1 Introduction

The intent of stability and control analysis is to evaluate the time response of a plane to perturbations from a steady flight condition. The perturbations can originate with the environment, for instance from a gust of wind, or from the actuation of a control.
The mathematical representation of the response is a complex matter, which requires some simplifying assumptions. Essentially, only small perturbations about the steady flight conditions are considered.
The theoretical aspects of flight dynamics and stability analysis can be found in reference [1]. The purpose of this document is:

- to provide a short and much simplified description of the flight dynamics for users not familiar with the theory
- to explain the choices made in XFLR5
- and to describe the analysis procedure.

Note : the mathematical concepts and formulas presented hereafter are not absolutely necessary to the understanding of the physics of flight dynamics. They are provided as information and background for those users interested in investigating further the concepts.

2 Method

2.1 Theory

XFLR5 follows the method proposed by Etkin in ref [1].
With this type of analysis, longitudinal and lateral dynamics are independent and are evaluated separately.

2.2 Frames of reference

Three different reference frames come into consideration in stability analysis : the geometric axes, the body axes and the stability axes. These are defined in Figure 1.

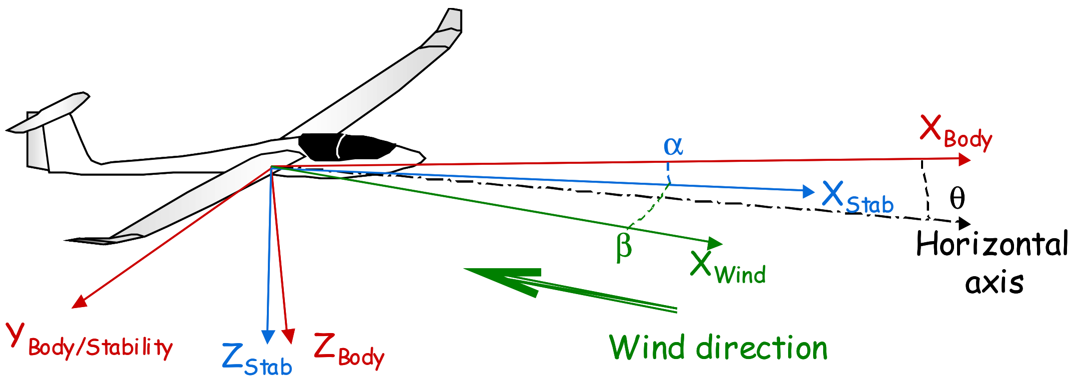


Figure 1 – Body and stability axes

Body axes:

The term body axes is generic and refers to any frame which is fixed to the body, and is therefore not an inertial frame of reference. A usual, but not universal, convention is as follows:

- the X_b-axis is aligned with the fuselage nose;
- the Z_b-axis is in the plane of symmetry, and points downwards;
- the Y_b-axis is perpendicular to the XZ-plane and points starboard.

Geometric axes:

This is the reference frame in which the geometry is defined.

- the X_s-axis is aligned "backwards"
- the Z_s-axis is in the plane of symmetry, and points upwards;
- the Y_s-axis is perpendicular to the xz-plane and points starboard.

The geometric axes are a special case of body axes.

Stability axes:

This is the frame in which the movement in the steady state conditions is most conveniently described:

- the x-axis is the projection of the velocity vector on the body's xz-plane; this axis therefore points forward
- the z-axis points downwards;
- the y-axis points starboard

The point of origin of the frame is the plane's centre of gravity CoG.
The stability axes are a special case of body axes.

Notes:

- In horizontal level flight, the axis $X_{stability}$ is horizontal
- Since sideslip in XFLR5 is simulated by rotation of the structure around the inertial Z_E axis, the wind axes are the same as the stability axes even if the sideslip is non zero.
- In equilibrium conditions, the stability axes are fixed to the body, and therefore are not an inertial frame.

XFLR5 follows the recommendation of [1] and performs all calculations in stability axes.

2.3 Coordinates, position, velocity, and rotation vector

The position of the body in stability axes is defined in some inertial frame of reference by the position of its origin $O(X_r, X_r, Z_r)$, and by the rotation defined by Euler angles (φ, θ, ψ).

Let V(U,V,W) be the body's velocity vector, and let ω(P,Q,R) be the body's rotation vector, both defined in the stability axes.

Additionally, assume that the plane is in equilibrium flight, for instance :

- steady level flight with no sideslip
- banked circle turn

– looping at constant speed (difficult to imagine, but no matter)

The state of the body/plane is defined by the set of variables (X_r, Y_r, Z_r, U, V, W, P, Q, R)

Since we shall be considering only small variations about the steady state conditions, each variable can be defined by an average value and a perturbation around this mean value. For instance:

$$U = U_0 + u$$

The subscript 0 refers to the steady flight state conditions. U_0 for instance is the speed in level flight along the stability x-axis.

The purpose of stability analysis is to calculate the time response of the flight variables in response to small perturbations.

2.4 Flight constraints

The stability derivatives are computed about equilibrium conditions. The conditions that are considered are level or banked horizontal flight. Using terminology from AVL :

- α : angle of attack
- β : sideslip angle
- C_L : Lift coefficient, calculated from the geometry, α and β.
- φ : arbitrary bank angle, positive to the right
- m : mass
- g : gravity acceleration
- ρ : air density
- S : reference area

The constraints are :

$U_0 = \mathrm{sqrt}(2mg/\rho S C_L \cos\varphi)$	airspeed
$R_0 = V_0^2/g \tan\varphi$	turn radius, positive for right turn
$W_0 = V_0/R$	turn rate, positive for right turn
$p_0 = 0$	roll rate, zero for steady turn
$q_0 = W_0 \sin\varphi$	pitch rate, positive nose upward
$r_0 = W_0 \cos\varphi$	yaw rate, positive for right turn

Type 2 analysis in XFLR5 only considers the condition φ=0. This condition is relaxed for stability analysis.

2.5 State description

The plane's state at any instant is given by a set of 8 variables. Four variables describe the longitudinal state:

- **u** is the variation of speed along the x-axis : $\mathbf{U = U_0 + u}$
- **w** is the variation of speed along the z-axis
- **q** is the pitch rate, i.e. the rotation vector around the y-axis
- $\boldsymbol{\theta}$ is the pitch angle, i.e. the angle between the stability x-axis and the horizontal flight line

Four variables describe the lateral dynamics:

- **v** is the variation of speed along the w-axis
- **p** is the roll rate, i.e. the rotation vector around the x-axis
- **r** is the yaw rate, i.e. the rotation vector around the z-axis
- **φ** is the bank angle, i.e. the angle between the stability y-axis and the horizontal flight line

The position defined by (X_r, Y_r, Z_r) does not come into consideration when studying flight dynamics, since the behaviour is not expected to depend on absolute position. The variation of gravity and density with altitude is negligible for model aircraft and is not taken into account.

In lateral dynamics, the heading ψ does not appear in the equations. In other words, lateral stability is independent of sideslip.

2.6 Analysis procedure

The stability analysis follows the following steps:

1. Define the geometry
2. Define the mass, inertia and CoG of each component of the plane. Two sub-options
   a. Enter the mass of the wing or body, and let XFLR5 estimate the inertia and CoG
   b. Enter those values manually
3. Define a stability analysis (Shift+F6).
   If no active controls are defined, the analysis will be run for the base geometry.
   If controls are defined, then the stability data may be calculated in a sequence for a range of control parameter, and a polar curve may be generated
4. Run the analysis for some control parameter. The code will
   a. Search for an angle of attack such that Cm=0, and will exit with a warning if unsuccessful
   b. Calculate the trim speed to achieve steady state flight
   c. Evaluate the stability derivatives,
   d. Build the state matrices,
   e. Extract the eigenvalues, and will exit with a warning if unsuccessful,
   f. Store the data in an OpPoint (optional) and in the polar object.
5. Visualize the results.

2.7 Input

2.7.1 Description

In input, the analysis takes

- the plane's geometry
- the plane's mass, CoG and inertia tensor, defined in geometrical body axes.
- the parameters defined by the stability analysis
- the position for the controls : wing and elevator tilt angles, flap positions, etc.
- the type of steady flight to be considered : steady level flight or steady banked turn.

2.7.2 Inertia estimations

A calculation form is provided to evaluate approximately the CoG position and the inertia tensor associated to the geometry. The evaluation should not be understood as anything else than a rough order of magnitude (ROM). The inertia of the plane is the sum of the inertia of each object and of the additional point masses.

2.7.2.1 Object inertias

The inertia of each object, i.e. wing or body, is evaluated in the dialog form for this object. It includes the volume inertia from the structural masses, and the inertia of point masses.

The volume inertia is evaluated based on the input mass, and on the geometrical data defining the object. It is evaluated in the geometrical coordinate system, with origin at each object's CoG.

The evaluation is based on the following assumptions.

- For the body, the mass is distributed uniformly in the external surface, and this surface is assumed to have a uniform thickness. The body is divided in N_b elementary sections along the x-axis. The weight is concentrated at the center of the cross section, as illustrated in Figure 2.
- For the wing, the mass is assumed to be distributed uniformly in the wing volume along the span. It is modelled as N_w point masses concentrated at the quarter-chord point of distributed sections along the span, as illustrated in Figure 3. The location of the mass at the quarter-chord point is a simplifying assumption, which is consistent with manufacturing techniques known to the author.

2.7.2.2 Point masses

Parts such as actuators, battery, nose lead, or receiver should be modelled separately as point masses, and not be included in the evaluation of the volume inertia.

2.7.2.3 Total inertia

The total inertia for a plane is the sum of the inertias of the object making up the plane, and of point masses. It is expressed in the reference frame defined by the plane's CoG and by the geometrical axes.

The transport of the inertia tensor from object CoG to plane CoG is done by application of Huyghens/Steiner theorem.

2.7.2.4 Notes

– The mass defined for wings and bodies is not the one used for Type 2 calculations. The mass for type 2 is defined by the Analysis/Polar setting.

– The distribution of point masses should be adjusted to obtain the targeted position of the CoG. Otherwise, because of the approximations made in the automatic evaluation of volume inertia, a strict transposition of the "real" position of masses may result in an incorrect position of the plane's CoG.

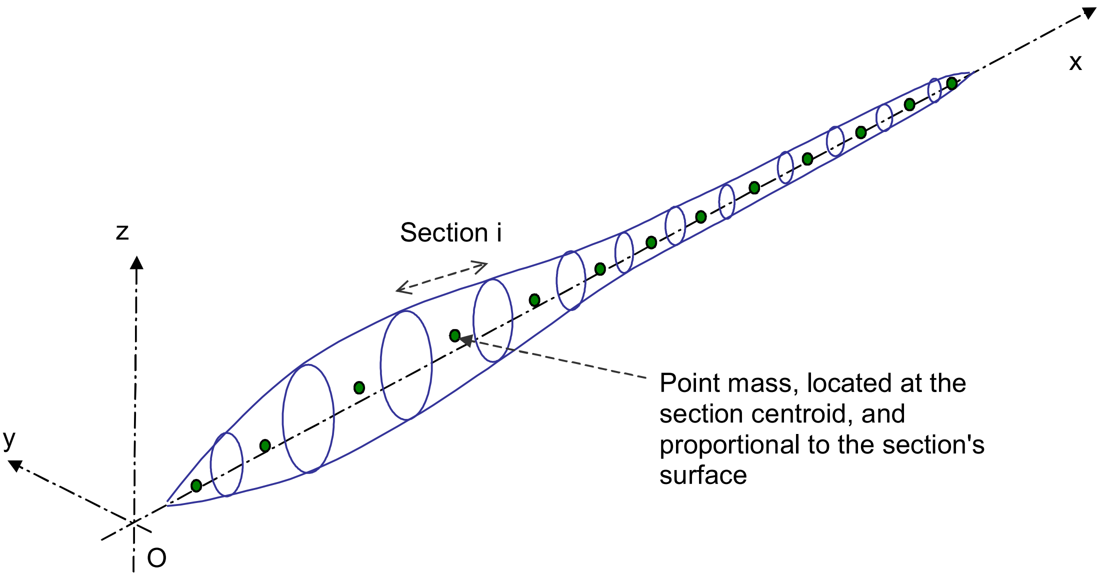


Figure 2 – Mass representation for the body

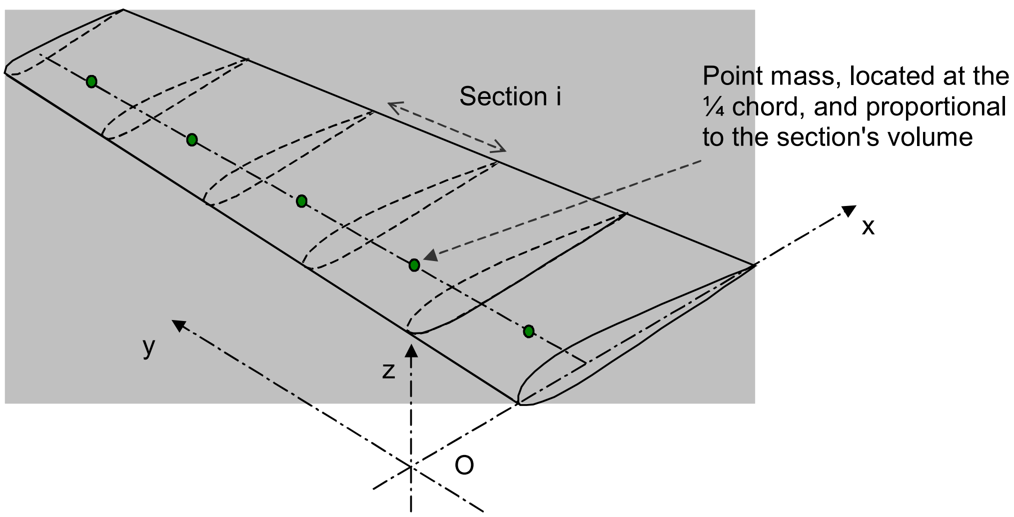


Figure 3 – Mass representation for the wing

2.7.3 Stability polar parameters

The "stability polars" replace the former "control polars". The main difference is that the position of the CoG is no longer a variable, but is instead determined by the distribution of masses in the plane.

The Stability Polar object takes for input

- the fluid's density and dynamic viscosity
- the type of reference length and area for the calculation of aerodynamic parameters
- the selection of either viscous or inviscid analysis
- the control variables to include in the analysis

2.7.4 Control variables

The polar points can be calculated for different state of control variables. These variables are:

- The tilting of the wing about the y-axis
- The tilting of the elevator about the y-axis
- the rotation of the main wing's flaps about their hinge axis
- the rotation of the elevator's flaps about their hinge axis
- the rotation of the fin/rudder's flaps about their hinge axis

Notes:

- The positive direction of rotation is positive by right-hand rule, i.e.
  - for the wing and elevator, a positive value will move the leading edge upwards and the trailing edge downwards
  - for a wing or elevator flap, a positive control value will move the trailing edge downwards
  - for the rudder, a positive control value will move the trailing edge to starboard
- To represent ailerons rotating in opposite directions, the min and max values of the controls for each wing's aileron should be opposite.
- The initial value of the control's angle is not taken into account in the analysis. For instance, if the flap has been defined with foils with non-zero flap angles, the initial angles will be cancelled before setting the position of the control.
  Similarly, the tilt angle defined for the wing or flap in the plane definition is cancelled before application of the control variable.
- For a control polar, all the parameters vary simultaneously in accordance with the value of the control parameter "c":

Control variable = (1-c) x Control_Min_position + c x Control_Max_position

- The rotation of controls is not represented in the 3D view

2.8 Output

In output, the code provides results for longitudinal and lateral dynamics :

- The dimensional stability and control derivatives
- The non-dimensional stability derivatives
- the time response for a step input
- the eigenvalues and eigenvectors for the four longitudinal modes and the four lateral modes.

2.8.1 Stability derivatives

The stability derivatives describe the change to a force or moment in response to a variation of a flight variable. For instance, the variation of the axial force resulting from a change in axial speed is:

$$\frac{\partial F_X}{\partial u} = \frac{1}{2}\rho \frac{\partial u_0^2}{\partial u}\ S\ C_X + \frac{1}{2}\rho u_0^2\ S\ \frac{\partial C_X}{\partial u} = \rho u_0\ S\ C_X + \frac{1}{2}\rho u_0^2\ S\ \frac{\partial C_X}{\partial u}$$

A usual convention is to use simplified notations:

$$\frac{\partial F_X}{\partial u} = X_u$$

$$\frac{\partial Cx}{\partial u} = Cx_u$$

with both derivatives being calculated in the steady state.

X_u is the dimensional stability derivative, and Cx_u is the non-dimensional stability derivative.

XFLR5 calculates the dimensional derivatives which are relevant at the scale of model sailplanes:

- In the longitudinal direction: (X_u, X_w, Z_u, Z_w, Z_q, M_u, M_w, M_q)
- In the lateral direction: (Y_v, Y_p, Y_r, L_v, L_p, L_r, N_v, N_p, N_r)

The non-dimensional derivatives are usually given in stability axes, and the derivatives w.r.t to v and w are provided instead w.r.t α and β. They are:

- In the longitudinal direction CL_a, CL_q, Cm_a, Cm_q,
- In the lateral direction : CY_b, CY_p, CY_r, Cl_b, Cl_p, Cl_r, Cn_b, Cn_p, Cn_r;

The definition of the non dimensional derivatives is

```
CLa = Zw*   u0       /(q*S);
CLq = Zq* 2.*u0      /(q*S*mac);
Cma = Mw*    u0      /(q*S*mac);
Cmq = Mq*(2.*u0/mac)/(q*S*mac);
CYb = Yv*    u0      /(q*S);
CYp = Yp* 2.*u0      /(q*S*b);
CYr = Yr* 2.*u0      /(q*S*b);
Clb = Lv*    u0      /(q*S*b);
Clp = Lp*(2.*u0/b)   /(q*S*b);
Clr = Lr*(2.*u0/b)   /(q*S*b);
Cnb = Nv*    u0      /(q*S*b);
Cnp = Np*(2.*u0/b)   /(q*S*b);
Cnr = Nr*(2.*u0/b)   /(q*S*b);
```

Where :

q is the dynamic pressure,
S is the reference Area
b is the reference Span
mac is the mean aerodynamic chord

The evaluation of the derivatives is an intermediate step in the calculation of the dynamic response. The derivative values are stored in the OpPoint object, and can be exported to a text file for use in other flight simulation codes.

2.8.2 Modes

Natural modes

From the mathematical point of view, the state matrix can be diagonalized for eigenvalues and eigenvectors. An eigenvalue is of the form

$$\lambda=\sigma+i\omega$$

where

σ is the damping constant, unit 1/s
ω is the circular natural frequency, unit rad/s

Any eigenvalue with a non-zero imaginary part ω, has a symmetric eigenvalue given by the its conjugate. This implies that the time response of a variable for such a mode is of the form:

$$x(t)=R\ e^{\sigma t}\cos(\omega t-\varphi)$$

where R and ϕ are constant values determined by the initial conditions.

The mode will be dynamically stable if the damping is negative, unstable otherwise. Dynamic stability means that after a perturbation, the plane will return progressively to its steady state flight.

Other definitions for oscillating modes:

$\omega_1=\sqrt{\lambda\bar{\lambda}}=\sqrt{\sigma^2+\omega^2}$ is the undamped natural circular frequency, unit rad/s

For damped modes, i.e. σ <0:

$\zeta = \frac{-\sigma}{\omega_1}$ is the damping ratio, without unit

- $\zeta>1$ if the mode is overdamped
- $\zeta=1$ if the mode is critically damped
- $\zeta<1$ if the mode is underdamped, i.e. oscillatory

If the damping is weak, i.e. $\zeta^2 \ll 1$, then $\omega_1 \approx \omega$.

The frequency of vibration of the mode (unit Hz) is determined by

$F = \omega / 2.\pi$

The time period (unit s) is

$T = 1/F = 2.\pi / \omega$

From the physics point of view, the eigenvalues and eigenvectors represent the natural modes on which the plane will tend to oscillate. For a standard well-defined problem, the modes will be:

In the longitudinal case

- two symmetric phugoid modes
- two symmetric short-period modes

In the lateral case

- a roll damping mode
- a spiral mode
- two symmetric dutch roll modes

<u>Root Locus plot</u>

The position of the eigenvalues may be represented in the complex plane, which is a convenient way to check visually the stability and frequency of the modes :

- Roots (=eigenvalues) lying on the left of the diagram with negative x value correspond to stable modes, those lying on the right with positive x-value are unstable. The further down the left is the root, the more stable is the mode.
- Roots with non zero imaginary part correspond to oscillating modes, those with zero imaginary part are non-oscillating.
  The further way is the root from the x-axis, the higher is the frequency of vibration.

<u>Mode shape</u>

The eigenvalue defines the mode's frequency and damping, and the eigenvector defines its shape.

It isn't an intuitive task to understand a mode shape from the eigenvector's components. Another more convenient way is to animate the mode in the 3D view.

Since the frequency and damping may be very different from one mode to the other, the time sampling and amplitude will need to be adjusted for each mode.

The mode amplitude R is arbitrary and has no physical significance. It may be adjusted to any scale for display purposes. In flight, a mode is seldom excited alone. Rather, an external perturbation will tend to generate a response on the different longitudinal and lateral modes. This can be modelled in the time response plot.

2.8.3 Time response

The time response is evaluated based on the flight dynamics equation. For instance, in the longitudinal case, this is expressed as

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \left[A_{long} \right] \cdot \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \left[B_{long} \right] \cdot \left[F(t) \right]$$

where:

- $[A_{long}]$ is the 4x4 longitudinal state matrix,
- $[B_{long}]$ is the 4xn control influence matrix, with n being the number of control variables
- $[F(t)]$ is nx1 matrix, giving the forced input history of each control variable

Similarly for lateral modes:

$$\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \left[A_{lat} \right] \cdot \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix} + \left[B_{lat} \right] \cdot \left[F(t) \right]$$

The time history of the state variables (u, w, q, θ) and (v, p, r, φ) can be calculated either

- as a consequence of perturbed initial conditions: this is the "Initial condition response"
- or as the consequence of control actuation vs. time : this is the "Forced response"

2.8.3.1 Initial condition response

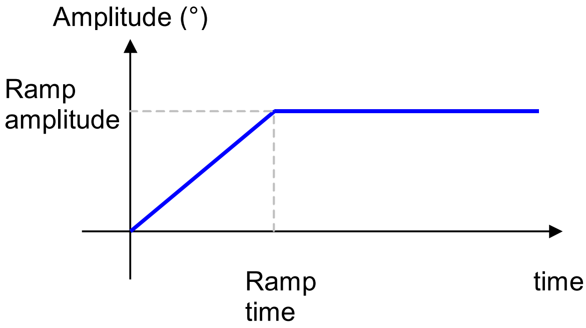
The input required is a step change from the steady state flight. For the longitudinal case, this input may be provided as any combination of values for u, w, and q. In the lateral case, it is input as a combination of values of v, p, and r.

The time response from perturbed initial conditions is solved exactly from the differential equation.

2.8.3.2 Open loop forced response

This type of analysis investigates the response of the plane to a change of a control parameter. Such parameters are typically a modification of thrust, or the actuation of a control surface such as the elevator, the rudder, or the ailerons. The modification of thrust is not considered in XFLR5.

The input required is a time history of a control parameter. XFLR5 only offers the possibility to simulate a linear ramp of a control in a finite time.



Although all control variables are set simultaneously to determine the steady state geometry and trim conditions, the variation of each may be set independently in the evaluation of the forced response. The ramp time however is the same for all control variables.

The time response open loop forced response is solved by a fourth order Runge-Kutta scheme.

3 References

[1] B. Etkin and L.D. Reid, Dynamics of Flight: Stability and Control. John Wiley and Sons, New York, NY, Third Edition, 1996.