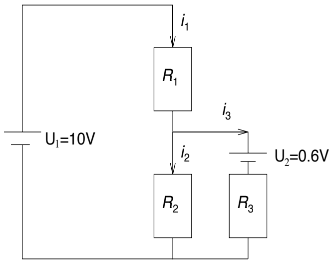
Exercise: Linear Systems

Numerical Analysis, FMNF10, 2021



x_i	1	2	4
f_i	2.5	4.5	9

Figure 1: Electrical network (left) and data for "Axa havregrynsgröt" (right).

Solving linear systems

1. A simple electrical network, see Figure 1, is described by the linear system of equations

$$\begin{cases} i_1 - i_2 - i_3 &= 0\,, \\ R_1 i_1 + R_2 i_2 &= U_1\,, \\ R_2 i_2 - R_3 i_3 &= U_2\,. \end{cases} \tag{1}$$

a) Write a MATLAB program which solves the system (1) for the unknown currents $[i_1, i_2, i_3]^T$ by Gaussian elimination. Try out your code for the case

$$[U_1, U_2] = [10V, 0.6V] \quad \text{and} \quad [R1, R_2, R_3] = [3\Omega, 2\Omega, 8\Omega]\,.$$

b) Modify your code in such a way that it solves a general linear system $Ax = b$, where A is an invertible $n \times n$-matrix.

c) Try out your code when `A=rand(10)` and `b=rand(10,1)`. Check that your result coincides with the solution given by Matlab's backslash operator, that is, `A\b`.

2. Consider the linear system $Ax = b$, where

$$A = \begin{bmatrix} 10.2 & 0 & -1.1 \\ 0.1 & 12.0 & 0 \\ 0.1 & 0.2 & -9.3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

a) Implement the Gauss–Seidel method and approximate the solution to the system with $x_0 = [1, 1, 1]^T$ and $N = 10$ iterations. Do the Gauss–Seidel iterations seem to converge? Compare with the solution given by `A\b`.

b) Give a sufficient condition for the Gauss–Seidel method to converge for a general system $Ax = b$. Check if your condition is fulfilled in the current example. Here, the MATLAB function `norm` is handy.

Linear System Analysis

3. Consider the linear system $Ax = b$, where

$$A = \begin{bmatrix} 210.5665 & 215.9568 & 375.3999 \\ 309.2944 & 317.1409 & 550.7982 \\ 227.6848 & 232.4327 & 403.2554 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -0.4816 \\ -0.7068 \\ -0.5182 \end{bmatrix}.$$

In order to obtain an understanding for the sensitivity of the solution process we will investigate the effect of perturbing the righthand side b by a vector δb. To this end, let $\bar{x}$ be the solution to the perturbed problem, that is, $A\bar{x} = b + \delta b$. A measure of the sensitivity can the be formulated as

$$k = \frac{\text{Relative error in output}}{\text{Relative error in input}} = \frac{\|\delta x\|_2/\|x\|_2}{\|\delta b\|_2/\|b\|_2} \tag{2}$$

where $\delta x = \bar{x} - x$, and $\|v\|_2 = (|v_1|^2 + |v_2|^2 + |v_3|^2)^{1/2}$. Here, a large value of k indicates that the system is sensitive to small perturbations.

a) Solve the original system $Ax = b$ to find x.

b) Generate a set of one thousand perturbations δb_i, with $|\delta b_i| \leq 0.01$. Compute the corresponding terms k_i, by (2), and find $k = \max_i k_i$. Is the system sensitive to perturbations?

c) Compute the condition number $\kappa(A) = \|A\|\|A^{-1}\|$ by using Matlab's function `cond`. How does the value of k compare to the matrix's condition number $\kappa(A)$?

Interpolation and Extrapolation (A lot more on this in Project 1)

4. The recipe for oatmeal porridge, according to an "Axa havregryn" bag, is given in Figure 1, where x is the number of servings (and the volume of oatmeal in dl), and y is the corresponding volume of water (also in dl).

a) Compute the quadratic interpolation polynomial

$$P_3(x) = c_0 + c_1 x + c_2 x^2$$

which interpolates the points in Figure 1.

b) Use the same polynomial to find how much water is needed to serve 100 people. Is the value realistic?

c) How much water should you take if you don't like oatmeal porridge? (That is, you want 0 servings.) Explain why this happens.