Financial Engineering and Risk Management

Review of matrices

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Matrices

- Matrices are rectangular arrays of real numbers
- Examples:
  - $\mathbf{A} = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 6 & 5 \end{bmatrix}$: 2×3 matrix
  - $\mathbf{B} = \begin{bmatrix} 2 & 3 & 7 \end{bmatrix}$: 1×3 matrix $\equiv$ row vector
  - $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$: $\mathbf{m} \times \mathbf{n}$ matrix ... $\mathbb{R}^{\mathbf{m} \times \mathbf{n}}$
  - $\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$ ... $n \times n$ **Identity** matrix
- Vectors are clearly also matrices

Matrix Operations: Transpose

- Transpose: $\mathbf{A} \in \mathbb{R}^{m \times d}$

$$\mathbf{A}^\top = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & a_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{md} \end{bmatrix}^\top = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{d2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1d} & a_{2d} & \cdots & a_{md} \end{bmatrix} \in \mathbb{R}^{d \times m}$$

  - Transpose of a row vector is a column vector

- **Example**:
  - $\mathbf{A} = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 6 & 5 \end{bmatrix}$: 2×3 matrix ... $\mathbf{A}^\top = \begin{bmatrix} 2 & 1 \\ 3 & 6 \\ 7 & 5 \end{bmatrix}$: 3×2 matrix
  - $\mathbf{v} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$: column vector ... $\mathbf{v}^\top = \begin{bmatrix} 2 & 6 & 4 \end{bmatrix}$: row vector

Matrix Operations: Multiplication

- Multiplication: $\mathbf{A} \in \mathbb{R}^{\mathbf{m}\times\mathbf{d}}$, $\mathbf{B} \in \mathbb{R}^{\mathbf{d}\times\mathbf{p}}$ then $\mathbf{C} = \mathbf{AB} \in \mathbb{R}^{\mathbf{m}\times\mathbf{p}}$

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{id} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{dj} \end{bmatrix}$$

  - row vector $\mathbf{v} \in \mathbb{R}^{1\times d}$ times column vector $\mathbf{w} \in \mathbb{R}^{d\times 1}$ is a scalar.
  - Identity times any matrix $\mathbf{AI}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$

- **Examples**:
  - $\begin{bmatrix} 2 & 3 & 7 \\ 1 & 6 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 2(2)+3(6)+7(4) \\ 1(2)+6(6)+5(4) \end{bmatrix} = \begin{bmatrix} 50 \\ 58 \end{bmatrix}$
  - ℓ_2 norm: $\left\| \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\|_2 = \sqrt{1^2 + (-2)^2} = \sqrt{\begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}} = \sqrt{\begin{bmatrix} 1 \\ -2 \end{bmatrix}^\top \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$
  - inner product: $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^\top \mathbf{w}$

Linear functions

- A function $f : \mathbb{R}^d \mapsto \mathbb{R}^m$ is linear if

$$f(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \quad \alpha, \beta \in \mathbb{R}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^d$$

- A function f is linear if and only if $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for matrix $\mathbf{A} \in \mathbb{R}^{m \times d}$

- **Examples**
  - $f(\mathbf{x}) : \mathbb{R}^3 \mapsto \mathbb{R}$: $f(\mathbf{x}) = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1 + 3x_2 + 4x_3$
  - $f(\mathbf{x}) : \mathbb{R}^3 \mapsto \mathbb{R}^2$: $f(\mathbf{x}) = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ x_1 + 2x_3 \end{bmatrix}$

- Linear constraints define sets of vectors that satisfy linear relationships
  - Linear equality: $\{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$ ... line, plane, etc.
  - Linear inequality: $\{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ ... half-space

Rank of a matrix

- column rank of $\mathbf{A} \in \mathbb{R}^{m \times d}$ = number of linearly independent columns
  - range($\mathbf{A}$) $= \{\mathbf{y} : \mathbf{y} = \mathbf{A}\mathbf{x}$ for some $\mathbf{x}\}$
  - column rank of $\mathbf{A}$ = size of basis for range($\mathbf{A}$)
  - column rank of $\mathbf{A} = m \Rightarrow$ range($\mathbf{A}$) $= \mathbb{R}^m$
- row rank of $\mathbf{A}$ = number of linearly independent rows
- **Fact**: row rank = column rank $\leq \min\{m, d\}$
- **Example**:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}, \quad \text{rank} = 1, \quad \text{range}(\mathbf{A}) = \left\{ \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix} : \lambda \in \mathbb{R} \right\}$$

- $\mathbf{A} \in \mathbb{R}^{\mathbf{n} \times \mathbf{n}}$ and $\mathbf{rank}(\mathbf{A}) = n \Rightarrow \mathbf{A}$ invertible, i.e. $\mathbf{A}^{-1} \in \mathbb{R}^{\mathbf{n} \times \mathbf{n}}$

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$