Volatility swaps made simple

Most articles on volatility products focus on the relatively straightforward variance swaps. Here, Oliver Brockhaus and Douglas Long take the subject further with a simple model of volatility swaps

Black-Scholes (1973) is the standard model for valuing liquid European-style vanilla options. It assumes the underlying asset follows a lognormal process with a constant volatility, regardless of any future moves in the asset or volatility. The fact that this is not observed in reality is evident from the market implied volatilities smile or smirk structure with respect to strike, and its term structure.

This article examines two fundamentally different approaches to this volatility smile[1]: deterministic volatility models (DVMs) and stochastic volatility models (SVMs). In general, the need for such models is not to price European-style vanilla options but to value other option types in the context of the market information provided by these option prices.

We assume that the underlying asset S_t follows the process:

$$\frac{dS_t}{S_t} = (r_t - d_t)dt + \sigma_t(\ldots)dZ_t^1 \tag{1}$$

in the risk-neutral world. The parameters r_t and d_t represent the risk-free interest rate and the asset dividend yield respectively. Volatility smile models are distinguished by the specification of $\sigma_t(\ldots)$.

The Black-Scholes model assumes that this volatility term is a constant, $\hat{\sigma}$ (implied volatility). Merton's (1973) extension for a term structure of volatility is $\sigma_t(\ldots) = \sigma_t$, with the implied volatility for an option of maturity T given by:

$$\hat{\sigma}_T^2 = \frac{1}{T}\int_0^T \sigma_u^2 du$$

Deterministic volatility models

The most natural extension to Black-Scholes is to generalise the asset process so that the asset volatility is a deterministic function of the asset level:

$$\sigma_t(\ldots) = \sigma_t(S_t) \tag{2}$$

This is known as local volatility, and is distinct from the implied volatility of Black-Scholes, which is a property of a traded option, whereas local volatility is a property of the asset process. In thus extending the model, the analytical tractability of Black-Scholes is lost but one has a single-asset process that consistently prices and hedges exotic options in the presence of the smile. The model is complete and if one assumes a continuum of market prices then the local volatility surface is unique. Practical issues involving the implementation of this model can be found in Brockhaus *et al* (1999) and references within.

Calibration requires determining the local volatility $\sigma_t(S_t)$ such that the European price $C_T(K)$ for a particular strike, K, and maturity, T, agrees with the price available in the market. Implied diffusion theory (Dupire, 1994) provides a theoretical basis for this; however, other models of local volatility have been considered (eg, constant elasticity of variance).

The implied volatility $\hat{\sigma}_T(K)$ can be regarded as a weighted average of $\sigma_t(S_t)$ along all the paths leading to K at T. Due to this averaging effect, the slope of $\sigma_t(S_t)$ is greater than that of the implied volatility surface in most markets.

Stochastic volatility models

An alternative approach would be to consider the volatility as another stochastic variable, and there is growing evidence to support this hypothesis.[2]

Here, we consider the Heston (1993) model. It is mean-reverting and has a correlation between volatility and asset level. European-style vanilla option prices are easily obtained by performing a one-dimensional numerical integral. The Heston asset process has a variance $\sigma_t^2(\ldots) = \sigma_t^2$ that follows a Cox, Ingersoll & Ross (1985) process:

$$d\sigma_t^2 = \kappa\left(\theta^2 - \sigma_t^2\right)dt + \gamma\sigma_t dZ_t^2 \tag{3}$$

It is specified by a short and a long volatility σ_0 and θ, the reversion speed $\kappa > 0$ and a volatility (of volatility) parameter $\gamma \geq 0$.

Assuming $\sigma_t < \theta$, this equation implies that, at a later time $t + dt$, the volatility will on average have risen, since $\theta^2 - \sigma_t^2 > 0$ and the expectation of $\gamma\sigma_t dZ_t^2$ is zero. Similarly, the volatility will fall if $\sigma_t > \theta$. The reversion speed determines the strength of the force pulling towards θ.

The correlation ρ between asset and volatility affects the volatility skew. For a positive correlation, for example, an increase in asset returns will on average be coupled with an increase in volatility, ie, the implied volatility curve increases with strike. This is a typical pattern for commodities markets. Equity markets, however, exhibit negative skewness, which may be reformulated in terms of a negative correlation. Exchange rate markets generally do not have a pronounced correlation.

The volatility parameter γ fattens both tails symmetrically and thus affects the kurtosis (smile) of the implied volatility. It also has the effect of diluting the reversion speed and flattens the skew over time. These effects are illustrated in figure 1.

Volatility contracts

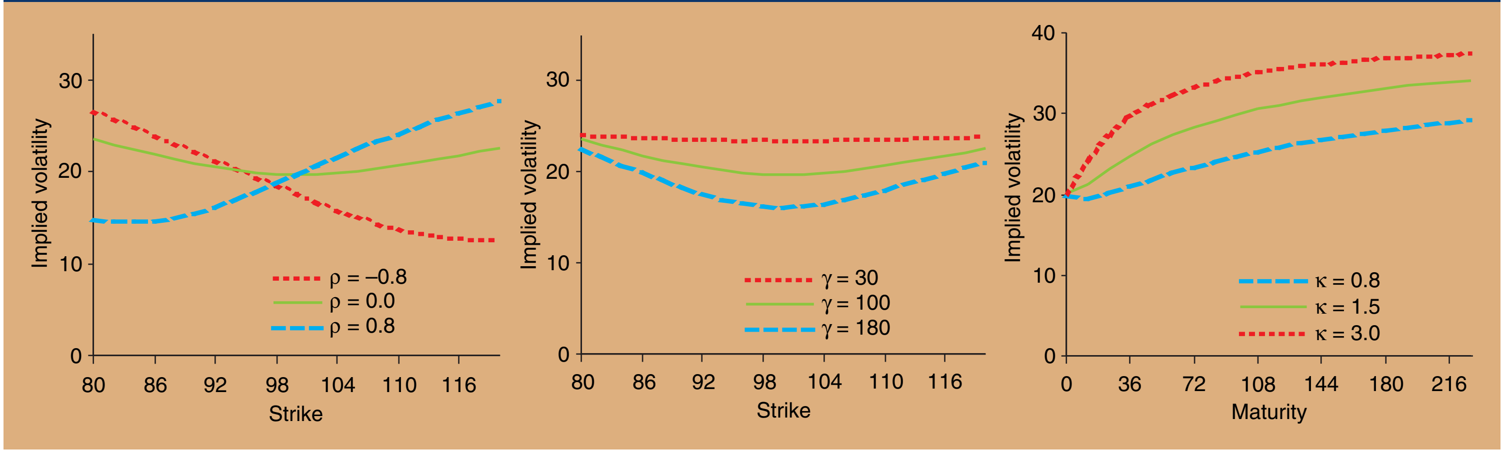
Derivatives traders are exposed to changes in the level and volatility of the underlying assets. Index future contracts are a convenient vehicle for hedging against index level moves, and hedging against index volatility is traditionally done by buying and selling options. This adjusts the volatility profile of the portfolio in a non-linear manner and induces additional exposure to the index level. A more natural approach would be to buy a forward volatility contract (or swap). These contracts, written on the realised volatility of returns, have only recently been available in the over-the-counter market, and by definition volatility smile models are crucial to their successful valuation and risk management. Currently, research has concentrated on realised variance swaps (Neuberger, 1994, Demeterfi *et al*, 1999, and Chriss & Morokoff, 1999), but attention has not focused on valuation techniques for the more fashionable but complicated volatility products available.

Given two assets S_t^1 and S_t^2, with $t \in [0,T]$ sampled on days $t_0 = 0 < t_1 < t_2 < \ldots < t_n = T$ between today and maturity T, the log return for each asset is:

[1] This is not an inclusive list, and in particular we do not look at Garch or jump models

[2] Time-series analysis on various markets suggests that it should be viewed as a random process exhibiting mean reversion and being correlated with asset returns

1. Correlation, volatility of volatility and mean-reversion behaviour



$$R_i = \log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) \qquad (i = 1,2,\ldots,n) \tag{4}$$

The realised variance is defined by a variance estimator for these returns, namely:

$$\mathrm{Var}_n(S) = \frac{n}{(n-1)T}\sum_{i=1}^{n}(R_i)^2 \tag{5}$$

Similarly, realised volatility, covariance and correlation can readily be approximated by:

$$\mathrm{Vol}_n(S) = \sqrt{\mathrm{Var}_n(S)} \tag{6}$$

$$\mathrm{Cov}_n\left(S^1,S^2\right) = \frac{n}{(n-1)T}\sum_{i=1}^{n}R_i^1R_i^2 \tag{7}$$

$$\mathrm{Corr}_n\left(S^1,S^2\right) = \frac{\mathrm{Cov}_n\left(S^1,S^2\right)}{\sqrt{\mathrm{Var}_n\left(S^1\right)}\sqrt{\mathrm{Var}_n\left(S^2\right)}} \tag{8}$$

We have assumed that the mean of the returns is of the order $1/n$ and can be neglected. Moreover, the scaling by n/T ensures that these quantities are annualised (daily) if the maturity T is expressed in years (days).

Variance swaps

It can be shown that the realised continuously sampled variance is given by:

$$V := \lim_{n\to\infty}\mathrm{Var}_n(S) = \frac{1}{T}\int_0^T \sigma_t^2(\ldots)dt \tag{9}$$

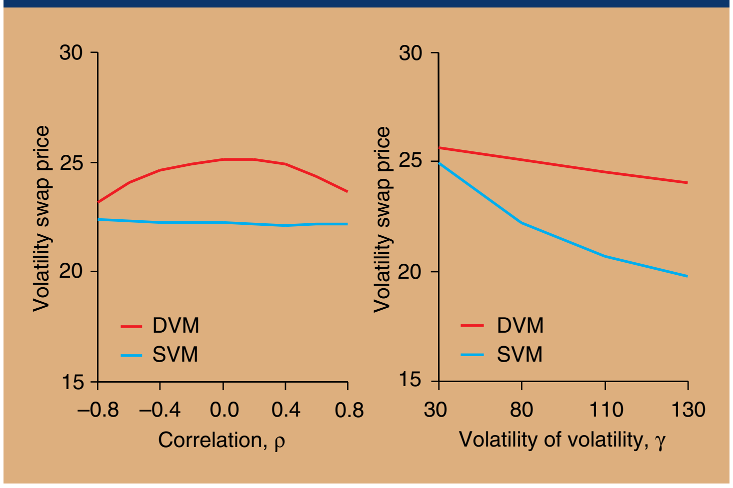
The original work by Dupire (1993), Neuberger (1994) and Demeterfi *et al* (1999) showed that the variance swap can be replicated (in a model-independent way) by a portfolio of options, forwards and bonds.

Volatility swaps

Pricing volatility swaps is more complicated than pricing variance swaps, although more desirable. For this reason, many financial institutions try to convince their clients to buy the variance swap.

Unlike the price of a variance swap, the price of a volatility swap varies considerably depending on the model used to price it. This is true even when the models used are calibrated to the market. A comparison between a DVM and an SVM shows that the price of the former is always higher. As can be seen from figure 2, the price difference increases with decreasing absolute asset-volatility correlation and with increasing volatility of volatility.[3] The origin of this behaviour is due to the Jensen inequality, and is further discussed and applied to other contract types in Brockhaus & Long (1999). The difference vanishes if there is no randomness in the volatility process ($\gamma = 0$) or if the correlation is perfect ($\rho = \pm 1$).

2. Model dependency of volatility swap



A model-independent approach can be derived using a Taylor expansion of the square root function around some V_0 close to the expected volatility swap value. As a first-order approximation to the volatility swap we have:

$$\sqrt{V} \approx \frac{V + V_0}{2\sqrt{V_0}} \qquad E\left[\sqrt{V}\right] \approx \frac{E[V] + V_0}{2\sqrt{V_0}} \tag{10}$$

Expanding to second order around V_0 leads to:

$$\sqrt{V} \approx \frac{V + V_0}{2\sqrt{V_0}} - \frac{(V - V_0)^2}{8V_0^{3/2}}$$

$$E\left[\sqrt{V}\right] \approx \frac{E[V] + V_0}{2\sqrt{V_0}} - \frac{\mathrm{Var}[V] + \left(E[V] - V_0\right)^2}{8V_0^{3/2}} \tag{11}$$

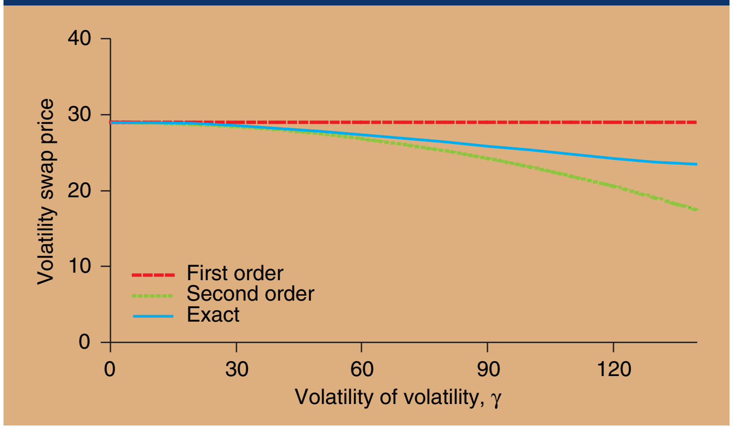
Since the variance swap value ($E[V]$) is the only volatility product with a model-independent price, only the first order provides a model-independent approximation. Additional assumptions on the volatility process have to be made to obtain the variance $\mathrm{Var}[V]$ of V and yield higher-order results.

Using the Heston (1993) model (3), the undiscounted value and variance of a variance swap is given by:

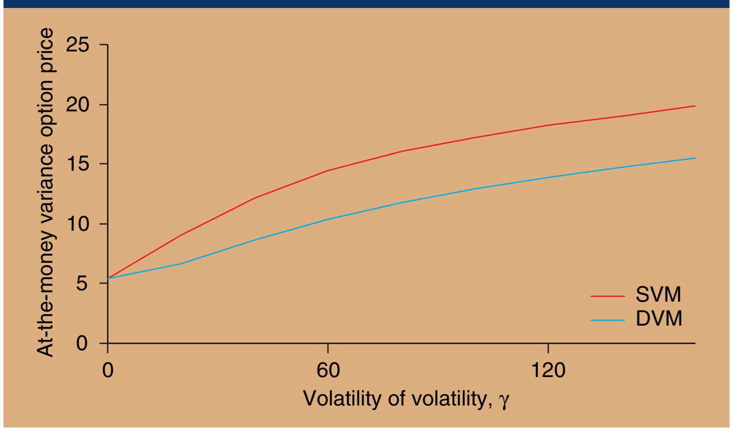
$$E[V] = \frac{1 - e^{-\kappa T}}{\kappa T}\left(\sigma_0^2 - \theta^2\right) + \theta^2 \tag{12}$$

[3] *All the numerical results in this article are generated using either Monte Carlo or finite difference techniques, calibrated to appropriate market data*

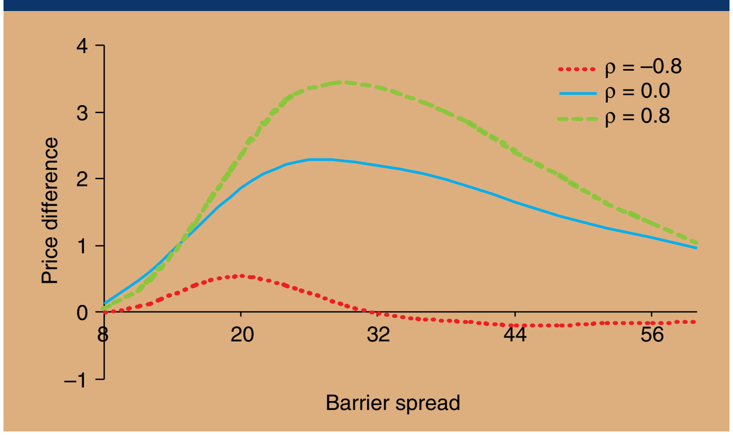
3. First- and second-order approximations for the volatility swap



4. At-the-money variance option price against volatility of volatility



5. Double barrier price difference against barrier spread



$$\mathrm{Var}[V] = \frac{\gamma^2 e^{-2\kappa T}}{2\kappa^3 T^2}\left[2\left(-1 + e^{2\kappa T} - 2e^{\kappa T}\kappa T\right)\left(\sigma_0^2 - \theta^2\right) + \left(-1 + 4e^{\kappa T} - 3e^{2\kappa T} + 2e^{2\kappa T}\kappa T\right)\theta^2\right] \quad (13)$$

In approximating volatility swaps only to first order, we do not capture the effects of stochastic volatility (volatility convexity). However, as shown in figure 2, such an effect has to be assumed.

The first- and second-order Taylor expansions (10, 11) provide two approximations that respectively overestimate and underestimate the true value $E[\sqrt{V}]$ of the swap. This can be seen in figure 3. In fact, $V_0 = E[V]$ provides the best point around which to expand in order to minimise the upper bound. These two prices can be used as candidates for the offer and bid prices for the volatility contract.

Higher-order moments can be obtained by the numerical calculation of multiple derivatives of the Laplace transform $f(\lambda)$ of the variance swap V. This is given by the function:

$$f(\lambda) = E\left[e^{-\lambda V}\right] = Ae^{-\lambda\sigma_0^2 B} \quad (14)$$

with A and B being functions of $\varphi = \sqrt{\kappa^2 + 2\lambda\gamma^2}$:

$$A = \left\{\frac{2\varphi e^{(\varphi+\kappa)T/2}}{(\varphi+\kappa)\left(e^{\varphi T}-1\right)+2\varphi}\right\}^{2\kappa\theta^2/\gamma^2} \qquad B = \frac{2\left(e^{\varphi T}-1\right)}{(\varphi+\kappa)\left(e^{\varphi T}-1\right)+2\varphi}$$

(see Cox, Ingersoll & Ross, 1985). This allows us to calculate the Taylor expansion of a volatility swap to all orders and hence obtain an exact result.

Alternatively, we have the value as the integral:

$$E\left[\sqrt{V}\right] = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1-f(\lambda)}{\lambda^{3/2}}\,d\lambda$$

as shown in Brockhaus & Long (1999) and Brockhaus *et al* (2000), which follows from a result by Yor (1999).

Covariance and correlation swaps

Options dependent on exchange rate movements, such as those paying in a currency different from the underlying currency, have an exposure to movements of the correlation between the asset and the exchange rate. This risk is not removed by hedging either the vega, gamma or cross-gamma exposure of a portfolio. However, it may be eliminated by using covariance swaps.

Pricing covariance swaps, from a theoretical point of view, is similar to pricing variance swaps:

$$\mathrm{Cov}_n = \frac{1}{2}\left\{-\mathrm{Var}_n\left(S^1\right) - \mathrm{Var}_n\left(S^2\right) + \mathrm{Var}_n\left(S^1S^2\right)\right\} \quad (15)$$

No assumptions about the volatility processes are made. As with the variance swap, the price of a covariance swap is given as a portfolio of bonds as well as forwards and options on S^1, S^2 and S^1S^2.

In practice, such options are often illiquid, and therefore are not suitable for this analysis. This is the case if both assets are equities, but when one or both assets are exchange rates, then there is a chance to obtain sufficient market data (Carr & Madan, 1999).

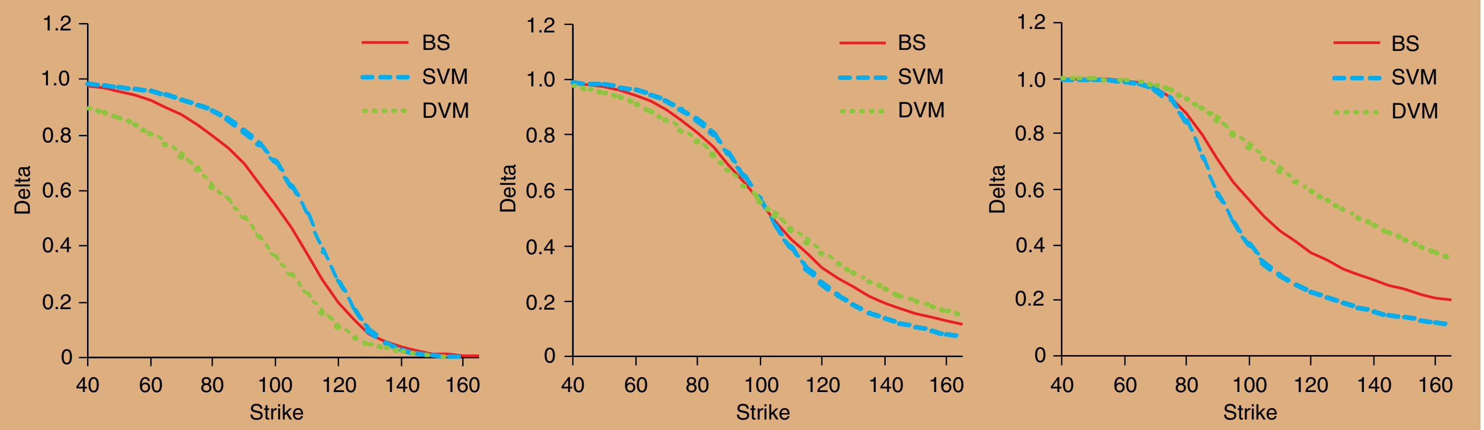
Options exposure to volatility convexity

Knowing option prices for all strikes at a given maturity allows one to calculate the implied distribution of the asset level at that maturity. This is independent of the choice of volatility smile model and, as such, European-style options on the asset level will not be sensitive to this choice. For example, the value of a European-style barrier option is the same under both these models. However, as discussed later, hedging strategy is sensitive to the choice of model.

The situation is different when we consider path-dependent options. Here, conditional probabilities become important and the different smile models will generally result in different prices. This can be seen clearly for at-the-money call variance options (see figure 4), where, although the value of the variance swap is model-independent, the option on it is not. With $\gamma = 0$, the SVM is complete and the prices are once again equal. As with the volatility swap, higher-order moments would allow for closed-form analytical approximations.

Figure 5 shows the price for an at-the-money double knock-out barrier option against the barrier spread (upper barrier level minus lower barrier level) for different values of the correlation, ρ. These options are sensitive to the forward volatility skew and, as such, are sensitive to volatility dynamics. Obviously, as the barrier spread drops to zero so does the price, and when the spread becomes large the two models are again in

6. Call delta in various markets



agreement (the option is then a plain European-style option).

As a general rule, a convex (concave) payout of the variance gives a DVM price that is lower (greater) than or equal to the SVM price. This can be seen in the volatility swap (concave) or the variance option (convex).

Hedging

Volatility smile models are important when hedging even European-style vanilla options, which all models should price consistently. This is particularly evident in figure 6, which shows the delta for three differently skewed markets ($\rho = -0.8, 0, 0.8$ respectively). In each case, one of the volatility models over- hedges and the other under-hedges Black-Scholes. These differences originate from the implied volatility smile dynamics that is intrinsically assumed by these models (Derman, 1999, and Brockhaus & Long, 1999).

In an SVM, there is no traded asset that is perfectly correlated with the volatility source of risk, and the market is incomplete. Delta hedging in such a model generally underperforms the Black-Scholes model. The situation can be improved by delta-sigma hedging, in which the Black-Scholes hedging portfolio is complemented with any other traded option on the same asset.

Calibration of SVM

We now provide a novel and appealing alternative to the standard calibration procedure of finding the set of parameters (σ_0, θ, κ, γ, ρ) that minimises a given utility function.

Model-independent prices of variance swaps can be obtained for all maturities from the market prices of European-style vanilla options.[4] The Heston (1993) model also provides a closed-form solution (12) that is a function of the term structure parameters, (σ_0, θ, κ), only. Therefore, given option prices at three different maturities, one can find analytical solutions for these terms (one could use more maturities and find a best fit). This is a particularly interesting approach as one is calibrating the volatility process to a variance contract. The remaining skew parameters are then obtained by the standard procedure.

Conclusion

There has been much work published on the relatively straightforward variance swap, and little on other contracts dependent on realised volatility. In the market, volatility swaps, and not variance swaps, are in more demand. We have addressed this situation, provided an analytical approximation for the valuation of volatility swaps and analysed other options with volatility exposure. This has been done within the framework of two volatility smile models, and we have shown that the choice of model can be crucial. ■

[4] *Accurate valuation requires a wide range of calls and put, in particular at-the-money puts*

Oliver Brockhaus and Douglas Long are vice-presidents in the global quantitative research team of Deutsche Bank's equity division in London.

They would like to thank Marcus Overhaus, Andrew Ferraris and Michael Farkas for many useful discussions

e-mail: Oliver.Brockhaus@db.com
Douglas.Long@db.com

REFERENCES

Black F and M Scholes, 1973
The pricing of options and corporate liabilities
Journal of Political Economy 81, pages 637–659

Brockhaus O, M Farkas, A Ferraris, D Long and M Overhaus, 2000
Equity derivatives and market risk models
Risk Books

Brockhaus O, A Ferraris, C Gallus, D Long, R Martin and M Overhaus, 1999
Modelling and hedging equity derivatives
Risk Books

Brockhaus O and D Long, 1999
Modelling, hedging and pricing volatility
Journal of Derivatives, submitted

Carr P and D Madan, 1999
Currency covariance contracting
Working paper

Chriss N and W Morokoff, 1999
Market risk for variance swaps
Risk October, pages 55–59

Cox J, J Ingersoll and S Ross, 1985
A theory of the term structure of interest rates
Econometrica 53, pages 385–407

Derman E, 1999
Regimes of volatility
Risk April, pages 55–59

Demeterfi K, E Derman, M Kamal and J Zou, 1999
A guide to variance swaps
Risk June, pages 54–59

Dupire B, 1993
Model art risk
Risk September, pages 118–120

Dupire B, 1994
Pricing with a smile
Risk January, pages 18–20

Heston S, 1993
A closed-form solution for options with stochastic volatility with applications to bond and currency options
Review of Financial Studies 6, pages 327–343

Merton R, 1973
Theory of rational option pricing
Bell Journal of Economic Management Science 4, pages 141–183

Neuberger A, 1994
The log contract
Journal of Derivatives, winter, pages 74–79

Yor M, 1999
Private communication to author