Stat 450

Chapter 7: Sampling distributions

Fall 2016

We turn now in full force to studying **sampling distributions**. Let's define some terms:

- **Population**: Collection of all elements of interest.
- **Parameter**: A quantity (or quantities) that, for a given population, is fixed and that is used as the value of a variable in some general distribution or frequency function to make it descriptive of that population. (E.g., μ is the mean of a normal distribution; (α, β) govern the Gamma distribution and are used to define the mean and variance.)
- **Sample**: A collection of elements drawn from the population and observed. In these notes, we will be considering univariate realizations Y_1, Y_2,...,Y_n drawn independently and identically from the population.
- **Statistic**: A function of the observable random variables in a sample.

These are important, because they bring us to the definition of a **sampling distribution**. A **sampling distribution** is the distribution of a statistic across repeated samples taken from the population. What are sampling distributions used for?

- Finding the mean and variance of a statistic; this is how *bias* and *mean-squared error* are defined (more later)
- Hypothesis testing
- Finding confidence intervals

Facts when sample is drawn from any distribution:

1. $\bar{Y}$ is independent of the residuals $(Y_1 - \bar{Y}), (Y_2 - \bar{Y}), ..., (Y_n - \bar{Y})$. This fact is best proved with linear algebra (see Chapter 13). However, any given residual is not independent of another residual; i.e. $(Y_i - \bar{Y})$ is not independent of $(Y_j - \bar{Y})$, since $\sum_{i=1}^{n}(Y_i - \bar{Y}) = 0$.
2. The previous point implies that $\bar{Y}$ and s^2, the sample variance, are independent.
3. $E(\bar{Y}) = \mu$ and $Var(\bar{Y}) = \sigma^2/n$. Here, μ is the mean of the distribution and σ^2 is the variance of the distribution (note these might be functions of other parameters governing the distribution; e.g. $\mu = \alpha\beta$ if the distribution is $GAM(\alpha, \beta)$). This is a result from Chapter 4.
4. If n is "large", then $\bar{Y} \sim N(\mu, \sigma^2/n)$. This is a result of the **central limit theorem.** We will prove this at the end of this handout.

Facts when sample is drawn from a $N(\mu, \sigma^2)$ distribution:

1. **No matter what size n is**, $\bar{Y} \sim N(\mu, \sigma^2/n)$, where μ is the mean of the normal distribution and σ^2 is the variance of the normal distribution.
2. The following, useful for obtaining confidence intervals and doing hypothesis tests for σ^2 :

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$

3. The following, useful for obtaining confidence intervals and doing hypothesis tests for μ:

$$\frac{\bar{Y} - \mu}{s/\sqrt{n}} \sim t_{n-1}$$

4. If $Y_1, Y_2, ..., Y_n$ is an i.i.d. sample from a $N(\mu_Y, \sigma_Y^2)$ distribution; and $X_1, X_2, ..., X_m$ is a sample drawn i.i.d from a $N(\mu_X, \sigma_X^2)$ distribution; then the ratio of the sample variances scaled as follows, $\frac{s_X^2/\sigma_X^2}{s_Y^2/\sigma_Y^2}$, follows an $F_{m-1,n-1}$ distribution.

We will prove facts 1-4 when the sample (or samples, in the case of #4) is drawn from a normal population. We will then prove the central limit theorem, which guarantees normality of $\bar{Y}$ for large n when the sample comes from *any* distribution.

Proof of 1

Proof of 2

Preliminaries:

A. If $Y \sim N(\mu, \sigma^2)$, then $Z = \frac{Y-\mu}{\sigma} \sim N(0,1)$
B. If $Z \sim N(0,1)$, then $U = Z^2 \sim \chi_1^2 \equiv GAM\left(\alpha = \frac{1}{2}, \beta = 2\right)$ (Proved using pdf method in Chapter 6).

Proof of A, using the fact that $M_{aY+b} = e^{tb} M_Y(at)$:

(Proof of 2, continued)

Using #2 to derive confidence intervals for σ^2

Below is some R code to simulate a sample of size n from a $N(0, \sigma^2 = 4)$ population. The function calculates and returns a single 95% confidence interval. We then replicate this function many times to obtain many confidence intervals, 95% of which should cover the true σ^2:

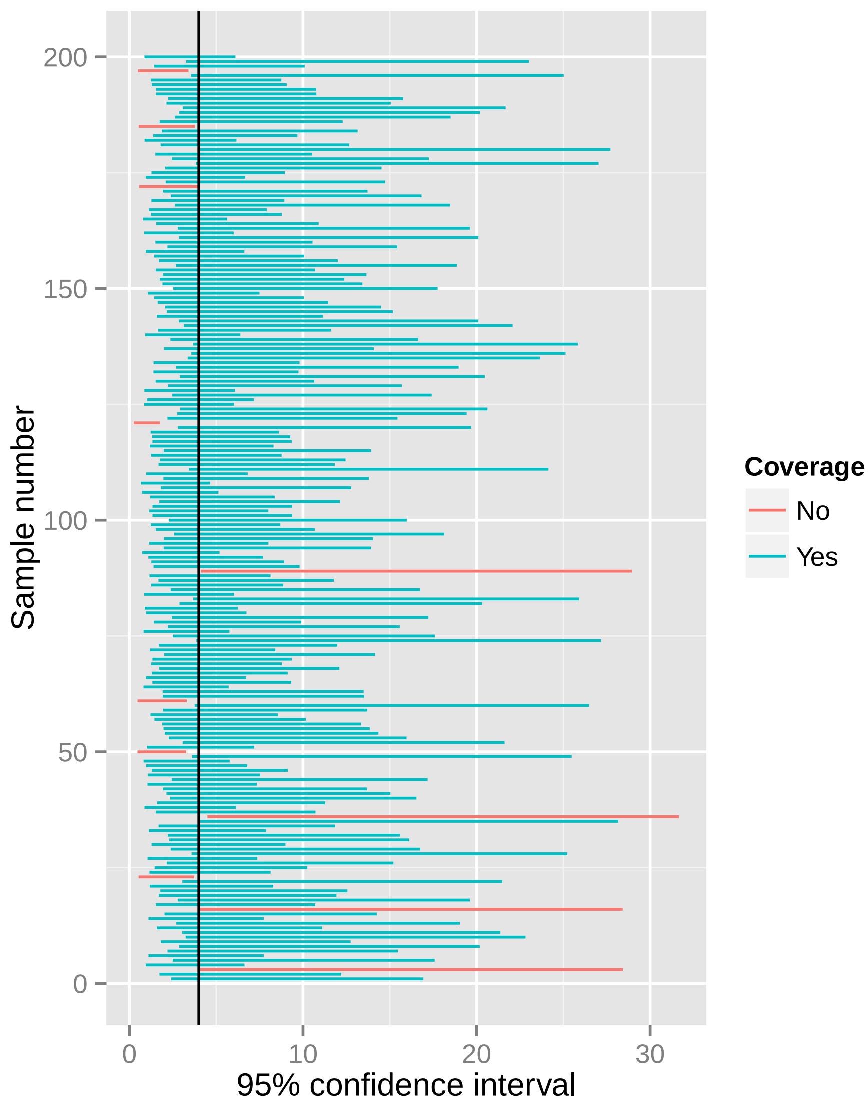
```r
#Write code to get sample, calculate 95% confidence interval
get.one.ci <- function(n){
  one.sample <- rnorm(n, mean = 0, sd = sqrt(4))
  s2 <- var(one.sample)
  lower <-   (n-1)*s2/qchisq(0.975,n-1)
  upper <-   (n-1)*s2/qchisq(0.025,n-1)
  ci <- c(lower,upper)
  return(ci)
}

#Given a 95% confidence interval, and a value of the true sigma^2, does the interval cover sigma^2?
covers.sigma2 <- function(ci,sigma2) {
  cover <- ifelse(ci[1] < sigma2 & sigma2 < ci[2],'Yes','No')
  return(cover)
}

#Gather 200 samples and corresponding confidence intervals, and calculate coverage:
set.seed(24211)
many.ci <-replicate(200,get.one.ci(n=10),simplify='matrix')
df <- data.frame(t(many.ci))
df$Coverage <- apply(df,1,covers.sigma2,sigma2=4)
df$Sample <- 1:nrow(df)
table(df$Coverage)/200
```

```
##
##    No   Yes
## 0.055 0.945
```

```
##Plot the results
library(ggplot2)
ggplot(data = df) +
    geom_segment(aes(x = X1, xend = X2, y = Sample, yend = Sample,color=Coverage)) +
  geom_vline(xintercept=4) + xlab('95% confidence interval') + ylab('Sample number')
```



Using #2 for hypothesis testing

EXAMPLE: Quality control. On a production line, consistency of performance is very important. For example, suppose a machine is calibrated to fill 12-ounce Coke bottles very precisely. The machine is supposed to fill each bottle to be 12 ounces, but may have slight variations from bottle-to-bottle. Specifically, suppose the distribution of *actual* bottle fills is intended to follow a normal distribution with mean $\mu = 12$ and standard deviation of $\sigma^2 = 0.01$. If there is evidence that $\sigma^2 > 0.01$, the machine will need to be recalibrated. This then becomes a problem of testing:

$$H_0 : \sigma^2 = 0.01$$
$$H_a : \sigma^2 > 0.01$$

Suppose a sample of $n = 20$ bottles is taken from the production line; how large will s^2 need to convincingly suggest the machine needs to be recalibrated? This involves finding the sampling distribution of s^2 (or some appropriate scaled version thereof), to find what values of s^2 would be very unusual if H_0 were true.

Proof of #3

Here, we want to prove that, if $Z \sim N(0,1)$, and $W \sim \chi^2_\nu$, that:

$$T = \frac{Z}{\sqrt{W/\nu}} \text{``} \equiv \text{''} \frac{N(0,1)}{\sqrt{\chi^2_\nu/\nu}}$$

follows a *t-distribution* with ν degrees of freedom. Pdf of the t-distribution:

$$f_T(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} \quad ; -\infty < t < \infty$$

We will proceed as follows:

A. Show T follows a t_ν distribution.
B. Let $Y_1,\ldots,Y_n$ be an i.i.d. sample from a $N(\mu,\sigma^2)$ distribution. Let:

$$T = \frac{\bar{Y} - \mu}{s/\sqrt{n}}$$

Show that this can be written as $Z/\sqrt{W/(n-1)}$ where $Z \sim N(0,1)$ and $W \sim \chi^2_{n-1}$, and hence that $T \sim t_{n-1}$.

Proof of A

(Proof of A, continued)

B. Let $Y_1, \ldots, Y_n$ be an i.i.d. sample from a $N(\mu, \sigma^2)$ distribution. Let:

$$T = \frac{\bar{Y} - \mu}{s/\sqrt{n}}$$

Show that this can be written as $Z/\sqrt{W/(n-1)}$ where $Z \sim N(0,1)$ and $W \sim \chi^2_{n-1}$, and hence that $T \sim t_{n-1}$.

Deriving 95% confidence intervals for μ

Suppose $Y_1, Y_2, ..., Y_n$ is an i.i.d. sample drawn from a $N(\mu, \sigma^2)$ population. Derive a 95% confidence interval for μ.

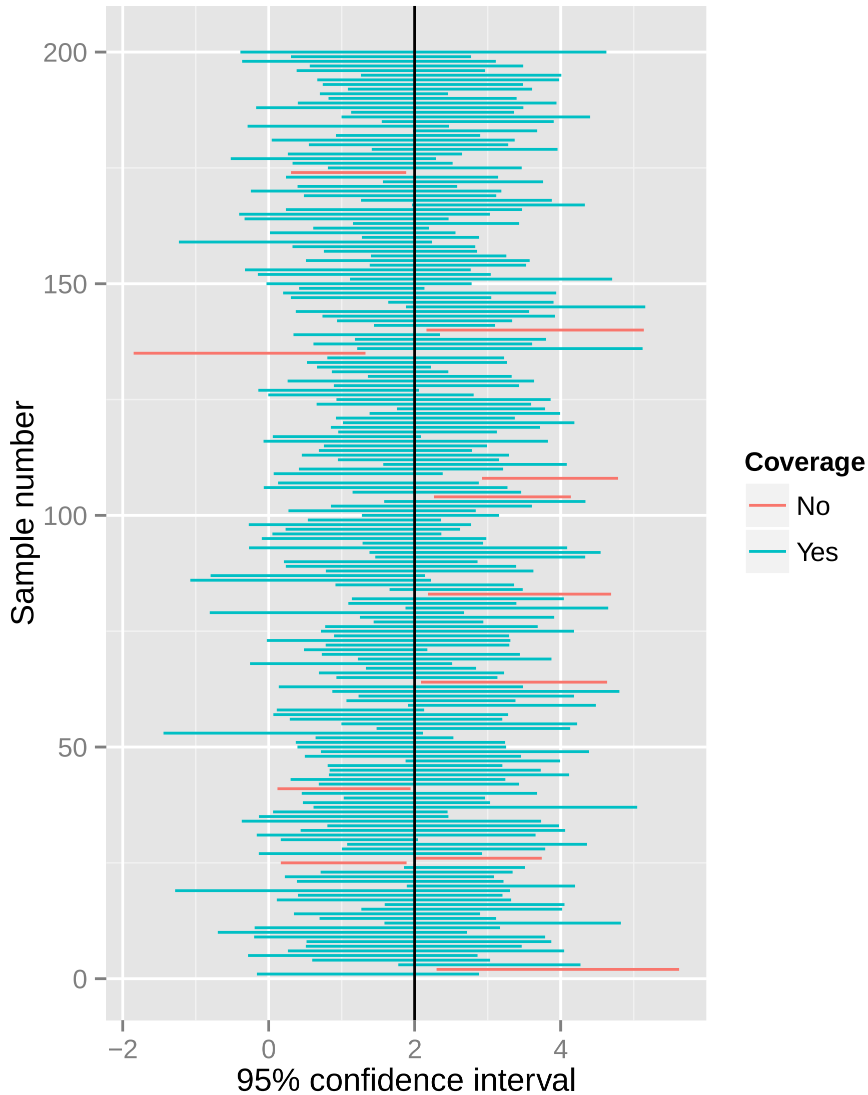
```r
#Write code to get sample, calculate 95% confidence interval
get.one.ci <- function(n){
  one.sample <- rnorm(n, mean = 2, sd = sqrt(4))
  ybar <- mean(one.sample)
  s <- sd(one.sample)
  lower <-   ybar - qt(0.975,n-1)*s/sqrt(n)
  upper <-   ybar + qt(0.975,n-1)*s/sqrt(n)
  ci <- c(lower,upper)
  return(ci)
}

#Given a 95% confidence interval, and a value of the true mu, does the interval cover mu?
covers.mu <- function(ci,mu) {
  cover <- ifelse(ci[1] < mu & mu < ci[2],'Yes','No')
  return(cover)
}

#Gather 200 samples and corresponding confidence intervals, and calculate coverage:
set.seed(24111)
many.ci <-replicate(200,get.one.ci(n=10),simplify='matrix')
df <- data.frame(t(many.ci))
df$Coverage <- apply(df,1,covers.mu,mu=2)
df$Sample <- 1:nrow(df)
table(df$Coverage)/200
```

```
##
##    No   Yes
## 0.055 0.945
```

```
##Plot the results
library(ggplot2)
ggplot(data = df) +
    geom_segment(aes(x = X1, xend = X2, y = Sample, yend = Sample,color=Coverage)) +
  geom_vline(xintercept=2) + xlab('95% confidence interval') + ylab('Sample number')
```



Showing #4

If $Y_1, Y_2, ..., Y_n$ is an i.i.d. sample from a $N(\mu_Y, \sigma_Y^2)$ distribution; and $X_1, X_2, ..., X_m$ is a sample drawn i.i.d from a $N(\mu_X, \sigma_X^2)$ distribution; then the ratio of the sample variances scaled as follows, $\frac{s_X^2/\sigma_X^2}{s_Y^2/\sigma_Y^2}$, follows an $F_{m-1,n-1}$ distribution.

A. First, we need to prove that **in general**, if $U \sim \chi_p^2$ and $V \sim \chi_q^2$, then $W = \frac{U/p}{V/q} \sim F_{p,q}$ where:

$$f_W(w) = \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \left(\frac{p}{q}\right)^{p/2} w^{p/2-1} \left(1 + \frac{p}{q}w\right)^{-\left(\frac{p+q}{2}\right)} ; w > 0$$

B. Then, we need to prove that $\frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2}$ " $\equiv$ " $\frac{\chi_{m-1}^2/(m-1)}{\chi_{n-1}^2/(n-1)}$

Proof of A:

Proof of B:

Usage: hypothesis testing for equality of two population variances

Suppose we have $X_1, X_2, ..., X_m$ drawn i.i.d. $\sim N(\mu_X, \sigma_X^2)$ and $Y_1, Y_2, ..., Y_n \sim N(\mu_Y, \sigma_Y^2)$. We are interested in testing whether the two population variances are equal, e.g.:

$$H_0 : \sigma_X^2 = \sigma_Y^2$$
$$H_a : \sigma_X^2 \neq \sigma_Y^2$$

How can we derive a test for these hypotheses?

The central limit theorem

One of the most important theorems in statistics, the central limit theorem (CLT) guarantees normality of $\bar{Y}$ for large n, no matter what distribution the individual Y_i themselves came from.

Here is the theorem in all its glory:

Let $Y_1, Y_2, ..., Y_n$ be i.i.d. random variables with $E(Y_i) = \mu$ and $Var(Y_i) = \sigma^2 < \infty$. *Note that no assumptions are made about normality of the individual Y_i!* Let:

$$U_n = \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} = \sqrt{n}\left(\frac{\bar{Y} - \mu}{\sigma}\right).$$

Then, as $n \to \infty$, $U_n \to_d N(0, 1)$.

The statement $\to_d$ means "converges in distribution." Essentially what this means is that, as $n \to \infty$,

$$P(U_n \leq u) \to \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt,$$

i.e. the CDF of a standard normal.

A couple points of clarification:

No matter what distribution the Y_i come from:

1. No matter what size n, $E(\bar{Y}) = \mu$ and $Var(\bar{Y}) = \sigma^2/\sqrt{n}$. This is Chapter 4 stuff.
2. What the CLT gives us is *normality* of the $\bar{Y}$ for large n.

Before proving this, let's investigate the CLT via simulations. We'll take repeated samples of $EXP(\beta = 5)$ random variables, of various sizes. Note from here:

- $\mu = E(Y_i) = \beta = 5$
- $\sigma^2 = Var(Y_i) = \beta^2 = 25$
- $E(\bar{Y}) = \mu = 5$ for all n
- $Var(\bar{Y}) = \sigma^2/n = 25/n$ for all n
- $\bar{Y}$ are normal for *large n only*

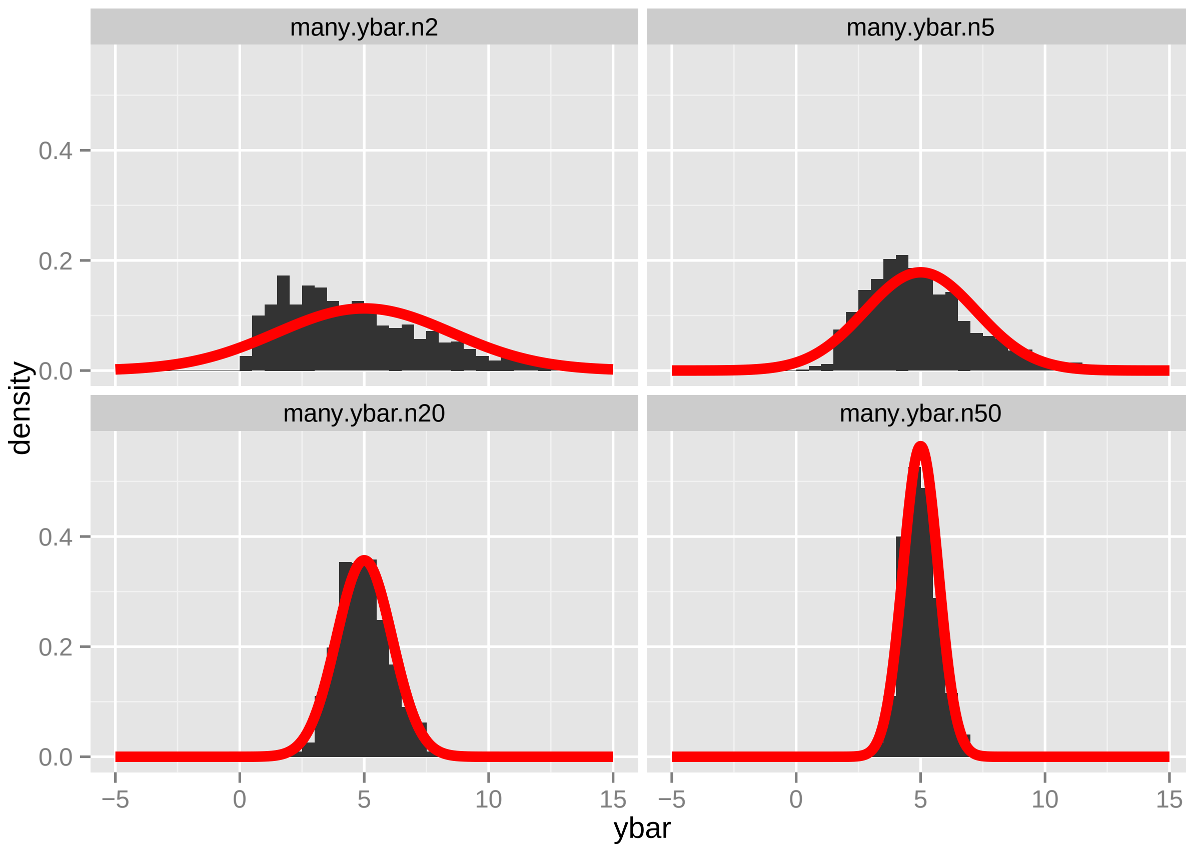
```r
get.one.ybar <- function(n){
  one.sample <- rexp(n, rate = 1/5)
  ybar <- mean(one.sample)
  return(ybar)
}
set.seed(12345)
many.ybar.n2 <- replicate(1000,get.one.ybar(n=2))
many.ybar.n5 <- replicate(1000,get.one.ybar(n=5))
many.ybar.n20 <- replicate(1000,get.one.ybar(n=20))
many.ybar.n50 <- replicate(1000,get.one.ybar(n=50))
df <- data.frame(many.ybar.n2,many.ybar.n5,many.ybar.n20,many.ybar.n50)
apply(df,2,mean)  #Should all be ~5:
```

```
##  many.ybar.n2  many.ybar.n5 many.ybar.n20 many.ybar.n50
##      4.928742      5.009634      5.001159      4.974126
```

```r
apply(df,2,var) #Should be decreasing:
```

```
##  many.ybar.n2  many.ybar.n5 many.ybar.n20 many.ybar.n50
##    11.8196198     4.6614922     1.1739804     0.4983923
```

```r
library(tidyr)
df2 <- gather(df, key = 'SampleSize',value = 'ybar')
xseq <- seq(-5,15,l=1000)
df2$xseq <- rep(xseq,4)
df2$yseq <- c(dnorm(xseq, mean = 5, sd = sqrt(25/2)),
              dnorm(xseq, mean = 5, sd = sqrt(25/5)),
              dnorm(xseq, mean = 5, sd = sqrt(25/20)),
              dnorm(xseq, mean = 5, sd = sqrt(25/50)))
ggplot(data = df2) +
  geom_histogram(aes(x = ybar, y = ..density..),binwidth =.5) +
  geom_line(aes(x = xseq, y = yseq),color='red',size=2) +
  facet_wrap(~SampleSize) + xlim(c(-5,15))
```



Proof of the CLT: preliminaries

To prove the CLT, we will use the method of MGFs. Before we embark, recall a couple important definitions and facts from calculus:

Definition: A function $f(n)$ is $o(n)$ ("little oh of n") if it goes to 0 faster than n does. Specifically, if $\lim_{n\to\infty} nf(n) \to 0$.

Examples: $f(n) = \frac{1}{n^2} = o(n)$; $f(n) = \frac{1}{\sqrt{n}} \neq o(n)$.

We also need the following facts:

- *Fact #1, from calculus*: For any t, $\left(1 + \frac{t}{n} + o(n)\right)^n \to e^t$.
- *Fact #2, from earlier this semester*: Let $M_Y(t)$ be the MGF of Y; then $M_{aY+b}(t) = e^{bt}M_Y(at)$.
- *Fact #3, from earlier this semester*: If $Y_1, Y_2, ..., Y_n$ are i.i.d. and $S_n = \sum_1^n Y_i$, then $M_{S_n}(t) = M_Y(t)^n$.

Given these facts, here is what we want to prove:

The CLT, technically stated: Let $Y_1, Y_2, ..., Y_n$ be an i.i.d. sample with $|E(Y)| = |\mu| < \infty$ and $0 < E(Y^2) < \infty$. Let:

$$U_n = \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} = \frac{\frac{1}{n}\sum_{i=1}^n (Y_i - \mu)}{\sigma/\sqrt{n}} = \frac{\sum_{i=1}^n (Y_i - \mu)}{\sqrt{n}\sigma} = \frac{\sum_{i=1}^n X_i}{\sqrt{n}\sigma},$$

where $X_i = (Y_i - \mu)$. Show that $M_{U_n}(t) \to e^{t^2/2}$ as $n \to \infty$, where $e^{t^2/2}$ is the MGF of a $N(0,1)$ distribution; hence showing that $U_n \to_d N(0,1)$.

PROOF:

Proof of CLT, continued