Problem 1

Let X and Y be i.i.d. Expo(λ), and $T = \log(X/Y)$. Find the CDF and PDF of T.

$$f_X(x) = \lambda e^{-\lambda x} \qquad f_Y(y) = \lambda e^{-\lambda y}$$

$$P(T \le t) = P\left(\ln\left(\frac{X}{Y}\right) \le t\right) = P\left(\frac{X}{Y} \le e^t\right) = P(X \le e^t Y)$$

$$P(X \le e^t Y) = \int_0^\infty \int_0^{e^t y} f_X(x) f_Y(y)\, dx\, dy$$

$$= \int_0^\infty f_Y(y) \int_0^{e^t y} \lambda e^{-\lambda x}\, dx\, dy$$

$$= \int_0^\infty f_Y(y) \left(-e^{-\lambda x} \Big|_0^{e^t y}\right) dy$$

$$= \int_0^\infty f_Y(y) \left(-e^{-\lambda e^t y} + 1\right) dy$$

$$= \int_0^\infty \lambda e^{-\lambda y} \left(1 - e^{-\lambda y e^t}\right) dy$$

$$= \int_0^\infty \lambda e^{-\lambda y} dy - \int_0^\infty -\lambda e^{-\lambda y e^t - \lambda y} dy$$

= 1 SINCE THIS IS THE PDF OF Expo(λ)

PULL CONSTANT OUT

$$= 1 + \lambda \int_0^\infty e^{y(-\lambda e^t - \lambda)} dy$$

$$u = y(-\lambda e^t - \lambda)$$

$$du = (-\lambda e^t - \lambda)\,dy$$

$$= 1 + \frac{\lambda}{-\lambda e^t - \lambda} \int_0^\infty e^u \, du$$

$$= 1 + \frac{\lambda}{-\lambda e^t - \lambda} \left(e^u \Big|_0^\infty \right)$$

$$= 1 + \frac{\lambda}{-\lambda e^t - \lambda} \left(e^{y(-\lambda e^t - \lambda)} \Big|_0^\infty \right)$$

$$= 1 + \frac{\lambda}{-\lambda e^t - \lambda} (0 - 1)$$

$$= 1 + \frac{-\lambda}{\lambda e^t + \lambda} = 1 + \frac{-\lambda}{\lambda(e^t + 1)} = 1 - \frac{1}{e^t + 1}$$

$$= \frac{e^t + 1}{e^t + 1} - \frac{1}{e^t + 1} = \frac{e^t}{e^t + 1}$$

CDF: $F_T(t) = \dfrac{e^t}{e^t + 1}$ For $t \in \mathbb{R}$

0 OTHERWISE

$g(x) = e^t \quad g'(x) = e^t$

$h(x) = e^t + 1 \quad h'(x) = e^t$

PDF: $f_T(t) = F_T(t) \dfrac{d}{dt} = \dfrac{e^t}{e^t + 1} \dfrac{d}{dt}$

$$= \frac{e^t(e^t + 1) - e^t e^t}{(e^t + 1)^2} = \frac{e^{2t} + e^t - e^{2t}}{(e^t + 1)^2}$$

$$= \frac{e^t}{(e^t + 1)^2}$$ For $t \in \mathbb{R}$

0 OTHERWISE

Problem 2

Let X and Y be i.i.d. $\textsf{Expo}(\lambda)$, and transform them to $T = X + Y$, $W = X/Y$.

(a) Find the joint PDF of T and W. Are they independent?

(b) Find the marginal PDFs of T and W.

a) $f_{T,W}(t,w) = f_{X,Y}(x,y) \left| \frac{\partial(x,y)}{\partial(t,w)} \right|$

$W = \frac{X}{Y}$ $\quad T = X + Y$

$X = T - Y$

$X = WY$

$Y = T - X$

$Y = \frac{X}{W}$

$T = WY + Y = Y(W+1)$

$\Rightarrow Y = \frac{T}{w+1}$

$\Rightarrow X = w\left(\frac{T}{w+1}\right) = \frac{wT}{w+1}$

$\frac{\partial x}{\partial t} = \frac{wT}{w+1} \frac{d}{dt} = \frac{w}{w+1}$

$\frac{\partial x}{\partial w} = T\left(\frac{w}{w+1}\right)\frac{d}{dw} = T\left(\frac{w+1-1}{(w+1)^2}\right)$

$= \frac{T}{(w+1)^2}$

$\frac{\partial y}{\partial t} = \frac{T}{w+1}\frac{d}{dt} = \frac{1}{w+1}$

$\frac{\partial y}{\partial w} = \frac{T}{w+1}\frac{d}{dw} = \frac{-T}{(w+1}$

$$\left|\frac{\partial(x,y)}{\partial(t,w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{w}{w+1} & \frac{T}{(w+1)^2} \\ \frac{1}{w+1} & \frac{-T}{(w+1)^2} \end{vmatrix}$$

$$= \left| \frac{w}{w+1} \cdot \frac{-T}{(w+1)^2} - \frac{T}{(w+1)^2} \cdot \frac{1}{w+1} \right| = \left| \frac{-wT}{(w+1)^3} - \frac{T}{(w+1)} \right|$$

$$= \left| \frac{-wT - T}{(w+1)^3} \right| = \left| \frac{-T(w+1)}{(w+1)^3} \right| = \left| \frac{-T}{(w+1)^2} \right| = \frac{T}{(w+1)^2}$$

$$f_{T,W}(t,w) = f_X(x) f_Y(y) |\vec{J}|$$
$$= \lambda e^{-\lambda x} \lambda e^{-\lambda y} |\vec{J}|$$
$$= \lambda^2 e^{-\lambda(x+y)} |\vec{J}| = \begin{cases} \lambda^2 e^{-\lambda t} \dfrac{t}{(w+1)^2} & \text{FOR } 0<T<\infty;\ 0<TW<\infty \\ 0 & \text{OTHERWISE} \end{cases}$$

INDEPENCE TEST BELOW

(b) $$f_T(t) = \int_0^\infty \lambda^2 e^{-\lambda t} \frac{t}{(w+1)^2} dw = \lambda^2 e^{-\lambda t} t \int_0^\infty \frac{1}{(w+1)^2} dw$$
$$= \lambda^2 e^{-\lambda t} t \left(\frac{-1}{w+1} \Big|_0^\infty \right)$$
$$= \lambda^2 e^{-\lambda t} t (0 - (-1))$$
$$= \lambda^2 e^{-\lambda t} t \quad \text{FOR } t \geq 0;\ 0 \text{ OTHERWISE}$$

$$f_W(w) = \int_0^\infty \lambda^2 e^{-\lambda t} \frac{t}{(w+1)} dt = \lambda^2 \frac{1}{w+1} \int_0^\infty t e^{-\lambda t} dt$$
$$= \frac{\lambda^2}{w+1} \left(\frac{e^{-\lambda t}(\lambda t + 1)}{\lambda^2} \Big|_0^\infty \right)$$
$$= \frac{\lambda^2}{w+1} \cdot \left(\frac{1}{\lambda^2} \right) = \begin{cases} \dfrac{1}{w+1} & \text{FOR } w \geq 0 \\ 0 & \text{OTHERWISE} \end{cases}$$

INDEPENDENCE IF $f_{T,W}(t,w) = f_T(t) f_W(w)$

$$\lambda^2 e^{-\lambda t} \frac{t}{(w+1)^2} = \lambda^2 e^{-\lambda t} t \cdot \frac{1}{w+1} \quad \therefore \text{ INDEPENDENT}$$

Problem 3

Let $U \sim \text{Unif}(0, 1)$ and $X \sim \text{Expo}(\lambda)$, independently. Find the PDF of $U + X$.

LET $T = U + X$

$U = T - X$

THEOREM 8.2.1 $\Rightarrow f_T(t) = \int_{-\infty}^{\infty} f_U(t-x) f_X(x) dx$

$$f_U(u) = \frac{1}{1-0} = \begin{cases} 1 & \text{FOR } u \in (0,1) \\ 0 & \text{OTHERWISE} \end{cases}$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{FOR } x > 0 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$\int_0^t 1 \cdot \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x} \Big|_0^t$$

$$= 1 - e^{-\lambda t} - (1-1) = \begin{cases} 1 - e^{-\lambda t} & t > 0 \\ 0 & \text{OTHERWISE} \end{cases}$$

Problem 4

Let X and Y be i.i.d. Expo(λ). Use a convolution integral to show that the PDF of $L = X - Y$ is

$$f_L(l) = \frac{\lambda}{2} e^{-\lambda |l|},$$

for all real l.

$L = X - Y$

$$f_L(l) = \frac{\lambda}{2} e^{-\lambda |l|} \quad l \in \mathbb{R}$$

AS $X \perp\!\!\!\perp Y$, WE CAN APPLY THEOREM 8.2.1:

$$f_L(l) = \int_0^\infty f_Y(x - l) f_X(x) \, dx$$

$$= \int_0^\infty \lambda e^{-\lambda(x - l)} \lambda e^{-\lambda x} \, dx$$

$$= \int_0^\infty \lambda^2 e^{-\lambda x + \lambda l - \lambda x} \, dx$$

$$= \lambda e^{-\lambda l} \int_0^\infty \lambda e^{-\lambda x - \lambda x} \, dx$$

$$= \lambda e^{-\lambda l} \int_0^\infty \lambda e^{-2\lambda x} \, dx$$

$$= \frac{\lambda}{2} e^{-\lambda l} \underbrace{\int_0^\infty 2\lambda e^{-2\lambda x} \, dx}_{= \text{Expo}(2\lambda), \text{ WHICH INTEGRATES TO } 1}$$

$$= \frac{\lambda}{2} e^{-\lambda l} \quad \text{FOR } l > 0$$

$$= \begin{cases} \frac{\lambda}{2} e^{-\lambda|l|} & \text{FOR } l \in \mathbb{R} \\ 0 & \text{OTHERWISE} \end{cases}$$

ABSOLUTE VALUE TO ACCOUNT FOR ALL l, NOT JUST POSITIVE

Problem 5

Use a convolution integral to show that if $X \sim \mathrm{N}(\mu_1, \sigma^2)$ and $Y \sim \mathrm{N}(\mu_2, \sigma^2)$ are independent, then

$$T = X + Y \sim \mathrm{N}(\mu_1 + \mu_2, 2\sigma^2).$$

You can use a standardization (location-scale) idea to reduce to the standard Normal case before setting up the integral. Hint: complete the square.

$$f_T(t) = \int_{-\infty}^{\infty} f_Y(t-x) f_X(x)\,dx$$

FORMULA TO FIND THE PDF OF $T = X + Y$ WHERE $X \perp\!\!\!\perp X$

FIRST, STANDARDIZE X AND Y SUCH THAT

$$V = \frac{X - \mu_1}{\sigma}, \quad W = \frac{Y - \mu_2}{\sigma}, \quad \text{AND} \quad Z = V + W$$

THUS,

$$f_Z(z) = \int_{-\infty}^{\infty} f_V(z-v) f_W(v)\,dv$$

THE PDFs OF V AND W ARE BOTH THAT OF A STANDARD NORMAL: $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

PLUGGING IN TO f_Z GIVES:

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-v)^2}{2}}\,dv$$

PULL THE CONSTANT OUT AND COMBINE EXPONENTS

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(v^2+(z-v)^2)}\,dv$$

COMPLETE THE SQUARE FOR THE EXPONENT

$$-\frac{1}{2}v^2 - \frac{1}{2}(z-v)^2 = -\frac{1}{2}v^2 - \frac{1}{2}(z^2-2zv+v^2)$$

$$= -\frac{1}{2}v^2 - \frac{1}{2}z^2 - zv - \frac{1}{2}v^2$$

$$= -v^2 + zv - \frac{1}{2}z^2$$

$$= -\left(v^2 - zv + \frac{1}{2}z^2\right)$$

$$= -\left(v^2 - zv + \frac{1}{4}z^2 - \frac{1}{4}z^2 + \frac{1}{2}z^2\right)$$

$$= -\left[\left(v-\frac{1}{2}z\right)^2 + \frac{1}{4}z^2\right]$$

$$= -\left(v-\frac{1}{2}z\right)^2 - \frac{1}{4}z^2$$

PLUG IN RESULTS:

$$f_Z(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(v-\frac{1}{2}z\right)^2 - \frac{1}{4}z^2}\,dv$$

PULL OUT $e^{-\frac{1}{4}z^2}$ FROM THE INTEGRAL

$$= \frac{e^{-\frac{1}{4}z^2}}{2\pi}\int_{-\infty}^{\infty} e^{-\left(v-\frac{1}{2}z\right)^2}\,dv$$

$u = v - \frac{1}{2}z \quad du = dv$

$$= \frac{e^{-\frac{1}{4}z^2}}{2\sqrt{\pi}}\underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\left(\frac{\sqrt{2}}{2}\right)} e^{-\frac{1}{2\left(\frac{\sqrt{2}}{2}\right)}u^2}\,du}$$

THIS IS EQUIVALENT TO $N\left(0, \frac{\sqrt{2}}{2}\right)$ AND INTEGRATES TO 1.

$$= \frac{1}{2\sqrt{\pi}} e^{-\frac{z^2}{4}} \sim N(0,2); \quad Z = V + W \sim N(0,2)$$

UNDO THE LOCATION-SCALE TRANSFORMATION:

$$T = X + Y = \sigma V + \mu_1 + \sigma W + \mu_2 = \sigma(V+W) + (\mu_1 + \mu_2)$$

HENCE, $T = X + Y \sim N(\mu_1 + \mu_2, 2\sigma^2)$

Problem 6

Let W_1 and W_2 be two random variables with the joint distribution:

$$\mathsf{P}(W_1 \leq w_1, W_2 \leq w_2) = \int_{-\infty}^{w_1} \int_{-\infty}^{w_2} \frac{1}{2\pi} \exp\left[-\frac{1}{2}(x^2 + y^2)\right] \mathrm{d}x\,\mathrm{d}y.$$

Consider two other random variables $Z_1 = |W_1|$ and $Z_2 = |W_2|$. In words, Z_1 is the absolute value of W_1, Z_2 is the absolute value of W_2.

(a) Show that Z_1 is independent of Z_2.

(b) Show that Z_1 and Z_2 have the same distribution, and find that distribution.

(a) $$P(W_1 \leq w_1, W_2 \leq w_2) = \int_{-\infty}^{w_1} \int_{-\infty}^{w_2} \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} dx\,dy$$

$$= \int_{-\infty}^{w_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \int_{-\infty}^{w_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= P(W_1 \leq w_1) P(W_2 \leq w_2)$$

GIVEN THIS, WE KNOW W_1 AND W_2 ARE INDEPENDENT STANDARD NORMALS

IN THE CASE OF Z_1 AND Z_2:

$$P(Z_1 \leq w_1, Z_2 \leq w_2) = P(-w_1 \leq Z_1 \leq w_1, -w_2 \leq Z_2 \leq w_2)$$

$$= \int_{-w_1}^{w_1} \int_{-w_2}^{w_2} \frac{1}{\sqrt{2\pi}} e^{(-\frac{1}{2}(x^2+y^2))} dx\,dy$$

$$= \int_{-w_1}^{w_1} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \int_{-w_2}^{w_2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$= P(-w_1 \le Z_1 \le w_1)\, P(-w_2 \le Z_2 \le w_2)$$

$$= P(Z_1 \le w_1)\, P(Z_2 \le w_2)$$

$\therefore$ INDEPENDENCE OF Z_1 AND Z_2

(b) Z_1 AND Z_2 ARE IDENTICAL EXCEPT FOR THEIR USE OF x OR y SO WE CAN JUST SOLVE THE PDF OF ONE TO GET THE OTHER:

$$P(Z_1 \le w_1) = P(-w_1 \le W_1 \le w_1) = P(W_1 \le w_1) - P(W_1 \le -w_1)$$

$$f_{Z_1}(y) = f_{W_1}(y) + f_{W_1}(-y)$$

$$= 2 f_{W_1}(y) \quad \text{DUE TO SYMMETRY OF } \mathcal{N} \text{ DIST}$$

$$= 2\left(\frac{1}{\sqrt{2\pi}} e^{-y^2/2}\right)$$

$$= \begin{cases} \sqrt{\frac{2}{\pi}}\, e^{-y^2/2} & \text{FOR } y \ge 0 \\ 0 & \text{OTHERWISE} \end{cases}$$

AND, SINCE THEY ARE IDENTICAL:

$$f_{Z_2}(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\, e^{-x^2/2} & \text{FOR } x \ge 0 \\ 0 & \text{OTHERWISE} \end{cases}$$