

Financial Engineering & Risk Management

Review of Multivariate Distributions

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Multivariate Distributions I

Let $\mathbf{X} = (X_1 \dots X_n)^\top$ be an n -dimensional vector of random variables.

Definition. For all $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, the **joint cumulative distribution function** (CDF) of \mathbf{X} satisfies

$$F_{\mathbf{X}}(\mathbf{x}) = F_{\mathbf{X}}(x_1, \dots, x_n) = P(X_1 \leq x_1, \dots, X_n \leq x_n).$$

Definition. For a fixed i , the **marginal CDF** of X_i satisfies

$$F_{X_i}(x_i) = F_{\mathbf{X}}(\infty, \dots, \infty, x_i, \infty, \dots, \infty).$$

It is straightforward to generalize the previous definition to **joint marginal** distributions. For example, the joint marginal distribution of X_i and X_j satisfies

$$F_{ij}(x_i, x_j) = F_{\mathbf{X}}(\infty, \dots, \infty, x_i, \infty, \dots, \infty, x_j, \infty, \dots, \infty).$$

We also say that \mathbf{X} has **joint PDF** $f_{\mathbf{X}}(\cdot, \dots, \cdot)$ if

$$F_{\mathbf{X}}(x_1, \dots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{\mathbf{X}}(u_1, \dots, u_n) du_1 \dots du_n.$$

Multivariate Distributions II

Definition. If $\mathbf{X}_1 = (X_1, \dots, X_k)^\top$ and $\mathbf{X}_2 = (X_{k+1} \dots X_n)^\top$ is a partition of \mathbf{X} then the **conditional** CDF of \mathbf{X}_2 given \mathbf{X}_1 satisfies

$$F_{\mathbf{X}_2|\mathbf{X}_1}(\mathbf{x}_2 | \mathbf{x}_1) = P(\mathbf{X}_2 \leq \mathbf{x}_2 | \mathbf{X}_1 = \mathbf{x}_1).$$

If \mathbf{X} has a PDF, $f_{\mathbf{X}}(\cdot)$, then the **conditional PDF** of \mathbf{X}_2 given \mathbf{X}_1 satisfies

$$f_{\mathbf{X}_2|\mathbf{X}_1}(\mathbf{x}_2 | \mathbf{x}_1) = \frac{f_{\mathbf{X}}(\mathbf{x})}{f_{\mathbf{X}_1}(\mathbf{x}_1)} = \frac{f_{\mathbf{X}_1|\mathbf{X}_2}(\mathbf{x}_1 | \mathbf{x}_2)f_{\mathbf{X}_2}(\mathbf{x}_2)}{f_{\mathbf{X}_1}(\mathbf{x}_1)} \quad (1)$$

and the conditional CDF is then given by

$$F_{\mathbf{X}_2|\mathbf{X}_1}(\mathbf{x}_2|\mathbf{x}_1) = \int_{-\infty}^{x_{k+1}} \cdots \int_{-\infty}^{x_n} \frac{f_{\mathbf{X}}(x_1, \dots, x_k, u_{k+1}, \dots, u_n)}{f_{\mathbf{X}_1}(\mathbf{x}_1)} du_{k+1} \dots du_n$$

where $f_{\mathbf{X}_1}(\cdot)$ is the joint marginal PDF of \mathbf{X}_1 which is given by

$$f_{\mathbf{X}_1}(x_1, \dots, x_k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1, \dots, x_k, u_{k+1}, \dots, u_n) du_{k+1} \dots du_n.$$

Independence

Definition. We say the collection \mathbf{X} is **independent** if the joint CDF can be factored into the product of the marginal CDFs so that

$$F_{\mathbf{X}}(x_1, \dots, x_n) = F_{X_1}(x_1) \dots F_{X_n}(x_n).$$

If \mathbf{X} has a PDF, $f_{\mathbf{X}}(\cdot)$ then independence implies that the PDF also factorizes into the product of marginal PDFs so that

$$f_{\mathbf{X}}(\mathbf{x}) = f_{X_1}(x_1) \dots f_{X_n}(x_n).$$

Can also see from (1) that if \mathbf{X}_1 and \mathbf{X}_2 are independent then

$$f_{\mathbf{X}_2|\mathbf{X}_1}(\mathbf{x}_2 | \mathbf{x}_1) = \frac{f_{\mathbf{X}}(\mathbf{x})}{f_{\mathbf{X}_1}(\mathbf{x}_1)} = \frac{f_{\mathbf{X}_1}(\mathbf{x}_1)f_{\mathbf{X}_2}(\mathbf{x}_2)}{f_{\mathbf{X}_1}(\mathbf{x}_1)} = f_{\mathbf{X}_2}(\mathbf{x}_2)$$

– so having information about \mathbf{X}_1 tells you nothing about \mathbf{X}_2 .

Implications of Independence

Let X and Y be independent random variables. Then for any events, A and B ,

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B) \quad (2)$$

More generally, for any function, $f(\cdot)$ and $g(\cdot)$, independence of X and Y implies

$$E[f(X)g(Y)] = E[f(X)]E[g(Y)]. \quad (3)$$

In fact, (2) follows from (3) since

$$\begin{aligned} P(X \in A, Y \in B) &= E[1_{\{X \in A\}}1_{\{Y \in B\}}] \\ &= E[1_{\{X \in A\}}]E[1_{\{Y \in B\}}] \quad \text{by (3)} \\ &= P(X \in A)P(Y \in B). \end{aligned}$$

Implications of Independence

More generally, if X_1, \dots, X_n are independent random variables then

$$\mathbb{E}[f_1(X_1)f_2(X_2)\cdots f_n(X_n)] = \mathbb{E}[f_1(X_1)]\mathbb{E}[f_2(X_2)]\cdots \mathbb{E}[f_n(X_n)].$$

Random variables can also be **conditionally independent**. For example, we say X and Y are conditionally independent given Z if

$$\mathbb{E}[f(X)g(Y) | Z] = \mathbb{E}[f(X) | Z] \mathbb{E}[g(Y) | Z].$$

– used in the (in)famous **Gaussian copula** model for pricing CDOs!

In particular, let D_i be the event that the i^{th} bond in a portfolio **defaults**.

Not reasonable to assume that the D_i 's are independent. Why?

But maybe they are **conditionally** independent given Z so that

$$\mathbb{P}(D_1, \dots, D_n | Z) = \mathbb{P}(D_1 | Z) \cdots \mathbb{P}(D_n | Z)$$

– often easy to compute this.

The Mean Vector and Covariance Matrix

The **mean** vector of \mathbf{X} is given by

$$\mathbf{E}[\mathbf{X}] := (\mathbf{E}[X_1] \ \dots \ \mathbf{E}[X_n])^\top$$

and the **covariance** matrix of \mathbf{X} satisfies

$$\mathbf{\Sigma} := \text{Cov}(\mathbf{X}) := \mathbf{E} [(\mathbf{X} - \mathbf{E}[\mathbf{X}]) (\mathbf{X} - \mathbf{E}[\mathbf{X}])^\top]$$

so that the $(i, j)^{th}$ element of $\mathbf{\Sigma}$ is simply the covariance of X_i and X_j .

The covariance matrix is **symmetric** and its diagonal elements satisfy $\Sigma_{i,i} \geq 0$.

It is also **positive semi-definite** so that $\mathbf{x}^\top \mathbf{\Sigma} \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

The **correlation** matrix, $\rho(\mathbf{X})$, has $(i, j)^{th}$ element $\rho_{ij} := \text{Corr}(X_i, X_j)$

- it is also symmetric, positive semi-definite and has 1's along the diagonal.

Variances and Covariances

For any matrix $\mathbf{A} \in \mathbb{R}^{k \times n}$ and vector $\mathbf{a} \in \mathbb{R}^k$ we have

$$\mathbf{E}[\mathbf{A}\mathbf{X} + \mathbf{a}] = \mathbf{A}\mathbf{E}[\mathbf{X}] + \mathbf{a} \quad (4)$$

$$\text{Cov}(\mathbf{A}\mathbf{X} + \mathbf{a}) = \mathbf{A} \text{Cov}(\mathbf{X}) \mathbf{A}^\top. \quad (5)$$

Note that (5) implies

$$\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(X, Y).$$

If X and Y independent, then $\text{Cov}(X, Y) = 0$

– but converse not true in general.

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The Multivariate Normal Distribution

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The Multivariate Normal Distribution I

If the n -dimensional vector \mathbf{X} is multivariate normal with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ then we write

$$\mathbf{X} \sim \text{MN}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}).$$

The PDF of \mathbf{X} is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} |\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}$$

where $|\cdot|$ denotes the determinant.

Standard multivariate normal has $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}_n$, the $n \times n$ identity matrix
- in this case the X_i 's are **independent**.

The **moment generating function** (MGF) of \mathbf{X} satisfies

$$\phi_{\mathbf{X}}(\mathbf{s}) = \mathbb{E} \left[e^{\mathbf{s}^\top \mathbf{X}} \right] = e^{\mathbf{s}^\top \boldsymbol{\mu} + \frac{1}{2} \mathbf{s}^\top \boldsymbol{\Sigma} \mathbf{s}}.$$

The Multivariate Normal Distribution II

Recall our partition of \mathbf{X} into $\mathbf{X}_1 = (X_1 \dots X_k)^\top$ and $\mathbf{X}_2 = (X_{k+1} \dots X_n)^\top$.

Can extend this notation naturally so that

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}.$$

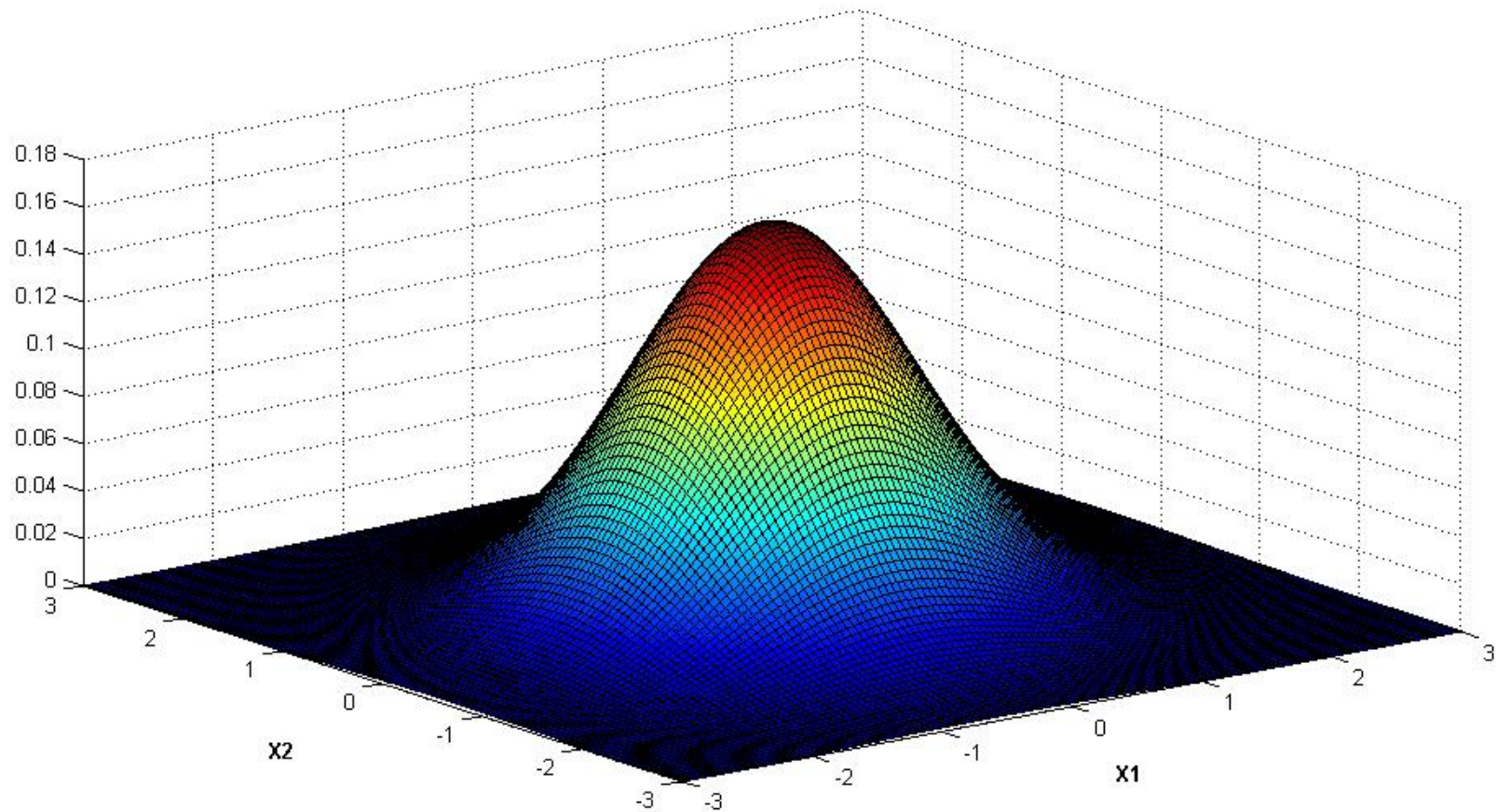
are the mean vector and covariance matrix of $(\mathbf{X}_1, \mathbf{X}_2)$.

Then have following results on marginal and conditional distributions of \mathbf{X} :

Marginal Distribution

The marginal distribution of a multivariate normal random vector is itself normal. In particular, $\mathbf{X}_i \sim \text{MN}(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_{ii})$, for $i = 1, 2$.

The Bivariate Normal PDF



The Bivariate Normal PDF

The Multivariate Normal Distribution III

Conditional Distribution

Assuming Σ is positive definite, the conditional distribution of a multivariate normal distribution is also a multivariate normal distribution. In particular,

$$\mathbf{X}_2 \mid \mathbf{X}_1 = \mathbf{x}_1 \sim \text{MN}(\boldsymbol{\mu}_{2.1}, \Sigma_{2.1})$$

where $\boldsymbol{\mu}_{2.1} = \boldsymbol{\mu}_2 + \Sigma_{21} \Sigma_{11}^{-1} (\mathbf{x}_1 - \boldsymbol{\mu}_1)$ and $\Sigma_{2.1} = \Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}$.

Linear Combinations

A linear combination, $\mathbf{A}\mathbf{X} + \mathbf{a}$, of a multivariate normal random vector, \mathbf{X} , is normally distributed with mean vector, $\mathbf{A}\mathbf{E}[\mathbf{X}] + \mathbf{a}$, and covariance matrix, $\mathbf{A} \text{Cov}(\mathbf{X}) \mathbf{A}^\top$.

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Introduction to Martingales

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Martingales

Definition. A random process, $\{X_n : 0 \leq n \leq \infty\}$, is a **martingale** with respect to the information filtration, \mathcal{F}_n , and probability distribution, P , if

1. $E^P[|X_n|] < \infty$ for all $n \geq 0$
2. $E^P[X_{n+m}|\mathcal{F}_n] = X_n$ for all $n, m \geq 0$.

Martingales are used to model **fair games** and have a rich history in the modeling of gambling problems.

We define a **submartingale** by replacing condition #2 with

$$E^P[X_{n+m}|\mathcal{F}_n] \geq X_n \quad \text{for all } n, m \geq 0.$$

And we define a **supermartingale** by replacing condition #2 with

$$E^P[X_{n+m}|\mathcal{F}_n] \leq X_n \quad \text{for all } n, m \geq 0.$$

A martingale is both a submartingale and a supermartingale.

Constructing a Martingale from a Random Walk

Let $S_n := \sum_{i=1}^n X_i$ be a random walk where the X_i 's are IID with mean μ .

Let $M_n := S_n - n\mu$. Then M_n is a martingale because:

$$\begin{aligned} \mathbb{E}_n[M_{n+m}] &= \mathbb{E}_n \left[\sum_{i=1}^{n+m} X_i - (n+m)\mu \right] \\ &= \mathbb{E}_n \left[\sum_{i=1}^{n+m} X_i \right] - (n+m)\mu \\ &= \sum_{i=1}^n X_i + \mathbb{E}_n \left[\sum_{i=n+1}^{n+m} X_i \right] - (n+m)\mu \\ &= \sum_{i=1}^n X_i + m\mu - (n+m)\mu = M_n. \end{aligned}$$

A Martingale Betting Strategy

Let X_1, X_2, \dots be IID random variables with

$$P(X_i = 1) = P(X_i = -1) = \frac{1}{2}.$$

Can imagine X_i representing the result of coin-flipping game:

- Win \$1 if coin comes up heads
- Lose \$1 if coin comes up tails

Consider now a **doubling strategy** where we keep doubling the bet until we eventually win. Once we win, we stop and our initial bet is \$1.

First note that size of bet on n^{th} play is 2^{n-1}

– assuming we're still playing at time n .

Let W_n denote total winnings after n coin tosses assuming $W_0 = 0$.

Then W_n is a martingale!

A Martingale Betting Strategy

To see this, first note that $W_n \in \{1, -2^n + 1\}$ for all n . Why?

1. Suppose we win for first time on n^{th} bet. Then

$$\begin{aligned} W_n &= -(1 + 2 + \cdots + 2^{n-2}) + 2^{n-1} \\ &= -(2^{n-1} - 1) + 2^{n-1} \\ &= 1 \end{aligned}$$

2. If we have not yet won after n bets then

$$\begin{aligned} W_n &= -(1 + 2 + \cdots + 2^{n-1}) \\ &= -2^n + 1. \end{aligned}$$

To show W_n is a martingale only need to show $E[W_{n+1} | W_n] = W_n$
– then follows by [iterated expectations](#) that $E[W_{n+m} | W_n] = W_n$.

A Martingale Betting Strategy

There are two cases to consider:

1: $W_n = 1$: then $P(W_{n+1} = 1 | W_n = 1) = 1$ so

$$E[W_{n+1} | W_n = 1] = 1 = W_n \quad (6)$$

2: $W_n = -2^n + 1$: bet 2^n on $(n+1)^{th}$ toss so $W_{n+1} \in \{1, -2^{n+1} + 1\}$.

Clear that

$$\begin{aligned} P(W_{n+1} = 1 | W_n = -2^n + 1) &= 1/2 \\ P(W_{n+1} = -2^{n+1} + 1 | W_n = -2^n + 1) &= 1/2 \end{aligned}$$

so that

$$\begin{aligned} E[W_{n+1} | W_n = -2^n + 1] &= (1/2)1 + (1/2)(-2^{n+1} + 1) \\ &= -2^n + 1 = W_n. \end{aligned} \quad (7)$$

From (6) and (7) we see that $E[W_{n+1} | W_n] = W_n$.

Polya's Urn

Consider an urn which contains **red** balls and **green** balls.
Initially there is just one green ball and one red ball in the urn.

At each time step a ball is chosen randomly from the urn:

1. If ball is red, then it's returned to the urn with an **additional** red ball.
2. If ball is green, then it's returned to the urn with an **additional** green ball.

Let X_n denote the number of red balls in the urn after n draws. Then

$$\begin{aligned}P(X_{n+1} = k + 1 \mid X_n = k) &= \frac{k}{n + 2} \\P(X_{n+1} = k \mid X_n = k) &= \frac{n + 2 - k}{n + 2}.\end{aligned}$$

Show that $M_n := X_n / (n + 2)$ is a martingale.

(These martingale examples taken from *"Introduction to Stochastic Processes"*
(Chapman & Hall) by Gregory F. Lawler.)