Lecture 7 Logistic Regression I

ECE 625: Data Analysis and Knowledge Discovery

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Outline

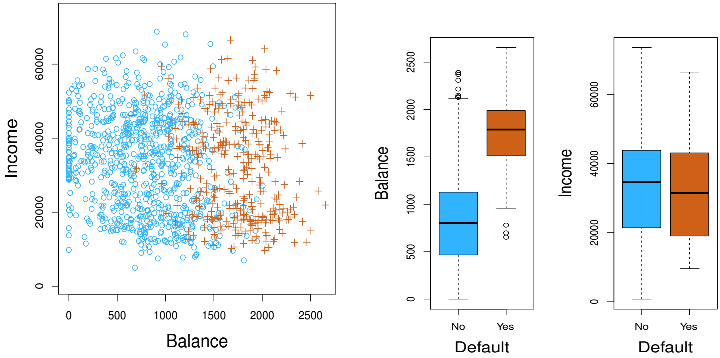
Regression is for quantitative response

Qualitative Response

For classification

- There are many qualitative response taking values in an unordered set $\mathcal{C}$ such as `eye color` $\in$ `{brown; blue; green}`.
- Given a feature vector X and a qualitative response Y taking values in the set $\mathcal{C}$, the classification task is to build a function $C(X)$ (learn a rule) that takes as input the feature vector X and predicts its value for Y; i.e. $C(X) \in \mathcal{C}$.
- Often we are more interested in estimating the probabilities that X belongs to each category in $\mathcal{C}$.
- For example, it is more valuable to have an estimate of the probability that an insurance claim is fraudulent, than a classification fraudulent or not.

Cedit Card Default



Individuals who defaulted in a given month in orange, and did not in blue.

Linear Regression Model

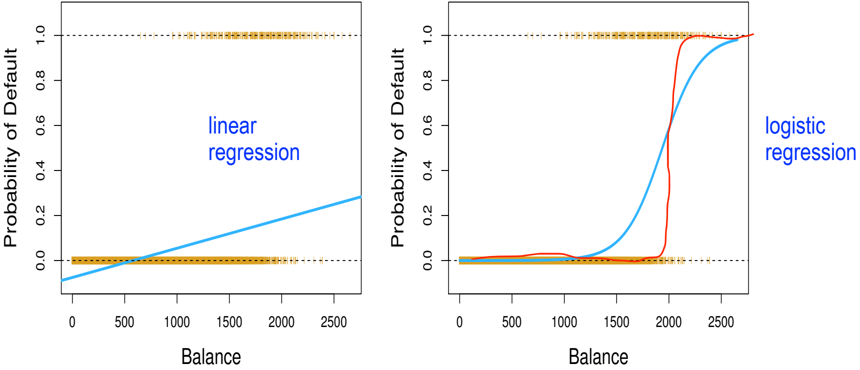
- Suppose for the Default classification task that we code

$$Y = \begin{cases} 1 & \text{if Yes} \\ 0 & \text{if No} \end{cases}.$$

Can we simply perform a linear regression of Y on X and classify as Yes if $\hat{Y} > 0.5$?

- In this case of a binary outcome, linear regression does a good job as a classifier, and is equivalent to linear discriminant analysis which we discuss later.
- Since in the population $E(Y|X = x) = \Pr(Y = 1|X = x)$, we might think that regression is perfect for this task.
- However, linear regression might produce probabilities less than zero or bigger than one. Logistic regression is more appropriate.

Credit data example



The orange marks indicate the response Y, either 0 or 1. Linear regression does not estimate $\Pr(Y = 1|X)$ well. Logistic regression seems well suited to the task.

Logistic Regression

1-p(X) = Pr(Y=0|X)

- Denote $p(X) = \Pr(Y = 1|X)$ consider using balance to predict default. Logistic regression uses the form

$$\Pr(Y{=}1|X) = p(X) = \frac{e^{\beta_0+\beta_1 X}}{1+e^{\beta_0+\beta_1 X}}.$$

We have a linear model here

- It is easy to see that no matter what values β_0, β_1 or X take, $p(X)$ will have values between 0 and 1.
- A bit of rearrangement gives

$$\log\left(\frac{p(X)}{1-p(X)}\right) = \beta_0 + \beta_1 X.$$

- This monotone transformation is called the log odds or logit transformation of $p(X)$.

Training samples: (x1, 0), (x2, 1), (x3, 1), (x4, 0), ...
likelihood under (beta0, beta1): (1-p(x1)) p(x2) p(x3) (1-p(x4))...

Estimation

- We use maximum likelihood to estimate the parameters.

$$l(\beta_0, \beta_1) = \Pi_{i:y_i=1} p(x_i) \Pi_{i:y_i=0}(1 - p(x_i)).$$

- This likelihood gives the probability of the observed zeros and ones in the data. We pick β_0 and β_1 to maximize the likelihood of the observed data. Let $\beta = (\beta_0, \beta_1)^\mathsf{T}$. The log-likelihood is

training samples

looks like a cap

$$\begin{aligned}\log \ell(\beta) &= \sum_{i=1}^{N} \{y_i \log p(x_i; \beta) + (1 - y_i) \log(1 - p(x_i; \beta))\} \\ &= \sum_{i=1}^{N} \left\{ y_i \beta^\mathsf{T} x_i - \log(1 + e^{\beta^\mathsf{T} x_i}) \right\}\end{aligned}$$

- which is concave. Setting the derivative to zero, we get

$$\partial \ell(\beta)/\partial \beta = \sum_{i=1}^{N} x_i (y_i - p(x_i; \beta)) = 0.$$

Estimation

- Can use Newton's Method to solve for the roots in the above nonlinear equations. or use hill climbing
- Most statistical packages can fit linear logistic regression models by maximum likelihood.
- In R we use the glm function. to do logistic regression

response must be 0/1

```
> glm.fit=glm(default~balance,data=defaultData,family=binomial)
> summary(glm.fit)
Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept) -1.065e+01  3.612e-01  -29.49   <2e-16 ***
balance      5.499e-03  2.204e-04   24.95   <2e-16 ***
---
```

means logistic regression

small p-value will
reject null hypothesis:
this variable is useful

Interpretation

- Interpreting what β_1 means is not very easy with logistic regression, simply because we are predicting $\Pr(Y = 1|X)$ and not Y.
- If $\beta_1 = 0$, this means that there is no relationship between Y and X.
- If $\beta_1 > 0$, this means that when X gets larger so does the probability that $Y = 1$.
- If $\beta_1 < 0$, this means that when X gets larger, the probability that $Y = 1$ gets smaller.
- But how much bigger or smaller depends on where we are on the slope.

increase in the middle is always steeper than a linear regression
see the figure on page 6

Hypothesis Testing

- We still want to perform a hypothesis test to see whether we can be sure that are β_0 and β_1 significantly different from zero.
- We use a z test instead of a t test, but of course that doesn't change the way we interpret the p-value
- Here the p-value for balance is very small, and β_1 is positive, so we are sure that if the balance increase, then the probability of default will increase as well.

```
              Estimate Std. Error z value Pr(>|z|)
(Intercept) -1.065e+01  3.612e-01  -29.49   <2e-16 ***
balance      5.499e-03  2.204e-04   24.95   <2e-16 ***
---
Signif. codes:  0 ?***? 0.001 ?**? 0.01 ?*? 0.05 ?.? 0.1 ? ? 1
```

Prediction

- What is our estimated probability of `default` for someone with a balance of 1000?

logit

$$\hat{p}(X) = \frac{e^{\hat{\beta}_0+\hat{\beta}_1 X}}{1+e^{\hat{\beta}_0+\hat{\beta}_1 X}} = \frac{e^{-10.5613+0.0055\times 1000}}{1+e^{-10.5613+0.0055\times 1000}} = 0.006.$$

Final classification is 0

- The predicted probability of default for an individual with a balance of \$1000 is less than 1%.
- For a balance of \$2000, the probability is much higher, and equals to 0.586(58.6%).

$$\hat{p}(X) = \frac{e^{\hat{\beta}_0+\hat{\beta}_1 X}}{1+e^{\hat{\beta}_0+\hat{\beta}_1 X}} = \frac{e^{-10.5613+0.0055\times 2000}}{1+e^{-10.5613+0.0055\times 2000}} = 0.586.$$

if >0.5, classify to 1

```
> predict(glm.fit, list(balance = c(1000,2000)),type="response")
          1           2
0.005752145 0.585769370
```

p(x), where x= 1000 or 2000

Summary and Remark

- Introduction
- Logistic regression and estimation
- Hypothesis testing and prediction
- Read textbook Chapter 4 and `R` code
- Do `R` lab