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Maria Debora Braga

Risk-Based Approaches to Asset Allocation

Concepts and Practical Applications

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ISSN 2193-1720 ISSN 2193-1739 (electronic)
SpringerBriefs in Finance
ISBN 978-3-319-24380-1 ISBN 978-3-319-24382-5 (eBook)
DOI 10.1007/978-3-319-24382-5

Library of Congress Control Number: 2015952768

JEL Classification: G10, G11, G12, G23

Springer Cham Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media
(www.springer.com)

This book is dedicated to my parents, Luigi and Giaele (Anna).
Their lifestyle and belief in simple and authentic values have given me the strength and guidance all these years to go anywhere and meet people from all walks of life.

Contents

Chapter 1
Introduction

One of the major topics in the area of asset management is asset allocation. The fundamental reason is that for investors it is crucial to obtain an answer to the question about how they should invest their funds among different asset types or asset classes.

Harry Markowitz suggested a formal and elegant method to come to a solution that became then popularly referred to as *Mean-Variance Optimization* or *Mean-Variance Analysis.* It consists of a quantitative technique that identifies the best asset allocation solutions on the basis of the trade-off between risk and return. However, a large amount of financial literature has successively documented that, in spite of its theoretical appeal and rationality, it does not work quite so well in practice.

In Chap. 2, we review the disappointing characteristics of the Markowitz portfolios that make them unreliable for asset managers. Most importantly, we outline that they are consequences of estimation errors in the optimization input parameters. In line with existing literature, we assign especially to estimation errors in the expected returns, the most deteriorating impact on the optimized portfolios. Therefore, it should be clear that the major criticisms do not concern so much the model itself, but rather the quality of information it uses.

Since the 1980s, dissatisfaction regarding portfolios based on the Markowitz optimization has resulted in the search for alternative asset allocation solutions. More precisely, we are aware that various approaches have been developed to address the problem of estimation risk and reduce the impact of estimation errors on portfolio weights and performance. They are represented by Bayesian methods and heuristic methods. These responses, through their multiple implementations, have provided some progress in the portfolio construction processes even if some authors believe they require further efforts to improve and effectively compete with a naïve diversification rule. Interestingly, we emphasize that both Bayesian and heuristic methods certainly do not "revolutionize" the mean-variance framework which continues to serve as a "reference point".

Using these considerations as a starting point, we have decided to devote Chaps. 3 and 4 to asset allocation approaches that provide an alternative, admittedly partial and, until few years ago, unusual response to the problem of estimation risk affecting the *Mean-Variance Optimization*: remove or drop the expected return

M.D. Braga, *Risk-Based Approaches to Asset Allocation*,
SpringerBriefs in Finance, DOI 10.1007/978-3-319-24382-5_1

estimates from the inputs for the portfolio construction process considering that related estimation errors are the most crucial. These solutions are called risk-based asset allocation approaches but they can also be referred to as *μ-free strategies*. We include in this group the following asset allocation methods: risk parity, equally-weighted, global minimum-variance and most diversified portfolio approach.

Chapters 3 and 4 focus on a comprehensive analysis of the aforementioned risk-based approaches. The work provides the theory, the math, the algorithms necessary for understanding and implementing such solutions. Since it is obvious that omitting expected returns from the set of inputs leads to a focus on the risk dimension (which justifies the name given to these methods), we highlight fundamental notions about risk budgeting as a core tool for asset managers interested in these approaches and we include a review of them in Chap. 3. It is not viable to build or interpret a risk-based portfolio without being familiar with the concepts of risk allocation or risk decomposition, marginal risk and risk contribution.

Although Chaps. 2–4 are deeply theoretical, they are continuously supplemented with examples in an attempt to allow readers to see how everything can be applied in practice. We believe that to handle with portfolio construction or optimization problems it is necessary to make use of an adequate modeling language and environment. For instance, the applications proposed throughout the book are carried out using Matlab.

The final part of the book, Chap. 5, provides an extensive empirical investigation of the risk-based asset allocation approaches discussed in the previous chapters. The analysis is undertaken using three different real datasets that allow to present applications concerning two problems of allocation inside a specific, though large, asset class and a problem of allocation among multiple asset classes. In order to provide a deeper knowledge of the main features of these strategies that can be helpful for investment practice, we adopt three different evaluation criteria in the comparative analysis: financial efficiency, level of diversification and asset allocation stability.

We hope that academics and practitioners will appreciate this book for its style of presentation and its contents. We would like to thank Professor Andrea Resti for his detailed and constructive comments regarding the drafts of the book. Needless to say, we are responsible for any shortcomings. We are grateful for the hospitality of Cass Business School (City University London), where a large part of this book was written.

Chapter 2
The Traditional Approach to Asset Allocation

Abstract After presenting the preliminary definitions and statistics that are necessary to correctly formulate portfolio risk and return, the chapter illustrates the appropriate way of constructing portfolios according to the Markowitz model, also named *Mean-Variance Optimization*. Its application requires optimization inputs (expected returns, risks, correlations) that are not observable ex-ante, thus they have to be estimated usually from past data. Then the standard implementation of Markowitz's model follows the "plug-in" rule. In practice, the estimated parameters are processed into the *mean-variance optimizer* that treats them as if they were the true parameters. This renders the portfolio construction process deterministic as the parameters uncertainty is completely neglected. This chapter gives evidence and clarification of the different undesirable features and deficiencies of the optimized portfolios (counter-intuitive nature, instability, erroneously supposed uniqueness and poor out-of-sample performance) triggered by implementing the *Mean-Variance Optimization* without recognizing the existence of estimation risk inherent in the input parameters. Amongst other, estimation errors in expected returns are the most crucial and costly ones.

Keywords Asset allocation · Bayesian approaches · Efficient frontier · Estimation risk · Heuristic approaches · Mean-Variance Optimization · Markowitz model · Parameters uncertainty · Plug-in rule · Portfolio instability

2.1 A Short Review of the Traditional Markowitz Model for Asset Allocation

Markowitz's seminal works (1952, 1959) provide the standard model to solve asset allocation problems. It is called *Mean-Variance Optimization* or *Mean-Variance Analysis* and is generally regarded as the cornerstone of *Modern Portfolio Theory*. Its foundations consist of the two following main ideas that have become central in finance literature and practice:

M.D. Braga, *Risk-Based Approaches to Asset Allocation*,
SpringerBriefs in Finance, DOI 10.1007/978-3-319-24382-5_2

- diversification is the ideal way of managing risk. Markowitz proves formally that the benefits of diversification depend on correlations/covariances among asset classes and not solely on their different stand alone risk. This, of course, also means that diversification is not only affected by the number of the selected asset classes in a portfolio;
- investors act (or should act), when taking decisions to allocate wealth, in a two dimensional space. Markowitz wrote (1952): "the investor does (or should) consider expected return a desirable thing and variance of return an undesirable thing". This is a novel idea and obviously not consistent with the choice of selecting the portfolio with the maximum expected return that was previously recommended to his contributions.

Based on the second idea, the so called mean-variance criteria or rule is formulated: let A and B be two different portfolios, denote with μ_A and μ_B their expected return and with σ_A and σ_B their expected risk, then A is dominating B or must be preferred to B if $\mu_A \geq \mu_B$ and $\sigma_A \leq \sigma_B$ with at least one strong inequality satisfied.

Before illustrating the appropriate way of constructing portfolios according to the Markowitz model, we present the preliminary definitions and statistics that are necessary to correctly formulate the return and risk of a portfolio.

The portfolio weights are given in an $N \times 1$ vector ($\mathbf{w}$) where w_i is the percentage holding of asset class i. Note that the following condition must be satisfied: $\sum_{i=1}^{N} w_i = 1$.

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \cdots \\ \cdots \\ w_i \\ \cdots \\ \cdots \\ w_N \end{bmatrix}$$

We then denote with $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$, respectively, the vectors $N \times 1$ of the expected returns and standard deviations for the selected asset classes in the investment universe, where μ_i is the return of asset class i and σ_i is its risk.

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \cdots \\ \cdots \\ \mu_i \\ \cdots \\ \cdots \\ \mu_N \end{bmatrix} \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \cdots \\ \cdots \\ \sigma_i \\ \cdots \\ \cdots \\ \sigma_N \end{bmatrix}$$

Finally, we summarize the relevant correlation coefficients in the $N \times N$ matrix **C**. We denote with $\boldsymbol{\Sigma}$ the corresponding covariance matrix. Obviously ρ_{ij} and σ_{ij} are the generic correlation and covariance parameters between asset class i and j such that $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$ and $\sigma_{ii} = \sigma_i^2$.

$$\mathbf{C} = \begin{bmatrix} \rho_{11} & \rho_{12} & \cdots & \cdots & \rho_{1i} & \cdots & \cdots & \rho_{1N} \\ \rho_{21} & \rho_{22} & \cdots & \cdots & \rho_{2i} & \cdots & \cdots & \rho_{2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \rho_{i1} & \rho_{i2} & \cdots & \cdots & \rho_{ii} & \cdots & \cdots & \rho_{iN} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \rho_{N1} & \rho_{N2} & \cdots & \cdots & \rho_{Ni} & \cdots & \cdots & \rho_{NN} \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \cdots & \sigma_{1i} & \cdots & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \cdots & \cdots & \sigma_{2i} & \cdots & \cdots & \sigma_{2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{i1} & \sigma_{i2} & \cdots & \cdots & \sigma_{ii} & \cdots & \cdots & \sigma_{iN} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \cdots & \sigma_{Ni} & \cdots & \cdots & \sigma_{NN} \end{bmatrix}$$

Using these notations and statistics, μ_P, the expected return of a portfolio, is given by:

$$\mu_P = \sum_{i=1}^{N} \mu_i w_i \tag{2.1}$$

that is equivalent, in matrix form, to the following expression:

$$\mu_P = \mathbf{w}'\boldsymbol{\mu} \tag{2.2}$$

The variance of the portfolio, σ_P^2, is computed, alternatively, by expressions (2.3) or by (2.4):

$$\sigma_P^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}\,;\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij}\,;\sum_{i=1}^{N}(w_i\sigma_i)^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} w_i w_j \sigma_{ij} \tag{2.3}$$

$$\sigma_P^2 = \mathbf{w}'\Sigma\mathbf{w} \tag{2.4}$$

According to Markowitz's works, the investor's asset allocation problem entails choosing the portfolio weights such that, for a given expected return target, the portfolio variance or standard deviation is kept to a minimum. The portfolios corresponding to this description are called *mean-variance efficient portfolios* because they present the best return and risk pairs. In practice these portfolios must be sought by running an optimization algorithm usually referred to as *Mean-Variance Optimization*. Mathematically, it consists of three fundamental components. The first one is the objective function given by the generic formulation of the portfolio variance, thus it is a quadratic objective function to be minimized. The second element is the set of the unknown variables representing the optimal portfolio weights to identify and upon which the objective function is dependent. The last component is the set of constraints. Naturally, this includes the return constraint. Beyond this, also equality and inequality constraints are included respectively to impose that portfolio weights add up to one (referred to as budget constraint) and to restrict short selling (referred to as non-negativity constraint or long-only constraint).

In formal terms, the *Mean-Variance Optimization* proposed by Markowitz is written as follows:

$$\begin{aligned}
&\underset{w*}{Min}\left(\sum_{i=1}^{N}(w_i\sigma_i)^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij}\right)\\
&s.t.\\
&\sum_{i=1}^{N}\mu_i w_i = \mu_P^*\\
&\sum_{i=1}^{N} w_i = 1\\
&w_i \geq 0
\end{aligned} \tag{2.5}$$

Table 2.1 Inputs for Mean-Variance Optimization

Expected returns (%)	Standard deviations (%)		Asset class 1	Asset class 2	Asset class 3	Asset class 4
2.50	5.00	Asset class 1	1	0.40	0.30	0.10
3.00	5.50	Asset class 2	0.40	1	0.50	0.40
6.00	8.00	Asset class 3	0.30	0.50	1	0.80
7.50	10.00	Asset class 4	0.10	0.40	0.80	1

The same problem in matrix form is stated as follows:

$$\begin{aligned}
&\underset{\mathbf{w}*}{Min}\, \mathbf{w}'\Sigma\mathbf{w} \\
&s.t. \\
&\mathbf{w}'\boldsymbol{\mu} = \mu_P^* \\
&\mathbf{w}'\mathbf{e} = 1 \\
&[w] \geq 0
\end{aligned} \tag{2.6}$$

where $\mathbf{e}$ is a vector $N \times 1$ of ones.

Given the type of restrictions imposed on portfolio weights, the optimization problem above cannot be solved analytically. Therefore solutions for $w*$ are found using numerical techniques that are of an iterative nature.

Now that the classical *Mean-Variance Optimization* has been underlined, it is worthwhile looking at an example. For this purpose, we consider an investment universe of four asset classes. Their expected returns, standard deviations and correlations are summarized in Table 2.1. The position that asset classes have in a risk-return space is marked with a circle in Fig. 2.1. Our first action is to randomly simulate 1000 portfolios and plot them in Fig. 2.1 as well. In this way we are able to provide an approximate illustration of the feasible portfolios. Next, we run Markowitz's algorithm (for different choices of μ_P^*) and obtain the set of the optimized portfolios. As previously stated, the optimized portfolios are called *mean-variance efficient portfolios* given that we have found the combination of assets that, for a particular level of risk, is expected to give the highest return or, equivalently, the combination of assets that, for a particular level of expected return, bears the lowest risk.

The set of all the optimized portfolios is shown by the concave function in Fig. 2.1 that is called *Mean-Variance Efficient Frontier* or simply *Efficient Frontier*.[1] We observe that our asset classes are (with one exception) below the

[1]For further information, we observe that the same *Efficient Frontier* can be derived using different but equivalent formulations of the *Mean-Variance Optimization* problem. We refer to the expected

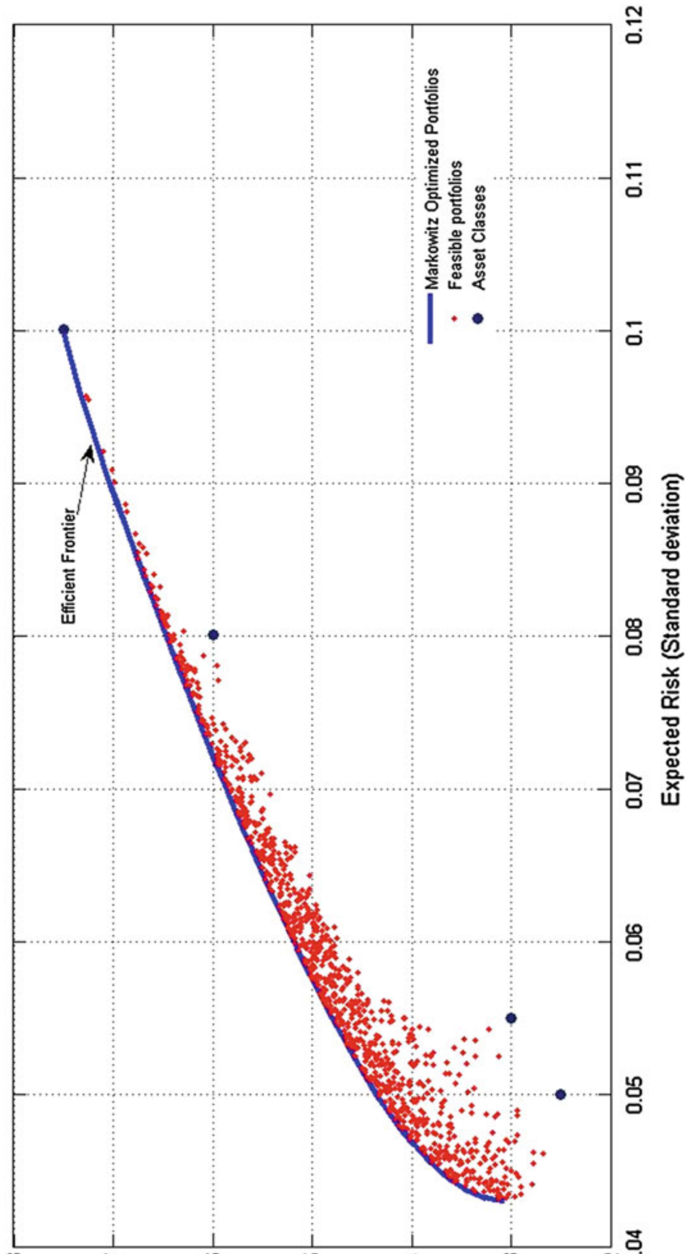

Fig. 2.1 Feasible portfolios and Mean-Variance Efficient Frontier

Efficient Frontier and that the dominating portfolios computed by the *Mean-Variance Optimization* allow a significant improvement of the risk-return profile with respect to the individual constituents.

2.2 An Analysis of Markowitz's Portfolios: Drawbacks and Motivations

The *Mean-Variance Optimization* illustrated in Sect. 2.1 provides a clear, rational framework to face the portfolio construction problem of investors. Its application requires knowledge of both expected returns and covariance matrix of asset classes returns or, equivalently, of expected returns, risks and correlations.

In reality, these model parameters are not observable ex-ante, thus they must be estimated. This task is commonly accomplished with the download of time series of historical returns for the selected asset classes and then consequently with the statistical computation of the relevant parameters using past data. This means that the following sample estimates for expected returns ($\hat{\mu}_i$), risks ($\hat{\sigma}_i$) and covariances or correlations (respectively, $\hat{\Sigma}_{ij}$ or $\hat{\rho}_{ij}$) are considered:[2]

$$\hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} R_{t,i} \tag{2.7}$$

$$\hat{\sigma}_i = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\left(R_{t,i} - \hat{\mu}_i\right)^2} \tag{2.8}$$

$$\begin{aligned}\hat{\Sigma}_{ij} &= \frac{1}{T-1}\sum_{t=1}^{T}\left(R_{t,i} - \hat{\mu}_i\right)\cdot\left(R_{t,j} - \hat{\mu}_j\right) \\ \text{or } \hat{\rho}_{ij} &= \frac{\frac{1}{T-1}\sum_{t=1}^{T}\left(R_{t,i} - \hat{\mu}_i\right)\cdot\left(R_{t,j} - \hat{\mu}_j\right)}{\hat{\sigma}_i\hat{\sigma}_j}\end{aligned} \tag{2.9}$$

(Footnote 1 continued)

return maximization formulation and to the risk aversion formulation. In the first case, the formulation of the portfolio return becomes the objective function to maximize and the level of risk becomes a constraint. In the second case, the trade-off between risk and return is explicitly modelled using a utility function to maximize for different risk aversion coefficients.

[2]The exclusive consideration for these statistical moments shows that the multivariate normality is assumed by the Markowitz model. Alternatively, it can be assumed that investors have a quadratic utility function. The result is the same because in this case the utility is a function of the portfolio mean and standard deviation.

The standard implementation of the Markowitz model then usually follows an approach named by Kan and Zhou (2007) the "plug-in" rule. In practice, the estimated parameters are processed into the *mean-variance optimizer* that treats them as if they were the true parameters. Handling the asset allocation problem in this way has an important and serious implication: the procedure becomes totally deterministic as the parameters uncertainty is completely neglected. However, in the portfolio optimization context (as in many other financial situations) the rising of estimation risk is almost unavoidable. From a logical point of view, it can be defined as the (very real) possibility to make estimation errors that is to record a difference between any parameter's estimated value and that parameter's true value. Of course, we are not able to know a priori how far we are from the true parameters. As indicated by a wide number of authors, Fernandes et al. (2012), Drobetz (2001), Eichorn et al. (1998), Grauer and Shen (2000), Herold and Maurer (2006), Jobson and Korkie (1981), Jorion (1985, 1992), Michaud (1989), implementing the *Mean-Variance Analysis* without recognizing the existence of estimation risk inherent in the input parameters has a huge impact on the optimized portfolios and leads to several undesirable features and deficiencies. They can be summarized in the their counter-intuitive nature, instability, erroneously supposed uniqueness and poor out-of-sample performance.

With reference to the above mentioned, the issue deserving attention is the lack of diversification of optimal portfolios which paradoxically contrasts with one of the main ideas of Markowitz. Efficient allocations frequently completely exclude some asset classes of the chosen investment universe and give extremely large weights to some others. They also show sudden shifts along the *Efficient Frontier*. In addition, not so rarely the most northeast point of the *Mean-Variance Efficient Frontier* lacks completely diversification and is equivalent to a mono-asset portfolio.[3] These are consequences of the high discriminatory and impassive manner adopted by the portfolio construction method in dealing with data inputs which are considered extremely accurate. The mean-variance optimizer tends to heavily weigh those asset classes that show high estimated returns compared to low variances (or standard deviations) and negative correlations. These are also the asset classes that are most likely to suffer from large estimation errors. For this reason, Michaud (1989) wrote: "The unintuitive character of many optimized portfolios can be traced to the fact that MV optimizers are, in a fundamental sense, estimation-error maximizers".

The second major drawback of *mean-variance efficient portfolios* is their instability. This attribute defines the high sensitivity of portfolio allocations to small changes in the estimated parameters. Especially, in the case we have in the investment universe couples of asset classes with similar risk-return profile, small perturbations may completely alter the distribution of the portfolio weights because

[3]This situation is verified in the simple asset allocation example presented in Sect. 2.1. The extreme portfolio on the right side of the *Efficient Frontier* in Fig. 2.1 is 100 % composed of the fourth asset class.

they may cause an alternation between the dominant and the dominated asset class. This circumstance proves that optimized portfolios are not reasonable under different set of plausible input parameters. Since they are estimated from historical observation of asset classes' returns, the occurrence of perturbations must be taken for granted and therefore the problem of instability has a practical impact.[4] The literature (Best and Grauer 1991) has suggested that especially changes in expected returns tend to impact dramatically on optimal portfolio weights.

The third limitation consists of failing to recognize the non-uniqueness of optimal portfolios. As sharply noted by Michaud (1989) "Optimizers, in general, produce, a unique optimal portfolio for a given level of risk. This appearance of exactness is highly misleading, however. The uniqueness of the solution depends on the erroneous assumption that the inputs are without statistical estimation error". Hence, in practice, it would be helpful, in order to reach the goal of identifying recommended portfolio structures, to consider many statistically equivalent portfolios and take the average weights resulting from their different compositions.[5]

The last but potentially the most serious drawback affecting *mean-variance efficient portfolios* is the poor out-of-sample performance: outside the sample period used to estimate input parameters, classical Markowitz's portfolios show a considerable deterioration of performance with respect to the expected one and the same occurs in terms of risk-adjusted performance. As observed by DeMiguel et al. (2009) and Jobson and Korkie (1981), the extent to which they fall short of the original targeted performance is such that very simple investment criteria can dominate *Mean-Variance Optimization*. Thus, this means that the "plug-in rule" cannot be validated.

Ignoring the possible misspecification of the input parameters has major consequences especially if we refer to expected returns. Sample means are subject to large fluctuations and their predictive power seems poor. The errors in variances and covariances/correlations are considered less costly than estimation errors in the expected returns. Chopra and Ziemba (1993) found that errors in mean are at least 10 times as important as errors in variances and errors in variances are about twice as important as errors in covariances.

To prove the impact of fluctuations of sample estimates on the identification of the optimal portfolios, we consider an investment universe including 9 asset classes[6]

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[4]Perturbations can simply result from the use of a rolling-window approach to get estimates of input parameters from historical returns.

[5]Michaud and Michaud (2008) propose the Resampled Efficiency technique. It allows, by means of Monte Carlo methods, to implement a simulation approach that draws numerous random samples of returns from a multivariate normal distribution described by the vector of means and covariance matrix derived from the actual sample of historical returns. Different sets of new input parameters can then be calculated from the simulated sample of returns and based on them many statistically equivalent portfolios can be obtained.

[6]They are represented by the following benchmarks: Jpm Euro Cash 12 M, Jpm Emu Government All Maturities, CGBI WGBI World non Euro All Maturities, Bofa ML Global Broad Corporate, MSCI Emu, MSCI North America, MSCI Pacific ex Japan Free, MSCI Emerging Markets.

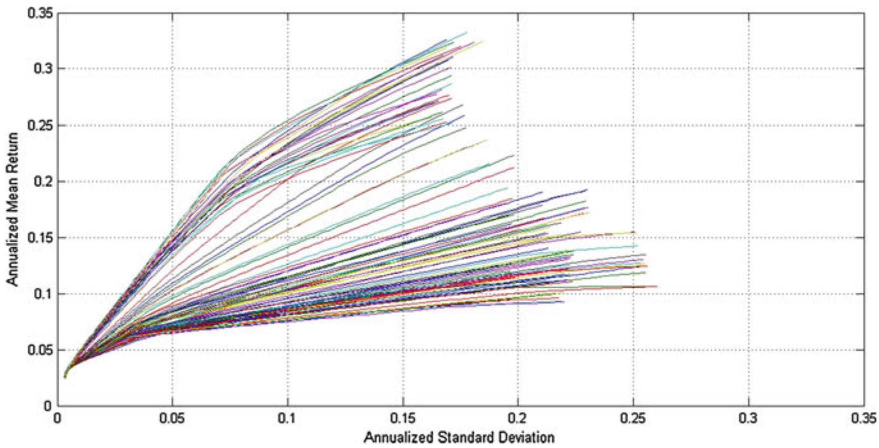

Fig. 2.2 Analysis of the fluctuating behaviour of historical Mean-Variance Efficient Frontiers with 10 years estimation window

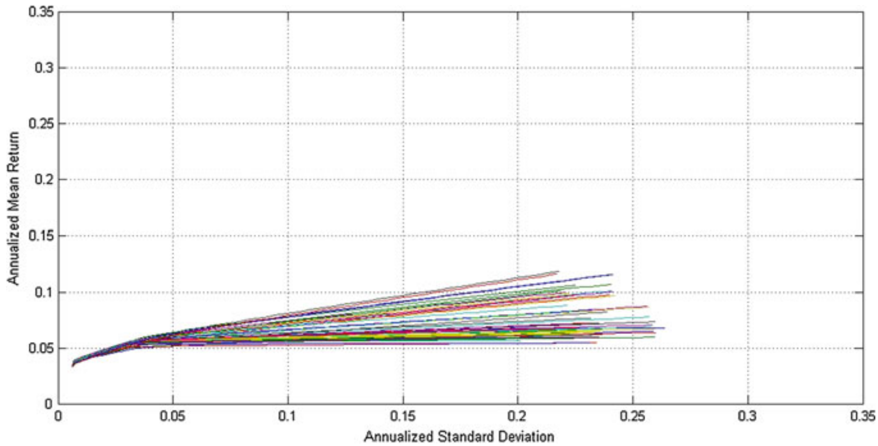

Fig. 2.3 Analysis of the fluctuating behaviour of historical Mean-Variance Efficient Frontiers with 12 years estimation window

for which 216 euro-denominated monthly returns are available from January 1997 to December 2014. We calculate sample estimates using a rolling estimation window of 120 months (10 years) and then we run the *Mean-Variance Optimization* exploiting the "plug-in" rule. We obtain the 97 historical or classical *Efficient Frontiers* shown in Fig. 2.2; each frontier consists of 100 optimized portfolios. We note that the range of return displayed in the historical *Efficient Frontiers* goes from 2.45 to 33.24 % while the range displayed by risk is from 0.30 to 26.01 %. Recalling our preceding consideration that uncertainty in the sample means is more critical than uncertainty in variances (or standard deviations) and covariances (or correlations), we repeat the exercise using a rolling estimation window of 144 months (12 years). In this case, we obtain the 73 historical *Efficient Frontiers* represented in Fig. 2.3 that shows, relatively to Fig. 2.2, a strong alteration especially in the range of returns.

It must be emphasized that instability in the optimal risk-return pairs has a tendency to decrease with an increase in the number of the sample observations in the rolling window and has a tendency to increase with size of the investment universe. This means the historical *Efficient Frontiers* would appear more/less compacted should we use a small/large set of asset classes and a larger/smaller estimation window.

Another way of demonstrating the impact of fluctuations in estimated input parameters is to by illustrating the composition of portfolios that are associated to the same rank relative to the 100 portfolios computed in each of the 97 historical *Efficient Frontiers* of our first application of the "plug-in" rule. We propose this exercise with reference to portfolios ranked as 25th and 50th. Figure 2.4 provides their 97 asset allocations. This confirms the important differences in portfolio weights using rolling samples of historical data.

The analysis and the discussion developed in this section demonstrate the severe drawbacks suffered by the Markowitz portfolios that prevent them from having an investment value and from being externally marketable by the asset managers. The motivation for these limitations is not a fault in the analytical formulation of the optimization algorithm or any other intrinsic or theoretical error. The cause is of an operative nature and refers to the way the optimizer uses investment information delivered by means of the optimization inputs. It works as if it had exactly the correct inputs available without accounting for any estimation error. Therefore, the fundamental problem is that the level of power of the optimization procedure in managing investment information is far greater than the quality of the inputs. In other words, we observe an over-fitting of inaccurate information represented by sample estimates. Michaud and Michaud (2008) contributed to this argument writing: "The procedure overuses statistically estimated information and magnifies the impact of estimation errors. It is not simply a matter of garbage in, garbage out, but rather a molehill of garbage in, a mountain of garbage out".

Given the implications of the presence of estimation risk for investment portfolios, several attempts have been made to overcome them. From the mid 1980s onwards, alternative methodologies have been developed to deal with the problem of parameters uncertainty. They can be distinguished between heuristic and Bayesian approaches. The first group includes using the Resampling technique patented by Michaud and Michaud (2008) and the application of additional weight constraints (besides the classical long-only and budget constraints) suggested by Frost and Savarino (1988) and Jagannathan and Ma (2003). Bayesian approaches can be implemented in different ways. One application consists in the use of shrinkage estimators that propose estimation of expected returns by shrinking the sample mean toward a common mean (Jorion 1985). A second application is represented by the

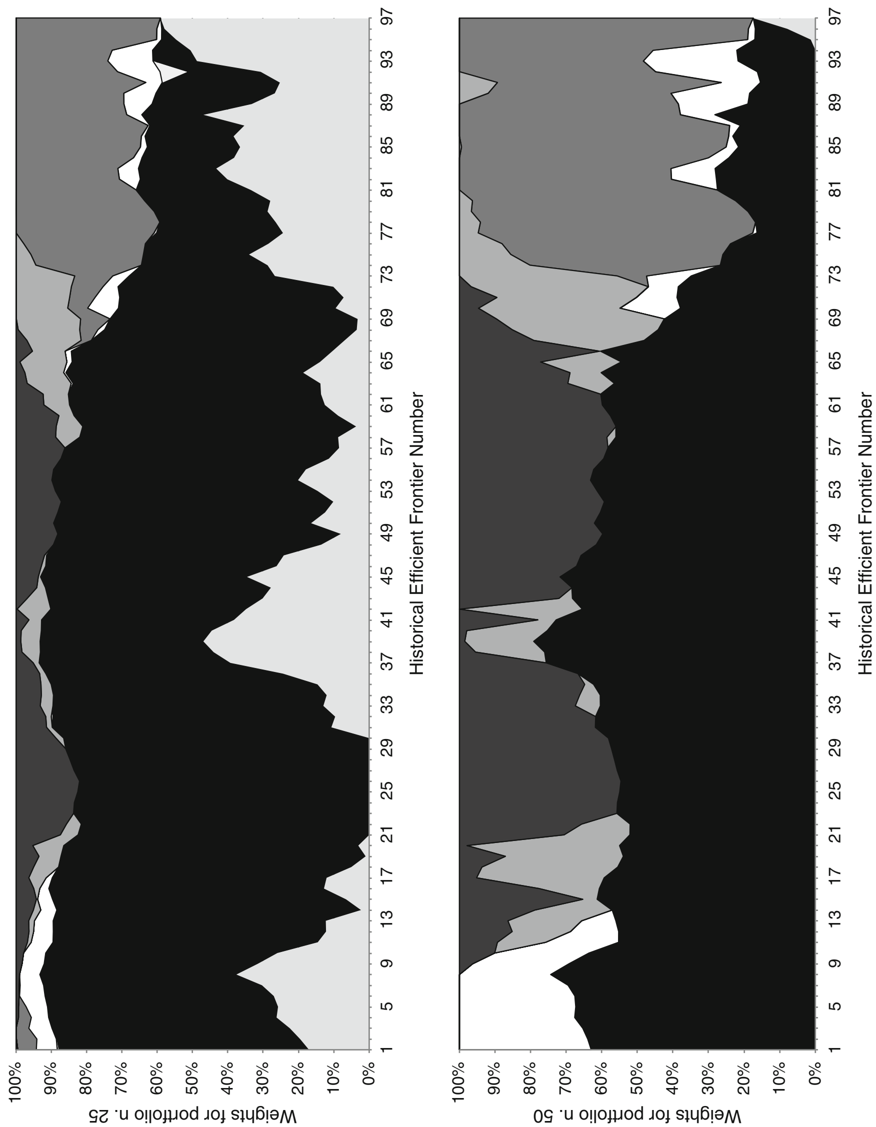

Fig. 2.4 Composition for portfolios number 25 and 50 of the historical Efficient Frontiers

Black and Litterman model (1992) which combines two sources of information (equilibrium returns and investor's views) to obtain predictive returns.

In practice, heuristic approaches and Bayesian approaches take different actions in response to the impact of the unavoidable imprecision of estimated inputs, especially of expected returns, on optimal portfolios construction. The former acts on the model side to make the optimization process less deterministic or to force it to produce more diversified portfolios. The latter acts on the estimation side to improve expected returns estimates with respect to their sample estimates. Obviously, it also makes sense to use methodologies that combine features of heuristic and Bayesian approaches as proposed by Fernandes et al. (2012).

In conclusion, it is worthwhile emphasizing that neither heuristic approaches nor Bayesian approaches modify the classical *mean-variance portfolio optimization* framework. They try to make its response more reliable and robust by making some improvements concerning how inputs are used or produced.

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Chapter 3
Risk-Based Approaches to Asset Allocation: The Case for Risk Parity

Abstract This chapter initially underlines the distinguishing features of the so called risk-based strategies for asset allocation in comparison with the *Mean-Variance Analysis* and proposes two criteria which highlight possible discrepancies among risk-based strategies. This is followed by a detailed illustration and interpretation of the theoretical concepts and tools of risk budgeting literature used to set up or analyse risk-based portfolios. We take into consideration the marginal risk, the total risk contribution and the percentage total risk contribution. The rest of the chapter offers an in depth discussion of the risk parity strategy, starting from the rudimental version, also known as naïve risk parity or inverse volatility strategy, to the optimal one which can really build a portfolio such that risk contributions from the different asset classes to the portfolio overall risk are equalized. We explore the use of leverage combined with a risk parity strategy. The chapter concludes giving attention to the potential evolutions of the risk parity strategy. In particular, we look at points of attractiveness and shortcomings of a potential new version of risk parity strategy that considers "risk factors" rather than asset classes as the building blocks of a portfolio construction approach.

Keywords μ-free strategies · Euler's principle · Inverse volatility strategy · Levered strategy · Marginal risk · Risk allocation · Risk budgeting · Risk contribution · Risk parity · Risk factors

3.1 The Distinguishing Characteristics of the Risk-Based Approaches

The practical problems of the Markowitz's approach for asset allocation described in Chap. 2 have given rise to numerous attempts both from academics and practitioners to address them. Section 2.2 focused on the Bayesian and heuristic approaches that have taken place to deal with estimation errors, especially in

M.D. Braga, *Risk-Based Approaches to Asset Allocation*,
SpringerBriefs in Finance, DOI 10.1007/978-3-319-24382-5_3

expected returns, given that we proved they strongly influence the resulting optimal portfolios structure causing very sensitive and extreme allocations.

We noted that these methods maintain the original Markowitz framework: they attempt to optimize the trade-off between the mean and the standard deviation (variance) of portfolio returns. Also the set of the required optimization inputs is unchanged relative to Markowitz's model.

The global financial crisis which started in 2008, has brought a novelty in this field of asset management. It has promoted several efforts to elaborate new approaches or to revive existing approaches for portfolio construction that remove the mean-variance setting. They are collectively called risk-based asset allocation approaches or strategies. The fundamental and common characteristic across portfolio compositions recommended by these strategies is that they are obtained without including expected returns among the inputs. For this reason, risk-based asset allocation approaches have recently been labelled as *μ-free strategies*, given that μ traditionally denotes expected returns (Braga 2015).

In light of the criticism of the Markowitz model regarding excessive exposure to estimation risk in expected returns, risk-based asset allocation strategies have reacted by giving less room to estimation errors by making the portfolio construction totally insensitive to expected returns estimates through their elimination. Indeed, their technical implementation does not require any explicit or implicit effort to model, measure or predict returns.

In absence of the performance dimension, the *μ-free strategies* tend to place at the heart of the portfolio construction process the risk dimension as documented in Chaps. 3 and 4. In other words, they focus their efforts on diversifying or minimizing portfolio risk; the latter goal represents the reason why sometimes these strategies are also mentioned as low-volatility portfolio construction methods (Chow et al. 2014).

A further difference between the family of risk-based strategies and the classical *Mean-Variance Optimization* is that implementing the former does not allow for the construction of a set of recommended portfolios for different desired levels of risk but only one recommended asset allocation solution. Therefore, the targeting of a specified risk level requires the asset manager or the investor to understand the portion or amount of funds to borrow or lend.

Different solutions corresponding to the above description have been suggested in finance literature and have received increasing attention in the marketplace. They include:

- the risk parity strategy;
- the equally-weighted strategy;
- the minimum-variance strategy;
- the most diversified portfolio strategy.

The first one is comprehensively explored in this chapter while the remaining strategies are examined in Chap. 4.

On the one hand the risk-based strategies share the distinguishing characteristics relative to the classical mean-variance framework just highlighted, but on the other hand they differentiate from each other when considering additional aspects.

Table 3.1 Risk-based strategies comparison

Risk-based strategy	Portfolio construction technique	Investment universe inclusion in the portfolio
Risk parity	Objective function optimization	Total inclusion
Equally-weighted	Heuristic rule-based	Total inclusion
Minimum-variance	Objective function optimization	Partial inclusion
Most diversified portfolio	Objective function optimization	Partial inclusion

Two are worth considering. The first refers to the tools required to implement a particular strategy and the second refers to the degree of inclusion in the suggested portfolios of the asset classes belonging to the specified investment universe. According to the first criterion, we can distinguish between risk-based strategies that use an objective function to be optimized and risk-based strategies that rely on a heuristic rule to build portfolios. The second criterion makes a distinction between *μ-free strategies* that include in the recommended portfolio a portion of the asset classes in the investment universe and those that include all of them. Table 3.1 summarizes these differentiating aspects for the cited risk-based strategies.

3.2 The Theoretical Background and Argument for Risk Parity

The first risk-based strategy we are going to examine in the next sections of this chapter is risk parity. This strategy recommends to build the portfolio so that its risk is equally distributed among asset classes: thus attention is primarily devoted to risk allocation and dollar or wealth allocation among asset classes is consequent.

In light of this premise, we need to learn how to attribute total portfolio risk to its individual asset class exposures or, said differently, how to answer the question: how much asset class i contributes to overall risk? In order to do this, we have to analyse and then implement the tools for risk decomposition provided by the literature on risk budgeting. However, a good starting point is to focus on Euler's theorem or principle given that it provides the appropriate background for such tools (Denault 2001; Tasche 2008; Roncalli 2014).

According to Euler's principle, if RM is a generic risk measure that can be expressed as a continuously differentiable function of a weights vector, $RM = f(\mathbf{w})$, and it is homogeneous of degree 1 (or linearly homogeneous) in the sense that the following equation holds for all $\lambda > 0$:

$$RM(\lambda \mathbf{w}) = \lambda RM(\mathbf{w}) \tag{3.1}$$

and therefore increasing (for instance, duplicating) or decreasing (for instance, halving) the scale of the portfolio increases or decreases the risk measure in the same magnitude, then the risk measure satisfies the equation:

$$RM(\mathbf{w}) = \sum_{i=1}^{N} w_i \cdot \frac{\partial RM}{\partial w_i} \tag{3.2}$$

The addends in Eq. (3.2), usually called Euler contributions, are already risk contributions. They satisfy a full allocation property which is fundamental for the design of risk parity strategy and to understand the distribution among different portfolio constituents of the overall risk.

On the basis of the Euler's principle, it is possible to identify the two fundamental tools that are needed when carrying out a risk decomposition together with the respective symbols:

- MR_i, the marginal risk or (more extensively) the marginal risk contribution for each asset class exposure i defined as $\frac{\partial RM}{\partial w_i}$, that is the first derivative of the selected portfolio risk measure with respect to its weight w_i;
- TRC_i, the risk contribution sometimes also mentioned as component risk or total risk contribution of asset class i defined as $w_i \cdot \frac{\partial RM}{\partial w_i}$, that is the product of the allocation to asset class i with its marginal risk.

Having given definitions both for marginal risk and risk contribution, we can now provide their interpretation.

The marginal risk tells us the variation caused in the portfolio risk by an infinitesimal change in an asset class weight under the implicit assumption that each position's change is funded from cash.[1] The above interpretation as the sensitivity of the risk measure to a small movement of w_i can be written formally as:

$$MR_i = \lim_{\Delta w_i \to 0} \frac{RM(w_1, w_2, \ldots, w_i + \Delta w_i, \ldots, w_N) - RM(w_1, w_2, \ldots, w_i, \ldots, w_N)}{\Delta w_i} \tag{3.3}$$

where Δw_i is an increment in w_i. If Δw_i is small enough (infinitesimal), we infer that:

$$RM(w_1, w_2, \ldots, w_i + \Delta w_i, \ldots, w_N) = RM(w_1, w_2, \ldots, w_i, \ldots, w_N) + \Delta w_i \frac{\partial RM}{\partial w_i} \tag{3.4}$$

The risk contribution or the total risk contribution is the load on total risk attributable to the position w_i. As shown before, it is simply defined as the individual weight of asset class i times MR_i:

[1]The assumption of cash funding in calculating the marginal risk seems strong. However, we have to admit it is more acceptable in the context of finding a policy portfolio that is a strategic asset allocation solution than it would be in the context of tactical asset allocation or active portfolio management. On the issue, see Scherer (2015).

$$TRC_i = w_i \cdot \frac{\partial RM}{\partial w_i} \tag{3.5}$$

Since risk contributions add up to the overall portfolio risk, dividing the component risk by the total risk leads to the percentage total risk contribution for each asset class exposure, $PTRC_i$. Formally, we have:

$$PTRC_i = \frac{w_i \cdot \frac{\partial RM}{\partial w_i}}{RM} = \frac{TRC_i}{RM} \tag{3.6}$$

As the standard deviation is the traditional measure of portfolio risk, we propose the definition of the concepts described before with reference to this standard measure. As a starting point, we consider a portfolio consisting of just two asset classes A and B. In this case we have the following portfolio standard deviation:

$$\sigma_P = \sqrt{(\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A \sigma_B w_A w_B \rho_{A,B}} \tag{3.7}$$

To determine the marginal risk of A we take the first derivative with respect to w_A that is given by:

$$\begin{aligned} MR_{\sigma\_A} = \frac{\partial \sigma_P}{\partial w_A} &= \frac{2\sigma_A^2 w_A + 2\sigma_A \sigma_B w_B \rho_{A,B}}{2\sqrt{(\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A \sigma_B w_A w_B \rho_{A,B}}} \\ &= \frac{\sigma_A^2 w_A + \sigma_A \sigma_B w_B \rho_{A,B}}{\sqrt{(\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A \sigma_B w_A w_B \rho_{A,B}}} \\ &= \frac{\sigma_A^2 w_A + cov_{A,B} w_B}{\sqrt{(\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A \sigma_B w_A w_B \rho_{A,B}}} \end{aligned} \tag{3.8}$$

For the marginal risk of B, we have:

$$\begin{aligned} MR_{\sigma\_B} = \frac{\partial \sigma_P}{\partial w_B} &= \frac{2\sigma_B^2 w_B + 2\sigma_A \sigma_B w_A \rho_{A,B}}{2\sqrt{(\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A \sigma_B w_A w_B \rho_{A,B}}} \\ &= \frac{\sigma_B^2 w_B + \sigma_A \sigma_B w_A \rho_{A,B}}{\sqrt{(\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A \sigma_B w_A w_B \rho_{A,B}}} \\ &= \frac{\sigma_B^2 w_B + cov_{A,B} w_A}{\sqrt{(\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A \sigma_B w_A w_B \rho_{A,B}}} \end{aligned} \tag{3.9}$$

Since the risk contribution is the product of each weight with the respective marginal risk, for the two-asset case we can write:

$$TRC_{\sigma\_A} = w_A \cdot \frac{\partial \sigma_P}{\partial w_A} = \frac{\sigma_A^2 w_A^2 + cov_{A,B} w_A w_B}{\sqrt{(\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A \sigma_B w_A w_B \rho_{A,B}}} \quad (3.10)$$

$$TRC_{\sigma\_B} = w_B \cdot \frac{\partial \sigma_P}{\partial w_B} = \frac{\sigma_B^2 w_B^2 + cov_{A,B} w_A w_B}{\sqrt{(\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A \sigma_B w_A w_B \rho_{A,B}}} \quad (3.11)$$

with TRC_{σ_A} and TRC_{σ_B} satisfying the equality:

$$\sigma_P = TRC_{\sigma\_A} + TRC_{\sigma\_B} \quad (3.12)$$

The definition of the concepts that were just described admits a natural extension to portfolios with N asset classes (where $N > 2$). We first present the generalization of the marginal risk of asset class i for the portfolio volatility (MR_{σ_i}) by Eq. (3.13):

$$MR_{\sigma\_i} = \frac{\partial \sigma_P}{\partial w_i} = \frac{\sigma_i^2 w_i + \sum_{j \neq i}^{N} w_j \sigma_i \sigma_j \rho_{ij}}{\sigma_P} = \frac{\sigma_i^2 w_i + \sum_{j \neq i}^{N} w_j \sigma_{ij}}{\sigma_P} = \frac{\sum_{j=1}^{N} w_j \sigma_{ij}}{\sigma_P} \quad (3.13)$$

where $j = 1,\ldots, i,\ldots, N$ in the last step of the formula.

According to (3.13), the marginal risk for i depends both on the asset class weight and on the covariances between i and each j weighted by the "size" of j. Therefore we can infer that it may happen that an increase in w_i doesn't result in rising portfolio risk if asset class i has negative covariance with the other asset classes and their relative weight is not too negligible.

It is now straightforward to extend the definition of total risk contribution for asset class i (TRC_{σ_i}):

$$\begin{aligned} TRC_{\sigma\_i} &= w_i \cdot \frac{\partial \sigma_P}{\partial w_i} = w_i \cdot \frac{\sigma_i^2 w_i + \sum_{j \neq i}^{N} w_j \sigma_i \sigma_j \rho_{ij}}{\sigma_P} = w_i \cdot \frac{\sigma_i^2 w_i + \sum_{j \neq i}^{N} w_j \sigma_{ij}}{\sigma_P} \\ &= w_i \cdot \frac{\sum_{j=1}^{N} w_j \sigma_{ij}}{\sigma_P} = \frac{\sum_{j=1}^{N} w_j w_i \sigma_{ij}}{\sigma_P} \end{aligned} \quad (3.14)$$

In case we use the matrix notation, we can summarize the marginal risks in a vector (the gradient) $N \times 1$ given by:

$$\nabla \mathbf{MR}_\sigma = \frac{\boldsymbol{\Sigma} \mathbf{w}}{\sqrt{\mathbf{w}' \boldsymbol{\Sigma} \mathbf{w}}} \quad (3.15)$$

while the single marginal risk is defined as follows:

$$MR_{\sigma\_i} = \frac{(\boldsymbol{\Sigma}\mathbf{w})_i}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}} \tag{3.16}$$

where $(\boldsymbol{\Sigma}\mathbf{w})_i$ denotes the ith row of the column vector resulting from the product of $\boldsymbol{\Sigma}$ with $\mathbf{w}$.

The risk contribution of the ith asset class in matrix form is then:

$$TRC_{\sigma\_i} = w_i \cdot \frac{(\boldsymbol{\Sigma}\mathbf{w})_i}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}} \tag{3.17}$$

Finally, on the basis of the full allocation property, we can show the generalized version of the risk (standard deviation) of the portfolio as the sum of the total risk contributions:

$$\sigma_P = \sum_{i=1}^{N} TRC_{\sigma\_i} = \mathbf{w}' \frac{\boldsymbol{\Sigma}\mathbf{w}}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}} = \mathbf{w}'\nabla\mathbf{MR}_\sigma \tag{3.18}$$

The necessary tools for the development of the risk parity strategy we will see in the next sections now have been provided.[2]

Since the strategy relies on the risk allocation argument as opposed to the traditional dollar allocation, at this point we need to conclude giving evidence, as remarked by Qian (2005, 2011), that seemingly well-diversified or well-balanced allocations can hide concentrated risk exposures.

[2]The concepts and measurements of marginal risks and total risk contributions can also be referred to different measures for portfolio risk, for example Value at Risk (*VaR*) and Expected Shortfall (*ES*). Assuming a Gaussian context and a parametric estimation approach, such risk measures are strongly related to volatility and so the extension is simple. For clarity on this point, we note that marginal risks for position i in case *VaR* or *ES* measures are used with a confidence level α (for example 95 % or 99 %) can be written under the zero-mean hypothesis as follows, respectively:

$$MR_{VaR\_i} = N^{-1}_{1-\alpha} \cdot \frac{\sum_{j=1}^{N} w_j \sigma_{ij}}{\sigma_P}$$

$$MR_{ES\_i} = \frac{1}{1-\alpha} \cdot \frac{\sum_{j=1}^{N} w_j \sigma_{ij}}{\sigma_P} \cdot \phi\left(N^{-1}_{1-\alpha}\right)$$

where N^{-1}_{α} is the inverse of the standard normal cumulative distribution function and ϕ indicates the probability density function of a Normal distribution. Moving from marginal risks to risk contributions, as usual, just requires the product with w_i. We observe that both marginal risks and risk contributions are different from the ones obtained in the decomposition of portfolio volatility, but what remains unchanged are the percentage total risk contributions. Considering our notation, we assume VaR and ES are expressed as losses keeping their natural sign.

Let us imagine that a European investor has a portfolio that allocates 60 % of its capital to the Eurozone government bonds and 40 % of its capital to world equities: we assume these investments can be represented respectively by the JPM Emu Government All Maturities index and by the MSCI World Index. From January 2005 to December 2014 an annualized standard deviation of 4.02 and 12.47 % was calculated for the two indices considering euro-denominated returns. The correlation between the two was −0.12. The portfolio might have appeared quite well distributed between its constituents but it has to be assessed from the perspective of risk allocation. For this purpose, we display in Table 3.2, in addition to our data, also the results from a risk decomposition exercise concerning the 5.27 % annualized portfolio standard deviation.

The table indicates that the portfolio 60 % bonds and 40 % equity in terms of asset allocation was 15.73 % bonds and 84.27 % equity portfolio in terms of risk allocation. So, from this perspective, it was a highly concentrated portfolio. The equity component dominated the risk of the portfolio even if it was a smaller fraction of the aforementioned.

3.3 The Naïve Risk Parity Strategy

The simplest and most rudimental form of risk parity strategy is commonly referred to as naïve risk parity (Lee 2011; Maillard et al. 2010). This denomination can be explained by the simplified way used to identify the weight to allocate to each asset class. As a matter of fact, recommended weights are obtained without considering correlation information among asset classes and this makes naïve risk parity computationally very attractive.

Table 3.2 An example of risk decomposition for portfolio standard deviation

Parameters	Parameters computation
$w_{JPM\ Emu\ Gov.\ All\ Mats}$	60 %
$w_{MSCI\ World}$	40 %
$\sigma_{JPM\ Emu\ Gov.\ All\ Mats}$	4.02 %
$\sigma_{MSCI\ World}$	12.47 %
$MR_{\sigma\_JPM\ Emu\ Gov.\ All\ Mats}$	$\frac{4.02\,\%^2 \cdot 60\,\% + 4.02\,\% \cdot 12.47\,\% \cdot 40\,\% \cdot (-0.12)}{\sqrt{(4.02\,\% \cdot 60\,\%)^2 + (12.47\,\% \cdot 40\,\%)^2 + 2 \cdot 4.02\,\% \cdot 12.47\,\% \cdot 60\,\% \cdot 40\,\% \cdot (-0.12)}} = 0.013824$
$MR_{\sigma\_MSCI\ World}$	$\frac{12.47\,\%^2 \cdot 40\,\% + 4.02\,\% \cdot 12.47\,\% \cdot 60\,\% \cdot (-0.12)}{\sqrt{(4.02\,\% \cdot 60\,\%)^2 + (12.47\,\% \cdot 40\,\%)^2 + 2 \cdot 4.02\,\% \cdot 12.47\,\% \cdot 60\,\% \cdot 40\,\% \cdot (-0.12)}} = 0.111103$
$TRC_{\sigma\_JPM\ Emu\ Gov.\ All\ Mats}$	$60\,\% \cdot 0.013824 = 0.83\,\%$
$TRC_{\sigma\_MSCI\ World}$	$40\,\% \cdot 0.111103 = 4.44\,\%$
$PTRC_{\sigma\_JPM\ Emu\ Gov.\ All\ Mats}$	$0.83\,\%/5.27\,\% = 15.73\,\%$
$PTRC_{\sigma\_MSCI\ World}$	$4.44\,\%/5.27\,\% = 84.27\,\%$

In detail, the rule it follows for computing portfolio weights suggests an exposure for each asset class proportional to the inverse of its standard deviation; it is then normalized to guarantee the weights sum to 1. Accordingly, the analytical expression for asset class weight w_i is written as:

$$w_i = \frac{\sigma_i^{-1}}{\sum_{j=1}^{N} \sigma_j^{-1}} = \frac{\frac{1}{\sigma_i}}{\sum_{j=1}^{N} \frac{1}{\sigma_j}} \tag{3.19}$$

As a consequence of the Eq. (3.19), the higher (lower) is the volatility of asset class i, the lower (higher) is its weight in the portfolio. The rule used to implement the strategy motivates the alternative denomination of naïve risk parity as inverse volatility strategy.

In spite of its attractive computational features, it seems unlikely that naïve risk parity provides a true homogeneity in asset class contributions to the portfolio standard deviation. Indeed, there are just two cases where the naïve risk parity portfolio is consistent and equivalent with the optimal risk parity portfolio, that is it provides a perfect balance in terms of risk contributions:

- in the case of a bivariate investment universe, thus in case the portfolio constituents are just two;
- in the case of uniform correlations for every possible couples of asset classes.

It is intuitive that under these conditions, ignoring correlations has no impact.

In order to easily understand that equal risk contributions are obtained in the two-asset class case without taking into account correlation, let us look at the example illustrated in Sect. 3.1. In this case, the unique solution the naïve risk parity would propose is given by:

$$\begin{bmatrix} w_{JPM\ Emu\ Gov.All\ Mats} \\ w_{MSCI\ World} \end{bmatrix} = \begin{bmatrix} \dfrac{1/4.02\%}{1/4.02\% + 1/12.47\%} \\ \\ \dfrac{1/12.47\%}{1/4.02\% + 1/12.47\%} \end{bmatrix} = \begin{bmatrix} 75.6\% \\ 24.4\% \end{bmatrix} \tag{3.20}$$

With this structure, the portfolio risk goes from 5.27 to 4.033 %. Table 3.3 summarizes the main figures concerning risk decomposition.

Table 3.3 Risk decomposition for naïve risk parity portfolio in a two-asset class case

	JPM Emu Gov. All Mats	MSCI World
Naïve risk parity portfolio weights (%)	75.6	24.4
Marginal risk	0.026666	0.082717
Risk contribution (%)	2.016	2.016
Percentage risk contribution (%)	50	50

These results show that for the two-asset class case the naïve risk parity solution for portfolio allocations to A and B can be written as:

$$\begin{bmatrix} w_A \\ w_B \end{bmatrix} = \begin{bmatrix} \dfrac{\sigma_A^{-1}}{\sigma_A^{-1}+\sigma_B^{-1}} \\ \dfrac{\sigma_B^{-1}}{\sigma_A^{-1}+\sigma_B^{-1}} \end{bmatrix} = \begin{bmatrix} \dfrac{1/\sigma_A}{1/\sigma_A+1/\sigma_B} \\ \\ \dfrac{1/\sigma_B}{1/\sigma_A+1/\sigma_B} \end{bmatrix} \tag{3.21}$$

By presenting w_A simply as w and w_B as (1-w), the allocations given in (3.21) are the only ones that satisfy the condition that the two risk contributions written in (3.22) are equal:

$$\frac{1}{\sigma_P}\begin{cases} \sigma_A^2 w^2 + w(1-w)\sigma_A\sigma_B\rho_{A,B} \\ \sigma_B^2(1-w)^2 + w(1-w)\sigma_A\sigma_B\rho_{A,B} \end{cases} \tag{3.22}$$

By illustrating the case of constant correlations, which means $\rho_{ij} = \rho\ \forall i,j$, we note that the marginal risk of asset class i becomes:

$$MR_{\sigma\_i} = \frac{\sigma_i^2 w_i + \rho \sum_{j\neq i}^{N} w_j\sigma_i\sigma_j}{\sigma_P} \tag{3.23}$$

Then we get the total risk contribution from asset class i just multiplying (3.23) by w_i:

$$TRC_{\sigma\_i} = \frac{\sigma_i^2 w_i^2 + \rho \sum_{j\neq i}^{N} w_j w_i\sigma_i\sigma_j}{\sigma_P} \tag{3.24}$$

We can rearrange (3.24) and obtain the following expression:

$$TRC_{\sigma\_i} = \frac{\sigma_i w_i\Big((1-\rho)\sigma_i w_i + \rho\sum_j^N w_j\sigma_j\Big)}{\sigma_P} \tag{3.25}$$

Considering the relationship that should exist among risk contributions:

$$\frac{\sigma_i w_i\big((1-\rho)\sigma_i w_i + \rho\sum_{h=1}^N w_h\sigma_h\big)}{\sigma_P} = \frac{\sigma_j w_j\big((1-\rho)\sigma_j w_j + \rho\sum_{h=1}^N w_h\sigma_h\big)}{\sigma_P} \tag{3.26}$$

it is easy to deduce that the fact it is verified depends on the equality between $\sigma_i w_i$ and $\sigma_j w_j$. Recalling the existence of the budget constraint, it is necessary that the generic weight w_i is inversely related to its volatility.

At this point, in order to draw a conclusion about the risk-based strategy discussed in this section, we can say that, even if naïve risk parity is not a sound and authentic risk parity strategy (except in two scenarios), it leads to the intuition that the application of this rudimental strategy will provide solutions similar to those originating from the optimal risk parity we illustrate in Sect. 3.4 when correlations tend to show convergence to common, or rather, strongly similar values.

3.4 The Optimal Risk Parity Strategy

The truest version of risk parity strategy is optimal risk parity. With this expression, we refer to the risk-based strategy that, among practitioners, is popular simply with the term risk parity and in the case of academics is frequently called equally weighted risk contribution strategy or portfolio (abbreviated as ERC portfolio).

Optimal risk parity strategy is deeply rooted in finance literature and ideas about risk budgeting for investment processes. It is well known that risk budgeting recommends portfolio analysis in terms of risk contributions rather than in terms of portfolio weights. In the portfolio construction phase, this means that the starting point in finding the appropriate portfolio is specifying the risk contributions preferred by the investor or the asset manager. Then, the resulting portfolio has asset classes' weights and, as a consequence, a capital allocation driven by the targeted risk allocation. Thus, in a portfolio construction process consistent with the risk budgeting approach, risk contributions (and second moments estimates) are used as inputs and portfolio weights are the outputs.[3]

To give a better representation of the portfolio construction methodology in the risk budgeting perspective, we consider a supposed set of risk budgets ($BD_1,\ldots, BD_i,\ldots, BD_N$)[4] and the specification of the desired risk contributions ($TRC_1,\ldots, TRC_i,\ldots, TRC_N$) to a general risk measure RM.[5] The problem of finding out the set of weights ($w_1,\ldots, w_i,\ldots, w_N$) such that the risk contributions match the preferred risk budgets takes the form of the following non linear mathematical system:[6]

[3]We note that in the classic *Mean-Variance Optimization* the portfolio weights are the main outputs and risk contributions are a secondary output determined by the first.

[4]Risk budgets (BD_i) are indicated in relative terms (that is they are expressed in %), not as nominal values.

[5]From Sect. 3.2, we remember that a general TRC_i corresponds to $w_i \cdot \frac{\partial RM}{\partial w_i}$.

[6]In spite of the general form of the mathematical system we have adopted, we observe that with reference to the constraint on the risk budgets, Roncalli (2014) considers not very reasonable to set one risk budget to zero, it would be better to reduce the investment universe excluding the corresponding asset in that case. For this reason, he suggests a strict positivity constraint on the BD_i.

$$\begin{cases} w_1 \cdot \dfrac{\partial RM}{\partial w_1} = BD_1 \cdot RM \\ \vdots \\ w_i \cdot \dfrac{\partial RM}{\partial w_i} = BD_i \cdot RM \\ \vdots \\ w_N \cdot \dfrac{\partial RM}{\partial w_N} = BD_N \cdot RM \\ \\ BD_1, \ldots, BD_i, \ldots, BD_N \geq 0 \\ BD_1 + \cdots + BD_i + \cdots + BD_N = 1 \\ w_1, \ldots, w_i, \ldots, w_N \geq 0 \\ w_1 + \cdots + w_i + \cdots + w_N = 1 \end{cases} \tag{3.27}$$

However, regarding optimal risk parity, it is not possible to analytically solve a mathematical system like the one in (3.27). For this reason, we will subsequently focus on a different way to formalize the same problem and will give an intuitive explanation as how solutions are identified.

Before we look at these technical aspects, we illustrate qualitatively the optimal risk parity. The main idea inspiring this strategy is that of preventing one or few asset classes from having a dominant role in driving portfolio risk, typically measured by the standard deviation. Consequently, the strategy seeks to equalize risk contributions from the different asset classes in the investment universe at least on an ex-ante basis. In other words, the optimal risk parity strategy wants the portfolio to be equally weighted or perfectly balanced in terms of risk allocations.

Regarding the technical issues, it should be clear from the above description that implementing optimal risk parity strategy is equivalent to finding out a risk budgeting portfolio with uniform (and obviously strictly positive) risk budgets assigned to each asset class. To formalize the concepts just expressed, the mathematical system in (3.27) must be rewritten as shown in (3.28):

$$\begin{cases} w_i \cdot \dfrac{\partial \sigma_P}{\partial w_i} = BD \cdot \sigma_P \\ BD = {}^{1}\!/_{N} \\ \sum\limits_{i=1}^{N} w_i = 1 \\ w_i \geq 0 \end{cases} \tag{3.28}$$

As mentioned before, it is difficult to find explicitly the solutions w_i for the risk parity portfolio in a general context where standard deviations and correlations are not homogeneous among portfolio constituents.[7] Maillard et al. (2010) have provided some intuitions on the appropriate weights for the ERC portfolio but again they do not lead to closed-form solutions. In particular, they first recall the definition of β_i as follows:

$$\beta_i = \frac{\sum_{j=1}^{N} w_j \sigma_{ij}}{\sigma_P^2} \tag{3.29}$$

Second, they link this definition with the one concerning risk contribution:

$$TRC_{\sigma\_i} = w_i \beta_i \sigma_P \tag{3.30}$$

Considering the condition of homogeneous risk contributions for the optimal risk parity strategy equivalent to σ_P/N, they conclude that:

$$w_i = \frac{\beta_i^{-1}}{\sum_{j=1}^{N} \beta_j^{-1}} = \frac{\beta_i^{-1}}{N} = \frac{1}{N\beta_i} \tag{3.31}$$

According to (3.31), the weight of asset i belonging to a risk parity portfolio should be inversely proportional to its beta: the higher the beta, the lower the weight and vice versa. Obviously, this also implies that an asset class with high individual volatility and/or with high correlation with the other asset classes becomes penalized in the portfolio allocations. Despite the interesting insight, Eq. (3.31) doesn't make the problem of finding the portfolio weights for the optimal risk parity strategy analytically tractable. The reason is that asset allocation w_i is function of itself or, said differently, the beta cannot be found without the portfolio weights, but they have to depend on beta. The problem that arises is referred by Chaves et al. (2012), Maillard et al. (2010) and Roncalli (2014) as an endogeneity problem.

Taking into account both the problem just mentioned and the impossibility to get a closed-form solution for the system in (3.28), it makes sense to search differently the weights for an ERC portfolio.

A suitable approach requires first to rewrite the non-linear system in (3.28) as a constrained optimization problem. Its form is given in (3.32):

$$\begin{aligned} & w^* = \arg\min f(w) \\ & s.t. \sum_{i=1}^{N} w_i = 1 \\ & 0 \leq w_i \leq 1 \end{aligned} \tag{3.32}$$

[7]Explicit solutions are admissible just in the bivariate case and in the case of constant correlation. See Sect. 3.3.

Secondly, a reasonable objective function *f(w)* to minimise is required. For this purpose, it is necessary to remark the condition that must be satisfied by each asset class allocation in an optimal risk parity portfolio. Considering that the main goal of the strategy is to distribute the same risk budget/contribution to each component, so that none has a dominant role on the portfolio risk, the condition can be formally translated as follows:

$$w_i \cdot \frac{\partial \sigma_P}{\partial w_i} = w_j \cdot \frac{\partial \sigma_P}{\partial w_j} \; \forall i, j \tag{3.33}$$

Accordingly, the objective function to be minimised is then given by the following mathematical expression:

$$f(w) = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(w_i \cdot \frac{\partial \sigma_P}{\partial w_i} - w_j \cdot \frac{\partial \sigma_P}{\partial w_j} \right)^2 \tag{3.34}$$

Now, the optimization problem to solve for $w*$ (the optimal risk parity portfolio weights) can be fully stated as follows:

$$\begin{aligned} & \underset{w*}{Min} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(w_i \cdot \frac{\partial \sigma_P}{\partial w_i} - w_j \frac{\partial \sigma_P}{\partial w_j} \right)^2 \\ & s.t. \\ & \sum_{i=1}^{N} w_i = 1 \\ & 0 \le w_i \le 1 \end{aligned} \tag{3.35}$$

We note that (3.35) is a problem of constrained nonlinear programming for which analytical solutions are not available and thus we must find them using a numerical algorithm that implements an iterative process. Maillard et al. (2010) and Roncalli (2014) recommend the use of the Sequential Quadratic Programming Algorithm (SQP).

This method generates approximated solutions that allow convergence to the minimum of the nonlinear optimization problem. So, SQP produces a sequence of solutions that get closer and closer to the right one $w*$. The tentative solutions are obtained using the current iterate simultaneously to replace the original objective function with a quadratic problem approximation and to approximate the constraint functions after linearizing them.[8]

[8]It is useful to keep in mind that try to model a constrained nonlinear optimization problem as a quadratic programming subproblem means write the second-order Taylor expansion of *f(w)*. Since it is an *N*-dimensional function, the expansion involves the gradient and the hessian of the function. In particular, the quadratic subproblem at the current approximation $\mathbf{w_s}$ has the form:

We observe that, besides the Sequential Quadratic Programming Algorithm, other authors (Chaves et al. 2012) have proposed different approaches to compute the appropriate allocations for an ERC portfolio presenting them as an important simplification for implementing the optimal risk parity strategy. It has to be noted they are based on matrix algebra; neither do they give a specification of an objective function nor do they allow for the insertion of weight constraints.

So far we have described the theoretical concepts and the tools on which the optimal risk parity rests. Next we apply this strategy for illustrative purpose to an investment universe of five risky asset classes denoted A, B, C, D and E. Their volatilities are respectively 4, 7, 15, 18 and 20 %. The correlation coefficients between couples are provided in Table 3.4 and Table 3.5 provides the corresponding covariance matrix.

We compute the weights for the optimal risk parity portfolio using the Sequential Quadratic Programming Algorithm, therefore we follow Maillard et al. (2010) and Roncalli (2014). The solutions are given in Table 3.6 and lead to 4.62 % portfolio standard deviation. Unsurprisingly, the optimal risk parity portfolio is invested in all asset classes. We observe substantial differences in the assigned wealth allocation. In particular, we note a significant exposure in asset class A and B, the ones with both the lower standard deviation and the lower correlation with C, D and F. Importantly, Table 3.6 gives evidence that the strategy implementation has determined the decomposition of the portfolio standard deviation into perfectly balanced separate components. Indeed, no asset class contributes more than its

Table 3.4 Correlation matrix for the numerical example

	A	B	C	D	E
A	1.00	0.30	−0.20	−0.10	−0.25
B	0.30	1.00	0.10	0.20	0.20
C	−0.20	0.10	1.00	0.70	0.75
D	−0.10	0.20	0.70	1.00	0.85
E	−0.25	0.20	0.75	0.85	1.00

(Footnote 8 continued)

$$\begin{aligned}
&\underset{\mathbf{d}_w^*}{Min}\ \nabla f(\mathbf{w_s})\mathbf{d_w} + \frac{1}{2}\mathbf{d}_\mathbf{w}^{'}\nabla^2 f(\mathbf{w_s})\mathbf{d_w}\\
&s.t.\\
&\nabla h_f(\mathbf{w_s})\mathbf{d_w} + h_f(\mathbf{w_s}) = 0\ with\ f = 1,\ldots,F\\
&\nabla y_g(\mathbf{w_s})\mathbf{d_w} + y_g(\mathbf{w_s}) \geq 0\ with\ g = 1,\ldots,G
\end{aligned}$$

where $\mathbf{d_w} = \mathbf{w} - \mathbf{w_s}$.

The solution of the subproblem is used as a search direction to determine the next iterate. The final solution must satisfy the so called Karush-Kuhn–Tucker conditions that, in this case, are referred to a Lagrangian function that takes into account the constraints.

Table 3.5 Covariance matrix for the numerical example

	A	B	C	D	E
A	0.001600	0.000840	−0.001200	−0.000720	−0.002000
B	0.000840	0.004900	0.001050	0.002520	0.002800
C	−0.001200	0.001050	0.022500	0.018900	0.022500
D	−0.000720	0.002520	0.018900	0.032400	0.030600
E	−0.002000	0.002800	0.022500	0.030600	0.040000

Table 3.6 Optimal risk parity solutions for the numerical example

	Weights (%)	Marginal risk	Risk contribution	Percentage risk contribution (%)
A	55.41	0.016680	0.009241	20.00
B	21.55	0.042878	0.009241	20.00
C	9.49	0.097344	0.009241	20.00
D	6.88	0.134346	0.009241	20.00
E	6.67	0.138587	0.009241	20.00

peers (or competitors) to the overall volatility. A 20 % portion of it (that is 0.009241) can be assigned to each individual portfolio constituent.

For the sake of clarity, it is helpful to highlight the critical role played by correlations in our illustrative example. To this end, we simply compute the portfolio weights using the naïve risk parity, the simplified version of the strategy that ignores correlations.

It is immediately clear how different the portfolio would be. As a matter of fact, we would get as result the following allocations vector: 44.24, 25.28, 11.80, 9.83 and 8.85 %. Even if they are ranked in the same order of magnitude, the discrepancies with the column "Weights" in Table 3.6 are somewhat substantial. The naïve strategy has a larger volatility than the optimal risk parity strategy (5.62 % vs. 4.62 %) and from Table 3.7 we can easily note that it displays imbalances in terms of risk contribution. In particular, asset class A has the lowest share in the portfolio volatility which is intuitive given that its low correlations with the other asset classes are not considered in the decision on portfolio weights.

Table 3.7 Naïve risk parity solutions for the numerical example

	Weights (%)	Marginal risk	Risk contribution	Percentage risk contribution (%)
A	44.24	0.009440	0.004176	7.43
B	25.28	0.039647	0.010023	17.82
C	11.80	0.110917	0.013086	23.27
D	9.83	0.150092	0.014756	26.23
E	8.85	0.160476	0.014199	25.25

3.5 Risk Parity Strategy and Leverage

A basic premise we expressed in Sect. 3.1 is that risk-based strategies provide a single portfolio as output. However, we can argue that risk parity investors, in addition to preserving the equally weighting of risk contributions, want to make a choice on the level of the overall risk to take in order to influence the potential of return. This normal circumstance motivates the introduction of leverage in the discussion of risk parity strategy.

Indeed, matching a volatility target requires understanding the amount of investment or borrowing at the risk free rate that should be "added" to the starting risky portfolio, named for this case the "source portfolio".[9] By convention, an investment strategy that exploits financing at the risk free rate is said to be a levered strategy and unlevered otherwise. As noted by Anderson et al. (2012), Levell (2010), Qian (2005, 2011) and Ruban and Meles (2011), the first situation typically occurs in implementing risk parity strategy. There are two major reasons:

- traditional portfolios that are rebalanced to become truly risk balanced across asset classes frequently need to have a greater portion in less aggressive asset classes than the actual structure;
- empirical evidence has often suggested that some less volatile asset classes exhibit higher risk-adjusted performance (higher Sharpe Ratios) than riskier asset classes but also lower raw returns. As a consequence, the risk parity portfolio dominates the actual portfolio in terms of risk and return trade-off but underperforms in terms of pure return.

When these two facts take place, the desire to bring together the better reward-to-variability ratio of the risk parity portfolio and a higher performance may arise. To accomplish this objective, the risk parity strategy has to achieve a higher risk level without changing its Sharpe Ratio. Using leverage is the solution for this problem because it allows to shift the portfolio along the "risk parity line".

To provide a better understanding, we refer again to the two-asset class case examined in Sect. 3.3. We begin by presenting the relevant and additional data we need. We remember the JPM Emu Government All Maturities index and the MSCI World index had, from January 2005 to December 2014, an annualized standard deviation of 4.02 and 12.47 %, respectively. The former achieved an annualized mean return of 4.95 % and the latter of 8.37 %. If we assume an annualized cash (free risk) return of 1.68 % (approximated by the JPM Euro Cash 1 M), we conclude that both asset classes had a positive Sharpe Ratio, however it was better for the bond index than for the equity index: 0.81 versus 0.54.

Moving from the individual components to portfolios, we compute, for the 60/40 portfolio an average return and a standard deviation of 6.32 and 5.27 %, respectively, and for the risk parity portfolio a 5.78 % average return against 4.033 % volatility. Therefore, both portfolios had positive Sharpe Ratio, but it was higher for the risk

[9] See Anderson et al. (2014).

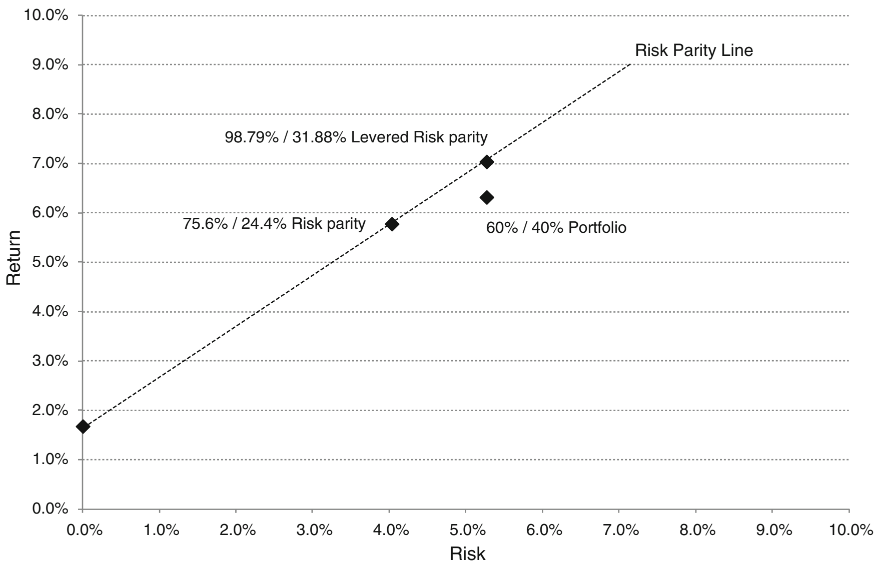

Fig. 3.1 Risk parity portfolio and leverage

parity portfolio than for the traditional portfolio, 1.02 versus 0.88. At this point, we can easily realize the risk parity portfolio had the best Sharpe Ratio but a lower return while the traditional portfolio had a lower Sharpe Ratio but a higher return.

According to what we said before, we have a situation in which applying leverage to a risk parity portfolio is the way to get the best of both alternatives. For example, to preserve risk parity and achieve a 5.27 % volatility (the same as the 60/40 portfolio), a leverage ratio of 1.31 (5.27 %/4.033 %) is required. The resulting levered risk parity portfolio borrows additional funds for 30.67 % of the capital initially available and arrives at an exposure of 98.79 % in the fixed income component and 31.88 % in the equity component. Its mean return is higher than that of the traditional portfolio considering approximately the leverage cost ($7.04\,\% = 4.95\,\% \cdot 98.79\,\% + 8.37\,\% \cdot 31.88\,\% - 30.67\,\% \cdot 1.68\,\%$) and preserves the risk parity risk-adjusted performance.

In a graphical illustration, this solution corresponds to a movement of the "source" 75.6 %/24.4 % risk parity portfolio up along the line which in a risk—return space passes both through the cash (risk-free rate) position and the risk parity portfolio. Figure 3.1 shows this movement. We can safely say that every combination of the cash position and the risk parity portfolio will stay on this line.

The shift just described on the Risk Parity Line resembles one of the possible movements on the Capital Market Line.[10] However this analogy is apparent because

[10]Similarly to the standard CAPM, Fig. 3.1 accepts the assumption that investors can lend and borrow money at the same rate, that is the risk-free rate.

the risky portfolio on the Capital Market Line is represented by the so called *market portfolio* that is a capitalization-weighted portfolio of all possible asset classes instead of a risk balanced portfolio. A frequently used synonym for *market portfolio* is *tangency portfolio* because, in a risk-return diagram, it is the portfolio corresponding to the point where the line from the risk free rate is tangent to the *Mean-Variance Efficient Frontier* and therefore it has the highest Sharpe Ratio.

To be precise, the equivalence between *market portfolio* and *tangency portfolio* can be assumed ex-ante. It is useful to remember that Asness et al. (2012) show that the market portfolio is quite different from the *tangency portfolio* on the basis of the realized return and volatility. They determine that, ex-post, the *tangency portfolio* includes a larger fraction of "safer" assets than supposed ex-ante and that a risk parity portfolio serves as an approximation of the *tangency portfolio* which is more plausible than the one of the market portfolio.[11]

As previously stated, leverage is a fundamental tool to "travel" along the Risk Parity Line. However, in addition to its potential, consequent dangers must also be taken into account. Given that leverage means to be financed, we have to acknowledge that, in extreme market conditions, the roll-over of loans is subject to uncertainty and/or the possibility to be forced to execute distressed sales because of significant changes of the borrowing rates and of the assets value. Furthermore, it can be taken for granted that leverage not only amplifies returns, it also exacerbates losses and the impact of tail events.

To conclude, it is important to emphasize that we share the idea of levered risk parity portfolio adopted by Qian (2011), that is leverage applied to the entire "source" portfolio. This contrasts with Ruban and Melas (2011) who apply leverage exclusively to the fixed income (low-volatility) component of the "source" portfolio.[12]

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[11]Evidently, this real experience contrasts with the predictions of *Modern Portfolio Theory* that would expect the higher risk-adjusted performance from the market and not from a portfolio that is largely exposed to "safer" asset classes. However, some years ago, Frazzini and Pedersen (2013) and Asness et al. (2012) have proposed the leverage aversion argument as a theoretical explanation for this real case. According to this argument, the significant pressure in the market coming from leverage averse investors (by choice or by regulation) implies that they prefer to select a portfolio with risky assets rather than levering low-risk positions in the search for more interesting return targets. As a consequence of their pressure on the demand for riskier assets, the expected return of such assets is reduced while disregarded low-risk assets can trade at lower prices thus offering higher returns with a consequent improvement of their risk-adjusted performance. Those investors who are less leverage averse and apply leverage can benefit from the overweighting of "safer" assets in their portfolio.

[12]If borrowing is added exclusively to the low-risk component, the amount of leverage (*LEV*) that is necessary can be written, according to Ruban and Meles (2011), as follows:

$$LEV = \frac{\sigma_{Equity} \cdot w_{Equity}}{\sigma_{Bond} \cdot \left(1 - w_{Equity}\right)}$$

In our example, where the equity component is 3.10 times more volatile than the fixed income component and has a 40 % original allocation, the fixed income component must accordingly be levered 2.07 times. The resulting portfolio would reach the equalization between risk contributions

3.6 Risk Parity Strategy and the Modern Portfolio Theory Framework

Asset managers are aware that risk parity represents an approach for a policy portfolio construction that does not adhere to the classic mean-variance framework. This is proven by the explicit exclusion of expected returns from the set of the optimization inputs.

Although a risk parity portfolio is undoubtedly identified by exploiting solely parameters that state the risk characteristics of the asset classes, we observe, in accord with Lee (2011), that it is evaluated ex-post using risk-adjusted performance measures (principally the Sharpe Ratio). Thus we acknowledge that implicitly risk parity investors expect a sort of reward from seeking the perfect risk dispersion among asset classes.

The aforesaid circumstances call for an attempt to put risk parity in relation to the *Mean-Variance Optimization* or, more broadly, to *Modern Portfolio Theory* on a theoretical basis. It therefore makes sense to investigate whether a portfolio with asset class weights such that each contributes equally to the portfolio risk is ex-ante optimal in a mean-variance sense. A reasonable starting point to address the issue is to recall that, in the context of *Modern Portfolio Theory*, when a risk-free asset is available, the optimal risky portfolio to select (and to combine alternatively with risk-free borrowing or lending) is the *tangency portfolio*. As stated in Sect. 3.5, it is the portfolio with the highest Sharpe Ratio so it is also called the *maximum Sharpe Ratio portfolio*. At this point, it is easy to understand that the above issue on the optimality of a risk parity portfolio is equivalent to examine whether it can correspond to the *maximum Sharpe Ratio portfolio*.

We learn from several authors (for example, Fabozzi et al. 2007; Maillard et al. 2010; Martellini 2008; Scherer 2015) that the composition (w_{MSR}) of the *tangency portfolio* or *maximum Sharpe Ratio portfolio* is given by:

$$w_{MSR} = \frac{\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - R_{free}\mathbf{e}\right)}{\mathbf{e}'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - R_{free}\mathbf{e}\right)} \tag{3.36}$$

where R_{free} is the risk-free rate and $\mathbf{e}$ is a vector $N \times 1$ of ones. Equation (3.36) indicates that the relative proportions of risky components in the *tangency portfolio* depend directly both on their respective expected risk premiums or their expected excess returns over the risk free rate and on their degree of precision (inverse of the covariance matrix).

(Footnote 12 continued)

and would have a higher volatility. It would include a 124.08 % exposure in the fixed income component and a 40 % exposure in equity.

Roncalli (2014) and Scherer (2015) noted that the *tangency portfolio* satisfies the condition that the ratio between marginal excess return and marginal risk is the same for all constituents. Mathematically it can be written as follows:

$$\frac{\partial\left(\mu_P - R_{free}\right)/\partial w_i}{\partial\sigma_P/\partial w_i} = \frac{\partial\left(\mu_P - R_{free}\right)/\partial w_j}{\partial\sigma_P/\partial w_j}\ \forall i,j \tag{3.37}$$

The concept formalized in (3.37) is important. It means that the impact on return of an infinitesimal increase in the weight of a portfolio constituent is "penalised" by an additional risk from an extremely small extension of its position in the same way across asset classes. Obviously, in this case, any change in the portfolio composition cannot provide an improvement in the portfolio risk-adjusted performance. Therefore, the ratio in (3.37) is also the Sharpe Ratio of the *tangency portfolio* that every investor should prefer to hold when the opportunity to borrow or lend at the risk-free rate exists.

With reference to this particular portfolio, the authors mentioned before remark that, as a consequence, it also verifies the following relationship:

$$\boldsymbol{\mu} - R_{free}\mathbf{e} = \left(\frac{\mu_P - R_{free}}{\sigma_P}\right)\frac{\Sigma\mathbf{w}}{\sigma_P} \tag{3.38}$$

where we note the presence of the gradient vector. According to (3.38), the excess return expected from each component implied by the allocations of the *tangency portfolio* is proportional to its marginal risk.

Considering both Eqs. (3.37) and (3.38), we can infer the two necessary conditions for the ex-ante optimality of the ERC or risk parity portfolio under the *Modern Portfolio Theory*. They can be stated as follows:

- pair-wise correlations among asset classes are the same or, equivalently, the correlation matrix is constant;
- all asset classes have the same Sharpe Ratio.

Here we should remember from Sect. 3.3 that, under the first condition, the inverse volatility strategy provides the ERC portfolio. The second condition requires that the linear function (from the risk free rate) between the volatility and the expected return of asset class i has the same slope for all asset classes. According to what we have just said, the individual excess return can be written as follows:

$$\mu_i - R_{free} = \frac{\frac{1}{\sum_{j=1}^{N}\frac{1}{\sigma_j}} \cdot \sum_{j=1}^{N}\left(\mu_i - R_{free}\right) \cdot \frac{1}{\sigma_i}}{N \cdot \frac{1}{\sum_{j=1}^{N}\frac{1}{\sigma_j}}}\sigma_i \tag{3.39}$$

where the fraction gives the common Sharpe Ratio.

Having established the conditions for the risk parity portfolio to be efficient in the sense of *Modern Portfolio Theory*, it is easy to understand they are extremely demanding and selective. In practice, they assume a world in which asset classes are extremely redundant. For this reason, it must be taken for granted they are violated in reality. Therefore, risk parity asset managers or investors should be aware that their portfolio, defined disregarding expected returns, doesn't fall ex-ante on the Capital Market Line and, consequently, they cannot claim it to be the *tangency portfolio*.

3.7 Potential Evolution of Risk Parity Strategy

This chapter focuses on risk parity strategy referring to the version that has become popular with the global financial crisis and its aftermath and that was originally developed mainly by Qian (2005, 2006) and Maillard et al. (2010). Recently, some clues of innovations concerning the strategy have started to emerge even if a deep exploration of their practical implementations and implications is still lacking.

We believe that one innovation worth mentioning refers to what the strategy considers as the primary building blocks of a strategic or policy portfolio. It is well known that in normal implementation, the investment universe is defined in terms of asset classes. Indeed, risk parity strategy determines the necessary individual allocation for asset classes in order to equalize their contribution to overall portfolio risk.

In this regard, recent contributions (Bender et al. 2010; Bhansali 2011; Lhore et al. 2014; Page and Taborsky 2011; Roncalli and Weisang 2012) have fostered the use of "risk factors" instead of asset classes as new potential building blocks. Obviously, the proposal has not emerged by chance. It has theoretical foundations in influential works like the ones by Ross (1976) on arbitrage pricing theory, by Fama and French (1992) and by Carhart (1997).

In a nutshell, the underlying idea is that asset class returns and variability are driven by common risk factors. This means that each asset class can have embedded loadings on several factors and can share some of them with one or more other asset classes that are apparently distinct.

Given this starting point, proponents of "risk factors" as portfolio building blocks assert that an approach to portfolio construction based on asset classes is likely to imply some opacity and misunderstanding of the real sources of risk and also is likely to lead to some unwanted and hidden risk overlaps. This thinking is extremely clear both from Bhansali (2011): "Assets are simply carriers of risk factor exposures in various mixes. Ignoring the risk factor content and fixating on assets themselves can result in opacity and tail risks…" and from the example proposed by Deguest et al. (2013) who write: "For example, convertible bond returns are subject to equity risk, volatility risk, interest rate risk and credit risk. As a consequence, analyzing the optimal allocation to such hybrid securities as part of a broad bond portfolio is not likely to lead to particularly useful insights. Conversely, a seemingly well-diversified allocation to many asset classes that essentially load on the same risk factor (e.g., equity risk) can eventually generate a portfolio with very concentrated risk exposure."

Our digression regarding the intuition at the back of the emerging innovation in favour of the use of "risk factors" in the portfolio construction process, requires the provision of various elements necessary for the implementation of what has been sometimes called a "paradigm shift" (from asset classes to "risk factors"). In order to identify them, it is important to keep in mind that herein we are going to combine factor-based asset allocation and the risk parity strategy. In other words, the ultimate goal is to select a portfolio such that its exposure to the different risk factors is well-diversified and well-balanced.

Given this premise, we can outline, in a schematic way, the following necessary steps or tools for a factor-based risk parity:

- decomposition of the returns variance of the asset classes in the investment universe in terms of factor exposures and consequent interpretation of the portfolio risks by means of risk factor exposures;
- definition of factor weights such that their contribution to the total portfolio variance matches a given risk budget (that should be the same across factors).

It is not our intent here to go in depth about the above points. We just focus on essentials. With reference to the first point, we assume that the asset class returns are linearly related to K systematic factors. This can be written in matrix notation as follows:

$$\mathbf{R}_t = \mathbf{L}\mathbf{F}_t + \boldsymbol{\varepsilon}_t \tag{3.40}$$

where:

$\mathbf{R}_t$ vector $N \times 1$ of N asset class returns in time t;
$\mathbf{L}$ matrix $N \times K$ of factor exposures;
$\mathbf{F}_t$ vector $K \times 1$ of K factor returns at time t;
$\boldsymbol{\varepsilon}_t$ vector $N \times 1$ of non systematic asset class returns

At this point, also the asset class covariances can be explained using the covariance among the common factors. Indeed, we can write:

$$\boldsymbol{\Sigma} = \mathbf{L}\boldsymbol{\Omega}\mathbf{L}' + \boldsymbol{\Omega}_\varepsilon \tag{3.41}$$

where:

$\boldsymbol{\Sigma}$ matrix $N \times N$ of asset class covariances;
$\boldsymbol{\Omega}$ $K \times K$ covariance matrix of factor returns;
$\boldsymbol{\Omega}_\varepsilon$ $N \times N$ covariance matrix (diagonal matrix) of non-systematic asset class returns

This leads to a new expression for the overall portfolio risk that consists of two parts given that $\boldsymbol{\varepsilon}_t$ and $\mathbf{F}_t$ are uncorrelated:

$$\sigma_P^2 = \mathbf{w}'\mathbf{L}\boldsymbol{\Omega}\mathbf{L}'\mathbf{w} + \mathbf{w}'\boldsymbol{\Omega}_{\varepsilon}\mathbf{w} \tag{3.42}$$

Moving to the second step, it is necessary, in analogy with what we did in order to implement the optimal risk parity at the asset class level, to get the sensitivity of the portfolio risk to a small increase of a given risk factor exposure which can be calculated by taking the first derivative and, if repeated for the different "risk factors", allows us to obtain the gradient vector ($K \times 1$) summarizing the marginal risk factor contributions. This is denoted by the symbol $\nabla\mathbf{MR\_RF}_{\sigma}$ and is shown in (3.43).

$$\nabla\mathbf{MR\_RF}_{\sigma} = \frac{\partial\sigma_P}{\partial(\mathbf{L}'\mathbf{w})} = \frac{\boldsymbol{\Omega}\mathbf{L}'\mathbf{w}}{\sigma_P} \tag{3.43}$$

In the case of factor-based risk parity, the risk contributions from "risk factors" approximate the overall portfolio risk.[13] This circumstance can formally be written as follows:

$$\sigma_P \approx \nabla\mathbf{MR\_RF}_{\sigma}' \cdot \mathbf{L}'\mathbf{w} \tag{3.44}$$

The final task for factor-based risk parity is solving an optimization problem in order to define the appropriate allocations for the original constituents (asset classes) of the investment universe such that the contribution of each factor to the total portfolio standard deviation is the same for all factors.

A risk parity solution based on risk budgets with respect to "risk factors" rather than the asset classes appears to be attractive for its focus on the profound drivers of reward and variability for investors embedded in asset classes that group together individual securities. However, we have to acknowledge it is also subject to some shortcomings or difficult tasks that can restrain asset managers from its practical use. One weak point is the difficulty of obtaining a unique and/or a solution from an optimization problem that tries to match risk factor contribution to the preferred risk budget or that makes attempts to minimize the risk concentration between the factors.[14] A quite difficult task can be represented by the choice of the type of "risk factors" to consider and the measurement of exposures to them. The main alternatives consist in resorting to macroeconomic risk factors or statistical factors.[15]

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[13]The reason is that we are ignoring specific risks.

[14]These difficulties have been highlighted and explained in details by Roncalli and Weisang (2012) and by Deguest et al. (2013).

[15]The author who established the original concept of risk parity as a portfolio construction process, Qian, declared a preference for macroeconomic risk factors. He has argued that *"…a portfolio with true risk parity should have balanced risk exposure to the economic risks of growth and inflation. As a consequence, it should have balanced (but not necessarily equal) risk contribution from three risk sources: equity, interest rates, and inflation"*. See Qian (2013).

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Chapter 4
The Different Risk-Based Approaches to Asset Allocation

Abstract In this chapter, we aim to unite concepts, analytical analysis and illustrative examples regarding the risk-based strategies that do not exploit, in their implementation, tools associated with risk budgeting literature. Firstly, we discuss the equally-weighted approach. It is the most straightforward risk-based strategy because there are no parameter estimates involved considering that its implementation does not entail an optimization problem to be solved. The reason for this is very simple: all asset classes are given the same weight. Then, we explore the global minimum-variance approach which is the only risk-based strategy that, with absolute certainty, recommends a portfolio located on the Efficient Frontier: the portfolio at the most left end. Finally, we take into consideration the most diversified portfolio approach. This is the asset allocation strategy that identifies the long-only portfolio with the highest diversification ratio, as introduced by Choueifaty and Coignard (2008). Concerning the application of each of the mentioned strategies, we provide an analysis, in terms of implicit risk allocation, of the recommended portfolio. We observe that the way such strategies manage diversification generally does not allow balanced risk allocation.

Keywords Concentration ratio · Diversification ratio · Equally-weighted strategy · Marginal risk · Minimum-variance strategy · Most diversified portfolio · Risk contribution

4.1 Not Only Risk Parity, What Else?

The previous chapter focused on a detailed investigation of the risk parity strategy, the most sophisticated of all the risk-based strategies. Before the explosion of the financial crisis in 2007–2008, financial literature had already given attention to and revisited more traditional strategies belonging to the same "family": the equally-weighted strategy and the global minimum-variance strategy. They will be discussed later with a newcomer, the most diversified portfolio approach, which is almost simultaneous with the optimal risk parity.

M.D. Braga, *Risk-Based Approaches to Asset Allocation*,
SpringerBriefs in Finance, DOI 10.1007/978-3-319-24382-5_4

The characteristics that are common to all risk-based strategies have been highlighted in Sect. 3.1, however, it is also important to note that the strategies considered in this chapter are in a sense separated from the optimal risk parity. The main reason is that the methodologies they propose to solve the investor's asset allocation problem do not require typical tools of risk budgeting and/or risk decomposition in the implementation process; evidently we are referring to marginal risk, total risk contribution and percentage total risk contribution. At most, these tools can be used to investigate the properties of portfolios recommended by such strategies.

Generally speaking all these strategies want to achieve the benefits of diversification. In spite of this common insight at the heart of the different portfolio construction methods, the above risk-based approaches rely on different meanings of that concept, as we will investigate later. According to the equally-weighted strategy, the benefits of diversification can be gained just following a heuristic process within where the decision about the extension of the investment universe is crucial because the larger it is, the higher the benefits of diversification can be. Diversification necessarily means that one should hold a portfolio that is expected to have the minimum standard deviation when we consider the global minimum-variance approach. Lastly, when we discuss the most diversified portfolio approach, we realize that obtaining the benefits of diversification is interpreted as a pursuit of risk saving without necessarily minimizing risk in absolute terms.

An analysis of these portfolio construction approaches enables us to venture out with a redundant, but still interesting consideration: although all approaches manage diversification, the extraction of the way each asset class is responsible for the risk that is inherent in the portfolio escape.

4.2 The Equally-Weighted Approach

The most straightforward risk-based strategy to illustrate and to put into practice as well is represented by the equally-weighted approach. According to it, the portfolio allocations have only to be driven by the number of asset classes in the investment universe. This is obvious because the asset allocation problem is solved using the so called "1 over N" rule which prescribes that:

$$w_i = w_j = \frac{1}{N} \quad \forall i, j \tag{4.1}$$

Hence, when the equally-weighted approach is used, all asset classes are given an identical and static weight.[1] We can draw a sort of parallelism from this point between the equally-weighted strategy and the optimal risk parity strategy: the first

[1]We note that when an equally-weighted strategy is rebalanced, the weights are reset to *1/N*, thus always to the same weights as the starting ones. This in not generally verified for the other risk-based strategies. See Chap. 5.

one recommends the less concentrated portfolio in terms of weights, the second one suggests the less concentrated portfolio in terms of risk. Said differently, the former applies the "1 over N" rule to asset allocation while the latter applies the same concept to risk allocation.

Looking at (4.1), we can immediately see that the equally-weighted approach requires no investigation of the distribution of asset class returns. The portfolio allocation is determined regardless of any statistical estimate for returns, risks and correlations. Obviously, it involves no parameter estimate because its implementation does not entail an optimization problem to solve. Therefore, we cannot identify an objective function to minimize or maximize for the case of equally-weighted strategy.

Given this evidence, it is extremely difficult to assume a condition of ex-ante optimality, in a mean-variance framework, for the equally-weighted portfolio. It is certain that very demanding and improbable conditions would be required in order to assert that it is ex-ante an efficient portfolio according to Markowitz: equal expected returns, equal volatilities and uniform correlations among asset classes. In ex-post or empirical analysis we find contrasting results on this strategy: in a very extensive study, based on several asset allocation models and different datasets, DeMiguel et al. (2009) conclude that none of the theoretically more robust asset allocation models was consistently better out-of sample than the "1 over N" rule, but an opposite conclusion comes from Kritzman et al. (2010).

Apart from the discussion regarding the optimality ex-ante versus ex-post of the equally-weighted strategy, the considerations expressed before may also have another consequence. More precisely, they can stimulate some hesitation or suspicion about the inclusion of the equally-weighted approach among the risk-based strategies given that the estimates for the statistical second moments of asset classes too are not taken into consideration when the strategy is implemented. In order to provide an explanation for such inclusion, it is both useful and interesting to develop an idea about what investors try to achieve when they use the equally-weighted strategy or, simply, about the justification for using the equally-weighted approach.

Regarding this point, we observe that behavioural finance represents a reference point because making decisions according to the "1 over N" rule can have a psychological basis.[2] First, this way of splitting the savings investment can be due to the belief of strong fallacies in the estimates of input parameters so that it is better to assume, a priori, that they are equal across asset classes and act consequently. Second, the equally-weighted portfolio can be a reasonable solution to avoid a feeling of regret that would occur if the investment decision consisted in a clear bet that comes out to be erroneous or suboptimal rather than in a mechanical and apparently neutral splitting. Finally, this naïve diversification can be the result of the search for variety when several choices have to be made simultaneously under uncertain conditions.

[2]See Fisher and Statman (1997a, b), Benartzi and Thaler (2001), Windcliff and Boyle (2004).

Table 4.1 Equally-weighted approach solutions for the numerical example

	Weights (%)	Marginal risk	Risk contribution	Percentage risk contribution (%)
A	20.00	−0.002948	−0.000590	−0.59
B	20.00	0.024125	0.004825	4.81
C	20.00	0.126998	0.025400	25.30
D	20.00	0.166741	0.033348	33.22
E	20.00	0.187061	0.037412	37.26

After providing these elements, the reason why the equally-weighted approach is included among risk-based strategies, albeit it works without any risk estimate, is because it is an attempt of defence toward risk (alternatively interpreted just as uncertainty or negative event), a sort of (unsophisticated) risk management tool. Certainly, the equally-weighted approach is motivated neither by a focus on a return target nor by the idea of being able to exploit particular performance-generating skills.

Nevertheless, the assumption that using this simple and heuristic method guarantees diversification can be misleading: it depends on the characteristics of asset classes that the strategy does not take into account. For instance, if their risk is extremely different, the equally-weighted scheme can lead to concentrated risk loadings.

In order to clarify the above point, we provide in (Table 4.1) the results, in the case the equally-weighted approach is implemented, from a risk decomposition analysis referred to the simplified investment universe of five asset classes we used in Sect. 3.4.

It is clear that even if the portfolio is perfectly balanced in terms of capital allocation, its risk is mainly driven by asset classes C, D and E while asset class A plays a little hedging effect; asset class B is a minor risk contributor. The equally-weighted portfolio is much more volatile than the optimal risk parity portfolio: 10.04 % standard deviation versus 4.62 %. We observe that, in the particular case of the equally-weighted portfolio, the formula for the portfolio standard deviation ($\sigma_P^{EW\cdot}$) becomes:

$$\sigma_p^{EW\cdot} = \sqrt{\sum_{i=1}^{N}\left(\frac{1}{N}\right)^2\sigma_i^2 + 2\sum_{j\neq i}^{N}\left(\frac{1}{N}\right)\left(\frac{1}{N}\right)\sigma_i\sigma_j\rho_{ij}} = \sqrt{\left(\frac{1}{N}\right)^2\left(\sum_{i=1}^{N}\sigma_i^2 + 2\sum_{j\neq i}^{N}\sigma_i\sigma_j\rho_{ij}\right)}$$
$$= \left(\frac{1}{N}\right)\cdot\sqrt{\left(\sum_{i=1}^{N}\sigma_i^2 + 2\sum_{j\neq i}^{N}\sigma_i\sigma_j\rho_{ij}\right)} = \left(\frac{1}{N}\right)\cdot\sqrt{\left(\sum_{i=1}^{N}\sigma_i^2 + \sum_{i=1}^{N}\sum_{j\neq i}^{N}\sigma_i\sigma_j\rho_{ij}\right)} \tag{4.2}$$

Also the previous expression (3.14) for the total risk contributions ($TRC^{EW}_{\sigma\_i}$) can be rewritten in a slightly different form as it is shown here:

$$TRC^{EW}_{\sigma\_i} = \frac{1}{N} \cdot \frac{\sigma_i^2 \frac{1}{N} + \sum_{j \neq i}^{N} \frac{1}{N} \sigma_i \sigma_j \rho_{ij}}{\sigma_P^{EW.}} = \frac{\sigma_i^2 + \sum_{j \neq i}^{N} \sigma_i \sigma_j \rho_{ij}}{N^2 \sigma_{P.}^{EW.}} \tag{4.3}$$

According to (4.3), the marginal risk, for example, of asset class A, could be obtained as follows:

$$TRC^{EW.}_{\sigma\_A} = \frac{\begin{array}{c} 4\%^2 + (4\% \cdot 7\% \cdot 0.30) + (4\% \cdot 15\% \cdot (-0.20)) + \\ + (4\% \cdot 18\% \cdot (-0.10)) + (4\% \cdot 20\% \cdot (-0.25)) \end{array}}{5^2 \cdot 10.04\%} = -0.000590 \tag{4.4}$$

4.3 The Global Minimum-Variance Approach

The global minimum-variance approach is a risk-based strategy that definitely recommends a portfolio located on the *Efficient Frontier*. This could be interpreted as conflicting with the general description of risk-based strategies presented in Sect. 3.1 but the problem is over when we clarify that it is a portfolio that plays a unique role. The so called global minimum-variance portfolio (GMVP) is the portfolio from which the concave curve of the *Efficient Frontier* starts, therefore, in the risk-return space where the *Efficient Frontier* is graphed, it is at the left most end. So, the global minimum-variance portfolio is the portfolio that, given an investment universe, is expected to have the lowest possible volatility.[3]

In addition to its special "location", the most important issue to point out when we refer to the global minimum-variance portfolio is the way it is determined. About this, we can deduce that the asset class weights that generate the lowest ex-ante portfolio standard deviation are determined performing an optimization which only exploits the estimates for risk and correlations of the asset classes available in the investment universe.[4] In other words, it is determined merely using an estimated covariance matrix. It is well-known this condition does not apply to all the other optimal portfolios on the *Efficient Frontier* which are strongly dependent also on the expected returns estimates. This is the reason why the global

[3] See Elton et al. (2014), Fabozzi et al. (2007), Gilli et al. (2011).

[4] For clarity, we remember that since standard deviation is a "derivative" of variance (it is simply its square root), using the conventional approach of minimizing the portfolio variance leads actually to the minimum volatility portfolio.

minimum-variance approach (and the GMVP as well) belongs legitimately to the group of risk-based strategies for asset allocation.

From a practical point of view, if we want the GMVP to be both fully invested and restricted from short-selling, the optimization we have to perform entails the use of the formula for portfolio variance as the objective function to be minimized and the inclusion of the two traditional constraints. So our optimization problem takes the following form:

$$\underset{w*}{Min}\left(\sum_{i=1}^{N}\left(w_i\sigma_i\right)^2+\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} w_iw_j\sigma_i\sigma_j\rho_{ij}\right) \tag{4.5}$$

If we adopt the matrix notation, it can be rewritten as follows:

$$\begin{aligned}&\underset{\mathbf{w}*}{Min}\,\mathbf{w}'\Sigma\mathbf{w}\\ &s.t.\\ &\mathbf{w}'\mathbf{e}=1\\ &[w]\geq 0\end{aligned} \tag{4.6}$$

The optimization models in (4.5) and (4.6) are quadratic programming problems for which solutions can be fund only with an iterative procedure, that is numerically. More precisely, it is the inequality constraint that prevents us from searching the asset class weights analytically.[5]

Academic literature has acknowledged good out-of-sample performances to the GMVP resulting from the above optimization problem versus capitalization weighted indices.[6] Apart from the fact that authors have generally considered a problem of securities selection rather than asset allocation issue, we point out that here the global minimum-variance approach has to be seen as a competing approach

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[5]When the same objective function and exclusively the linear equality constraint (the budget constraint) are considered, which is equivalent to allow the GMVP to be a long-short portfolio, the optimization problem has a closed-form solution. Actually, in this scenario, the problem has to be arranged as a Lagrange function written as follows:

$$L(w,\lambda)=\mathbf{w}'\Sigma\mathbf{w}-\lambda(\mathbf{w}'\mathbf{e}-\mathbf{1})$$

where λ is the Lagrange coefficient or multiplier.

Then the first derivatives with respect to the asset allocation weights and the multiplier must be set to zero. The analytical solution obtained in this case is written as follows:

$$\mathbf{w}^*=\frac{\Sigma^{-1}\mathbf{e}}{\mathbf{e}'\Sigma^{-1}\mathbf{e}}$$

.

[6]See Clarke et al. (2006, 2011), (Haugen and Baker 1991), Scherer (2011).

Table 4.2 Global minimum-variance approach solutions for the numerical example

	Weights (%)	Marginal risk	Risk contribution	Percentage risk contribution (%)
A	82.20	0.035308	0.029024	82.20
B	8.26	0.035308	0.002916	8.26
C	6.61	0.035308	0.002333	6.61
D	0.00	0.049911	0.000000	0.00
E	2.93	0.035308	0.001035	2.93

with different risk-based strategies. For this reason, it makes sense to refer its application to the identical simplified example we are considering for the other *μ-free strategies*.

In Table 4.2, we report the asset allocation recommended by the global minimum-variance approach. It entails a 3.53 % portfolio standard deviation which is substantially lower than the volatility of the optimal risk parity portfolio. The global minimum-variance portfolio is heavily concentrated in asset class A. It is the one with the lowest volatility and with negative correlations with 3 of the remaining 4 other asset classes. It is worth noting that giving extreme loadings to one asset class is quite a general characteristic of the global minimum-variance approach. For this reason, the strategy is likely to be particularly sensitive to volatilities and correlations estimates.

In Table 4.2, we also show the results from the risk decomposition exercise. Certainly, such an exercise provides some interesting insights or validations. First, we have evidence that the global minimum-variance approach admits the possibility of partial inclusion of the asset classes available in the selected investment universe (in our example asset class D has been assigned a zero weight). Second, the marginal risk contribution is the same for all asset classes, except for the asset classes not included in the recommended asset allocation. Actually, for the global minimum-variance portfolio we can formally write:

$$\frac{\partial \sigma_P}{\partial w_i} = \frac{\partial \sigma_P}{\partial w_j} \quad \forall i, j \tag{4.7}$$

On this point, Scherer (2015) states: “If this were not so, we could always find a pair of assets where slightly increasing one holding while at the same time reducing the other would result in lowered risk”. Third, the equality of marginal risks does not signify equality in total risk contributions given that asset classes have different weights. Fourth, we note homogeneity between the portfolio weight for an asset class in the GMVP and the percentage contribution to the overall risk from that asset class. The last point suggests that, albeit the GMVP is labelled as a diversified portfolio because it shows the strongest “knocking down” of portfolio risk, it cannot be considered well diversified from the point of view of capital allocation or risk allocation.

Table 4.3 Modified correlation matrix for the numerical example

	A	B	C	D	E
A	1.00	0.45	−0.05	0.05	−0.10
B	0.45	1.00	0.25	0.35	0.35
C	−0.05	0.25	1.00	0.85	0.90
D	0.05	0.35	0.85	1.00	1.00
E	−0.10	0.35	0.90	1.00	1.00

Table 4.4 Global minimum-variance approach solutions for the modified numerical example

	Weights (%)	Marginal risk	Risk contribution	Percentage risk contribution (%)
A	90.86	0.038092	0.034609	90.86
B	1.91	0.038092	0.000726	1.91
C	6.23	0.038092	0.002375	6.23
D	0.00	0.057833	0.000000	0.00
E	1.00	0.038092	0.000382	1.00

Given that we have focussed our attention on a long-only GMVP, we can intuitively understand that its standard deviation must increase with individual volatilities and/or correlations increases. For clarity of presentation, we hypothesise that all the relevant correlations increase by 0.15. Thus, we must replace Table 3.4 in Sect. 3.4 with Table 4.3.

In Table 4.4 we provide the new results for the asset allocation of the GMVP and for the risk budgeting exercise. As expected, the new GMVP has a higher standard deviation (3.81 vs. 3.53 %). We can also observe that its composition has become more polarized.

4.4 The Most Diversified Portfolio Approach

Among the risk-based portfolio construction methods, the most diversified portfolio approach is the most recent one. Originally introduced by Choueifaty and Coignard (2008), it aims to enhance the diversification potential of a portfolio. To achieve this goal, the authors propose an asset allocation strategy that identifies the long-only portfolio with the highest diversification ratio which, for this reason, is called maximum diversification portfolio, usually denoted by *MDP*.

The diversification ratio of a portfolio (DR_P) is defined as the ratio of the weighted average of asset classes' standard deviations to the actual volatility of the portfolio with those asset classes and allocations. We can formally write the diversification ratio as follows:

$$DR_P = \frac{\sum_{i=1}^{N} w_i \sigma_i}{\sqrt{\sum_{i=1}^{N} \sigma_i^2 w_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} w_i w_j \sigma_{ij}}} = \frac{\sum_{i=1}^{N} \sigma_i w_i}{\sigma_P} \tag{4.8}$$

Using the matrix notation, it is written alternatively as:

$$DR_P = \frac{\mathbf{w}'\boldsymbol{\sigma}}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}} \text{ or } \frac{\mathbf{w}'\sqrt{diag(\boldsymbol{\Sigma})}}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}} \tag{4.9}$$

where $\boldsymbol{\sigma}$ is an $N \times 1$ vector of asset classes' volatilities and $diag(\boldsymbol{\Sigma})$ indicates the vector $N \times 1$ of diagonal elements of the covariance matrix.

From the above definitions, we understand that, in essence, the diversification ratio quantifies how much higher the risk of the portfolio would be if all constituents were perfectly correlated. This is clear from the fact that the numerator of the diversification ratio computes the volatility of a given portfolio if all pair-wise correlations were equal to one, which corresponds to the maximum value theoretically admissible for portfolio standard deviation. Therefore, the diversification ratio provides an indirect measurement of the benefits of diversification in terms of risk saving. We have also to point out, as previously done by Meucci (2009), that it gives a quantification of diversification in relative terms, not on an absolute basis, considering that it emphasizes how far two measures of volatility are for the same portfolio. In this sense, the most diversified portfolio approach is different from the global minimum-variance approach: in both cases, investors and/or asset managers are interested in saving on risk, but in the latter case this goal is interpreted in absolute terms taking the lowest risk, given the selected universe of asset classes.

Two characteristics can intuitively be deduced from what we have just said and from the formalization of the diversification ratio for a long-only portfolio:

- it has a lower bound that equals 1. This value would be recorded for a mono-asset portfolio or for a portfolio really consisting of perfect positively correlated assets because, in this case, numerator and denominator in (4.8) and (4.9) would be the same. Consequently, a diversification close to 1 means the portfolio is poorly diversified;
- it has not a standard upper bound; the higher it is, the more diversified the portfolio is.

The most diversified approach consists in the inclusion of the diversification ratio in an objective function to be maximized by means of an appropriate selection of the portfolio weights (the unknown variables of the problem). Therefore, the risk-based strategy we discuss can be represented as an optimization problem in the following way:

$$\underset{w*}{Max}\left(\frac{\sum_{i=1}^{N} w_i\sigma_i}{\sqrt{\sum_{i=1}^{N}\sigma_i^2 w_i^2+\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} w_i w_j \sigma_{ij}}}\right) \tag{4.10}$$

In matrix notation, the same problem is written as follows:

$$\begin{aligned}&\underset{\mathbf{w}*}{Max}\frac{\mathbf{w}'\sqrt{diag(\mathbf{\Sigma})}}{\sqrt{\mathbf{w}'\mathbf{\Sigma}\mathbf{w}}}\\&s.t.\\&\mathbf{w}'\mathbf{e}=1\\&[w]\geq 0\end{aligned} \tag{4.11}$$

In the case of an investment universe of asset classes with the same volatility, running the above optimization algorithms leads to a solution corresponding to the global minimum-variance portfolio.[7]

Similarly to what we have seen for the optimal risk parity portfolio, it is possible to search for a link with *Modern Portfolio Theory* for the most diversified portfolio. On this point, the case to consider is the one where the expected excess returns of the asset classes are proportional to their volatility in the same way, so that is possible to write for each of them:

$$\mu_i - R_{free} = m\sigma_i \tag{4.12}$$

where m is a constant.

Under this condition, indentifying portfolio allocations by maximizing the Sharpe Ratio or the diversification ratio of the portfolio is equivalent. This means that in this particular case the maximum diversification portfolio would be also the *tangency portfolio*.[8]

We can now apply the risk-based approach with the data from previous sections of this chapter. The results of the example for the most diversified portfolio approach are summarized in Table 4.5.

The most diversified portfolio shows a volatility of 3.96 % while the weighted average of asset classes' standard deviations is 6.84 %. Thus the diversification

[7]For example, if we assume that all asset classes in our dataset have 7.00 % standard deviation, both the global minimum-variance approach and the most diversified portfolio approach recommend the following portfolio: 39.46 % A, 14.65 % B, 21.93 % C, 0.00 % D and 23.96 % E. This asset allocation has 4.05 % standard deviation and 1.73 maximum diversification ratio.

[8]The reason is that the diversification ratio is proportional to the asset classes' Sharpe Ratio which is homogeneous across them. See for demonstration Choueifaty et al. (2013) and Roncalli (2014).

Table 4.5 Most diversified portfolio approach solutions for the numerical example

	Weights (%)	Marginal risk	Risk contribution	Percentage risk contribution (%)
A	67.48	0.023143	0.015617	39.46
B	14.32	0.040500	0.005800	14.65
C	10.00	0.086787	0.008681	21.93
D	0.00	0.107964	0.000000	0.00
E	8.20	0.115716	0.009484	23.96

ratio is 1.73. As appropriate interpretation of the result, we deduce that if the 10 relevant pair-wise correlations were equal to 1, the risk of the same portfolio would be approximately 73 % higher than it really is.

This portfolio is less risky than the optimal risk parity portfolio (3.96 % standard deviation versus 4.62 %). Additionally, in contrast to it, the maximum diversification portfolio does not employ the entire investable set; we note that asset class D is excluded. The same applies to the global minimum-variance approach. Unsurprisingly, if we look at the columns “Weights” and “Percentage risk contribution” in Table 4.5, we can see that the search for the highest risk saving in relative terms achieves neither an equilibrated risk dispersion nor a balanced capital allocation.

A more in-depth investigation of the index from which the most diversified portfolio approach is driven is also feasible. In 2013, Choueifaty et al. proposed a decomposition of the maximum diversification ratio. According to this decomposition, the ratio includes two basic components:

– the volatility-weighted average correlation (ρ_{MDP});
– the concentration ratio or the volatility-weighted concentration ratio (CR_{MDP}).

The diversification ratio can be rewritten as follows:

$$DR_P = \frac{1}{\sqrt{\rho_{MDP} \cdot (1 - CR_{MDP}) + CR_{MDP}}} \tag{4.13}$$

therefore its value is inversely linked to both components.

To provide more details on them, we note that the volatility-weighted average correlation is given by:

$$\rho_{MDP} = \frac{\sum_{i \neq j}^{N} \left(w_i \sigma_i w_j \sigma_j\right) \rho_{ij}}{\sum_{i \neq j}^{N} \left(w_i \sigma_i w_j \sigma_j\right)} \tag{4.14}$$

Intuitively, we observe the numerator in (4.14) becomes smaller as the pair-wise correlations get lower. In an opposite and extreme case where correlations increase and become all equal to 1, the volatility weighted average correlation is also equal to 1. The same happens to the entire diversification ratio, regardless of the calculation of the concentration ratio.

The second component, the concentration ratio, is defined as:

$$CR_{MDP} = \frac{\sum_{i=1}^{N} (w_i \sigma_i)^2}{\left(\sum_{i=1}^{N} w_i \sigma_i\right)^2} \tag{4.15}$$

In (4.15) the weight of the asset classes is considered proportionally to their volatility. However, the concentration ratio does not take correlations into account, if they were uniform it would be the only component determining the diversification ratio. From the formalization, we deduce its value for a mono-asset portfolio is equal to one.

We can illustrate the calculations required by the diversification ratio decomposition using the data of the previous example. They are summarized in Table 4.6.

Table 4.6 Diversification ratio decomposition

Object	Calculation
Numerator of the volatility-weighted average correlation	(67.48 % × 4 % × 14.32 % × 7 %) × 0.30 + (67.48 % × 4 % × 10 % × 15 %) × (−0.20) + (67.48 % × 4 % × 0 % × 18 %) × (−0.10) + (67.48 % × 4 % × 8.20 % × 20 %) × (−0.25) + (14.32 % × 7 % × 10 % × 15 %) × 0.10 + (14.32 % × 7 % × 0 % × 18 %) × 0.20 + (14.32 % × 7 % × 8.20 % × 20 %) × 0.20 + (10 % × 15 % × 0 % × 18 %) × 0.70 + (10 % × 15 % × 8.20 % × 20 %) × 0.75 + (0 % × 18 % × 8.20 % × 20 %) × 0.85 = 0.000121923
Denominator of the volatility-weighted average correlation	(67.48 % × 4 % × 14.32 % × 7 %) + (67.48 % × 4 % × 10 % × 15 %) + (67.48 % × 4 % × 0 % × 18 %) + (67.48 % × 4 % × 8.20 % × 20 %) + (14.32 % × 7 % × 10 % × 15 %) + (14.32 % × 7 % × 0 % × 18 %) + (14.32 % × 7 % × 8.20 % × 20 %) + (10 % × 15 % × 0 % × 18 %) + (10 % × 15 % × 8.20 % × 20 %) + (0 % × 18 % × 8.20 % × 20 %) = 0.001678677
Numerator of the concentration ratio	$(67.48\,\% \times 4\,\%)^2 + (14.32\,\% \times 7\,\%)^2 + (10\,\% \times 15\,\%)^2$ $+ (0\,\% \times 18\,\%)^2 + (8.20\,\% \times 20\,\%)^2 = 0.001322872$
Denominator of the concentration ratio	((67.48 % × 4 %) + (14.32 % × 7 %) + (10 % × 15 %) + (0 % × 18 %) $+ (8.20\,\% \times 20\,\%))^2 = 0.004680227$
Volatility-weighted average correlation	=0.000121923/0.001678677 = 0.072630318
Concentration ratio	0.001322872/0.004680227 = 0.282651204

We conclude our example inserting the numbers we have obtained for the volatility-weighted correlation and the concentration ratio in the formulas (4.14) and (4.15). We have the proof that our decomposition of the diversification ratio is correct:

$$DR_P = \frac{1}{\sqrt{0.072630318 \times (1 - 0.282651204) + 0.282651204}} = 1.73 \tag{4.16}$$

We observe that both components are significantly far from the level 1 thus, unsurprisingly, the portfolio provides a substantial risk saving.

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Chapter 5
Application of the Risk-Based Approaches to Asset Allocation

Abstract We provide an empirical investigation of the risk-based approaches to asset allocation using three different datasets; two of them represent equity markets (Eurozone and Emerging Europe) while the third one represents a larger investment universe including equity and fixed income asset classes. These datasets differ for the number of asset classes included and for the market conditions they describe. Our empirical study consists of an out-of-sample analysis of the behaviour of the *μ-free strategies* implemented using a rolling-window procedure. Risk-based strategies are evaluated considering three different criteria: financial efficiency, level of diversification and asset allocation stability. Our results suggest that the equally weighted approach, thought celebrated by DeMiguel et al. (2009) as superior to several optimization models working in the classical mean-variance framework, cannot be firmly acknowledged as a dominant strategy when compared to other risk-based approaches in terms of financial efficiency. From this point of view, the optimal risk parity seems to be the risk-based approach that most keeps a constant position across different investment environments. It also provides a superior diversification in risk contributions and a competitive turnover. The risk based strategies that invest in a fraction of asset classes show a less stable ranking in terms of financial efficiency, important concentration both in weights and risk contributions and higher turnover.

Keywords Asset allocation stability · Drawdown · Financial efficiency · Out-of-sample analysis · Risk-adjusted performance measures · Serial correlation · Sharpe ratio · Shannon entropy measure · Sortino ratio · Farinelli-Tibiletti ratio

5.1 Description of the Datasets Considered for Asset Allocation Experiments

After having illustrated from a theoretical and analytical standpoint the different risk-based approaches to asset allocation, in this chapter we attempt to extend the existing literature with a broad empirical investigation of these strategies.

M.D. Braga, *Risk-Based Approaches to Asset Allocation*,
SpringerBriefs in Finance, DOI 10.1007/978-3-319-24382-5_5

For this purpose, we consider three different datasets. The first dataset, named "EUROZONE", consists of the monthly returns for the 11 countries sub-indices in the MSCI Emu calculated by Morgan Stanley Capital International. The sample period ranges from May 1996 through April 2015. The second dataset, labelled "EMERGING EUROPE", includes the 5 equity sub-indices in the MSCI Emerging Markets Europe index. The monthly returns cover the period from May 2001 through April 2015. The third dataset, named "GLOBAL", contains monthly returns from May 1997 to April 2015 for the 8 asset classes identified by the following indices: JPM Europe Government Bond, JPM United States Government Bond, JPM Embi Global Diversified Composite, BOFA Ml Global Broad Corporate, MSCI Europe, MSCI North America, MSCI Pacific Free, MSCI Emerging Markets. All returns in the chosen datasets are euro-denominated and calculated at total return level on the basis of the month-end value of the indices. The datasets characteristics are summarized in Table 5.1.

The investment universe is very different for the three datasets. Basically, with the use of the first and the second dataset, we explore risk-based investing in two specific equity markets: the Eurozone and the Emerging Europe. On the contrary, the third dataset uses a larger investment universe that includes both equity and fixed income asset classes. Therefore, our empirical tests in the next sections of this chapter deal with two problems of allocation within an asset class and with a problem of allocation among multiple asset classes.

As starting point, we explore the statistical properties of the monthly returns in our datasets initiating from the univariate statistics. We begin with the estimates of

Table 5.1 Description of the datasets used for the empirical investigation

Dataset	Description	Period	Frequency	Number of observations
DATASET "EUROZONE"	MSCI AUSTRIA, MSCI BELGIUM, MSCI FINLAND, MSCI FRANCE, MSCI GERMANY, MSCI GREECE, MSCI IRELAND, MSCI ITALY, MSCI NETHERLANDS, MSCI PORTUGAL, MSCI SPAIN	From 5/1996 to 4/2015	Monthly	228
DATASET "EMERGING EUROPE"	MSCI CZECH REPUBLIC, MSCI HUNGARY, MSCI POLAND, MSCI RUSSIA, MSCI TURKEY	From 5/2001 to 4/2015	Monthly	168
DATASET "GLOBAL"	JPM EUROPE GOVT. BOND, JPM UNITED STATES GOVT. BOND, JPM EMBI GLOBAL DIVERSIFIED COMPOSITE, BOFA ML GLB BROAD CORP., MSCI EUROPE, MSCI NORTH AMERICA, MSCI PACIFIC FREE, MSCI EMERGING MARKETS	From 5/1997 to 4/2015	Monthly	216

Table 5.2 Descriptive univariate statistics on monthly returns for dataset "EUROZONE"

Asset class	Mean (%)	Standard deviation (%)	Skewness	Kurtosis
MSCI AUSTRIA	0.512	6.583	–0.954	6.352
MSCI BELGIUM	0.774	5.550	–1.582	8.601
MSCI FINLAND	1.323	9.160	0.120	4.775
MSCI FRANCE	0.783	5.341	–0.442	3.420
MSCI GERMANY	0.881	6.289	–0.569	4.901
MSCI GREECE	0.177	9.899	0.212	4.684
MSCI IRELAND	0.341	6.289	–0.671	3.960
MSCI ITALY	0.620	6.212	0.090	3.599
MSCI NETHERLANDS	0.791	5.521	–0.763	4.325
MSCI PORTUGAL	0.485	5.790	–0.334	3.930
MSCI SPAIN	1.019	6.313	–0.290	4.044

Table 5.3 Descriptive univariate statistics on monthly returns for dataset "EMERGING EUROPE"

Asset class	Mean (%)	Standard deviation (%)	Skewness	Kurtosis
MSCI CZECH REPUBLIC	1.459	6.422	–0.038	3.665
MSCI HUNGARY	0.983	8.988	–0.376	4.531
MSCI POLAND	0.896	8.716	0.068	3.825
MSCI RUSSIA	1.163	9.246	–0.130	3.249
MSCI TURKEY	1.327	12.338	–0.059	3.574

Table 5.4 Descriptive univariate statistics on monthly returns for dataset "GLOBAL"

Asset class	Mean (%)	Standard deviation (%)	Skewness	Kurtosis
JPM EUROPE GOVT. BOND	0.499	1.117	0.073	3.080
JPM UNITED STATES GOVT. BOND	0.507	3.138	0.526	3.667
JPM EMBI GLOBAL DIVERS. COMP.	0.825	3.848	–1.679	14.156
BOFA ML GLB BROAD CORP.	0.508	2.059	0.363	3.261
MSCI EUROPE	0.667	4.725	–0.540	3.752
MSCI NORTH AMERICA	0.718	4.729	–0.480	3.441
MSCI PACIFIC FREE	0.423	5.018	–0.029	3.538
MSCI EMERGING MARKETS	0.780	6.693	–0.718	4.798

the first four statistical moments for each element in the datasets over the entire available sample period. Tables 5.2, 5.3 and 5.4 report these statistics.

The market conditions, in terms of return-risk combination, are rather different across the datasets. Just to give an evidence, the highest mean return, on a monthly basis, for an asset class in the first dataset is 1.323 %; it is 1.459 % in the second dataset and 0.825 % in the third dataset. The maximum monthly standard deviation is, respectively, 9.899, 12.338 and 6.693 %. The range of monthly volatilities in the first dataset is from 5.341 to 9.899 %; it goes from 6.422 to 12.338 % in the second dataset while for the third dataset it goes from 1.117 to 6.693 %.

As second step, we look at the sample estimates of multivariate parameters by reporting the correlation matrix for the chosen datasets in Tables 5.5, 5.6 and 5.7.

With reference to datasets 1 and 2, given that the equity indices in the correlation matrices reported in Tables 5.5 and 5.6 are constituents or sub-indices of the same geographical area benchmark, not so surprisingly all pair-wise correlations are positive. Nevertheless, we observe in the first dataset a larger variation than in the second dataset. Pair-wise correlations range between 0.388 and 0.909 for asset classes in the first dataset, and between 0.458 and 0.745 for asset classes in the second dataset. With regard to the third dataset, the global investment universe represented allows us to observe negative correlation parameters and a wider range of values from −0.134 to 0.850. Therefore, we can easily deduce that the last dataset exhibits major benefits of diversification.

The next step for each dataset was that of testing the Gaussian hypothesis. We start by performing two tests of the null hypothesis of normality for the univariate returns distributions. We apply the most widely used test due to Jarque and Bera (1980), named "JB test", that is based on the fact that skewness and kurtosis are, respectively, zero and three in a Gaussian distribution.[1] In addition to the "JB test", which is based on moments, we employ a test based on densities. Specifically, we consider the Lilliefors test, "LI test". It compares the empirical cumulative distribution function of returns with a normal distribution with mean and standard deviation equal to the sample estimates. We summarize the results from the tests for the dataset "EUROZONE", "EMERGING EUROPE" and "GLOBAL", respectively, in Tables 5.8, 5.9 and 5.10. The results for acceptance of the null hypothesis of normality at 5 % confidence level are shown in bold.

We also test for the null hypothesis that our datasets come from a multivariate normal distribution. To this end, we rely on the Mardia test that is based on the multivariate third and fourth moments statistics. The results are reported in Table 5.11.

To sum up the results from our tests, we observe that in general the normality assumption for our univariate return series cannot be accepted (except for the series included in the dataset "EMERGING EUROPE") and that the multivariate normality hypothesis has to be strongly rejected.

As last step in exploring the empirical behaviour of asset class returns, we consider the issue of time dependency. Initially, we believe it is crucial to

[1]The JB statistic is defined as: $JBstat = T\left[\frac{Skewness}{6} + \frac{(Kurtosis-3)^2}{24}\right]$ and is distributed as Chi-squared with 2 degrees of freedom. T is the sample size. We have to admit that *JB* test is more appropriate when the sample is very large, so it is a good idea to report also the results from a different test.

Table 5.5 Correlation matrix for time series of monthly returns in dataset "EUROZONE"

	MSCI AUSTRIA	MSCI BELGIUM	MSCI FINLAND	MSCI FRANCE	MSCI GERMANY	MSCI GREECE	MSCI IRELAND	MSCI ITALY	MSCI NETHERLANDS	MSCI PORTUGAL	MSCI SPAIN
MSCI AUSTRIA	**1**	0.667	0.388	0.646	0.611	0.567	0.585	0.614	0.662	0.557	0.580
MSCI BELGIUM	0.667	**1**	0.457	0.759	0.716	0.488	0.655	0.657	0.802	0.596	0.618
MSCI FINLAND	0.388	0.457	**1**	0.700	0.652	0.400	0.477	0.588	0.612	0.514	0.545
MSCI FRANCE	0.646	0.759	0.700	**1**	0.909	0.564	0.625	0.853	0.887	0.712	0.808
MSCI GERMANY	0.611	0.716	0.652	0.909	**1**	0.525	0.619	0.764	0.856	0.637	0.731
MSCI GREECE	0.567	0.488	0.400	0.564	0.525	**1**	0.416	0.595	0.504	0.524	0.610
MSCI IRELAND	0.585	0.655	0.477	0.625	0.619	0.416	**1**	0.542	0.672	0.484	0.536
MSCI ITALY	0.614	0.657	0.588	0.853	0.764	0.595	0.542	**1**	0.766	0.703	0.799
MSCI NETHERLANDS	0.662	0.802	0.612	0.887	0.856	0.504	0.672	0.766	**1**	0.638	0.725
MSCI PORTUGAL	0.557	0.596	0.514	0.712	0.637	0.524	0.484	0.703	0.638	**1**	0.721
MSCI SPAIN	0.580	0.618	0.545	0.808	0.731	0.610	0.536	0.799	0.725	0.721	**1**

Table 5.6 Correlation matrix for time series of monthly returns in dataset “EMERGING EUROPE”

	MSCI CZECH REPUBLIC	MSCI HUNGARY	MSCI POLAND	MSCI RUSSIA	MSCI TURKEY
MSCI CZECH REPUBLIC	**1**	0.652	0.712	0.582	0.528
MSCI HUNGARY	0.652	**1**	0.745	0.629	0.522
MSCI POLAND	0.712	0.745	**1**	0.568	0.569
MSCI RUSSIA	0.582	0.629	0.568	**1**	0.458
MSCI TURKEY	0.528	0.522	0.569	0.458	**1**

understand if returns of asset classes included in our datasets display significant serial correlation. To ascertain the aforesaid, we use the test based on the Ljung-Box Q statistic (abbreviated as LBQ).[2] In particular, we test for the null hypothesis at 5 % confidence level of no serial correlation in asset class returns computed up to the fifth lag[3] against the alternative hypothesis that at least one autocorrelation coefficient is different from zero. The results for this analysis of linear dependence between our asset class returns and their past values are shown in Tables 5.12, 5.13 and 5.14. Our results do not reveal significant linear dependencies in returns; actually the null hypothesis is frequently accepted. This can be viewed as a further support for risk-based asset allocation strategies. Indeed, these results suggest that returns generally do not move in a predictable fashion, so the exclusion of expected returns from the set of asset allocation inputs can make sense.

We execute the same test, based on the Ljung-Box Q statistic, on the squared returns. In this way, we attempt to ascertain serial dependence in the variability, rather than in the level, of returns. Actually, from our results we learn of time dependence in volatility for the majority of our time series. In fact, the null hypothesis of no autocorrelation is frequently rejected. This is especially true if we refer to the datasets “EUROZONE” and “EMERGING EUROPE”. It is a proof that there is heteroskedasticity in the returns data. This feature is not negligible. For the purpose of implementing asset allocation approaches focused on the risk dimension, it can reasonably legitimate the use of time-varying volatility models to appropriately describe a process of conditional volatility. Only some series in the dataset “GLOBAL” seem to show time invariant volatility (Tables 5.15, 5.16 and 5.17).

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[2]The Ljung-Box Q statistic is defined in our analysis as follows: $LBQstatistic(5) = T(T+2)\sum_{z=1}^{5}\frac{1}{T-z}\rho_z^2$ where ρ_z indicates the autocorrelation of order z. The Ljung-Box Q statistic is asymptotically distributed as a Chi-squared with degrees of freedom equal to number z of lags considered.

[3]The choice to test for linear dependence up to the fifth lag is not random. We followed some studies (see Tsay 2002) which suggest to use a number of lags approximately equal to $\ln(T)$.

Table 5.7 Correlation matrix for time series of monthly returns in dataset "GLOBAL"

	JPM EUROPE GOVT. BOND	JPM UNITED STATES GOVT. BOND	JPM EMBI GLOBAL DIVERS. COMPOSITE	BOFA ML GLB BROAD CORP.	MSCI EUROPE	MSCI NORTH AMERICA	MSCI PACIFIC FREE	MSCI EMERGING MARKETS
JPM EUROPE GOVT. BOND	**1**	0.518	0.237	0.543	−0.134	−0.063	−0.022	−0.129
JPM UNITED STATES GOVT. BOND	0.518	**1**	0.538	0.850	−0.122	0.195	0.122	−0.062
JPM EMBI GLOBAL DIVERS. COMPOSITE	0.237	0.538	**1**	0.696	0.437	0.619	0.552	0.598
BOFA ML GLB BROAD CORP.	0.543	0.850	0.696	**1**	0.176	0.425	0.413	0.228
MSCI EUROPE	−0.134	−0.122	0.437	0.176	**1**	0.824	0.649	0.756
MSCI NORTH AMERICA	−0.063	0.195	0.619	0.425	0.824	**1**	0.694	0.730
MSCI PACIFIC FREE	−0.022	0.122	0.552	0.413	0.649	0.694	**1**	0.739
MSCI EMERGING MARKETS	−0.129	−0.062	0.598	0.228	0.756	0.730	0.739	**1**

Table 5.8 Test for normality on monthly returns–dataset "EUROZONE"

	JB test		LI test	
	Stat.	p-value (%)	Stat.	p-value (%)
MSCI AUSTRIA	141.296	0.100	0.097	0.100
MSCI BELGIUM	393.102	0.100	0.110	0.100
MSCI FINLAND	30.491	0.100	0.072	0.593
MSCI FRANCE	9.104	1.831	0.064	2.593
MSCI GERMANY	46.660	0.100	0.075	0.324
MSCI GREECE	28.655	0.100	0.061	3.790
MSCI IRELAND	25.894	0.110	0.063	2.863
MSCI ITALY	3.719	**12.034**	0.042	**43.111**
MSCI NETHERLANDS	38.785	0.100	0.083	0.100
MSCI PORTUGAL	12.470	0.852	0.048	**23.721**
MSCI SPAIN	13.562	0.690	0.071	0.720

Table 5.9 Test for normality on monthly returns–dataset "EMERGING EUROPE"

	JB test		LI test	
	Stat.	p-value (%)	Stat.	p-value (%)
MSCI CZECH REPUBLIC	3.131	**15.485**	0.039	**50.000**
MSCI HUNGARY	20.375	0.265	0.048	**44.824**
MSCI POLAND	4.899	**6.616**	0.065	**8.376**
MSCI RUSSIA	0.905	**50.000**	0.039	**50.000**
MSCI TURKEY	2.406	**23.810**	0.057	**19.859**

Table 5.10 Test for normality on monthly returns–dataset "GLOBAL"

	JB test		LI test	
	Stat.	p-value (%)	Stat.	p-value (%)
JPM EUROPE GOVT. BOND	0.252	**50.000**	0.063	3.797
JPM UNITED STATES GOVT. BOND	13.957	0.653	0.057	**8.944**
JPM EMBI GLOBAL DIVERS. COMP.	1221.682	0.100	0.061	4.749
BOFA ML GLB BROAD CORP.	5.362	**5.659**	0.053	**14.341**
MSCI EUROPE	15.580	0.489	0.079	0.219
MSCI NORTH AMERICA	10.061	1.464	0.077	0.356
MSCI PACIFIC FREE	2.639	**21.691**	0.058	**7.901**
MSCI EMERGING MARKETS	47.632	0.100	0.063	3.909

Table 5.11 Mardia tests for multivariate normality on monthly returns

	Stat.	Critical value	p-value (%)
Dataset "EUROZONE"	857.7404	327.5117	0.000
Dataset "EMERGING EUROPE"	74.3569	50.9985	0.018
Dataset "GLOBAL"	2370.6555	147.6735	0.000

Table 5.12 Analysis of serial correlation in returns–dataset "EUROZONE"

	LBQ on returns	
	Stat.	p-value (%)
MSCI AUSTRIA	21.174	0.075
MSCI BELGIUM	26.751	0.006
MSCI FINLAND	11.458	4.302
MSCI FRANCE	4.413	**49.157**
MSCI GERMANY	1.788	**87.767**
MSCI GREECE	10.979	**5.179**
MSCI IRELAND	18.224	0.268
MSCI ITALY	4.216	**51.874**
MSCI NETHERLANDS	4.590	**46.791**
MSCI PORTUGAL	11.718	3.886
MSCI SPAIN	5.682	**33.845**

Table 5.13 Analysis of serial correlation in returns–dataset "EMERGING EUROPE"

	LBQ on returns	
	Stat.	p-value (%)
MSCI CZECH REPUBLIC	7.546	**18.310**
MSCI HUNGARY	10.362	**6.560**
MSCI POLAND	4.983	**41.800**
MSCI RUSSIA	8.563	**12.784**
MSCI TURKEY	2.853	**72.257**

Table 5.14 Analysis of serial correlation in returns–dataset "GLOBAL"

	LBQ on returns	
	Stat.	p-value (%)
JPM EUROPE GOVT. BOND	3.817	**57.601**
JPM UNITED STATES GOVT. BOND	2.648	**75.413**
JPM EMBI GLOBAL DIVERS. COMP.	1.771	**87.986**
BOFA ML GLB BROAD CORP.	4.575	**46.997**
MSCI EUROPE	8.593	**12.643**
MSCI NORTH AMERICA	5.109	**40.267**
MSCI PACIFIC FREE	13.847	1.661
MSCI EMERGING MARKETS	10.810	**5.528**

Table 5.15 Analysis of serial correlation in squared returns–dataset "EUROZONE"

	LBQ on squared returns	
	Stat.	p-value (%)
MSCI AUSTRIA	74.011	0.000
MSCI BELGIUM	46.690	0.000
MSCI FINLAND	37.065	0.000
MSCI FRANCE	19.929	0.129
MSCI GERMANY	13.854	1.656
MSCI GREECE	14.239	1.416
MSCI IRELAND	75.346	0.000
MSCI ITALY	8.615	**12.545**
MSCI NETHERLANDS	32.281	0.001
MSCI PORTUGAL	11.801	3.761
MSCI SPAIN	9.754	**8.253**

Table 5.16 Analysis of serial correlation in squared returns–dataset "EMERGING EUROPE"

	LBQ on squared returns	
	Stat.	p-value (%)
MSCI CZECH REPUBLIC	16.608	0.531
MSCI HUNGARY	12.120	3.318
MSCI POLAND	30.827	0.001
MSCI RUSSIA	34.804	0.000
MSCI TURKEY	18.918	0.199

Table 5.17 Analysis of serial correlation in squared returns–dataset "GLOBAL"

	LBQ on squared returns	
	Stat.	p-value (%)
JPM EUROPE GOVT. BOND	2.896	**71.602**
JPM UNITED STATES GOVT. BOND	12.441	2.922
JPM EMBI GLOBAL DIVERS. COMP.	2.480	**77.951**
BOFA ML GLB BROAD CORP.	6.357	**27.303**
MSCI EUROPE	26.295	0.008
MSCI NORTH AMERICA	39.531	0.000
MSCI PACIFIC FREE	19.497	0.155
MSCI EMERGING MARKETS	7.493	**18.650**

5.2 Implementation of the Empirical Investigation

The primary goal of our empirical investigation is to evaluate the risk-based approaches to asset allocation in order to obtain helpful information for investment practice. We consider the optimal risk parity strategy, the equally-weighted approach, the global minimum-variance approach and the most diversified portfolio approach, in one world the portfolio strategies that react to the problem of estimation risk in the portfolio construction procedures removing the "performance dimension" from the set of inputs.

In contrast to some previous studies (Maillard et al. 2010; Chaves et al. 2011, 2012; Anderson et al. 2012), we do not deem it of interest to evaluate a competition between the *μ-free strategies* and portfolios that are ex-ante mean-variance efficient, but exposed to the likely mis-specification of expected returns. We focus exclusively on the risk-based strategies to get a better understanding of their practical implications for financial portfolios from several and relevant perspectives. We also aim to understand if the equally-weighted approach deserves to be celebrated as a prevailing investment rule over the remaining risk-based strategies after it has gained the reputation of a winner versus asset allocation approaches that work in a classical mean-variance framework as documented by DeMiguel et al. (2009).

To achieve the stated goal, our empirical study has to be implemented across the three datasets that were described in Sect. 5.1 under realistic conditions. We remember that, with the exception of the equally-weighted approach, the remaining *μ-free strategies* require as input only a risk model that is a covariance matrix or variances (standard deviations) and correlations. The realism could not be maintained if the portfolio weights recommended by each risk-based strategy were computed using risk estimates derived from the entire sample period available for each dataset and if they were then used to explore how the strategies behave during the same historical sample. In fact, these in-sample solutions would assume perfect knowledge of forward-looking information in computing portfolio weights and, as a consequence, would focus on non-investable strategies. Moreover, such methodology would disregard the time-varying behaviour of risk over the sample period that was detected in Sect. 5.1 (with relevant exceptions in the third dataset) using the Ljung-Box Q statistic.

Given this premise, it becomes clear that our empirical investigation must consists of an out-of-sample analysis. We believe that implementing a rolling window procedure is a prominent and appropriate approach to this end.

Specifically, given a T-month-long dataset of asset class returns (with $T = 228$, 168 and 216, respectively, for dataset 1, 2 and 3), we choose an estimation window of length L set to 60 months. With these observations, we estimate the parameters needed to implement a particular strategy. Then, the estimated parameters are used to determine the relative portfolio weights according to the algorithms and/or the algebraic procedures illustrated in Chaps. 3 and 4. These portfolio allocations are considered as the starting point to compute, with monthly frequency, the portfolio returns for the next 6 months. Precisely, we assume to manage the portfolio according to the buy and hold technique over this time interval. Afterwards, the estimation window is rolled 6 months forward, the input parameters are re-estimated and new portfolio allocations, based another time on 5 years of monthly data and free of fluctuating due to the asset class performances for the following 6 months, are determined. This process is continued until the estimation windows have exploited the data that reach the end of the sample period in each dataset. We can thus say that in our empirical analysis the risk-based strategies rely on 5-years-long estimation periods and are rebalanced semi-annually.

For sake of clarity, we provide an example. With reference to the "EUROZONE" dataset, we first use the historical returns for the 11 asset classes from May 1996 through April 2001 to estimate the input parameters and obtain portfolio weights for the first asset allocation experiment according to the different *μ-free strategies*. Using the asset class returns from May 2001 through October 2001, we calculate the first 6 months out-of-sample returns for our initial risk-based portfolios. Then, the second composition for these portfolios is obtained using the estimation period from November 1996 to October 2001. These new allocations generate out-of-sample returns from November 2001 to April 2002.

Once completed, the procedure for the empirical investigation described above generates a series of non-overlapping monthly out-of-sample returns for each risk-based asset allocation approach with different time-span across the three datasets. Generally speaking, the length of the series of out-of-sample returns is calculated as T-L. However, we prefer to summarize the final outcome of the rolling window procedure in Table 5.18 by reporting:

- the number of feasible asset allocation experiments using each dataset;
- the number of out-of-sample returns obtained from each dataset;
- the historical period covered by the out-of-sample returns generated from each dataset.[4]

For a complete presentation of our empirical analysis, there are only two basic aspects not pointed out so far. The first one is that for all the asset allocation experiments carried out, we make use of the same constraints: non-negative portfolio weights that sum to one. This is a necessary condition in order to analyse and evaluate the risk-based strategies in a fair way. The second aspect is that our

Table 5.18 Details on the out-of-sample analysis

	Number of asset allocation experiments	Number of out-of-sample monthly returns	Extension of the out-of-sample period
Dataset "EUROZONE"	29	168	From May 2001 through April 2015
Dataset "EMERGING EUROPE"	19	108	From May 2006 through April 2015
Dataset "GLOBAL"	27	156	From May 2002 through April 2015

[4]Since the estimation windows exploit all the observations available until the end of the sample period in each dataset, which is set to April 2015, we are short of the out-of-sample returns for the last asset allocation experiment.

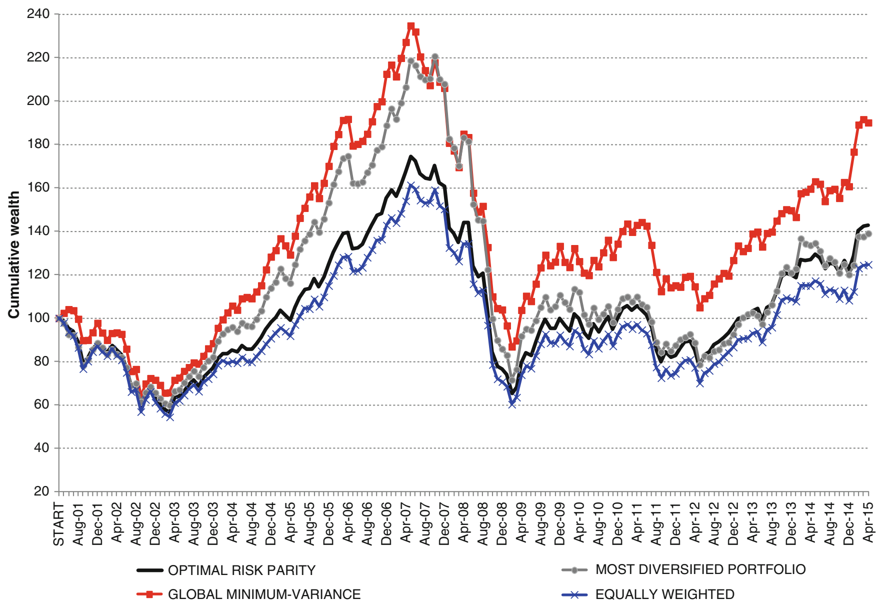

Fig. 5.1 Wealth evolution from risk-based asset allocation approaches using the dataset "EUROZONE"

empirical investigation does not suffer from a look-ahead bias. In other words, portfolio solutions are computed in each experiment without using forward-looking information. Consequently, the out-of-sample returns generated by these asset allocation experiments represent performances of investable strategies. In the following figures (Figs. 5.1, 5.2 and 5.3), we show the wealth evolution they would have produced in the out-of-sample period assuming 100 € as initial capital invested.

We also implement some statistical tests on the out-of-sample returns of each risk-based strategy resulting from the use of each dataset. As was the case for the time series of asset class returns, we try to understand if they are approximately normally distributed on a monthly basis. The results from the Jarque-Bera test and from the Lilliefors test are reported in Table 5.19. From them we learn that the Gaussian distribution can be considered an acceptable assumption for the returns resulting from strategies applied to the dataset "GLOBAL" even if the majority of the asset class involved does not obey this probabilistic distribution.[5] By contrast, the null hypothesis has to be rejected at the 5 % significant level for strategies implemented with dataset "EUROZONE" and "EMERGING EUROPE", with only the exception of the global minimum-variance approach in the latter (though normality was accepted for individual asset classes).[6]

[5]See Table 5.10 in Sect. 5.1.

[6]See Table 5.9 in Sect. 5.1.

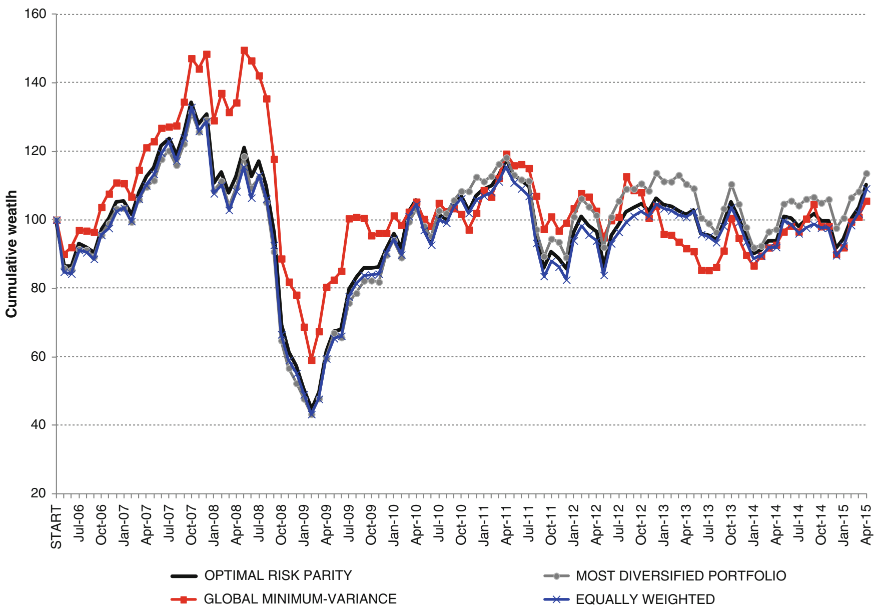

Fig. 5.2 Wealth evolution from risk-based asset allocation approaches using the dataset "EMERGING EUROPE"

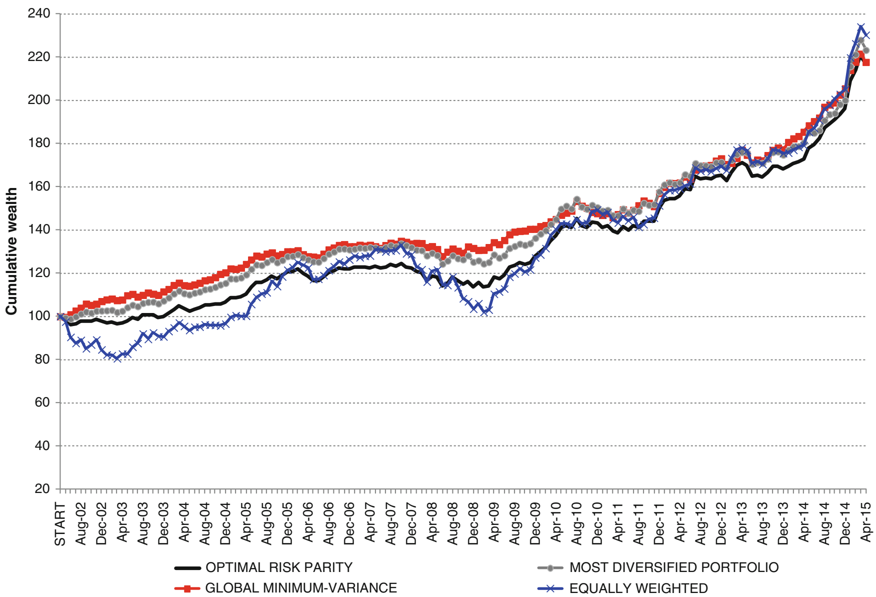

Fig. 5.3 Wealth evolution from risk-based asset allocation approaches using the dataset "GLOBAL"

Table 5.19 Normality tests on out-of-sample returns from risk-based asset allocation approaches

	JB test		LI test	
	Stat.	p-value (%)	Stat.	p-value (%)
DATASET "EUROZONE"				
Optimal risk parity	32.345	0.100	0.076	1.931
Global minimum-variance	27.030	0.121	0.080	1.093
Most diversified portfolio	23.586	0.177	0.076	1.860
Equally-weighted	27.843	0.110	0.074	2.698
DATASET "EMERGING EUROPE"				
Optimal risk parity	33.382	0.100	0.087	4.565
Global minimum-variance	19.839	0.339	0.067	**26.655**
Most diversified portfolio	36.904	0.100	0.087	4.329
Equally-weighted	30.584	0.110	0.088	3.827
DATASET "GLOBAL"				
Optimal risk parity	9.992	1.562	0.041	**50.000**
Global minimum-variance	2.681	**19.857**	0.059	**21.540**
Most diversified portfolio	145.615	0.100	0.063	**14.248**
Equally-weighted	5.339	**5.478**	0.067	**9.130**

Since the portfolio allocations according to each strategy are computed using estimates from partially overlapping rolling windows, we analyse for serial correlation up to the fifth lag both at the level of returns and at the level of squared returns. According to the results from test based on the Ljung-Box Q statistic shown in Table 5.20, significant autocorrelation is found in the out-of-sample returns, in the squared version as well, of risk based strategies used for investments in the Eurozone or in the area of Emerging Europe. However, time dependency is not detected in returns and volatility of the asset allocation approaches regarding the dataset "GLOBAL". Based on our statistical tests, we have the conditions to consider the sequences of out-of-sample returns generated by this last dataset as independent and identically distributed (*iid*) processes.

5.3 Methodology and Criteria for Evaluating the Risk-Based Approaches

The way we implement our empirical study is closely related to the article by DeMiguel et al. (2009) and also inspired by their findings.

The authors came to the conclusion that naively diversified portfolios (e.g. equally-weighted portfolios) are superior on a risk-adjusted basis both to the market capitalization-weighted portfolio and to portfolios that ex-ante are constructed to be mean-variance efficient and, consequently, they should be recommended to investors. In detail, using seven different empirical datasets and fourteen models for

Table 5.20 Tests for serial correlation in out-of-sample returns from risk-based asset allocation approaches

	LBQ on returns		LBQ on squared returns	
	Stat.	p-value (%)	Stat.	p-value (%)
DATASET "EUROZONE"				
Optimal risk parity	15.965	0.694	18.946	0.197
Global minimum-variance	18.645	0.224	12.189	3.229
Most diversified portfolio	21.591	0.063	19.621	0.147
Equally-weighted	14.772	1.138	21.193	0.074
DATASET "EMERGING EUROPE"				
Optimal risk parity	12.329	3.055	14.742	1.152
Global minimum-variance	11.318	4.543	25.796	0.010
Most diversified portfolio	12.643	2.696	12.359	3.019
Equally-weighted	10.863	**5.416**	12.187	3.232
DATASET "GLOBAL"				
Optimal risk parity	6.992	**22.121**	2.652	**75.348**
Global minimum-variance	6.600	**25.215**	5.187	**3.344**
Most diversified portfolio	4.135	**53.015**	3.765	**58.366**
Equally-weighted	7.779	**16.884**	6.399	**26.927**

asset allocation, DeMiguel et al. (2009) found out that optimization models proposed in the literature to deal with the issue of estimation risk cannot consistently deliver better risk-adjusted performance than that of the "1 over N" portfolio which, according to the authors, means that, out-of-sample, the advantage from optimized diversification is more than offset by estimation errors and that the improvements in the estimation of the moments of financial returns are not yet sufficient. In particular, they observed that more efforts are required as regards the estimation of expected returns.

The authors' conclusion inspires the idea of considering the equally-weighted approach as a benchmark also in our contribution. In other words, the "1 over N" portfolio can be used to assess other investment strategies that are not explicitly designed to improve the estimates of the required inputs or to better manage these estimates (as documented in the prevailing existing literature), but rather that opt to dismiss the most critical input; we know this is the case for the risk-based strategies we have examined in the previous chapters. Therefore, our empirical study can be helpful in order to understand if the equally-weighted approach, the most simplified investment strategy, can be considered systematically superior (or inferior) with respect to other asset allocation approaches that are somewhat simplified as well.

Regarding the implementation of our empirical investigation, similarly to DeMiguel et al. (2009), we believe it is not advisable to confine the analysis of the risk-based strategies to a single criteria, it is definitely better to consider different evaluation dimensions. To select them, it is useful to keep in mind that investors look for several characteristics: they like efficient, diversified and stable portfolios. Thus the fundamental criteria to adopt are the following:

- financial efficiency;
- level of diversification;
- asset allocation stability.

The second criteria was ignored by DeMiguel et al. (2009), but its lack would be inappropriate in this study. The mentioned criteria are illustrated according to this sequence and are quantified in the next section using an expanded set of tools with respect to the existing empirical literature.

Concerning the first evaluation dimension, financial efficiency, we can see how it is motivated by the need or interest to capture simultaneously return and risk of a strategy and compare this trade-off across the other investment options when there are or there can be differences in the values for return or risk involved. Paying attention to financial efficiency with regard to investment strategies that have completely excluded the expected returns from the portfolio construction process may seem illogical because we note the objective function (or its absence) chosen, ex-ante, by the different risk-based approaches and the performance evaluation, ex-post, are inconsistent. However, we have to keep in mind that the exclusion of the performance dimension from strategies' set up is motivated by the intention to mitigate estimation risk and this in no way means completely disregarding the reward achieved. So it is reasonable to acknowledge some space to risk-adjusted performance measures in our empirical investigation.

As a preliminary comparison, we explore separately the return and risk characteristics of the strategies. With regard to the first quantity, we merely propose the cumulative performance and measures of geometric and arithmetic return both in the monthly and annualized version.

Concerning risk, we are aware that several ideas or conceptions are plausible. Obviously, we consider the traditional standard deviation of returns without forgetting that it is a proper measure of variability when returns are normally distributed and when investors are disposed to treat as a contribution to risk any returns that fall on either side of the mean return. On this point, previous studies (Demey et al. 2010; Maillard et al. 2010) have observed that the volatilities of (long-only) risk-based portfolios are ranked according to this order:[7]

$$\sigma_{GMVP} \leq \sigma_{ORP} \leq \sigma_{EW} \tag{5.1}$$

where $\sigma_{GMVP}, \sigma_{ORP}, \sigma_{EW}$ represent the volatility, respectively, of the global minimum-variance portfolio, of the optimal risk parity strategy, of the equally-weighted approach.[8]

[7]The most diversified portfolio approach was not included in the mentioned studies.

[8]For sake of completeness, we remember that Maillard et al. (2010) attempt to provide also a "theoretical" justification for this ranking. They view the optimal risk parity portfolio as intermediate between the minimum-variance portfolio and the equally-weighted portfolio in the sense

In addition to standard deviation, we compute measures of risk that overcome its criticisms. We refer to measures that focus on the variability of underperformance with respect to an exogenous threshold level that is considered as a Minimum Acceptable Return (*MAR*). They are called downside risk measures and are mainly addressed in the contributions by Nawrocki (1999) and Sortino et al. (1999).

Drawdown measures are also included in our analysis. Specifically, we consider the maximum drawdown, that concentrates on a single event, and the average value of drawdowns we can count over the entire out-of-sample path for each strategy.

Once the separated analysis of performance and risk is terminated, in order to reveal the efficiency of the risk-based portfolios we turn our attention to the category of reward-to-variability ratios or, equivalently, of risk-adjusted performance measures. From this class, we select the most popular index that is the Sharpe Ratio. For its computation, we use monthly returns from JPM Euro Cash 1 Month index as a proxy for the risk free rate. In fact, as well-known, the Sharpe Ratio (*SR*) is defined as the difference between the mean strategy return and the mean risk free return divided by the standard deviation of the strategy returns. Formally:

$$SR_k = \frac{\bar{R}_k - \bar{R}_{rf}}{\sigma_k} \tag{5.2}$$

where:

$\bar{R}_k$ arithmetic mean return for strategy k;
$\bar{R}_{rf}$ arithmetic mean return for the risk free proxy;
σ_k standard deviation of returns for strategy k

Naturally, the Sharpe Ratio is not the best fitting ratio in presence of non-Gaussian distributions (as we observed the standard deviation, capturing variability around a central tendency, is not necessarily the most correct way to interpret risk in such situations), so we also use ratios that modify the risk measure in order to integrate the downside risk or to consider as a penalty element the drawdowns encountered by each strategy. Furthermore, it is useful to point out that the frequent violation of the normal distribution assumption is not the unique reason for the selected risk-adjusted performance measures. Everyone can observe that risk-based strategies have attracted many investors in conjunction with a growing fear of collapses and crashes. So, the stronger need for protection and the increasing demand for control of bad risk and not just for the instability around the mean can be seen as a rationale for using additional and different ratios in the ex-post analysis.

(Footnote 8 continued)

that the authors consider all these portfolios as a form of variance-minimizing portfolio subject to a constraint of diversification in terms of asset class weights with a different power: not restrictive for the minimum-variance portfolio, extremely restrictive for the equally-weighted portfolio and sufficiently restrictive for the optimal risk parity portfolio.

Regarding the above, we include the Sortino Ratio, with *MAR* represented by both the risk free return (as in the original version) and a null return and also the Performance Ulcer Index proposed by Martin and McCann (1989). They are calculated, respectively, as follows:

$$Sortino\,Ratio_k = \frac{\bar{R}_k - MAR}{\sqrt{\frac{1}{T}\sum_{t=1}^{T}\min\left(R_{t,k} - MAR;0\right)^2}} \tag{5.3}$$

$$Performance\ Ulcer\ Index_k = \frac{\bar{R}_k - \bar{R}_{rf}}{\sqrt{\frac{\left(\sum_{t=1}^{T} DD_{t,k}^2\right)}{T}}} \tag{5.4}$$

where:

$R_{t,k}$	return for strategy k in period t within the sample period of length T;
$DD_{t,k}$	drawdown for strategy k computed in period t

Even if the Sortino Ratio and the Performance Ulcer Index are able to overcome the problem of "Gaussian dependency" and recognize the existence of good and bad risk, they do not allow for a subjective and different emphasis of favourable events (overperforming a threshold) and unfavourable ones (underperforming a threshold) and they do not incorporate one-sided measure for reward valuation.[9] Thus the calculation of a reward-to-risk ratio in the set-up proposed by Farinelli and Tibiletti (2008) which can be written in the following general form is necessary:

$$\Phi^{p;q}_{threshold}(R_k) = \frac{E^{1/p}\left(\left((R_k - threshold)^+\right)^p\right)}{E^{1/q}\left(\left((R_k - threshold)^-\right)^q\right)} \tag{5.5}$$

where E indicates the expected value and p and q (>0) are called, respectively, right and left orders of the ratio. It is important to point out that the higher the order the higher is the weight given to large deviations relative to small deviations above (in evaluating the reward in the numerator) and under (in evaluating risk in the denominator) the threshold level.

We calculate the Farinelli-Tibiletti Ratio with two common choices for the orders p and q and a third formalization that is not very familiar but can be considered proper for investors and/or asset managers using *μ-free strategies*. In every implementation, the threshold level or *MAR* is represented through the risk free return.

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[9]The theoretical motivation for the use of one-sided type measures in the ratios that include the two key quantities, risk and reward, comes from behavioral finance which argues that investors do not share a unilateral risk aversion in rewarding and losing (Kahneman and Tversky 1979).

In detail, our first choice is for the Upside Potential Ratio (*UPR*) by Sortino et al. (1999) which corresponds to $\Phi^{1,2}_{threshold}$. It is given by:

$$UPR = \Phi^{1;2}_{threshold} = \frac{\frac{1}{T}\sum_{t=1}^{T}\max\left(R_{t,k} - MAR; 0\right)}{\sqrt{\frac{1}{T}\sum_{t=1}^{T}\min\left(R_{t,k} - MAR; 0\right)^2}} \tag{5.6}$$

The second choice is for the Omega Index (*Ω-ratio*) introduced by Cascon et al. (2002) which corresponds to $\Phi^{1;1}_{threshold}$ and is calculated as follows:

$$\Omega - ratio = \Phi^{1;1}_{threshold} = \frac{\frac{1}{T}\sum_{t=1}^{T}\max\left(R_{t,k} - MAR; 0\right)}{\frac{1}{T}\sum_{t=1}^{T}\left|\min\left(R_{t,k} - MAR; 0\right)\right|} \tag{5.7}$$

The last implementation of the Farinelli-Tibiletti Ratio corresponds to $\Phi^{0.5;2}_{threshold}$. With this parametrization, we believe the investor is strongly concerned that large deviations below the *MAR* can occur and is not looking for excellent outcomes above *MAR*. In other words, our choice assumes the investor's desire for "safety" is much stronger than the desire for exceptional performance. Formally this solution for the Farinelli-Tibiletti Ratio is expressed as:

$$\Phi^{0.5;2}_{threshold} = \frac{\left[\frac{1}{T}\sum_{t=1}^{T}\left(\max\left(R_{t,k} - MAR; 0\right)^{0.5}\right)\right]^2}{\sqrt{\frac{1}{T}\sum_{t=1}^{T}\min\left(R_{t,k} - MAR; 0\right)^2}} \tag{5.8}$$

After listing the tools selected to investigate financial efficiency, attention is devoted to the second evaluation criteria we have chosen: the level of diversification or, conversely, of concentration of the risk-based portfolios. This aspect is measured, for each strategy k, both with regard to asset class weights and with regard to percentage total risk contributions. For this purpose, we use different quantities. We start with the Shannon Entropy measure (*SE*) which takes this form:

$$SE_k(\mathbf{w}) = -\sum_{i=1}^{N} w_{i,k}\ln w_{i,k} \tag{5.9}$$

or the following one when the degree of diversification refers to the portfolios risk allocation:

$$SE_k(\mathbf{PTRC}) = -\sum_{i=1}^{N} PTRC_{i,k}\ln PTRC_{i,k} \tag{5.10}$$

When portfolio allocations or percentage total risk contributions are identical across asset classes, the Shannon Entropy measure has its maximum value, *ln(N)*.

The other extreme value, that is zero, occurs when $w_i = 1$ or $PTRC_i = 1$ for one i and zero for the remaining asset classes.

Another interesting statistic we consider for each strategy k is the Gini coefficient (G). For its computation we follow Chaves et al. (2012). First we sort in ascendant order the asset class weights or the asset class percentage total risk contributions and then we, respectively, calculate:

$$G_k(\mathbf{w}) = \frac{2}{N} \cdot \sum_{i=1}^{N} i \cdot \left(w_{i,k} - \bar{w}_k\right) \tag{5.11}$$

$$G_k(\mathbf{PTRC}) = \frac{2}{N} \cdot \sum_{i=1}^{N} i \cdot \left(PTRC_{i,k} - \overline{PTRC}_k\right) \tag{5.12}$$

The last measure for the level of diversification we take into account is the Herfindahl index(H). We use a normalized version (H^*) that is defined as follows:

$$H_k^*(\mathbf{w}) = \frac{\left(H_k(\mathbf{w}) - \frac{1}{N}\right)}{1 - \frac{1}{N}} \text{with } H_k(\mathbf{w}) = \sum_{i=1}^{N} w_{i,k}^2 \tag{5.13}$$

or, as shown in (5.14) when we pay attention to the percentage total risk contributions:

$$H_k^*(\mathbf{PTRC}) = \frac{\left(H_k(\mathbf{PTRC}) - \frac{1}{N}\right)}{1 - \frac{1}{N}} \text{with } H_k(\mathbf{PTRC}) = \sum_{i=1}^{N} PTRC_{i,k}^2 \tag{5.14}$$

According to our definitions, both the Gini coefficient and the Herfindahl index range between zero and one. They have an opposite interpretation compared to the Shannon Entropy measure: lower values for these statistics rather than higher values mean stronger diversification. More precisely, zero means perfect equality or dissemination of weight allocations/risk contributions across asset classes, while one means perfect inequality that is maximum weight or risk concentration in one asset class.

Regardless of the specific statistic considered, we have to clarify that we carry out numerous asset allocation experiments for each strategy k based on a given dataset. This is the reason why we report average values for the selected measures of diversification/concentration calculated at each rebalance date of the portfolio recommended by each strategy within a given investment universe.

The last evaluation dimension included in our empirical analysis is that of portfolio stability. In order to capture this feature, we compute, for each strategy and for each dataset, the average value across the asset allocation experiments of the turnover which, evidently, is inversely related to portfolio stability. In practice, the average turnover for strategy k is defined as follows:

$$\mathit{Average\ Turnover}_k = \frac{1}{NR-1} \sum_{t\_reb=1}^{NR-1} \sum_{1}^{N} \left(\left| w_{k,i,t\_reb^+} - w_{k,i,t\_reb^-} \right| \right) \tag{5.15}$$

where:

NR	total number of asset allocation experiments when using a given dataset[10];
t_reb	order of a given asset allocation experiment in their entire sequence;
w_{k,i,t_reb^+}	portfolio weight in asset class i under strategy k just after re-estimation of portfolio allocation;
w_{k,i,t_reb^-}	desired portfolio weight in asset class i under strategy k before rebalancing

The $\mathit{Average\ Turnover}_k$ defined above can be interpreted as the average percentage of portfolio wealth that needs to be traded at each rebalancing date in order to properly implement each *μ-free strategy*. Generally speaking, high values for turnover mean higher transaction costs and more complexity of implementing a strategy.[11] In addition to the average turnover, we also report the maximum turnover recorded by each strategy.

5.4 Main Findings and Conclusions

In this section, we give the results we obtained from the examination of the various risk-based strategies within the different datasets being considered. We start by displaying the statistics on the performance in Table 5.21.

As a preliminary thought, we can understand from observing the results for cumulative wealth in combination with Figs. 5.1, 5.2 and 5.3 in the previous section, that the market conditions at the back of the out-of-sample period are rather different for the three various datasets. However, we find that it is not possible to identify an identical dominant strategy, in terms of performance, for all datasets. The top performer is the global minimum-variance approach, the most diversified portfolio approach and the equally-weighted approach when, respectively, the dataset "EUROZONE", "EMERGING EUROPE" and "GLOBAL" is used.[12]

[10]In Eq. (5.15), the average turnover is computed considering *NR-1* terms in the mean because, at the beginning, each risk-based portfolio is already allocated coherently with each *μ-free strategy*.

[11]An average turnover higher than 100 % implies that the sum of the absolute value of asset sales and purchases overcomes the entire portfolio wealth.

[12]Based on Figs. 5.1, 5.2 and 5.3, we recognize December 2008 as a turning point for the risk-based strategies we implement. It is easy to see, the top performer with reference to the full sample period

Table 5.21 Performance statistics for the out-of-sample analysis of the risk-based asset allocation approaches

	Optimal risk parity (%)	Global minimum-variance (%)	Most diversified portfolio (%)	Equally-weighted (%)
DATASET "EUROZONE"				
Cumulative performance	42.5790	90.0731	39.0154	24.7337
Geometric annualized return	2.5661	4.6943	2.3809	1.5912
Geometric monthly return	0.2114	0.3830	0.1963	0.1316
Arithmetic annualized return	4.1650	6.1817	4.1905	3.2651
Arithmetic monthly return	0.3471	0.5151	0.3492	0.2721
DATASET "EMERGING EUROPE"				
Cumulative performance	10.2223	5.5297	13.5632	9.1054
Geometric annualized return	1.0873	0.5998	1.4232	0.9730
Geometric monthly return	0.0902	0.0498	0.1178	0.0807
Arithmetic annualized return	4.0538	3.1022	4.4487	4.1057
Arithmetic monthly return	0.3378	0.2585	0.3707	0.3421
DATASET "GLOBAL"				
Cumulative performance	116.7543	117.6719	123.2116	130.2302
Geometric annualized return	6.1313	6.1658	6.3713	6.6249
Geometric monthly return	0.4971	0.4998	0.5160	0.5360
Arithmetic annualized return	6.0988	6.0685	6.3045	6.7835
Arithmetic monthly return	0.5082	0.5057	0.5254	0.5653

In Table 5.22, we compare the *μ-free strategies* in terms of risk admitting the different interpretations illustrated in Sect. 5.3. Looking at standard deviation, our

(Footnote 12 continued)

does not keep this position constantly. For example, with reference to the dataset "EMERGING EUROPE", the most diversified portfolio approach is the top performer over the full sample period, but we can verify it was the worst performer from May 2006 to December 2008. The same happens for the equally-weighted strategy with reference to the dataset "GLOBAL": it is the worst performer in the time interval May 2002–December 2008, but it wins the competition over the entire sample period. This evidence encourages, as a future research opportunity, an empirical investigation to implement considering sub-samples rather than a unique out-of-sample period.

Table 5.22 Risk statistics for the out-of-sample analysis of the risk-based asset allocation approaches

	Optimal risk parity (%)	Global minimum-variance (%)	Most diversified portfolio (%)	Equally-weighted (%)
DATASET "EUROZONE"				
Monthly standard deviation	5.1536	5.0915	5.4884	5.2434
Annualized standard deviation	17.8526	17.6376	19.0125	18.1638
Monthly downside deviation (Threshold = MAR = Free risk)	3.9359	3.8131	4.1019	4.0252
Monthly downside deviation (Threshold = 0 %)	3.8210	3.7011	3.9872	3.9096
Maximum drawdown	–62.6669	–62.9959	–67.5589	–62.6239
Number of drawdowns	147	139	144	151
Average drawdown	–31.1278	–31.9845	–36.4822	–31.6969
DATASET "EMERGING EUROPE"				
Monthly standard deviation	6.9683	6.4278	7.0363	7.1601
Annualized standard deviation	24.1390	22.2664	24.3745	24.8034
Monthly downside deviation (Threshold = MAR = Free risk)	5.1082	4.6250	5.1279	5.2416
Monthly downside deviation (Threshold = 0 %)	5.0064	4.5329	5.0248	5.1385
Maximum drawdown	–66.6867	–60.5033	–67.0390	–67.4580
Number of drawdowns	98	95	99	99
Average drawdown	–26.2714	–30.2791	–23.3865	–26.9070
DATASET "GLOBAL"				
Monthly standard deviation	1.5011	1.0897	1.3793	2.4282
Annualized standard deviation	5.1999	3.7749	4.7779	8.4115
Monthly downside deviation (Threshold = MAR = Free risk)	0.8642	0.6060	0.7273	1.6310
Monthly downside deviation (Threshold = 0 %)	0.7645	0.5157	0.6397	1.5210
Maximum drawdown	–8.8967	–5.2520	–7.0405	–23.4195
Number of drawdowns	99	77	84	109
Average drawdown	–2.4733	–1.6202	–2.1156	–6.7580

empirical study leads to a validation of the ranking stated by Demey et al. (2010) and Maillard et al. (2010). In contrast to previous studies we have also included in our analysis the most diversified portfolio approach which documents the impossibility of assigning a stable rank to this recent risk-based strategy in comparison with the others. To be precise, with reference to datasets considered, we obtain this sequence for the volatility:

$$\sigma_{GMVP} \leq \sigma_{ORP} \leq \sigma_{EQ} \leq \sigma_{MDP} \text{ (dataset ``EURОZONE'')} \tag{5.16}$$

$$\sigma_{GMVP} \leq \sigma_{ORP} \leq \sigma_{MDP} \leq \sigma_{EW} \text{ (dataset ``EMERGING EUROPE'')} \tag{5.17}$$

$$\sigma_{GMVP} \leq \sigma_{MDP} \leq \sigma_{ORP} \leq \sigma_{EW} \text{ (dataset ``GLOBAL'')} \tag{5.18}$$

The above order remains the same when we consider one-side measure of variability like the downside risk whether the MAR is equal to the risk-free return or whether it is a zero return.

When we turn our interest to the drawdown statistics, it does not matter which dataset we consider, we observe that, among the risk-based strategies, the equally-weighted approach exhibits the highest number of drawdowns. Moreover, it shows the highest value of the average drawdown and the heaviest value of maximum drawdown when the dataset “GLOBAL” is considered.

Once we have measured separately the returns and risks from each risk-based asset allocation approach, we can evaluate their composite effect in terms of the efficiency of the portfolios they recommend. The results obtained for the aforementioned evaluation dimension are provided in Table 5.23.

According to these results, the global minimum-variance portfolio outperforms the other risk-based portfolios, when we consider the dataset “EUROZONE” and “GLOBAL”, with a monthly Sharpe Ratio of 0.0688 and 0.3250, respectively, quite distantly followed by the Sharpe Ratio of the optimal risk parity strategy and the most diversified portfolio approach. With reference to the dataset “EMERGING EUROPE”, the Sharpe Ratio of the minimum-variance portfolio collapses at the last position in the ranking with a monthly value of 0.0202, while the most diversified portfolio shows, with 0.0344, the highest value. It is noteworthy the weak position in terms of Sharpe Ratio of the equally-weighted approach within every dataset: it is in the third position with regard to the dataset “EMERGING EUROPE” and it is the worst strategy for the datasets “EUROZONE” and “GLOBAL”.

As we said in Sect. 5.3, in addition to the Sharpe Ratio, we considered “alternative” risk-adjusted performance measures. However, if we take a closer look at the Sortino Ratio, we recognize a totally identical ranking to that of Sharpe Ratio, regardless of the dataset considered. When we use the reward-to-risk ratios in the set-up proposed by Farinelli and Tibiletti (2008) we focus on the Omega Index, the Upside Potential Ratio and to the $\Phi^{0.5;2}_{threshold}$. We observe no changes in the order of the efficiency measures if we refer to the dataset “GLOBAL” and only minor changes if we refer to the dataset “EUROZONE” (alternation of position between the most diversified portfolio approach and the optimal risk parity portfolio). The results we obtain using this kind of measure contrast with the ones obtained from the Sharpe Ratio within our second dataset (“EMERGING EUROPE”): the Upside Potential Ratio gives the top position to the global minimum-variance portfolio which was the last one for the Sharpe Ratio and, according to the $\Phi^{0.5;2}_{threshold}$, the equally weighted approach overcomes the other strategies even if it was only the third classified on the basis of the Sharpe Ratio.

Table 5.23 Financial efficiency analysis for the risk-based asset allocation approaches

	Optimal risk parity	Global minimum-variance	Most diversified portfolio	Equally-weighted
DATASET "EUROZONE"				
Sharpe Ratio	0.0354	0.0688	0.0336	0.0205
Sortino Ratio (rhreshold = MAR = Free risk)	0.0463	0.0919	0.0449	0.0266
Sortino Ratio (Threshold = MAR = 0 %)	0.0908	0.1392	0.0876	0.0696
Performance Ulcer Index (or Martin ratio)	0.0055	0.0106	0.0048	0.0032
Omega Index (Threshold = MAR = free risk)	1.0954	1.1920	1.0898	1.0542
Upside Potential Ratio (Threshold = MAR = Free risk)	0.5316	0.5703	0.5457	0.5183
Farinelli-Tibiletti Ratio (with parametrization p = 0.5; q = 2, Threshold = MAR = Free risk)	0.2663	0.2967	0.2694	0.2544
DATASET "EMERGING EUROPE"				
Sharpe Ratio	0.0301	0.0202	0.0344	0.0299
Sortino Ratio (Threshold = MAR = Free risk)	0.0410	0.0281	0.0473	0.0408
Sortino Ratio (Threshold = MAR = 0 %)	0.0675	0.0570	0.0738	0.0666
Performance Ulcer Index (or Martin ratio)	0.0075	0.0042	0.0094	0.0075
Omega Index (Threshold = MAR = Free risk)	1.0852	1.0564	1.0988	1.0846
Upside Potential Ratio (Threshold = MAR = Free risk)	0.5221	0.5269	0.5255	0.5226
Farinelli-Tibiletti ratio (with parametrization p = 0.5; q = 2, Threshold = MAR = Free risk)	0.2414	0.2268	0.2364	0.2434
DATASET "GLOBAL"				
Sharpe Ratio	0.2376	0.3250	0.2710	0.1704
Sortino ratio (Threshold = MAR = Free risk)	0.4127	0.5844	0.5140	0.2537
Sortino Ratio (Threshold = MAR = 0 %)	0.6648	0.9806	0.8213	0.3717
Performance Ulcer Index (or Martin Ratio)	0.1341	0.2428	0.1789	0.0537
Omega Index (Threshold = MAR = Free risk)	1.8181	2.1745	2.0513	1.5495
Upside Potential Ratio (Threshold = MAR = Free risk)	0.9171	1.0820	1.0028	0.7153
Farinelli-Tibiletti ratio (with parametrization p = 0.5; q = 2, Threshold = MAR = Free risk)	0.4600	0.6164	0.5080	0.3716

Table 5.24 Level of diversification in portfolio weights for the risk-based asset allocation approaches (*average values*)

	Optimal risk parity	Global minimum-variance	Most diversified portfolio	Equally-weighted
DATASET "EUROZONE"				
Shannon entropy measure	2.3724	1.0687	1.5523	2.3979
Gini coefficient	0.1177	0.7788	0.6362	0.0000
Normalized Herfindhal index	0.0053	0.3503	0.1620	0.0000
DATASET "EMERGING EUROPE"				
	Optimal risk parity	Global minimum-variance	Most diversified portfolio	Equally-weighted
Shannon entropy measure	1.5887	0.5311	1.3854	1.6094
Gini coefficient	0.1070	0.7019	0.3057	0.0000
Normalized Herfindhal index	0.0105	0.6321	0.0852	0.0000
DATASET "GLOBAL"				
	Optimal risk parity	Global minimum-variance	Most diversified portfolio	Equally-weighted
Shannon entropy measure	1.6939	0.4001	1.0018	2.0794
Gini coefficient	0.4165	0.8387	0.7134	0.0000
Normalized Herfindhal index	0.1631	0.7699	0.3916	0.0000

The next step consists of a discussion of the evidences concerning the second evaluation dimension: the level of diversification for the risk-based portfolios.

Firstly, the perspective of portfolio weights is examined. The results are reported in Table 5.24 while Appendix 1, at the end of this chapter, using area graphs, shows the composition of portfolios recommended by each risk-based strategy across the asset allocation experiments.

As expected, the level of asset class weights dissemination or diversification is most prominent for the equally-weighted approach. Across our datasets, it shows the highest Shannon Entropy measure and the lowest Herfindhal index and Gini coefficient. It is closely followed by the optimal risk parity strategy while the most diversified portfolio and, even more, the global minimum-variance approach exhibit strong concentration.[13] We have to remember they are both approaches that admit

[13]With reference to the statistics about the level of diversification/concentration of the risk-based portfolios, we have to underline we compute ex-ante measures considering that the reported results refer to each rebalance date. In other words, we do not examine the possible changeable behavior of these measures during the 6 months until portfolios are rebalanced again. For instance, the

risk-based portfolios be invested in a fraction of available asset classes. The graphs in Appendix 1 confirm that this "option" has been exploited especially by the global minimum-variance approach. As a matter of fact, the range of asset classes mixed up by the global minimum-variance asset allocation experiments goes from 2 to 8 for the dataset "EUROZONE", from 1 to 4 for the dataset "EMERGING EUROPE" and from 2 to 5 for the dataset "GLOBAL". The most frequent value, the mode, is 4 for the first dataset, and 3 for the others. With regard to the most diversified portfolio, the number of asset classes used in the asset allocation experiments goes from 4 to 7 for the dataset "EUROZONE", from 3 to 5 for the dataset "EMERGING EUROPE" and from 4 to 5 for the dataset "GLOBAL". The mode is 5 asset classes for the first and second datasets, 4 for the last one.

These additional elements strengthen the idea that the minimum-variance portfolios tend to be more extreme compared with the most diversified portfolio. We can reasonably argue that, since the goal of the global minimum-variance strategy consists of the minimization of risk in absolute terms rather than in relative terms, it tends to preferably select very few asset classes according, firstly, to the inverse of the individual risk and, secondly, to the inverse of correlation with other asset classes. These considerations provide an explanation, for example, for the significant concentration of the portfolio in the MSCI Czech Republic, when we refer to the dataset "EMERGING EUROPE", and in the JPM Europe Government Bond, followed distantly by the MSCI Europe, when the dataset "GLOBAL" is considered.

The analysis of the level of diversification from the perspective of risk contributions is carried out in Table 5.25. As it is intuitive to understand, the sources of risk are better disseminated or spread out over all asset classes in the case of the optimal risk parity strategy. We have a proof of the highest value of the Shannon Entropy measure and the lowest value for the Gini coefficient and the Herfindhal index.

To make our comments more robust, we can add that, across all datasets, the following sequences of the measures we have selected to investigate the level of diversification/concentration in the risk allocations of the risk-based portfolios are validated:

$$SE_{ORP}(PTRC) > SE_{EW}(PTRC) > SE_{MDP}(PTRC) > SE_{GMVP}(PTRC) \tag{5.19}$$

$$G_{ORP}(PTRC) < G_{EW}(PTRC) < G_{MDP}(PTRC) < G_{GMVP}(PTRC) \tag{5.20}$$

$$H^*_{ORP}(PTRC) < H^*_{EW}(PTRC) < H^*_{MDP}(PTRC) < H^*_{GMVP}(PTRC) \tag{5.21}$$

(Footnote 13 continued)

equally-weighted approach shows a value of zero for the Gini coefficient and the Herfindhal index and a value equal to $\ln(N)$ for the Shannon Entropy measure at each rebalance date, but just after that and during the following 6 months, the original "1 over N" portfolio weights drift because of the price movements until they are rebalanced again.

Table 5.25 Level of diversification in risk contributions for the risk-based asset allocation approaches (*average values*)

	Optimal risk parity	Global minimum-variance	Most diversified portfolio	Equally-weighted
DATASET "EUROZONE"				
Shannon entropy measure	2.3979	1.0687	1.5603	2.3585
Gini coefficient	0.0002	0.7788	0.6352	0.1494
Normalized Herfindhal index	0.0000	0.3503	0.1528	0.0086
DATASET "EMERGING EUROPE"				
Shannon entropy measure	1.6094	0.5311	1.3777	1.5822
Gini coefficient	0.0000	0.7019	0.3113	0.1243
Normalized Herfindhal index	0.0000	0.6321	0.0899	0.0131
DATASET "GLOBAL"				
Shannon entropy measure	2.0790	0.4001	1.2645	1.9463
Gini coefficient	0.0057	0.8387	0.6253	0.2578
Normalized Herfindhal index	0.0001	0.7699	0.2183	0.0323

In the following Tables 5.26, 5.27 and 5.28 designed for each dataset, we report for any asset class in the investment universe considered to implement each strategy, the elements listed below:

- the mean percentage risk contribution from an asset class across the asset allocation experiments;
- the standard deviation of the percentage risk contributions from an asset class across the asset allocation experiments[14];
- the minimum value observed for the percentage risk contribution of an asset class across the asset allocation experiments;
- the maximum value observed for the percentage risk contribution of an asset class across the asset allocation experiments.

Looking at these Tables, we observe that the portfolio that most effectively contrasts the overall risk has its volatility strongly dominated by few drivers (asset classes). In any dataset, for almost all asset classes the minimum value for the percentage risk contribution to the volatility of the global minimum-variance

[14]This value should be zero when we consider the optimal risk parity strategy because, theoretically, the minimum, the maximum and the mean percentage risk contribution should be identical. In practice, the numerical procedure we implement with a given number of iteration couldn't converge to a perfect risk allocation equilibrium in some asset allocation experiments when we used the dataset "EUROZONE" and the dataset "GLOBAL".

Table 5.26 Risk contribution analysis for dataset "EUROZONE"

	Mean (%)	Standard deviation (%)	Minimum (%)	Maximum (%)
Risk contribution analysis—optimal risk parity				
MSCI AUSTRIA	9.09	0.01	9.05	9.11
MSCI BELGIUM	9.09	0.00	9.07	9.09
MSCI FINLAND	9.09	0.00	9.09	9.11
MSCI FRANCE	9.09	0.00	9.08	9.10
MSCI GERMANY	9.09	0.00	9.08	9.10
MSCI GREECE	9.09	0.01	9.09	9.12
MSCI IRELAND	9.09	0.00	9.08	9.09
MSCI ITALY	9.09	0.01	9.06	9.10
MSCI NETHERLANDS	9.09	0.00	9.08	9.11
MSCI PORTUGAL	9.09	0.00	9.08	9.10
MSCI SPAIN	9.09	0.00	9.07	9.10
Risk contribution analysis—global minimum variance				
MSCI AUSTRIA	21.13	26.73	0.00	66.32
MSCI BELGIUM	13.82	21.49	0.00	59.67
MSCI FINLAND	0.12	0.40	0.00	1.60
MSCI FRANCE	18.05	27.12	0.00	72.50
MSCI GERMANY	0.00	0.00	0.00	0.00
MSCI GREECE	0.54	1.49	0.00	5.97
MSCI IRELAND	8.34	9.14	0.00	36.89
MSCI ITALY	6.56	11.35	0.00	37.97
MSCI NETHERLANDS	3.57	7.54	0.00	32.12
MSCI PORTUGAL	26.26	13.98	0.00	60.03
MSCI SPAIN	1.61	4.06	0.00	16.71
Risk contribution analysis—most diversified portfolio				
MSCI AUSTRIA	13.95	13.41	0.00	33.56
MSCI BELGIUM	3.37	5.41	0.00	15.39
MSCI FINLAND	22.61	8.81	2.97	34.83
MSCI FRANCE	0.00	0.00	0.00	0.00
MSCI GERMANY	0.30	0.89	0.00	4.33
MSCI GREECE	19.90	6.75	0.00	27.69
MSCI IRELAND	19.12	13.18	0.00	36.68
MSCI ITALY	0.19	0.73	0.00	3.33
MSCI NETHERLANDS	0.00	0.00	0.00	0.00
MSCI PORTUGAL	15.27	5.19	5.50	26.10
MSCI SPAIN	5.29	6.62	0.00	19.09
Risk contribution analysis—equally-weighted				
MSCI AUSTRIA	10.84	0.82	9.69	12.19
MSCI BELGIUM	6.43	1.46	5.06	8.70
MSCI FINLAND	9.18	0.57	8.20	10.27
MSCI FRANCE	8.08	0.25	7.71	8.44
MSCI GERMANY	8.11	0.20	7.71	8.51

(continued)

Table 5.26 (continued)

	Mean (%)	Standard deviation (%)	Minimum (%)	Maximum (%)
Risk contribution analysis—optimal risk parity				
MSCI GREECE	15.63	1.49	13.31	17.58
MSCI IRELAND	7.16	0.70	6.36	8.31
MSCI ITALY	10.64	0.75	9.46	11.30
MSCI NETHERLANDS	7.23	0.46	6.57	8.00
MSCI PORTUGAL	6.79	0.40	6.06	7.48
MSCI SPAIN	9.91	0.90	8.51	10.85

Table 5.27 Risk contribution analysis for dataset "EMERGING EUROPE"

	Mean (%)	Standard deviation (%)	Minimum (%)	Maximum (%)
Risk contribution analysis—optimal risk parity				
MSCI CZECH REPUBLIC	20.00	0.00	20.00	20.00
MSCI HUNGARY	20.00	0.00	20.00	20.00
MSCI POLAND	20.00	0.00	20.00	20.00
MSCI RUSSIA	20.00	0.00	20.00	20.00
MSCI TURKEY	20.00	0.00	20.00	20.00
Risk contribution analysis—global minimum variance				
MSCI CZECH REPUBLIC	81.24	12.99	62.81	100.00
MSCI HUNGARY	3.43	6.89	0.00	25.64
MSCI POLAND	0.81	3.07	0.00	13.33
MSCI RUSSIA	10.36	8.51	0.00	25.48
MSCI TURKEY	4.16	6.28	0.00	22.71
Risk contribution analysis—most diversified portfolio				
MSCI CZECH REPUBLIC	17.78	7.32	9.65	33.11
MSCI HUNGARY	7.79	8.08	0.00	22.63
MSCI POLAND	11.22	7.30	0.00	22.88
MSCI RUSSIA	31.97	4.19	21.93	37.15
MSCI TURKEY	31.24	6.40	18.89	39.57
Risk contribution analysis—equally-weighted				
MSCI CZECH REPUBLIC	12.82	1.17	11.23	14.82
MSCI HUNGARY	26.44	0.97	25.01	28.20
MSCI POLAND	20.92	0.78	19.95	22.33
MSCI RUSSIA	19.05	1.71	16.81	22.67
MSCI TURKEY	20.77	1.96	16.03	22.54

portfolio is zero. Concerning this strategy, we also note high standard deviation around the mean percentage risk contribution of the asset classes. This feature can be interpreted as a proof of the natural attitude of the minimum-variance portfolio to take bets which is opposite and conflicting with the thinking behind optimal risk parity. The distinctiveness of this situation with respect to the traditional asset management is that bets are not prompted by the search of outperformance but

Table 5.28 Risk contribution analysis for dataset "GLOBAL"

	Mean (%)	Standard deviation (%)	Minimum (%)	Maximum (%)
Risk contribution analysis—optimal risk parity				
JPM EUROPE GOVT. BOND	12.08	0.84	9.69	12.50
JPM UNITED STATES GOVT. BOND	12.59	0.20	12.44	13.20
JPM EMBI GLOBAL DIVERS. COMPOSITE	12.60	0.20	12.50	13.18
BOFA ML GLB BROAD CORP.	12.52	0.09	12.26	12.79
MSCI EUROPE	12.51	0.12	12.17	12.80
MSCI NORTH AMERICA	12.57	0.13	12.49	12.92
MSCI PACIFIC FREE	12.58	0.16	12.50	13.01
MSCI EMERGING MARKETS	12.57	0.14	12.49	13.01
Risk contribution analysis—global minimum variance				
JPM EUROPE GOVT. BOND	88.73	4.68	77.19	97.08
JPM UNITED STATES GOVT. BOND	0.00	0.00	0.00	0.00
JPM EMBI GLOBAL DIVERS. COMPOSITE	0.00	0.00	0.00	0.00
BOFA ML GLB BROAD CORP.	0.00	0.00	0.00	0.00
MSCI EUROPE	8.03	2.81	0.00	15.08
MSCI NORTH AMERICA	2.67	3.60	0.00	12.21
MSCI PACIFIC FREE	0.35	0.74	0.00	2.35
MSCI EMERGING MARKETS	0.22	0.55	0.00	2.16
Risk contribution analysis—most diversified portfolio				
JPM EUROPE GOVT. BOND	27.39	14.50	3.65	43.05
JPM UNITED STATES GOVT. BOND	24.29	15.09	5.47	45.20
JPM EMBI GLOBAL DIVERS. COMPOSITE	0.00	0.00	0.00	0.00
BOFA ML GLB BROAD CORP.	0.00	0.00	0.00	0.00
MSCI EUROPE	31.83	10.47	1.89	46.60
MSCI NORTH AMERICA	0.29	0.76	0.00	2.89
MSCI PACIFIC FREE	5.24	6.34	0.00	18.90
MSCI EMERGING MARKETS	10.95	9.78	0.00	29.45
Risk contribution analysis—equally-weighted				
JPM EUROPE GOVT. BOND	2.82	1.22	1.07	5.00
JPM UNITED STATES GOVT. BOND	7.29	3.96	1.77	13.59
JPM EMBI GLOBAL DIVERS. COMPOSITE	12.84	1.34	10.82	14.56
BOFA ML GLB BROAD CORP.	9.00	1.47	6.78	11.64
MSCI EUROPE	13.65	3.44	9.15	18.33
MSCI NORTH AMERICA	15.91	2.00	12.26	18.79
MSCI PACIFIC FREE	17.66	0.52	16.85	18.68
MSCI EMERGING MARKETS	20.83	2.20	16.21	24.02

simply by the search of risk minimization. The most diversified portfolio approach shares, even if in a less pronounced way, similar characteristics of the global minimum-variance portfolio.

Table 5.29 Turnover analysis for the risk-based asset allocation approaches

	Optimal risk parity (%)	Global minimum-variance (%)	Most diversified portfolio (%)	Equally-weighted (%)
DATASET "EUROZONE"				
Average turnover	8.1003	37.3058	24.7091	6.8854
Maximum turnover	16.7386	109.8580	77.6447	17.1511
DATASET "EMERGING EUROPE"				
Average turnover	8.7011	17.0713	19.1418	8.3205
Maximum turnover	14.1777	45.5376	31.7702	13.6049
DATASET "GLOBAL"				
Average turnover	6.6407	4.9143	11.3808	6.0407
Maximum turnover	13.2031	12.3298	36.0108	18.4843

The final evaluation criteria we selected is the asset allocation stability. The related results in terms of average turnover and maximum turnover are shown in Table 5.29.

We can see that, generally speaking, these values are higher for portfolios from risk-based strategies that cannot be invested in some asset classes. We are referring to the most diversified portfolio approach and, even more resolutely, to the global minimum-variance approach. This last solution recommends portfolio compositions that can be significantly modified at distance of some rebalance dates. Only with the use of the dataset "GLOBAL", the global minimum-variance portfolio has lower turnover than the other approaches. This is the dataset without any relevant time-varying behaviour in the out-of-sample raw and squared returns of the component asset classes. It is also the dataset for which the minimum-variance portfolios have constantly resulted as being strongly invested in the JPM Europe Government Bond.

Going back to general considerations, we can affirm that the equally-weighted approach is usually successful in reducing the average turnover closely followed by the optimal risk parity strategy. The two are competing in terms of maximum turnover.

At this point, after that all the selected evaluation dimensions for the risk-based asset allocation strategies have been discussed and examined across the three datasets, we attempt to draw below some concluding and interesting considerations:

- the equally weighted approach, which outperformed according to the results by DeMiguel et al. (2009) several optimization models working in the classical mean-variance framework, cannot be firmly acknowledged as a dominant asset allocation approach relative to other risk-based approaches. In our empirical investigation it was recurrently found as the strategy with the lowest financial efficiency;
- the global minimum-variance approach tends to have, in terms of financial efficiency, an unstable, or to be more precise, an extreme relative position across the datasets. It was very promising when we used the dataset "EUROZONE"

and "GLOBAL", but it felt to the bottom of the ranking when we used the dataset "EMERGING EUROPE";
- the optimal risk parity is the risk-based asset allocation approach that most keeps a constant position across all datasets. It repeatedly ranked as second or third according to a wide set of risk-adjusted performance measures. It also provides superior diversification in asset class risk contribution, at least ex ante and, in the perspective of practical implementation, requires a minor amount of trading compared to the global minimum-variance portfolio and the most diversified portfolio approach;
- the strategies that, for construction, have to be invested in all asset classes of the investment universe considered show preferable properties both in terms of lower turnover and in terms of major risk diversification/dissemination. This feature together with their portfolio composition represents a proof that such strategies have an implicit constraint against taking marked bets;
- the realized volatility computed on the out-of-sample returns can be ranked in the order indicated by Maillard et al. (2010) for all datasets but, in addition, we show that for the most diversified portfolio it is possible to have a standard deviation either higher or lower than the standard deviation of the equally-weighted approach and the optimal risk parity portfolio.

The empirical investigation developed in this chapter can be extended and improved in several directions. One would be to consider estimation windows for the input parameters of different length, different frequency of rebalance and improved estimation techniques for time-varying second moments since we observed repeatedly evidences of heteroskedasticity in asset class returns. Transaction costs due to trading activity could be integrated in the analysis. The number of datasets considered could be increased, even if we have already taken into account investment universes that differ both in terms of the number of asset classes included and in terms of the market conditions represented. The strategies could be compared with reference to subsamples periods. Although limited, we believe the proposed application is nevertheless useful for investment practice for providing relevant information regarding the main distinguishing characteristics of risk-based asset allocation approaches.

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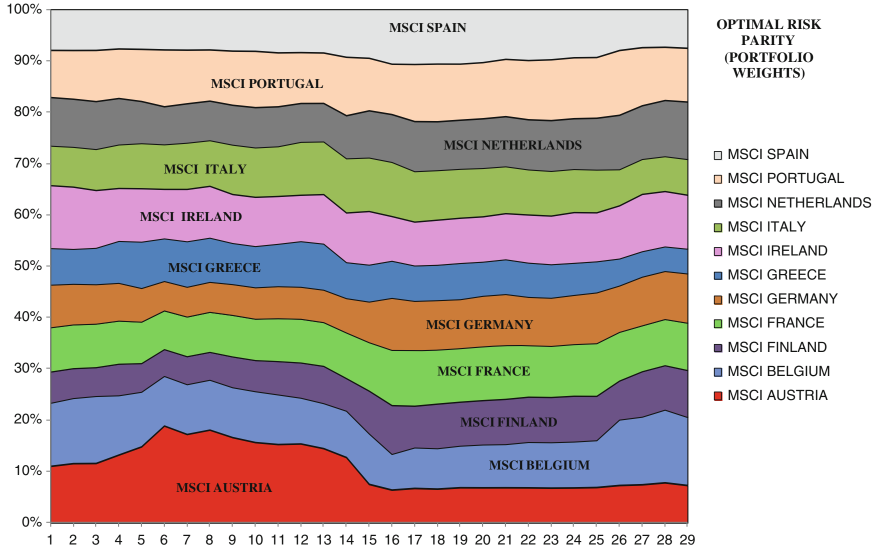
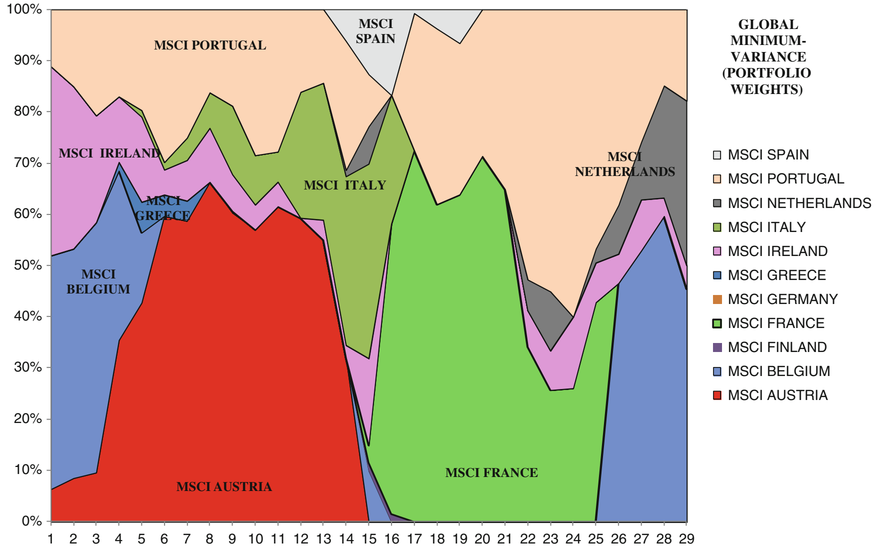
Appendix
Risk-Based Portfolio Compositions Across the Asset Allocation Experiments

Dataset "Eurozone"



M.D. Braga, *Risk-Based Approaches to Asset Allocation*,
SpringerBriefs in Finance, DOI 10.1007/978-3-319-24382-5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MSCI IRELAND
MSCI GREECE
MSCI ITALY
MSCI NETHERLANDS
MSCI BELGIUM
MSCI AUSTRIA
MSCI FRANCE
MSCI SPAIN
MSCI PORTUGAL
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MSCI GERMANY
MSCI FRANCE
MSCI FINLAND
MSCI BELGIUM
MSCI AUSTRIA
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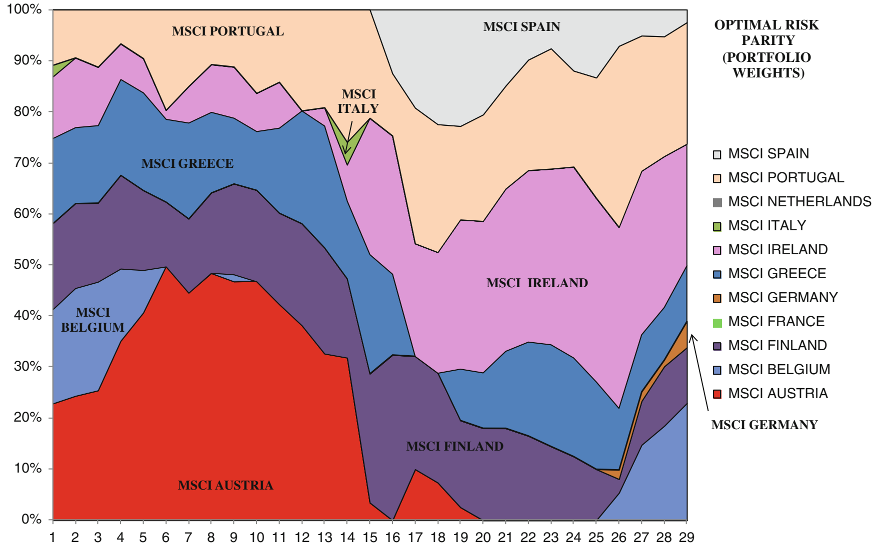
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MSCI FINLAND
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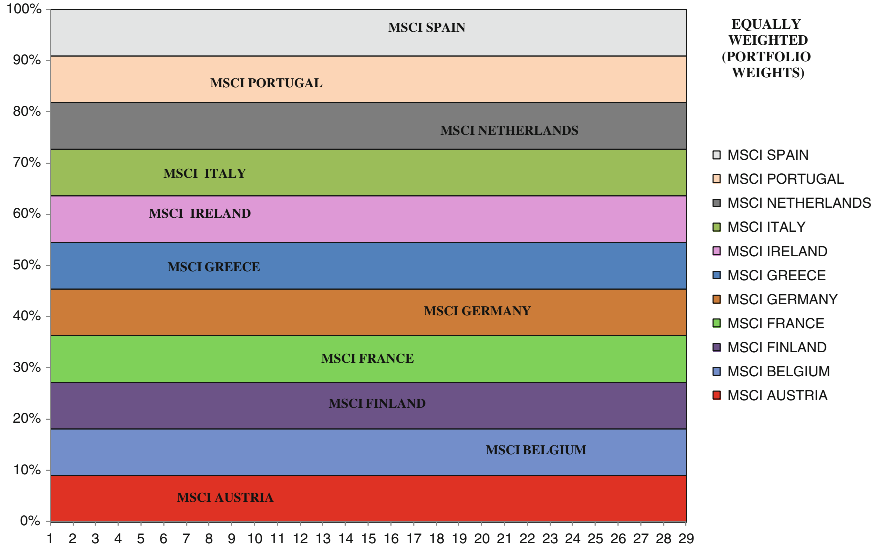

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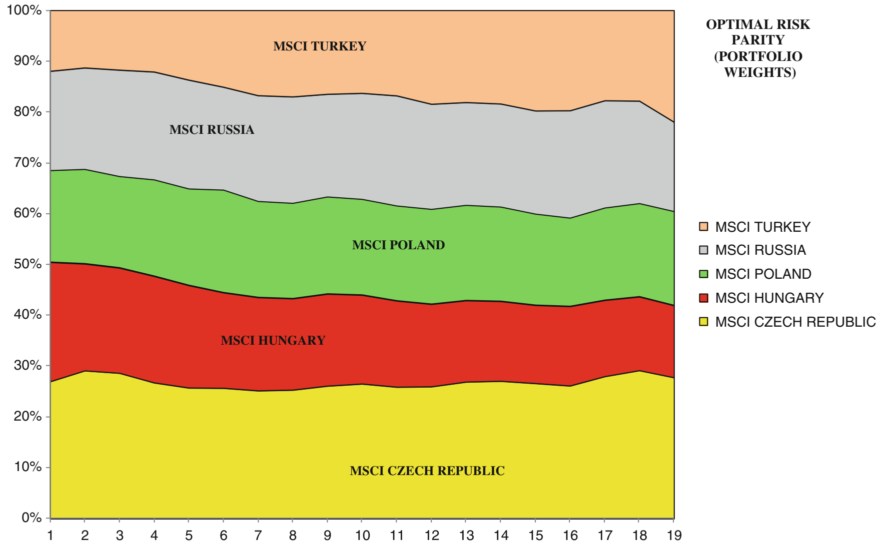

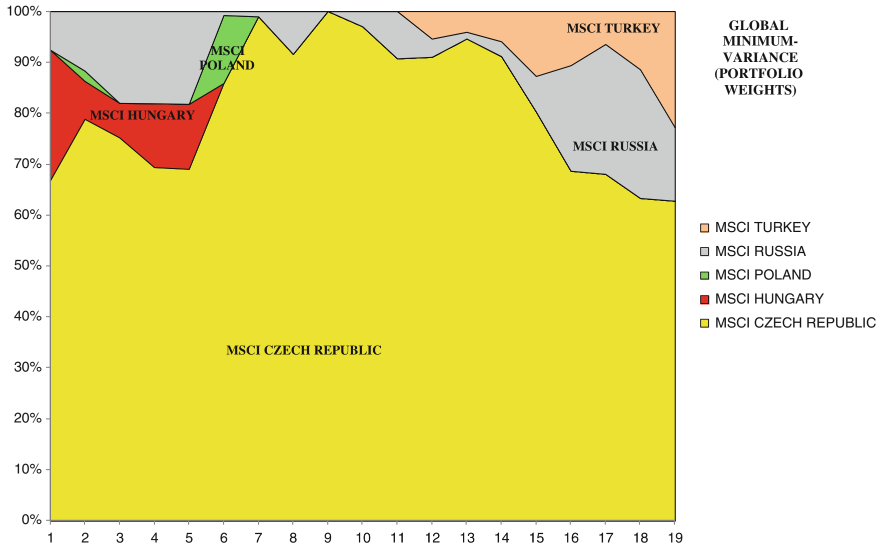
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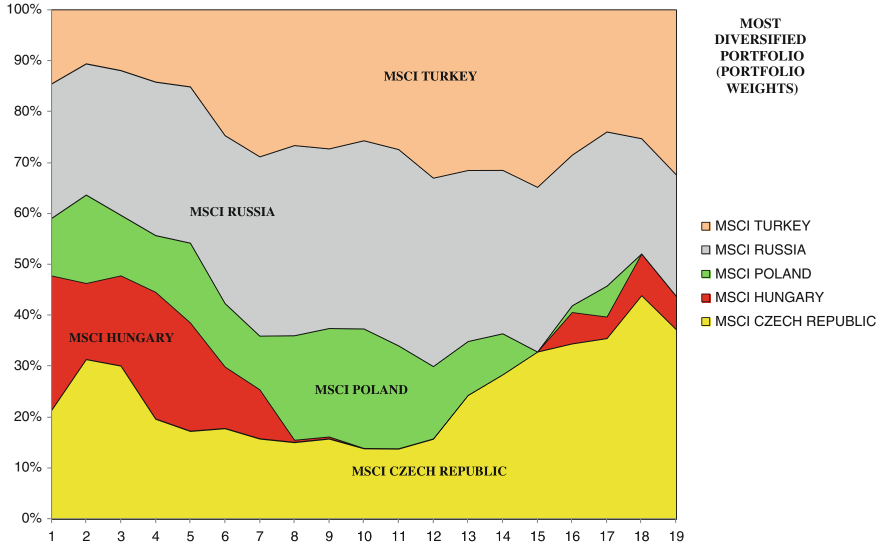
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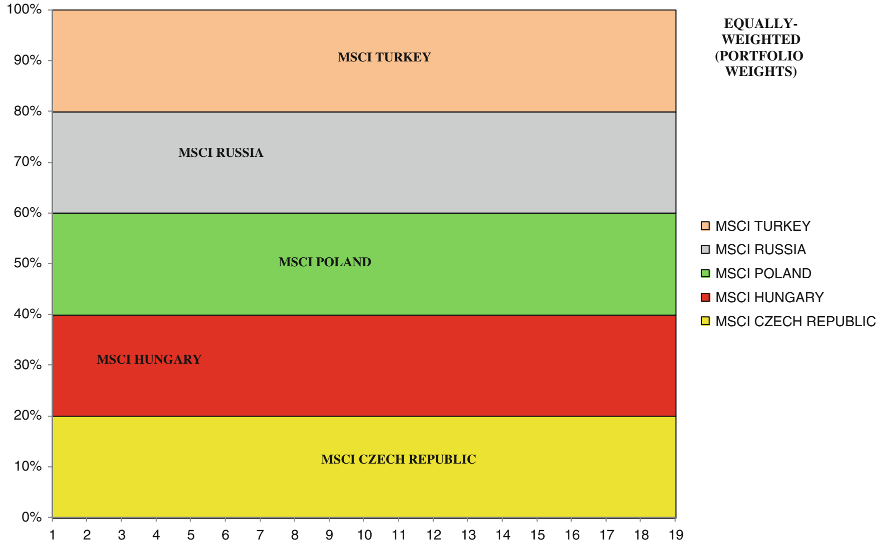
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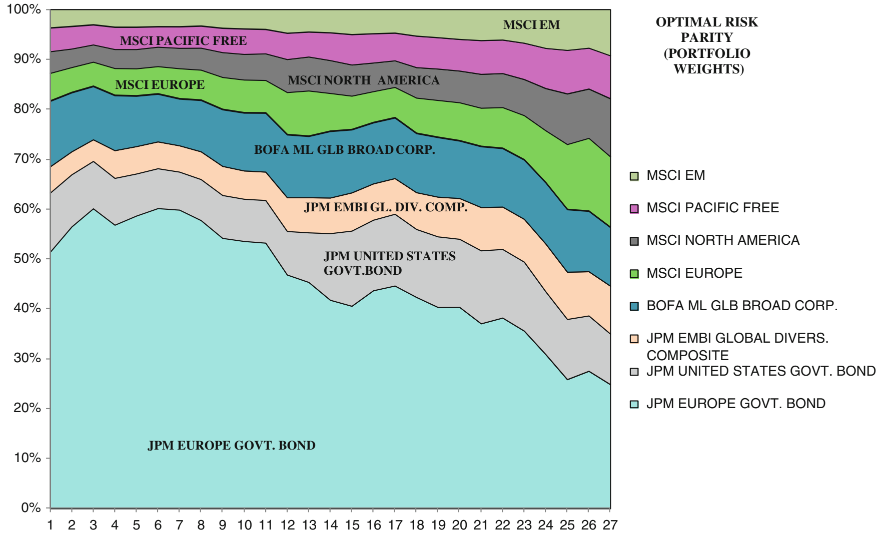
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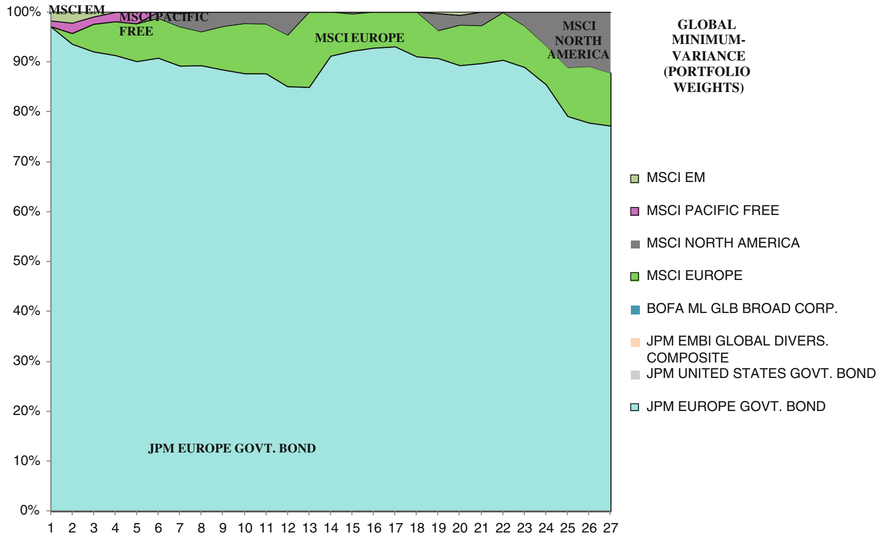
GLOBAL MINIMUM-VARIANCE (PORTFOLIO WEIGHTS)
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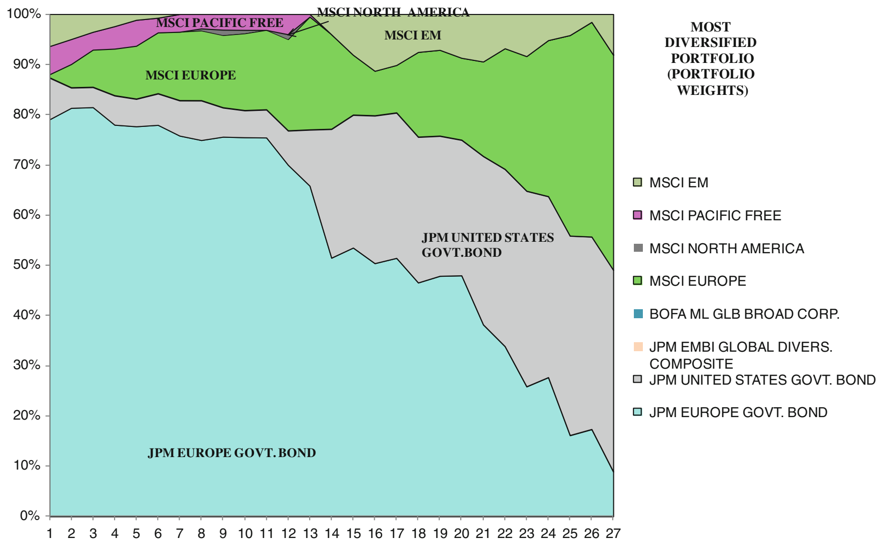
MOST DIVERSIFIED PORTFOLIO (PORTFOLIO WEIGHTS)
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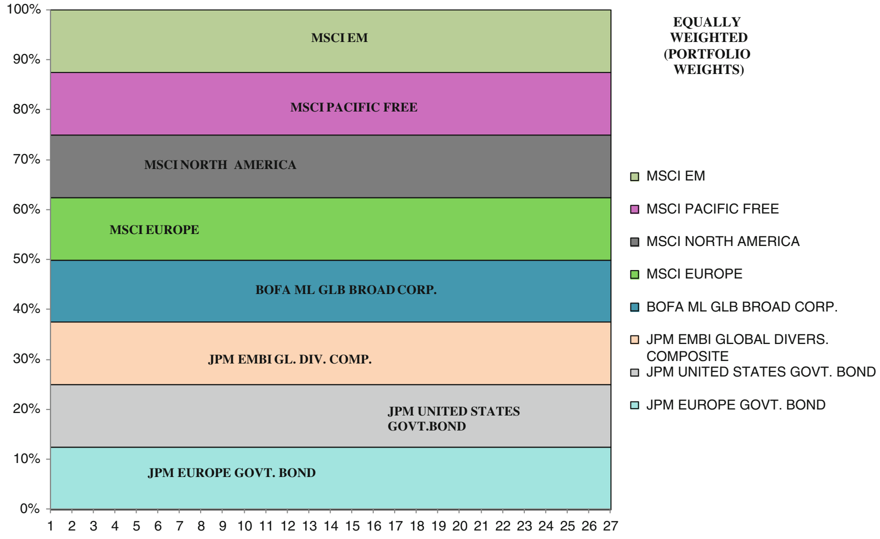
EQUALLY WEIGHTED (PORTFOLIO WEIGHTS)
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