Financial Engineering & Risk Management

Review of Basic Probability

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Discrete Random Variables

Definition. The cumulative distribution function (CDF), $F(\cdot)$, of a random variable, X, is defined by

$$\mathsf{F}(x) := \mathsf{P}(X \leq x).$$

Definition. A discrete random variable, X, has probability mass function (PMF), $p(\cdot)$, if $p(x) \geq 0$ and for all events A we have

$$\mathsf{P}(X \in A) = \sum_{x \in A} p(x).$$

Definition. The expected value of a discrete random variable, X, is given by

$$\mathsf{E}[X] \; := \; \sum_i x_i \, p(x_i).$$

Definition. The variance of any random variable, X, is defined as

$$\begin{aligned} \mathsf{Var}(X) \quad &:= \quad \mathsf{E}\left[(X - \mathsf{E}[X])^2\right] \\ &= \quad \mathsf{E}[X^2] \; - \; \mathsf{E}[X]^2. \end{aligned}$$

The Binomial Distribution

We say X has a binomial distribution, or $X \sim \mathsf{Bin}(n, p)$, if

$$\mathsf{P}(X = r) = \left(\begin{array}{c} n \\ r \end{array} \right) p^r (1-p)^{n-r}.$$

For example, X might represent the number of heads in n independent coin tosses, where $p = \mathsf{P}(\text{head})$. The mean and variance of the binomial distribution satisfy

$$\begin{aligned} \mathsf{E}[X] &= np \\ \mathsf{Var}(X) &= np(1-p). \end{aligned}$$

A Financial Application

- Suppose a fund manager outperforms the market in a given year with probability p and that she underperforms the market with probability $1-p$.
- She has a track record of 10 years and has outperformed the market in 8 of the 10 years.
- Moreover, performance in any one year is independent of performance in other years.

Question: How likely is a track record as good as this if the fund manager had no skill so that $p = 1/2$?

Answer: Let X be the number of outperforming years. Since the fund manager has no skill, $X \sim \text{Bin}(n = 10, p = 1/2)$ and

$$\text{P}(X \geq 8) \;=\; \sum_{r=8}^{n} \binom{n}{r} p^r (1-p)^{n-r}$$

Question: Suppose there are M fund managers? How well should the best one do over the 10-year period if none of them had any skill?

The Poisson Distribution

We say X has a Poisson(λ) distribution if

$$\mathsf{P}(X = r) = \frac{\lambda^r e^{-\lambda}}{r!}.$$

$$\mathsf{E}[X] \;=\; \lambda \quad \text{and} \quad \mathsf{Var}(X) = \lambda.$$

For example, the mean is calculated as

$$\begin{aligned}
\mathsf{E}[X] \;=\; \sum_{r=0}^{\infty} r\,\mathsf{P}(X = r) \;=\; \sum_{r=0}^{\infty} r\,\frac{\lambda^r e^{-\lambda}}{r!} \;&=\; \sum_{r=1}^{\infty} r\,\frac{\lambda^r e^{-\lambda}}{r!} \\
&=\; \lambda \sum_{r=1}^{\infty} \frac{\lambda^{r-1} e^{-\lambda}}{(r-1)!} \\
&=\; \lambda \sum_{r=0}^{\infty} \frac{\lambda^r e^{-\lambda}}{r!} \;=\; \lambda.
\end{aligned}$$

Bayes' Theorem

Let A and B be two events for which $\mathsf{P}(B) \neq 0$. Then

$$\begin{aligned}
\mathsf{P}(A \mid B) &= \frac{\mathsf{P}(A \bigcap B)}{\mathsf{P}(B)} \\
&= \frac{\mathsf{P}(B \mid A)\mathsf{P}(A)}{P(B)} \\
&= \frac{\mathsf{P}(B \mid A)\mathsf{P}(A)}{\sum_j \mathsf{P}(B \mid A_j)\mathsf{P}(A_j)}
\end{aligned}$$

where the A_j's form a partition of the sample-space.

An Example: Tossing Two Fair 6-Sided Dice

Y_2							
	6	7	8	9	10	11	12
	5	6	7	8	9	10	11
	4	5	6	7	8	9	10
Y_2	3	4	5	6	7	8	9
	2	3	4	5	6	7	8
	1	2	3	4	5	6	7
		1	2	3	4	5	6
					Y_1		

Table : $X = Y_1 + Y_2$

- Let Y_1 and Y_2 be the outcomes of tossing two fair dice independently of one another.
- Let $X := Y_1 + Y_2$. Question: What is $\mathrm{P}(Y_1 \geq 4 \mid X \geq 8)$?

Continuous Random Variables

Definition. A continuous random variable, X, has probability density function (PDF), $f(\cdot)$, if $f(x) \geq 0$ and for all events A

$$\mathsf{P}(X \in A) = \int_A f(y)\ dy.$$

The CDF and PDF are related by

$$\mathsf{F}(x) = \int_{-\infty}^{x} f(y)\ dy.$$

It is often convenient to observe that

$$\mathsf{P}\left(X \in \left(x - \frac{\epsilon}{2},\ x + \frac{\epsilon}{2}\right)\right)\ \approx\ \epsilon f(x)$$

The Normal Distribution

We say X has a Normal distribution, or $X \sim \mathsf{N}(\mu, \sigma^2)$, if

$$f(x) \;=\; \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

The mean and variance of the normal distribution satisfy

$$\begin{aligned} \mathsf{E}[X] &= \mu \\ \mathsf{Var}(X) &= \sigma^2. \end{aligned}$$

The Log-Normal Distribution

We say X has a log-normal distribution, or $X \sim \mathsf{LN}(\mu, \sigma^2)$, if

$$\log(X) \sim \mathsf{N}(\mu, \sigma^2).$$

The mean and variance of the log-normal distribution satisfy

$$\begin{aligned} \mathsf{E}[X] &= \exp(\mu + \sigma^2/2) \\ \mathsf{Var}(X) &= \exp(2\mu + \sigma^2)\ (\exp(\sigma^2) - 1). \end{aligned}$$

The log-normal distribution plays a very important in financial applications.

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Review of Conditional Expectations and Variances

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Conditional Expectations and Variances

Let X and Y be two random variables.

The conditional expectation identity says

$$\mathsf{E}[X] = \mathsf{E}\left[\mathsf{E}[X|\,Y]\right]$$

and the conditional variance identity says

$$\mathsf{Var}(X) = \mathsf{Var}(\mathsf{E}[X|\,Y]) + \mathsf{E}[\mathsf{Var}(X|\,Y)].$$

Note that $\mathsf{E}[X|\,Y]$ and $\mathsf{Var}(X|\,Y)$ are both functions of Y and are therefore random variables themselves.

A Random Sum of Random Variables

Let $W = X_1 + X_2 + \ldots + X_N$ where the X_i's are IID with mean μ_x and variance σ_x^2, and where N is also a random variable, independent of the X_i's.

Question: What is $\mathsf{E}[W]$?

Answer: The conditional expectation identity implies

$$
\begin{aligned}
\mathsf{E}[W] &= \mathsf{E}\left[\mathsf{E}\left[\sum_{i=1}^{N} X_i \,|\, N\right]\right] \\
&= \mathsf{E}\left[N\mu_x\right] = \mu_x \,\mathsf{E}\left[N\right].
\end{aligned}
$$

Question: What is $\mathsf{Var}(W)$?

Answer: The conditional variance identity implies

$$
\begin{aligned}
\mathsf{Var}(W) &= \mathsf{Var}(\mathsf{E}[W|N]) + \mathsf{E}[\mathsf{Var}(W|N)] \\
&= \mathsf{Var}(\mu_x N) + \mathsf{E}[N\sigma_x^2] \\
&= \mu_x^2 \mathsf{Var}(N) + \sigma_x^2\,\mathsf{E}[N].
\end{aligned}
$$

An Example: Chickens and Eggs

A hen lays N eggs where $N \sim \text{Poisson}(\lambda)$. Each egg hatches and yields a chicken with probability p, independently of the other eggs and N. Let K be the number of chickens.

Question: What is $\mathsf{E}[K|N]$?

Answer: We can use indicator functions to answer this question.

In particular, can write $K = \sum_{i=1}^{N} 1_{H_i}$ where H_i is the event that the i^{th} egg hatches. Therefore

$$1_{H_i} = \begin{cases} 1, & \text{if } i^{th} \text{ egg hatches;} \\ 0, & \text{otherwise.} \end{cases}$$

Also clear that $\mathsf{E}[1_{H_i}] = 1 \times p + 0 \times (1-p) = p$ so that

$$\mathsf{E}[K|N] = \mathsf{E}\left[\sum_{i=1}^{N} 1_{H_i} \,|\, N\right] = \sum_{i=1}^{N} \mathsf{E}\left[1_{H_i}\right] = Np.$$

Conditional expectation formula then gives $\mathsf{E}[K] = \mathsf{E}[\mathsf{E}[K|N]] = \mathsf{E}[Np] = \lambda p$.