SOME NUMERICAL EXPERIMENTS FOR AN $M \times J$ FLOW SHOP AND ITS DECISION-THEORETICAL ASPECTS*

J. Heller

AEC Computing and Applied Mathematics Center, New York University, New York, New York

(Received May 25, 1959)

Numerical experiments for $M \times J$ deterministic flow shops performed on an IBM 704 digital computer lead to the conclusion that the schedule times are approximately normally distributed for large numbers of jobs The meaning of this result in the decision-theoretical problem of sampling for a minimum is discussed Examples of these results for 10×100 and 10×20 schedules are given

BECAUSE of the lack of success in finding a practical algorithm for the solution of complex scheduling problems, simulation has become the vogue In many scheduling simulation experiments the complete complexity of the problem is introduced with the consequence that some of the important theoretical aspects of scheduling problems are lost In this paper we describe some simple numerical experiments carried out on an IBM-704 digital computer to determine the distribution of schedule times over all possible schedules

For simplicity we choose the many-machine version of book-printing, book-binding scheduling discussed by Johnson [4] The processing times of each individual job on each machine are taken as nonnegative integers, therefore we are dealing with deterministic flow-shop scheduling

These experiments are motivated by some mathematical studies of flow-shop and machine-shop scheduling [2 3] By combinatorial means rooted partly in a lattice-theory framework, it has been shown that although there are many possible schedules there are far less different schedule times Because of the relatively small number of different schedule times for any given set of processing times, we might expect the probability distribution of the schedule times over the set of all schedules to have a 'simple' form The numerical experiments described below show that the distribution of schedule times is normal,[1] theoretical analysis indicates[2] that the schedule times are asymptotically normally distributed[1] for schedules with large numbers of jobs

* This work was carried out under Contract AT(30-1)-1480 with the United States Atomic Energy Commission

This result is of importance in simulation, for simulation in the context of scheduling theory tries many schedules and chooses the 'best' schedule, i e , the minimum of some function of the schedule time * In terms of decision theory the number of samples, i e , simulations of a schedule, taken before settling for the best found schedule so far depends on the cost of sampling and the possible loss incurred in settling for a particular schedule Decision theory affords us a systematic procedure by which we can answer the question How many samples should we take before the probable cost of taking another sample will be more than the probable gain in finding a better schedule?

In order to answer this question it is not necessary to know the distribution of schedule times over the set of possible schedules in this case we have a decision problem under uncertainty [5] If we know the distribution of schedule times over the set of possible schedules, we have a simpler problem in this case we have a decision problem under risk [5] In practice the problem under risk is preferred because we can make estimates of the number of samples needed before sampling [Cf reference 5 for a general discussion, reference 2 for the specific example in scheduling, and reference 6 for the solution of the general problem with a known distribution]

SAMPLING EXPERIMENTS USING A DIGITAL COMPUTER

SIMULATION experiments, generally speaking, are used to sample from some subset of possible situations and settle for the 'best' found sample In our case we are going to sample different schedules in a flow shop with a view to finding a 'best' schedule and also with a view to deciding if random sampling is of value in solving sequencing problems This second use of simulation is an attempt to understand some of the properties of sequencing The reason for choosing a simple problem is connected with the understanding aspect of simulation if there is a pattern to our experiments, we may be able to explain these observations for flow shop scheduling, and if we are lucky we may be able to extend these results to other sequencing problems and possibly to all finite problems † In this paper we will simply describe our experiments and suggest a few conjectures to explain these results

In all the experiments we numbered the jobs to be processed in an

* Various optimizers ve are minimum schedule time, minimum in-service inventory, minimum total o r-due time, etc

† At the time of the initiation of these experiments it was hoped that a pattern for the flow shop would become evident Now that the experiments are completed and the mathematical foundations of these patterns are evident, it is clear that many functions defined on a finite set have the pattern, and that this property is a consequence of an approximation to the central limit theorem for a simple periodic Markov chain

TABLE I
Processing Times

Job	Machine									
	1	2	3	4	5	6	7	8	9	10
1	1	1	1	4	3	5	5	7	6	4
2	2	5	4	3	1	9	5	4	7	0
3	5	6	8	4	4	2	5	6	7	5
4	4	1	5	6	5	7	9	2	6	2
5	4	4	2	7	3	6	5	2	4	1
6	7	6	2	5	4	1	4	7	5	5
7	8	5	8	7	9	5	3	5	1	5
8	4	2	5	8	9	9	4	7	5	8
9	2	7	4	2	5	4	5	8	4	3
10	6	5	1	9	4	4	7	6	5	1
11	5	4	7	3	9	1	4	7	3	2
12	2	4	9	2	4	5	2	1	4	2
13	4	0	1	2	2	3	1	4	2	8
14	1	2	5	7	8	6	2	1	4	8
15	6	4	5	1	2	4	5	6	2	9
16	4	5	3	1	8	7	0	1	4	6
17	7	3	1	4	7	0	4	1	5	6
18	5	2	4	1	2	7	5	3	2	3
19	8	6	8	5	7	4	2	5	9	5
20	4	5	3	5	7	9	2	4	5	8
21	3	5	7	9	6	2	4	4	7	3
22	0	2	4	5	4	7	4	5	4	8
23	4	2	5	7	4	5	3	2	8	5
24	7	8	2	1	9	6	7	8	4	1
25	4	8	5	2	6	8	9	5	8	5
26	4	5	7	2	3	7	3	6	5	4
27	4	2	1	5	1	3	5	6	5	5
28	5	8	5	7	8	2	5	8	3	5
29	5	4	5	4	5	7	6	2	5	9
30	8	2	1	5	5	6	5	8	7	5
31	8	3	5	9	5	4	5	2	4	2
32	8	5	2	5	7	6	2	8	9	5
33	3	7	4	6	8	2	4	5	2	3
34	5	5	4	7	9	8	2	5	2	5
35	5	2	5	2	5	4	8	2	1	3
36	5	5	9	5	4	9	8	5	3	5
37	2	1	2	1	4	3	3	5	2	6
38	8	8	4	7	2	6	8	6	3	5
39	9	7	5	8	5	6	5	8	9	4
40	5	6	9	6	5	3	1	8	7	4
41	6	4	7	4	3	6	1	4	5	8
42	4	3	7	5	1	9	2	4	2	5
43	4	2	8	7	3	4	9	8	7	4
44	2	5	9	4	2	5	3	0	4	7
45	9	5	4	2	3	7	0	2	1	6
46	2	3	2	5	1	0	8	9	5	3
47	5	2	7	9	4	3	6	2	5	0
48	7	8	2	1	4	7	5	8	9	4
49	1	4	2	3	6	8	2	4	7	5
50	2	5	4	5	6	8	4	1	7	5

TABLE I—*Continued*

Job	Machine 1	2	3	4	5	6	7	8	9	10
51	8	3	0	2	5	6	8	2	9	4
52	7	2	4	3	6	2	9	4	1	8
53	3	5	7	5	3	8	6	4	8	1
54	5	0	5	6	0	0	2	4	7	8
55	1	9	5	2	4	7	5	0	2	5
56	0	2	9	6	1	4	0	0	5	2
57	0	2	5	8	3	6	9	1	2	4
58	7	9	6	3	5	1	7	5	4	5
59	4	3	5	2	1	4	9	7	4	1
60	0	3	5	2	4	9	4	7	5	4
61	7	8	5	6	3	9	8	7	4	6
62	1	9	6	7	0	2	4	8	3	6
63	6	1	2	0	3	5	4	1	7	3
64	6	5	1	4	9	7	3	5	6	4
65	1	8	2	6	9	4	7	5	8	4
66	0	1	6	2	9	4	8	5	7	6
67	4	2	5	6	8	5	6	4	1	4
68	3	4	5	8	4	1	2	3	6	8
69	9	8	2	3	1	4	0	2	4	5
70	4	3	2	5	6	4	1	8	9	2
71	5	7	1	2	6	8	2	3	4	7
72	2	1	4	3	8	4	6	2	4	5
73	6	7	9	2	4	3	2	5	6	7
74	2	4	0	2	5	3	4	7	8	6
75	2	7	5	4	3	1	5	6	0	2
76	3	5	7	0	2	4	5	2	5	7
77	2	4	5	3	4	7	8	3	2	4
78	9	9	1	4	5	7	6	5	3	2
79	8	2	5	2	2	5	1	5	7	8
80	4	5	2	4	7	9	5	4	2	4
81	5	5	1	2	4	2	3	8	5	1
82	0	2	9	5	4	2	5	9	6	5
83	1	2	3	5	6	2	4	0	2	5
84	4	2	3	5	4	2	3	5	4	2
85	4	4	5	8	9	8	5	2	8	3
86	4	2	3	3	2	5	8	8	1	2
87	5	3	6	2	5	6	4	7	9	3
88	4	2	3	6	8	5	3	4	7	2
89	5	8	8	3	5	6	5	6	5	2
90	5	6	4	2	5	4	6	5	8	4
91	2	1	4	7	4	5	9	8	5	6
92	2	1	4	6	5	8	6	1	3	5
93	9	5	1	3	5	7	9	1	2	5
94	3	5	4	9	7	2	6	5	2	1
95	2	5	0	3	2	4	7	8	9	5
96	5	3	5	7	9	2	4	5	5	8
97	6	3	1	5	0	1	4	8	9	8
98	3	0	4	3	7	2	6	9	4	1
99	1	7	4	2	2	4	5	0	6	9
100	1	7	8	4	6	5	4	8	5	2

arbitrary manner calling them Job 1, Job 2, , Job J The chosen order of the jobs through each machine was determined by generating J random numbers uniformly distributed between 0 and 1 and then ordering the jobs according to the ordering of the random numbers For example, if five random numbers were 0 01214, 0 31256, 0 11255, 0 84257, and 0 38142 we would order the jobs thusly Job 4, Job 5, Job 2, Job 3, and then Job 1 In the case where the job ordering was different on each machine,

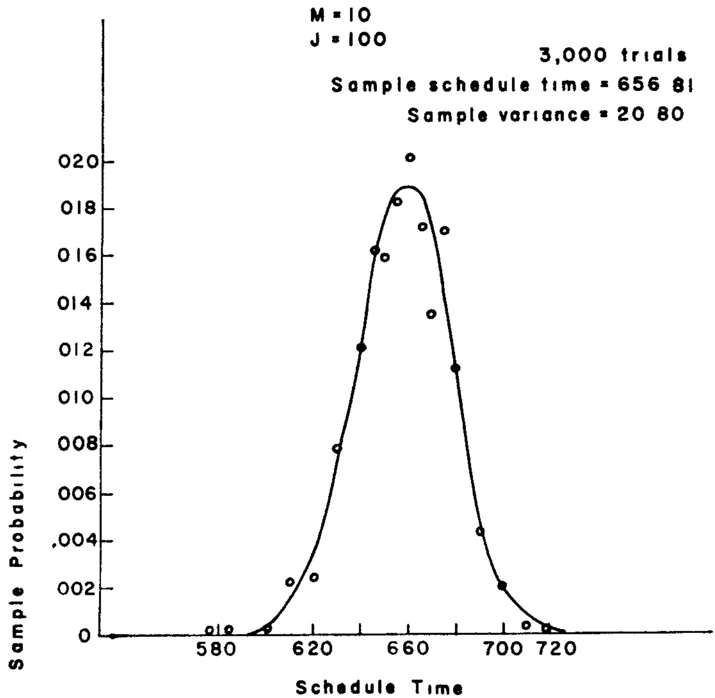

Figure 1

J random numbers were generated for each machine and the objects were ordered as described above

After the order of the jobs was determined, we computed the schedule time, using the formula for the idle time given in reference 2 The number of times each schedule time occurred was recorded, and each time a schedule time was found smaller than or equal to the previous schedule times, its schedule, i e , order of processing the jobs, was recorded

The output of the experiment was the ratio of the number of schedule times found divided by the number of samples, i e , the sample probability of the occurrence of a schedule time and the minimum schedule(s) found in the sample

No attempt was made to check if any sample was used before, for the checking would be prohibitive in machine time and the occurrence of two random samples having the same schedule is very small *

TABLE II

ORDERING FOR THE MINIMUM SCHEDULE TIME FOR THE PROCESSING TIMES LISTED IN TABLE I

85	64	52	92	3	56	68	2	53	79	67	9	13	20	50	25	27	46	62	8
74	47	95	48	23	54	30	97	57	15	73	17	31	6	33	51	19	39	96	22
91	99	32	90	58	98	41	80	75	60	29	40	35	70	43	5	100	28	65	93
84	42	45	36	37	1	21	49	69	55	88	81	76	11	89	18	87	94	12	71
83	44	61	66	78	16	82	38	86	7	14	59	72	24	63	4	26	77	10	34

The first experiment involved 100 jobs and 10 machines The processing times were randomly chosen and ranged between 0 and 9 units of time (Table I) The distribution of schedule times plotted against schedule time after 3000 samples is given in Fig 1 Only some of the sample prob-

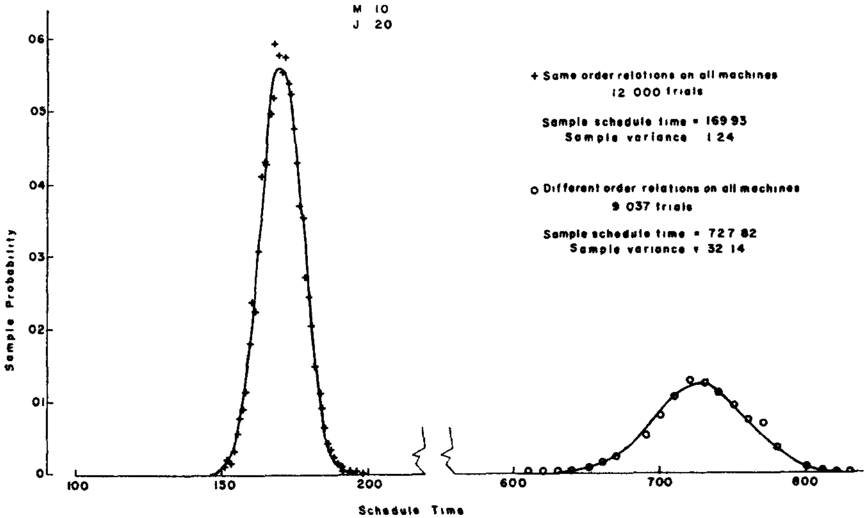

Figure 2

abilities were plotted, we found some schedule times at integer values from the schedule time of 606 to the schedule time of 707 Only near the tails of the distribution did we fail to find some schedule times The curve drawn through the points is the normal frequency curve with the expected value taken as the sample expected schedule time, and the variance taken as the sample variance of the schedule times

* Its probability $\leq 1/J!$

A lower bound for a flow shop has been given,[8] which could possibly be obtained In the course of getting samples in the above case, we found that the minimum schedule time was obtained at trial 1,154 The ordering of the objects through the machines is given in Table II

The second experiments involved sampling of schedules involving 20 jobs and 10 machines, where the processing times are those of the first 20 jobs in Table I. The first sampling was from the subset of schedules in which each job was processed in the same order, the second sampling was from the total set of orderings, i e , each machine's order of processing the jobs was independently chosen in the above-described random fashion

The curves drawn through the points are the normal distribution, where the parameters are the sample parameters Only a few of the sample probabilities are plotted almost all integer values were found between the maximum and minimum sample-schedule times

We observe in the above case that if we were looking for the minimum schedule, we would sample from those schedules which have the same ordering on all machines In reference 3 a reason for this result is given (Lemma 22)

From these experiments we would infer that the schedule times are normally distributed over the set of schedules. A partial explanation of this result is given in reference 3, and is based on an approximation to the central limit theorem for a particular periodic Markov-chain

REFERENCES

1 W Feller, "Introduction to Theory of Probability," Wiley, New York, 1957

2 J Heller, "Combinatorial, Probabilistic, and Statistical Aspects of an $M \times J$ Scheduling Problem," NYO-2540, AEC Research and Development Report, 1959 (Out of Print)

3 ——, "Combinatorial Properties of Machine Shop Scheduling," NYO-2879, AEC Research and Development Report, 1959

4 S M Johnson, "Optimal Two- and Three-Stage Production Schedules with Setup Time Included," *Naval Res Log Quart* 1 (1954)

5 R D Luce and H Raiffa, "Games and Decisions," Wiley, New York, 1957

6 H Robbins, Abstract to appear in *J Math Stat*, results to be presented in the Fourth Berkeley Symposium on Probability and Statistics

7 W E Smith, "Various Optimizers for Single-Stage Production," *Naval Res Log Quart* 3, No 1 (1956)