Lecture 4 Linear Regression II

ECE 625: Data Analysis and Knowledge Discovery

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Outline

Multiple Linear Regression

Estimation and Inference

Indicator Variables

Summary and Remark

Multiple Linear Regression

- Multiple Linear Regression has more than one covariate,

$$Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p + \varepsilon,$$

  where usually $\varepsilon \sim N(0, \sigma^2)$.
- We interpret β_j as the average effect on Y due to one unit of increase in X_j, while holding all the other covariates fixed.
- In the advertising example, the model becomes

$$\text{Sales} = \beta_0 + \beta_1 \times \text{TV} + \beta_2 \times \text{Radio} + \beta_3 \times \text{Newspaper} + \varepsilon.$$

Coefficient Interpretation

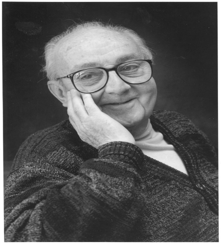
- The ideal scenario is when the predictors are uncorrelated — a balanced design.
  - Each coefficient can be estimated and tested separately.
  - Interpretations such as a unit change in X_j is associated with a β_j change in Y , while all the other variables stay fixed, are possible.
- Correlations amongst predictors cause problems.
  - The variance of all coefficient tends to increase, sometimes dramatically.
  - Interpretations become hazardous — when X_j changes, everything else changes.

The woes of regression coefficients

Data Analysis and Regression, Mosteller and Tukey 1977

- A regression coefficient β_j estimates the expected change in Y per unit change in X_j, with all other predictors held fixed. But predictors usually change together!
- Example: Y total amount of change in your pocket; $X_1 = \#$ of coins; $X_2 = \#$ of quarters and loonies. By itself, regression coefficient of Y on X_2 will be > 0. But how about with X_1 in model?
- $Y =$ number of tackles by a football player in a season; W and H are his weight and height. Fitted regression model is $Y = \beta_0 + 0.50W - 0.10H$. How do we interpret $\hat{\beta}_2 < 0$?

Two famous quotes



In real world, we are just analyzing data passively

1919 - 2013 (aged 93)

- Essentially, all models are wrong, but some are useful.
  George Box
- The only way to find out what will happen when a complex system is disturbed is to disturb the system, not merely to observe it passively.
  Fred Mosteller and John Tukey, paraphrasing George Box

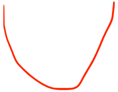
Coefficient estimation

- Given the estimates $\hat{\beta}_0, \hat{\beta}_1, \cdots,$ and $\hat{\beta}_p$, the estimated regression line is

$$y = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \cdots + \hat{\beta}_p x_p.$$

- We estimate all the coefficients β_i, $i = 0, 1, \cdots, p$ as the values that minimize the sum of squared residuals

Residual Sum of Squares $$\mathrm{RSS} = \sum_{i=1,\ldots,n} (y_i - \hat{y}_i)^2,$$



convex optimization

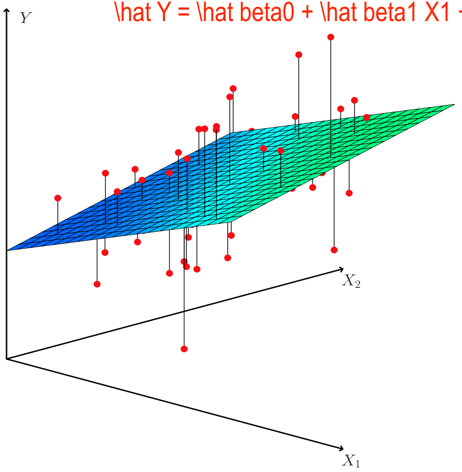
where $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \cdots + \hat{\beta}_p x_p$ is the predicted values.

- This can be done using software. The values $\hat{\beta}_0, \hat{\beta}_1, \cdots,$ and $\hat{\beta}_p$ that minimize RSS are the multiple least squares regression coefficient estimates.

Estimation Example

Y = beta0 + beta1 X1 + beta2 X2 + e

Prediction is the hyperplane:
\hat Y = \hat beta0 + \hat beta1 X1 + \hat beta2 X2



Inference

feature
covariate

- Is at least one predictor useful? Use F statistic:

H0: all beta =0 (none of the predictors is useful)

$$F = \frac{(\text{TSS} - \text{RSS})/p}{\text{RSS}/(n-p-1)} \sim F_{p,n-p-1}.$$

leads to a p-value
small: reject H0 (at least one predictor is useful)

- What about an individual coefficient, say if β_i useful? Use t statistic

H0: beta_i = 0 (predictor i is not useful)

$$t = \frac{\hat{\beta}_i - 0}{\text{SE}\left(\hat{\beta}_i\right)} \sim t_{n-p-1}.$$

leads to a p-value
small: reject H0 (beta_i useful)

- For given $x_1, \cdots, x_p$, what is the prediction interval (PI) of the corresponding y? PI is the CI of (\hat y+e) that includes the effect of noise e
- What about the confidence interval (CI) of y? CI is the CI of \hat y
- What is the difference — PI, individual and CI, average, PI is wider than CI.

Advertising example

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  2.938889   0.311908   9.422   <2e-16 ***
TV           0.045765   0.001395  32.809   <2e-16 ***
Radio        0.188530   0.008611  21.893   <2e-16 ***
Newspaper   -0.001037   0.005871  -0.177     0.86
---
Signif. codes:  0 ?***? 0.001 ?**? 0.01 ?*? 0.05 ?.? 0.1 ? ? 1

Residual standard error: 1.686 on 196 degrees of freedom
Multiple R-squared:  0.8972,Adjusted R-squared:  0.8956
F-statistic: 570.3 on 3 and 196 DF,  p-value: < 2.2e-16

> predict(TVadlm, newdata, interval="c", level=0.95)
       fit      lwr      upr
1 20.52397 19.99627 21.05168
> predict(TVadlm, newdata, interval="p", level=0.95)
       fit      lwr      upr
1 20.52397 17.15828 23.88967
```

with 95% probability, y is in [19.99627 21.05168]

confidence interval

prediction interval

Indicator Variables

- Some predictors are not quantitative but are qualitative, taking discrete values. These are also called categorical variables.
- Example: investigate difference in credit card balance between males and females, ignoring the other variables. We create a new variable,

$$x_i = \begin{cases} 1 & \text{if } i\text{-th person is female,} \\ 0 & \text{if } i\text{-th person is male} \end{cases}.$$

- Resulting model

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i = \begin{cases} \beta_0 + \beta_1 + \varepsilon_i & \text{if } i\text{-th person is female,} \\ \beta_0 + \varepsilon_i & \text{if } i\text{-th person is male} \end{cases}.$$

- Interpretation and more than two levels (categories)?

Indicator Variables

- In general, if we have k levels, we need $(k-1)$ indicator variables.
- For example, we have 3 categories — $A, B,$ and C for a covariate x,

$$x_A = \begin{cases} 1 & \text{if } x \text{ is A,} \\ 0 & \text{if } x \text{ is not A} \end{cases}; \quad x_B = \begin{cases} 1 & \text{if } x \text{ is B,} \\ 0 & \text{if } x \text{ is not B} \end{cases}.$$

- If x is C, then $x_A = x_B = 0$. We call C the baseline or default category.
- β_A is the contrast between A and C and β_B is the contrast between B and C.

Summary and Remark

- Multiple linear regression
- Estimation and inference
- Indicator variables
- Read textbook Chapter 3
- Do R lab