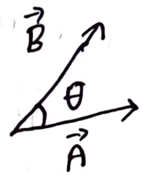
Product of vector

Product

- Dot (Scalar) product
- Cross (vector product)

Dot Product :- Dot product of $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A} \cdot \vec{B}$;



$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$

Now, $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$

$\vec{B} \cdot \vec{A} = |\vec{A}||\vec{B}| \cos(-\theta) = |\vec{A}| \cdot |\vec{B}| \cos\theta$

$\therefore \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$ [Dot product is commutative]

As, $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cdot \cos\theta$

If $\theta = \frac{\pi}{2}$, then $\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cos\frac{\pi}{2} = 0$

and if $\theta = 0$, then, $\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cos\theta = |\vec{A}||\vec{B}|$

So, if $\hat{i}, \hat{j}, \hat{k}$ are, unit vector along mutually perpendicular axes,

$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

So, $\vec{A}$ and $\vec{B}$ are Orthogonal if and only if

$\vec{A} \cdot \vec{B} = 0$

Let's m, n two scalar, and $\vec{a}, \vec{b}$ two vector.

$$(m.\vec{a})(n.\vec{b}) = n\vec{a}.m\vec{b} \text{ (associative law)}$$

Now, consider three vector, $\vec{a}, \vec{b}, \vec{c}$.

$$\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c} \text{ (distributive law)}$$

Let's two vector, $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

we have to find, $\vec{A}.\vec{B}$

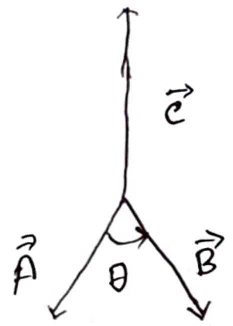
$$\therefore \vec{A}.\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

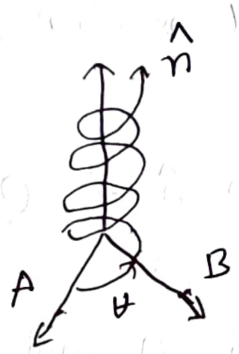
Schwarz inequality :- As $|\cos\theta| \leq 1$

$$|\vec{A}.\vec{B}| \leq |\vec{A}||\vec{B}|$$

Cross or Vector Product :-

Let $\vec{A}, \vec{B}$ two vector and $\vec{c}$ their cross or vector product.





$\vec{C}$ is defined as $\vec{C} = \vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta \; \hat{n}$

Where, $\hat{n}$ is unit vector along $\vec{A} \times \vec{B}$, $\hat{n}$ is determined from right hand screw rule.

Characteristics of vector product :-

i) $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ [Fails commutative law]

Explaination :-

$\vec{A} \times \vec{B} = |\vec{A}|\,|\vec{B}| \sin\theta \; \hat{n}$

$\vec{B} \times \vec{A} = |\vec{B}|\,|\vec{A}| \sin(-\theta)\,\hat{n} = -\,|\vec{B}|\,|\vec{A}| \sin\theta \; \hat{n}$

$\therefore \vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$

ii) Vector or cross product of two parallel or colinear vector is always 0.

for co-linear vector, $\theta = 0$

$\therefore \sin\theta = 0$ $\quad\therefore$ their cross product is 0.

iii) Cross product of perpendicular vector;

Let's $\vec{A}$ and $\vec{B}$ are perpendicular.

$$\vec{A} \times \vec{B} = |\vec{A}|\,|\vec{B}| \sin \frac{\pi}{2}\, \hat{n} = |\vec{A}|\,|\vec{B}|\, \hat{n}$$

iv) Cross product of equal vector.;

Let^r a vector, $\vec{A}$.

$$\vec{A} \times \vec{A} = |\vec{A}|\,|\vec{A}| \sin 0\, \hat{n} = 0$$

v) Let m is scalar, and $\vec{A}$ and $\vec{B}$ are vector.

$$(m\vec{A}) \times \vec{B} = m\,|\vec{A}|\,|\vec{B}| \sin\theta = |\vec{A}|\,|m\vec{B}| \sin\theta = \vec{A} \times (m\vec{B})$$

$\therefore$ It obeys associative law.

vi) If $\vec{A}$, $\vec{B}$ and $\vec{C}$ three vector.

$$\vec{A} \times (\vec{B}+\vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

vii) If $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

and $\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Product of three vector

Scalar triple product :-

Let $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$

$\vec{C} = C_x \hat{i} + C_y \hat{j} + C_z \hat{k}$

To find scalar triple product, we need to find $\vec{A} \cdot (\vec{B} \times \vec{C})$;

Let find $(\vec{B} \times \vec{C})$

$$(\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= (B_y C_z - B_z C_y)\hat{i} + (B_z C_x - B_x C_z)\hat{j} + (B_x C_y - B_y C_x)\hat{k}$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = A_x (B_y C_z - B_z C_y) + A_y (B_z C_x - B_x C_z) + A_z (B_x C_y - B_y C_x)$$

$$\therefore \vec{A} \cdot (\vec{B} \times \vec{C}) = [\vec{A}\,\vec{B}\,\vec{C}] = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= - \begin{vmatrix} B_x & B_y & B_z \\ A_x & A_y & A_z \\ C_x & C_y & C_z \end{vmatrix} = -[\vec{B}\,\vec{A}\,\vec{C}]$$

$$\therefore [\vec{A}\,\vec{B}\,\vec{C}] = -[\vec{B}\,\vec{A}\,\vec{C}]$$

Vector product of three vector :-

Let $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$

$\vec{C} = C_x \hat{i} + C_y \hat{j} + C_z \hat{k}$

We have to find $\vec{A} \times (\vec{B} \times \vec{C})$

$$\vec{A} \times (\vec{B} \times \vec{C}) = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) \times (C_x \hat{i} + C_y \hat{j} + C_z \hat{k})$$

$$= (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times [(B_y C_z - B_z C_y)\hat{i} + (B_z C_x - B_x C_z)\hat{j} + (B_x C_y - B_y C_x)\hat{k}]$$

$$= [A_y (B_x C_y - B_y C_x) - A_z (B_z C_x - B_x C_z)]\hat{i}$$

$$\cancel{+[A_z (B_z C_x}$$

$$+ [A_z (B_y C_z - B_z C_y) - A_x (B_x C_y - B_y C_x)]\hat{j}$$

$$+ [A_x (B_z C_x - B_x C_z) - A_y (B_y C_z - B_z C_y)]\hat{k}$$

$$= (A_x C_x + A_y C_y + A_z C_z)(B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) - (A_x B_x + A_y B_y + A_z B_z)(C_x \hat{i} + C_y \hat{j} + C_z \hat{k})$$

$$= (\vec{A} \cdot \vec{C}) \cdot \vec{B} - (\vec{A} \cdot \vec{B}) \cdot \vec{C}$$

Problem and Solution

1) Prove that, $(\vec{A} \times \vec{B})^2 = (AB)^2 - (\vec{A} \cdot \vec{B})^2$

Solution :-

$$(\vec{A} \times \vec{B})^2 = (AB \sin\theta)^2$$
$$= (A^2B)^2 \sin^2\theta$$
$$= A^2B^2 [1 - \cos^2\theta]$$
$$= A^2B^2 - A^2B^2 \cos^2\theta$$
$$= (AB)^2 - (\vec{A} \cdot \vec{B})^2 \quad \text{(Proved)}$$

2) Find a vector $\vec{A}$, that is perpendicular to

$\vec{U} = 2\hat{i} + \hat{j} - \hat{k}$

$\vec{V} = \hat{i} - \hat{j} + \hat{k}$.

Solution :- As, $\vec{A}$ is perpendicular to $\vec{U}$ and $\vec{V}$.

$$\therefore \vec{A} = \vec{U} \times \vec{V}$$
$$= (2\hat{i} + \hat{j} - \hat{k}) \times (\hat{i} - \hat{j} + \hat{k})$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$
$$= (1-1)\hat{i} + (-1.1 - 2.1)\hat{j} + (2.-1 - 1.1)\hat{k}$$
$$= -3\hat{j} - 3\hat{k}$$

$\therefore$ We get $\vec{A} = -3\hat{i} - 3\hat{k}$.

3) The magnetic induction $\vec{B}$ is defined by the Lorentz force equation,

$$\vec{F} = q(\vec{v} \times \vec{B})$$

Carrying out three experiment, we find that if

$$\vec{v} = \hat{i}, \quad \frac{\vec{F}}{q} = 2\hat{k} - 4\hat{j}$$

$$\vec{v} = \hat{j}, \quad \frac{\vec{F}}{q} = 4\hat{i} - \hat{k}$$

$$\vec{v} = \hat{k}, \quad \frac{\vec{F}}{q} = [illegible]\,\hat{j} - 2\hat{i}$$

From three separate experiment, calculat the magnetic induction.

Solution :- Let the magnetic induction, $\vec{B} = x\hat{i} + y\hat{j} + z\hat{k}$.

Now, to find the value of $\vec{B}$, we have to find the value of x, y and z.

Now, $\frac{\vec{F}}{q} = \vec{v} \times \vec{B}$

Let find $\vec{v} \times \vec{B}$ when $\vec{v} = \hat{i}$

$$\vec{v} \times \vec{B} = (\hat{i}) \times (x\hat{i} + y\hat{j} + z\hat{k}) = 2\hat{k} - 4\hat{j}$$

$$\Rightarrow y\hat{k} - z\hat{j} = 2\hat{k} - 4\hat{j} \quad \text{--- (1)}$$

again, $\vec{v} \times \vec{B}$ when $\vec{v} = \hat{j}$

$$\vec{v} \times \vec{B} = (\hat{j}) \times (x\hat{i} + y\hat{j} + z\hat{k}) = 4\hat{i} - \hat{k}$$

$$\Rightarrow z\hat{i} - x\hat{k} = 4\hat{i} - \hat{k} \quad \text{--- (2)}$$

again, $\vec{v} = \hat{k}$;

$$\vec{v} \times \vec{B} = (\hat{k})(x\hat{i} + y\hat{j} + z\hat{k}) = \hat{j} - 2\hat{i}$$

$$\Rightarrow x\hat{j} - y\hat{i} = \hat{j} - 2\hat{i} \quad \text{——} \quad (3)$$

From, equation 1, 2 and 3 we get,

$x = 1$, $y = 2$ and $z = 4$.

$\therefore$ The magnetic induction $\vec{B} = \hat{i} + 2\hat{j} + 4\hat{k}$.