# The Power of Amortization on Scheduling with Explorable Uncertainty\*

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**Abstract.** In this work, we study a scheduling problem with explorable uncertainty. Each job comes with an upper limit of its processing time, which could be potentially reduced by testing the job, which also takes time. The objective is to schedule all jobs on a single machine with a minimum total completion time. The challenge lies in deciding which jobs to test and the order of testing/processing jobs.

The online problem was first introduced with unit testing time [6,7] and later generalized to variable testing times [1]. For this general setting, the upper bounds of the competitive ratio are shown to be 4 and 3.3794 for deterministic and randomized online algorithms [1]; while the lower bounds for unit testing time stands [6,7], which are 1.8546 (deterministic) and 1.6257 (randomized).

We continue the study on variable testing times setting. We first enhance the analysis framework in [1] and improve the competitive ratio of the deterministic algorithm in [1] from 4 to  $1+\sqrt{2}\approx 2.4143$ . Using the new analysis framework, we propose a new deterministic algorithm that further improves the competitive ratio to 2.316513. The new framework also enables us to develop a randomized algorithm improving the expected competitive ratio from 3.3794 to 2.152271.

**Keywords:** Explorable uncertainty, Online scheduling algorithms, Total completion time, Competitive analysis, Amortized analysis

# 1 Introduction

In this work, we study the single-machine Scheduling with Uncertain Processing time (SUP) problem with the minimized total completion time objective. We are given n jobs, where each job has a testing time  $t_j$  and an upper limit  $u_j$  of its real processing time  $p_j \in [0, u_j]$ . A job j can be executed (without testing), taking  $u_j$  time units. A job j can also be tested using  $t_j$  time units, and after it is tested, it takes  $p_j$  time to execute. Note that any algorithm needs to test a job j beforehand to run it in time  $p_j$ . The online algorithm does not know the

<sup>\*</sup> This is a full version of the papers [18] and [19].

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exact value of  $p_j$  unless it tests the job. On the other hand, the optimal offline algorithm knows in advance each  $p_j$  even before testing. Therefore, the optimal strategy is to test job j if and only if  $t_j + p_j \le u_j$  and execute the shortest job first, where the processing time of a job j is  $\min\{t_j + p_j, u_j\}$  [1,6,7]. However, since the online algorithm only learns about  $p_j$  after testing j, the challenge to the online algorithm is to decide which jobs to test and the order of tasks that could be testing, execution, or execution-untested.

It is typical to study uncertainty in scheduling problems, for example, in the worst case scenario for online or stochastic optimization. Kahan [17] has introduced a novel notion of explorable uncertainty where queries can be used to obtain additional information with a cost. The model of scheduling with explorable uncertainty studied in this paper was introduced by Dürr et al. recently [6,7]. In this model, job processing times are uncertain in the sense that only an upper limit of the processing time is known, and can be reduced potentially by testing the job, which takes a testing time that may vary according to the job. An online algorithm does not know the real processing time before testing the job, whereas an optimal offline algorithm has the full knowledge of the uncertain data.

One of the motivations to study scheduling with uncertain processing time is clinic scheduling [4, 20]. Without a pre-diagnosis, it is safer to assign each treatment the maximum time it may need. With pre-diagnosis, the precise time a patient needs can be identified, which can improve the performance of the scheduling. Other applications are, as mentioned in [6, 7], code optimization [3], compression for file transmission over network [24], fault diagnosis in maintenance environments [21]. Application in distributed databases with centralized master server [22] is also discussed in [1].

In addition to its practical motivations, the model of explorable uncertainty also blurs the line between offline and online problems by allowing a restricted uncertain input. It enables us to investigate how uncertainty influences online decision quality in a more quantitative way. The concept of exploring uncertainty has raised a lot of attention and has been studied on different problems, such as sorting [15], finding the median [12], identifying a set with the minimum-weight among a given collection of feasible sets [9], finding shortest paths [11], computing minimum spanning trees [16], etc. More recent work and a survey can be found in [8,11,14]. Note that in many of the works, the aim of the algorithm is to find the optimal solution with the minimum number of testings for the uncertain input, comparing against the optimal number of testings.

Another closely related model is Pandora's box problem [5,10,23], which was based on the secretary problem, that was first proposed by Weitzman [23]. In this problem, each candidate (that is, the box) has an independent probability distribution for the reward value. To know the exact reward a candidate can provide, one can open the box and learn its realized reward. More specifically, at any time, an algorithm can either open a box, or select a candidate and terminate the game. However, opening a box costs a price. The goal of the algorithm is to maximize the reward from the selected candidate minus the total cost of opening boxes. The Pandora's box problem is a foundational framework

for studying how the cost of revealing uncertainty affects the decision quality. More importantly, it suggests what information to acquire next after gaining some pieces of information.

Previous works. For the SUP problem, Dürr et al. studied the case where all jobs have the same testing time [6,7]. In the paper, the authors proposed a Threshold algorithm for the special instances. For the competitive analysis, the authors proposed a delicate *instance-reduction* framework. Using this framework, the authors showed that the worst case instance of Threshold has a special format. An upper bound of the competitive ratio of 2 of Threshold is obtained by the ratio of the special format instance. Using the instance-reduction framework, the authors also showed that when all jobs have the same testing time and the same upper limit, there exists a 1.9338-competitive Beat algorithm. The authors provided a lower bound of 1.8546 for any deterministic online algorithm. For randomized algorithms, the authors showed that the expected competitive ratio is between 1.6257 and 1.7453.

Later, Albers and Eckl studied a more general case where jobs have variable testing time [1]. In the paper, the authors proposed a classic and elegant framework where the completion time of an algorithm is divided into contribution segments by the jobs executed prior to it. For the jobs with "correct" execution order as they are in the optimal solution, their total contribution to the total completion time is charged to twice the optimal cost by the fact that the algorithm does not pay too much for wrong decisions of testing a job or not. For the jobs with "wrong" execution order, their total contribution to the total completion time is charged to another twice the optimal cost using a comparison tree method, which is bound with the proposed  $(\alpha, \beta)$ -SORT algorithm. The authors also provide a preemptive 3.2361-competitive algorithm and an expected 3.3794-competitive randomized algorithm.

In the works [1,6,7], the objective of minimizing the maximum completion time on a single machine was also studied. For the uniform-testing-time setting, Dürr et al. [6,7] proposed a  $\phi$ -competitive deterministic algorithm and a  $\frac{4}{3}$ -competitive randomized algorithm, where both algorithms are optimal. For a more general setting, Albers and Eckl [1] showed that variable testing time does not increase the competitive ratios of online algorithms.

Our contribution. We first analyze the  $(\alpha, \beta)$ -SORT algorithm proposed in the work [1] in a more amortized sense. Instead of charging the jobs in the correct order and in the wrong order to the optimal cost separately, we manage to partition the tasks into groups and charge the total cost in each of the groups to the optimal cost regarding the group. The introduction of amortization to the analysis creates room for improving the competitive ratio by adjusting the values of  $\alpha$  and  $\beta$ . The possibility of picking  $\alpha > 1$  helps balance the penalty incurred by making a wrong guess on testing a job or not. On the other hand, the room for different  $\beta$  values allows one to differently prioritize the tasks that provide extra information and the tasks that immediately decide a completion time for a job. By this new analysis and the room of choosing different values of  $\alpha$  and  $\beta$ , we improve the upper bound of the competitive ratio of  $(\alpha, \beta)$ -SORT

	Testing time	Upper limit	Upper Bound	Lower bound
Deterministic	1	Uniform	1.9338 [6,7]	1.8546 [6,7]
		Variable	2 [6,7]	
	Variable	Variable	$4 [1] \rightarrow 2.414 \text{ (Theorem 3)}$	
			<b>2.316513</b> (Theorem 7)	
		(Prmp.)	3.2361 [1]	
			<b>2.316513</b> (Theorem 7)	
Randomized	1	Variable	1.7453 [6,7]	1.6257 [6,7]
	Variable	Variable	3.3794 [1]	
			<b>2.152271</b> (Theorem 11)	

Table 1: Summary of the results. The results from this work are bold and in red.

from 4 to  $1 + \sqrt{2}$ . With the power of amortization, we improve the algorithm by further prioritizing different tasks using different parameters. The new algorithm, PCP<sub> $\alpha,\beta$ </sub>, is 2.316513-competitive. This algorithm is extended to a randomized version with an expected competitive ratio of 2.152271. Finally, we show that under the current problem setting, preempting the execution of jobs does not help in gaining a better algorithm. A summary of the results can be found in Table 1.

**Paper organization.** In Section 2, we introduce the notation used in this paper. We also review the algorithm and analysis of the  $(\alpha, \beta)$ -SORT algorithm proposed in the work [1]. In Section 3, we elaborate on how amortized analysis helps to improve the competitive analysis of  $(\alpha, \beta)$ -SORT (Subsection 3.1). Upon the new framework, we propose a better algorithm,  $PCP_{\alpha,\beta}$ , in Subsection 3.2. In Subsection 3.3, we argue that the power of preemption is limited in the current model. Finally, we show how amortization helps to improve the performance of randomized algorithms.

# 2 Preliminary

Given n jobs  $1, 2, \dots, n$ , each job j has a testing time  $t_j$  and an upper limit  $u_j$  of its real processing time  $p_j \in [0, u_j]$ . A job j can be executed-untested in  $u_j$  time units or be tested using  $t_j$  time units and then executed in  $p_j$  time units. Note that if a job is tested, it does not need to be executed immediately. That is, for a tested job, there can be tasks regarding other jobs between its testing and its execution.

We denote by  $p_j^A$  the time spent by an algorithm A on job j, i.e.,  $p_j^A = t_j + p_j$  if A tests j, and  $p_j^A = u_j$  otherwise. Similarly, we denote by  $p_j^*$  the time spent by OPT, the optimal algorithm. Since OPT knows  $p_j$  in advance, it can decide optimally whether to test a job, i.e.,  $p_j^* = \min\{u_j, t_j + p_j\}$ , and execute the jobs

in the ascending order of  $p_j^*$ . We denote by cost(A) the total completion time of any algorithm A.

The tasks regarding a job j are the testing, execution, or execution-untested of j (taking  $t_j$ ,  $p_j$ , or  $u_j$ , respectively). We follow the notation in the work of Albers and Eckl [1] and denote c(k,j) as the contribution of job k in the completion time of job j in the online schedule A. That is, c(k,j) is the total time of the tasks regarding job k before the completion time of job j. The completion time of job j in the schedule A is then  $\sum_{k=1}^{n} c(k,j)$ . Similarly, we define  $c^*(k,j)$  as the contribution of job k in the completion time of job j in the optimal schedule. As observed, OPT schedules in the order of  $p^*$ ,  $c^*(k,j) = 0$  if k is executed after j in the optimal schedule, and  $c^*(k,j) = p_k^*$  otherwise.

We denote by  $i <_o j$  if the optimal schedule executes job i before job j. We also define  $i >_o j$  and  $i =_o j$  similarly (in the latter case, job i and job j are the same job). The completion time of job j in the optimal schedule is denoted by  $c_j^* = \sum_{i \le_o j} p_i^*$ . The total completion time of the optimal schedule is then  $\sum_{j=1}^n c_j^*$ . Note that there is an optimal strategy where  $p_i^* \le p_j^*$  if  $i \le_o j$ .

# 2.1 Review $(\alpha, \beta)$ -SORT algorithm [1]

For completeness, we summarise the  $(\alpha, \beta)$ -SORT algorithm and its analysis proposed in the work of Albers and Eckl [1].

Intuitively, the algorithm tests a job j if and only if  $u_j \geq \alpha \cdot t_j$ . Depending on whether a job is tested or not, the job is transformed into one task (execution-untested task) or two tasks (testing task and execution task). These tasks are then maintained in a priority queue for the algorithm to decide their processing order. More specifically, a testing task has a weight of  $\beta \cdot t_j$ , an execution task has a weight of  $p_j$ , and an execution-untested task has a weight of  $u_j$ . (See Algorithm 1.) After the tasks regarding the jobs are inserted into the queue, the algorithm executes the tasks in the queue and deletes the executed tasks, starting from the task with the shortest (weighted) time. If the task is a testing of a job j, the resulting  $p_j$  is inserted into the queue after testing. (See Algorithm 2.) Intuitively, both  $\alpha$  and  $\beta$  are at least 1. The precise values of  $\alpha$  and  $\beta$  will be decided later based on the analysis.

**Analysis** [1]. Recall that c(k, j) is the contribution of job k of the completion time of job j, and the completion time of job j is  $c_j^A = \sum_{k=1}^n c(k, j)$ . The key

# **Algorithm 1** $(\alpha, \beta)$ -SORT algorithm [1]

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Initialize a priority queue Q for j=1,2,3,\cdots,n do if u_j\geq \alpha\cdot t_j then Insert a testing task with weight \beta\cdot t_j into Q else Insert an execution-untested task with weight u_j into Q Queue-Execution(Q) \triangleright See Algorithm 2
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## **Algorithm 2** Procedure Queue-Execution (Q)

idea of the analysis is that given job j, partitioning the jobs (say, k) that are executed before j into two groups,  $k \leq_o j$  or  $k >_o j$ . Since the algorithm only tests a job j when  $u_j \geq \alpha t_j$ ,  $p_k^A \leq \max\{\alpha, 1 + \frac{1}{\alpha}\} \cdot p_k^*$ . Therefore, the total cost incurred by the first group of jobs is at most  $\max\{\alpha, 1 + \frac{1}{\alpha}\} \cdot cost(\text{OPT})$ . Note that the ratio, in this case, reflects the penalty to the algorithm that makes a wrong guess on testing a job or not.

For the second group of jobs, the authors proposed a classic and elegant comparison tree framework to charge each c(k,j) with  $k>_o j$  to the time that the optimal schedule spends on job j. More specifically,  $c(k,j) \leq \max\{(1+\frac{1}{\beta})\alpha, 1+\frac{1}{\alpha}, 1+\beta\} \cdot p_j^*$  for any k and j. Hence, the total cost incurred by the second group of jobs can be charged to  $\max\{(1+\frac{1}{\beta})\alpha, 1+\frac{1}{\alpha}, 1+\beta\} \cdot cost(\text{OPT})$ .

By summing up the c(k, j) values for all pairs of k and j, the total completion time of the algorithm is at most

$$\max\{\alpha, 1 + \frac{1}{\alpha}\} + \max\{(1 + \frac{1}{\beta}) \cdot \alpha, 1 + \frac{1}{\alpha}, 1 + \beta\}.$$

When  $\alpha = \beta = 1$  (which is the optimal selection), the competitive ratio is 4.

## 2.2 Our observation

As stated by Albers and Eckl [1],  $\alpha=\beta=1$  is the optimal choice in their analysis framework. Therefore, it is not possible to find a better  $\alpha$  and  $\beta$  to tighten the competitive ratio under the current framework. However, the framework can be improved via observations.

For example, given that  $\alpha=\beta=1$ , consider two jobs k and j, where  $(t_k,u_k,p_k)=(1+\varepsilon,1+3\varepsilon,1+3\varepsilon)$  and  $(t_j,u_j,p_j)=(1,1+4\varepsilon,1+2\varepsilon)$ . By the  $(\alpha,\beta)$ -SORT algorithm, both k and j are tested. The order of the tasks regarding these two jobs is  $t_j$ ,  $t_k$ ,  $p_j$ , and finally  $p_k$ . On the other hand, in the optimal schedule,  $p_k^*=u_k=1+3\varepsilon$  and  $p_j^*=u_j=1+4\varepsilon$ . Since  $k\leq_o j$ , as shown in Figure 1, both c(k,j) and c(j,k) are charged to  $2p_k^*$ , separately. Note that although  $c(k,j)=t_k$  in this example, the worst-case nature of the analysis framework fails to capture the fact that the contribution from the tasks

regarding k to the completion time of j is even smaller than  $p_k^*$ . This observation motivates us to establish a new analysis framework.

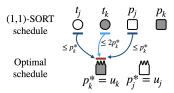


Fig. 1: An example where  $p_k^*$  is charged four times. The light blue and dark blue segments represent c(k,j) and c(j,k), respectively. The red segment represents  $p_k^*$ .

# 3 Deterministic algorithms

In this section, we first enhance the framework by equipping it with amortized analysis in Subsection 3.1. Using amortized arguments, for any two jobs  $k \leq_o j$ , we manage to charge the sum of c(k,j) + c(j,k) to  $p_k^*$ . The new framework not only improves the competitive ratio but also creates room for adjusting  $\alpha$  and  $\beta$ .

Finally, in Subsection 3.2, we improve the  $(\alpha, \beta)$ -SORT algorithm based on our enhanced framework.

## 3.1 Amortization

We first bound c(k, j) + c(j, k) for all pairs of jobs k and j with  $k \leq_o j$  by a function  $r(\alpha, \beta) \cdot c^*(k, j)$ . Then, we can conclude that the algorithm is  $r(\alpha, \beta)$ -competitive by the following argument:

$$cost((\alpha, \beta)\text{-SORT}) = \sum_{j=1}^{n} \sum_{k=1}^{n} c(k, j) = \sum_{j=1}^{n} (\sum_{k <_{o} j} (c(k, j) + c(j, k)) + c(j, j))$$

$$\leq \sum_{j=1}^{n} r(\alpha, \beta) \cdot (\sum_{k <_{o} j} c^{*}(k, j) + c^{*}(j, j)) = r(\alpha, \beta) \cdot cost(\text{OPT})$$

To bound c(k,j) + c(j,k) by the cost of tasks k, we first observe that it is impossible that  $c(k,j) = p_k^A$  and  $c(j,k) = p_j^A$  at the same time. More specifically, depending on whether the jobs k and j are tested or not, the last task regarding these two jobs does not contribute to c(k,j) + c(j,k). Furthermore, the order of these jobs' tasks in the priority queue provides a scheme to charge the cost of the tasks regarding j to the cost of tasks regarding k.

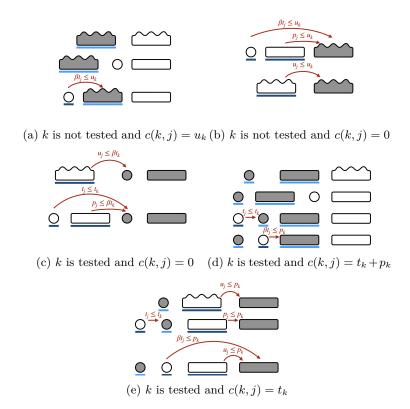


Fig. 2: The red arrows illustrate how to charge c(k,j)+c(j,k) to the cost of tasks regarding k. Each row in the sub-figures is a permutation of how the tasks are executed. The circles and rectangles are testing tasks and execution tasks after testing, respectively. The rectangles with curly tops are execution tasks without testing. The tasks in gray are from the job k, and the tasks in white are from the job j. The light blue and dark blue line segments under the tasks represent the contribution c(k,j) and c(j,k), respectively.

Figure 2 shows how the charging is done. Each row in the subfigures is a permutation of how the tasks regarding job j and k are executed. The gray objects are tasks regarding k, and the white objects are tasks regarding j. The circles, rectangles, and rectangles with the wavy top are testing tasks, execution tasks, and execution-untested tasks, respectively. The horizontal lines present the values of c(k,j) (light blue) and c(j,k) (dark blue). The red arrows indicate how the cost of a task regarding j is charged to that of a task regarding k according to the order of the tasks in the priority queue. The charging c(k,j) + c(j,k) to the cost of tasks regarding k results in Lemmas 1 and 2.

**Lemma 1.** If  $(\alpha, \beta)$ -SORT does not test job k,

$$c(k,j) + c(j,k) \le (1 + \frac{1}{\beta})u_k.$$

*Proof.* Since job k is not tested,  $c(k,j) = u_k$  or c(k,j) = 0. The case  $c(k,j) = u_k$  only happens when job k is executed untested before j is finished. Thus,  $c(j,k) \le t_j$  (see Figure 2a). In this case,  $t_j$  is executed before  $u_k$  because  $\beta t_j \le u_k$ . Overall,  $c(k,j) + c(j,k) \le (1 + \frac{1}{\beta})u_k$ 

If c(k,j) = 0, all the tasks regarding job j are done by  $u_k$  (see Figure 2b). Hence,  $c(j,k) \leq \max\{u_j,t_j+p_j\}$ . In the first case, j is executed untested before k because  $u_j \leq u_k$ . In the second case, both the (weighted) testing and processing time of job j are less than  $u_k$ . Therefore, in the worst case,  $c(k,j) + c(j,k) \leq (1+\frac{1}{\beta})u_k$ .

**Lemma 2.** If  $(\alpha, \beta)$ -SORT tests job k,

$$c(k,j) + c(j,k) \le \max\{2t_k + p_k, (1+\beta)t_k, t_k + (1+\frac{1}{\beta})p_k\}.$$

Proof. Since job k is tested, c(k,j) = 0,  $t_k + p_k$ , or  $t_k$ . The case c(k,j) = 0 happens only when all tasks regarding job j are done before testing k (see Figure 2c). Therefore,  $c(j,k) = p_j^A$ , which is  $u_j$  or  $t_j + p_j$ . In the first case,  $u_j \leq \beta t_k$ . In the second case,  $t_j \leq t_k$ , and  $p_j \leq \beta t_k$ . Overall,  $c(k,j) + c(j,k) \leq (1+\beta)t_k$  in this case.

The case  $c(k,j)=t_k+p_k$  happens only when  $p_k$  is finished before the last task regarding job j (see Figure 2d). Therefore,  $c(j,k) \leq t_j$ . In this case,  $t_j \leq \max\{t_k,\frac{p_k}{\beta}\}$ . Overall,  $c(k,j)+c(j,k)\leq \max\{2t_k+p_k,t_k+(1+\frac{1}{\beta})p_k\}$ .

If  $c(k,j)=t_k, c(j,k) \leq p_j^A$  since all tasks regarding job j finish before the last task regarding job k (see Figure 2e). If  $c(j,k)=u_j$ , the execution of j finished before k because  $u_j \leq p_k$ . If  $p_j^A=t_j+p_j, \ t_j \leq \max\{t_k, \frac{p_k}{\beta}\}$ , and  $p_j \leq p_k$ . Overall,  $c(k,j)+c(j,k) \leq \max\{2t_k+p_k,t_k+(1+\frac{1}{\beta})p_k\}$ .

Now, we can bound the competitive ratio of the  $(\alpha, \beta)$ -SORT (Theorem 3). The idea is, depending on whether job k is tested or not by the optimal schedule, the expressions in Lemmas 1 and 2 can be written as a function of  $\alpha$ ,  $\beta$ , and  $p_k^*$ . By selecting the values of  $\alpha$  and  $\beta$  carefully, we can balance the worst case ratio in the scenario where k is executed-untested by the algorithm (Lemma 1) and that in the scenario where k is tested by the algorithm (Lemma 2).

**Theorem 3.** The competitive ratio of  $(\alpha, \beta)$ -SORT is at most

$$\max\{\alpha(1+\frac{1}{\beta}), 1+\frac{1}{\alpha}+\frac{1}{\beta}, 1+\beta, 2, 1+\frac{2}{\alpha}\}\tag{1}$$

*Proof.* For simplicity, denote  $r(\alpha, \beta)$  as  $\max\{(1 + \frac{1}{\beta})u_k, 2t_k + p_k, (1 + \beta)t_k, t_k + (1 + \frac{1}{\beta})p_k\}$ . Recall that the  $(\alpha, \beta)$ -SORT only tests job k when  $u_k \geq \alpha t_k$ , and  $p_k \leq u_k$  for all jobs k. The argument is divided into two cases according to  $p_k^*$ .

Case 1:  $p_k^* = t_k + p_k$ . In this case,  $\frac{r(\alpha,\beta)}{p_k^*} \le \max\{\frac{(1+\frac{1}{\beta})u_k}{t_k + p_k}, 2, 1+\beta, 1+\frac{1}{\beta}\} \le \max\{(1+\frac{1}{\beta})\alpha, 2, 1+\beta, 1+\frac{1}{\beta}\}$ .

Case 2: 
$$p_k^* = u_k$$
. In this case,  $\frac{r(\alpha,\beta)}{p_k^*} \le \max\{1 + \frac{1}{\beta}, \frac{2t_k + p_k}{u_k}, \frac{(1+\beta)t_k}{u_k}, \frac{t_k + (1+\frac{1}{\beta})p_k}{u_k}\} \le \max\{1 + \frac{1}{\beta}, \frac{2}{\alpha} + 1, \frac{1+\beta}{\alpha}, \frac{1}{\alpha} + 1 + \frac{1}{\beta}\}.$ 

Note that by Theorem 3, the  $(\alpha, \beta)$ -SORT algorithm is 3-competitive when  $\alpha = \beta = 1$ , which matches the observation in Figure 1.

Our analysis framework provides room for adjusting the values of  $\alpha$  and  $\beta$ . By selecting the values of  $\alpha$  and  $\beta$ , we can tune the cost of tasks regarding k that is charged. By selecting a value of  $\alpha$  other than 1, we can balance the penalty of making a wrong decision on testing a job or not. The capability of selecting a value of  $\beta$  other than 1 allows us to prioritize the testing tasks (which are scaled by  $\beta$ ) and the execution tasks (which immediately decide a completion time of a job). Finally, the performance of the algorithm is tuned by finding the best values of  $\alpha$  and  $\beta$ .

**Corollary 4.** By choosing  $\alpha = \beta = \sqrt{2}$ ,  $(\alpha, \beta)$ -SORT algorithm is  $(1 + \sqrt{2})$ -competitive. This choice of  $\alpha$  and  $\beta$  is optimal for expression (1).

*Proof.* By setting  $\alpha = \beta = \sqrt{2}$ , all  $\alpha(1 + \frac{1}{\beta})$ ,  $1 + \frac{1}{\alpha} + \frac{1}{\beta}$ ,  $1 + \beta$ , and  $1 + \frac{2}{\alpha}$  are equal to  $1 + \sqrt{2}$ . That is, the choice minimizes  $\max\{\alpha(1 + \frac{1}{\beta}), 1 + \frac{1}{\alpha} + \frac{1}{\beta}, 1 + \beta, 2, 1 + \frac{2}{\alpha}\}$ . Thus the corollary follows.

However, recall that the parameter  $\alpha$  encodes the penalty for making a wrong guess on testing a job or not. When  $\alpha = \sqrt{2}$ , the penalty for testing a job we should not test is more expensive than that for executing-untested a job that we should test. It inspires us to improve the algorithm further.

#### 3.2 An improved algorithm

Surprisingly, the introduction of amortization even sheds light on further improvement of the algorithm. We propose a new algorithm,  $Prioritizing\text{-}Certain\text{-}Processing\text{-}time\ }(PCP_{\alpha,\beta})$ . The main difference between  $PCP_{\alpha,\beta}$  and  $(\alpha,\beta)\text{-}SORT$  is that in the  $PCP_{\alpha,\beta}$  algorithm after a job j is tested, an item with weight  $t_j+p_j$  is inserted into the queue instead of  $p_j$  (see Algorithm 3). Intuitively, we prioritize a job by its certain (total) processing time  $p_j^A$ , which can be  $t_j+p_j$  or  $u_j$ . Then, we can charge the total cost of tasks regarding a wrong-ordered j to  $\beta t_k$  or  $p_k^A$  all at once.

The new algorithm  $PCP_{\alpha,\beta}$  (Algorithm 1 combined with Algorithm 3) has an improved estimation of c(k,j)+c(j,k) when  $c(j,k)=t_j+p_j$ . However, when there is only one task regarding j contributing to c(j,k), the estimation of c(k,j)+c(j,k) may increase. Formally, we have the following two lemmas.

**Lemma 5.** Given two jobs  $k \leq_o j$ , if  $PCP_{\alpha,\beta}$  does not test job k,

$$c(k,j) + c(j,k) \le (1 + \frac{1}{\beta})u_k.$$

## Algorithm 3 Procedure Updated Queue-Execution (Q)

*Proof.* Since job k is not tested,  $c(k,j) = u_k$  or c(k,j) = 0. The case  $c(k,j) = u_k$  only happens when job k is executed untested before j is finished. Thus,  $c(j,k) \leq t_j$ . In this case,  $t_j$  is executed before  $u_k$  because  $\beta t_j \leq u_k$ . Overall,  $c(k,j) + c(j,k) \leq (1 + \frac{1}{\beta})u_k$ 

If c(k,j)=0, all the tasks regarding job j are finished before  $u_k$ . Hence,  $c(j,k) \leq \max\{u_j,t_j+p_j\}$ . In the first case, j is executed untested before k because  $u_j \leq u_k$ . In the second case,  $t_j+p_j \leq u_k$ . Therefore, in the worst case,  $c(k,j)+c(j,k) \leq (1+\frac{1}{\beta})u_k$ .

**Lemma 6.** Given two jobs  $k \leq_o j$ , if  $PCP_{\alpha,\beta}$  tests job k,

$$c(k,j) + c(j,k) \le \max\{2t_k + p_k, \beta t_k, (1 + \frac{1}{\beta})(t_k + p_k)\}.$$

*Proof.* Since job k is tested, c(k,j) = 0,  $t_k + p_k$ , or  $t_k$ . The case c(k,j) = 0 happens only when all tasks regarding job j are done before testing k. Therefore,  $c(j,k) = p_j^A$ , which is  $u_j$  or  $t_j + p_j$ . In the first case,  $u_j \leq \beta t_k$ . In the second case,  $t_j + p_j \leq \beta t_k$ . Overall,  $c(k,j) + c(j,k) \leq \beta t_k$  in this case.

The case  $c(k,j)=t_k+p_k$  happens only when  $p_k$  is finished before the last task regarding job j. Therefore,  $c(j,k)\leq t_j$ . In this case,  $t_j\leq \max\{t_k,\frac{t_k+p_k}{\beta}\}$ . Overall,  $c(k,j)+c(j,k)\leq \max\{2t_k+p_k,t_k+(1+\frac{1}{\beta})(t_k+p_k)\}$ .

If  $c(k,j) = t_k$ ,  $c(j,k) \le p_j^A$  since all tasks regarding job j finish before the last task regarding job k. No matter optimal schedule tests j or not,  $p_j^A \le t_k + p_k$ . Overall,  $c(k,j) + c(j,k) \le 2t_k + p_k$ .

Similar to the proof of Theorem 3, we have the following competitiveness results of the  $PCP_{\alpha,\beta}$  algorithm.

**Theorem 7.** The competitive ratio of  $PCP_{\alpha,\beta}$  is at most

$$\max\{\alpha(1+\frac{1}{\beta}), 1+\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\alpha\beta}, \beta, 2, 1+\frac{2}{\alpha}\} . \tag{2}$$

Corollary 8. By choosing  $\alpha = \frac{1+\sqrt{5}}{2}$  and  $\beta = \frac{1+\sqrt{5}+\sqrt{2(7+5\sqrt{5})}}{4}$ , the competitive ratio of  $PCP_{\alpha,\beta}$  is  $\frac{1+\sqrt{5}+\sqrt{2(7+5\sqrt{5})}}{4} \leq 2.316513$ . This choice of  $\alpha$  and  $\beta$  is optimal for expression (2).

Proof. We consider  $\alpha(1+\frac{1}{\beta})$ ,  $1+\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\alpha\beta}$ , and  $\beta$  in the max $\{\ldots\}$ . Let S be the set of these three items. By making the first two items of S equal, we obtain  $\alpha=\frac{1+\sqrt{5}}{2}$ . We then make the second and the third items of S equal, and with the obtained value of  $\alpha$  plugged in, we get  $\beta=\frac{1+\sqrt{5}+\sqrt{2(7+5\sqrt{5})}}{4}$ . With these specified values of  $\alpha$  and  $\beta$ , the items in S are all equal, and they are the largest items in the max $\{\ldots\}$ . Since  $\alpha$  and  $\beta$  are located in both numerators and denominators in S, any changes to  $\alpha$  and  $\beta$  yield one of the items larger. Thus no other choices of  $\alpha$  and  $\beta$  can provide a smaller value of the max $\{\ldots\}$ .

The selection of golden ratio  $\alpha$  balances the penalty of making a wrong guess for testing a job or not.

Note that using the analysis proposed in the work of Albers and Eckl [1] on the new algorithm that put  $t_j + p_j$  back to the priority list after testing job j, the competitive ratio is  $\max\{\alpha, 1 + \frac{1}{\alpha}\} + \max\{\alpha, 1 + \frac{1}{\alpha}, \beta\}$ . The best choice of the values is  $\alpha = \phi$  and  $\beta \in [1, \phi]$ , and the competitive ratio is at most  $2\phi$ .

#### 3.3 Preemption

We show that preempting the tasks does not improve the competitive ratio. Intuitively, we show that given an algorithm A that generates a preemptive schedule, we can find another algorithm B that is capable of simulating A and performs the necessary merging of preempted parts. The simulation may make the timing of A's schedule gain extra information about the real processing times earlier due to the advance of a testing task. However, a non-trivial A can only perform better by receiving the information earlier. Thus, B's non-preemptive schedule has a total completion time at most that of A's schedule.

**Lemma 9.** In the SUP problem, if there is an algorithm that generates a preemptive schedule, then we can always find another algorithm that generates a non-preemptive schedule and performs as well as the previous algorithm in terms of competitive ratios.

Proof. In a preemptive schedule, a job may be divided into multiple in-contiguous parts  $s_1, s_2, \cdots$ . In order to obtain a corresponding non-preemptive schedule, one may need to reschedule these parts such that they are executed together. One way to do so is to right-merge each of these parts. Consider two parts  $s_i$  and  $s_{i+1}$  and the sequence of tasks S located in between  $s_i$  and  $s_{i+1}$ . A right-merging of  $s_i$  changes the subsequence of tasks from  $(s_i, S, s_{i+1})$  to  $(S, s_i, s_{i+1})$ . This operation varies the total completion time non-increasingly, since only the completion times of the jobs corresponding to S have been changed, and they cannot increase. By right-merging all the parts for each job, we can obtain a non-preemptive schedule with equal or smaller total completion time. In the following paragraphs, we prove that such right-merging is always possible. More precisely, we show that: given an algorithm S that generates a preemptive schedule, we can find another algorithm S that is capable of simulating S and performs necessary

right-merging. The schedule generated by B is non-preemptive and has a total completion time at most that of A. Thus the lemma follows.

Algorithm B will simulate algorithm A and perform a right-merging for each part of preempted tasks that A generates. Algorithm A may change its behavior based on the results of testing tasks, and B must follow A's behavior carefully. We elaborate on B's behavior in the following two cases. For ease of analysis, we assume without loss of generality that B only performs a right-merging of  $s_i$  and leaves the other parts still preempted. One can apply the arguments repeatedly and complete the proof.

Suppose S does not contain any testing tasks. Let t be the last testing task located before  $s_i$  and t' be the first testing task located after  $s_{i+1}$ , which means the subsequence containing these tasks is  $(t, \ldots, s_i, S, s_{i+1}, \ldots, t')$ . At the moment immediately after t is executed, B is able to simulate A's behavior between tand t', and thus B can perform a right-merging of  $s_i$  and execute the other tasks in between t and t' accordingly. On the other hand, suppose S contains testing tasks. Let t be the last testing task in S and t' be the first testing task located after  $s_{i+1}$ , which means the subsequence containing these tasks is  $(s_i, S, s_{i+1}, ..., t')$ where  $t \in S$ . In order to perform a right-merging of  $s_i$ , B needs to postpone the execution of  $s_i$ . This makes the testing tasks in S being executed earlier, and A may change its behavior and become another algorithm, denoted by A', due to the advance of the test results. We note that, for any non-trivial A', it performs no worse than A with respect to the tasks located before t (excepting the postponed  $s_i$ ). The detailed behavior of B is as follows. It first postpones  $s_i$  and sees if A changes its behavior. If A does not change its behavior, B simulates A until t. Otherwise, B simulates A' until t. Algorithm B also executes the tasks assigned in the simulation (except  $s_i$ ). After that, B performs a right-merging of  $s_i$  and executes the other tasks in between t and t' accordingly.

## 4 Randomized algorithm

The amortization also helps improve the performance of randomized algorithms. We combine the  $PCP_{\alpha,\beta}$  algorithm with the framework in the work of Albers and Eckl [1], where instead of using a fixed threshold  $\alpha$ , a job j is tested with probability  $\mathbb{P}_j$ , which is a function of  $u_j$ ,  $t_j$ , and  $\beta$ .

Our randomized algorithm. For any job j with  $\frac{u_j}{t_j} < 1$  or  $\frac{u_j}{t_j} > 3$ , we insert  $u_j$  or  $\beta t_j$  into the queue, respectively. For any job j with  $1 \le \frac{u_j}{t_j} \le 3$ , we insert  $\beta t_j$  into the queue with probability  $\mathbb{P}_j$  and insert  $u_j$  with probability  $1 - \mathbb{P}_j$ . Once a testing task  $t_j$  is executed, we insert  $t_j + p_j$  into the queue. (See Algorithms 4 and 3.)

**Lemma 10.** The expected total completion time of the n jobs is at most

$$\sum_{j} \sum_{k \le j} (1 + \frac{1}{\beta}) u_k (1 - \mathbb{P}_k) + \max\{2t_k + p_k, \beta t_k, (1 + \frac{1}{\beta})(t_k + p_k)\} \mathbb{P}_k,$$

where  $\mathbb{P}_k$  is the probability that job k is tested.

## **Algorithm 4** Rand-PCP $_{\beta}$ algorithm

Initialize a priority queue Q for  $j=1,2,3,\cdots,n$  do Let  $r_j\leftarrow \frac{u_j}{t_j}$  if  $r_j<1$  then  $\mathbb{P}_j\leftarrow 0$  else if  $r_j>3$  then  $\mathbb{P}_j\leftarrow 1$  else  $\mathbb{P}_j=\frac{3r_j^2-3r_j}{3r_j^2-4r_j+3}$ 

Choose one of  $\beta t_j$  and  $u_j$  randomly with probability  $\mathbb{P}_j$  for  $\beta t_j$  and  $1 - \mathbb{P}_j$  for  $u_j$ Insert a testing task with weight  $\beta t_j$  into Q if  $\beta t_j$  is chosen, and insert an execution-untested task with weight  $u_j$  into Q otherwise

#### Updated Queue-Execution(Q)

⊳ See Algorithm 3

*Proof.* By Lemma 5 and Lemma 6,  $\mathbb{E}[c(k,j) + c(j,k) \mid k \text{ is tested}] \leq (1 + \frac{1}{\beta})u_k$ , and  $\mathbb{E}[c(k,j) + c(j,k) \mid k \text{ is not tested}] \leq \max\{2t_k + p_k, \beta t_k, (1 + \frac{1}{\beta})(t_k + p_k)\}$ . Therefore,  $\mathbb{E}[c(k,j) + c(j,k)] = \sum_j \sum_{k \leq j} (1 + \frac{1}{\beta})u_k (1 - \mathbb{P}_k) + \max\{2t_k + p_k, \beta t_k, (1 + \frac{1}{\beta})(t_k + p_k)\}\mathbb{P}_k$ .

**Theorem 11.** Let  $r_k$  denote  $\frac{u_k}{t_k}$ . The expected competitive ratio of Rand-PCP $_{\beta}$  is at most

$$\max_{k} \frac{(1 + \frac{1}{\beta})u_{k}(1 - \mathbb{P}_{k}) + \max\{2t_{k} + p_{k}, \beta t_{k}, (1 + \frac{1}{\beta})(t_{k} + p_{k})\}\mathbb{P}_{k}}{p_{k}^{*}}, where$$

$$\mathbb{P}_k = \frac{(\beta+1)(r_k-1)}{\beta(\max\{\frac{2}{r_k}+1,\frac{\beta}{r_k},(1+\frac{1}{\beta})(1+\frac{1}{r_k})\} - \max\{2,\beta,1+\frac{1}{\beta}\} + r_k-1) + r_k-1}$$

if  $r_k \in [1,3]$ ,  $\mathbb{P}_k = 0$  if  $r_k < 1$ , and  $\mathbb{P}_k = 1$  if  $r_k > 3$ .

Proof.

$$\mathbb{E}[\sum_{j} C_{j}] = \mathbb{E}[\sum_{j} \sum_{k \leq oj} c(k, j) + c(j, k)] = \sum_{j} \sum_{k \leq oj} \mathbb{E}[c(k, j) + c(j, k)]$$

$$= \sum_{j} \sum_{k \leq oj} \frac{(1 + \frac{1}{\beta})u_{k}(1 - \mathbb{P}_{k}) + \max\{2t_{k} + p_{k}, \beta t_{k}, (1 + \frac{1}{\beta})(t_{k} + p_{k})\}\mathbb{P}_{k}}{p_{k}^{*}} \cdot p_{k}^{*}$$

$$\leq \max_{k} \frac{(1 + \frac{1}{\beta})u_{k}(1 - \mathbb{P}_{k}) + \max\{2t_{k} + p_{k}, \beta t_{k}, (1 + \frac{1}{\beta})(t_{k} + p_{k})\}\mathbb{P}_{k}}{p_{k}^{*}} \cdot cost(OPT)$$

Next, we explain how to find  $\mathbb{P}_k$ . There are two cases of  $p_k^*$ : 1)  $p_k^* = u_k$ , and 2)  $p_k^* = t_k + p_k$ . In the first case, the expected competitive ratio is at most

$$(1 + \frac{1}{\beta})(1 - \mathbb{P}_k) + \max\{\frac{2}{r_k} + 1, \frac{\beta}{r_k}, (1 + \frac{1}{\beta})(1 + \frac{1}{r_k})\}\mathbb{P}_k$$
 (3)

Similarly, in the second case, the expected competitive ratio is at most

$$(1 + \frac{1}{\beta})r_k(1 - \mathbb{P}_k) + \max\{2, \beta, 1 + \frac{1}{\beta}\}\mathbb{P}_k$$
 (4)

We optimize  $\mathbb{P}_k$  by letting expressions 3 and 4 equal. And  $\mathbb{P}_k$  is obtained as stated.

Corollary 12. By choosing  $\beta=2$ , Rand-PCP $_{\beta}$  has expected competitive ratio  $\frac{3(7+3\sqrt{6})}{20} \leq 2.152271$ . In this case, job j is tested in a probability of  $\frac{3r_j^2-3r_j}{3r_j^2-4r_j+3}$  for  $1 \leq r_j \leq 3$ , where  $r_j = \frac{u_j}{t_i}$ .

Proof. We show that for any  $r_j$ , both expressions 3 and 4 are at most  $\frac{3(7+3\sqrt{6})}{20}$ . If  $r_j>1$ ,  $\mathbb{P}_j=0$  by the algorithm. Expression 3 is  $\frac{3}{2}$ , and expression 4 is  $\frac{3}{2}r_j\leq \frac{3}{2}$ . If  $r_j<3$ ,  $\mathbb{P}_j=1$  by the algorithm. Expression 3 is  $\frac{3}{2}(1+\frac{1}{r_j})\leq \frac{3}{2}(1+\frac{1}{3})=2$ , and expression 4 is 2. Otherwise,  $1\leq r_j\leq 3$ . To find the max of expressions 3 and 4, we make the two expressions equal. This gives us  $\mathbb{P}_j=\frac{3r_j^2-3r_j}{3r_j^2-4r_j+3}$ . By plugging  $\mathbb{P}_j$  back to expression 3, we obtain  $\frac{9r_j^2-3r_j}{6r_j^2-8r_j+6}$ . This function has the maximum  $\frac{3(7+3\sqrt{6})}{20}$ , which happens at  $r_j=1+\sqrt{\frac{2}{3}}\approx 1.816497$ .

**Lemma 13.** The choice of  $\beta = 2$  minimizes the expected competitive ratio in the analysis among all possible  $\beta > 0$ .

*Proof.* Given any  $\beta > 0$ , we show that there exists an  $r_j$  such that the max of expressions 3 and 4 is at least  $\frac{3(7+3\sqrt{6})}{20}$ , and the lemma follows. To find the max of the two expressions, we use  $\mathbb{P}_j$  stated in Theorem 11. For ease of analysis, we explicitly list the following two expressions that will be referred frequently in the proof.

$$\max\{\frac{2}{r_i} + 1, \frac{\beta}{r_i}, (1 + \frac{1}{\beta})(1 + \frac{1}{r_i})\}\tag{5}$$

$$\max\{2, \beta, 1 + \frac{1}{\beta}\}\tag{6}$$

We consider four cases of  $\beta$ . If  $0<\beta\leq 1$ , we set  $r_j=2$ . In this case, expression 5 is  $\frac{3}{2}(1+\frac{1}{\beta})$  and expression 6 is  $1+\frac{1}{\beta}$ . The max of expressions 3 and 4 is  $\frac{4}{3}+\frac{4}{3\beta}\geq \frac{4}{3}+\frac{4}{3}\geq 2.66$ . For  $1<\beta\leq 2$ , we set  $r_j=1+\sqrt{\frac{2}{3}}$ . We have that expression 5 is  $(1+\frac{1}{\beta})(4-\sqrt{6})$ , and expression 6 is 2. The max of expressions 3 and 4 is  $\frac{(1+\beta)(\sqrt{6}\beta+\sqrt{6}+6)}{2\beta((3-\sqrt{6})\beta-\sqrt{6}+6)}$ . This function decreases in the domain  $1<\beta\leq 2$  when  $\beta$  goes larger. So, the function has the minimum in the domain at  $\beta=2$ , which is  $\frac{3(7+3\sqrt{6})}{20}$ . If  $2<\beta\leq 2+\sqrt{\frac{2}{3}}$ , we also set  $r_j=1+\sqrt{\frac{2}{3}}$ . In this case, expression 5 is  $(4-\sqrt{6})(1+\frac{1}{\beta})$  and expression 6 is  $\beta$ . The max of expressions 3 and 4 is  $\frac{(\beta+1)(3\beta^2-(\sqrt{6}+6)(\beta+1))}{\beta(3\beta^2+(2\sqrt{6}-12)(\beta+1))}$ . This function increases in the domain  $2<\beta\leq 2+\sqrt{\frac{2}{3}}$ 

when  $\beta$  goes larger. So, the function has the minimum in the domain at  $\beta=2$ , which is also  $\frac{3(7+3\sqrt{6})}{20}$ . Otherwise,  $\beta>2+\sqrt{\frac{2}{3}}$ . We set  $r_j=\beta-1$ . We have that expression 5 is  $\frac{\beta+1}{\beta-1}$ , and expression 6 is  $\beta$ . The max of expressions 3 and 4 is  $\frac{\beta^2-1}{2}$ . This function increases in the domain  $\beta\geq 2+\sqrt{\frac{2}{3}}$  when  $\beta$  goes larger. So, the function has the minimum in the domain at  $\beta=2+\sqrt{\frac{2}{3}}$ , which is  $\frac{11+4\sqrt{6}}{6}\geq 3.46$ .

## 5 Conclusion

In this work, we study a scheduling problem with explorable uncertainty. We enhance the analysis framework proposed in the work [1] by introducing amortized perspectives. Using the enhanced analysis framework, we are able to balance the penalty incurred by different wrong decisions of the online algorithm. In the end, we improve the competitive ratio significantly from 4 to 2.316513 (deterministic) and from 3.3794 to 2.152271 (randomized). An immediate open problem is if one can further improve the competitive ratio by a deeper level of amortization.

Additionally, we show that preemption does not improve the competitive ratio in the current problem setting, where all jobs are available at first. It may not be true in the fully online setting, where jobs can arrive at any time. Thus, another open problem is to study the problem in the fully online model.

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# A Appendix

For parallel machines, Gong et al. [13] proposed a framework based on Albers and Eckl's work. When m tends to infinity, the ratio of their algorithm is 2.92706. The authors employed the lower bound that the optimal cost  $\geq \max\{\sum_j p_j^*, \frac{1}{m}\sum_j jp_j^* + (\frac{1}{2} - \frac{1}{2m})\sum_j p_j^*\}^3$ , where  $p_j^* = \min\{u_j, t_j + p_j\}$ , and proposed a  $2\phi$ -competitive algorithm, adapted from SORT, for variable testing time case and a  $\phi + \frac{\phi+1}{2} \cdot (1-\frac{1}{m})$ -competitive algorithm for uniform testing time case. The algorithms proposed by the authors were modified from the  $(\alpha, \beta)$ -SORT with a more sophisticated ordering of the tasks (testing, execution, and execution-untested). More specifically, Gong et al. also use the priority list of the jobs where the jobs j after testing have keys  $t_j + p_j$ .

<sup>&</sup>lt;sup>3</sup> The order of j is according to the execution order of the jobs in the optimal schedule.

The challenge of applying the priority-queue based algorithms is the precedence constraints between the testings and actual executions.

An unlucky case in the parallel machines setting. Consider only two available jobs k and j, where the algorithm tests k and executes untested j, and  $\beta \cdot t_k \leq t_k + p_k \leq u_j$ . In the scenario where two machines  $m_1$  and  $m_2$  are available at time  $\tau_1$  and  $\tau_2 \geq \tau_1$ , respectively,  $t_k$  will be assigned to  $m_1$ , and  $p_j$  will be available at time  $\tau_1 + t_k$ . If  $\tau_2 < \tau_1 + t_k$ , a no-wait priority-queue based algorithm will assign  $u_j$  to  $m_2$ , which violates the real priority, where  $p_k$  should be assigned before  $u_j$  since  $t_k + p_k < u_j$ .

To deal with the priority violation issue, the authors of [13] found the following important property:

**Lemma 14.** (Recasting Lemma 3 in [13]) For any list scheduling algorithm, if a task  $o_1$  precedes another task  $o_2$  in the list used by the algorithm, then the starting time of  $o_2$  is on or after the starting time of the continuous processed segment containing  $o_1$ .

Recall the unlucky example and treat  $p_k$  as  $o_1$  and  $u_j$  as  $o_2$ . By this lemma, once  $p_k$  becomes available, it must be executed right away. Otherwise, the continuous processed segment containing  $p_k$  is only the time spend on  $p_k$  itself, and it contradicts the lemma.

The authors of [13] also defined *last segments* of jobs:

**Definition 15.** (Recasting Definition 4 in [13]) Given a list scheduling algorithm, the last segment of a job j is

- the time spent on  $u_j$  if j is executed untested,
- the time spent on  $t_j + p_j$  if j is tested, and the tasks  $t_j$  and  $p_j$  are on the same machine without another job in between, or
- the time spent on  $p_i$  otherwise.

By Lemma 14, the authors redefined the *contribution* of job k to the completion time of job j, c(k,j), by the time the algorithm spent on k before the starting time of the last segment of j. By the greedy nature, the completion time of job j,  $C_j \leq \frac{1}{m} \sum_{k \neq j} c(k,j) + p_j^A$ .

Applying the amortized analysis on parallel machines. Our analysis can be further combined with the multiprocessor environment framework by Gong [13]:

**Theorem 16.** There is an  $2.31652(\frac{1}{2} + \frac{1}{2m}) + \phi(1 - \frac{1}{m})$ -competitive algorithm for the SEU problem with objective minimizing the total completion time on m parallel machines. When m tends to infinity, the competitive ratio is 2.77630.

*Proof.* The proof is similar to the proof of Theorem 7. More specifically, we need to deal with the cases in Lemma 5 and Lemma 6 when some  $t_i + p_i$  is involved. In the following, let  $a(p_i)$  be the time when the task  $p_i$  is finished. Namely, the time when  $t_i$  is completed. Similarly, let  $s(\cdot)$  be the starting time of the task  $\cdot$ , which can be any testing, actual execution, or untested execution.

- If k is untested by  $PCP_{\alpha,\beta}$ : The priority is violated if j is tested and  $t_j + p_j \le u_k$ . In this case, the proof of Lemma 5 still holds if  $a(p_j) \le s(u_k)$ . That is,  $c(k,j) + c(j,k) \le t_j + p_j \le u_k$ . On the other hand, if  $a(p_j) > s(u_k)$ , the priority is violated. By Lemma 14, there is no gap between  $t_j$  and  $p_j$ , and  $s(u_k) \in [s(t_j), s(p_j))$ . Therefore, c(k,j) = 0 and  $c(j,k) \le t_j$ . Overall,  $c(k,j) + c(j,k) \le t_j \le \frac{u_k}{\beta}$ . In both subcases,  $c(k,j) + c(j,k) \le u_k$ .
- If k is tested by  $P\tilde{C}P_{\alpha,\beta}$ :
  - If j is untested and  $t_k + p_k \le u_j$ : When  $a(p_k) \le s(u_j)$ , the original bound  $c(k, j) + c(j, k) \le (t_k + p_k) + 0 \le t_k + p_k$  still holds. When  $a(p_k) > s(u_j)$ , by Lemma 14, c(j, k) = 0 and  $c(k, j) \le t_k$ . Therefore,  $c(k, j) + c(j, k) \le t_k$ .
  - If j is tested and  $t_k \leq t_j$ : Since  $t_k \leq t_j$ ,  $a(p_k) \leq a(p_j)$ . Therefore, the unlucky scenarios can happen only when  $a(p_k) > s(t_j)$  and  $t_k + p_k \leq \beta \cdot t_j$  or  $t_j + p_j \leq t_k + p_k$ . In the first case, c(j,k) = 0 and  $c(k,j) \leq t_k$  by Lemma 14. In the second case, job j has a single connected segment, and  $c(k,j) + c(j,k) \leq t_k + t_j \leq t_k + \frac{1}{2}(t_k + p_k)$ .
  - $c(k,j)+c(j,k) \leq t_k+t_j \leq t_k+\frac{1}{\beta}(t_k+p_k).$  If j is tested and  $t_j \leq t_k$ : This case is symmetric to the previous case. When  $t_j+p_j \leq \beta \cdot t_k$  but  $a(p_j) > s(t_k)$ , by Lemma 14,  $c(k,j)+c(j,k) \leq 0+t_j \leq t_k$ . When  $t_j+p_j \geq \beta \cdot t_k$  and  $t_k+p_k \leq t_j+p_j$  but  $a(p_k) > s(p_j)$ , by Lemma 14,  $c(k,j)+c(j,k) \leq t_k+t_j \leq 2t_k$ .

From the above discussions, the violation of priority can be compensated by Lemma 14, and the dominant cases fall in the non-violation cases, which is the case in the single machine setting.

Therefore, let R be  $\max\{\alpha(1+\frac{1}{\beta}), 1+\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\alpha\beta}, \beta, 2, 1+\frac{2}{\alpha}\}$  and r be  $\max\{\alpha, 1+\frac{1}{\alpha}\}$ . The cost of  $PCP_{\alpha,\beta}$  is

$$\begin{split} \sum_{j} C_{j}^{A} &\leq \sum_{j} \left( \frac{1}{m} \sum_{k \neq j} c(k, j) + p_{j}^{A} \right) \\ &= \sum_{j} \left( \frac{1}{m} \sum_{k} c(k, j) + (1 - \frac{1}{m}) p_{j}^{A} \right) \\ &\leq \frac{1}{m} \sum_{j} \sum_{k < j} (c(k, j) + c(j, k)) + \sum_{j} (1 - \frac{1}{m}) p_{j}^{A} \\ &\leq \frac{1}{m} \cdot R \cdot \sum_{j} \sum_{k < j} p_{k}^{*} + r \cdot (1 - \frac{1}{m}) \sum_{j} p_{j}^{*} \\ &= R \cdot \left( \frac{1}{m} \sum_{j} \sum_{k < j} p_{k}^{*} + (\frac{1}{2} - \frac{1}{2m}) \sum_{j} p_{j}^{*} \right) + \left( r \cdot (1 - \frac{1}{m}) - R \cdot (\frac{1}{2} - \frac{1}{2m}) \right) \sum_{j} p_{j}^{*} \\ &\leq R \cdot \text{OPT} + \left( r \cdot (1 - \frac{1}{m}) - R \cdot (\frac{1}{2} - \frac{1}{2m}) \right) \cdot \text{OPT} \\ &= \left( R \cdot (\frac{1}{2} + \frac{1}{2m}) + r \cdot (1 - \frac{1}{m}) \right) \cdot \text{OPT}. \end{split}$$

The ratios R and r can be decided by our choice of  $\alpha$  and  $\beta$ .

Next, we further investigate the performance of  $PCP_{\alpha,\beta}$  when all jobs have unit testing times. Note that in this case, the schedule has a special form, where all untested jobs are executed before any testing if  $\alpha < \beta$ . After all the untested executions, the algorithm tests all remaining jobs. Any tested jobs j are executed immediately if  $p_j \leq \beta - 1$  and are postponed to the end otherwise. Thus, we adjust  $PCP_{\alpha,\beta}$  a bit by removing the parameter  $\beta$ . The resulting algorithm first executes all untested jobs, then tests all remaining jobs, and then executes all actual processing times in a non-decreasing order.

**Theorem 17.** If the testing times of all jobs are uniform, there is an  $\max\{2, \alpha, 1 + \frac{2}{\alpha}\} \cdot (\frac{1}{2} + \frac{1}{2m}) + \max\{\alpha, 1 + \frac{1}{\alpha}\} \cdot (1 - \frac{1}{m})$ -competitive algorithm for the SEU problem with objective minimizing the total completion time on m parallel machines, where  $\alpha \geq 1$ . When m tends to infinity, the competitive ratio is 2.73606.

*Proof.* Consider jobs k and j where  $k \leq_o j$ . If at least one of them is untested, that is,  $u_k \leq \alpha$  or  $u_j \leq \alpha$ , the  $c(k,j) + c(j,k) \leq u_k$ .

Now, we consider the case where  $u_k \geq \alpha$  and  $u_j \geq \alpha$ . Again, we denote the available time of  $p_j$  by  $a(p_j)$  and the starting time of task  $\cdot$  by  $s(\cdot)$ . If  $p_k \leq p_j$  but  $a(p_k) > s(p_j)$ , it must happen when  $t_k$  and  $t_j$  are assigned to two different machines, and  $s(t_j) < s(t_k)$ . By Lemma 14,  $t_k$  and  $p_k$  form a continuous segment, and  $s(p_j) \in (s(t_k), s(t_k) + 1)$ . Therefore,  $c(k, j) + c(j, k) \leq 1 + 1 = 2$  On the other hand, if the priority constraint is not violated,  $c(k, j) + c(j, k) \leq (1 + p_k) + 1 = 2 + p_k$ . The case where  $p_k \geq p_j$  is symmetric.

Let  $R = \max\{\alpha, 2, 1 + \frac{2}{\alpha}\}$  and  $r = \max\{\alpha, 1 + \frac{1}{\alpha}\}$ . By the same techniques in the proof of Theorem 3 and Theorem 17, the cost of the algorithm is at most

$$R \cdot (\frac{1}{2} + \frac{1}{2m}) + r \cdot (1 - \frac{1}{m}).$$

Interestingly, the later results on uniform testing time suggest that when m=1 and 2, the parameter  $\alpha$  should be chosen as 2 and  $\sqrt{3}$ , respectively. And it shows that the competitive ratio of the algorithm is 2 and 2.48206, respectively. Note that in Theorems 16 and 17, the ratios when m=1 match the current best results on a single machine.

Our contribution on makespan. In addition to the total completion time, we also study the objective of minimizing the makespan, where the highest load of machines is minimized. Albers and Eckl [2] examined this problem in the non-preemptive setting, in which once a job is tested, its actual processing time must follow its testing time immediately on the same machine where the job is tested. By making upper limits extremely large and forcing algorithms to test all jobs, the authors derived a competitive ratio lower bound of  $2 - \frac{1}{m}$ . However, the adversary overlooked the impact of uncertain processing time on the competitive ratio. We carefully select an appropriate upper limit that puts online algorithms in a predicament and thus improves the lower bound.

**Theorem 18.** For SEU problem under the non-preemptive setting and aiming at minimizing the makespan, the competitive ratio is at least  $2 - \frac{1}{2m}$  even for uniform testing time and uniform upper limit.

*Proof.* We present an adversary that makes any algorithm A be at least (2 - 1/(2m))-competitive.

The adversary generates an instance that depends on the behavior of A. It first releases 2m(m-1)+1 jobs, each with testing time  $t_j=1$  and upper limit  $u_j=2m$ . Due to the large amount of jobs, A must assign at least one of them to time 2(m-1)+1 or later. This is because assigning the jobs, each occupies at least time 1, on m machines requires  $\lceil (2m(m-1)+1)/m \rceil = 2(m-1)+1$  time. Afterwards, the adversary sets all  $p_j=0$  for all jobs j except job j', which is any job picked by the adversary from the jobs assigned to time 2(m-1)+1 or later. Depending on A's behavior on j', there are two cases:

Case 1: A tests j'. In this case, the adversary sets  $p_{j'} = 2m$ , i.e., it makes j' have the processing time equal to the upper limit. Since jobs must be scheduled non-preemptively, the completion time of j' is at least  $2(m-1)+1+p_{j'}=4m-1$ . In contrast, OPT does not test j' and schedules it solely on one of the machines. The other 2m(m-1) jobs are all tested and assigned evenly on the other machines. The OPT schedule has load 2m for all machines. Thus, the competitive ratio of A is at least (4m-1)/(2m)=2-1/(2m).

Case 2: A does not test j'. The completion time of j' is at least  $2(m-1)+u_{j'}=4m-2$ . To benefit OPT, the adversary sets  $p_{j'}=0$ . OPT tests all the jobs and assigns them evenly. The makespan of OPT is  $\lceil (2m(m-1)+1)/m \rceil = 2m-1$ . Thus, the competitive ratio of A is at least (4m-2)/(2m-1) = 2.

Finally, the lower bound of competitive ratio is obtained by the smaller one of the two cases, which is 2 - 1/(2m).

The lower bound indicates that SEU problem with makespan minimization is more uncertain than its counterpart in the pure online model, where a  $(2-\frac{1}{m})$ -competitive algorithm exists.