DIVIDEND MODELING : MOVING FORWARD

DETERMINISTIC AND STOCHASTIC CASH DIVIDENDS

PASCAL DELANOË

First Version: May 15, 2015

This Version: December 30, 2017

Abstract

The idea of this paper is to present different ways to model dividends. The first part evokes the most widespread model for dividend modeling, which considers that the dividend function is affine in the spot. The second part focuses on Forward Market models, i.e. models where we directly diffuse the forwards, and how these models are well suited to model dividends, even when stochastic rates are introduced, which is difficult to handle in an affine model. Eventually, we compare these two approaches for various dividend dependent products (Autocalls, Forward forwards, Dividend Options), and different vol models, mainly local volatility or stochastic volatility. We also mention in this article different techniques related to dividend modeling : the calibration to American Vanillas or the (already known) dividend averaging method.

1. An introduction to dividends

Dividends are a way for the management of a company to share the benefits among its shareholders. There are different kinds of dividends : the shareholder can either receive some cash (this is a "cash dividend") or be granted additional shares ("stock dividend"). We will restrict our analysis to "cash dividends" (note however that "cash "describes here the way the dividend is being paid, but this "cash dividend" has nothing to do with modeling and shouldn't therefore be mixed up with another "cash dividend", which is the fixed amount that appears in the affine modeling of dividends, and which will be one of the main focuses of this paper).

Two dates are financially important for dividends : the ex-div date (or ex-date) and the payment date (or payable) date. The ex-div date is the date at which the shareholder loses its right on the current dividend. The payment date is the date at which the dividend is paid, provided that the shareholder was the owner of the shares the day before the ex date and was on the records of the company. The ex-div date is therefore before the payment date. Another (not financially important) date is the so-called record date on which the company looks at its records to see who the shareholders of the company are. This record date is after the ex-date and before the payment date.

When it comes to consider the pricing of Forwards, the stochastic interest rates risk for the period between the ex-div date and the payment date is generally not very important so that the amount paid at the payment date can be discounted up to the ex-date using a deterministic interest rate. We can then consider that this discounted amount is the "real" dividend and see that we won't lose in generality by considering that the ex-div date and the payment date are equal.

From what was previously said, it is clear that, in order to avoid arbitrages, the stock should be reduced of the dividend fixed amount on the morning of the ex div date. If the stock was to decrease less than this value, it would be interesting to buy the stock at the close the day before the ex div date and sell it at the open on the ex div date. If the decrease was more important than the fixed amount, the reverse operation would be profitable. However, this "arbitrage" is only valid if we suppose that there is no jump risk at the open (which is clearly not true, or more accurately, true to a certain -quantitative- extent). We will not consider jump risk throughout this article.

Another important point is important when it comes to talk about dividends, and it is taxation. We indeed know that some taxes are taken out from the dividend amounts. They may differ from one place to the other, so that the market generally thinks in terms of average taxation. Market practitioners generally use an all-in ratio to account for this taxation. The net dividend amount is considered to be the all-in (something generally worth 90% or 95%) times the gross amount. This might be also considered as something that doesn't involve too complicated modeling since we should only think in terms of net dividends. However, the issue with this all-in is that it is not something that is observable. As a consequence, for products which are directly written on dividends, like dividend swaps, the amounts that are mentioned in the product are gross dividends, not net dividends. This is why we can't replace our dividends by their net amount in all the situations.

However, we will omit this point, and will consider in the following that there is no difference between gross dividend and net dividend.

2. An introduction to dividend modeling

Now that we have explained some generic things to know about dividends, let's focus on the modeling. The first point to mention is that the date at which the dividend is granted is accurate in time : the stock will lose its dividend on the ex-div date. This is why dividends are considered to be paid discretely, at predefined dates, and not continuously as in most academics papers.

We will then write under the risk neutral measure :

$$dS_t = (r_t - q_t)S_t dt - \sum_i Div_i 1_{t=t_i} + \sigma_t S_t dW_t$$

(with r_t the (possibly stochastic) interest rate, q_t the (possibly stochastic) repo rate) rather than :

$$dS_t = (r_t - q_t)S_t dt - Div_t dt + \sigma_t S_t dW_t$$

Note that here the amount Div_i is stochastic. However, it progressively becomes deterministic. Indeed, before the ex-div date, we omitted another important date : the declaration date or the announcement date, which is the date at which the amount and the dividend specific dates are announced. After this announcement date, this dividend amount is a fixed amount. Therefore, when we come closer to t, D_t becomes a deterministic amount.

When we analyze long term dividends, it is clear that we don't know much things about them (we can however have some information using dividend swaps and dividend options on indexes). But it is reasonable to think that if the company performs well, shares will go up, and growing profits will enable the management to distribute more dividends. Reversely, if it performs badly, its shares will go down and the company may reduce or even cut its dividends. It seems therefore natural to consider that spots and dividends are positively correlated. This can also be observed from historical data (see for example [8]). Since we want a simple model, an easy way to mimic this positive correlation is to write the dividend as :$Div_i = D_i + d_i S_{t_i}$, with D_i and d_i deterministic. This is known as the affine model for dividends. Here, D_i is referred to as the cash dividend, and d_i is referred to as the proportional dividend.

The affine model will now write :

$$dS_t = r_t S_t dt - \sum_i (D_i + d_i S_t) 1_{t=t_i} + \sigma_t S_t dW_t$$

where σ_t is totally generic (i.e. stochastic) here. Note that we consider here that, if the cash dividend and the proportional dividends are due on the same date (which is by definition the case), the way we write our model make us consider that the cash dividend is paid after the proportional dividend. However, it may be more interesting to consider that the cash dividend is paid prior to the proportional dividend, in particular because it reduces the probability of S_t to reach zero. In order to do this we can consider that D_i is paid at t_i^- and d_i is paid at t_i^+.

Note that this affine model is the one that is the most referenced in the practitioners' literature. The first part will therefore sound familiar to the reader, because it is very much inspired (and sometimes simple quotation) of some well-known papers such as the one by Hans Buehler ([4]) or Henry-Labordere([10]).

3. The forward in the affine model

The first interesting thing to note about this affine model is the fact that it has a simple (closed) formula for forwards. Let's indeed choose an arbitrary $T > t$.

We can easily derive the following formula for the forward (in view of the possible discontinuities at t and T, we will consider the forward F_{t+}^{T+} here) if we consider that interest rates and repos are deterministic:

$$F_{t+}^{T+} = \frac{(S_t - D(t,T))d(t,T)}{A(t,T)}$$

with :

$$\begin{aligned} D(t,T) &= \sum_{t<t_i\leq T} \frac{D_i e^{-\int_t^{t_i}(r_s-q_s)ds}}{\prod_{t<t_j\leq t_i}(1-d_j)} \\ d(t,T) &= \prod_{t<t_i\leq T}(1-d_i) \\ A(t,T) &= e^{-\int_t^T(r_s-q_s)ds} \end{aligned}$$

where D_i refers to the cash dividend, d_i to the proportional dividend, r_s to the interest rate and q_s to the repo rate.

What is interesting about this formula is that, for a given t, the quantities $D(t,T)$ and $d(t,T)$ only need to be updated for each dividend dates and this can be done in a recursive manner :

$$\begin{aligned} d(t,T_{i+1}) &= (1-d_{i+1})d(t,T_i) \\ D(t,T_{i+1}) &= D(t,T_i) + \frac{D_{i+1}e^{-\int_t^{t_{i+1}}(r_s-q_s)ds}}{d(t,T_{i+1})} \end{aligned}$$

Between two dividend dates ($T_i < T < T_{i+1}$), the only quantity that has to be updated is A:

$$A(t,T) = A(t,T_i)e^{-\int_{T_i}^T(r_s-q_s)ds}$$

Once these quantities are calculated, the forward can be instantaneously computed.

What is also interesting to write in this model is the diffusion of the forward (which has no drift in the T-forward measure, which is the spot measure when rates are deterministic) :

$$
\begin{aligned}
dF_t^T &= (\ldots)dt + \frac{dS_t d(t,T)}{A(t,T)} \\
&= (\ldots)dt + \frac{dS_t F_t^T}{S_t - D(t,T)} \\
&= \frac{F_t^T}{S_t - D(t,T)} \sigma_t S_t dW_t
\end{aligned}
$$

Written in terms of spot and forward vols, it can be written as :

$$\sigma_F = \frac{\sigma_S S_t}{S_t - D(t,T)}$$

which gives us some ideas that will be used when we will have to define a dynamics for our forwards.

Note also the implications of this formula. Since we need to have, in order to avoid arbitrages :

$$F_t^T \geq 0$$

This inequality implies that, $\forall t > 0$:

$$S_t \geq \underset{T>t}{max} D(t,T)$$

Therefore, it is natural to introduce the time $T^*(t)$ as the time at which this maximum is reached - or the first of these times if it's not unique (it will only be referred to as T^* when $t = 0$). This is also naturally the date at which the forward is minimum (in the affine dividend model).

4. Looking for closed formulas...

The model that is currently used by most banks for exotics is the full proportional model (i.e. where $D_i = 0$). This model is indeed extremely simple to handle because proportional dividends don't introduce any jumps in the implied vols before and after the dividend date. It is considered a more challenging task to price exotics using cash dividend models. However, when it comes to consider vanillas, some clever approximations have been introduced. Indeed, the first thing that some people tried to do in order to better take into account cash dividends was to find closed formulas in the most simple model, i.e. the model where the volatility above is constant (we will speak of this model as a spot volatility model with cash dividends, and σ will be the constant spot volatility) :

$$dS_t = r_t S_t dt - \sum_i D_i 1_{t=t_i} + \sigma S_t dW_t$$

Among the papers written on the subject, and to our knowledge, the best known papers are the one by :

- (Michael) Bos and Vandermark ([2])
- (Remco) Bos, Gairat and Shepeleva ([1])
- Gocsei and Sahel ([7])

What is the purpose of having closed formulas for such a model ? There are two main purposes : first, for very illiquid stocks, they enable to mark some smooth volatilities (i.e. spot volatilities rather than BS volatilities) that are more intuitive to the traders since they exclude the volatility due to the dividends. The other interest of these formulas is that they can be used for interpolation or extrapolation purposes.

We briefly present the above papers :

4.1. Bos and Vandermark (BV)

Let's denote C(T,K) the value of a dividend with strike K. The idea here consists in deriving a formula that can handle the two extreme cases : the dividend falls right now (at time 0) or it falls at time T (i.e. the maturity of the option).

If the dividend falls at time 0, only the spot will be impacted, and we can apply a BS formula to the spot just after the dividend :

$$C(t_0^-, T, S_0, K) = BS(t_0^+, T, S_0 - D, K, \sigma)$$

If the dividend falls at time T, the strike is impacted, since we have :

$$\begin{aligned} C(t_0, T+, S_0, K) &= \mathbb{E}^{\mathbf{Q}}((S_{T+} - K)^+) \\ &= \mathbb{E}^{\mathbf{Q}}((S_{T-} - (D+K))^+) \\ &= BS(t_0, T-, S_0, K+D, \sigma) \end{aligned}$$

so only the strike should be changed in the formula.

Therefore, it may sound reasonable to write in the general case :

$$\begin{aligned} C(t, T-, S_{t+}, K) &= BS(S_t - D_n(t), K + D_f(t)e^{\int_t^T r_s ds}) \\ D_n(t) &= \sum_{t<t_i<T} \frac{T-t_i}{T} D_i e^{-\int_t^{t_i} r_s ds} \\ D_f(t) &= \sum_{t<t_i<T} \frac{t_i}{T} D_i e^{-\int_t^{t_i} r_s ds} \end{aligned}$$

(here, following BV's article, we use "n" for near and "f" for far). The affine form here ($\frac{t_i}{T}$ and $\frac{T-t_i}{T}$) is justified in the article using a first order approximation.

This formula has many advantages : it is consistent with the dividend conditions at time 0 and T, is extremely simple to implement and its computation is very quick. However, it is know to perform poorly for (far) in the money and out of the money options.

This is partly due to the fact that it uses a Black-Scholes formula where the spot and the strike are changed, but not the volatility.

4.2. Bos Gairat Shepeleva (BGS)

This approach underlines that the previous formula is not accurate enough for out of the money put and calls. It therefore introduces a skew correction, and uses a most likely path (or brownian bridge) technique to infer the implied volatility that needs to be used for pricing the vanilla option.

It uses a Black-Scholes formula where not only the spot and the strike, but also the volatility are changed.

4.3. Gocsei and Sahel (GS)

This article extends the Bos-Vandermark method and gives an approximate formula of the form:

$$C(S, K) \quad = \quad BS(S*, K*, \sigma)$$

with $S*$ and $K*$ some polynomial functions in the cash dividend values. The coefficients of these functions can be calculated in closed forms (but recursively). The order of approximation can be extended to an arbitrary value, but the complexity increases very quickly with the order.

As in Bos Vandermark, it uses a Black-Scholes formula where the spot and the strike are changed, but not the volatility.

What is interesting in this article is that the expansion for low vols of the first order approximation of this formula is exactly BV's formula. It also naturally introduces some expression of the form $\mathcal{N}(d_1) - \mathcal{N}(d_1 - \frac{\sigma}{\sqrt{T}} t)$ that can also be seen in BGS, which explains in some way the origin of these terms.

There is another very accurate formula by Henry-Labordere that is based on the dividend averaging technique that we will speak of later in the paper. In this formula, the strike, the forward and the volatility are changed.

5. The pivot technique

5.1. Buehler's analysis, or the pivot technique with $T = \infty$

The closed formulas above are interesting. There are rather quick to calculate (and clearly quicker than approaches like PDE or Monte Carlo), but are only approximated formulas. The approach presented by Buehler ([4]) enables to avoid these approximations. This approach can both help have an idea of the right vol to choose when no data is available and be a simple and accurate tool for interpolation/extrapolation purposes. We will see that it consists in selecting a special forward F_t^T with fixed maturity T (the fixed maturity forward chosen in Buehler's paper was F_t^∞) and writing the vol of this process : we will call this forward the "pivot" forward. The advantage here is that this forward is a continuous process (at least in an affine dividend model where the spot jumps are only due to dividends). We will see afterwards that this pivot forward can be a real forward (if $t < T$) or an "artificial" forward (if $T < t$). Apart from the fact that, in his original paper, Buehler considered the pivot to be at infinity (so that the pivot forward in his case is always a real forward), he also introduces the possibility for the stock to default. We won't consider this case here.

A very similar approach has been presented in Henry-Labordere's paper [10]. Here, the idea consists in introducing a continuous local martingale with can be chosen amongst a general family of process that depends on the choice of two parameters : A_0 and B_0. This approach is the most generic one, and we will see that our formulation is simply a particular case.

We will show here how Buehler's and Henry-Labordere's papers can be linked.

Note that Buehler's paper denotes :

$$S_t \quad = \quad (F_t - D_t)X_t + D_t$$

with D_t the sum of reinvested dividends and X_t a non-negative martingale.

Using our previous expression for the forward, the previous formula can also be written as:

$$S_t \quad = \quad (F_0^t - D(t, \infty))X_t + D(t, \infty)$$

Equivalently, we can say that :

$$\begin{aligned} X_t &= \frac{S_t - D_t}{F_t - D_t} \text{ in Buehler} \\ &= \frac{S_t - D(t,\infty)}{F_0^t - D(t,\infty)} \text{ in this paper} \end{aligned}$$

Buehler proposed a way to prove that the quantity above is a non-negative martingale. Another proof consists in noting that:

$$\begin{aligned} X_t &= \frac{S_t - D(t,\infty)}{F_0^t - D(t,\infty)} \\ &= \frac{S_t - D(t,\infty)}{\mathbb{E}_0^{\mathbf{Q}}(S_t) - D(t,\infty)} \\ &= \frac{(S_t - D(t,\infty))\, d(t,\infty)}{A(t,\infty)} \frac{A(t,\infty)}{\left(\mathbb{E}_0^{\mathbf{Q}}(S_t) - D(t,\infty)\right) d(t,\infty)} \\ &= \frac{(S_t - D(t,\infty))\, d(t,\infty)}{A(t,\infty)} \frac{1}{\mathbb{E}_0^{\mathbf{Q}}\left(\frac{(S_t - D(t,\infty))d(t,\infty)}{A(t,\infty)}\right)} \\ &= \frac{F_t^\infty}{\mathbb{E}_0^{\mathbf{Q}}(F_t^\infty)} \\ &= \frac{F_t^\infty}{F_0^\infty} \end{aligned}$$

Since the forward is a non-negative martingale, so is X_t (note that here, we considered that our rates are deterministic).

In other terms, we can say that $X_t = \frac{F_t^T}{F_0^T}$ for $T = \infty$.

It is therefore easy to extend this formula and write in a general context :

$$S_t = (F_0^t - D(t,T))X_t + D(t,T)$$

with $X_t = \frac{F_t^T}{F_0^T}$ a non-negative martingale with mean 1.

In the following, T will be called the pivot.

Any arbitrage free model follows the equation above. However, it must be noted that the previous process may not be arbitrage free. Indeed, let's consider the time $T^*(t)$ for which $D(t,T)$ is maximum for a given t (it is not necessarily ∞, for example, in the case where there are no more dividends after a certain date). In this case, we have :

$$\begin{aligned} F_t^{T^*(t)} &= \frac{(S_t - D(t,T^*(t)))d(t,T^*(t))}{A(t,T^*(t))} \\ &= \frac{((F_0^t - D(t,\infty))X_t + D(t,\infty) - D(t,T^*(t)))d(t,T^*(t))}{A(t,T^*(t))} \end{aligned}$$

If the process X reaches zero, we therefore obtain, for $X = 0$:

$$\begin{aligned} F_t^{T^*(t)} &= \frac{(D(t,\infty) - D(t,T^*(t)))d(t,T^*(t))}{A(t,T^*(t))} \\ &\leq 0 \end{aligned}$$

For example, if there is only one cash dividend, and if rates are strictly positive (but such that the infinite forward is not infinite), the short term forward $F(t_{ex}^{-}, t_{ex}^{+})$ will be negative if $X = 0$.

As a consequence, the process above may not be totally arbitrage-free. Only the case where $T = T^*$ would theoretically enable to avoid such arbitrages if X is not chosen carefully. However, in practice, if the pivot date is chosen to be infinite, the probabilities of having negative forwards are very low. In a more general context (i.e. with a generic pivot date T), a way to remedy this issue is to pay special attention to the process X so that it doesn't reach certain levels.

The most widespread criticism again the model with $\overline{T} = \infty$ is linked to the fact that a far away dividend is going to have an influence on short term vols, which seems unrealistic in practice : a more sensible choice seems therefore to consider the case $\overline{T} = 0$. However, for generic models, the forward F_t^T is only defined for $T > t$. But the affine model has this property that we can consider an "artificial" forward which verify the same relation. In this case ($\overline{T} = 0$), the probability that the output process might be negative may no be worth zero (it is clear that, if we choose the same X, the case where $\overline{T} = 0$ will generate more negative paths than the case $\overline{T} = \infty$). But we can always work on the implied volatility of the process X_t (for example by canceling the implied vol under a certain level so that such issues are avoided).

5.2. The artificial forward

We have underlined above that the formula for F_t^T is only available when $T \geq t$. Since we may want to write $S_t = (F_0^t - D(t,T))X_t + D(t,T)$ for $T < t$, we have to extend this formula so that we can take : $T < t$. This "artificial" forward therefore follows the following definition:

$$F_{t+}^{T+} = \frac{(S_t + \tilde{D}(t,T))\tilde{d}(t,T)}{A(t,T)}$$

with (if $t_1 < t_2 < \ldots < t_n$ the dividend dates):

$$\begin{aligned}
\tilde{D}(t,T) &= \sum_{T<t_i\leq t} D_i \prod_{t\geq t_j>t_i} (1-d_j)e^{-\int_t^{t_i}(r_s-q_s)ds} \\
&= \frac{D(T,t)d(T,t)}{A(T,t)} \\
\tilde{d}(t,T) &= \prod_{T<t_i\leq t} \frac{1}{1-d_i} \\
&= \frac{1}{d(T,t)}
\end{aligned}$$

It can be noticed that this artificial forward, even if it is not a financial product, is still a martingale in the risk neutral measure if we have an affine dividend model. This artificial forward is constructed by reinvesting all the "flows" associated to the stock under the risk neutral measure : rates (subtracted), repos (added), and cash or proportional dividends (added).

If $D(t,T)$ and $d(t,T)$ haven't already been calculated, we can also underline the fact that it is easy to calculate the terms $\tilde{D}$ and $\tilde{d}$ recursively for a given T. We indeed have the following formulas :

$$\begin{aligned}
\tilde{D}(t_{i+1},T) &= \tilde{D}(t_i,T)(1-d_{i+1})e^{\int_{t_i}^{t_{i+1}}(r_s-q_s)ds} + D_{i+1} \\
\tilde{d}(t_{i+1},T) &= \frac{1}{1-d_{i+1}}\tilde{d}(t_i,T)
\end{aligned}$$

and can therefore easily derive the lognormal vol of the (artificial) forward from the lognormal vol of the spot :

$$\sigma_F = \sigma_S \frac{S_t}{S_t + \tilde{D}(t,T)}$$

5.3. The Pivot Technique with $\overline{T} = 0$

What is interesting about a forward for a specified time T is that it doesn't suffer the discontinuities that we have to face for the spot. That's why we are going to choose a specific pivot T. How to choose this pivot ? Let's remember the three issues that cash dividends try to solve :

- Fit of the term structure of Forwards
- Proper management of vanilla discontinuities at ex div dates
- Positivity of all the Forwards

The two first points are naturally solved by the approach followed by Buehler, which consists in writing that :

$$S_t = (F_0^t - D(t,\overline{T}))X_t + D(t,\overline{T})$$

which is what we call the Pivot Technique (with $\overline{T}$ the pivot). Three natural choices are available : $\overline{T} = 0$, $\overline{T} = \infty$ and $\overline{T} = T^*(0)$. In order to ease the choice of X so that it doesn't introduce arbitrage, the choice of $\overline{T} = \infty$ and $\overline{T} = T^*(0)$ is more appropriate. However, this has the main drawback that it makes the volatility at any time t (or at time $t < T^*$) depend from dividend variations posterior to that date.

This is why the choice of $\overline{T} = 0$ may be more relevant to practitioners. It is for example the choice that has been presented in [10].

Indeed, the 0-Forward corresponds to the particular case where $A_0 = 0$ and $B_0 = 1$ in [10], which is the most natural choice. Following Henry-Labordere's paper, we indeed observe that the equality $A_0 = 1 - B_0$ ensures that the local martingale f_t begins at S_0. In view of the multiplicative nature of B_t, the choice of $B_0 = 1$ is natural, even if the choice of any constant is relevant : the important thing here is that the expression of f_t doesn't depend on the dividends after t.

We also see that more generally, we can link the general pivot forward to Henry-Labordere's approach writing that $F_t^{\overline{T}} = f_t$ with $A_0 = \frac{D(0,\overline{T})}{S_0}$ and $B_0 = \frac{A(0,\overline{T})}{d(0,\overline{T})}$.

6. The affine dividend framework

6.1. Dupire Equation for the pivot forward

Since the Forward $F_t^{\overline{T}}$ is a continuous quantity, it seems more simple to model it first. Also, since the forward must remain positive, it's natural to write that it follows a process of the form.

$$\frac{dF_t^{\overline{T}}}{F_t^{\overline{T}}} = \sigma(t, F_t^{\overline{T}})\sqrt{V_t}dW_t$$

Let's simplify further for the moment, considering only the local vol process. We will then write :

$$\frac{dF_t^{\overline{T}}}{F_t^{\overline{T}}} = \sigma_{F_t^{\overline{T}}}(t, F_t^{\overline{T}})dW_t$$

or equivalently :

$$\frac{dF_T^{\overline{T}}}{F_T^{\overline{T}}} = \sigma_{F^{\overline{T}}}(T, F_T^{\overline{T}})dW_T$$

We can use the previous diffusion to derive the traditional Dupire formula in a deterministic rates world (for the undiscounted call on $F_T^{\overline{T}}$ denoted $C(T,K)$):

$$\begin{aligned}
dC(T,K) &= \frac{\partial C}{\partial T}dT \\
&= d\mathbb{E}^{\mathbf{Q}}((F_T^{\overline{T}} - K)^+) \\
&= \mathbb{E}^{\mathbf{Q}}(dF_T^{\overline{T}} 1_{F_T^{\overline{T}} > K}) \\
&+ \frac{1}{2}\mathbb{E}^{\mathbf{Q}}(d < F_T^{\overline{T}} > 1_{F_T^{\overline{T}} = K}) \\
&= \frac{1}{2}K^2\sigma^2_{F^{\overline{T}}}(T,K)\frac{\partial^2 C}{\partial K^2}dT
\end{aligned}$$

or :

$$\frac{\partial C}{\partial T} = \frac{1}{2}K^2\sigma^2_{F^{\overline{T}}}(T,K)\frac{\partial^2 C}{\partial K^2}$$

which gives the local volatility of the forward pivot :

$$\sigma^L_{F^{\overline{T}}}(T,K) = \sqrt{\frac{\frac{\partial C}{\partial T}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}}$$

Following Gatheral, we can obtain a similar formula, which links the implied vol instead of the prices :

$$\sigma^L_{F^{\overline{T}}}(T,K) = \sqrt{\frac{\partial_T w}{\left(1 - \frac{y}{w}\partial_y w + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w^2}\right)(\partial_y w)^2 + \frac{1}{2}\partial_y^2 w\right)}}$$

with w the total variance of the pivot forward and y the log-moneyness in terms of pivot forward.

6.2. Infer the pivot forward vols from the spot vols

If we have the implied volatility of the $\overline{T}$-Forward (which may be "artificial"), it is therefore easy to derive its local volatility. However, we first need to deduce from the market the values of this implied vanillas of the $\overline{T}$-Forward for the observed pillars at which we observe the vanillas on the spot. Once more, this can be done using the simple linear relation that we have between the $\overline{T}$-Forward and the Spot.

Let's explain how we can imply the implied vol of the $\overline{T}$-Forward from the implied vol of the spot for Market pillars (when interest rates are deterministic). We can write, (if $F_T^{\overline{T}} = \frac{\left(S_T - D(T,\overline{T})\right)d(T,\overline{T})}{A(T,\overline{T})}$) :

$$\begin{aligned}\mathbb{E}^{\mathbf{Q}}((S_T-K)^+) &= \mathbb{E}^{\mathbf{Q}}\left(\left(\frac{A(T,\overline{T})F_T^{\overline{T}}}{d(T,\overline{T})}+D(T,\overline{T})-K\right)^+\right)\\ &= \frac{A(T,\overline{T})}{d(T,\overline{T})}\mathbb{E}^{\mathbf{Q}}\left(\left(F_T^{\overline{T}}-\frac{\left(K-D(T,\overline{T})\right)d(T,\overline{T})}{A(T,\overline{T})}\right)^+\right)\end{aligned}$$

But, since we have:

$$F_T^{\overline{T}}=\frac{\left(S_T-D(T,\overline{T})\right)d(T,\overline{T})}{A(T,\overline{T})}$$

Taking the expectation given that $S_T=F_T^T$ and that A, B and b are deterministic leads to :

$$F_0^{\overline{T}}=\frac{\left(F_0^T-D(T,\overline{T})\right)d(T,\overline{T})}{A(T,\overline{T})}$$

Using this equality in the relation above, we can eventually write in terms of Black-Scholes formula:

$$F_0^T\mathcal{N}(d_1)-K\mathcal{N}(d_2)=\left(F_0^T-D(T,\overline{T})\right)\mathcal{N}(\tilde{d}_1)-\left(K-D(T,\overline{T})\right)\mathcal{N}(\tilde{d}_2)$$

with :

$$\begin{aligned}d_1 &= \frac{log(\frac{F_0^T}{K})}{\sigma_{F^T}(T,K)\sqrt{T}}+\frac{\sigma_{F^T}(T,K)\sqrt{T}}{2}\\ d_2 &= d_1-\sigma_{F^T}(T,K)\sqrt{T}\\ \tilde{d}_1 &= \frac{log\left(\frac{F_0^T-D(T,\overline{T})}{K-D(T,\overline{T})}\right)}{\sigma_{F^{\overline{T}}}(T,\tilde{K})\sqrt{T}}+\frac{\sigma_{F^{\overline{T}}}(T,\tilde{K})\sqrt{T}}{2}\\ \tilde{d}_2 &= \tilde{d}_1-\sigma_{F^{\overline{T}}}(T,\tilde{K})\sqrt{T}\end{aligned}$$

with $\tilde{K}=\frac{\left(K-D(T,\overline{T})\right)b(T,\overline{T})}{A(T,\overline{T})}$
.

Note that we have chosen the formula $F_T^{\overline{T}}=\frac{\left(S_T-D(T,\overline{T})\right)d(T,\overline{T})}{A(T,\overline{T})}$, but we could also use the formula available for the artificial forward ($F_T^{\overline{T}}=\frac{\left(S_T+\tilde{D}(T,\overline{T})\right)\tilde{d}(T,\overline{T})}{A(T,\overline{T})}$) would exactly lead to the same kind of result, replacing $D(T,\overline{T})$ by $-\tilde{D}(T,\overline{T})$.

This equation can be solved in $\sigma_{F^{\overline{T}}}(T,\tilde{K})$ using a standard implied vol.

Looking at the expression above, it is clear that if $D(T,\overline{T})=0$, the vols are unchanged : the spot vol and the forward vol are identical if we only have proportional dividends.

We have already underlined that, if the formulas above can be written for any $\overline{T}$, we will prefer to write it with $\overline{T}=0$. This formula helps us understand how the vol of the (artificial) forward can be zero under a certain level due to the fact that the spot can go below zero. Therefore, the fact that Puts with zero strike are worth zero on the left hand side explains the fact that the process X_t introduced before can't go below certain (positive in the case of the artificial forward) levels.

7. Adding stochastic vol in the affine dividend model

We will see in this paragraph that it is extremely easy to add some stochastic volatility on top of the cash dividends (but it is difficult to add stochastic rates on top of cash dividends - such an approach is presented in [9]).

If we consider that the process that we are interested in is the artificial forward, we can simply write that :

$$\frac{dF_t^{\overline{T}}}{F_t^{\overline{T}}} = \hat{\sigma}_{F^{\overline{T}}}(t, F_t^{\overline{T}})\sqrt{V_t}dW_t$$

where V_t is a well-chosen volatility process (OU, Heston, lognormal,...).

We then know that we have the following formula for :

$$\hat{\sigma}_{F^{\overline{T}}}(T, K) = \frac{\sigma_{F^{\overline{T}}}(T, K)}{\mathbb{E}^{\mathbf{Q}}(V_T | F_T^{\overline{T}} = K)}$$

It's therefore extremely simple to calibrate a LSV model with cash dividends following the below steps:

- calculate the implied volatilities for the pivot forward process
- deduce the local volatilities for the pivot forward process
- deduce the local volatilities adjustments (used in a LSV model) for the pivot forward process using the formula above

For example, in the (LSV) Heston case (with deterministic rates, so that the forward is martingale under the spot measure), the model written on the pivot forward will follow the below diffusion:

$$\begin{aligned}
\frac{dF_t^{\overline{T}}}{F_t^{\overline{T}}} &= \sigma(t, F_t^{\overline{T}})\sqrt{V_t}dW_t^F \\
dV_t &= -\kappa(V_t - V_\infty)dt + \nu\sqrt{V_t}dW_t^V \\
d < W_t^F, W_t^V > &= \rho dt
\end{aligned}$$

8. Looking for closed formulas with Stochastic vol... : dividend averaging

It is also possible to infer some approximate analytical formulas in the case where the volatilities are stochastic (case of Heston with cash dividends written on the spot rather than on the 0-Forward as above for example).

Indeed, Grandchamp presented in [8] the dividend averaging technique. This technique has also been presented in [10] at about the same time. The idea of these authors consisted in noting that (with the notations of this article) :

$$\begin{aligned}
dF_t^T &= \sigma_S\left(F_t^T + \frac{D(t,T)d(t,T)}{A(t,T)}\right) \\
dF_t^T &= \sigma_S\gamma_t^T\left(\beta_t^T F_t^T + (1-\beta_t^T)F_0^T\right)dW_t
\end{aligned}$$

with:

$$\begin{aligned}\gamma_t^T &= 1+\frac{D(t,T)d(t,T)}{A(t,T)F_0^T}\\ \beta_t^T &= \frac{1}{\gamma_t^T}\end{aligned}$$

where σ_S is the (possibly stochastic) vol of the spot.

Using skew averaging method (cf [12]), they wrote that :

$$dF_t^T = \sigma_S\gamma_t^T\left(\overline{\beta^T}F_t^T+(1-\overline{\beta^T})F_0^T\right)dW_t$$

with :

$$\overline{\beta^T} = \frac{1}{\int_0^T\frac{1}{(\beta_t^T)^2}\int_0^t\frac{1}{(\beta_s^T)^2}dsdt}\int_0^T\frac{1}{\beta_t^T}\int_0^t\frac{1}{(\beta_s^T)^2}dsdt$$

as in Piterbarg. Grandchamp also noted that this result was similar to BGS since the methods were similar (approximation of displaced lognormal diffusions).

Henry-Labordere gave a closed approximate formula in the flat vol case and Grandchamp mentioned that this formula was appropriate to obtain approximate formulas not only in the Black-Scholes model case but also for a wide range of stochastic models.

9. How to avoid arbitrages ?

We have seen that a model doesn't present any arbitrage if, at each time t, we have:

$$\begin{aligned}S_t &\geq \underset{T>t}{max}D(t,T)\\ &\triangleq D_{sup}(t)\end{aligned}$$

This information should already be present in the initial smile (there are some positive values under which - if we believe in our model - the puts, and hence the implied vols - should be worth zero). These thresholds, directly deduced from the future flows of cash dividends, will naturally introduce some dependence to future dividends (this dependence will however be extremely small due to the fact that it applies to deep far OTM puts).

Another way to consider the problem would be to cancel the local volatility for 0-forwards under the level : $K = \frac{\left(D_{sup}(t)+\tilde{D}(t,0)\right)\tilde{d}(t,0)}{A(t,0)}$. This also reintroduces some dependence to the future dividends since these levels are going to move when far away dividends move. However, for numerical reasons, these bounds could be recalculated once in a while. They may remain one or two arbitrageable paths out of a few thousands, but these can be easily solved by introducing a specific treatment for low spots : this will change the marginal distributions (which is why it is generally preferable to "rectify" directly the implied volatility rather than the spot) but the change will be extremely small.

10. Advantages of affine dividend model

Apart from the fact that this model is not more time consuming than a model with proportional dividends in PDE (which will also require a relevant interpolation, e.g. a spline function for the interpolation in strike), we can underline that, in Monte Carlo, it enables to avoid stopping at each dividend date, which might be cumbersome when considering an index with many cash dividends for example : indeed, we can diffuse directly the 0-forward and infer the spot using a post treatment after each path for each date where the spot is needed for the evaluation of the product.

10.1. How to fit American options with such a model ?

There are some stocks for which the only observables vanillas are American options (T_i, K). We know that these products are very sensitive to dividend assumptions. We also know that introducing some dividends has the consequence that American options are not identically equal to European options (for Calls, they are already different even without dividends for Puts), so that it is not possible to use the traditional Dupire formula. However, we can use the following fixed point algorithm:

$$\sigma^{E,(n+1)}(t,T_i,K) = \sigma^{E,(n)}(t,T_i,K) + \frac{C^{A,mkt}(t,T_i,K) - C^{A,(n)}(t,T_i,K)}{\partial_\sigma C^{A,(n)}(t,T_i,K)}$$

or even (for a speed-up):

$$\sigma^{E,(n+1)}(t,T_i,K) = \sigma^{E,(n)}(t,T_i,K) + \frac{C^{A,mkt}(t,T_i,K) - C^{A,(n)}(t,T_i,K)}{\partial_\sigma C^{E,(n)}(t,T_i,K)}$$

The idea here consists in beginning with a smart guess for the european implied vols (for example by implying the European vols from the American prices but there are also better ones). This enables us to price our American options using a standard algorithm (like in [3]). We can then correct the european vanilla prices with the difference between the model American calls and the market American Calls by adding this normalized term to our new estimation for the implied vols of european options. Then, we use Gatheral's formula to infer the local vols, calculate our new American vanilla prices using a PDE, correct the European vanillas, etc.

The reader can refer to [13] to get more intuition about how the dividends influence the American option prices.

11. Flaws of the affine dividend model

11.1. Dynamic issues : case of forward start variance swaps

Let's consider a 1Y in 10Y Variance Swap on Stock (for which there is a dividend correction which makes the Variance swap the average value of the Spot Vol rather than the Forward Vol). If we price it using an affine dividend model, the dividends for these maturities (between 10Y and 11Y) will typically be fully proportional. Therefore, this variance swap will be worth the log contract (we omit the stochastic rates here since we are only interested in the dividend risk, even if it's clear that for such maturities, a stochastic rate adjustment must be added). However, if we enter in this product, after ten years, it will be a 1Y Variance Swap, for which cash dividends are considered. Therefore, the Variance Swap won't be worth the value of the log contract, but will typically be worth less than this value (since spot vol is less than forward vol when there are some cash dividends). As a consequence, a position short a 1Y in 10Y Variance Swap and long the associated (forward start) log contract will generate a positive carry : there is an arbitrage here.

11.2. Incapacity to handle dividend transitions for long term vanilla options

A flaw that it is indirectly linked to the analysis of the forward variance swaps that we underlined in the previous paragraph is that the dividend transitions at dividend dates are not well defined. If the market was liquid enough, and if all the banks used an affine dividend model with 100% proportionality in the long term to mark their long term vanillas, it would be possible to "arbitrage" the market trading options before and after the dividend at certain strikes. However, this point may be mitigated by the fact that cash dividends are stochastic (so that no real "arbitrage" can be made, unless some dividend swaps are used to hedge the dividend risk, but the jump risk would remain) and also (partially) by the fact that we can consider that, in the long term, even the dividend dates are stochastic (we don't consider this stochasticity in this paper).

11.3. Dividend Cuts

All the practitioners know very well that cash dividends can be cut (the recent crisis proved it), this probability shouldn't therefore be underestimated. It is therefore not totally unreasonable to think that there is no real absolute threshold under which the Puts are worth zero (even without considering default).

11.4. Forward Forwards

Forward forward are worth less with cash dividends than with proportional dividends. Some long dated forward Forwards could therefore be arbitraged with dividend swaps.

In order to solve these well-known issues, we therefore introduce the Forward Market Models. These models are clearly more time consuming than the affine dividend model, and are therefore not ready to be used in production risk management. However, these may provide a powerful benchmarking model.

12. The Forward Market Model framework

12.1. Forward diffusion with stochastic dividend and stochastic rates

We consider a forward in a generic model where we introduce stochastic cash dividends and stochastic rates. In such a model, the diffusion of the spot will write :

$$dS_t = (r_t - q_t)S_t dt - \sum_i D_{t_i} 1_{t=t_i} + \sigma_t dW_t$$

We know that in the most generic setting, the forward is going to be :

$$\begin{aligned} F_t^T &= \mathbb{E}_t^{\mathbf{Q^T}}(S_T) \\ &= \frac{\mathbb{E}_t^{\mathbf{Q}}(e^{-\int_t^T r_s ds} S_T)}{\mathbb{E}_t^{\mathbf{Q}}(e^{-\int_t^T r_s ds})} \\ &= \mathbb{E}_t^{\mathbf{Q}}(e^{-\int_t^T (r_s - f(t,s))ds} S_T) \end{aligned}$$

where $f(t, s)$ refers to the instantaneous forward rate for s at date t.

But, considering the quantity : $Y_t = e^{-\int_0^t (r_s - f(0,s))ds} S_t$, we can see that it follows the following diffusion :

$$dY_t = (-r_t + f(0,t))Y_t dt + (r_t - q_t)Y_t dt - \sum_i D_{t_i} e^{-\int_0^t (r_s - f(0,s))ds} 1_{t=t_i} + (\ldots).dW_t$$

so that, calculating the expectancy $Z_t = \mathbb{E}^{\mathbf{Q}}(Y_t)$, we obtain, using Dufresne's trick :

$$dZ_t = (f(0,t) - q_t)Z_t dt - \sum_i \mathbb{E}^{\mathbf{Q}}\left(D_{t_i} e^{-\int_0^t (r_s - f(0,s))ds} 1_{t=t_i}\right)$$

which gives us the value of the forward Z_T :

$$Z_T = S_0 e^{\int_0^T (f(0,s) - q_s)ds} - \sum_{0<t_i<T} \mathbb{E}^{\mathbf{Q}}\left(D_{t_i} e^{-\int_0^{t_i} (r_s - f(0,s))ds}\right) e^{\int_{t_i}^T (f(0,s) - q_s)ds}$$

We therefore have in general:

$$F_t^T = S_t e^{\int_t^T (f(t,s)-q_s)ds} - \sum_{t<t_i<T} \mathbb{E}_t^{\mathbf{Q}}\left(D_{t_i} e^{-\int_t^{t_i}(r_s - f(t,s))ds}\right) e^{\int_{t_i}^T (f(t,s)-q_s)ds}$$

We can then infer the equation of the forward in such model in the T-forward probability (since F_t^T is a martingale under this probability):

$$\begin{aligned} dF_t^T &= \sigma_S S_t e^{\int_t^T (f(t,s)-q_s)ds} dW_t^{S,T} - S_t e^{\int_t^T (f(t,s)-q_s)ds} \sigma_B(t,T) dW_t^{B,T} \\ &- \sum_{t<t_i<T} e^{\int_{t_i}^T (f(t,s)-q_s)ds} \sigma_S \sigma_D(t,t_i) \mathbb{E}_t^{\mathbf{Q}}\left(D_{t_i} e^{-\int_t^{t_i}(r_s - f(t,s))ds}\right) dW_t^{D,T} \\ &+ \sum_{t<t_i<T} \mathbb{E}_t^{\mathbf{Q}}\left(D_{t_i} e^{-\int_t^{t_i}(r_s - f(t,s))ds}\right) (\sigma_B(t,T) - \sigma_B(t,t_i)) e^{\int_{t_i}^T (f(t,s)-q_s)ds} dW_t^{B,T} \end{aligned}$$

with:

$$\begin{aligned} dW_t^{S,T} &= dW_t^S - \rho_{S,B}\sigma_B(t,T)dt \\ dW_t^{B,T} &= dW_t^B - \sigma_B(t,T)dt \\ dW_t^{D,T} &= dW_t^D - \rho_{D,B}\sigma_B(t,T)dt \end{aligned}$$

and where $\sigma_B(t,T)$ stands for the volatility of the zero-coupon bond (it is equal to in a Hull-White model for example) and where we have considered that the diffusion of the cash dividends is the following (we could replace σ_S with a general quantity $\alpha_t \sigma_S$ but will keep $\alpha = 1$ so that this model looks like the proportional dividend model that is used by practitioners) :

$$d\mathbb{E}_t^{\mathbf{Q}}\left(D_{t_i} e^{-\int_t^{t_i}(r_s - f(t,s))ds}\right) = \mathbb{E}_t^{\mathbf{Q}}\left(D_{t_i} e^{-\int_t^{t_i}(r_s - f(t,s))ds}\right) \sigma_S \sigma_D(t,t_i) dW_t^D + (\ldots)dt$$

where σ_S stands for the (possibly stochastic) volatility of the spot at time t.
Note however that our diffusion is far too complicated for the moment : we want to simplify it to make it usable.

Another issue with such a model is that it is self-referencing (since the cash dividends can be extracted from the forward term structure for short term horizons) and not necessarily free of arbitrage. However, it has the advantage that, since this is a Forward Market Model, the term structure of Forwards will be fitted by construction.

In order to make this mode usable, we therefore need to simplify the diffusion of the forward. For example, we would like to introduce a time $\overline{t_D}(t,T)$ such that :

$$\sum_{t<t_i<T} e^{\int_{t_i}^T (f(t,s)-q_s)ds} \sigma_D(t,t_i) \mathbb{E}_t^{\mathbf{Q}}\left(D_{t_i} e^{-\int_t^{t_i}(r_s - f(t,s))ds}\right) \simeq \sigma_D(t,\overline{t_D}) \sum_{t<t_i<T} e^{\int_{t_i}^T (f(t,s)-q_s)ds} \mathbb{E}_t^{\mathbf{Q}}\left(D_{t_i} e^{-\int_t^{t_i}(r_s - f(t,s))ds}\right)$$

A natural way to verify this approximation is to choose $\overline{t_D}$ so that the above equation is an equality seen from 0 (and considering that stochastic dividends and stochastic rates are independent only to obtain this approximation of what a reasonable $\overline{t_D}$ is):

$$\sigma_D(t,\overline{t_D}) = \frac{\sum_{t<t_i<T} e^{\int_{t_i}^T (f(0,s)-q_s)ds} \sigma_D(t,t_i) D_{t_i}^0}{\sum_{t<t_i<T} e^{\int_{t_i}^T (f(0,s)-q_s)ds} D_{t_i}^0}$$

where $D_{t_i}^0$ denotes the cash dividend seen from time 0. Stated otherwise, we can write that:

$$\overline{t_D}(t,T) \triangleq t + g^{-1}\left(\frac{\sum_{t<t_i<T} e^{\int_{t_i}^T (f(0,s)-q_s)ds} D_{t_i}^0 g(t_i - t)}{\sum_{t<t_i<T} e^{\int_{t_i}^T (f(0,s)-q_s)ds} D_{t_i}^0}\right)$$

where $\sigma_D(t,t_i) = g(t_i - t)$. $\overline{t_D}$ can be understood as a duration of the ex-div dates between dates t and T. Concerning the stochastic rates, a good approximation in a Hull White model, that is particularly relevant when mean reversion are low, consists in writing that $\sigma_B(t,T) - \sigma_B(t,t_i) \simeq \sigma_B(t_i,T)$.

Eventually, we can remark that we have the following equality :

$$\sum_{t<t_i<T} \mathbb{E}_t^{\mathbf{Q}}\left(D_{t_i} e^{-\int_t^{t_i}(r_s - f(t,s))ds}\right) e^{\int_{t_i}^T (f(t,s)-q_s)ds} = S_t e^{\int_t^T (f(t,s)-q_s)ds} - F_t^T$$

As a conclusion, if we introduce the following diffusion of the forward :

$$\begin{aligned} dF_t^T &= S_t e^{\int_t^T (f(t,s)-q_s)ds}\left(\sigma_S dW_t^{S,T} - \sigma_B(t,T) dW_t^{B,T}\right) \\ &- \left(S_t e^{\int_t^T (f(t,s)-q_s)ds} - F_t^T\right)\left(\sigma_S \sigma_D(t,\overline{t_D}) dW_t^{D,T} - \sigma_B(\overline{t_R},T) dW_t^{B,T}\right) \end{aligned}$$

where $\overline{t_R}$ has been defined similarly to $\overline{t_D}$ for the rates part, i.e.:

$$\overline{t_R}(t,T) \triangleq t + h^{-1}\left(\frac{\sum_{t<t_i<T} e^{\int_{t_i}^T (f(0,s)-q_s)ds} D_{t_i}^0 h(t_i - t)}{\sum_{t<t_i<T} e^{\int_{t_i}^T (f(0,s)-q_s)ds} D_{t_i}^0}\right)$$

where, in the case of a Hull-White process for example:

$$h(x) = -\sigma_R \frac{1 - e^{-\kappa x}}{\kappa}$$

we can say that this new "cash dividend inspired" diffusion of the forward will be extremely relevant to simulate stochastic cash dividends together with stochastic rates.

Note also that this diffusion is consistent with the diffusion of the forward that was defined earlier in the case where rates and cash dividends are deterministic (no proportional dividends) :

$$\begin{aligned} dF_t^T &= S_t e^{\int_t^T (f(t,s)-q_s)ds} \sigma_S dW_t^{S,T} \\ &= \sigma_S \left(F_t^T + \frac{B(t,T)}{A(t,T)}\right) dW_t^{S,T} \end{aligned}$$

It can also be seen that this diffusion is also consistent with the diffusion of the forward in the case where dividends are proportional if $\sigma_D = 1$ and $\rho_{S,D} = 1$ since we have in this case:

$$
\begin{aligned}
dF_t^T &= \frac{F_t^T}{\Pi_{t<t_i<T}(1-d_i)}\sigma_S dW_t^S + F_t^T\left(1 - \frac{1}{\Pi_{t<t_i<T}(1-d_i)}\right)\sigma_S dW_t^S \\
&= \sigma_S F_t^T dW_t^S
\end{aligned}
$$

which means the the forward vol equals the spot vol here. It also degenerates in the expected diffusion with proportional dividends and stochastic rates, if we have $\overline{t_R} = t$, on top of the two equalities $\sigma_D = 1$ and $\rho_{S,D} = 1$ since we have in this case:

$$
\frac{dF_t^T}{F_t^T} = \left(\sigma_S dW_t^{S,T} - \sigma_B(t,T) dW_t^{B,T}\right)
$$

12.2. Choice of σ_D

There are different choices which may sound relevant for σ_D (theoretically, for the model to be consistent, we only need to ensure that $\sigma_D(t,T) \underset{T \to t}{\to} 0$). We could for example choose :

$$
\sigma_D(t,T) = \frac{1 - e^{-\lambda_D(T-t)}}{\lambda_D}
$$

But this form has the drawback that it has a sharp slope for small $T-t$. However, the volatility of the cash dividend is generally null for the first year or at least for the first few months, due to the fact that dividends have been announced. After that, it progressively increases. Therefore, we want to choose a more flexible form. We therefore consider the function (for $T > t$):

$$
\begin{aligned}
\sigma_D(t,T) &= g(T-t) \\
&= \frac{1 - e^{-\alpha(T-t)}}{1 + e^{-\alpha(T-t-t_{min})}}
\end{aligned}
$$

Here is the shape of this function with $t_{min} = 2.5$ and $\alpha = 2$ (fast convergence) or with $t_{min} = 4$ and $\alpha = 1$ (slow convergence). We compare this (at time 0) to the behavior that are obtained from a deterministic cash dividend model and a deterministic proportional dividend model :

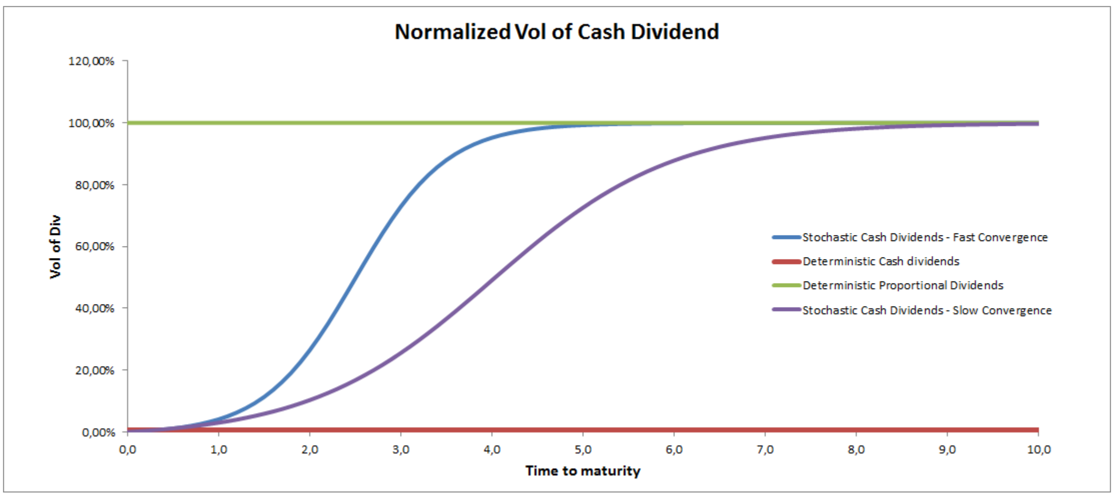

Figure 1: Shape of $\sigma_D(0,T) = g(T)$

If we choose t_{min} and α in a relevant manner, we can make the volatility of the cash dividend tend to zero when $T-t$ goes below a certain maturity, and the cash dividend then becomes deterministic.

12.3. Avoid arbitrages

We want to "kill" the vol when the forwards approach zero, i.e. to make it decrease to zero fast enough, so that our forwards won't become negative. This operation will tend to decorrelate the forwards. However, we think that's it's preferable to do so than to create arbitrages. This is already a technique that is (implicitely) used in the Black-scholes model (where the normal vol is $\sigma(x) = x\sigma_{BS}$) or the CIR process (where the normal vol is $\sigma(x) = \sqrt{x}\sigma_{CIR}$). If our normal vol behaves like x^{α}) with $\alpha \geq 1$, the spot (or the forward) won't reach zero (at least in a continuous setting) and we will avoid any arbitrage.

In order to do so, we will add a further term $f\left(\frac{F_t^T}{F_0^T}\right)$ to kill our volatility :

$$dF_t^T = \left(S_t e^{\int_t^T (f(t,s)-q_s)ds}(\sigma_S dW_t^{S,T} - \sigma_B(t,T)dW_t^{B,T}) + (F_t^T - S_t e^{\int_t^T (f(t,s)-q_s)ds})\left(\sigma_S\sigma_D(t,\overline{t_D})dW_t^{D,T} - \sigma_B(\overline{t_R},T)dW_t^{B,T}\right)\right) f\left(\frac{F_t^T}{F_0^T}\right)$$

which can also be written:

$$dF_t^T = \left(\frac{S_t e^{-\int_t^T q_s ds}}{B(t,T)}(\sigma_S dW_t^{S,T} - \sigma_B(t,T)dW_t^{B,T}) + \left(F_t^T - \frac{S_t e^{-\int_t^T q_s ds}}{B(t,T)}\right)\left(\sigma_S\sigma_D(t,\overline{t_D})dW_t^{D,T} - \sigma_B(\overline{t_R},T)dW_t^{B,T}\right)\right) f\left(\frac{F_t^T}{F_0^T}\right)$$

where $B(t,T)$ stands for the (possibly stochastic) Zero Coupon bond.

We can for example choose the two attenuating functions equal to (for $x \in [0;\alpha]$):

$$\begin{aligned}
f(x) &= \frac{2}{\alpha^3}x^2\left(\frac{3\alpha}{2} - x\right) \\
f(x) &= \frac{1}{2} + \frac{1}{\pi}Arctan\left(\beta\left(\frac{1}{(x-\alpha)^{2n}} - \frac{1}{x^{2n}}\right)\right)
\end{aligned}$$

with suitable α, β and n (typically, $n=1$, $\alpha = 2\%$ and $\beta = 1e^{-4}$).

Concerning the stochastic rates component, we remind that we have, in a HJM model:

$$\begin{aligned}
B(t,T) &= \frac{B(0,T)}{B(0,t)}\frac{e^{\int_0^t \sigma_B(s,T)dW_s - \frac{1}{2}\int_0^t \sigma_B^2(s,T)ds}}{e^{\int_0^t \sigma_B(s,t)dW_s - \frac{1}{2}\int_0^t \sigma_B^2(s,t)ds}} \\
&= \frac{B(0,T)}{B(0,t)}\frac{U(t,T)}{U(t,t)}
\end{aligned}$$

where we have introduced :

$$U(t,T) \triangleq e^{\int_0^t \sigma_B(s,T)dW_s^B - \frac{1}{2}\int_0^t \sigma_B^2(s,T)ds}$$

(note that $U(t_{i+1},T)$ can be calculated iteratively from $U(t_i,T)$ and an integrated brownian between t_i and t_{i+1}).

This equality directly follows the fact that our Zero Coupon Bond diffusion writes:

$$\frac{dB(t,T)}{B(t,T)} = r_t dt + \sigma_B(t,T)dW_t^B$$

12.4. Achieving decorrelation ?

We see that introducing could consider that dividends are stochastic (i.e. "really" stochastic in the sense that their stochasticity doesn't only depend on the stochasticity of the spot): we could add some extra factors that would introduce some more decorrelation between the different forwards like in Libor Market Models or HJM Models with several factors. However, this wouldn't add much complexity in terms of modeling here and it is not our purpose to study the impact of such specifications.

12.5. Spot Diffusion in a Forward Market Model

Let's consider that we have (with a multi-dimensional brownian here):

$$\frac{dF_t^T}{F_t^T} = \sigma_t^{F^T}.(dW_t - \sigma_B(t,T)dt)$$

Since we have : $S_t = F_t^t$, standard Ito calculus gives :

$$dS_t = dF_t^T|_{T=t} + \partial_T F_t^T|_{T=t} dt$$

We can therefore readily obtain:

$$dS_t = S_t \sigma_t^S.dW_t + \partial_T F_t^T|_{T=t} dt$$

Indeed, from the expression of the diffusion of the forward, and since we have :

$$\begin{array}{lcl} \sigma_B(t,T) & \underset{T\to t}{\to} & 0 \\ \sigma_D(t,T) & \underset{T\to t}{\to} & 0 \end{array}$$

we conclude that :

$$\begin{array}{lcl} \sigma_t^{F^T} & \underset{T\to t}{\to} & \sigma_t^{F^t} \\ & \underset{T\to t}{\to} & \sigma_S \end{array}$$

so that the two volatilities (spot and forward) coincide on the short term : our Forward Market Model is fully consistent.

12.6. Local Stochastic Vol in a Forward Market Model

We have previously seen that it is very relevant to use a Forward Market Model, in order to make it easier to introduce Stochastic dividends. We haven't yet spoken of the spot volatility : it can clearly be any kind of model. However, to make it easier to calibrate the vanillas, we will generally consider Local Stochastic Vol Models (the Local Vol model being a particular case of these models), i.e. write : $\sigma_S = \sqrt{V_t}\sigma(t, S_t)$.

The other volatilities (i.e. σ_B and σ_D) will typically be calibrated to other markets (swaptions for interest rates or dividend options for dividends).

Once we have specified our spot volatility, we now need to calibrate it.

How can we do so ? One more time, let's derive the local formula following Dupire's idea, i.e. let's derive the Market observables, which are the vanillas, with respect to their parameters strike and maturity. However, we will consider a "normalized" vanilla here, i.e.:

$$\begin{aligned}\tilde{C}(T,K) &= \mathbb{E}_t^{\mathbf{Q}}\left(e^{-\int_t^T (r_s - f(t,s))ds}\left(\frac{S_T}{F_t^T} - K\right)^+\right)\\ &= \frac{C(T, KF_t^T)}{B(t,T)F_t^T}\end{aligned}$$

We therefore write, using the Ito-Tanaka formula :

$$\begin{aligned}d\tilde{C}(T,K) &= \partial_T \tilde{C} dT\\ &= \mathbb{E}_t^{\mathbf{Q}}\left(e^{-\int_t^T (r_s - f(t,s))ds}\left((f(t,T) - r_T)\left(\frac{S_T}{F_t^T} - K\right)^+ + d\left(\frac{S_T}{F_t^T}\right)1_{\frac{S_T}{F_t^T} > K} + \frac{1}{2} d < \frac{S_T}{F_t^T} > 1_{\frac{S_T}{F_t^T} = K}\right)\right) dT\end{aligned}$$

but we have :

$$\begin{aligned}d\left(\frac{S_T}{F_t^T}\right) &= \left(\frac{dS_T}{S_T} - \frac{dF_t^T}{F_t^T}\right)\frac{S_T}{F_t^T}\\ &= \left(\sigma_S dW_T + \frac{\partial_{T'} F_T^{T'}|_{T'=T}}{S_T} dT - \frac{\partial_T F_t^T}{F_t^T} dT\right)\frac{S_T}{F_t^T}\end{aligned}$$

Since $\sigma_S = \sqrt{V_t}\sigma(T, \frac{S_T}{F_t^T})$ and $S_T = F_T^T$, we are therefore left with :

$$\partial_T \tilde{C} = \mathbb{E}_t^{\mathbf{Q}}\left(e^{-\int_t^T (r_s - f(t,s))ds}\left((f(t,T) - r_T)\left(\frac{S_T}{F_t^T} - K\right)^+ + \left(\frac{\partial_{T'} F_T^{T'}|_{T'=T}}{F_T^T} - \frac{\partial_T F_t^T}{F_t^T}\right)\frac{S_T}{F_t^T} 1_{\frac{S_T}{F_t^T} > K} + \frac{1}{2} K^2 V_T \sigma^2(T,K) 1_{\frac{S_T}{F_t^T} = K}\right)\right)$$

Since we also know that :

$$\frac{\mathbb{E}_t^{\mathbf{Q}}\left(e^{-\int_t^T (r_s - f(t,s))ds} V_T 1_{\frac{S_T}{F_t^T} = K}\right)}{\mathbb{E}_t^{\mathbf{Q}}\left(e^{-\int_t^T (r_s - f(t,s))ds} 1_{\frac{S_T}{F_t^T} = K}\right)} = \frac{\mathbb{E}_t^{\mathbf{Q}}\left(e^{-\int_t^T (r_s - f(t,s))ds} V_T 1_{\frac{S_T}{F_t^T} = K}\right)}{\frac{\partial^2 \tilde{C}}{\partial K^2}} = \mathbb{E}_t^{\mathbf{Q^T}}\left(V_T \middle| \frac{S_T}{F_t^T} = K\right)$$

We can eventually conclude that:

$$\begin{aligned}\partial_T \tilde{C} &= \mathbb{E}_t^{\mathbf{Q}}\left(e^{-\int_t^T (r_s - f(t,s))ds}\left((f(t,T) - r_T)\left(\frac{S_T}{F_t^T} - K\right)^+ + \left(\frac{\partial_{T'} F_T^{T'}|_{T'=T}}{F_T^T} - \frac{\partial_T F_t^T}{F_t^T}\right)\frac{S_T}{F_t^T} 1_{\frac{S_T}{F_t^T} > K}\right)\right)\\ &+ \frac{1}{2} K^2 \sigma^2(T,K) \mathbb{E}_t^{\mathbf{Q^T}}\left(V_T \middle| \frac{S_T}{F_t^T} = K\right)\frac{\partial^2 \tilde{C}}{\partial K^2}\end{aligned}$$

which can also be expressed in terms of local volatilities as :

$$\sigma^2(T,K) = \frac{\partial_T \tilde{C} + \mathbb{E}_t^{\mathbf{Q}}\left(e^{-\int_t^T (r_s - f(t,s))ds}(r_T - f(t,T))\left(\frac{S_T}{F_t^T} - K\right)^+\right) + \mathbb{E}_t^{\mathbf{Q}}\left(e^{-\int_t^T (r_s - f(t,s))ds}\left(\frac{\partial_T F_t^T}{F_t^T} - \frac{\partial_{T'} F_T^{T'}|_{T'=T}}{F_T^T}\right)\frac{S_T}{F_t^T} 1_{\frac{S_T}{F_t^T} > K}\right)}{\frac{1}{2} K^2 \mathbb{E}_t^{\mathbf{Q^T}}\left(V_T \middle| \frac{S_T}{F_t^T} = K\right)\frac{\partial^2 \tilde{C}}{\partial K^2}}$$

where $f(t,T)$ refers to the instantaneous forward rate.

13. Calibration

It is important to underline here that particle methods can't be used in order to calibrate Forward Market Models due to the stochastic cash dividends that are introduced in the model since we don't know how the implied volatility is going to evolve at the ex-div date (unless in the case where cash or proportional dividends are deterministic). In order to calibrate this model, we therefore need to use an approach such as the one proposed by Reghai in [14] or to an alternative approach, the discrete local fixed point method, with converges faster that the latter method as it is explained in [6].

14. Dividend conditions

The formula above is quite generic and we would like to use it in order to understand the dividend conditions that we would use if we were to define a cash dividend model or proportional dividend model where we would directly diffuse each of the dividends like in e.g. [5] (and if we want all our vanilla prices - even between pillar dates - to remain the same).

We must underline that this formula is directly linked to a Black-Scholes set-up which, in view of its multiplicative nature, seems very inappropriate to handle cash dividends which are of additive nature (which is why the general diffusion of the forward is a displaced lognormal - property that is used in BGS or in the Dividend averaging technique for example). Indeed, since dividends are paid at discrete dates, and supposing that implied volatilities are continuous at dividend dates, we will write that we have :

$$\mathbb{E}_t^{\mathbf{Q^T}}\left(\frac{\partial_{T'}F_T^{T'}|_{T'=T}}{F_T^T}\frac{S_T}{F_t^T}1_{\frac{S_T}{F_t^T}>K}\right)=\mathbb{E}_t^{\mathbf{Q^T}}\left(\frac{\partial_T F_t^T}{F_t^T}\frac{S_T}{F_t^T}1_{\frac{S_T}{F_t^T}>K}\right)$$

In a model with stochastic proportional dividends, we deduce from (a derivation of) this formula that we need to have :

$$\mathbb{E}_t^{\mathbf{Q^T}}\left(d_T^{Sto}\middle|\frac{S_T}{F_t^T}=K\right)=d_T^{Det}$$

This is a necessary (and sufficient) condition that we need to fulfill in any stochastic proportional dividend model if we wan't to keep the same local vol without changing our prices.

For cash dividends, following the same methodology, we won't obtain that:

$$\mathbb{E}^{\mathbf{Q^T}}\left(D_T^{Sto}\middle|\frac{S_T}{F_t^T}=K\right)=D_T^{Det}$$

However, this equation can be derived by simply writing the transition of the call prices just before and after the ex-div date. Why is it so ? Simply because a cash dividend introduces a jump in the implied vol parameter. Therefore, in the equation above, the $\partial_T w$ term can't be neglected anymore.

15. Product Study : Dividend sensitive products

We now have several models which have different ways of introducing dividends that all fit the whole term structure of forwards and the vanilla on their observable pillars (note that a proportional dividend model can't be consistent with a cash dividend model between pillars, and this statement is quite generic when it comes to consider different dividend models):

- Local volatility with proportional dividends (PD-LV)

- Stochastic volatility with proportional dividends (PD-SV)
- Local volatility with cash dividends (0-forward model) (CD-LV)
- Stochastic volatility with cash dividends (0-forward model) (CD-SV)
- Local volatility with stochastic cash dividends (SCD-LV)
- Stochastic volatility with stochastic cash dividends (SCD-SV)

Our stochastic volatility model is a Local Stochastic Vol model (we however refer as SV rather than LSV) such that:

$$\xi_t^t = e^{\int_0^t \nu e^{-\kappa(t-s)} dW_s^V - \frac{1}{2}\int_0^t \nu^2 e^{-2\kappa(t-s)} ds}$$

We list below the parameters that we have considered for our analysis :

Parameter	Value
κ	95.61%
ν	150%
$\rho_{S,V}$	-54%
$\rho_{S,D}$	90%
$\rho_{V,D}$	-50%

Concerning the underlying that we use, it has a vanilla surface that is typical of Eurostoxx, the rate is flat equal to 2%, the dividends with a cash amount of 5% of the current spot (considered to be 100) fall each year at 6M, 1Y6M, 2Y6M, etc.

In the following, we will present the different prices that these models can provide. These models all provide some prices that can be quite different for dividend sensitive products.

15.1. Sanity check : calibration to Vanillas on market pillars

We present here smiles that prove that we fit accurately the volatility on the market observable pillars. All the benchmarks are accurately fitted. For simplicity, we however present only three of them (3M, 1Y and 5Y):

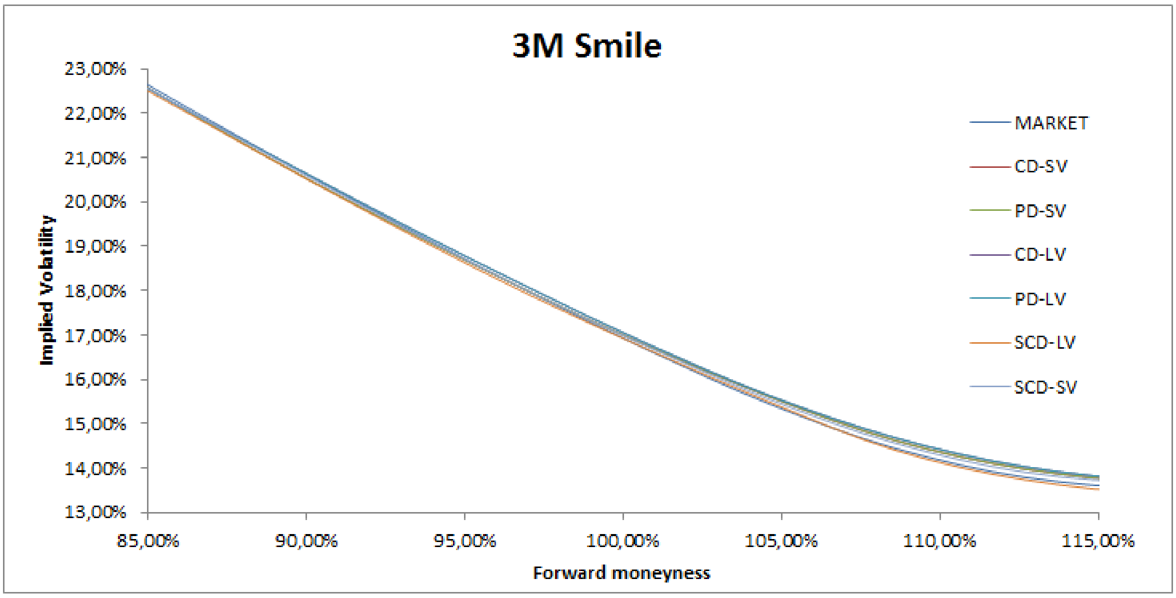

Figure 2: 3M Vanilla Smile Fit with various dividend models

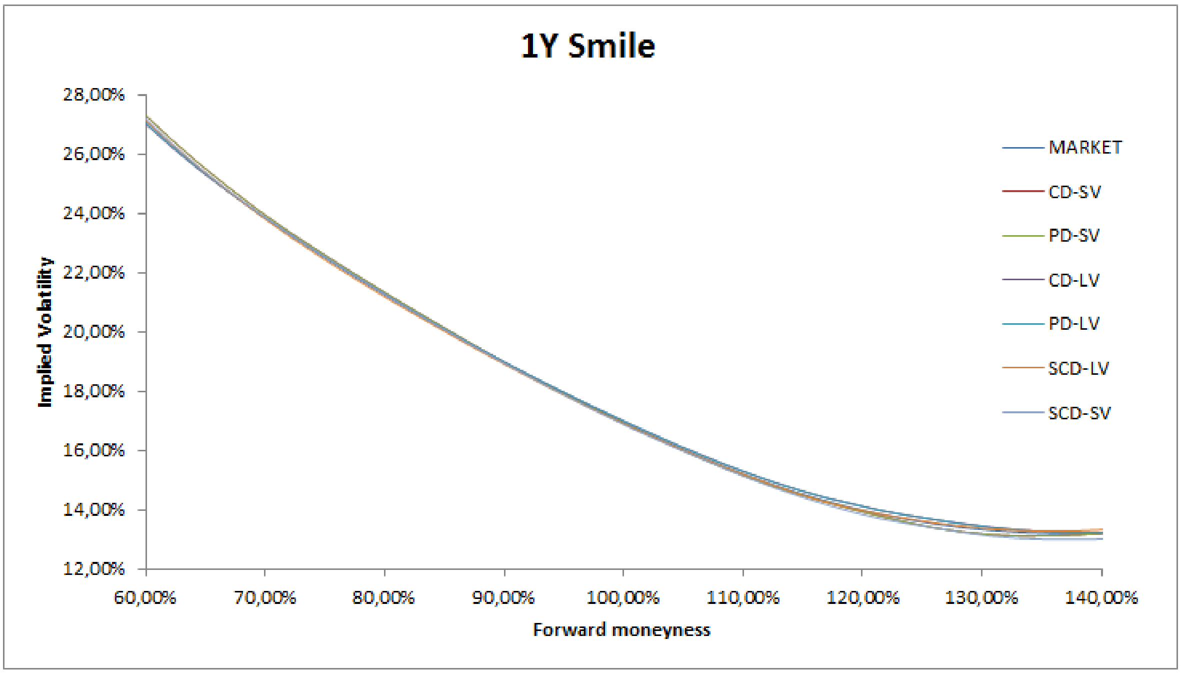

Figure 3: 1Y Vanilla Smile Fit with various dividend models

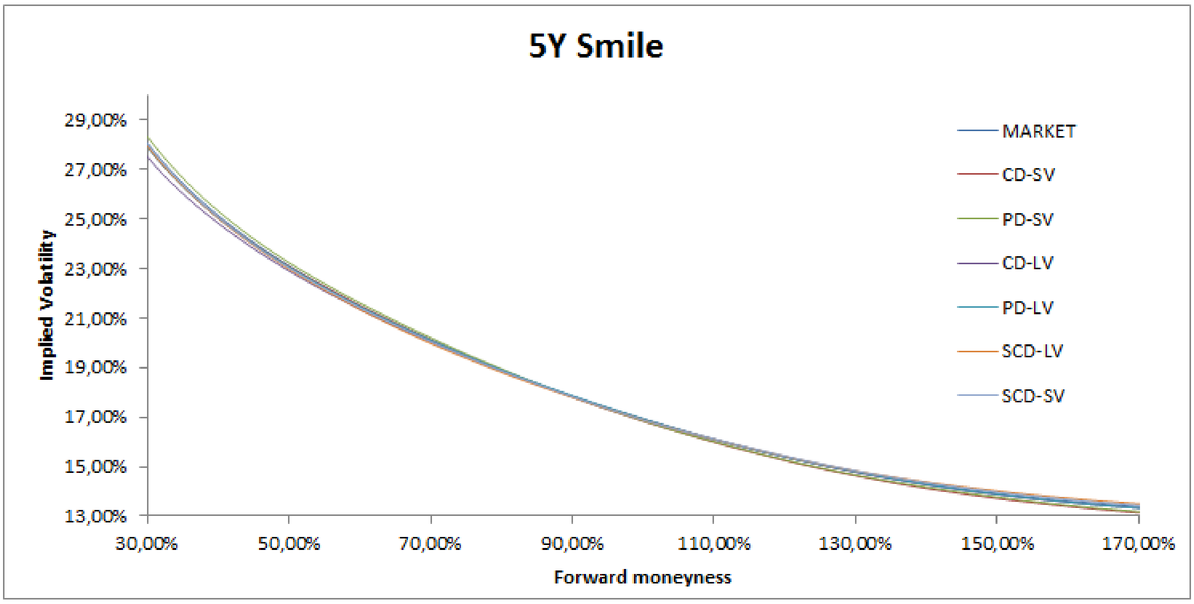

Figure 4: 5Y Vanilla Smile Fit with various dividend models

15.2. Vanillas between pillars : difference in interpolation

If vanilla prices are by definition identical on the observable pillars, they may differ when it comes to evaluate vanillas at intermediate dates. Indeed, different dividend modeling will provide different vol interpolations between these pillars. Here, we look at the smiles that are obtained in our different models just after the ex div dates, at 6M, 2Y6M, 4Y6M:

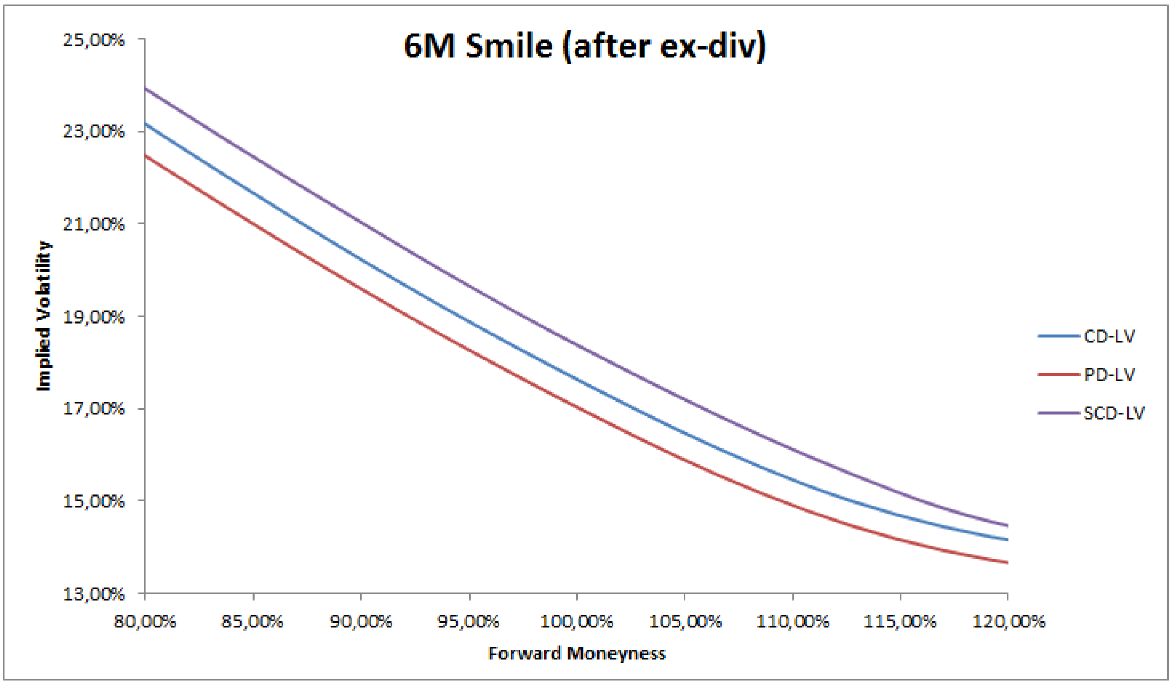

Figure 5: 6M Vanilla Smile after ex-div date

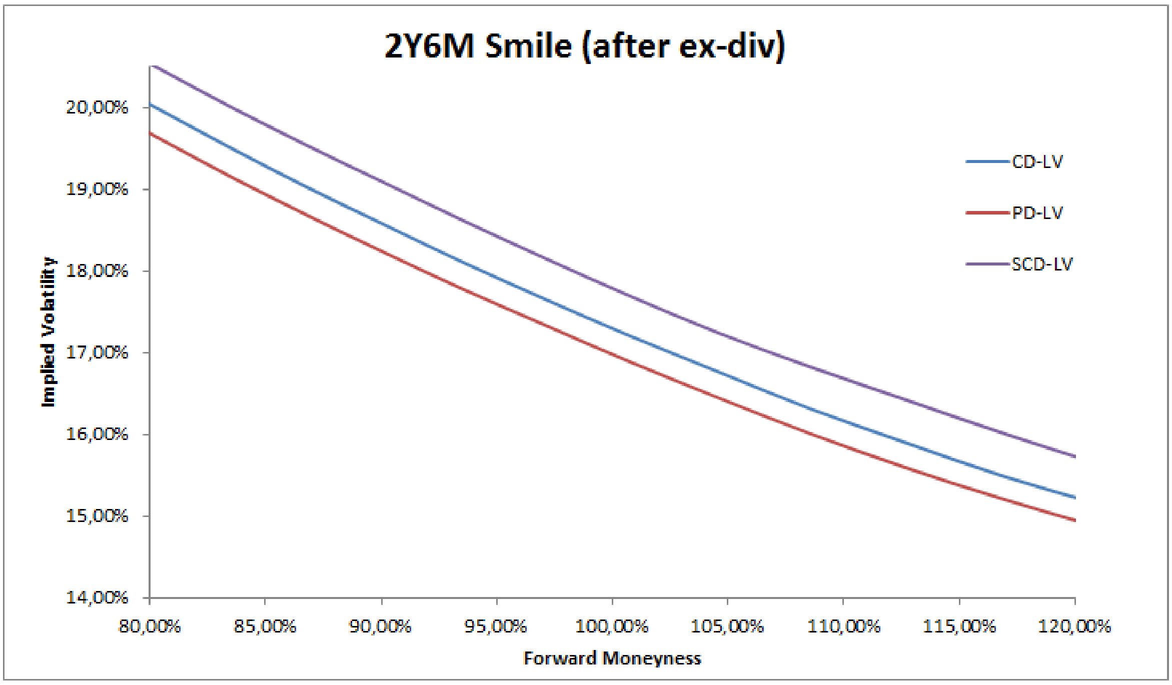

Figure 6: 2Y6M Vanilla Smile after ex-div date

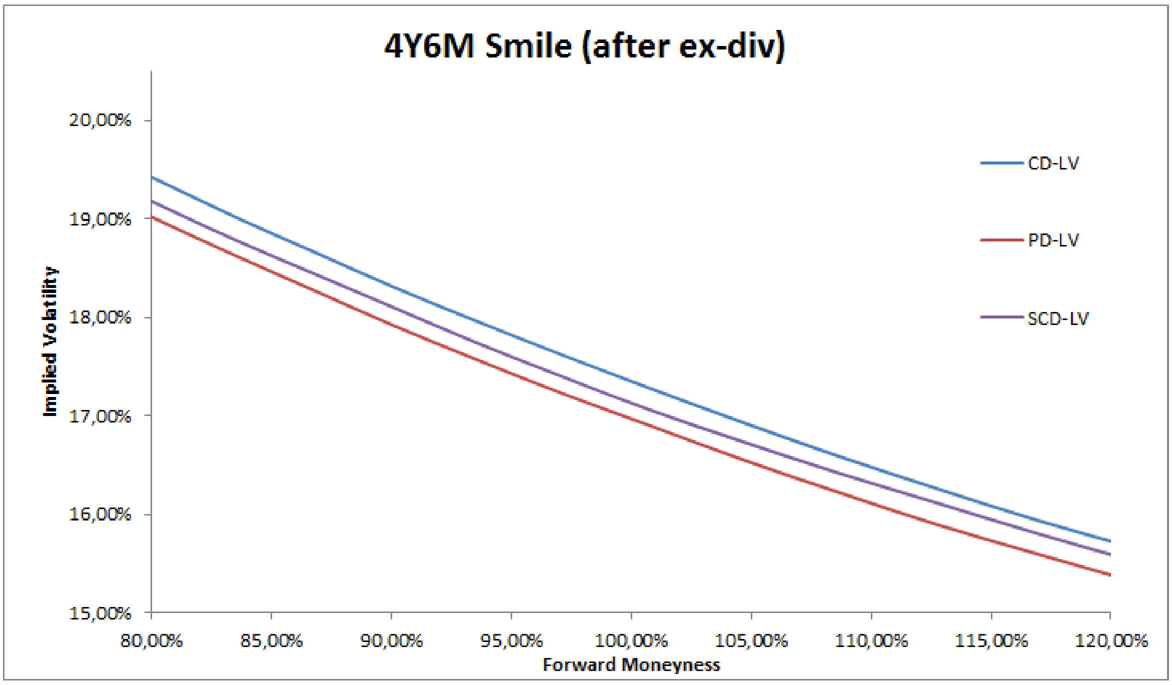

Figure 7: 4Y6M Vanilla Smile after ex-div date

We can see that the Forward Market Model with Stochastic Volatility enables to reproduce the interpolation that we have in a cash dividend model.

15.3. Variance Products

We present in this section the results that we have for long term Forward Starting Variance Swaps and Volatility Swaps :

When it comes to consider variance swaps on stocks, the dividend is considered to be reinvested. This is done in order to capture only the spot volatility and not the forward volatility. This is not strictly true : indeed, due to the fact that there are some differences between net and gross dividends, the variance swaps doesn't exactly capture the spot volatility (the effect generally remains limited). This correction is generally only done for Variance products on stocks, but not for variance swaps on indexes. In the following, we consider the case where dividends are reinvested.

We present in the following tab the prices that we obtain for long term forward variance swaps and long term forward Volatility Swaps (i.e. 1Y in 2Y, 1Y in 3Y, 1Y in 4Y swaps) for different dividend models :

Variance Product	CD-LV	PD-LV	CD-SV	PD-SV	SCD-LV	SCD-SV
1Y in 2Y Variance Swap	15.26%	19.93%	15.23%	19.98%	16.05%	16.17%
1Y in 3Y Variance Swap	14.10%	19.79%	13.93%	19.77%	16.00%	15.85%
1Y in 4Y Variance Swap	11.47%	19.63%	11.54%	19.71%	15.50%	15.56%
1Y in 2Y Volatility Swap	14.20%	17.46%	13.22%	16.41%	14.69%	13.90%
1Y in 3Y Volatility Swap	12.34%	17.24%	11.47%	16.12%	14.22%	13.15%
1Y in 4Y Volatility Swap	9.99%	17.02%	9.50%	15.93%	13.57%	12.59%

We see that, as expected, the forward Variance Swaps and Volatility Swaps are lower with cash dividends than with proportional dividends : from a modeling point of view, proportional dividends are therefore not adapted for long maturities. We also see that the correction associated to a deterministic cash dividend model is too important compared to a stochastic cash dividend model : only stochastic cash dividends (i.e. Forward Market Model here) are adapted to price these products.

15.4. Forward Forwards

We know that Forward Forwards, that pay $\frac{S_{T_{i+1}}}{S_{T_i}}$ are dividend sensitive products : they are higher with proportional dividends and lower with cash dividends. Let's present how these products behave in our different models :

Products	PD-LV	PD-SV	CD-LV	CD-SV	SCD-LV	SCD-SV
1Y Forward	97.01%	97.01%	97.02%	97.01%	97.00%	97.02%
1Y in 1Y Forward Forward	96.82%	96.80%	96.63%	96.61%	96.65%	96.64%
1Y in 2Y Forward Forward	96.75%	96.72%	96.22%	96.20%	96.23%	96.21%
1Y in 3Y Forward Forward	96.40%	96.41%	95.54%	95.52%	95.99%	95.95%
1Y in 4Y Forward Forward	96.33%	96.22%	94.93%	94.82%	95.71%	95.69%

We see that the results are as expected. We can see that with a traditional mix of cash dividends and proportional dividends, the long term Forward Forwards are not correctly priced.

15.5. Dividend options

We present here the dividend smiles that are obtained with a cash dividend model, a proportional dividend model and a Forward Market Model (or Stochastic Cash Dividends model with local vol here):

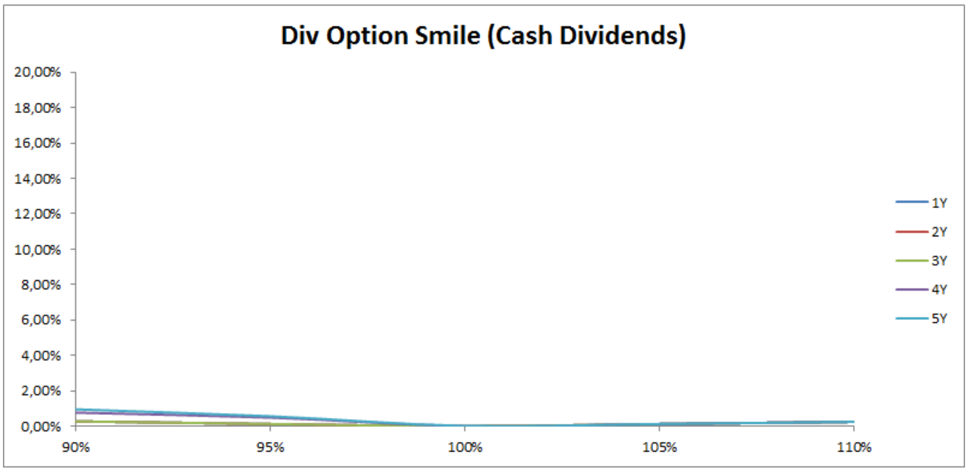

Figure 8: Dividend Smile with a cash dividend model

As expected, there is no volatility of the div in such a model.

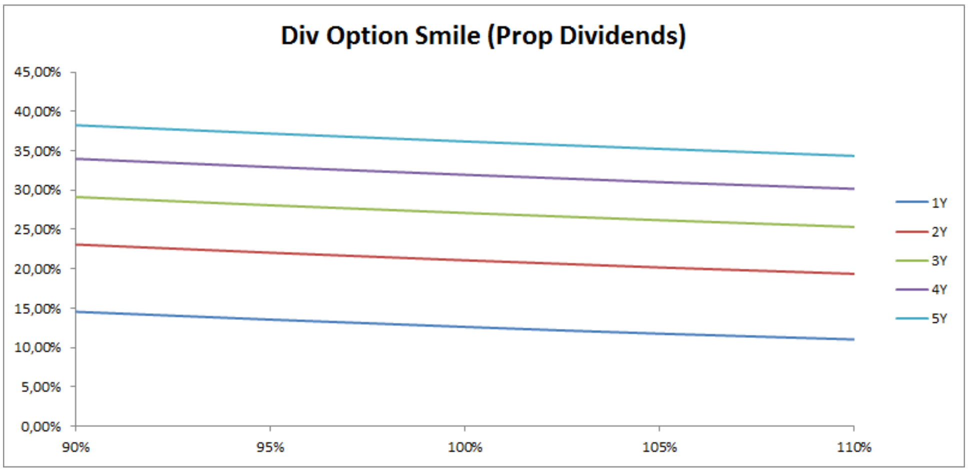

Figure 9: Dividend Smile with a proportional dividend model

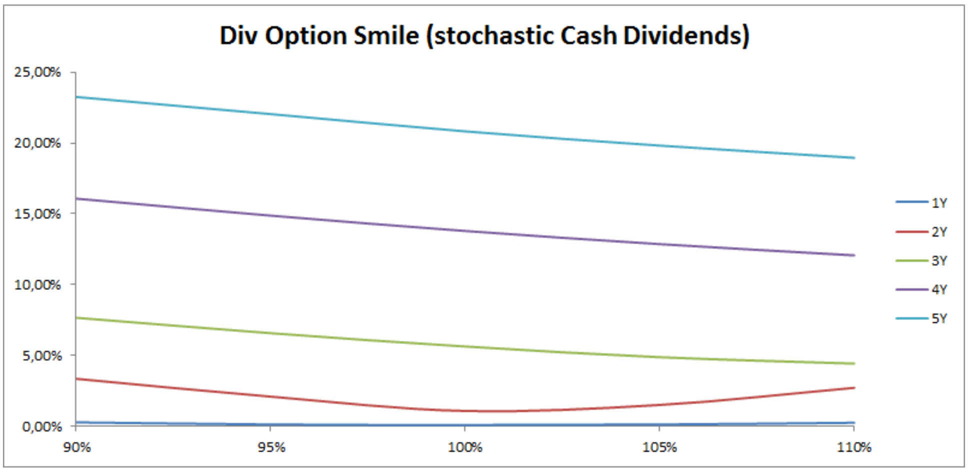

Figure 10: Dividend Smile with a stochastic cash dividend model

We can see that both smiles are decreasing in strike and increasing in time : this is a standard feature.

In practice, dividend options are observable. A model with a mix between cash and proportional dividends can be easily fitted to these. The various versions of Forward Market models that we introduced are also able to fit this smile by playing on the vol of div term. Clearly the prices of dividend exotics can only be compared between two models can only be compared if their dividend smiles match.

15.6. Exotic products : Autocall structure, KO Div Swaps

We focus in this section on two products that are sensitive to the assumptions made on the comovements of spot and div : a standard autocallable structure, and several KO Dividend Swaps.

For the autocallable structure (a 5Y autocallable that can knock out quarterly at 100% with a terminal short position in an ATM Put option Down In at maturity at 60%).

Products	PD-LV	PD-SV	CD-LV	CD-SV	SCD-LV	SCD-SV
5Y Autocallable	91.82%	92.13%	91.22%	91.54%	91.42%	91.71%

We can see that the price of the autocallable structure is less expensive with cash dividends than it is with proportional dividends. This is mainly due to the following proof (where we denote p for proportional and c for cash - for simplicity, we don't consider stochastic rates here) :

$$\mathbb{E}_0^{\mathbf{Q}}\left(S_T^{(p)}1_{S_{T_1}^{(p)}<K}\right) - \mathbb{E}_0^{\mathbf{Q}}\left(S_T^{(c)}1_{S_{T_1}^{(c)}<K}\right) = \mathbb{E}_0^{\mathbf{Q}}\left((F_{T_1}^T)^{(p)}1_{S_{T_1}^{(p)}<K}\right) - \mathbb{E}_0^{\mathbf{Q}}\left((F_{T_1}^T)^{(c)}1_{S_{T_1}^{(c)}<K}\right)$$

But we have :

$$\begin{aligned}
(F_{T_1}^T)^{(c)} &= \frac{S_{T_1}^{(c)} - B(T_1,T)}{A(T_1,T)} \\
(F_{T_1}^T)^{(p)} &= \frac{S_{T_1}^{(p)} b(T_1,T)}{A(T_1,T)}
\end{aligned}$$

and $F_0^{T_1} - B(T_1,T) = F_0^{T_1} b(T_1,T)$.

which eventually leads to (in view of the equality of the marginals at T_1):

$$\begin{aligned}
\mathbb{E}_0^{\mathbf{Q}}\left((F_{T_1}^T)^{(p)}1_{S_{T_1}^{(p)}<K}\right)-\mathbb{E}_0^{\mathbf{Q}}\left((F_{T_1}^T)^{(c)}1_{S_{T_1}^{(c)}<K}\right) &= \frac{1}{A(T_1,T)}\left(\frac{F_0^{T_1}-B(T_1,T)}{F_0^{T_1}}\mathbb{E}_0^{\mathbf{Q}}\left(S_{T_1}^{(p)}1_{S_{T_1}^{(p)}<K}\right)-\mathbb{E}_0^{\mathbf{Q}}\left(S_{T_1}^{(c)}1_{S_{T_1}^{(c)}<K}\right)+B(T_1,T)\mathbb{E}_0^{\mathbf{Q}}\left(1_{S_{T_1}^{(c)}<K}\right)\right)\\
&= \frac{B(T_1,T)}{A(T_1,T)}\left(-\frac{\mathbb{E}_0^{\mathbf{Q}}\left(S_{T_1}^{(p)}1_{S_{T_1}^{(p)}<K}\right)}{\mathbb{E}_0^{\mathbf{Q}}\left(S_{T_1}^{(p)}\right)}+\mathbb{E}_0^{\mathbf{Q}}\left(1_{S_{T_1}^{(c)}<K}\right)\right)\\
&= \frac{B(T_1,T)}{A(T_1,T)}\frac{\mathbb{E}_0^{\mathbf{Q}}\left(1_{S_{T_1}^{(c)}<K}\right)}{\mathbb{E}_0^{\mathbf{Q}}\left(S_{T_1}^{(c)}\right)}\left(\mathbb{E}_0^{\mathbf{Q}}\left(S_{T_1}^{(c)}\right)-\frac{\mathbb{E}_0^{\mathbf{Q}}\left(S_{T_1}^{(c)}1_{S_{T_1}^{(c)}<K}\right)}{\mathbb{E}_0^{\mathbf{Q}}\left(1_{S_{T_1}^{(c)}<K}\right)}\right)\\
&= \frac{B(T_1,T)}{A(T_1,T)}\frac{\mathbb{E}_0^{\mathbf{Q}}\left(1_{S_{T_1}^{(c)}<K}\right)}{\mathbb{E}_0^{\mathbf{Q}}\left(S_{T_1}^{(c)}\right)}\left(\mathbb{E}_0^{\mathbf{Q}}\left(S_{T_1}^{(c)}\right)-\mathbb{E}_0^{\mathbf{Q}}\left(S_{T_1}^{(c)}\middle|S_{T_1}^{(c)}<K\right)\right)\\
&\geq 0
\end{aligned}$$

the last inequality comes from the fact that we have the general inequality $\mathbb{E}(X) \geq \mathbb{E}(X|X < K)$.

The conditional forward is therefore higher with proportional dividends, and since this autocallable is long the spot (its delta is positive), the autocallable is more expensive with proportional dividends.

We now consider an exotic product on dividend, i.e. a knock out dividend swap whose fair strike is worth :

$$K=\frac{\mathbb{E}_0^{\mathbf{Q}}\left(e^{-\int_0^{T_2}r_s ds}1_{\underset{T_1<t\leq T_2}{min}S_t<B}\left(\sum_{T_1<t\leq T_2}D_i\right)\right)}{\mathbb{E}_0^{\mathbf{Q}}\left(e^{-\int_0^{T_2}r_s ds}1_{\underset{T_1<t\leq T_2}{min}S_t<B}\right)}$$

We have the following prices for two different dividend models (a mixed model with both cash and proportional dividends and a Forward Market Model) with different vol models (local vol and stochastic vol) which have been calibrated so that they both give the same spot smiles (like explained above) and the same dividend smiles:

Products	Mixed-LV	Mixed-SV	SCD-LV	SCD-SV
1Y KO Div Swap at 80%	5.0	5.0	5.0	5.0
1Y in 1Y KO Div Swap at 80%	4.94	4.94	4.97	4.98
1Y in 2Y KO Div Swap at 80%	4.75	4.76	4.89	4.93
1Y in 3Y KO Div Swap at 80%	4.43	4.48	4.73	4.84
1Y in 4Y KO Div Swap at 80%	4.19	4.27	4.58	4.78

Concerning the prices of KO Dividend Swaps, we see that they behave in a similar manner when we use proportional dividends or stochastic cash dividends.

15.7. Negative dividends probability

We have underlined that, in the stochastic cash dividend model, we don't monitor directly the dividends, but rather the forward. Even if the diffusion is "cash dividend inpired", we may therefore produce some negative cash dividends. However, this effect is quite limited. Indeed, in practice, this may only happen in the long term (after 4Y, with a high dividend yield here), and even in that case, we find with the above parameters a probability of 0.10% of having a negative dividend.

16. Conclusion

We have presented in this paper different ways to introduce cash dividends, either deterministic or stochastic. Deterministic cash dividends can be added without introducing any additional complexity to the models that are generally used in production risk management, either local volatility models or stochastic volatility models. However, many products are sensitive to stochastic dividends, so that an approach that mixes cash (for the short term) and proportional dividends (for the long term) is generally more relevant.

This representation is however improper to model the fact that the Black-scholes volatility jumps at dividend dates even in the long term : only cash dividend models can have this feature. We have therefore presented a model which is a Forward Market Model, that, despite the fact that it's more time consuming, is the most generic model in order to tackle the general problem of stochastic cash dividends. This approach is powerful and can also be used to mix stochastic cash dividend and stochastic rates.

Acknowledgements I am grateful to Nicolas Grandchamp des Raux and Marc Laurencin for their valuable feedbacks.

References

[1] R. Bos, A. Gairat, A. Shepeleva, *Dealing with discrete dividends*, Risk magazine, Jan. 2003.

[2] M. Bos, S. Vandermark, *Finessing fixed dividends*, Risk magazine, Sept. 2002

[3] M. Brennan, E. Schwartz, *The Valuation of American Put Options*, The Journal of Finance, Sept. 1977.

[4] H. Buehler, *Volatility and Dividends - Volatility Modeling with Cash Dividends and Simple Credit Risk*, 2010.

[5] H. Buehler, A. Dhouibi, D. Sluys, *Stochastic Proportional Dividends*, 2010.

[6] P. Delanoë, *Discrete Local Fixed Point Methods: A Fast and Generic Way to Calibrate Option Smiles*, May 2015.

[7] A. Gocsei, F. Sahel, *Analysis of the sensitivity to discrete dividends : A new approach for pricing vanillas* , 2010.

[8] N. Grandchamp des Raux, *Back to the dividend model*, ICBI Conference, 2008.

[9] J. Guyon, P. Henry-Labordere, *The Smile Calibration Problem Solved*, July 2011, available at SSRN: http://ssrn.com/abstract=1885032

[10] P. Henry-Labordere, *Calibration of local stochastic volatility models to market smiles : a Monte Carlo approach*, Risk magazine, Sept. 2009.

[11] P. Jäckel, *By Implication*, available at http://www.jaeckel.org

[12] V. Piterbarg, *Time To Smile*, Risk magazine, May 2005.

[13] A. Reghai, *Decoding the American Vanilla Prices*, October 2013.

[14] A. Reghai, *The hybrid most likely path*, Risk magazine, April 2006.