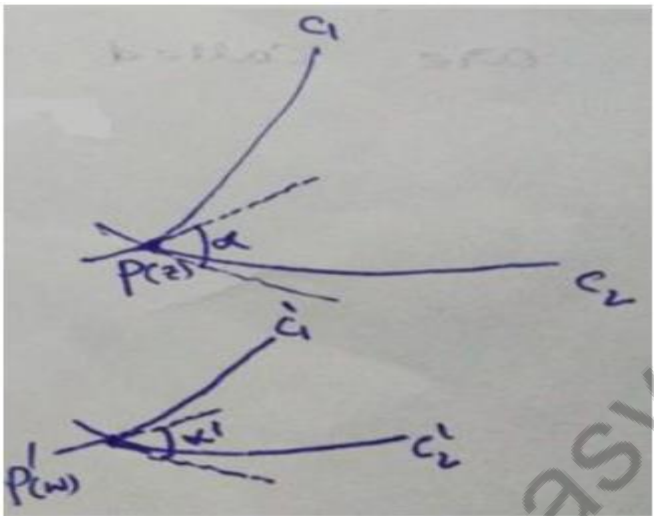
MODULE – 02

CONFORMAL TRANSFORMATION:

Let c_1 and c_2 be any two curves in the z-plane intersecting at z_0, suppose the transformation $w = f(z)$ transforms the curves c_1 and c_2 to the curves $c_1' \text{ and } c_2'$ respectively which intersect at a point $w_0 = f(z_0)$ in the w-plane. Then the transformation is said to be conformal if the angle between c_1 and c_2 is equal to the angle between $c_1' \text{ and } c_2'$ in both magnitude and direction.



SOME STANDARD TRANSFORMATION:

1. **Discuss the transformation** $w = z^2$

$$w = z^2 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

$$u + iv = (x + iy)^2$$

$$= (x^2 - y^2) + i2xy$$

$$\therefore\ u = x^2 - y^2 \quad ; \quad v = 2xy \ \ldots\ldots\ldots\ (2)$$

case(i): consider a line parallel to y-axis i.e., x=a, a is a constant

eqn(2) becomes

$$u = a^2 - y^2,\ v = 2ay$$

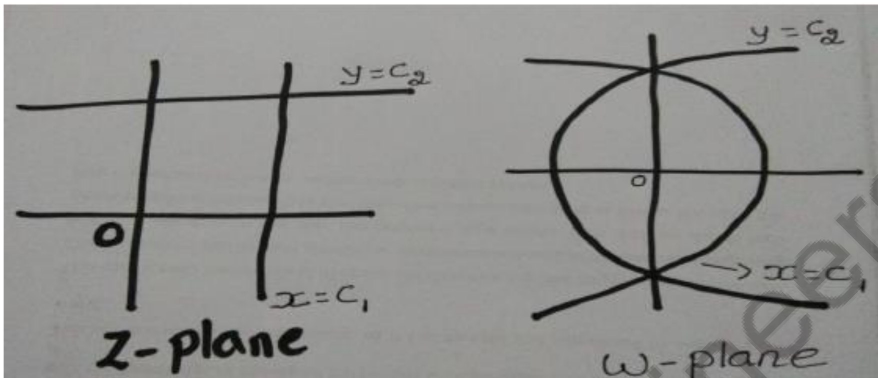
$$\text{or } v^2 = 4a^2y^2 = 4a^2(a^2 - u) = -4a^2(u - a^2)$$

This represents parabola in the w-plane along negative u-axis vertex at the point $(a^2, 0)$

Case(ii): consider a straight line parallel to x-axis i.e., y = b, b is a constant $u = x^2 - b^2$, v=2bx

Or $v^2 = 4b^2x^2 = 4b^2(u + b^2)$

This represents parabola in the w-plane along positive u axis, vertes at the point (b^2,0)



CONCLUSION: Straight line parallel to coordinate axes in z – plane transforms parabolas in w-plane under the transformation $w = z^2$

Case (iii)

Determine the images of $|z| = r$

Taking $z = re^{i\theta}$

$\therefore w = r^2 e^{2i\theta}$

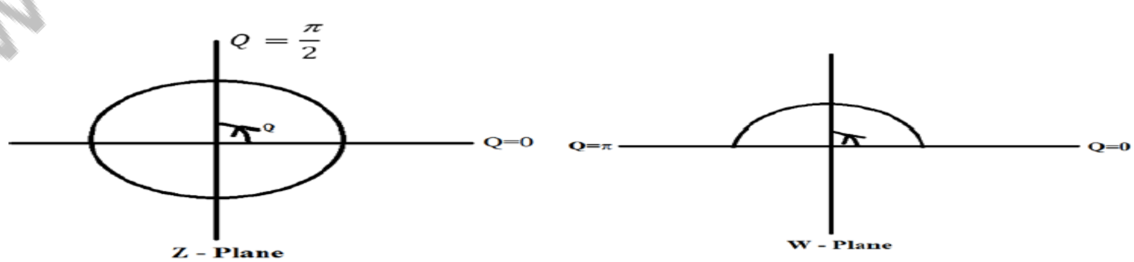
$w = R\,e^{i\phi}$ ....(5) Where $R=r^2$

$\phi = 2\theta$

$|w| = R$

$\therefore$ The angles at the origin are doubled under the mapping $w=z^2$.

The first quadrant of the z-plane $0 \le \theta \le \frac{\pi}{2}$ is mapped upon the entire upper half of the w-plane



2. Discuss the transformation W = e^z

$\cancel{w \quad e^z}$

$u + iv = e^{(x+iy)} \quad = e^x (\cos y + i \sin y)$

$u = e^x \cos y,$

$v = e^x \sin y$ ....(1)

Squaring and adding (1)

$u^2 + v^2 = e^{2x}$......................(2)

Dividing

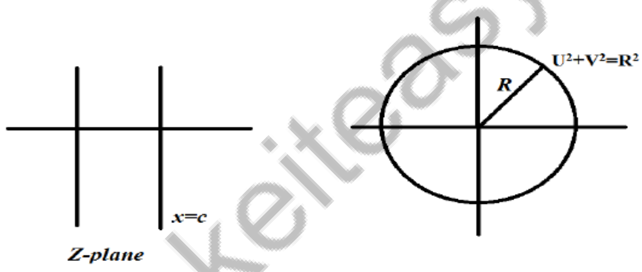
$\frac{v}{u} = \tan y$................(3)

Case(i): consider a straight line parallel to x-axis i.e., x = a, a is a constant

Therefore eqn(2 becomes)

$u^2 + v^2 = e^{2a} = r^2$

This represents a circle with centre origin and radius r in the w-plane



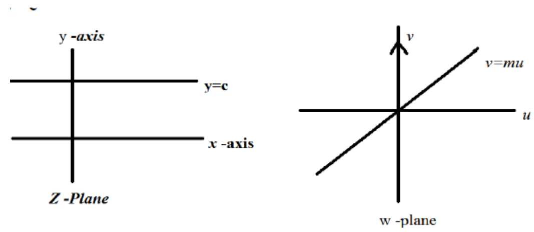
Case(ii): consider a straight line parallel to x-axis i.e., y = b, b is a constant

Therefore eqn(2 becomes)

$\frac{v}{u} = \tan b. = m(say)$

$v = um$

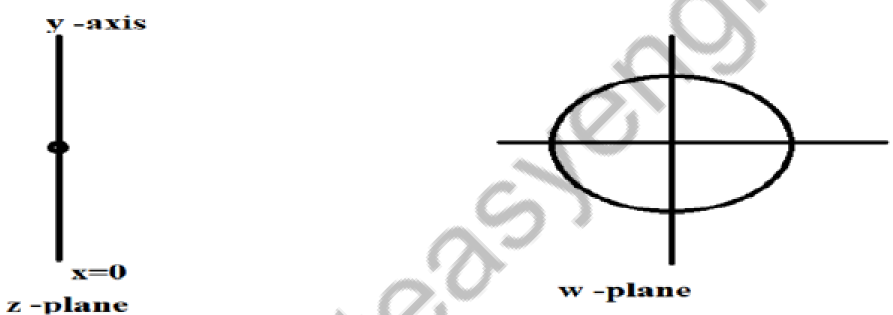
This represents a straight line passing through" the origin in the w-plane



Observation:

(1) Since $e^z \neq 0$, for all z, the point $w=0$ is not an image of any point z.

(2) Suppose $c=0$ ie. $x=0$ means that the y - axis in the z - plane is mapped onto the unit circle $u^2+v^2=1$



CONCLUSION: Straight line parallel to coordinate axes in z – plane transforms circle with centre origin and straight line passing through" origin respectively in w-plane under the transformation $w = e^z$

2) Discuss the transformation $w = z + \frac{1}{z}, z \neq 0$

Solution: The given transformation is conformal except at the points $z = \pm 1$.

since $\frac{dw}{dz} = 1 - \frac{1}{z^2} = 0$ for $z = \pm 1$

$w = re^{i\theta} + \frac{1}{r}e^{-i\theta}$

$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + \left(r - \frac{1}{r}\right)\sin\theta$

Equating real and imaginary parts we get

$$\left.\begin{aligned} u &= \left(r + \frac{1}{r}\right)\cos\theta \\ v &= \left(r - \frac{1}{r}\right)\sin\theta \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Case (i):- Find the images of circle, r=constant ie. r=c, represents a circle with constant Radius.

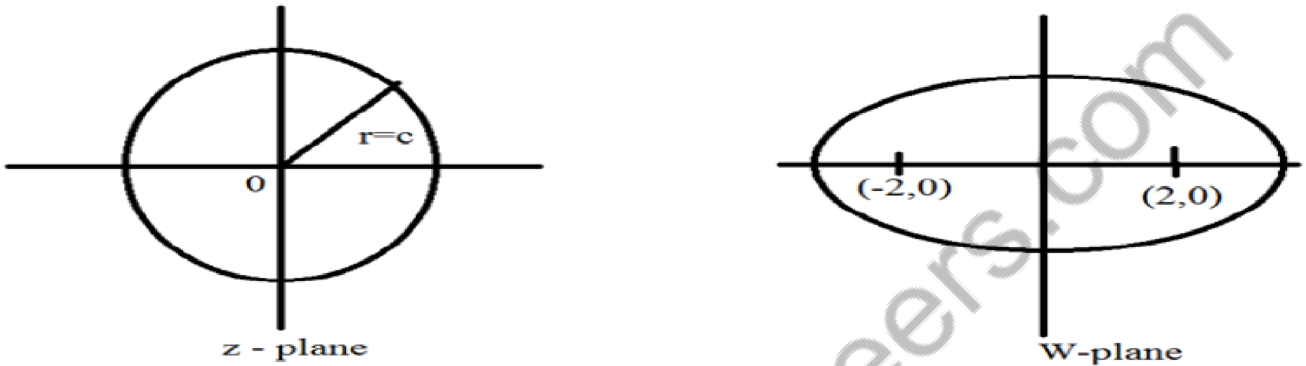
$\cos\theta = \frac{u}{a}$ where a=constant

$\sin\theta = \frac{v}{b}$ where b=constant

$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$

Equation given by (2) represents ellipses whose principal axes lie in u and v axes and have the length 2a and 2b respectively with foci$(\pm 2,0)$

Thus the circle r=constant is mapped onto ellipses under the transformation $w = z + \frac{1}{z}$



Case (ii) Find the images of line θ=constant, passing through origin, ie. θ=C

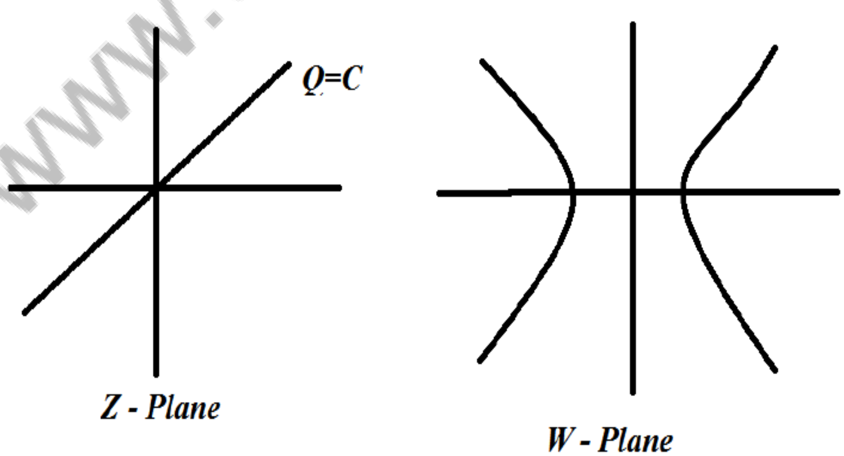
From (1) $\quad \frac{u}{a} = \left(r + \frac{1}{r}\right)$ Where $a = \cos c$

$\frac{v}{b} = \left(r - \frac{1}{r}\right)$ where $b = \sin c$

$\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1$....................(3) where A=2a and B=2b

Equation given by (3) represents hyperbalas in the w- plane.

Thus lines θ=constant is mapped onto hyperbolas under w=z+$\frac{1}{z}$



BILINEAR TRANSFORMATION OR MOBIUS TRANSFORMATION

The transformation $w = \frac{az+b}{cz+d}$, where a,b,c,d are real or complex constants such that $ad - bc \neq 0$ is called a bilinear transformation

CROSS RATIO:

Cross ratio of four points z_1, z_2, z_3, z_4 defined by $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$

INVARIANT POINTS OR FIXED POINTS

If a point z maps ont itself i.e., w = z then the point is called invariant point or a fixed point of the bilinear transformation

PROPERTIES OF BLT

1. The cross ratio of a set of 4 points remain invariant under a BLT
   $$\frac{(W_1 - W_2)(W_3 - W_4)}{(W_2 - W_3)(W_4 - W_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$
2. Every BLT map circles or straight lines in z-plane into circles or straight lines in w- plane

PROBLEMS:

1. **Find the BLT that maps the points (0,-i,-1) of z-plane onto the points (i,1,0) of w- plane respectively**.

Soln.

Given (z_1, z_2, z_3, z_4)= (z , 0. –i, 0)

(w_1, w_2, w_3, w_4)= (w, i, 1, 0)

Using cross ratio of 4 points

$$\frac{(w-i)(1-0)}{(i-1)(0-w)} = \frac{(z-0)(-i+1)}{(0+i)(-1-z)}$$

$$\frac{w-i}{w} = \frac{(1-i)^2}{-i} \cdot \frac{z}{z+1}$$

$$\frac{w-i}{w} = 2. \frac{z}{z+1}$$

$$wz - iz + w - i = 2zw$$

$$w = \frac{iz+1}{-z+1}$$

This is the required BLT

2. **Find the BLT that maps the points (∞, i, 0) of z-plane onto the points (-1, -i, 1) of w-plane respectively.Also find the invariant points.**

Soln.

Given $(z_1, z_2, z_3, z_4) = (z, \infty, i, 0)$

$(w_1, w_2, w_3, w_4) = (w, -1, -i, 1)$

Using cross ratio of 4 points and $z_2 \to \infty \; \& 1/z_2 \to 0$

$$\frac{(w+1)(-1-i)}{(-1+i)(1-w)} = \frac{(-1)(i)}{(1)(-z)}$$

$$\frac{(w+1)(1+i)}{(1-i)(1-w)} = \frac{i}{z}$$

$$zwi + zi + wz + z = i + 1 - wi - w$$

$$(w+1)(i+1)z = (i+1)(1-w)$$

$$w = \frac{-z+1}{z+1}$$

This is the required BLT

$$put\ w = z$$

$$z = \frac{-z+1}{z+1}$$

$$z^2 + 2z - 1 = 0$$

$$z = -1 \pm \sqrt{2}$$

Are invariant points.

3. **Find the BLT that maps the points (i,1,-1) of z-plane onto the points (1, 0,∞) of w- plane respectively.Also find the invariant points.**

Solution: The required bilinear transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)w_3}{w_3\left(\frac{w}{w_3}-w_3\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \quad \because \frac{w}{w_3} \to 0 \quad w_3 \to \infty$$

$$\frac{(w-1)(0-1)}{(0-1)(0-1)}=\frac{(z-i)(1+1)}{(z+1)(1-i)}$$

$$-(w-1)=\frac{(z-i)(1+1)}{(z+1)(1-i)}$$

$$-w+1=\frac{2(z-i)}{(z+1)(1-i)}$$

$$w=1-\frac{2(z-i)}{(z+1)(1-i)}$$

$$w=\frac{(z+1)(1-i)-2(z-i)}{(z+1)(1-i)}$$

$$=\frac{z-iz+1-i-z-iz+1+i}{(z+1)(1-i)}$$

$$w=\frac{(z+1)(1-i)-2(z-i)}{(z+1)(1-i)}$$

$$=\frac{z-iz+1-i-2z+2i}{(z+1)(1-i)}$$

$$=\frac{-z-iz+1+i}{(z+1)(1-i)}$$

$$=\frac{(1-z)+i(1-z)}{(z+1)(1-i)}$$

$$=\frac{(1-z)(1+i)}{(z+1)(1-i)}$$

$$=\frac{(1-z)}{(1+z)}\frac{(1+i)}{(1-i)}\frac{(1+i)}{(1+i)}$$

$$w=\frac{(1-z)}{(1+z)}\cdot\frac{2i}{2} \qquad (1+i)^2=2i$$

$$1-i^2=2$$

$$w=\frac{i(1-z)}{1+z}\text{...........}(*) \text{ is a required bilinear transform}$$

To find the invariant points of bilinear transform

Taking w=2 in equation (*)

$$z=\frac{i(1-z)}{(1+z)}$$

$$z^2+z=i-iz$$

$$z^2+(1+i)z-i=0$$

$$z=\frac{-(1+i)\pm\sqrt{(1+i)^2-4(-i)}}{2}$$

$$=\frac{-(1+i)\pm\sqrt{2i+4i}}{2}$$

$$=\frac{-(1+i)\pm\sqrt{6i}}{2}$$

$\therefore z_1=\dfrac{-(1+i)+\sqrt{6i}}{2}$, $\quad z_2=\dfrac{-(1+i)-\sqrt{6i}}{2}$ are invarient points.

4. **Find the BLT that maps the points (-1, i,1) of z-plane onto the points (1,i,-1) of w- plane respectively**.

Solution: Let $z_1 = -1, \quad z_2 = i \quad z_3 = 1$

$$w_1 = -1, \quad w_2 = i \quad w_3 = 1$$

The required bilinear transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-1)(i+1)}{(w+1)(i-1)} = \frac{(z+1)(i-1)}{(z-1)(i+1)}$$

$$\frac{(w-1)}{(w+1)} = \frac{(z+1)}{(z-1)} \cdot \frac{(i-1)^2}{(i+1)^2}$$

$$= \frac{(z+1)}{(z-1)} \times \frac{(-2i)}{(2i)}$$

$$\frac{(w-1)}{(w+1)} = -\frac{(1+z)}{(z-1)}$$

$$\frac{(w-1)}{(w+1)} = \frac{(1+z)}{(1-z)}$$

$$(w-1)(1-z) = (w+1)(1+z)$$

$$w - wz - 1 + z = w + wz + 1 + z$$

$$-2wz - 2 = 0$$

$$-2wz = 2$$

$w = -\frac{1}{z}$ this is the requird transformation

COMPLEX INTEGRATION:

Consider xy-plane is taken as complex plane then the point p(x,y) on this curve corresponding to the complex number z=x-iy

The equation z = z(t) where t is parameter is reffered to as the equation of curve in the complex form.

Ex: as t varies over the interval and x=a cast y=a sint then the complex form of the equation of circle is

$z = x = iy$

$z = a\cos t = i \sin t$

$= a(cast + i \sin t$

$z = ae^{it} \qquad 0 \le t \le 2\pi$

Represent a circle leaving center at origin. And radius is equal to a.

Consider a continuous function f^z of complex variable z=x=iy defined at all points of curve from p to Q

Divide the curve in to n equal parts by taking points p= $p(z_0)$, $p_1(z_1)$, $p_2(z_2)$...

$p_{k-1}(Z_{k-1})..p_n(z_n) = Q$ on the came C

The complex line integral along the path C is defined by $\int_c f(z)dz.$

If C is closed curve then $\oint_c f(z)dz.$

<u>LINE INTEGRAL OF A COMPLEX VALUED FUNCTION</u>

Let "D" is the region of complex plane and f(z)=u(x,y)+iv(x,y) be complex valued function defind on „D" Let C be the curve in „D" then (z=x+iy dz=dx+idy)

$\int_c f(z)dz. = \int_c a(x, y) + iv(x, y)(dx = idy)$

$\int_c f(z)dz. = \int_c u(x, y)dx - v(x, y)dy + i\int_c vdx + udy$

$\int_c f(z)dz = \int_c (udx - vdy) + i\int_c (vdx + udy)$

Is called the line integral of f(z) along the carve C

PROBLEMS:

1. Evaluate $\int_c z^2 dz$ Along the straight line from z=0 to z=3+i

Solution ; a) $\int_c z^2 dz = \int_{y=0}^{3+i} z^2 dz$

Here z is varies from o to 3+i

Z = x+iy

Z=0 =) (0.0) (3,1)

Equation of the line gaining in $\frac{y-y_0}{x-x_0} = \frac{y_1-y_0}{x_1-x_0}$

$$y = \frac{x}{3}$$

$$dx = 3.dy$$

$$f(z) = z^2 =)(x+iy)^2 = x^2 - y^2 + i2xy$$

$$\int_c z^2 dz = \int_{(0.0)}^{(3.1)} (x^2 - y^2)dx - 2xydy + i\int_{(0.0)}^{(3.1)} [2xydx + 1x^2 - y^2)dy$$

Convert these integral in to ltu variable y or x respects to integral wi y from 0 to1 and x from 0to3

Use variable „y‟

$$\int_c z^2 dz = \int_0^1 [9y^2 - y^2)3dy - 2(3y)ydy] + i\int_0^1 [2(3y)y3dy + i9y^2 - y^2)dy]$$

$$= \int_{y=0}^{1} (24y^2 - 6y^2)dy - +i\int_0^1 (18y^2 + 8y^2)dy$$

$$i\int_0^1 18y^2 dy + i\int_0^1 26y^2 dy$$

$$= 18\left[\frac{y^2}{3}\right]_0^1 + 26i\left[\frac{y^3}{3}\right]_0^1 = 6 + \frac{26}{3}i$$

$$\int_c z^2 dz = 6 + \frac{26}{3}i$$

Along the given path.

.

2. Evaluate $\int_{(0.3)}^{(2,40} (2y = x^{2)dx=93x-y)dy}$ along the parabola x=2t, y=$t^2 = 3$

X varies from 0to2 and here

X=2t =) 0 =) 2t =) t=0

X=2t =) 02=) 2t =) t=1

T variesfrom 0to 1

$$\int_0^1 \left[(t^2+3) = 4t^2 \right] .dt + \left[t - (t^2+3) = 4t^2 \right] .t.dt$$

$$\int_0^1 (24t^2 - 2t^2 - 6t + 12)at$$

$$= 8t^3 - \frac{t^4}{2} - 3t^2 = 12t \int_0^1$$

$$=) 8 - \frac{1}{2} - 3 + 12 - (0)$$

$$=) \left(6 -^{1-} 6 + 24 \right] 2 =) \frac{33}{2}$$

Example-3: Evaluate $\int_{(0,3)}^{(2,4)} (2y + x^2)dx + (3x - y)dy$ along

(i) The parabola $x = 2t$ and $y = t^2 + 3$

(ii) The straight line from (0,3) to (2,4)

Solution :

(i) Along x=2t and $y = t^2 + 3$, from the given limit, x $\rightarrow$ 0 to 2

and $y \rightarrow 3$ *to* 4. Compute limit for t ie.

x	$t=\frac{x}{2}$	y	$t=\sqrt{y-3}$
0	0	3	0
2	1	4	1

Here t varies from 0 to 1, as x varies from 0 to 2 and y varies from 3 to 4

$\therefore x = 2t \quad dx = 2dt$

$y = t^2 + 3 \quad$ dy=2tdt

Let $I = \int_{(0,3)}^{(2,4)} (2y + x^2)dx + (3x - y)dy$

$$I = \int_{t=0}^{1} \left[2(t^2+3) + 4t^2\right]2dt + \left[6t - t^2 - 3\right]2t\, dt$$

$$= 2\int_{t=0}^{1} \left[6t^2 + 6\right]dt + \left[6t^2 - t^3 - 3t\right]dt$$

$$= 2\int_{t=0}^{1} \left[6t^2 + 6 + 6t^2 - t^3 - 3t\right]dt$$

$$= 2\int_{t=0}^{1} \left[12t^2 - 3t - t^3 + 6\right]dt$$

$$= 2\left[\frac{12t^3}{3} - \frac{3t^2}{2} - \frac{t^4}{4} + 6t\right]_0^1$$

$$= 2\left[\frac{12}{3} - \frac{3}{2} - \frac{1}{4} + 6\right]$$

$$= 2\left[10 - \frac{7}{4}\right]$$

$$= 2\frac{[40-7]}{4}$$

$$= \frac{33}{2}$$

(ii) Along straight line from $(0,3)$ to $(2,4)$.

Equation of line joining the points $(0,3)$ to $(2,4)$

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{(x_2-x_1)}$$

$$\frac{y-3}{x-0}=\frac{4-3}{(2-0)}$$

$$\frac{y-3}{x}=\frac{1}{2}$$

$$x=2y-6 \qquad \text{or} \qquad y=\frac{1}{2}[x+6]$$

Let $I=\int_{(0,3)}^{(2,4)}(2y+x^2)dx+(3x-y)dy$ -------(1)

Taking $y=\frac{1}{2}(x+6)$ $\quad \therefore dy=\frac{dx}{2}$ and x varies from 0 to 2

$$I=\int_0^2\left[2.\frac{1}{2}(x+6)+x^2\right]dx+\left[3x-\frac{1}{2}(x+6)\right]\frac{dx}{2}$$

$$=\int_0^2(x^2+x+6)dx+(6x-x-6)\quad\frac{dx}{4}$$

$$=\frac{1}{4}\int_0^2\left[4x^2+4x+24+5x-6\right]dx$$

$$=\frac{1}{4}\int_0^2\left(4x^2+9x+18\right)dx$$

$$=\frac{1}{4}\left[4\frac{x^3}{3}+9\frac{x^2}{2}+18x\right]_0^2$$

$$=\frac{1}{4}\left[4\times\frac{8}{3}+9\times\frac{4}{2}+36\right]$$

$$=\frac{1}{4}\left[\frac{32}{3}+18+36\right]$$

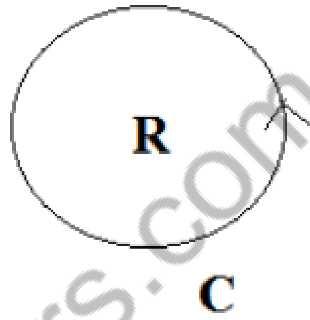
$$=\frac{1}{4}\left[\frac{32+54+108}{3}\right]$$

$$=\frac{194}{12}$$

$$=\frac{97}{6}$$

Cauchy's Theorem

Statement: If f(z) is analytic function and $f'(z)$ is continuous at all points inside and on a simple closed curve C then $\int_C f(z)dz = 0$



Proof : Let $f(z)=u+iv$ and $z = x+iy$,
$dz = dx + idy$ as usual.

Then

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy) \text{-------}(1)$$

The given curve in the complex plane is a simple closed curve C

Greens Theorem states that

$$\int_C Mdx + Ndy = \iint_A \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy,$$ Where A is a region bounded by A

Applying this theorem on RHS of (1) we obtain

$$\int_C f(z)dz = \iint_A \left[\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right] dx\, dy + i\iint_A \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dx\, dy$$

Since $f(z)$ is analytic, we have Cauchy Riemann Equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\int_C f(z)dz = \iint_A \left[-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right] dx\, dy + i\iint_A \left[\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right] dx\, dy$$

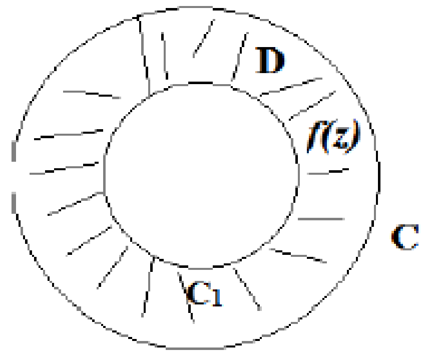
$= 0$ This proves Cauchy's Theorem

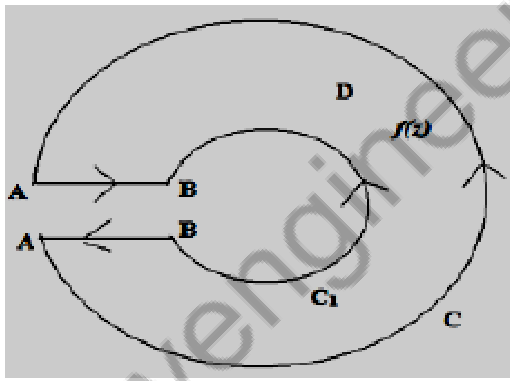
Extension of Cauchy's Theorem:

If $f(z)$ is analytic in the region D between two simple closed curve C and C_1, then

$$\int_C f(z)dz = \int_{C_1} f(z)dz$$

To Prove this, we need to introduced the cross cut AB, say





Now $f(z)$ is analytic at all points inside and on a simple closed curve

□ $: C \cup AB \cup C_1 \cup BA$, By Cauchy's Theorem

$$\int_{□} f(z)dz = 0$$

$$\int_{C \cup AB \cup C_1 \cup BA} f(z)dz = 0$$

$$\int_C f(z)dz + \int_{AB} f(z)dz + \int_{C_1} f(z)dz + \int_{BA} f(z)dz = 0$$

$$\int_C f(z)dz + \int_{AB} f(z)dz + \int_{-C_1} f(z)dz + \int_{-AB} f(z)dz = 0$$

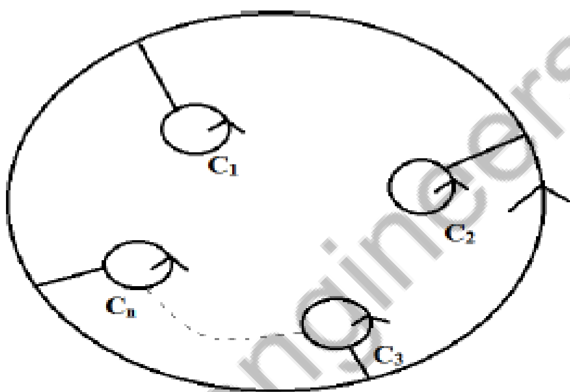
$$\int_C f(z)dz + \int_{AB} f(z)dz - \int_{C_1} f(z)dz - \int_{AB} f(z)dz = 0$$

$$\int_C f(z)dz - \int_{C_1} f(z)dz = 0$$

$$\int_C f(z)dz = \int_{C_1} f(z)dz$$

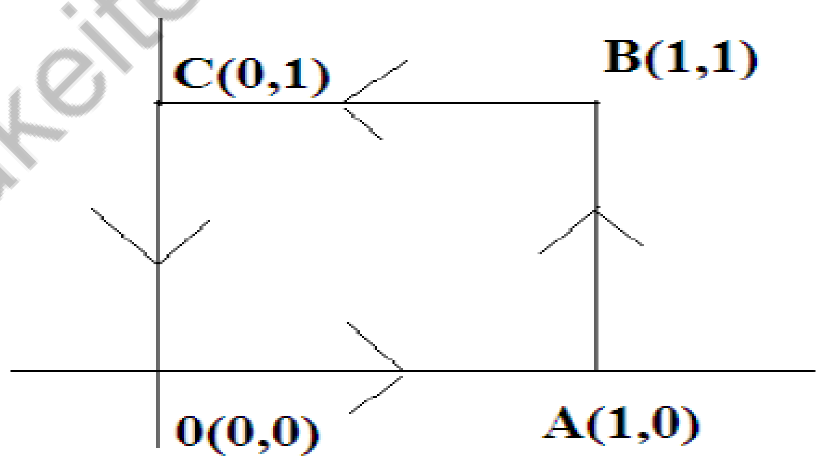
If C_1, C_2, C_3............C_n be any n number of closed curves with in C then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \ldots\ldots\ldots\ldots + \int_{C_n} f(z)dz$$



Example: Verify Cauchy's Theorem for the function $f(z) = z^2$ where C is the square having vertices (0,0), (1,0), (1,1),(0,1).

Solution:



Here the given curve C is the square in the Complex plane as shown in the above figure.

Since $f(z)=z^2$ is analytic everywhere in the complex plane, it is analytic at all points inside and on the curve C.

By Cauchy's Theorem

$$\int_C f(z)dz=0$$

$$\int_C z^2dz=0\ldots\ldots\ldots\ldots\ldots\ldots(*)$$

$$\int_C z^2dz=\int_{OA} z^2dz+\int_{AB} z^2dz+\int_{BC} z^2dz+\int_{CO} z^2dz$$

$$\int_C z^2dz=\int_{(0,0)}^{(1,0)} z^2dz+\int_{(1,0)}^{(1,1)} z^2dz+\int_{(1,1)}^{(0,1)} z^2dz+\int_{(0,1)}^{(0,0)} z^2dz\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Consider $\int_{(0,0)}^{(1,0)} z^2dz=\int_{(0,0)}^{(1,0)}(x+iy)^2(dx+idy)$

Here $y=0\quad\therefore dy=0$ and x varies from 0 to 1

$$=\int_{x=0}^{1}(x+io)^2(dx+o)$$

$$=\int_{x=0}^{1}x^2dx$$

$$=\frac{1}{3}\ldots\ldots\ldots\ldots\ldots(2)$$

Consider $\int_{(1,0)}^{(1,1)} z^2dz=\int_{(1,0)}^{(1,1)}(x+iy)^2(dx+idy)$

Here $x=1,\ dx=0$ and y varies from 0 to 1

$$=\int_{y=0}^{1}(1+iy)^2(idy)$$

$$= i \int_{y=0}^{1} (1+iy)^2$$

$$= i\left[\frac{(1+iy)^3}{3i}\right]_0^1 \qquad\qquad (1+i)^2 = 2i$$

$$= \frac{1}{3}\left[(1+i)^3 - 1\right]$$

$$= \frac{1}{3}\left[(1+i)(2i) - 1\right]$$

$$= \frac{1}{3}\left[2i - 2 - 1\right]$$

$$= \frac{1}{3}\left[2i - 3\right]$$

$$= \frac{2}{3}i - 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Consider $\int_{(1,1)}^{(0,1)} (x+iy)^2 (dx + idy)$

Here $y = 1,\ dy = 0$ and x varies from 1 to 0

$$= \int_{x=1}^{0} (x+i)^2\, dx$$

$$= \left.\frac{(x+i)^3}{3}\right|_1^0$$

$$= \frac{1}{3}\left[i^3 - (1+i)^3\right]$$

$$= \frac{1}{3}\left[-i - (1+i)2i\right]$$

$$= \frac{1}{3}\left[-i - 2i + 2\right]$$

$$= \frac{1}{3}\left[-3i + 2\right]$$

$$= -i + \frac{2}{3} \ldots\ldots\ldots\ldots (4)$$

Consider $\int_{(0,1)}^{(0,0)} z^2 dz = \int_{(0,1)}^{(0,0)} (x+iy)^2(dx+idy)$

Here $x=0$, $dx=0$ and y varies from 1 to 0

$$= \int_{y=1}^{0} (iy)^2 idy$$

$$= -i\left[\frac{y^3}{3}\right]_1^0$$

$$= -i\left[0-\frac{1}{3}\right]$$

$$= \frac{i}{3}..............(5)$$

Substitute 2,3,4&5 on RHS of (1)

$$\int_C z^2 dz = \frac{1}{3}+\frac{2i}{3}-1+\frac{2}{3}-i+\frac{i}{3}$$

$$= -\frac{2}{3}+\frac{2i}{3}+\frac{2}{3}-\frac{2i}{3}$$

$= 0$ Hence Cauchy's Theorem verified

If C is the circle $|z|=1$ verify Cauchy's Theorem for $f(z)=z^3$

Example-2:

Show that $\int_C |z|^2 dz = i-1$, where C is the square having vertices (0,0)(1,0)(1,1)(0,1).

Give the reason for Cauchy's theorem not being satisfied.

Solution:-

$$\int_C |z|^2 dz = \int_{0A} |z|^2 dz + \int_{AB} |z|^2 dz + \int_{BC} |z|^2 dz + \int_{C0} |z|^2 dz$$

$$= \int_{(0,0)}^{(1,0)} (x^2+y^2)(dx+idy) + \int_{(1,0)}^{(1,1)} (x^2+y^2)(dx+idy) + \int_{(1,1)}^{(0,1)} (x^2+y^2)(dx+idy) + \int_{(0,1)}^{(0,0)} (x^2+y^2)(dx+idy)$$

$$= \int_{x=0}^{1} x^2 dx + \int_{y=0}^{1} (1+y^2) i dy + \int_{x=1}^{0} (x^2+1) dx + \int_{y=1}^{0} y^2 . i dy$$

$$= \frac{1}{3} + i\left(\frac{4}{3}\right) - \frac{4}{3} - \frac{i}{3}$$

$$= -1 + i$$

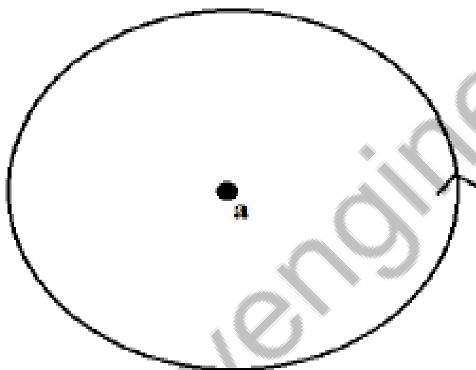
$\therefore \int_C |z|^2 = i - 1 \neq 0$. Hence Cauchy's Theorem is not verified since $f(z) = |z|^2 = x^2 + y^2$

ie. $u + iv = x^2 + y^2$ is not analytic. The necessary conditions $u_x = v_y$, $u_y = -v_x$ are not satisfied. This is the reason for Cauchy's Theorem not being satisfied.

Cauchy's Integral formula:

Statement: If $f(z)$ is analytic within and on a closed curve C and if a is any point within C, then $f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}dz$

Proof: Consider a closed curve C with 'a' is a point within C



Consider function $\frac{f(z)}{(z-a)}$ which is a analytic at all points within C except at $z=a$.

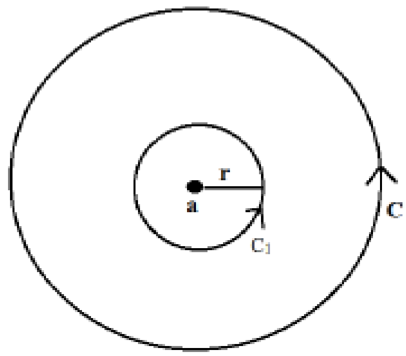
with the point 'a' as centre and radius r, draw a small circle C_1

lying entirely within C

Now $\frac{f(z)}{(z-a)}$ being analytic in the region

enclosed by C_1 and C, we have by Cauchy's Theorem

$$\int_C \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)}{(z-a)}dz$$



For any point z on C_1, $z - a = re^{i\theta}$

and $dz = i\,re^{i\theta}d\theta \qquad \therefore z = a + re^{i\theta}$

Where θ varies from 0 to 2π

$$\int_C \frac{f(z)}{(z-a)}dz = \int_0^{2\pi} \frac{f(a+r\,e^{i\theta})}{r\,e^{i\theta}}.ire^{i\theta}d\theta$$

$$= i\int_0^{2\pi} f(a+r\,e^{i\theta})d\theta$$

in the limiting form, as the circle C_1 shrinks to the point 'a' ie as $r \to 0$,

The above line integral approach to

$$\int_C \frac{f(z)}{(z-a)}dz = i\int_0^{2\pi} f(a)d\theta$$

$$= i\,f(a)\int_0^{2\pi} d\theta$$

$$= 2\pi i.\,f(a)$$

$$\therefore f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}dz,$$

which is the desired Cauchy's Integral formula

Note:- Generalized the Cauchy's Integral formula:

(i) $f'(a)=\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{(z-a)^2}dz$

(ii) $f''(a)=\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{(z-a)^3}dz$ and so on

$$f^n(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

Note:- In view of solving problems we consider Cauchy's integral formula as

$$\int_c \frac{f(z)}{(z-a)}dz=\begin{cases}2\pi i\, f(a) & \text{if } a \text{ is inside } C\\ 0 & \text{if } a \text{ is outside } C\end{cases}$$

Problems on Cauchy's Integral formula:

Example-1:

Evaluate $\displaystyle\int_c \frac{e^z}{(z+i\pi)}dz$ over each of the following regions C:

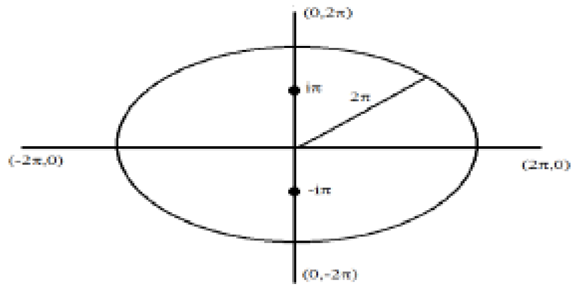
(i) $|z|=2\pi$ (ii) $|z|=\dfrac{\pi}{2}$ (iii) $|z-1|=1$

Solution:

$$\int_c \frac{e^z}{(z+i\pi)}dx=\int_c \frac{f(z)}{[z-(-)i\pi]}dz$$

where f(z)=e^z, which is analytic everywhere in the complex plane

(i) $|z| = 2\pi$ is a circle centre at the origin and radius 2π



$$\int_C \frac{e^z}{(z+i\pi)}dz = \int_C \frac{f(z)}{[z-(-i\pi)]}dz$$

Here the point $a = -i\pi$ lies inside the circle $|z| = 2\pi$ and $f(z) = e^z$ is analytic within and on the circle $|z| = 2\pi$. By Cauchy's Integral Formula

$= 2\pi i\, f(-i\pi)$

$= 2\pi i\, e^{-i\pi}$

$= 2\pi i[\cos\pi - i\sin\pi]$

$= -2\pi i$

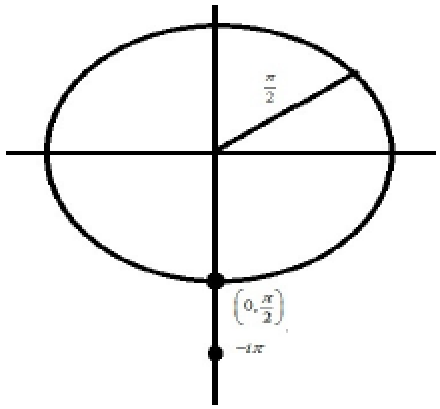
(ii) $|z| = \dfrac{\pi}{2}$ is a circle centre at the origin and radius $\dfrac{\pi}{2}$

$$\int_C \frac{e^z}{(z+i\pi)}dz = \int_C \frac{f(z)}{[z-(-i\pi)]}dz$$

Here point a=-iπ lies outside the circle

circle $|z| = \dfrac{\pi}{2}$, by Cauchy,s Integral

formula $\displaystyle\int_C \frac{e^z}{(z+i\pi)} = 0$



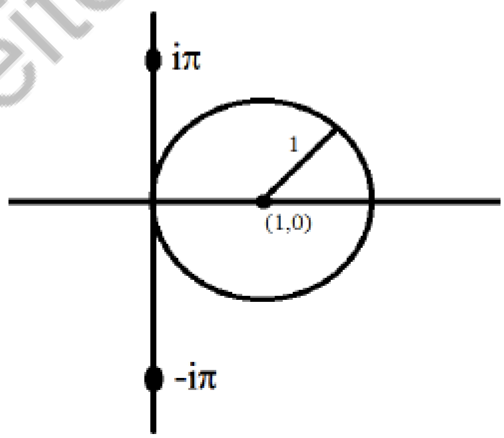
(iii) $|z-1|=1$ is a circle centre at the point (1.0) and radius 1.

$$\int_C \frac{e^z}{(z+i\pi)}dz = \int_C \frac{f(z)}{[z-(-i\pi)]}dz$$

Here point a=-iπ lies outside the circle

$|z-1|=1$ by Cauchu's Integral formula

$$\int_C \frac{e^z}{(z+i\pi)}dz = 0$$



Example- 02

Evaluate using Cauchy's integral formula:

(i) $\int_C \frac{e^{2z}}{(z+1)(z-2)}dz$ where C represents the circle $|z|=3$.

Solution: $\int_C \frac{e^{2z}}{(z+1)(z-2)}dz = \int_C \frac{f(z)}{(z+1)(z-2)}dz$..................(1)

Where $f(z)=e^{2z}$ which is analytic every where in the complex plane.

Consider $\frac{1}{(z+1)(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z-2)}$

$1 = A(z-2) + B(z+1)$

put $z=2$, $B=\frac{1}{3}$

put $z=-1$ $A=-\frac{1}{3}$

$$\frac{1}{(z+1)(z-2)} = \frac{-\frac{1}{3}}{(z+1)} + \frac{\frac{1}{3}}{(z-2)}$$

$$=\frac{1}{3}\left[\frac{1}{(z-2)} - \frac{1}{(z+1)}\right]................(2)$$

using (2) in (1) we get

$$\int_C \frac{e^{2z}}{(z+1)(z-2)}dz = \int_C f(z).\frac{1}{3}\left[\frac{1}{(z-2)} - \frac{1}{(z+1)}\right]dz$$

$$=\frac{1}{3}\left\{\int_C \frac{f(z)}{(z-2)}dz - \int_C \frac{f(z)}{[z-(1)]}dz\right\}....................(*)$$

$|z|=3$ is a circle centre at the origin and radius 3

$$=\frac{1}{3}\left\{\int_C \frac{f(z)}{(z-2)}dz - \int_C \frac{f(z)}{[z-(-1)]}dz\right\}$$

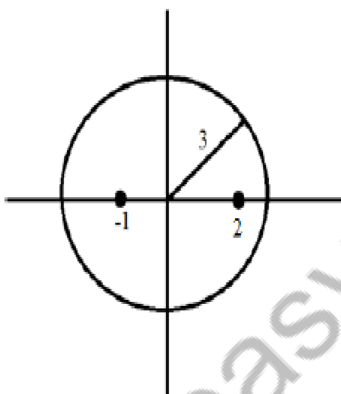
here point a=2, a=-1 both lies inside the circle $|z|$=3

$$=\frac{1}{3}\ 2\pi i f(2) - \frac{1}{3}\ 2\pi i f(-1)$$

$$=\frac{1}{3}\ 2\pi i\, e^{4} - \frac{1}{3}\ 2\pi i\, e^{-2}$$

$$=\frac{1}{3}\ 2\pi i\left[e^{4} - e^{-2}\right]$$

$$=\frac{2\pi i}{3}\left[e^{4} - e^{-2}\right]$$



Problems :

1. Verify Cauchy's theorem for the function f(z) = z^2 where 'C'is squair leaving vertices (0,0) (1,0)91,1)(0,1)

$$\int_{oA} z^2 dz + \int_{AB} z^2 dz + \int_{BC} z^2 dz + \int_{co} z^2 dz = 0$$

$$z^2 dz = (x+iy)^2(dx+idy)$$

$$x^2 dx$$

$$\int_{oA} z^2 dz = \int_{x=0} z^2 dx =) \frac{x^3}{3} \int_0^1 =) \frac{1}{3}$$

$$z^2 =) (x+iy)^2 (ddx + idy)$$

$$=) (1+i2y-y^2) idy$$

$$\int_{AB} z^2 dz = i \int_{y=0}^{1} (1-y^2+i2y) dy$$

$$\int_{AB} z^2 dz = \frac{2!}{3} - 1$$

$$\int_{BC} z^2 dz = \int_1^0 (x^2+2ix-1) dz$$

$$\int_{BC} z^2 dz = \frac{2}{3} - i$$

$$\int_{CO} z^2 dz = \int_1^0 (-y^2) idy$$

$$\int_{CO} z^2 dz = \frac{i}{3}$$

Adding (1),(2) ,(3), (4)

$$\int_C z^2 dz = \frac{1}{3} + \frac{2i}{3} - 1 + \frac{2}{3} - i + \frac{i}{3}$$

$$= 0$$

2. Evaluate $\int_C \frac{c^z}{z+iy\Gamma} dz$ over each contour C $|z-1|=1$

$$2\pi i = \int_c \frac{f(z)}{(z-a)} dz.$$

$$f(z) = e^z,\ |z-1| = 1$$

Soln: we have f(a) $\quad a = 1, r = 1$

$$\int_c \frac{f(z)}{(z-a)} dz = 0$$

$$when\ |z-1| = 1$$