATM volatility, $\mathcal{S} \equiv \mathcal{S}(\sigma_0)$ and $\mathcal{C} \equiv \mathcal{C}(\sigma_0)$, our empirical investigation shows that this dependence is very weak, so we do not consider it here. The dimension of these variables is as follows: $\lfloor \sigma_0 \rfloor = T^{-1/2}$, $\lfloor \mathcal{S} \rfloor = T^{-1/2}$, $\lfloor \mathcal{C} \rfloor = T^{-1}$.

Subsequently, $\boldsymbol{\sigma}$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(x)$, will stand for the function defined by (2.1), σ will stand for a constant volatility parameter, σ_0 will stand for the ATM volatility. We assume that vanilla options are priced using the implied volatility $\boldsymbol{\sigma}$ without making any explicit assumptions on the dynamics of underlying price S and ATM volatility σ_0.

2.1 Implication

A convenient way to characterize the market information about the ATM volatility, the skew and the convexity is to consider volatilities for options with fixed time-to-maturity time τ and three strikes $\{K(\tau)\} = \{(1-k)S(\tau), S(\tau), (1+k)S(\tau)\}$, where k is typically either 5% or 10%. This volatilities are inffered from implied volatilities of available options as follows:

$$\sigma^2_{BSM}(\tau, K(\tau)) = \frac{T_2 - \tau}{T_2 - T_1}\sigma^2_{BSM}(T_1, K(T_1)) + \frac{\tau - T_1}{T_2 - T_1}\sigma^2_{BSM}(T_2, K(T_2))$$

where T_1 and T_2 are annualized times-to-maturity of listed options such that $T_1 < \tau \leq T_2$. If listed options do not contain strike K, implied volatility $\sigma_{BSM}(T, K(T))$ is interpolated from implied volatilities at adjacent strikes with maturity time T.

The exact solution for σ_0, $\mathcal{S}$, $\mathcal{C}$ in equation (2.1), so that the implied volatilities at three strikes $\{K\}$ are matched, is given by:

$$\begin{aligned}
&\sigma_0 = \sigma_{BSM}(\tau, S(\tau)) \\
&\mathcal{S} = \frac{k_+^2 \sigma_{BSM}(\tau, (1-k)S(\tau)) - k_-^2 \sigma_{BSM}(\tau, (1+k)S(\tau)) - (k_+^2 - k_-^2)\sigma_{BSM}(\tau, S(\tau))}{k_+ k_- (k_+ - k_-)} \\
&\mathcal{C} = 2\sigma_0 \frac{k_+ \sigma_{BSM}(\tau, (1-k)S(\tau)) - k_- \sigma_{BSM}(\tau, (1+k)S(\tau)) - (k_+ - k_-)\sigma_{BSM}(\tau, S(\tau))}{k_+ k_- (k_+ - k_-)}
\end{aligned} \tag{2.2}$$

where $k_+ = \ln(1+k)$, $k_- = \ln(1-k)$.

If k is small, then $k_+ = -k_- \approx k$. Applying (2.2) with $k = 5\%$, we obtain:

$$\begin{aligned}
&\sigma_0 = \sigma_{BSM}(\tau, S(\tau)) \\
&\mathcal{S} = \frac{\sigma_{BSM}(\tau, 105\% S(\tau)) - \sigma_{BSM}(\tau, 95\% S(\tau))}{0.1} \\
&\mathcal{C} = \sigma_{BSM}(\tau, S(\tau)) \frac{\sigma_{BSM}(\tau, 105\% S(\tau)) + \sigma_{BSM}(\tau, 95\% S(\tau)) - 2\sigma_{BSM}(\tau, S(\tau))}{0.05^2}
\end{aligned} \tag{2.3}$$

Thus, σ_0 is the level of the ATM volatility, skew parameter $\mathcal{S}$ is the spread between the implied volatility of the out-of-the-money call and that of the out-of-the-money put (see Mixon (2010) for an interesting discussion of alternative measures of the implied volatility skew), convexity parameter $\mathcal{C}$ measures the convexity around the ATM volatility.

The top figure in Exhibit 3 shows time series of σ_0. The average value of σ_0 is 24.00% with the standard deviation of 11.21%, σ_0 follows a mean-reverting pattern rising during market sell-offs and falling during market rallies. The top figure in Exhibit 5 shows time series of $\mathcal{S}$ (using $k = 5\%$), whose average value is -68.31% with the standard deviation of 15.29%. Skew $\mathcal{S}$ also follows a mean-reverting pattern falling during market sell-offs and increasing during market rallies. The top figure in Exhibit 7 shows time series of $\mathcal{C}$ (using $k = 5\%$), whose average value is 49.93% with the standard deviation of 59.27%.

2.2 Justification

While the quadratic skew parametrization (2.1) can be considered as a simple tool to "connect the dots", it is related to the risk-neutral standard deviation, skewness, and kurtosis, of returns on the underlying asset. The following interesting result is derived by Backus-Foresi-Wu (2004). We denote by $\vartheta(T)\sqrt{T}$, $\gamma_1(T)$ $\gamma_2(T)$ the standard deviation, skewness, and kurtosis, respectively, of the log-forward price at time T under the risk-neutral measure. Backus-Foresi-Wu assume that vanilla options are valued using the Gram-Charlier expansion accounting for the first four moments and derive an approximation for the implied BSM volatility:

$$\sigma_{BSM}(d(x)) = \vartheta(T)\left(1 - \frac{\gamma_1(T)}{3!}d(x) - \frac{\gamma_2(T)}{4!}\left(1 - d(x)^2\right)\right)$$

where $d(x) = -x/(\vartheta(T)\sqrt{T}) + \vartheta(T)\sqrt{T}/2$.

Some rearrangement yields:

$$\begin{aligned}\sigma_{BSM}(d(x)) =& \vartheta(T) + \left(\frac{\gamma_1(T)}{3!\sqrt{T}} - \frac{\vartheta(T)\gamma_2(T)}{4!}\right)x + \frac{\gamma_2(T)}{4!\vartheta(T)T}x^2 \\ &+ \left(-\frac{\vartheta^2(T)\gamma_1(T)\sqrt{T}}{3!2} - (1+\vartheta^2(T)T)\frac{\vartheta(T)\gamma_2(T)}{4!}\right)\end{aligned} \tag{2.4}$$

Thus, skew parametrization (2.4) is similar to (2.1) with the only difference that the skew is a function of $d(x)$. Importantly, equation (2.4) illustrates the connection between the skewness and kurtosis of the density of log-returns and the implied volatility skew. We also note that, in equation (2.4) the skew and convexity are normalized by $\sqrt{T}$ and T, respectively. In equation (2.1), we do not apply this normalization for the sake of brevity. Since we assume short maturities the time-dependency of $\mathcal{S}$ and $\mathcal{C}$ is not essential. Finally, the kurtosis in front of x^2 term is normalized by $\vartheta(T)$, which we also assume by defining parametrization (2.1).

Furthermore, Zhou (2003) shows that the implied volatility skew generated by a stochastic local volatility model and the CEV local volatility model can be approximated by the quadratic volatility parametrization similar to (2.1) with coefficients $\mathcal{S}$ and $\mathcal{C}$ determined by the parameters of the volatility process. Thus, parametrization (2.1) can also be interpreted as a snapshot of the volatility skew implied by a specific model for the price dynamics.

3 P&L explain for vanilla options

Given that vanilla options are priced using the implied volatility $\boldsymbol{\sigma}$, the model value of a vanilla option, denoted by U, satisfies the following equation:

$$U(t, S, \sigma_0, \mathcal{S}, \mathcal{C}, K, T) = V(t, S, \boldsymbol{\sigma}, K, T) \tag{3.1}$$

where $V(t, S(T), \sigma, K, T)$ is the Black-Scholes-Merton (BSM) value, given by equation (8.1), for vanilla option with maturity time T and strike K on the asset with the forward price $S(T)$ valued using constant volatility σ.

We denote by Π the delta-hedged position in a vanilla option with value function U at time t:

$$\Pi(t, S, \sigma_0, \mathcal{S}, \mathcal{C}, K, T) = \frac{1}{\Gamma}\left[U(t, S, \sigma_0, \mathcal{S}, \mathcal{C}, K, T) - \Delta_S S\right] \tag{3.2}$$

where the notional of the position is normalized by the option cash gamma Γ, $\Gamma \equiv \Gamma(t, S, \boldsymbol{\sigma}, K, T)$, defined by equation (8.2). Here, Δ_S is the option delta obtained by differentiating equation (3.1):

$$\Delta_S = \frac{\partial}{\partial S} V(t, S, \sigma, K, T)\bigg|_{\sigma=\boldsymbol{\sigma}(x)} + \boldsymbol{\sigma}_S(x) V_\sigma(t, S, \sigma, K, T) \tag{3.3}$$

where $V_\sigma(t, S, \sigma, K, T)$ is the BSM option vega evaluated at implied volatility:

$$V_\sigma(t, S, \sigma, K, T) = \frac{\partial}{\partial \sigma} V(t, S, \sigma, K, T) \,|_{\sigma=\boldsymbol{\sigma}(x)}$$

and $\boldsymbol{\sigma}_S(x)$ is the volatility delta. Conceptually, the volatility delta is implied by the volatility parametrisation:

$$\boldsymbol{\sigma}_S(x) = -\left(\mathcal{S} + \frac{\mathcal{C}}{\sigma_0} x\right) \frac{1}{S} \tag{3.4}$$

In practice, an option hedger might use some other rule to specify $\boldsymbol{\sigma}_S(x)$, as we consider in equation (5.9). To model the effect of the volatility delta on the realized P&L, we introduce function $E(t, S, \delta S)$ defined by:

$$E(t, S, \delta S) = \frac{1}{\Gamma}\left[\left(\Delta^{(\text{realized})}\sigma(\delta S) - \Delta^{(\text{hedge})}\sigma(\delta S)\right) V_\sigma(t, S, \sigma, K, T)\right] \tag{3.5}$$

where $\Delta^{(\text{realized})}\sigma(\delta S)$ is the realized change in BSM volatility given change in the underlying price δS, $\Delta^{(\text{realized})}\sigma(\delta S) = \sigma_{BSM}(S + \delta S) - \sigma_{BSM}(S)$, and $\Delta^{(\text{hedge})}\sigma(\delta S)$, $\Delta^{(\text{hedge})}\sigma(\delta S) = \boldsymbol{\sigma}_S(x)\delta S$, is the change implied by computing the delta-hedge. We consider the delta-hedging error in more details in Section 5.3.

For brevity we assume zero interest rate and dividend yield, and no transaction costs. Given that the delta-hedged position is initiated at time t_0, the P&L at time $t = t_0 + \delta t$ given change is S, δS and in σ_0, $\delta\sigma_0$, is defined by:

$$\text{P\&L}(t_0, \delta t, S, \delta S, \sigma_0, \delta\sigma_0, K, T) \equiv \frac{1}{\Gamma}\left[\Pi(t_0 + \delta t, S + \delta S, \sigma_0 + \delta\sigma_0) - \Pi(t_0, S, \sigma_0)\right] \tag{3.6}$$

For the time being we ignore the delta-hedging error. By expanding value function $U(t_0+\delta t, S+\delta S, \sigma_0+\delta\sigma_0, \mathcal{S}, \mathcal{C}, K, T)$ in Taylor series in the first three variables (with the first order expansion in the time variable), we obtain:

$$\begin{aligned}\text{P\&L}(t_0, \delta t, S, \delta S, \sigma_0, \delta\sigma_0, K, T) &\approx \frac{1}{\Gamma}\left[U_t\delta t + \sigma_0 U_{\sigma_0}\left(\frac{\delta\sigma_0}{\sigma_0}\right)\right.\\ &\left.+\frac{1}{2}S^2U_{SS}\left(\frac{\delta S}{S}\right)^2 + \frac{1}{2}\sigma_0^2U_{\sigma_0\sigma_0}\left(\frac{\delta\sigma_0}{\sigma_0}\right)^2 + \sigma_0 SU_{\sigma_0 S}\left(\frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}\right)\right]\end{aligned} \tag{3.7}$$

where subscripts denote partial derivatives.

Our goal is to derive a simplified expression for equation (3.7) (a similar analysis is independently applied by Bergomi (2009)). All relevant computations are given in the Appendix. Plugging in obtained results, we approximate the P&L (with the approximation error $\Phi(T, x)$ defined by equation (3.10)) by:

$$\begin{aligned}\text{P\&L}(t_0, \delta t, S, \delta S, \sigma_0, \delta\sigma_0, K, T) &= -\sigma_0^2\widetilde{\Theta}\delta t + \widetilde{\Gamma}\left(\frac{\delta S}{S}\right)^2\\ &+\widetilde{\mathcal{V}}\left(\frac{\delta\sigma_0}{\sigma_0}\right) + \widetilde{\mathcal{V}\mathcal{V}}\left(\frac{\delta\sigma_0}{\sigma_0}\right)^2 + \widetilde{\Delta\mathcal{V}}\left(\frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}\right)\end{aligned} \tag{3.8}$$

where $\widetilde{\Theta}$, $\widetilde{\Gamma}$, $\widetilde{\mathcal{V}}$, $\widetilde{\mathcal{V}\mathcal{V}}$, and $\widetilde{\Delta\mathcal{V}}$ are defined as follows (taking $t_0 = 0$):

$$\begin{aligned}\widetilde{\Theta} &= 1 + 2\frac{\mathcal{S}}{\sigma_0}x + \left(\left(\frac{\mathcal{S}}{\sigma_0}\right)^2 + \frac{\mathcal{C}}{\sigma_0^2}\right)x^2\\ \widetilde{\Gamma} &= \left(1 + T\mathcal{C} - \frac{1}{4}\sigma_0^2T^2\mathcal{S}^2\right) - 2\frac{\mathcal{S}}{\sigma_0}\left(1 - \frac{1}{2}\mathcal{C}T\right)x + \left(3\left(\frac{\mathcal{S}}{\sigma_0}\right)^2 - 2\frac{\mathcal{C}}{\sigma_0^2}\left(1 - \frac{1}{4}\mathcal{C}T\right)\right)x^2\\ \widetilde{\mathcal{V}} &= 2\sigma_0^2T + 2\sigma_0T\mathcal{S}x\\ \widetilde{\mathcal{V}\mathcal{V}} &= -\frac{1}{4}\sigma_0^4T^2 + (1 + T\mathcal{C})\,x^2\\ \widetilde{\Delta\mathcal{V}} &= \sigma_0^2T\left(1 + \frac{1}{2}\sigma_0T\mathcal{S}\right) + (2 + \sigma_0T\mathcal{S} + 2T\mathcal{C})\,x - 4\frac{\mathcal{S}}{\sigma_0}\left(1 - \frac{1}{2}T\mathcal{C}\right)x^2\end{aligned} \tag{3.9}$$

One of the reasoning by deriving of the simplified P&L explain given by equation (3.8) is that sensitivities of vanilla options are proportional to the cash gamma Γ, as can be seen from equations (8.3). Therefore, the P&L can be linearised in terms of δt, $\frac{\delta S}{S}$, $\frac{\delta\sigma_0}{\sigma_0}$. Weights $\widetilde{\Gamma}$, $\widetilde{\Theta}$, $\widetilde{\mathcal{V}}$, $\widetilde{\mathcal{V}\mathcal{V}}$, and $\widetilde{\Delta\mathcal{V}}$ are quadratic functions of T and x.

For equation (3.8), the approximation error is given by function $\Phi(T,x)$:

$$\Phi(T,x) = O((\sigma_0^2 T)^3) + O((\sigma_0^2 T)^2 x) + O((\sigma_0^2 T)^2 x^2) + O(x^3) \qquad (3.10)$$

so that approximation (3.8) has the second order accuracy in maturity time T and log-moneyness x and the first order accuracy in the cross terms $\sigma_0^2 T x$ and $\sigma_0^2 T x^2$. From our computational experiments, we have found that there is a good agreement between (3.8) and (3.6) for a wide range of option strikes and maturities (up to one year).

3.1 Analysis

Before applying equation (3.8) for option strategies, we provide some initial analysis. We note that term $-\sigma_0^2 \widetilde{\Theta} \delta t$ is the carry cost of the position. For a long gamma position terms proportional to x and x^2 are much smaller than 1, so that the position is losing value due to time decay while accruing the realized return squared. For gamma-neutral positions, terms proportional to either x (as for the risk-reversal) or x^2 (as for the butterfly) dominate, so that, dependent on the magnitude of x and x^2 as well as the implied skew parametrization, the position can be very sensitive to changes in the ATM volatility and the covariance between returns in the underlying price and the ATM volatility. The risk to the delta-hedged position is associated with large realizations of $\left(\frac{\delta S}{S}\right)^2$, $\left(\frac{\delta \sigma_0}{\sigma_0}\right)$, $\left(\frac{\delta \sigma_0}{\sigma_0}\right)^2$, and $\left(\frac{\delta S}{S}\frac{\delta \sigma_0}{\sigma_0}\right)$.

[Exhibit 1 about here]

In Exhibit 1 we plot the frequencies of realized values of these quantities based on daily variations and average monthly variations, all quantities are annualized by multiplying by 252. We see that in case of the realized variance of the underlying price and the ATM volatility, $\left(\frac{\delta S}{S}\right)^2$ and $\left(\frac{\delta \sigma_0}{\sigma_0}\right)^2$, the daily realizations have heavy right tails. This implies that delta-hedged positions, with terms $\widetilde{\Gamma}$ and $\widetilde{\mathcal{V}\mathcal{V}}$ being negative and significant, can lead to large losses for option sellers (this, for example, happened to option dealers with short gamma positions during market sell-offs in November 2008 and May 2010 (see Cameron (2010))). The frequency of $\left(\frac{\delta S}{S}\frac{\delta \sigma_0}{\sigma_0}\right)$ has heavy left tail, implying that positions where term $\widetilde{\Delta\mathcal{V}}$ is negative and significant (as for a long position in the risk-reversal) benefit from large negative covariance between moves in the underlying price and the ATM volatility. Out of the three, the frequency of $\left(\frac{\delta \sigma_0}{\sigma_0}\right)$ is the most symmetric, implying that the position with sizeable vega term $\widetilde{\mathcal{V}}$ should not lead to tail risks in any direction. These observations confirm an old saying among volatility traders that "vega wounds, but gamma kills" (see Sinclair (2010), page 238).

4 Applications

We apply formula (3.8) to analyse the P&L of widely-spread option strategies involving trading in a portfolio of delta-hedged vanilla options. At this stage, we ignore the delta-hedging error E. By linearity, the portfolio P&L is the sum of component P&L-s, $\text{P\&L}_i$, weighted by notional amounts N_i:

$$\text{P\&L} = \sum_{i=1} N_i \text{P\&L}_i \tag{4.1}$$

4.1 Straddle

The straddle strategy includes a long position in an ATM call option and an ATM put option. The strategy is summarized in Table 1. The notional is a half of one unit normalized by the option cash-gamma. Position cash-gamma is the notional times the option cash-gamma and is equal to 1/2 for both options.

Type	Long/Short	Strike	x	Notional	Cash-gamma
Call	+1	$S(T)$	0	$N_c = \frac{1}{2\Gamma_c}$	$\frac{1}{2}$
Put	+1	$S(T)$	0	$N_p = \frac{1}{2\Gamma_p}$	$\frac{1}{2}$

Table 1: Straddle strategy

Applying formula (4.1) along with equation (3.8), we obtain:

$$\begin{aligned}\text{P\&L} =& -\sigma_0^2\delta t + \left(1 + \mathcal{C}T - \frac{1}{4}(\sigma_0 T\mathcal{S})^2\right)\left(\frac{\delta S}{S}\right)^2 + 2\sigma_0^2 T\left(\frac{\delta\sigma_0}{\sigma_0}\right) \\ &- \frac{1}{4}\sigma_0^4 T^2\left(\frac{\delta\sigma_0}{\sigma_0}\right)^2 + \sigma_0^2 T\left(1 + \frac{1}{2}\sigma_0 T\mathcal{S}\right)\left(\frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}\right)\end{aligned} \tag{4.2}$$

Ignoring terms of order $O(T)$, we obtain:

$$\text{P\&L} = -\sigma_0^2\delta t + \left(\frac{\delta S}{S}\right)^2 \tag{4.3}$$

Accordingly, for short-dated options, the P&L of the straddle is sensitive to the convexity parameter only in order T and to the skew parameter only in order T^2. The P&L is the most affected by the ATM volatility σ_0. The term $\sigma_0^2\delta t$ is the carry cost of the straddle strategy. The strategy is profitable if the realized variance is greater than the implied volatility squared.

[**Exhibit 2 about here**]

The top plot in Exhibit 2 shows the P&L of the straddle as function of δS computed exactly by means of equation (3.6) (denoted as "Exact") and approximately by means of equation (4.2) (The bottom plot is discussed in Section (5.2.1)). The relevant parameters are specified as follows: $\delta t = 1/250$, $T = 20/250$, $\sigma_0 = 0.15$, $\mathcal{S} = -0.68$, $\mathcal{C} = 0.5$ (these values correspond to the averages from the time series). The position size is 10000 option contracts. We see that the P&L obtained by the approximate formula is in good agreement with the exact result.

4.1.1 Break-even volatility

We consider a trading strategy under which the delta-hedged straddle with fixed maturity time T is rolled-over at the end of each day for the total number of M days. At the end of the m-th trading day, $m = 1, ..., M$, the previous position is unwound and a new trade for the delta-hedged straddle maturing at time T is entered; no delta-hedging is applied during the next day up to the end of the trading day. We note that the notional of the new trade is scaled by the cash-gamma so that the trade notional is constant for all days in the trading period. Using equation (4.3), the P&L of this strategy realized after M days is approximately given by:

$$\text{P\&L} = M\delta t\left(-\overline{\sigma_0^2} + \sigma_R^2\right) \tag{4.4}$$

where $\overline{\sigma_0^2}$ is the average ATM volatility squared:

$$\overline{\sigma_0^2} = \frac{1}{M}\sum_{m=1}^{M}\sigma_0^2(t_{m-1})$$

with $\sigma_0^2(t_m)$ being the implied ATM volatility at the beginning of m-th day, and σ_R^2 is the average realized variance:

$$\sigma_R^2 = \frac{1}{M}\sum_{m=1}^{M}\left(\frac{\delta S}{S}(t_m)\right)^2\frac{1}{\delta t}$$

with $\frac{\delta S}{S}(t_m) = \frac{S(t_m)-S(t_{m-1})}{S(t_{m-1})}$ being the realized return.

Thus, the strategy is profitable if $\sigma_R^2 > \overline{\sigma_0^2}$. Therefore the delta-hedged straddle can be considered as a bet on the spread between the implied ATM volatility and the realized ATM volatility.

[Exhibit 3 about here]

In Exhibit 3, the top figure shows time series of one month averages of $(\overline{\sigma_0^2})^{1/2}$ and $(\sigma_R^2)^{1/2}$. The bottom plot shows the average monthly P&L of the straddle ("Straddle P&L") computed using equation (6.2), on the left scale, and the difference between $(\overline{\sigma_0^2})^{1/2}$ and $(\sigma_R^2)^{1/2}$ ("Spread"), on the right scale. We see that typically, the implied volatility is traded at a premium to the realized volatility, implying that the P&L of the long position in the delta-hedged straddle is losing money most of the time. Notable exceptions include extreme sell-offs in October 2008, February 2009, and May 2010, when short volatility strategies (such as short position in a straddle) back-fired. In our sample the implied volatility overestimates the realized volatility by about 12%, which is close to the number reported by Bergomi (2004) (13%).

4.2 Risk-reversal

The risk-reversal strategy includes a long position in the put with strike $K = (1-k)S(T)$ and a short position in the call with strike $K = (1+k)S(T)$, where k is typically 5%. The strategy is summarized in Table 2. Option notionals are scaled by options cash-gamma and term $4k\sigma_0$.

Type	Long/Short	Strike	x	Notional	Cash-gamma
Call	-1	$(1+k)S(T)$	$\ln(1+k) \approx +k$	$N_c = \frac{1}{4k\sigma_0\Gamma_c}$	$-\frac{1}{4k\sigma_0}$
Put	$+1$	$(1-k)S(T)$	$\ln(1-k) \approx -k$	$N_p = \frac{1}{4k\sigma_0\Gamma_p}$	$\frac{1}{4k\sigma_0}$

Table 2: Risk-reversal strategy

Applying formula (3.8), we obtain:

$$\begin{aligned}\text{P\&L} = \mathcal{S}\delta t + \frac{\mathcal{S}}{\sigma_0^2}\left(1 - \frac{1}{2}T\mathcal{C}\right)\left(\frac{\delta S}{S}\right)^2 - T\mathcal{S}\left(\frac{\delta\sigma_0}{\sigma_0}\right) \\ - \left(\frac{1}{\sigma_0} + T\left(\frac{1}{2}\mathcal{S} + \frac{\mathcal{C}}{\sigma_0}\right)\right)\left(\frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}\right)\end{aligned} \tag{4.5}$$

Dropping terms of order $O(T)$, we obtain:

$$\text{P\&L} = \mathcal{S}\delta t + \frac{\mathcal{S}}{\sigma_0^2}\left(\frac{\delta S}{S}\right)^2 - \frac{1}{\sigma_0}\left(\frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}\right) \tag{4.6}$$

As a result, the P&L is highly sensitive to the skew parameter used to compute the delta-hedge. The realized P&L is influenced by the realized variance of the underlying price and covariance between returns in the underlying price and the ATM volatility. Given that $\mathcal{S} < 0$, the term $\mathcal{S}\delta t$ is the carry cost of the risk-reversal strategy. The strategy is profitable given a large negative realized covariance between the underlying and the ATM volatility, which should be higher than the first two terms in equation (4.6). We note that in equation (4.6) the second term is the realized variance times the skew, which is negative, so that the risk-reversal tends to be least profitable given a large realized variance and a small change in the ATM volatility.

[**Exhibit 4 about here**]

The top plot in Exhibit 4 shows the P&L of the risk-reversal (using $k = 5\%$) as function of δS computed exactly by means of equation (3.6) (denoted as "Exact") and approximately by means of equation (4.5) (The bottom plot is discussed in Section (5.2.1)). We see that the P&L obtained by the approximate formula is in good agreement with the exact result if $\delta S/S$ is more than -1%. For smaller values of $\delta S/S$, higher order terms dominate the P&L. At the same time, we see that the risk-reversal is pofitable only for large negative moves in the underlying price. For small returns, the carry cost $\mathcal{S}$ leads to persistant losses for the long position in the delta-hedged risk-reversal.

4.2.1 Break-even skew

Similarly to the analysis in Section (4.1.1), we assume that the delta-hedged risk-reversal trade maturing at time T is rolled-over at the end of each day for the total number of M days. Using equation (4.6), the realized P&L of this strategy is approximately given by:

$$\text{P\&L} = M\delta t\left(\overline{\mathcal{S}} + \overline{\mathcal{S}}\Sigma_1 - \Sigma_2\right) \tag{4.7}$$

where $\overline{\mathcal{S}}$ is the average implied skew:

$$\overline{\mathcal{S}} = \frac{1}{M}\sum_{m=1}^{M}\mathcal{S}(t_{m-1}),$$

with $\mathcal{S}(t_m)$ being the implied skew at the beginning of m-th day, and Σ_1 is computed by:

$$\Sigma_1 = \frac{1}{M}\sum_{m=1}^{M}\left(\frac{\delta S}{S}(t_m)\right)^2 \frac{1}{\sigma_0^2(t_{m-1})\delta t} \tag{4.8}$$

and Σ_2 is computed by:

$$\Sigma_2 = \frac{1}{M}\sum_{m=1}^{M}\left(\frac{\delta S}{S}(t_m)\frac{\delta\sigma_0}{\sigma_0}(t_m)\right) \frac{1}{\sigma_0(t_{m-1})\delta t}$$

The break-even skew $\mathcal{S}_{BE}$ is defined by:

$$\mathcal{S}_{BE} = \frac{\Sigma_2}{1+\Sigma_1} \tag{4.9}$$

The strategy is profitable if $\overline{S} > \mathcal{S}_{BE}$. Accounting that $\mathcal{S} < 0$, the strategy is profitable if the absolute value of the realized skew is greater than the implied: $|\overline{S}| < |\mathcal{S}_{BE}|$.

In equation (4.7), we apply the following approximation:

$$\frac{1}{M}\sum_{m=1}^{M}\mathcal{S}(t_{m-1})\left(\frac{\delta S}{S}(t_m)\right)^2 \frac{1}{\sigma_0^2(t_{m-1})\delta t} \approx \overline{S}\Sigma_1$$

noting that, based on our empirical analysis, the difference between the two is immaterial.

We note that in equation (4.9), Σ_1 can be considered an indicator of how the variance implied by the market predicts the realized variance of the underlying. Ideally, if the prediction is correct, then $\Sigma_1 = 100\%$. However, based on our data the historic average of Σ_1 is 97.28%, so that the implied variance over-estimates the actual variance by about 3% in relative terms. We have examined a simpler definition of the break-even skew: $\mathcal{S}_{BE} = (1/2)\Sigma_2$, however this estimator tends to have higher standard derivation, by about 100%, compared to the one proposed in equation (4.9).

Σ_2 can be interpreted as the covariance between the realized ATM volatility and the ratio of the volatility of realized return over the predicted ATM volatility. If the latter is correct and moves in the ATM volatility are perfectly inversely correlated with moves in returns, then $\Sigma_2 = -100\%$. In fact, we obtain that the historic average of Σ_2 is -110%.

[**Exhibit 5 about here**]

In Exhibit 5, the top figure shows time series of one month averages of $\overline{\mathcal{S}}$ and $\mathcal{S}_{BE}$. The bottom plot shows the average monthly P&L of the risk-reversal (”Risk-reversal P&L”) computed using equation (6.2), on the left scale, and the spread between the break-even and implied skew, $\mathcal{S}_{BE} - \overline{\mathcal{S}}$ (”Spread”), on the right scale. We see that typically, the implied skew $\overline{\mathcal{S}}$ is traded at a (negative) premium to the realized skew. As a result, the P&L of the long position in the delta-hedged straddle is losing money most of the times with only exceptions occurred during large sell-offs.

4.3 Butterfly

The butterfly strategy includes a long position in the call with strike $K = (1+k)S(T)$, a short position in the call with strike $K = S(T)$, a short position in the put with strike $K = S(T)$, and a long position in the put with strike $K = (1+k)S(T)$ (alternatively, the butterfly can be constructed by a long position in a call with strike $K = (1+k)S(T)$ and a call with strike $K = (1-k)S(T)$, and a short position in two calls with $K = S(T)$). The option notional is normalized by the cash-gamma and term $2k^2$. The strategy is summarized in Table 3.

Type	Long/Short	Strike	x	Notional	Cash-gamma
Call	$+1$	$(1+k)S(T)$	$+k$	$N_{c+} = \frac{1}{2\Gamma_{c+}k^2}$	$\frac{1}{2k^2}$
Call	-1	$S(T)$	0	$N_c = \frac{1}{2k^2\Gamma_c}$	$-\frac{1}{2k^2}$
Put	-1	$S(T)$	0	$N_p = \frac{1}{2k^2\Gamma_p}$	$-\frac{1}{2k^2}$
Put	$+1$	$(1-k)S(T)$	$-k$	$N_{p-} = \frac{1}{2k^2\Gamma_{p-}}$	$\frac{1}{2k^2}$

Table 3: Butterfly strategy

Applying formula (3.8), we obtain:

$$\begin{aligned} \text{P\&L} = &-(\mathcal{C}+\mathcal{S}^2)\delta t + \left(-2\left(1-\frac{1}{4}T\mathcal{C}\right)\frac{\mathcal{C}}{\sigma_0^2} + 3\left(\frac{\mathcal{S}}{\sigma_0}\right)^2\right)\left(\frac{\delta S}{S}\right)^2 \\ &+ (1+T\mathcal{C})\left(\frac{\delta\sigma_0}{\sigma_0}\right)^2 - 4\frac{\mathcal{S}}{\sigma_0}\left(1-\frac{1}{2}T\mathcal{C}\right)\left(\frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}\right) \end{aligned} \tag{4.10}$$

In the first order in T we obtain:

$$\text{P\&L} = -(\mathcal{C}+\mathcal{S}^2)\delta t + \left(-2\frac{\mathcal{C}}{\sigma_0^2} + 3\left(\frac{\mathcal{S}}{\sigma_0}\right)^2\right)\left(\frac{\delta S}{S}\right)^2 + \left(\frac{\delta\sigma_0}{\sigma_0}\right)^2 - 4\frac{\mathcal{S}}{\sigma_0}\left(\frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}\right) \tag{4.11}$$

Thus, the realized P&L of the butterfly is sensitive to the volatility skew parametrization. Given that $\mathcal{C} > 0$, the term $-(\mathcal{C} + \mathcal{S}^2)\delta t$ is the carry cost of the butterfly strategy. When the skew and the convexity parameters are large, the cost of carry will be very high so that the straddle strategy is profitable only for extreme large moves in the ATM volatility and a realized positive covariance between returns in the underlying price and the ATM volatility (which is very unlikely). If $\mathcal{C}$ and $\mathcal{S}$ are small, the P&L is mostly affected by the dynamics of the realized variance of the ATM volatility. Thus, the straddle strategy can be considered as a bet on the realized variance of the ATM volatility in the market with a small implied skew and convexity.

[**Exhibit 6 about here**]

The bottom plot in Exhibit 6 shows the P&L of the butterfly (using $k = 5\%$) as function of δS computed exactly by means of equation (3.6) (denoted as "Exact") and approximately by means of equation (4.10). We see that the P&L obtained by the approximate formula is in good agreement with the exact result if $\delta S/S$ is between -1% and 1%. For larger values of $\delta S/S$, higher order terms dominate the P&L. We see that the butterfly is pofitable only for large positive moves in the underlying price accompanied by large moves in the ATM volatility. For small returns, the carry cost $(\mathcal{C} + \mathcal{S}^2)$ leads to persistent small losses for the long position in the delta-hedged butterfly.

4.3.1 Break-even convexity

Similarly to the analysis in Section (4.1.1), we assume that the delta-hedged butterfly trade maturing at time T is rolled-over at the end of each trading day for the total number of M days. Using equation (4.11), the realized P&L of this strategy is approximately given by:

$$\text{P\&L} = M\delta t \left(-\overline{\mathcal{C}} - \overline{\mathcal{S}^2} - 2\overline{\mathcal{C}}\Sigma_1 + 3\Sigma_3 + \Sigma_4 - 4\Sigma_5 \right) \tag{4.12}$$

where $\overline{\mathcal{C}}$ is the average implied convexity:

$$\overline{\mathcal{C}} = \frac{1}{M} \sum_{m=1}^{M} \mathcal{C}(t_{m-1})$$

with $\mathcal{C}(t_m)$ being the implied convexity at the beginning of m-th day; $\overline{\mathcal{S}^2}$ is the average skew squared:

$$\overline{\mathcal{S}^2} = \frac{1}{M} \sum_{m=1}^{M} \mathcal{S}^2(t_{m-1})$$

Σ_1 is defined by (4.8), Σ_3 is computed by:

$$\Sigma_3 = \frac{1}{M}\sum_{m=1}^{M}\left(\frac{\delta S}{S}(t_m)\right)^2\left(\frac{\mathcal{S}(t_{m-1})}{\sigma_0(t_{m-1})}\right)^2\frac{1}{\delta t}$$

Σ_4 is computed by:

$$\Sigma_4 = \frac{1}{M}\sum_{m=1}^{M}\left(\frac{\delta\sigma_0}{\sigma_0}(t_m)\right)^2\frac{1}{\delta t},$$

and Σ_5 is computed by:

$$\Sigma_5 = \frac{1}{M}\sum_{m=1}^{M}\left(\frac{\delta S}{S}(t_m)\frac{\delta\sigma_0}{\sigma_0}(t_m)\right)\frac{\mathcal{S}(t_{m-1})}{\sigma_0(t_{m-1})\delta t}$$

In equation (4.12), we employ the following approximation:

$$\frac{1}{M}\sum_{m=1}^{M}\mathcal{C}(t_{m-1})\left(\frac{\delta S}{S}(t_m)\right)^2\frac{1}{\sigma_0^2(t_{m-1})\delta t} \approx \overline{\mathcal{C}}\Sigma_1$$

noting that, based on our empirical analysis, the difference between the two is immaterial.

In equation (4.12), Σ_3 is the ratio of the realized variance versus the implied variance multiplied by the skew squared. The historic average of Σ_3 is 49.55%. Σ_5 is the ratio of the implied covariance to the realized one. Its historic average is 80.02%. Σ_4 is the realized variance of the ATM volatility, its historic average is 166.59%.

The break-even convexity $\mathcal{C}_{BE}$ is defined by:

$$\mathcal{C}_{BE} = \frac{\overline{\mathcal{S}^2} + 3\Sigma_3 + \Sigma_4 - 4\Sigma_5}{1 + 2\Sigma_1} \tag{4.13}$$

so that the strategy is profitable if $\overline{C} < \mathcal{C}_{BE}$. If the convexity dominates the skew, $\mathcal{C} >> \mathcal{S}$, then:

$$\mathcal{C}_{BE} = \frac{\Sigma_4}{1 + 2\Sigma_1} \approx \frac{1}{3}\Sigma_4$$

so that the $\mathcal{C}_{BE}$ directly relates to the realized variance of the ATM volatility.

[**Exhibit 7 about here**]

In Exhibit 7, the top figure shows the time series of one month averages of $\overline{\mathcal{C}}$ and $\mathcal{C}_{BE}$, and the bottom figure shows the time series of the spread between the break-even and implied convexity, $\mathcal{C}_{BE} - \overline{\mathcal{C}}$. The bottom plot shows the average monthly P&L of the butterfly ("Butterfly P&L") computed using equation (6.2), on the left scale, and the difference between $\overline{\mathcal{C}}$ and $\mathcal{C}_{BE}$ ("Spread"), on the right scale. We see that most of the time, the implied convexity $\overline{\mathcal{C}}$ is traded at a (negative) premium to the realized convexity. Similarly, Bergomi (2004) finds that the implied volatility of the ATM volatility is typically overestimated by a factor of two compared to the realized volatility of volatility. As a result, the long position in the delta-hedged butterfly is bound to lose money most of the time.

5 Single factor dynamics

In the previous section we assumed no dependence between changes in the underlying price δS and change in ATM volatility $\delta\sigma_0$. In this section we consider the single factor model where a change in the ATM volatility σ_0 is a quadratic function of a change in the underlying price implied by the skew parametrization (2.1):

$$\delta\sigma_0 = \beta\mathcal{S}\frac{\delta S}{S} + \frac{1}{2}\alpha\frac{\mathcal{C}}{\sigma_0}\left(\frac{\delta S}{S}\right)^2 \tag{5.1}$$

where β and α are specified constants.

The model corresponds to the following regression model which can be estimated using the historical data:

$$\sigma_0(t_m) - \sigma_0(t_{m-1}) = \beta\mathcal{S}(t_{m-1})R(t_m) + \frac{1}{2}\alpha\frac{\mathcal{C}(t_{m-1})}{\sigma_0(t_{m-1})}R^2(t_m) + \epsilon_m \tag{5.2}$$

where $R(t_m) = (S(t_m) - S(t_{m-1}))/S(t_{m-1})$ and ϵ_m is standard normal random variable with zero mean and stationary variance.

Applying this model on the our data we obtain the following estimates:

$$\beta = 1.5788, \ \alpha = 3.2888 \tag{5.3}$$

with the R-square coefficient of 80.36%. Thus 80% of variations in changes in the implied ATM volatility is explained by changes in the underlying price.

We note that Gatheral-Kamal (2010) apply a similar regression model with $\alpha = 0$ for one-month options on the S&P500 index from 2 January 2001 to 9 February 2009 to obtain that $\beta = 1.55$ with the R-square coefficient of 77.4%. As a result, the regression model (5.1) appears to be stable over different time frames and serve as a useful tool to get additional insight.

5.1 Dynamics of implied volatility

We note that given a change in underlying price $S \to S + \delta S$, the moneyness changes by:

$$x(S + \delta S) - x(S) \approx -\frac{\delta S}{S} + \frac{1}{2}\left(\frac{\delta S}{S}\right)^2$$

Now, given the change of the ATM volatility (5.1), we consider the change in the volatility as function of x using parametrization (2.1) along with the above expansion, up to the second order:

$$\begin{aligned}
&\delta\boldsymbol{\sigma}(x) \equiv \boldsymbol{\sigma}(x(S + \delta S)) - \boldsymbol{\sigma}(x(S)) \\
&= \delta\sigma_0 - \left(\mathcal{S} + x\frac{\mathcal{C}}{\sigma_0}\right)\frac{\delta S}{S} + \frac{1}{2}\left(\mathcal{S} + \frac{\mathcal{C}}{\sigma_0}(1 + x)\right)\left(\frac{\delta S}{S}\right)^2 \\
&= \left((\beta - 1)\mathcal{S} - x\frac{\mathcal{C}}{\sigma_0}\right)\frac{\delta S}{S} + \frac{1}{2}\left(\mathcal{S} + \frac{\mathcal{C}}{\sigma_0}(1 + \alpha + x)\right)\left(\frac{\delta S}{S}\right)^2 + O\left(\left(\frac{\delta S}{S}\right)^3\right)
\end{aligned} \tag{5.4}$$

We discuss important conclusions (based on the classification of volatility regimes proposed by Derman (1999b)):
$\beta = 1$ roughly corresponds to the sticky-strike rule when the implied volatility is fixed to strike K and does not change following changes in the underlying price;
$\beta < 1$ roughly corresponds to the sticky-delta rule when the implied volatility is fixed to $K/S(T)$ and rises as underlying price drops;
$\beta > 1$ roughly corresponds to the sticky-local volatility rule when the implied volatility falls as underlying price drops.

In the equity market, when the underlying price is range-bounded, implied volatilities tend to be sticky-strike. When the underlying price follows trending patterns, implied volatilities tend to be sticky-delta. During volatile regimes, implied volatilities tend to be sticky-local volatility (for empirical studies see, for example, Alexander (2001), Daglish *et al* (2007) and Crépey (2004)).

5.2 Realized P&L

In this section, we analyse the P&L under the implied dynamics for the ATM volatility (5.1). We define the one-period realized return R and realized variance σ_R^2, respectively, by:

$$R = \frac{1}{\delta t}\left(\frac{\delta S}{S}\right), \; \sigma_R^2 = \frac{1}{\delta t}\left(\frac{\delta S}{S}\right)^2$$

and note that the current analysis is based on the assumption that σ_R^2 is small.

For the straddle, we use equation (4.3) along with (5.1) to obtain:

$$\text{P\&L} = \delta t\left[-\sigma_0^2 + 2\sigma_0 T\beta\mathcal{S}R + \left(1 + T(\sigma_0\beta\mathcal{S} + \mathcal{C}(1+\alpha)) - \frac{1}{4}T^2(\sigma_0\mathcal{S})^2\left(1-\beta\right)^2\right)\sigma_R^2\right] \tag{5.5}$$

The delta-hedged straddle is profitable in the first order if $\sigma_R^2 > \sigma_0^2$ so that the straddle is not noticeably affected by the dependence between returns in the underlying price and the ATM volatility. However, due to the skew effect there is dependence, which is first order in T, on the realized return arising from the vega term.

For the risk-reversal, we use equations (4.6) and (5.1) to obtain:

$$\text{P\&L} = \mathcal{S}\delta t\left(1 - \frac{\mathcal{S}}{\sigma_0}T\beta R - \left((\beta-1) + \frac{1}{2}T(\mathcal{C}(1+\alpha) + \beta(\mathcal{S}\sigma_0 + 2\mathcal{C}))\right)\frac{\sigma_R^2}{\sigma_0^2}\right) \tag{5.6}$$

and, in the first order:

$$\text{P\&L} = \mathcal{S}\delta t\left(1 - (\beta-1)\frac{\sigma_R^2}{\sigma_0^2}\right),$$

so that, given that $\mathcal{S} < 0$, the strategy is profitable for large values of the realized volatility and strong dependence, measured by β, between returns in the underlying price and the ATM volatility. Given that $\mathcal{S} < 0$, the long position in the risk-reversal is profitable if:

$$\beta > 1 + \frac{\sigma_0^2}{\sigma_R^2}$$

so that, in case $\sigma_0^2 \approx \sigma_R^2$, the regression coefficient should be above two, $\beta > 2$, for the risk-reversal to be profitable.

Finally, for the butterfly, we use equations (4.10) and (5.1) to obtain:

$$\text{P\&L} = \delta t \left[-(\mathcal{C}+\mathcal{S}^2) + \left(\left(\mathcal{S}^2((\beta-2)^2-1) - 2\mathcal{C}\right) + \frac{1}{2} T\mathcal{C}\left(\mathcal{C} + 2\mathcal{S}^2\beta(\beta+2)\right) \right) \frac{\sigma_R^2}{\sigma_0^2} \right] \tag{5.7}$$

and, in the first order in T:

$$\text{P\&L} = \delta t \left[-(\mathcal{C}+\mathcal{S}^2) + \left(\mathcal{S}^2((\beta-2)^2-1) - 2\mathcal{C}\right) \frac{\sigma_R^2}{\sigma_0^2} \right]$$

As a result, the butterfly depends on the relationship between the underlying price and the ATM volatility, but it is rather difficult to analyse it in general. Consider the following simplified cases. First if the convexity dominates the skew, $\mathcal{C} >> \mathcal{S}$, then the P&L is dominated by term $-\mathcal{C}(1+2\sigma_R^2/\sigma_0^2)$, so that the P&L is negative if $\mathcal{C} > 0$ (which is the case on most of the markets) and positive if otherwise. Therefore, the butterfly strategy can be considered as a play on the realized convexity in a market with insignificant skew effect. Second if the skew dominates the convexity, $\mathcal{S} >> \mathcal{C}$, then we can show that the P&L of the butterfly is negative if $\beta \in [2-\sqrt{1+\sigma_0^2/\sigma_R^2}, 2+\sqrt{1+\sigma_0^2/\sigma_R^2}]$, so that if $\sigma_0^2 \approx \sigma_R^2$ then $\beta \in [0.58, 3.41]$. Thus the strategy is profitable if β is either very small or very large.

5.2.1 Illustration

Here we analyse the impact of the dependence between the underlying price and ATM volatility moves. The bottom figure in Exhibit 2 shows the one-day P&L of the straddle computed using equation (3.6) with $\mathcal{S} = \mathcal{C} = 0$ ("Exact, Flat Vol"), the P&L computed using equation (3.6) with $\delta\sigma_0 = 0$ ("Exact, Delta vol =0"), the P&L computed using equation (3.6) with $\delta\sigma_0$ specified by the regression model (5.2) with parameter estimates in (5.3) ("Exact, Delta vol implied"), the P&L approximated using equation (5.5) with regression parameters (5.3). The relevant parameters are specified as follows: $\delta t = 1/252$, $T = 1/12$, $\sigma_0 = 0.24$, $\mathcal{S} = -0.68$, $\mathcal{C} = 0.50$. The bottom figure in Exhibit 4 shows the same computations for the risk-reversal using equation (5.6) for the approximate P&L. The bottom figure in Exhibit 6 shows identical computations for the butterfly using equation (5.7) for the approximate P&L. The position size is 10000 option contracts.

For the straddle, we see that the P&L is not significantly impacted by incorporating the skew and convexity, as can be seen by comparing the P&L

illustrated by "Flat Vol" and "Delta vol=0". However, under the negative relationship between moves in the underlying price and the ATM volatility, as implied by the regression model (5.2), the P&L of the straddle is negatively affected by positive moves in the underlying price and, in the opposite, when the underlying price falls the P&L increases faster. Our approximation (5.5) is in close agreement with the exact result.

For the risk-reversal, the P&L using the flat volatility is small for moderate values of $\delta S/S$. In contrast, under the skew parametrization and independence between returns in the underlying price and the volatility ("Exact, Delta vol =0"), the P&L is shifted more to the left because when $\delta S < 0$ the implied volatility for the put is assumed to fall more than the implied volatility for the call decreasing the profit. At the same time, when $\delta S > 0$ the the increase in the implied volatility of the call is larger than increase in the put volatility, which leads to larger loss. Under the regression model, when the underlying price falls, the implied volatility for the put option grows increasing the profit, while when the underlying price rises, the implied volatility for the call falls reducing the loss. Our approximation (5.6) is in agreement with the exact result for moderate values of $\delta S/S$ (-2% to 2%).

For the butterfly, the P&L under the independence between the underlying and volatility moves ("Exact, Delta vol =0"), when $\delta S < 0$, increases slower because the implied volatility of the put option falls, while, when $\delta S > 0$, it increases faster because the implied volatility of the call option grows. With the dependence, the P&L grows faster when the underlying price falls while increases slower when the underlying price rises. At the same time, all being the same, the underlying return $\delta S/S$ should be sizeable, with $|\delta S/S| > 4\%$ for the P&L to be positive. Thus, the long position in the butterfly consistently loses money most of the time, while occasionally benefiting from large moves in the underlying price. Our approximation (5.6) is in agreement with the exact result for moderate values of $\delta S/S$ (-2% to 2%).

5.3 Delta-hedging error

Now given the implied dynamics (5.1), we analyse the delta-hedging error induced by mis-specifying the option delta. Using equation (8.6) for option vega, we approximate the delta-hedging error, defined by equation (3.5) by:

$$E(t, S, \delta S) = 2\sigma_0 T \left(1 + \frac{\mathcal{S}}{\sigma_0} x\right) \Xi(t, S, \delta S) \tag{5.8}$$

where

$$\Xi(t, S, \delta S) = \left(\delta\boldsymbol{\sigma}(x) - \Delta^{(\text{hedge})}\sigma(\delta S)\right)$$

and $\delta\boldsymbol{\sigma}(x)$ is specified by equation (5.4).

We assume that the volatility delta $\boldsymbol{\sigma}_S(x)$, which is used to compute the option delta Δ_S in equation (3.3), is computed by

$$\boldsymbol{\sigma}_S^{(\text{hedge})}(x) = \left((\lambda - 1)\mathcal{S} - qx\frac{\mathcal{C}}{\sigma_0}\right)\frac{1}{S} \tag{5.9}$$

where λ and q are specified according to subjective view of an option hedger. Choice with $\lambda = 0$ and $q = 1$ corresponds to the case considered in the previous section. Here we analyse more general situations. Given equation (5.9), the implied change in the volatility is then given by

$$\Delta^{(\text{hedge})}\sigma(\delta S) = \left((\lambda - 1)\mathcal{S} - qx\frac{\mathcal{C}}{\sigma_0}\right)\frac{\delta S}{S} \tag{5.10}$$

Using equation (5.8) along with (5.10), we obtain that for the straddle strategy the delta-hedging error becomes:

$$E(t, S, \delta S) = 2\sigma_0 T\left[(\beta - \lambda)\mathcal{S}\left(\frac{\delta S}{S}\right) + \frac{1}{2}\left(\mathcal{S} + \frac{\mathcal{C}}{\sigma_0}(1 + \alpha)\right)\left(\frac{\delta S}{S}\right)^2\right]$$

thus, the variance of the delta-hedging error is minimized if the straddle is delta-hedged with λ close to the estimated slope of regression coefficient β.

By analogy, for the risk-reversal strategy, the delta-hedging error is given by:

$$\begin{aligned} E(t, S, \delta S) = -T&\left[\left((\beta - \lambda)\frac{\mathcal{S}^2}{\sigma_0} + (q - 1)\frac{\mathcal{C}}{\sigma_0}\right)\left(\frac{\delta S}{S}\right)\right. \\ &\left.+\frac{1}{2}\left(\frac{\mathcal{S}^2}{\sigma_0} + \left(1 + (1 + \alpha)\frac{\mathcal{S}}{\sigma_0}\right)\frac{\mathcal{C}}{\sigma_0}\right)\left(\frac{\delta S}{S}\right)^2\right] \end{aligned}$$

so that the variance of the delta-hedging error is minimized if λ is close to β and also $q = 1$.

Finally, for the butterfly strategy, the delta-hedging error is specified by:

$$E(t, S, \delta S) = T\frac{\mathcal{C}\mathcal{S}}{\sigma_0}\left[2(q - 1)\left(\frac{\delta S}{S}\right) + \left(\frac{\delta S}{S}\right)^2\right]$$

so that the variance is minimized if $q = 1$ or, in other words, the hedging is performed according to the implied volatility delta.

In section (6.2), we provide some further empirical analysis to support the above reached conclusion.

6 Empirical analysis

In this Section we analyse the daily P&L of delta-hedged option strategies obtained by using the empirical data.

6.1 P&L explain

We consider delta-hedged position in an option strategy at time t_n:

$$\Pi(t_n) = \sum_{i=1} \frac{N_i}{\Gamma_i(t_n)} \left[U(t_n, S(t_n), \sigma_0(t_n), \mathcal{S}(t_n), \mathcal{C}(t_n), K_i S(t_n), T) - S(t_n)\Delta_i(t_n)\right] \tag{6.1}$$

where $\Delta_i(t_n)$ is the delta at time t_n, N_i is the number of contracts in the i-th option scaled by option cash-gamma $\Gamma_i(t_n)$ computed at time t_n, K_i is the option strike proportional to the forward maturing at time T.

We assume that the delta-hedged position is initialized at the end of the n-th day and unwound at the end of the $n+1$-th day. The actual P&L realized at time t_{n+1} is then given by:

$$\begin{aligned} \text{P\&L}^{(R)}(t_{n+1}) = \sum_{i=1} \frac{N_i}{\Gamma_i(t_n)} &\left[U(t_{n+1}, S(t_{n+1}), \sigma_0(t_{n+1}), \mathcal{S}(t_{n+1}), \mathcal{C}(t_{n+1}), K_i S(t_n), T)\right. \\ &\left. -U(t_n, S(t_n), \sigma_0(t_n), \mathcal{S}(t_n), \mathcal{C}(t_n), K_i S(t_n), T) - (S(t_{n+1}) - S(t_n))\Delta_i(t_n)\right] \end{aligned} \tag{6.2}$$

We also consider the predicted P&L defined by:

$$\begin{aligned} \text{P\&L}^{(P)}(t_{n+1}) = \sum_{i=1} \frac{N_i}{\Gamma_i(t_n)} &\left[U(t_{n+1}, S(t_{n+1}), \sigma_0(t_{n+1}), \mathcal{S}(t_n), \mathcal{C}(t_n), K_i S(t_n), T)\right. \\ &\left. -U(t_n, S(t_n), \sigma_0(t_n), \mathcal{S}(t_n), \mathcal{C}(t_n), K_i S(t_n), T) - (S(t_{n+1}) - S(t_n))\Delta_i(t_n)\right] \end{aligned} \tag{6.3}$$

and finally we analyse the approximate P&L defined by:

$$\text{P\&L}^{(A)}(t_{n+1}) = \sum_{i=1} N_i \text{P\&L}(t_n, \delta t_n, S(t_n), \delta S_n, \sigma_0(t_n), \delta\sigma_0, K_i S(t_n), T) \tag{6.4}$$

where $\text{P\&L}(t, \delta t, S, \delta S, \sigma_0, \delta\sigma_0, K, T)$ is given by equation (3.8) with $\delta t_n = t_{n+1} - t_n$, $\delta S_n = S(t_{n+1}) - S(t_n)$, $\delta\sigma_0 = \sigma_0(t_{n+1}) - \sigma_0(t_n)$.

We note that the actual P&L reproduces the exact P&L from the delta-hedged position accounting both for changes in the underlying variables $S(t)$

and $\sigma_0(t)$ as well as day-to-day changes in parameter values of skew $\mathcal{S}$ and convexity $\mathcal{C}$. The predicted P&L produces the P&L that corresponds only to changes in the underlying variables $S(t)$ and $\sigma_0(t)$ assuming that the skew and convexity parameters stay fixed. The approximate P&L replicates the P&L assuming only changes in underlying variables $S(t)$ and $\sigma_0(t)$ and relying on the derived approximation for the P&L explain. We note that if the skew and smile parameters are stable day-to-day then the predicted P&L will be the same as the actual P&L. The approximate P&L is also based on the assumption that skew and smile parameters are stable day-to-day. In this case, the approximate P&L is expected to be close to the actual P&L. In the opposite case, if skew $\mathcal{S}$ and convexity $\mathcal{S}$ parameters are unstable, the approximate P&L is expected to be close to the predicted P&L.

We analyse the above considered straddle, risk-reversal and butterfly option strategies and, for each of the strategy, compute the three quantities P&L$^{(R)}$, P&L$^{(P)}$, P&L$^{(A)}$ for each day in the sample. In total we have 625 P&L realizations for the three strategies. The number of option contracts is 10000 for the straddle, 500 for the risk-reversal, and 50 for the butterfly (with this choice all three strategies have approximately equal notional cash-gamma).

In Exhibits 8, 9, and 10, we provide four plots for the straddle, risk-reversal and butterfly strategies, respectively. We assume that the option delta is computed using the implied delta given by equation (3.4). The top left plot shows the predicted P&L versus the actual P&L, the top right plot shows the approximate P&L versus the actual P&L, the bottom left plot shows the approximate P&L versus the predicted P&L, the bottom right plot shows the approximate P&L, which is computed using equation (5.5) with the regression model parameters specified by equation (5.3) versus the actual P&L. We note that if the predicted P&L exactly reproduces the actual P&L then all points will lay on the straight line through the origin, any difference between the two is due to the changes in the skew and convexity parameters. If the approximate P&L is close to the predicted then the expansion of the P&L to the second order has enough explanatory power and the quality of our approximation is good.

[**Exhibit 8 about here**]

From Exhibit 8, we see that the straddle is less sensitive to the changes in the skew and convexity parameters so that all three realized quantities for the P&L explain are very close to each other.

[**Exhibit 9 about here**]

From Exhibit 9, it follows that the risk-reversal is more sensitive to the changes in the skew and convexity parameters. As we see, the predicted P&L for the risk-reversal does not fit the actual as well as that for the straddle. The reason is variability in the skew parameter. At the same time, the approximate P&L fits the predicted P&L reasonably well indicating that the P&L can be explained by changes in the underlying and the ATM volatility in the second order, if the value of the skew parameter is stable stable.

[**Exhibit 10 about here**]

Finally, as we see from Exhibit 10, it is very difficult to explain the P&L of the butterfly, as the P&L is significantly affected by changes in skew and convexity parameters. The approximate P&L fits predicted P&L to some extent indicating the P&L can be explained by changes in the underlying price and the ATM volatility, but only if these changes are relatively small. In general, the P&L of the butterfly is very small for relatively small changes in the underlying price and ATM volatility, while it is largely affected by changes in the skew and convexity parameters.

By regressing the actual P&L of the straddle against the approximate P&L computed using the implied volatility dynamics (5.1), we find that the latter explains about 77% of the P&L variance. The regression for the the approximate P&L of the risk-reversal explains about 33% of the P&L variance. Finally, the regression for the butterfly suggest that less than 1% is explained by the approximate P&L computed using the implied volatility dynamics.

6.2 Delta-hedging error

Now we study the impact of the specifying of the option delta on the realized P&L. We use equation (6.2) to compute realizations of the P&L assuming that the option delta is computed using formula (5.10) with the following specifications: 1) hedging at implied volatility with $\lambda = 0$ and $q = 1$, 2) hedging assuming the sticky-strike volatility with $\lambda = 1$ and $q = 0$, 3) hedging assuming the implied dynamics (5.1) with parameter estimates (5.3) with $\lambda = 1.5788$ and $q = 0$. The number of contracts is specified in the same way as in the above section. The descriptive statistics of P&L realizations is reported in Exhibit 11.

We see that the volatility of the P&L for the straddle and risk-reversal is minimized when hedging with $\lambda = 1.6$, while the volatility of the P&L for the butterfly is minimized when hedging at the implied delta. We confirm analytical findings of Section 5.3.

7 Conclusion

We have derived an approximation for the P&L of a delta-hedged option position under the quadratic skew parametrization. We have applied this result to study the P&L of the straddle, the risk-reversal, and the butterfly. As a result, we have derived the break-even realized skew and convexity that equate the realized P&L of the risk-reversal and the butterfly, respectively, to zero. Similarly to the notion of "trading of the implied volatility versus the realized volatility" in the Black-Scholes-Merton (1973) framework, the break-even realized skew and convexity serve as tools to analyse the P&L of the straddle, the risk-reversal and the butterfly in the presence of the volatility skew.

Using time series data of the implied volatilities of the S&P 500 index, we have analysed realizations of P&L's of these trading strategies. We have found that the P&L of the straddle is the most sensitive to the ATM volatility and the realized variance, while it is little affected by the implied volatility parametrization. The P&L of the risk-reversal is the most sensitive to the skew parameter and, given that the skew and convexity parameters are stable, the P&L can be well explained by the second order changes in the underlying price and the ATM volatility. The P&L of the butterfly is the most sensitive to day-to-day changes in the skew and convexity parameters and, given that these parameters are stable, the P&L can also be well explained by the second order changes in the underlying price and the ATM volatility.

We have analysed the delta-hedging error arising from the sensitivity of the volatility change to the change in the underlying price, assumed by computing the option delta, and the realized change in the implied volatility given the change in the underlying price. We have have found that the volatility of the delta-hedging error for the straddle and the risk-reversal is minimized when they are hedged using the sensitivity implied by regression model (5.1) ($\lambda = \beta$ and $q = 0$ in equation (5.9)), while the volatility for the butterfly is minimized when the butterfly is hedged using the slope implied by the volatility parametrization ($\lambda = 0$ and $q = 1$ in equation (5.9)).

References

[1] Alexander, C. (2001), "A Guide to Financial Data Analysis," in *Wiley.*

[2] Cameron, L. (2010), "Goldman hit by short equity volatility position," *Risk* **7**, 26-30.

[3] Carr, P., Frequently Asked Questions in Option Pricing Theory. *Working paper*, 2005.

[4] Backus, D., Foresi, S. and Wu, L. (2004), "Accounting for biases in Black-Scholes," *Working paper.*

[5] Bergomi, L. (2004), "Smile Dynamics I," *Risk* **9**, 117-123.

[6] Bergomi, L. (2009), "Smile Dynamics IV," *Risk* **12**, 94-100.

[7] Black, F. and Scholes, M. (1973), "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* **81**, 637-659.

[8] Breeden, D. and Litzenberger, R. (1978), "Prices of state contingent claims implicit in option prices," *Journal of Business* **51(6)**, 621-651.

[9] Crépey, S. (2004), "Delta-hedging vega risk?," *Quantitative Finance*, **5**, 559-597.

[10] Chriss, N., Derman, E, and Kani, I. (1996), "Implied trinomial trees of the volatility smile," *Journal of Derivatives*, **3(4)**, 7-22.

[11] Daglish, T., Hull, J., and Suo, W. (2007), "Volatility surfaces: Theory, rules of thumb, and empirical evidence," *Quantitative Finance*, **7**, 507-524.

[12] Derman, E., and Kani, I. (1994), "The volatility smile and its implied tree," *Goldman Sachs Quantitative Strategies Research Notes.*

[13] Demeterfi, K., Derman, E., Kamal, M. and Zou, J. (1999), "More than you ever wanted to know about volatility swaps," *Goldman Sachs Quantitative Strategies Research Notes.*

[14] Derman, E. (1999a), " When You Cannot Hedge Continuously, The Corrections of Black-Scholes," Risk, **4**, 82-85.

[15] Derman, E. (1999b), "Regimes of Volatility," *Risk*, **4**, 55-59.

[16] Dumas, B., Fleming, J. and Whaley, R. (1998), "Implied Volatility Functions," *Journal of Finance*, **8**, 2059-2106.

[17] Dupire, B. (1994), "Pricing with a smile," *Risk*, **7**, 18-20.

[18] El Karoui, N., Jeanblanc, M., and Shreve, S. (1998) "Robustness of the Black and Scholes formula," *Mathematical Finance*, **2**, 93-126.

[19] Gatheral, J. and Kamal, M. (2010), "Implied Volatility Surface," in *Encyclopedia of Quantitative Finance, Wiley.*

[20] Jackwerth, J. K. (1999), "Option-Implied Risk-Neutral Distributions and Implied Binomial Trees," *Journal of Derivatives*, **7(2)**, 66-82.

[21] Lipton, A., *Mathematical methods for foreign exchange: A Financial Engineer's Approach*, 2001 (Singapore: World Scientific).

[22] Merton, R. (1973), "Theory of Rational Option Pricing," *The Bell Journal of Economics and Management Science* **4(1)**, 141-183.

[23] Mixon, S. (2010), "What Does Implied Volatility Skew Measure?" *Working paper*, ssrn.com/abstract=1618602.

[24] Rubinstein, M. (1994), "Implied binomial trees," *Journal of Finance*, **49**, 771-818.

[25] Sepp, A. (2010), "An Approximate Distribution of Delta-Hedging Errors in a Jump-Diffusion Model with Discrete Trading and Transaction Costs," *Quantitative Finance*, forthcoming.

[26] Sinclair, E. (2008), "Volatility Trading," *New York: Wiley.*

[27] Sinclair, E. (2010), "Option Trading: Pricing and Volatility Strategies and Techniques," *New York: Wiley.*

[28] Wilmott, P., *Paul Wilmott on Quantitative Finance*, 2006 (New York: Wiley, second edition).

[29] Zhou, F. (2003), "Black smirks," *Risk* **5**, 87-91.

8 Appendix. Derivation of equation (3.8)

We recall that the Black-Scholes-Merton (1973) value of a vanilla call or put option on the underlying with forward price S, strike K, and maturity time T using zero interest rate and constant volatility σ is given by:

$$V(t, S, \sigma, K, T) = \varpi \left(S\mathcal{N}(\varpi d_1(\sigma)) - K\mathcal{N}(\varpi d_2(\sigma))\right) \tag{8.1}$$

where $\mathcal{N}(y)$ is the cumulative density function of standard normal, $\varpi = +1$ for a call option and $\varpi = -1$ for a put option, $x = \ln(K/S)$ and

$$d_1(\sigma) = \frac{-x + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},\ d_2(\sigma) = \frac{-x - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

We define the option cash-gamma by (it is the same for both calls and puts):

$$\Gamma(\sigma) \equiv \Gamma(t, S, \sigma, K, T) \equiv \frac{1}{2}S^2 V_{SS} = \frac{Sn(d_1(\sigma))}{2\sigma\sqrt{T-t}} \tag{8.2}$$

where $n(y)$ is the standard normal density function.

For the sake of simplicity, subsequently we assume that $t = 0$. We use the following partial derivatives of the Black-Scholes-Merton formula for a vanilla call or put option $V(t, S, \sigma, K, T)$ with σ treated as a constant:

$$\begin{aligned}
V_t &\equiv -\frac{Sn(d_1(\sigma))\sigma}{2\sqrt{T}} = -\sigma^2\Gamma(\sigma) \\
V_{SS} &\equiv \frac{n(d_1(\sigma))}{S\sigma\sqrt{T}} = \frac{2\Gamma(\sigma)}{S^2} \\
V_\sigma &\equiv Sn(d_1(\sigma))\sqrt{T} = 2\sigma T\Gamma(\sigma) \\
V_{\sigma\sigma} &\equiv \frac{V_\sigma d_1(\sigma)d_2(\sigma)}{\sigma} = 2d_1(\sigma)d_2(\sigma)T\Gamma(\sigma) \\
V_{S\sigma} &\equiv -\frac{d_2(\sigma)n(d_1(\sigma))}{\sigma} = -2\frac{d_2(\sigma)\sqrt{T}\Gamma(\sigma)}{S}
\end{aligned} \tag{8.3}$$

Next we consider the partial derivatives of the model value U using equation (3.1) with σ treated as a function of S and σ_0:

$$\begin{aligned}
U_t &= V_t \\
U_S &= V_S + V_\sigma \boldsymbol{\sigma}_S \\
U_{SS} &= V_{SS} + 2V_{S\sigma}\boldsymbol{\sigma}_S + V_{\sigma\sigma}\boldsymbol{\sigma}_S^2 + V_\sigma \boldsymbol{\sigma}_{SS} \\
U_{\sigma_0} &= V_\sigma \boldsymbol{\sigma}_{\sigma_0} \\
U_{\sigma_0\sigma_0} &= V_{\sigma\sigma}\boldsymbol{\sigma}_{\sigma_0}^2 + V_\sigma \boldsymbol{\sigma}_{\sigma_0\sigma_0} \\
U_{S\sigma_0} &= V_{S\sigma}\boldsymbol{\sigma}_{\sigma_0} + V_{\sigma\sigma}\boldsymbol{\sigma}_{\sigma_0}\boldsymbol{\sigma}_S + V_\sigma \boldsymbol{\sigma}_{S\sigma_0}
\end{aligned} \tag{8.4}$$

where $\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}(x)$.

Partial derivatives of the implied volatility function (2.1) are given by:

$$\begin{aligned}
&\boldsymbol{\sigma}_S = -\frac{1}{S}\left(\mathcal{S} + \frac{\mathcal{C}}{\sigma_0}x\right), \ \boldsymbol{\sigma}_{SS} = \frac{1}{S^2}\left(\mathcal{S} + \frac{\mathcal{C}}{\sigma_0} + \frac{\mathcal{C}}{\sigma_0}x\right), \\
&\boldsymbol{\sigma}_{\sigma_0} = 1 - \frac{1}{2}\frac{\mathcal{C}}{\sigma_0^2}x^2, \ \boldsymbol{\sigma}_{\sigma_0\sigma_0} = \frac{\mathcal{C}}{\sigma_0^3}x^2, \ \boldsymbol{\sigma}_{S\sigma_0} = \frac{1}{S}\frac{\mathcal{C}}{\sigma_0^2}x
\end{aligned} \tag{8.5}$$

Now we combine (8.3), (8.4), and (8.5). We start with U_{SS} term. We keep Γ to be fixed as function of x and consider:

$$\begin{aligned}
U_{SS} &= \frac{2\Gamma(\boldsymbol{\sigma})}{S^2}\left(1 - 2d_2(\boldsymbol{\sigma})\sqrt{T}S\boldsymbol{\sigma}_S + d_1(\boldsymbol{\sigma})d_2(\boldsymbol{\sigma})TS^2\boldsymbol{\sigma}_S^2 + \boldsymbol{\sigma}TS^2\boldsymbol{\sigma}_{SS}\right) \\
&= \frac{2\Gamma(\boldsymbol{\sigma})}{S^2}\left(\widetilde{\Gamma} + \Phi(T,x)\right)
\end{aligned}$$

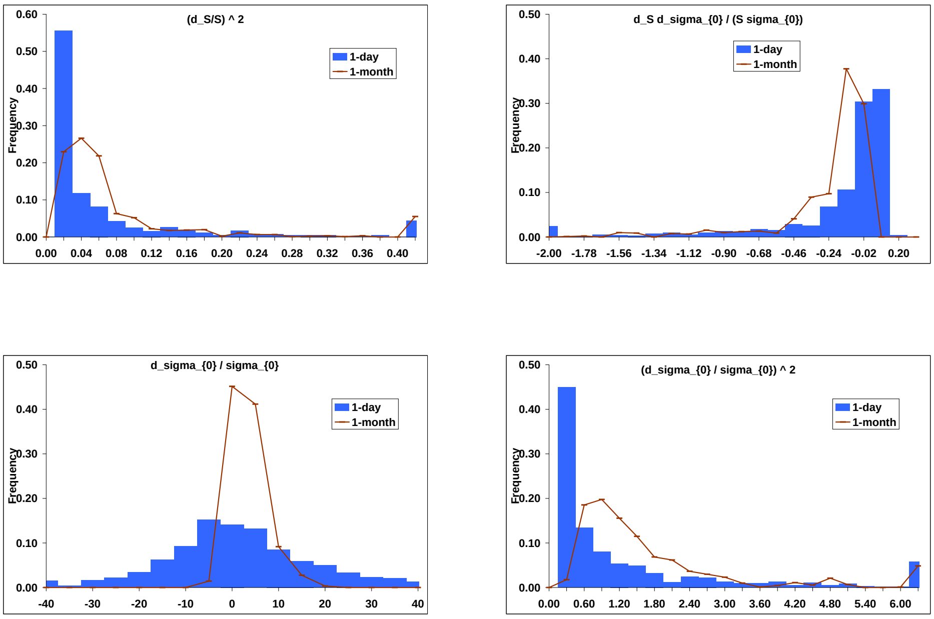
where in the first line we expand terms in the brackets in Taylor series around $x = 0$ (we emphasize that Γ is kept fixed as function of x) and keep only second order terms with the approximation error $\Phi(T,x)$ defined by equation (3.10) and $\widetilde{\Gamma}$ is defined by equation (3.9)

Applying the same analysis to other terms, we obtain:

$$\begin{aligned}
U_t &= -\Gamma(\boldsymbol{\sigma})\boldsymbol{\sigma}^2 = -\Gamma(\boldsymbol{\sigma})\left(\sigma_0^2\widetilde{\Theta} + \Phi(T,x)\right) \\
U_{\sigma_0} &= 2\Gamma(\boldsymbol{\sigma})T\boldsymbol{\sigma}\boldsymbol{\sigma}_{\sigma_0} = \frac{\Gamma(\boldsymbol{\sigma})}{\sigma_0}\left(\widetilde{\mathcal{V}} + \Phi(T,x)\right) \\
U_{\sigma_0\sigma_0} &= 2\Gamma(\boldsymbol{\sigma})\left(Td_1(\boldsymbol{\sigma})d_2(\boldsymbol{\sigma})\boldsymbol{\sigma}_{\sigma_0}^2 + T\boldsymbol{\sigma}\boldsymbol{\sigma}_{\sigma_0\sigma_0}\right) \\
&= \frac{2\Gamma(\boldsymbol{\sigma})}{\sigma_0^2}\left(\widetilde{\mathcal{V}\mathcal{V}} + \Phi(T,x)\right) \\
U_{S\sigma_0} &= 2\Gamma(\boldsymbol{\sigma})\left(-\frac{d_2(\boldsymbol{\sigma})\sqrt{T}}{S}\boldsymbol{\sigma}_{\sigma_0} + d_1(\boldsymbol{\sigma})d_2(\boldsymbol{\sigma})T\boldsymbol{\sigma}_{\sigma_0}\boldsymbol{\sigma}_S + \boldsymbol{\sigma}T\boldsymbol{\sigma}_{S\sigma_0}\right) \\
&= \frac{\Gamma(\boldsymbol{\sigma})}{\sigma_0 S}\left(\widetilde{\Delta\mathcal{V}} + \Phi(T,x)\right)
\end{aligned} \tag{8.6}$$

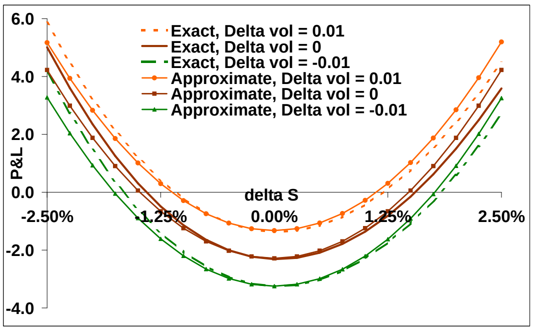
where $\widetilde{\Theta}$, $\widetilde{\mathcal{V}}$, $\widetilde{\mathcal{V}\mathcal{V}}$, $\widetilde{\Delta\mathcal{V}}$ are defined by equation (3.9).

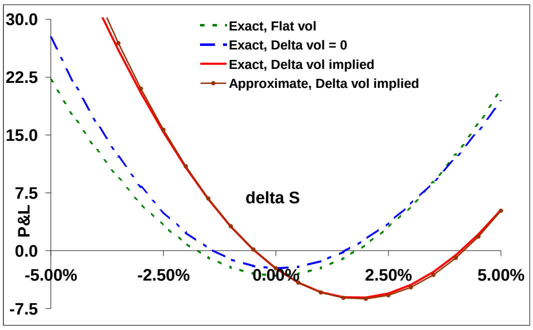
Exhibit 1. Empirical frequencies



The top left plot shows the empirical frequency of daily and the one-month average realizations of $\left(\frac{\delta S}{S}\right)^2$, the top right plot shows the corresponding frequency of $\left(\frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}\right)$, the bottom left plot shows the corresponding frequency of $\left(\frac{\delta\sigma_0}{\sigma_0}\right)$, and the bottom right plot shows the corresponding frequency of $\left(\frac{\delta\sigma_0}{\sigma_0}\right)^2$. All quantities are annualized by multiplying by 252.

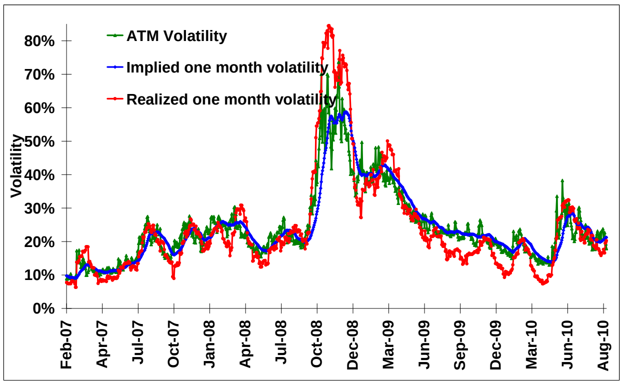
Exhibit 2. P&L of straddle

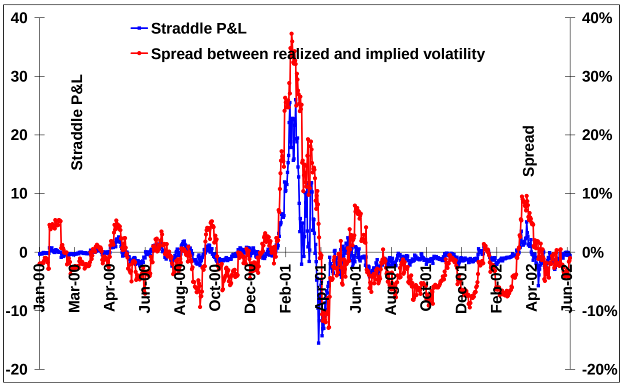




The top plot shows the one-day P&L of the straddle as function of $\delta S/S$ computed using equation (3.6) (denoted as "Exact") and approximately using equation (4.2) ("Approximate") for three values of $\delta\sigma_0/\sigma_0$: 1%, 0, -1%, the relevant parameters are specified as follows: $\delta t = 1/250$, $T = 1/12$, $\sigma_0 = 0.24$, $\mathcal{S} = -0.68$, $\mathcal{C} = 0.50$, notional of the position is $N = 10000$. The bottom plot shows the one-day P&L of the straddle computed using equation (3.6) with $\mathcal{S} = \mathcal{C} = 0$ ("Exact, Flat Vol"), the P&L computed using equation (3.6) with $\delta\sigma_0 = 0$ ("Exact, Delta vol =0"), the P&L computed using equation (3.6) with $\delta\sigma_0$ specified by the regression model (5.2) with parameter estimates in (5.3) ("Exact, Delta vol implied"), the P&L approximated using equation (5.5) with regression parameters (5.3).

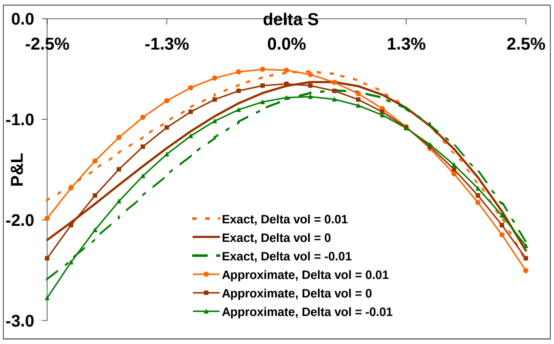
Exhibit 3. Realized volatility

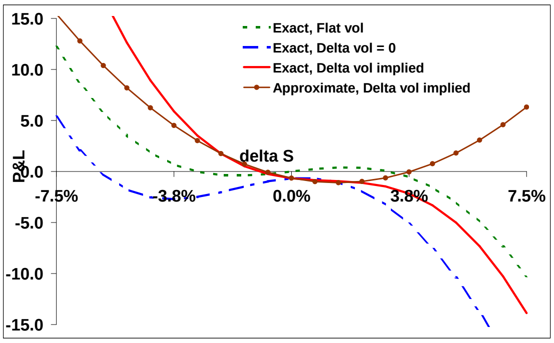




The top plot shows the daily time series of the ATM volatility $\sigma_0(t)$, one month average implied volatility $(\overline{\sigma_0^2})^{1/2}$ and average one month realized volatility $(\sigma_R^2)^{1/2}$ defined by equation (4.4). The bottom plot shows the average monthly P&L of the straddle ("Straddle P&L") computed using equation (6.2), on the left scale, and the difference between $(\overline{\sigma_0^2})^{1/2}$ and $(\sigma_R^2)^{1/2}$ ("Spread"), on the right scale.

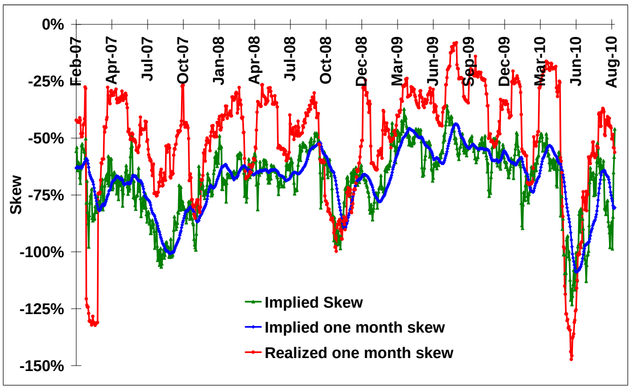
Exhibit 4. P&L of risk-reversal

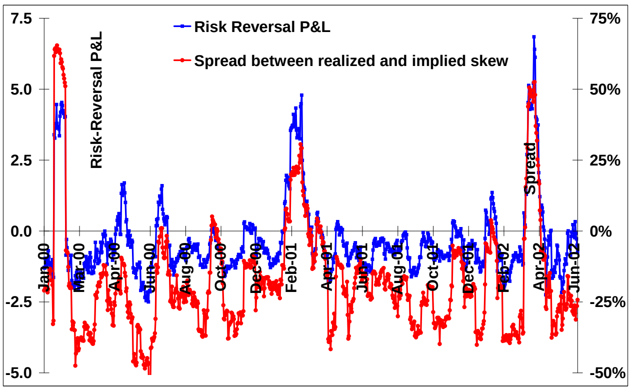




The top plot shows the one-day P&L of the risk-reversal as function of $\delta S/S$ computed using equation (3.6) (denoted as "Exact") and approximately using equation (4.5) ("Approximate") for three values of $\delta\sigma_0/\sigma_0$: 1%, 0, -1%, with relevant parameters as in Exhibit 2. The bottom plot shows the one-day P&L of the risk-reversal computed using equation (3.6) with $\mathcal{S} = \mathcal{C} = 0$ ("Exact, Flat Vol"), the P&L computed using equation (3.6) with $\delta\sigma_0 = 0$ ("Exact, Delta vol =0"), the P&L computed using equation (3.6) with $\delta\sigma_0$ specified by the regression model (5.2) with parameter estimates in (5.3) ("Exact, Delta vol implied"), the P&L approximated using equation (5.6) with regression parameters (5.3).

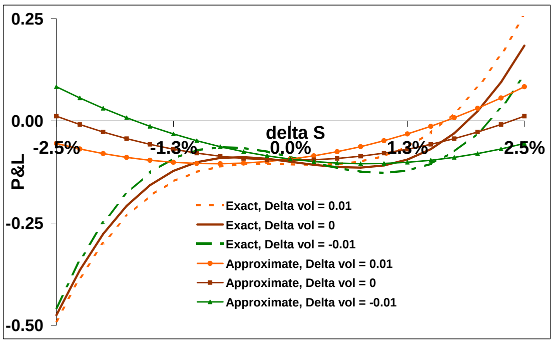
Exhibit 5. Realized skew

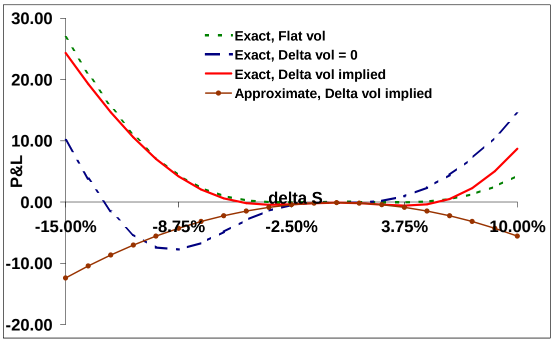




The top plot shows the daily time series of the implied skew $\mathcal{S}$, one month average implied skew $\overline{\mathcal{S}}$ defined by equation (4.7) and one month break-even skew $\mathcal{S}_{BE}$ defined by equation (4.9). The bottom plot shows the average monthly P&L of the risk-reversal ("Risk-reversal P&L") computed using equation (6.2), on the left scale, and the difference between $\overline{\mathcal{S}}$ and $\mathcal{S}_{BE}$ ("Spread"), on the right scale.

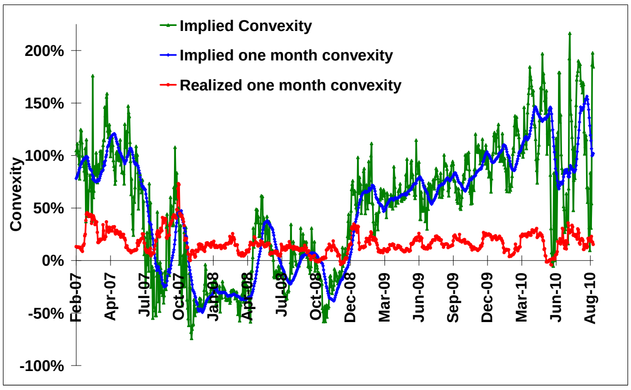
Exhibit 6. P&L of butterfly

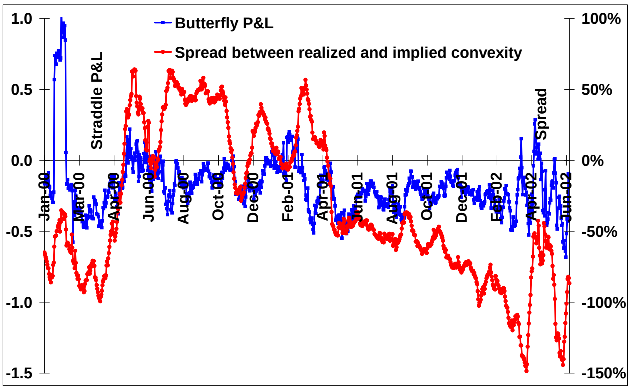




The top plot shows the one-day P&L of the butterfly as function of $\delta S/S$ computed using equation (3.6) (denoted as "Exact") and approximately using equation (4.10) ("Approximate") for three values of $\delta\sigma_0/\sigma_0$: 1%, 0, -1%, with relevant parameters as in Exhibit 2. The bottom plot shows the one-day P&L of the butterfly computed using equation (3.6) with $\mathcal{S} = \mathcal{C} = 0$ ("Exact, Flat Vol"), the P&L computed using equation (3.6) with $\delta\sigma_0 = 0$ ("Exact, Delta vol =0"), the P&L computed using equation (3.6) with $\delta\sigma_0$ specified by the regression model (5.2) with parameter estimates in (5.3) ("Exact, Delta vol implied"), the P&L approximated using equation (5.7) with regression parameters (5.3).

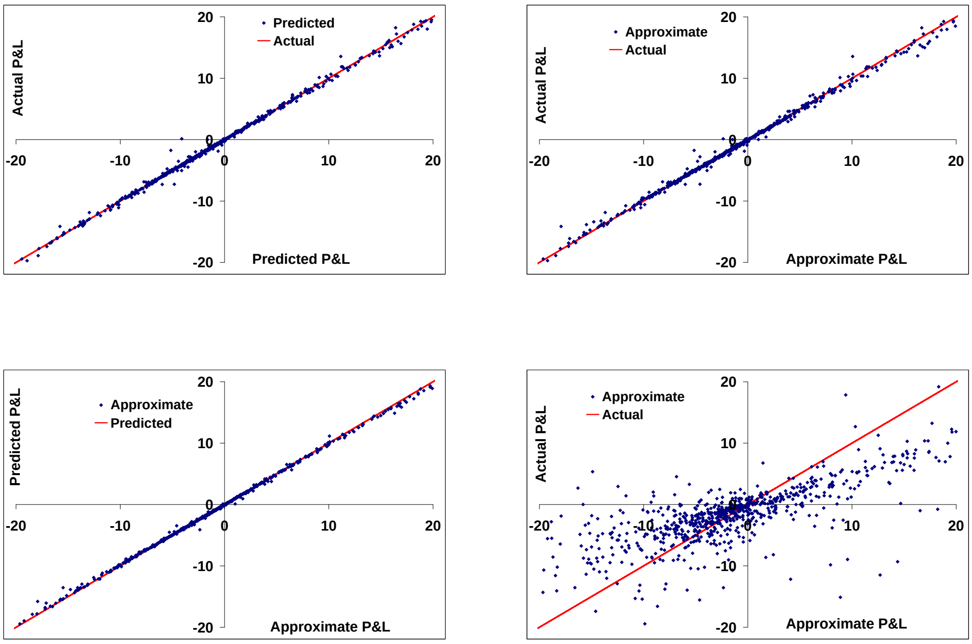
Exhibit 7. Realized convexity





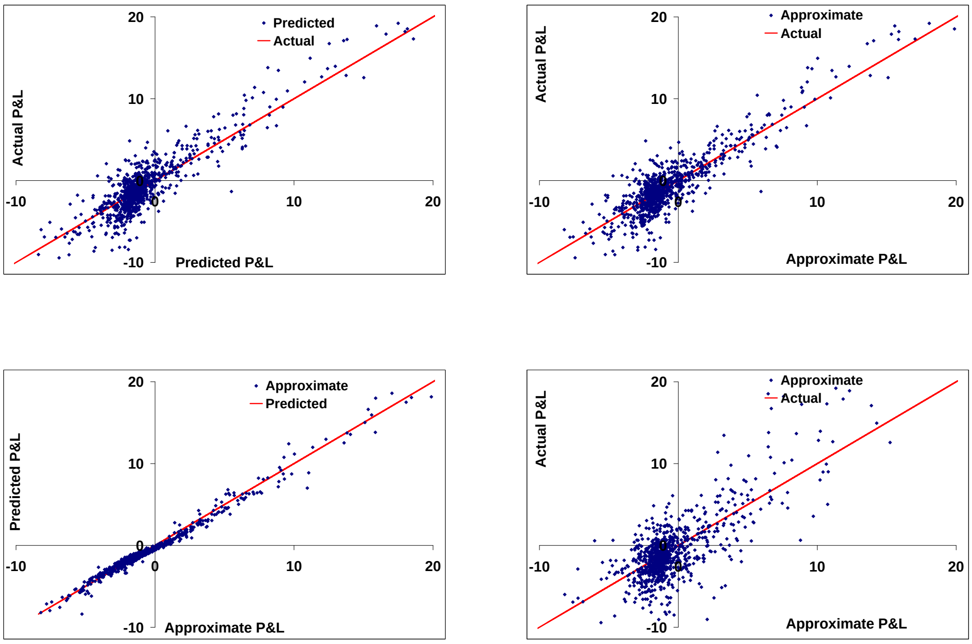
The top plot shows the daily time series of the implied convexity $\mathcal{C}$, one month average implied convexity $\overline{\mathcal{C}}$ defined by equation (4.12) and one month break-even convexity $\mathcal{C}_{BE}$ defined by equation (4.13). The bottom plot shows the average monthly P&L of the butterfly ("Butterfly P&L") computed using equation (6.2), on the left scale, and the difference between $\overline{\mathcal{C}}$ and $\mathcal{C}_{BE}$ ("Spread"), on the right scale.

Exhibit 8. P&L explain for straddle



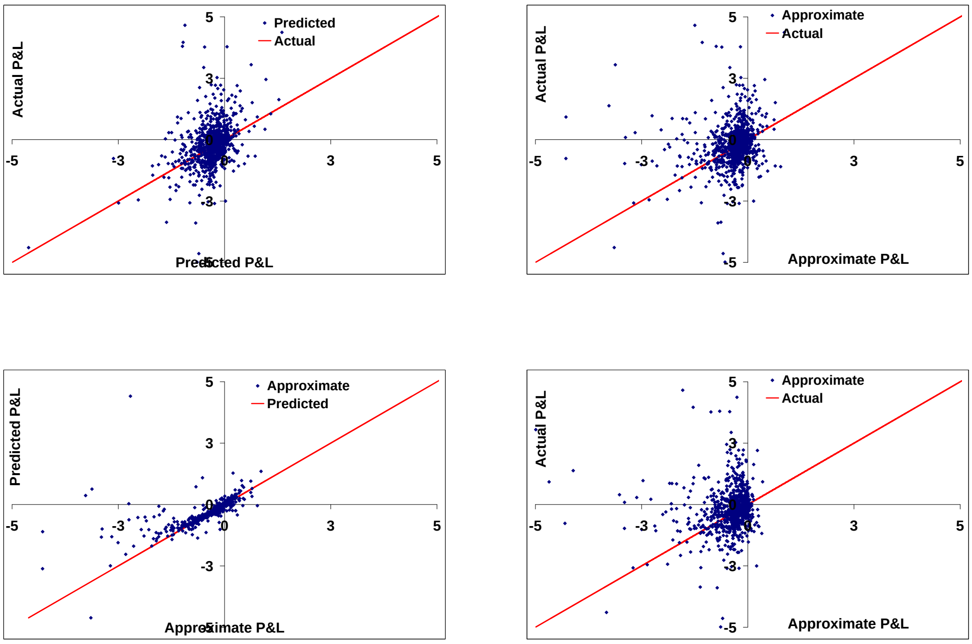
The top left figure shows the predicted P&L of the straddle computed using equation (6.3) versus the actual P&L computed using equation (6.2). The right left figure shows the approximate P&L of the straddle computed using equation (6.4) versus the actual P&L computed using equation (6.2). The bottom left figure shows the approximate P&L of the straddle computed using equation (6.4) versus the predicted P&L computed using equation (6.3). The bottom right plot shows the P&L of the straddle computed exactly and approximated using equation (5.5) with the regression model parameters specified by equation (5.3).

Exhibit 9. P&L explain for risk-reversal



The top left figure shows the predicted P&L of the risk-reversal computed using equation (6.3) versus the actual P&L computed using equation (6.2). The right left figure shows the approximate P&L of the risk-reversal computed using equation (6.4) versus the actual P&L computed using equation (6.2). The bottom figure shows the approximate P&L of the risk-reversal computed using equation (6.4) versus the predicted P&L computed using equation (6.3). The bottom right plot shows the P&L of the risk-reversal computed exactly and approximated using equation (5.5) with the regression model parameters specified by equation (5.6).

Exhibit 10. P&L explain for butterfly



The top left figure shows the predicted P&L of the butterfly computed using equation (6.3) versus the actual P&L computed using equation (6.2). The right left figure shows the approximate P&L of the butterfly computed using equation (6.4) versus the actual P&L computed using equation (6.2). The bottom figure shows the approximate P&L of the butterfly computed using equation (6.4) versus the predicted P&L computed using equation (6.3). The bottom right plot shows the P&L of the butterfly computed exactly and approximated using equation (5.5) with the regression model parameters specified by equation (5.7).

Exhibit 11. Descriptive statistics of the realized P&L

	Average	Stdev	Skew	Kurt	Max	Min
Straddle, Implied	-0.10	17.73	2.64	26.20	169.99	-115.68
Straddle, $\lambda = 1$	-0.06	11.69	3.18	29.17	115.31	-75.11
Straddle, $\lambda = 1.58$	-0.04	10.82	3.95	45.73	140.92	-66.78
Risk-Reversal, Implied	-0.35	5.73	7.46	104.46	98.70	-10.47
Risk-Reversal, $\lambda = 1$	-0.40	4.08	9.02	150.57	77.26	-10.19
Risk-Reversal, $\lambda = 1.58$	-0.42	3.98	8.64	148.01	75.09	-10.62
Butterfly, Implied	-0.18	1.06	4.07	52.78	15.17	-4.99
Butterfly, $\lambda = 1$	-0.18	1.12	2.25	23.56	12.57	-5.59
Butterfly, $\lambda = 1.58$	-0.18	1.15	1.85	18.96	12.10	-5.70

The descriptive statistics the realized daily P&L of the straddle, risk-reversal, an butterfly computed using equation (6.1) with option delta computed using equation (5.9) with $\lambda = 0$ and $q = 1$ (denoted as "Hedged at implied"), $\lambda = 1$ and $q = 0$ (denoted as "Hedged at lambda=1"), $\lambda = 1.5788$ and $q = 0$ (denoted as "Hedged at lambda=1.58"). The number of option contracts is 10000 for the straddle, 500 for the risk-reversal, and 50 for the butterfly (with this choice all three strategies have approximately equal notional cash-gamma).