Problem 1

Let X be a continuous random variable with PDF $f(x)$ and CDF $F(x)$. For a fixed number x_0 such that $F(x_0) < 1$, define the function:

$$g(x) = \begin{cases} \frac{f(x)}{1-F(x_0)}, & \text{for } x \geq x_0, \\ 0, & \text{for } x < x_0. \end{cases}$$

Prove that $g(x)$ is a PDF.

VALID PDFs ARE POSITIVE AND INTEGRATE TO 1.

$$g(x) = \frac{f(x)}{1-F(x_0)}$$

THIS IS A POSITIVE FUNCTION SINCE $f(x)$ IS ALREADY A VALID, POSITIVE PDF, $F(x_0) < 1$ SO $1-F(x_0)$ IS POSITIVE, AND $\frac{\text{POSITIVE}}{\text{POSITIVE}} = \text{POSITIVE}$

$$G(x) = \int_{x_0}^{\infty} \frac{f(x)}{1-F(x_0)}\,dx = (1-F(x_0))^{-1}\int_{x_0}^{\infty} f(x)\,dx$$

$\frac{1}{1-F(x_0)}$ IS A CONSTANT THAT CAN BE MOVED OUTSIDE THE INTEGRAL.

$$= (1-F(x_0))^{-1}\left[F(\infty) - F(x_0)\right]$$

$$= \frac{1-F(x_0)}{1-F(x_0)} = 1$$

$g(x)$ INTEGRATES TO 1

$\therefore$ $g(x)$ IS A VALID PDF.

Problem 2

Let X be a random variable with PDF

$$f(x) = \frac{1}{2}(1+x), \text{ for } -1 < x < 1.$$

(a) Find the PDF of the random variable $Y = X^2$.

(b) Find the mean $\mathsf{E}(Y)$ and the variance $\mathsf{Var}(Y)$ of the random variable Y.

(a) FIRST, FIND THE CDF $F_Y(y)$, THEN DIFFERENTIATE TO GET THE PDF:

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = P(X \le \sqrt{y}) - P(X \le -\sqrt{y})$$

$$= \int_{-\infty}^{\sqrt{y}} \frac{1}{2}(1+x)\,dx - \int_{-\infty}^{-\sqrt{y}} \frac{1}{2}(1+x)\,dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\sqrt{y}} (1+x)\,dx - \int_{-\infty}^{-\sqrt{y}} (1+x)\,dx\right]$$

$$= \frac{1}{2}\left[x + \frac{x^2}{2}\Big|_{-\infty}^{\sqrt{y}} - \left(x + \frac{x^2}{2}\Big|_{-\infty}^{-\sqrt{y}}\right)\right]$$

$$= \frac{1}{2}\left[\sqrt{y} + \frac{y}{2} - \left(-\sqrt{y} + \frac{y}{2}\right)\right] = \frac{1}{2}(2\sqrt{y}) = \sqrt{y}$$

$$f_Y(y) = \sqrt{y}\,\frac{d}{dy} = \begin{cases} \frac{1}{2\sqrt{y}} & \text{FOR } 0 < y < 1 \\ 0 & \text{OTHERWISE} \end{cases}$$

(b) $$E(Y) = \int_0^1 y \cdot \frac{1}{2\sqrt{y}}\,dy = \frac{1}{2}\int_0^1 \sqrt{y}\,dy = \frac{1}{2}\left[\frac{2}{3} y^{3/2}\Big|_0^1\right] = \frac{1}{3}$$

$$\mathrm{VAR}(y) = E(Y^2) - (EY)^2$$

$$E(Y^2) = \int_0^1 y^2 \frac{1}{2\sqrt{y}}\,dy = \frac{1}{2}\int_0^1 y^{3/2}\,dy = \frac{1}{2}\left(\frac{2}{5} y^{5/2}\Big|_0^1\right) = \frac{1}{5}$$

$$\mathrm{VAR}(y) = \frac{1}{5} - \left(\frac{1}{3}\right)^2 = \frac{9}{45} - \frac{5}{45} = \frac{4}{45}$$

Problem 3

Let X be a random variable with moment generating function $M_X(t)$, for $-h < t < h$. Prove that

(a) $P(X \geq a) \leq e^{-at}M_X(t)$, for $0 < t < h$.

(b) $P(X \leq a) \leq e^{-at}M_X(t)$, for $-h < t < 0$.

PROPOSITION 6.4.11 FOR MGF OF A LOCATION-SCALE TRANSFORMATION GIVES:

IF X HAS MGF $M(t)$, THEN THE MGF OF $a + bX$ IS:

$$E(e^{t(a+bx)}) = e^{at}E(e^{btx}) = e^{at}M(bt)$$

(a) GIVEN THE ABOVE, WE CAN SEE THE R.H.S AS A LOCATION TRANSFORMATION OF X (NO SCALING)

WHERE $e^{-at}M_x(t) = X - a$. IF $0 < t < h$, THEN:

CASE 1: WHEN $x \geq a$, $P(X \geq a) = 1$ AND $E(e^{t(X-a)}) = E(e^{t[\text{VALUE} \geq 0]})$. THE EXPECTATION OF AN EXPONENTIAL IS ≥ 1 SO $P(X \geq a) \leq e^{-at}M_x(t)$ HOLDS.

CASE 2: WHEN $x < a$, $P(X \geq a) = 0$ AND $E(e^{t(x-a)}) = E(e^{t(\text{VALUE} < 0)})$. THE EXPECTATION OF $e^{[\text{NEGATIVE}]}$ MUST BE IN THE INTERVAL $(0,1)$ SO $P(X \geq a) \leq e^{-at}M_x(t)$ HOLDS

NO OTHER CASES ARE POSSIBLE

(b) IF $-h < t < 0$, THEN:

CASE 1: WHEN $x \geq a$, $P(X \leq a) = 0$ AND $E(e^{t(X-a)}) = E(e^{t(\text{VALUE} \geq 0)})$. SINCE t IS NEGATIVE, WE END UP WITH $E(e^{[\text{NEGATIVE}]})$, WHICH MUST BE IN THE INTERVAL $(0,1)$. GIVEN THIS, $P(X \geq a) \leq e^{-at}M_x(t)$ HOLDS SINCE $0 \leq (0,1)$

CASE 2: WHEN $x < a$, $P(X \leq a) = 1$ AND $E(e^{t(X-a)}) = E(e^{t(\text{VALUE} < 0)}) = E(e^{(\text{POSITIVE})})$. THE EXPECTATION OF $e^{(\text{POSITIVE})}$ MUST BE > 1 SO $P(X \geq a) \leq e^{-at}M_x(t)$ HOLDS.

NO OTHER CASES ARE POSSIBLE

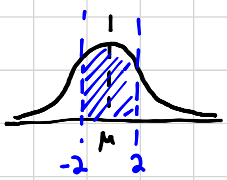
Problem 4

Let $X \sim \mathrm{N}(\mu, \sigma^2)$. Find the values of μ and σ^2 such that

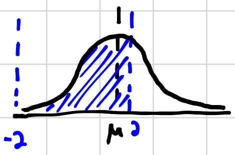
$$\mathrm{P}(|X| < 2) = \frac{1}{2}.$$

Prove or disprove that these values of μ and σ^2 are unique.

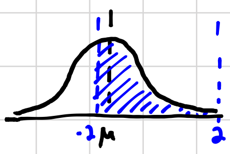
EXAMPLE DISTRIBUTIONS SATISFYING $X \sim N(\mu, \sigma^2)$ WHERE $P(|X|<2) = \frac{1}{2}$:



IN THIS CASE, $\mu = 0$ AND $P(\mu < X < 2) = 0.25$



HERE, $P(X<-2) = P(\mu < X < 2)$. NOTICE THAT μ CANNOT BE ≥ 2 SINCE THE SUPPORT EXTENDS TO ∞ AND $(-2, 2)$ IS FINITE. IN OTHER WORDS, IF $\mu = 2$, THE $P(-2<X<2) < \frac{1}{2}$



SIMILARLY, $P(X>2) = P(-2<X<\mu)$

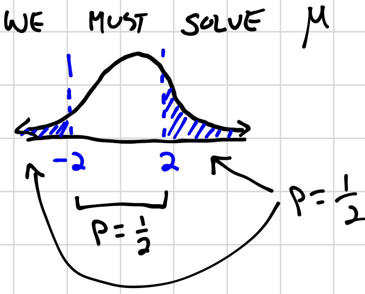
GIVEN THIS, THERE MUST EXIST AN INFINITE NUMBER OF μ AND σ^2 BOUNDED BY $-2 < \mu < 2$.

TO GET A FUNCTION THAT DESCRIBES THESE VALUES, WE MUST SOLVE μ AS A FUNCTION OF σ^2

$$P(|X|<2) = P(-2<X) + P(X>2) = \frac{1}{2}$$

$$= \int_{-\infty}^{-2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \int_{2}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{2}$$

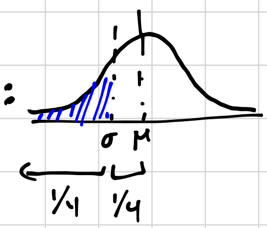
VISUALLY:



THAT'S IMPOSSIBLE IN THIS FORM. LET'S TRY SOLVING FOR σ^2 WHEN $\mu = 0$ AND SEE IF THAT GETS US ANYWHERE:

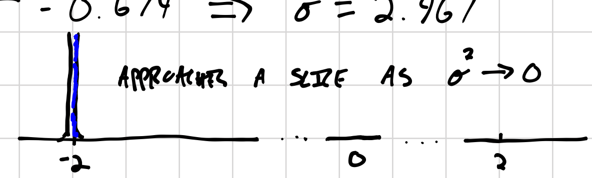
$$2P(Z<\sigma) = \frac{1}{2}$$

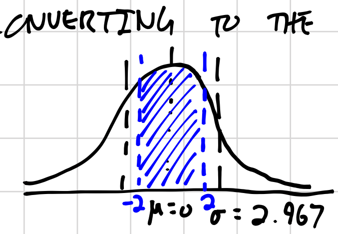
DUE TO SYMMETRY, WE CAN JUST SOLVE FOR THE LEFT TAIL OF:



A Z TABLE GIVES $\sigma = 0.674$.

CONVERTING TO THE X DISTRIBUTION, WE GET $\frac{2-0}{\sigma} = 0.674 \Rightarrow \sigma = 2.967$



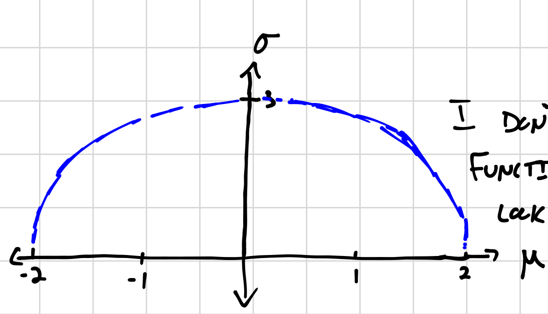


μ	σ^2
$\lim_{\mu \to -2}$	$\lim_{\sigma^2 \to 0}$
0	2.967^2
$\lim_{\mu \to 2}$	$\lim_{\sigma \to 0}$

THIS LOOKS LIKE THE PERIMETER OF THE ECLIPSE:

$$\frac{\mu^2}{2^2} + \frac{\sigma^2}{2.967^2} = 1$$

$$\Rightarrow \mu = \sqrt{4 - \frac{4\sigma^2}{2.967^2}} = 1.4835\sqrt{4-\sigma^2}$$



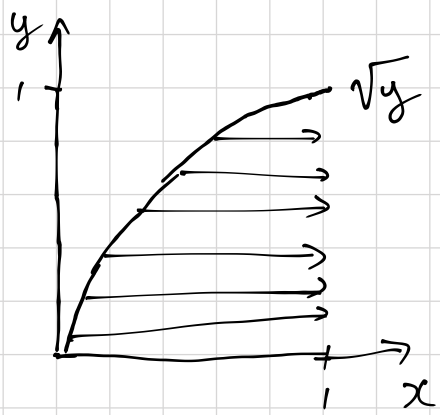
I DON'T KNOW WHAT THIS FUNCTION IS, BUT IT MUST LOOK LIKE THIS

IF THIS ISN'T RIGHT, IT HAS TO BE REALLY CLOSE, I WOULD THINK

Problem 5

Find $\mathsf{P}\left(X > \sqrt{Y}\right)$ if X and Y are jointly distributed with PDF

$$f(x, y) = x + y, \text{ for } 0 \le x \le 1, \quad 0 \le y \le 1.$$



X FROM $\sqrt{y}$ TO 1
y FROM 0 TO 1

$$P(X > \sqrt{y}) = \int_0^1 \int_{\sqrt{y}}^1 (x+y)\,dx\,dy$$

$$= \int_0^1 \left(\frac{x^2}{2} + xy \Big|_{\sqrt{y}}^{1} \right) dy$$

$$= \int_0^1 \left[\left(\frac{1^2}{2} + (1)y \right) - \left(\frac{\sqrt{y}^2}{2} + \sqrt{y} \cdot y \right) \right] dy = \int_0^1 \left[\frac{1}{2} + y - \frac{y}{2} - y^{3/2} \right] dy = \int_0^1 \left(\frac{1}{2} + \frac{y}{2} - y^{3/2} \right) dy$$

$$= \frac{1}{2}y + \frac{y^2}{4} - \frac{2y^{5/2}}{5} \Big|_0^1 = \frac{1}{2} + \frac{1}{4} - \frac{2}{5} = \frac{10}{20} + \frac{5}{20} - \frac{8}{20} = \boxed{\frac{7}{20}}$$

Problem 6

Suppose X and Y are independent $\mathsf{N}(0,1)$ random variables.

(a) Find $\mathsf{P}\left(X^2 < 1\right)$.

(b) Find $\mathsf{P}\left(X^2 + Y^2 < 1\right)$.

(a) $X^2 \sim \chi_1^2$ SINCE $X \sim N(0,1)$ AND X^2 IS A CHI-SQUARED DISTRIBUTION WITH 1 DF.

IN R, PCHISQ (1,1) GIVES 0.6827

(b) $(X^2+Y^2) \sim \chi_2^2$ SINCE BOTH X AND $Y \sim N(0,1)$ AND X^2+Y^2 IS A CHI-SQUARED DIST. WITH 2 DF.

IN R, PCHISQ (1,2) = 0.3935

Problem 7

Let $X \sim \mathrm{N}(\mu, \sigma^2)$, and let $Y \sim \mathrm{N}(\gamma, \sigma^2)$. Suppose X and Y are independent. Define two random variables: $U = X + Y$ and $V = X - Y$.

(a) Show that U and V are independent Normal random variables.

(b) Find the distribution of U.

(c) Find the distribution of V.

(a) SINCE X AND Y ARE INDEPENDENT NORMALS AND U AND V ARE LINEAR COMBINATIONS OF THEM, U AND V ARE BIVARIATE NORMALS SUCH THAT:

$$\begin{bmatrix} U \\ V \end{bmatrix} \sim \begin{bmatrix} E(U) \\ E(V) \end{bmatrix} \begin{bmatrix} Var(U) & Cov(V,U) \\ Cov(U,V) & Var(V) \end{bmatrix}$$

BIVARIATE NORMALS HAVE A SPECIAL PROPERTY WHERE UNCORRELATED IMPLIES INDEPENDENCE SO WE CAN PROVE INDEPENDENCE BY PROVING THEY ARE UNCORRELATED:

$$Cov(U,V) = Cov(X+Y, X-Y)$$

$$= Var(X) - Cov(X,Y) + Cov(Y,X) - Var(Y)$$

$$= \sigma^2 - Cov(X,Y) + Cov(Y,X) - \sigma^2$$

$$= -Cov(X,Y) + Cov(Y,X) \qquad Cov(X,Y) \text{ AND } Cov(Y,X) = 0 \text{ SINCE } X \perp\!\!\!\perp Y$$

$$= 0 \quad \therefore U \perp\!\!\!\perp V$$

(b) HOMEWORK 7, PROBLEM 5 PROVED THAT IF $X \sim N(\mu_1, \sigma^2) \perp\!\!\!\perp Y \sim N(\mu_2, \sigma^2)$, THEN

$$T = X + Y \sim N(\mu_1 + \mu_2, 2\sigma^2)$$

IN THIS CASE, $U = X + Y \sim N(\mu + \gamma, 2\sigma^2)$

(c) SIMILARLY, $V = X - Y \sim N(\mu - \gamma, 2\sigma^2)$

Problem 8

A random variable X is defined by the transformation $Z = \log(X)$, where the mean of the random variable Z is $\mathsf{E}(Z) = 0$. Is $\mathsf{E}(X)$ greater than, less than, or equal to 1?

JENSEN'S INEQUALITY GIVES $E(g(Z)) \geq g(E(Z))$ FOR CONVEX FUNCTIONS

IF $Z = \log(X)$, THEN $X = e^Z$ CAN BE THE FUNCTION $g(Z)$ SUCH THAT

$g(E(Z)) = g(0) = e^0 = 1$

SINCE $g(Z)$ IS CONVEX, WE CAN PLUG INTO JENSEN'S INEQUALITY:

$E(g(Z)) \geq 1$ AND SINCE $g(Z) = X$, WE HAVE:

$E(X) \geq 1$