MODULE - 01

CALCULUS OF COMPLEX FUNCTION

INTRODUCTION:

An extension of the concept of real numbers to accommodate complex numbers was evolved while considering solutions of equations like $x^2 + 1 = 0$. This equation cannot be satisfied for any real value of x. In fact, the solution of the equation $x^2 + 1 = 0$ is of the form $x = \pm\sqrt{-1}$. The square root of -1 cannot be a real no. because the square o any real no. is nonnegative. Similarly, there are any number of algebraic equations whose solutions involve square roots of negative numbers.

FUNCTION OF A COMPLEX VARIABLE:

If z = x+iy is a complex variable, then w = f(z) is called function of a complex variable. W = f(z) = u+iv where u = u(x,y), v = v(x,y). Hence for every point of (x,y) in z-plane, there corresponds (u,v) in w-plane

LIMIT OF A COMPLEX FUNCTION:

Complex value function f(z) defined in the neighbourhood of a point z_0 is said to have limit L as $z \to z_0$ if for all ε>0 however small, there exists a positive real number δ such that

$$|f(z) - L| < \varepsilon \text{ whenever } |z - z_0| < \delta ,$$

$$\text{i.e., } \lim_{z \to z_0} f(z) = L$$

CONTINUITY:

A function f(z) is said to be continuous at a point z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$

DIFFERENTIABILITY:

A function f(z) is said to be differentiable at a point z_0 if $\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists and this is unique

ANALYTIC FUNCTION:

A function f(z) is said to be analytic at a point z_0 if f(z) is differentiable at z_0 as well as at all points in a neighbourhood of z_0.

i.e., $f(z) = \lim_{\delta z \to 0} \dfrac{f(z + \delta z) - f(z)}{\delta z}$ exists and unique for all points in complex region.

NOTE:

Analytic function is also called as regular function or holomorphic function.

CAUCHY'S RIEMANN EQUATIONS OR C-R EQUATIONS IN CARTESIAN FORM:

Statement: If w = f(z) = u+iv is analytic function at the point z = x+iy, then there exists partial derivatives and satisfy the equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ called C-R equations in cartesian form.

Proof: By data f(z) is analytic at a point z = x+iy, there by definition of analytic function,

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \quad \text{……………(1)exists and unique.}$$

We have f(z) = u(x,y) +iv(x,y)

$$f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$$

$$z = x + iy$$

$$\delta z = \delta x + i\delta y$$

Substituting the above in (1) we get

$$f'(z) = \lim_{\delta z \to 0} \frac{\left[u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)\right] - \left[u(x,y) + iv(x,y)\right]}{\delta x + i\delta y} \quad \text{…………(2)}$$

Since $\delta z \to 0$, we have 2 possibilities.

Case(i): If δz is only real, then δy = 0

i.e., if $\delta z \to 0$ then $\delta x \to 0$

equn(2) becomes

$$f'(z) = \lim_{\delta x \to 0} \frac{\left[u(x+\delta x, y) - u(x,y)\right]}{\delta x} + i\frac{\left[v(x+\delta x, y) - v(x,y)\right]}{\delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{……………… (3)}$$

Case(ii): If δz is only imaginary, then δx = 0

i.e., if $\delta z \to 0$ then $\delta y \to 0$

equn(2) becomes

$$f'(z) = \frac{1}{i}\lim_{\delta y \to 0} \frac{\left[u(x, y+\delta y) - u(x,y)\right]}{\delta y} + \lim_{\delta y \to 0} \frac{\left[v(x, y+\delta y) - v(x,y)\right]}{\delta y}$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{……………… (4)}$$

Comparing real and imaginary parts of equations (3) and (4) we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{………………………proved}$$

CAUCHY'S RIEMANN EQUATIONS OR C-R EQUATIONS IN POLAR FORM:

Statement: If w = f(z) = u+iv is analytic function at the point $z = re^{i\theta}$, then there exists partial derivatives and satisfy the equations $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ & $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ called C-R equations in polar form.

Proof: By data f(z) is analytic at a point $z = re^{i\theta}$, there by definition of analytic function,

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \quad \text{……………(1)exists and unique.}$$

We have f(z) = u(r,θ) +iv(r,θ)

$$f(z+\delta z) = u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta)$$

$$\delta z = \delta r e^{i\theta} + ire^{i\theta}\delta\theta$$

Substituting the above in (1) we get

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta)] - [u(r,\theta) + iv(r,\theta)]}{\delta r e^{i\theta} + ire^{i\theta}\delta\theta} \text{………..(2)}$$

Since $\delta z \to 0$, we have 2 possibilities.

Case(i): If δz is only real, then δθ = 0

i.e., if $\delta z \to 0$ then $\delta r \to 0$

equn(2) becomes

$$f'(z) = e^{-i\theta}\lim_{\delta r \to 0} \frac{[u(r+\delta r, \theta) - u(r,\theta)]}{\delta r} + ie^{-i\theta}\lim_{\delta r \to 0} \frac{[v(r+\delta r, \theta) - v(r,\theta)]}{\delta r}$$

$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \text{……………. (3)}$$

Case(ii): If δz is only imaginary, then δr = 0

i.e., if $\delta z \to 0$ then $\delta\theta \to 0$

equn(2) becomes

$$f'(z) = \frac{e^{-i\theta}}{ir}\lim_{\delta\theta \to 0} \frac{[u(r, \theta+\delta\theta) - u(r,\theta)]}{\delta\theta} + i\frac{e^{-i\theta}}{ir}\lim_{\delta\theta \to 0} \frac{[v(r, \theta+\delta\theta) - v(r,\theta)]}{\delta\theta}$$

$$f'(z) = \frac{e^{-i\theta}}{ir}\left[\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right] \text{…………..(4)}$$

Comparing real and imaginary parts of equations (3) and (4) we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \text{ ……………… proved}$$

HARMONIC FUNCTION:

A function u is said to be harmonic function if it satisfies the Laplace equation.

i.e., $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = o$ in Cartesian form

$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = o$ in polar form.

Theorem:

Statement: The real and imaginary parts of an analytic function are harmonic.

Proof: Let f(z) = u(x,y)+iv(x,y)

Since f(z) is analytic, satisfies C-R equations

i.e., $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ……..(1) & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ …………(2)

differentiating (1) partially w.r.t x and (2) w.r.t y , we get

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ & $\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}$

Therefore $\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow$ u satisfies Laplace equation.

Hence real part u is harmonic.

differentiating (1) partially w.r.t y and (2) w.r.t x , we get

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial x^2}$ & $\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$

Equating the above equations $\frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2}$

$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \Rightarrow$ vsatisfies Laplace equation.

Hence imaginary part v is harmonic.

Polar form: If $f(z)=u(r,\ \theta)+i\ v(r,\ \theta)$ is an analytic function, then show that u and v satisfy Laplace's equation in polar form.

- Laplace equation in Polar form in two variables,

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0 \quad \text{and} \quad \frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}=0$$

We have C-R equation in polar form

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}----(1)$$

$$\frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}----(2)$$

Differentiate *(1)* with respect to r, $\frac{\partial^2 u}{\partial r^2}=-\frac{1}{r^2}\frac{\partial v}{\partial \theta}+\frac{1}{r}\frac{\partial^2 v}{\partial r\partial \theta}----(3)$

Differentiate *(2)* with respect to θ, $\frac{\partial^2 u}{\partial \theta^2}=-r\frac{\partial^2 v}{\partial \theta\partial r}----(4)$

using *(4)* and *(1)* on *RHS* of Equation *(3)*, we get

$$\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\left(\frac{\partial u}{\partial r}\right) + \frac{1}{r}\left(-\frac{1}{r}\frac{\partial^2 u}{\partial \theta}\right)$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

➢*Hence* u is Harmonic

From *(1)* we get, $\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial \theta}$

Differentiate with respect to θ $\frac{\partial^2 v}{\partial \theta^2} = r\frac{\partial^2 u}{\partial \theta \partial r} ---(5)$

From *(2)* we get $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} ----(6)$

Differentiate with respect to r $\frac{\partial^2 v}{\partial r^2} = +\frac{1}{r^2}\frac{\partial u}{\partial \theta} - \frac{1}{r}\frac{\partial^2 u}{\partial r \partial \theta} ----(7)$

using *(5)*,*(6)* on RHS of *(7)*

$$\frac{\partial^2 v}{\partial r^2} = \frac{1}{r}\left(-\frac{\partial v}{\partial r}\right) - \frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 v}{\partial \theta^2}\right) = 0$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

Hence v is Harmonic

Orthogonal System:

- Two curves are said to be orthogonal to each other when they intersect at right angles at each of their point of intersections.

Property: If $w=f(z)=u+iv$ be an analytic function then the family of curves $u(x,y)=c_1$ and $v(x,y)=c_2$ form an orthogonal system.

Solution: $f(z)=u+iv$ is an analytic functions.

$$\left.\begin{aligned}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\end{aligned}\right\}----C-\text{R}\,equation$$

$$u(x,y)=c_1$$

differentiate with respect to x, we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}=m_1----(2)$$

differentiate w.r.t, x we get

$$\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}=m_2---(3)$$

$$\therefore \quad m_1.m_2 = \frac{+\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \times \frac{+\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}$$

$$= \frac{\frac{\partial v}{\partial y}}{-\frac{\partial v}{\partial x}} \times \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} \quad \text{(By C-R Equations)}$$

$m_1.m_2 = -1$, form an orthogonal system

Polar form: Consider $u(r,\theta) = c_1 ---(1) \quad and \quad v(r,\theta) = c_2 ---(2)$

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r} \frac{\partial v}{\partial \theta} \\ \frac{\partial u}{\partial \theta} &= -r \frac{\partial v}{\partial r} \end{aligned}\right\} ----(3) \text{ C-R Equations}$$

differentiate (1) w.r.t. θ

$$\frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial r} \frac{dr}{d\theta} = 0$$

$$\frac{dr}{d\theta} = \frac{-\frac{\partial u}{\partial \theta}}{\frac{\partial u}{\partial r}} -----(4)$$

$\tan \phi_1 = \dfrac{r}{\frac{dr}{d\theta}}$ where ϕ_1 being the angle between the radius vector and the tangent to the curve(1)

$$\tan\phi_1 = \frac{r}{\dfrac{-\dfrac{\partial u}{\partial \theta}}{\dfrac{\partial u}{\partial r}}}$$

$$\tan\phi_1 = -\frac{r\dfrac{\partial u}{\partial r}}{\dfrac{\partial u}{\partial \theta}} ----(5)$$

Differentiate (2) w. r. t. θ

$$\frac{\partial v}{\partial \theta} + \frac{\partial v}{\partial r}\frac{dr}{d\theta} = 0$$

$$\frac{dr}{d\theta} = \frac{-\dfrac{\partial v}{\partial \theta}}{\dfrac{\partial v}{\partial r}}$$

$\tan\phi_2 = \dfrac{r}{\dfrac{dr}{d\theta}}$, where ϕ_2 being the angle between the radius and the tangent to the curve(2)

$$\tan\phi_1 \times \tan\phi_2 = \frac{r\dfrac{\partial u}{\partial r}}{\dfrac{\partial u}{\partial \theta}} \times \frac{r\dfrac{\partial v}{\partial r}}{\dfrac{\partial v}{\partial \theta}}$$

$$= \frac{r.\dfrac{1}{r}\dfrac{\partial v}{\partial \theta}}{-r\dfrac{\partial \theta}{\partial r}} \times \frac{r\dfrac{\partial v}{\partial r}}{\dfrac{\partial v}{\partial \theta}}$$

$$= -1 \text{ form an orthogonal system}$$

$$Note:\ (i)\ \sin(i\,x) = i\sin\text{h}\,x \quad or \quad \sin hx = \frac{1}{i}[\sin(i\,x)]$$

$$(ii)\ \cos(i\,x) = \cos hx$$

Example:1

Show that *f(z)=sin z* is analytic and hence find, $f'(z)$

Solution: *f(z) = sin(z)*

=sin(x+iy)

=sin(x)cos(iy)+cos(x)sin(iy)

f(z)=sin x cos hy + i cos x sin hy

Equating *real* and *imaginary* parts *u=sinx coshy* and *v=cosx sin hy--(1)*

u and *v* satisfies necessary conditions $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= \cos x \cos\text{h}\,y + i(-\sin x)\sin\text{h}\,y) ----(*)$$
$$= \cos(x)\cos(iy) - i\sin x.\frac{1}{i}\sin(iy)$$
$$= \cos(x)\cos(iy) - \sin x\sin(iy)$$
$$= \cos(x+iy)$$

$$f'(z) = \cos(z) \qquad \therefore \frac{d[\sin z]}{dz} = \cos z$$

or By Milne's Thomson method replace x by z and y by 0 in (*)

$$f'(z) = \cos(z).1 - 0 \qquad \therefore f'(z) = \cos(z) \quad or \quad \frac{d[\sin z]}{dz} = \cos z$$

2) *Show that* $w = z + e^z$ is analytic, hence find $\dfrac{dw}{dz}$

Solution : Let $w = f(z) = u + iv$.

$w = (x + e^x \cos y) + i(y + e^x \sin y)$

Equating real and imaginary parts

$u = (x + e^x \cos y),\ v = (y + e^x \sin y)$

u and v satisfies C-R equations

consider

$$\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= (1 + e^x \cos y) + i(e^x \sin y)$$

$$= 1 + e^x[\cos y + i\sin y] --- (1)$$

$$= 1 + e^x.e^{iy}$$

$$= 1 + e^z$$

$$\frac{d[z + e^z]}{dz} = 1 + e^z$$

Or By Milne's-Thomson method replace x by z and y by 0 in (1), we get derivative of $z + e^z$

Construction of Analytic Function:

Construction of analytic function $f(z)=u+iv$ when u or v or $u\pm v$ is given.

Example1: Find the Analytic Function *f(z)*, whose real part is $e^{2x}[x\cos 2y - y\sin 2y]$.

Solution:

$$\text{Given} \quad u = e^{2x}[x\cos 2y - y\sin 2y] ----(1)$$

Differentiate (1) w.r.t. x

$$\frac{\partial u}{\partial x} = e^{2x}\left[\cos 2y\right] + 2e^{2x}\left[x\cos 2y - y\sin 2y\right] ----(2)$$

Differentiate (1) w.r.t. y

$$\frac{\partial u}{\partial y} = e^{2x}\left[-2.x.\sin 2y - y.2\cos 2y - \sin 2y\right] ----(3)$$

$$\text{Consider} \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} ----(4)$$

By C-R Equations replace $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} ----(5)$$

using (2) and (3) on RHS (5)

$$f'(z) = e^{2x}\left[\cos 2y + 2x\cos 2y - 2y\sin 2y\right] + i\,e^{2x}\left[2x\sin 2y + 2y\cos 2y + \sin 2y\right]$$

By Milne's Method replace x by z and y by 0

$$f'(z) = e^{2z}[1+2z]$$

$$f'(z) = e^{2z} + 2e^{2z}.z$$

int $egrate$ we get

$$f(z) = \frac{1}{2}e^{2z} + 2\left[\frac{e^{2z}}{2}.z - \frac{e^{2z}}{4}\right] + c$$

$$f(z) = \frac{1}{2}e^{2z} + ze^{2z} - \frac{1}{2}e^{2z} + c$$

$$f(z) = ze^{2z} + c$$

2) Find the Analytic function whose real part is $\dfrac{\sin 2x}{\cos 2y - \cos 2x}$

$$Solution: u = \frac{\sin 2x}{\cosh 2y - \cos 2x} -----(1)$$

Differentiate w.r.t. x

$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x).2\cos 2x - \sin 2x[+2\sin 2x]}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2\cosh 2y \cos 2x - 2[\cos^2(2x) + \sin^2 2x]}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} -----(2)$$

Differentiate (1) w.r.t. y

$$\frac{\partial u}{\partial y} = \frac{\sin 2x[-(2\sinh 2y)]}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} -----(3)$$

Consider $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

By C-R equation replace $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ $\qquad f'(z) = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$

$$f'(z) = \frac{[2\cos 2x \cosh 2y - 2] + i2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

By Milne's Thomson method replace x by z and y by 0

$$f'(z) = \frac{2[\cos 2z - 1] + i.0}{(1 - \cos 2z)^2}$$

$$f'(z) = \frac{-2[1 - \cos 2z]}{(1 - \cos 2z)^2}$$

$$f'(z) = \frac{-2}{[1-\cos 2z]}$$

$$f'(z) = \frac{-2}{2\sin^2 z}$$

$$f'(z) = -\cos ec^2 z$$

$\text{int}\, ergate$

$$f(z) = +\cot z + c$$

3) Construct the analytic function whose imaginary part is $\left(r\text{-}\frac{1}{r}\right)\sin\theta,\ r \neq 0.$

Hence find the Real part.

Solution: Given $v = \left(r - \frac{1}{r}\right)\sin\theta ----(1)$

Differentiate (1) *w.r.t.* θ

$$\frac{\partial u}{\partial \theta} = \left(r - \frac{1}{r}\right)\cos\theta ----------(2)$$

Differentiate (1) *w.r.t.* r

$$\frac{\partial u}{\partial r} = \left(1 + \frac{1}{r^2}\right)\sin\theta ------------(3)$$

$$Consider\ f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] ----(4)$$

By C-R Equation replace $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ *on*

RHS of (4) *we get*

$$f'(z) = e^{-i\theta}\left[\frac{1}{r}\frac{\partial v}{\partial \theta} + i\frac{\partial v}{\partial r}\right]$$

$$f'(z) = e^{-i\theta}\left[\frac{1}{r}\left(r - \frac{1}{r}\right)\cos\theta + i\left(1 + \frac{1}{r^2}\right)\sin\theta\right]$$

By Milne's method replace r by z and θ by 0

$$f'(z) = e^{0}\left[\frac{1}{z}\left(z - \frac{1}{z}\right).1 + i.0\right]$$

$$f'(z) = \left(1 - \frac{1}{z^2}\right)$$

Integrate we get

$$f(z) = z + \frac{1}{z} + ic$$

To find real part: Consider $f(z) = re^{i\theta} + \frac{1}{re^{i\theta}} + ic$

$$u + iv = (r\cos\theta + ir\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta) + ic$$

$$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left[\left(r - \frac{1}{r}\right)\sin\theta + c\right]$$

$Equating$ real and imaginary parts

$$u=\left(r+\frac{1}{r}\right)\cos\theta$$

$$v=\left(r-\frac{1}{r}\right)\sin\theta+c \quad \text{to get actual imaginary part of an analytical function}$$

$$f(z)=u+iv \ \ taking \quad c=0$$

$$\therefore v=\left(r-\frac{1}{r}\right)\sin\theta$$

4) Find an analytic function $f(z)$ as a function of z

given that the sum of real and imaginary part is $x^3-y^3+3xy(x-y)$

$Solution$: The sum of real and imaginary part is given by

$$u+v=x^3-y^3+3xy(x-y)-------(1)$$

$Differentiate$ (1) $w.r.t.\ x$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}=3x^2-0+3xy+3y(x-y)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}=3x^2+3xy+3y(x-y)----(2)$$

$Differentiate$ (1) $w.r.t.\ y$

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}=0-3y^2+3xy(-1)+3x(x-y)$$

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}=-3y^2-3xy+3x(x-y)-------(3)$$

By C-R Equation replace $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad in(3)$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = -3y^2 - 3xy + 3x(x-y) ------(4)$$

Consider

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 3x^2 + 3xy + 3y(x-y)$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = -3y^2 - 3xy + 3x(x-y)$$

$$2\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + (x-y)3(x+y)$$

$$2\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 3x^2 - 3y^2$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 -----------(5)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 3x^2 + 3xy + 3y(x-y)$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = -3y^2 - 3xy + 3x(x-y)$$

$$2\frac{\partial v}{\partial x} = 3x^2 + 3y^2 + 6xy + (x-y).3(y-x)$$

$$=3x^2+3y^2+6xy-3(x-y)^2$$
$$=3x^2+3y^2+6xy-3x^2-3y^2+6xy$$
$$=12xy$$
$$\frac{\partial u}{\partial x}=6xy--------(6)$$

$$Consider\ f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$
$$=\left(3x^2-3y^2\right)+i6xy\left[by\ (5)\&(6)\right]$$
By Milne's Thomson method replace x by z and y by 0
$$f'(z)=3z^2$$
int $egrat$
$$f(z)=z^3+c$$

5) Find an analytic function f(z)-u+iv, given that u+v=$\frac{1}{r^2}\left[\cos 2\theta - \sin 2\theta\right], r \neq 0$

$$Solution: u + v = \frac{1}{r^2}\left[\cos 2\theta - \sin 2\theta\right] - - - - - (1)$$

Differentiate (1) *w.r.t. r*

$$\frac{\partial u}{\partial r} + \frac{\partial v}{\partial r} = -\frac{2}{r^3}\left[\cos 2\theta - \sin 2\theta\right] - - - - - (2)$$

Differentiate (1) *w.r.t.* θ

$$\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta} = \frac{2}{r^2}\left[-2\sin 2\theta - 2\cos 2\theta\right] - - - - - (3)$$

By C-R Equations

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial u}{\partial \theta} &= -r\frac{\partial v}{\partial r} \end{aligned}\right\} \text{in LHS of (3)}$$

$$-r\frac{\partial v}{\partial r} + r\frac{\partial u}{\partial r} = \frac{-2}{r^2}\left[\sin 2\theta + \cos 2\theta\right]$$

$$\frac{\partial u}{\partial r} - \frac{\partial v}{\partial r} = \frac{-2}{r^3}\left[\sin 2\theta + \cos 2\theta\right] - - - - - (4)$$

Consider

$$\frac{\partial u}{\partial r} + \frac{\partial v}{\partial r} = \frac{-2}{r^3}\left[\cos 2\theta - \sin 2\theta\right]$$

$$\frac{\partial u}{\partial r} - \frac{\partial v}{\partial r} = \frac{-2}{r^3}\left[\cos 2\theta + \sin 2\theta\right]$$

$$2\frac{\partial u}{\partial r} = \frac{-2}{r^3}\left[2\cos 2\theta\right]$$

$$\frac{\partial u}{\partial r} = \frac{-2}{r^3}\cos 2\theta - - - - - - - - - - - - - - (5)$$

Subtract (3)-(4) we get

$$2\frac{\partial u}{\partial r} = -\frac{2}{r^3}\left[-2\sin 2\theta\right]$$

$$\frac{\partial u}{\partial r} = \frac{2}{r^3}\sin 2\theta ----------(6)$$

$$Consider\ f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$$

$$f'(z) = e^{-i\theta}\left[-\frac{2}{r^3}\cos 2\theta + i\frac{2}{r^3}\sin 2\theta\right]$$

By Milne's Thomson method replace r by z and θ by 0

$$f'(z) = -\frac{2}{r^3}$$

int *egrate*

$$f(z) = -2\left(-\frac{1}{2z^2}\right) + \text{c}$$

$$f(z) = \frac{1}{z^2} + \text{c}$$

6) Show that $u=\left(r+\frac{1}{r}\right)\cos\theta$ is harmonic. find its harmonic conjugate and also corresponding analytic function.

Solution: Given $u=\left(r+\frac{1}{r}\right)\cos\theta --------(1)$

we shall show that u is a solution of Laplace's equation in two variables in polar form.

i.e $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}-0----------(2)$

Differentiate (1) w.r.t. r

$$\frac{\partial u}{\partial r}=\left(1-\frac{1}{r^2}\right)\cos\theta---------------(3)$$

Differentiate (3) w.r.t. r

$$\frac{\partial^2 u}{\partial r^2}=+\frac{2}{r^3}\cos\theta-----------------(4)$$

Differentiate (1) w.r.t. θ

$$\frac{\partial u}{\partial\theta}=\left(1+\frac{1}{r}\right)(-\sin\theta)---------------(5)$$

Differentiate (5) w.r.t. θ

$$\frac{\partial^2 u}{\partial\theta^2}=-\left(r+\frac{1}{r}\right)\cos\theta--------(6)$$

Consider

$$\begin{aligned}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}&=\frac{2}{r^3}\cos\theta+\frac{1}{r}\left(1-\frac{1}{r^2}\right)\cos\theta-\frac{1}{r^2}\left(r+\frac{1}{r}\right)\cos\theta\\&=\frac{2}{r^3}\cos\theta+\frac{1}{r}\cos\theta-\frac{1}{r^3}\cos\theta-\frac{1}{r}\cos\theta-\frac{1}{r^3}\cos\theta\\&=\frac{2}{r^3}\cos\theta-\frac{2}{r^3}\cos\theta\\&=0\end{aligned}$$

$\therefore u$ is solution of equation(2)

Hence u is harmonic function.

Consider

$$f'(z)=e^{-i\theta}\left[\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right]--------(7)$$

By C-R Equation $\frac{\partial u}{\partial \theta}$=-r $\frac{\partial v}{\partial r}$

$$\therefore\ replace\ \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}\ in\ (7)$$

$$f'(z)=e^{-i\theta}\left[\left(1-\frac{1}{r^2}\right)\cos\theta-\frac{i}{r}\left(r+\frac{1}{r}\right)\sin\theta\right]$$

By Milne's Thomson method replace r by z and θ by 0

$$f'(z)=\left(1-\frac{1}{z^2}\right)-i.o$$

$$f'(z)=\left(1-\frac{1}{z^2}\right)$$

Integrate

$$f(z)=z+\frac{1}{z}$$

To find harmonic Conjugate

consider $\quad u+iv=re^{i\theta}+\frac{1}{r}e^{-i\theta}$

$$u+iv=\left(r+\frac{1}{r}\right)\cos\theta+i\left(r-\frac{1}{r}\right)\sin\theta$$

Equating real and imaginary parts

$$\therefore u=\left(r+\frac{1}{r}\right)\cos\theta$$

$$v=\left(r-\frac{1}{r}\right)\sin\theta$$

which is required conjugate harmonic

7)Find the analytic function f(z) where imaginary part is $e^{x}(x\sin y + y\cos y)$

$So\ln : v = e^{x}\left(x\sin y + y\cos y\right)$

$\frac{\partial v}{\partial x} = e^{x}\left(x\sin y + y\cos y\right) + e^{x}\sin y$

$\frac{\partial v}{\partial y} = e^{x}\left(x\cos y + \cos y - y\sin y\right)$

we have $f^{1}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

By C-R equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$:f^{1}(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$

$f^{1}(z) = e^{x}\left[\left(x\cos y + \cos y - y\sin y\right) + i\left(x\sin y = y\cos y + \sin y\right)\right]$

$Put\ x = z,\ y = 0$

$f^{1}(z) = e^{z}\left[(z+1) + i(0)\right]$

$= (z+1)e^{z}$

Integrating w.r.t z

$f(z) = (z+1)e^{z} - (1)e^{z} + c$

$f(z) = ze^{z} + c$

8)Find the analytic function where real part is $e^x\left[\ x^2-y^2\ \ \cos y-2xy\sin y\right]$

Solution:

$$u=e^x\left[\ x^2-y^2\ \ \cos y-2xy\sin y\right]$$

$$\frac{\partial u}{\partial x}=e^x\left[\ x^2-y^2\ \ \cos y-2xy\sin y\right]+e^x\ \ 2x\cos y-2y\sin y$$

$$\frac{\partial u}{\partial y}\Big|(z.0)\ =e^z\left[z^2+2z\right]$$

$$\frac{\partial u}{\partial y}=e^x\left[-2y\cos y-(x^2-y^2)\sin y-2x\sin y-2xy\cos y\right]$$

$$\frac{\partial u}{\partial y}\Big|_{z,o}\ =e^z\ \ 0\ =0$$

$$\mathrm{f}^1(\mathrm{z})=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\Big|_{z,o}$$

$$=\frac{\partial u}{\partial x}+i\left(-\frac{\partial u}{\partial y}\right)\Big|_{z,o}\ \left[\text{By C-R equations}\right]$$

$$=\left(z^2+2z\right)e^z+0$$

$$\mathrm{f}^1(\mathrm{z})=\ z^2+2\mathrm{z}\ \ e^z$$

$$\mathrm{f}(\mathrm{z})=\int\ z^2+2\mathrm{z}\ \ e^z dz$$

$$=\ z^2+2\mathrm{z}\ \ e^z-\ \ 2\mathrm{z}+2\ \ e^z+(2)e^z+c$$

$$\mathrm{f}(\mathrm{z})=z^2e^z\ +c$$

9)If f(z)=u(x,y)+iv(x,y) is analytic, Show that $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 4|f^1(z)|^2$

Solution:

Let f(z) =u+iv is analytic:

$f^1(z)$ exist

$$f^1(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$|f^1(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2} = \left(\frac{\partial v}{\partial x}\right)^2$$

$$|f^1(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$$Also\ |f^1(z)|^2 = u^2 + v^2$$

Diff partially (2) w.r.t x

$$\frac{\partial}{\partial x}|f(z)| = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

Differentialing again w.r.t x we get

$$\frac{\partial^2}{\partial x^2}|f(z)|^2 = 2\left[u\ \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly we can obtain

$$\frac{\partial^2}{\partial y^2}|f(z)|^2 = 2\left[u\ \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

$(3) + (4) \Rightarrow$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2\left[u\ \left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right\} + v\left\{\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right\} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

Since u & v are harmonic & using C-R equation

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$, we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2\left[2\ \left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$= 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$= 4|f^1(z)|^2$$