||Jai Sri Gurudev||

S. J. C. INSTITUTE OF TECHNOLOGY, CHICKBALLAPUR

Department of Mathematics

LECTURE NOTES

Complex Analysis, Probability and Statistical Methods (18MAT41)

Prepared By:

Purushotham.P
Assistant Professor
SJC Institute of Technology
Email id: ppurushothamp48@gmail.com
Phone: 8197481658

COMPLEX ANALYSIS, PROBABILITY AND STATISTICAL METHOD

MODULE : 01

CALCULUS OF COMPLEX VARIABLES

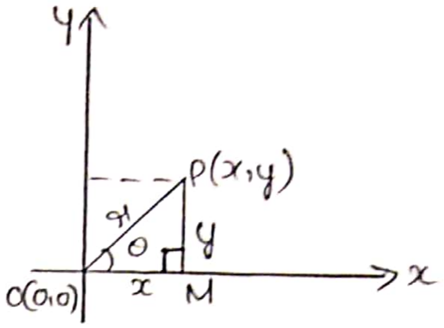
Definition:

let x, y be the two real values, then the number (or) variable $z = x+iy$ is called the complex variable (or) complex number. where the first part of z is called real part and the second part is called imaginary part. But both x and y are the real values.

$\boxed{z = x+iy}$ is a complex number in the Cartesian form and its conjugate complex is $\bar{z} = x-iy$ and $|z| = \sqrt{x^2+y^2}$

Geometrical representation:

Let $\vec{ox}$, $\vec{oy}$ be the real and imaginary axis, let 'p' be any point on the plane and 'M' be the foot of the perpendicular of 'p' on the real axis.



Let 'θ' be the angle b/n $\vec{op}$ and $\vec{ox}$. Then from the right angle Δ^{le} OPM.

$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$$

Substitute the above in a complex variable $z = x+iy$

we get, $z = r\cos\theta + ir\sin\theta$

$$z = r(\cos\theta + i\sin\theta)$$

$\boxed{z = r\,e^{i\theta}}$ is called the complex variable in polar form.

Complex valued function:

1. Suppose $u(x,y)$ and $v(x,y)$ be the two real function in the variables 'x' and 'y', then the complex valued function in the Cartesian form can be defined as

$$\omega = f(z) = u(x,y) + iv(x,y) = u+iv$$

2. Suppose $u(r,\theta)$ and $v(r,\theta)$ be the two real function in the variables of 'r' and 'θ', then the complex valued function in the polar form can be defined as

$$\omega = f(z) = u(r,\theta) + iv(r,\theta)$$

Some Important Results:

WKT,

$$e^{ix} = \cos x + i\sin x \rightarrow (1)$$

and $$e^{-ix} = \cos x - i\sin x \rightarrow (2)$$

from (1) + (2), $e^{ix} + e^{-ix} = 2\cos x$

$$\Rightarrow \cos x = \frac{e^{ix}+e^{-ix}}{2} \rightarrow (3)$$

$\Rightarrow$ (1) − (2), $e^{ix} - e^{ix} = 2i\sin x$

$$\Rightarrow \sin x = \frac{e^{ix}-e^{-ix}}{2i} \rightarrow (4)$$

$$\therefore (3) \Rightarrow \cos ix = \frac{e^{i^2x}+e^{-i^2x}}{2}$$

$$= \frac{e^{-x}+e^{x}}{2}$$

$$\Rightarrow \boxed{\cos ix = \cosh x}$$

$$(4) \Rightarrow \sin ix = \frac{e^{i^2x}-e^{-i^2x}}{2i}$$

$$= \frac{e^{-x}-e^{x}}{2i}$$

$$= \frac{-(e^{x}-e^{-x})}{2i}$$

$$= \frac{i^{\cancel{2}}(e^{x}-e^{-x})}{2\cancel{i}}$$

$$= \frac{i(e^{x}-e^{-x})}{2}$$

$$\Rightarrow \boxed{\sin ix = i\sinh x}$$

Some definitions:

* **Limit of a Complex variable:** Suppose 'z' be an Complex variable to the neighbourhood of 'z_0', then for any positive small quantity 'δ' (delta), $|z-z_0| \leq \delta$ is called the limit of a Complex variable.

* **Limit of a Complex valued function:** Suppose $\omega = f(z)$ be a complex valued function for any positive quantity 'ε', we have $|f(z) - f(z_0)| < \varepsilon$ (or) $|f(z) - l| < \varepsilon$ (or) $\lim_{z \to z_0} f(z) = l$ is called limit of a complex valued function.

* **Differentiability of a function:** Suppose $\omega = f(z)$ be a Complex valued function and is differentiable at z, then
$$f'(z) = \frac{d\omega}{dz} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}, \quad \delta z \neq 0,$$

* **Continuity:** A Complex valued function $f(z)$ is said to be a Continuous, if and only if $\lim_{z \to z_0} f(z) = f(z_0)$

* **Analytic of a function:**

let $\omega = f(z)$ be a Complex valued function and it is said to be an analytic function when $\omega = f(z)$ to be a differentiable at any point of z,
$$\text{i.e., } f'(z) = \frac{d\omega}{dz} = \lim_{\delta z \to 0} \left(\frac{f(z+\delta z) - f(z)}{\delta z} \right) \therefore \delta z \neq 0$$

It is also called regular (or) holomorphic function.

Theorem: Cauchy Rieman Equation in Cartesian form (or) C-R Equation.

Statement: The necessary Condition that the function $w = f(z) = u(x,y) + iv(x,y) = u+iv$ may be analytic at any point $z = x+iy$ is that, there exists 4 continuous first order partial derivatives,

i.e., $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ can satisfy the given equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Proof: Given, $w = f(z) = u(x,y) + iv(x,y) \rightarrow (1)$ is a complex valued function in the Cartesian form, for the complex variable $z = x+iy$.

$$\therefore \text{eq}^n (1) \Rightarrow f(x+iy) = u(x,y) + iv(x,y) \rightarrow (2)$$

and given $w = f(z)$ is an analytic function.

differentiating eq^n(2) w.r.t 'x' partially,

$$(2) \Rightarrow f'(x+iy) \cdot 1 = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\Rightarrow f'(x+iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \rightarrow (3)$$

diff. eq^n (2) w.r.t 'y' partially,

$$(2) \Rightarrow f'(x+iy)(i) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

$$\Rightarrow f'(x+iy) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow f'(x+iy) = \frac{i}{i^2}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow f'(x+iy) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow f'(x+iy) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \rightarrow (4)$$

$\therefore$ from (3) & (4)

$$\Rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Hence proved.

Cauchy Rieman Equation in polar form.

Statement: If $w=f(z) = u(r,\theta) + iv(r,\theta)$ is an analytic function at $z = re^{i\theta}$, then their exists continuous first order partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ can satisfy the equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof: Given $w=f(z) = u(r,\theta) + iv(r,\theta) \rightarrow (1)$ is an analytic function at $z = re^{i\theta}$.

$$\therefore \text{eq}^n (1) \Rightarrow w = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta) \rightarrow (2)$$

differentiating eq^n (2) wrt 'r' partially,

$$(2) \Rightarrow f'(re^{i\theta})e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

$$\Rightarrow e^{i\theta}\cdot f'(re^{i\theta}) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \rightarrow (3)$$

diff. eq^n (2) wrt 'θ' partially,

$$(2) \Rightarrow f'(re^{i\theta})\, ri\cdot e^{i\theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$\Rightarrow e^{i\theta}\cdot f'(re^{i\theta}) = \frac{1}{ri}\frac{\partial u}{\partial \theta} + \frac{\cancel{i}}{\cancel{i}r}\frac{\partial v}{\partial \theta}$$

$$\Rightarrow e^{i\theta}\cdot f'(re^{i\theta}) = \frac{i}{i^2 r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\Rightarrow e^{i\theta}\cdot f'(re^{i\theta}) = \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta} \rightarrow (4)$$

from eq^n (3) and (4)

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} - i\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Hence proved.

Derivation of analytic function:

Step 1: write the given Complex valued function $w = f(z)$

Step 2: write the $z = x+iy$ in the cartesian form and $z = re^{i\theta}$ in the polar form, we get

$f(z) = u(x,y) + iv(x,y)$ in Cartesian form and

$f(z) = u(r,\theta) + iv(r,\theta)$ in polar form.

Step 3: Identify u and v and verify the C-R equations in the respective forms.

Step 4: Find the derivative of $f(z)$ as

$f'(z) = u_x + iv_x$ in the Cartesian form and

$f'(z) = (u_r + iv_r)\cdot e^{-i\theta}$ in polar form.

Step 5: Substitute $x = z, y = 0$ in cartesian form and $r = z, \theta = 0$ in polar form in the required $f'(z)$.

Problems:

1. Verify the function $f(z) = z^2$ is analytic (or) not, hence find its derivative.

Soln Given, $f(z) = z^2 \rightarrow (1)$

let $z = x+iy$

$\therefore (1) \Rightarrow f(z) = (x+iy)^2$

$$f(z) = x^2 + i^2y^2 + 2ixy \qquad \because i^2 = -1$$

$$= x^2 - y^2 + 2ixy$$

$$f(z) = (x^2 - y^2) + i(2xy)$$

$$\Rightarrow f(z) = u(x,y) + iv(x,y)$$

$$u = x^2 - y^2, \quad v = 2xy$$

$$\therefore \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial x} = +2y, \quad \frac{\partial v}{\partial y} = 2x$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y$$

$\therefore$ The given $f(z)$ is satisfied C-R equation.

$\therefore f(z)$ is an analytic.

WKT, $f'(z) = u_x + iv_x$

$$\Rightarrow f'(z) = 2x + i2y$$

$$f'(z) = 2(x + iy) \rightarrow (2)$$

let $x = z, y = 0$

$$\therefore f'(z) = 2(z + 0)$$

$$\therefore \underline{\underline{f'(z) = 2z}}$$

2. Verify $f(z) = z^3$ is analytic (or) not, hence find its derivative.

Sol: Given, $f(z) = z^3 \rightarrow (1)$

let $z = re^{i\theta}$

$$\therefore (1) \Rightarrow f(z) = (re^{i\theta})^3$$

$$f(z) = r^3 \cdot e^{i3\theta}$$

$$\Rightarrow f(z) = r^3(\cos 3\theta + i\sin 3\theta)$$

$$f(z) = r^3\cos 3\theta + ir^3\sin 3\theta$$

$$\therefore f(z) = u(r,\theta) + iv(r,\theta)$$

$$u = r^3\cos 3\theta \quad , V = r^3 \sin 3\theta$$

$$\therefore \frac{\partial u}{\partial r} = 3r^2\cos 3\theta \quad , \frac{\partial u}{\partial \theta} = -3r^3 \sin 3\theta$$

$$\frac{\partial v}{\partial r} = 3r^2 \sin 3\theta \quad , \frac{\partial v}{\partial \theta} = 3r^3\cos 3\theta$$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad , \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

$\therefore f(z)$ is an analytic function.

$\therefore$ WKT, $f'(z) = e^{-i\theta}\left[u_r + iv_r\right]$

$$= e^{-i\theta}\left(3r^2\cos 3\theta + i\, 3r^2 \sin 3\theta\right)$$

let $r = z, \theta = 0$

$$\therefore f'(z) = 3z^2$$

3. Verify $f(z) = z^n$ is an analytic function (or) not, hence find its derivative for any positive integer 'n'.

Given, $f(z) = z^n$

let $z = re^{i\theta}$

$$f(z) = (re^{i\theta})^n = r^n . e^{in\theta}$$

$$\Rightarrow f(z) = r^n(\cos n\theta + i \sin n\theta)$$

$$f(z) = r^n\cos n\theta + ir^n . \sin n\theta$$

$$\therefore f(z) = u(r,\theta) + iv(r,\theta)$$

$$u = r^n\cos n\theta \quad , V = r^n . \sin n\theta$$

$$\therefore \frac{\partial u}{\partial r} = n . r^{n-1}\cos n\theta \quad , \frac{\partial u}{\partial \theta} = -n . r^n \sin n\theta$$

$$\frac{\partial v}{\partial r} = n . r^{n-1}\sin n\theta \quad , \frac{\partial v}{\partial \theta} = n . r^n . \cos n\theta$$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad , \frac{\partial v}{\partial r} = \frac{-1}{r} . \frac{\partial u}{\partial \theta}$$

$\therefore f(z)$ is an analytic function.

$\therefore$ WKT, $f'(z) = e^{-i\theta}\left(u_r + i v_r\right)$

$$f'(z) = e^{-i\theta}\left(n.r^{n-1}\cos n\theta + i n r^{n-1}.\sin n\theta\right)$$

let $r = z$, $\theta = 0$

$$f'(z) = n.z^{n-1}$$

4. verify $f(z) = \sin z$ is analytic (or) not, hence find its derivative.

Soln: Given, $f(z) = \sin z \rightarrow (1)$

let $z = x+iy$

$(1) \Rightarrow f(z) = \sin(x+iy) = \sin x.\cos(iy) + \cos x.\sin(iy)$

$\Rightarrow f(z) = \sin x.\cosh y + \cos x \sinh y$

$\Rightarrow f(z) = u(x,y) + i v(x,y)$

$u = \sin x.\cosh y$, $v = \cos x.\sinh y$

$\frac{\partial u}{\partial x} = \cos x.\cosh y$, $\frac{\partial u}{\partial y} = \sin x.\sinh y$

$\frac{\partial v}{\partial x} = -\sin x.\sinh y$, $\frac{\partial v}{\partial y} = \cos x.\cosh y$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\therefore f(z)$ is analytic

$\therefore$ WKT, $f'(z) = u_x + i v_x$

$\Rightarrow f'(z) = \cos x.\cosh y - i \sin x.\sinh y$

$\Rightarrow f'(z) = \cos x.\cos(iy) - i \sin x.\sin(iy)$

$\Rightarrow f'(z) = \cos(x+iy)$

$$f'(z) = \cos z$$

5. Verify $f(z) = \sin 2z$ is analytic (or) not, hence find its derivative.

Soln Given, $f(z) = \sin 2z \rightarrow (1)$

let $z = x+iy$

$(1) \Rightarrow f(z) = \sin 2(x+iy)$

$= \sin(2x+2iy)$

$f(z) = \sin(2x) \cdot \cos(2iy) + \cos(2x) \cdot \sin(2iy)$

$f(z) = \sin(2x) \cdot \cosh(2y) + \cos(2x) \cdot \sinh(2y)$

$\Rightarrow f(z) = u(x,y) + iv(x,y)$

$u = \sin(2x) \cdot \cosh(2y)$, $v = \cos 2x \cdot \sinh(2y)$

$\therefore \frac{\partial u}{\partial x} = 2\cos 2x \cdot \cosh(2y)$, $\frac{\partial u}{\partial y} = 2\sin(2x) \cdot \sinh(2y)$

$\frac{\partial v}{\partial x} = -2\sin(2x) \cdot \sinh(2y)$

$\frac{\partial v}{\partial y} = 2\cos(2x) \cdot \cosh(2y)$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\therefore f(z)$ is an analytic function.

$\Rightarrow f'(z) = u_x + i v_x$

$f'(z) = 2\cos 2x \cdot \cosh(2y) - i\, 2\sin(2x) \cdot \sinh(2y)$

$\Rightarrow f'(z) = 2\cos 2x \cdot \cos 2(iy) - i\, 2\sin(2x) \cdot \sin 2(iy)$

$\Rightarrow f'(z) = 2\left[\cos 2x \cdot \cos(2iy) - \sin(2x) \cdot \sin(2iy)\right]$

$\Rightarrow f'(z) = 2\cos(2x+2iy)$

$f'(z) = 2\cos 2(x+iy)$

$\Rightarrow f'(z) = 2\cos 2z$

6. verify the analytic function $f(z) = \log z$, hence find its derivative.

Soln Given, $f(z) = \log z \rightarrow (1)$

let $z = re^{i\theta}$

$(1) \Rightarrow f(z) = \log(re^{i\theta})$

$\Rightarrow f(z) = \log r + \log e^{i\theta}$

$\Rightarrow f(z) = \log r + i\theta \cdot \log_e e \qquad \because \log_e e = 1$

$f(z) = \log r + i\theta$

$\Rightarrow f(z) = u(r,\theta) + iv(r,\theta)$

$u = \log r, \; v = \theta$

$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{\partial u}{\partial \theta} = 0$

$\frac{\partial v}{\partial r} = 0, \quad \frac{\partial v}{\partial \theta} = 1$

$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$\therefore f(z)$ is analytic function.

$\Rightarrow f'(z) = (u_r + iv_r)e^{-i\theta}$

$f'(z) = e^{-i\theta}\left(\frac{1}{r} + i(0)\right)$

$\Rightarrow f'(z) = e^{-i\theta}\left(\frac{1}{r}\right)$

$\Rightarrow f'(z) = \frac{1}{re^{i\theta}}$

$\therefore f'(z) = \frac{1}{z}$

Harmonic property:

(1.) If $f(z) = u(x,y) + iv(x,y)$ is an analytic, then its real and imaginary parts are both harmonics.

i.e., $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

(2) If $f(z) = u(r,\theta) + iv(r,\theta)$ is an analytic, then its real and imaginary parts are both harmonics,

i.e, $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0$

and $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0$

Theorem :

If $f(z)$ is regular function of z, show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(f|z|^2) = 4\left(f'|z|^2\right).$$

Given, $f(z) = u(x,y) + iv(x,y) = u+iv$ is regular

$\Rightarrow f(z)$ is differentiable at any point of $z = x+iy$

$\Rightarrow f'(z) = u_x + iv_x$

and WKT, $f|z| = \sqrt{u^2+v^2}$

$f(|z|^2) = u^2+v^2 = \phi \rightarrow (1)$

$\therefore \quad f'|z| = \sqrt{u_x^2+v_x^2}$

$f'(|z|^2) = u_x^2 + v_x^2 \longrightarrow (2)$

$$LHS = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \cdot f(|z|^2)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \cdot \phi$$

$$LHS = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

$$\therefore \frac{\partial \phi}{\partial x} = 2u \cdot u_x + 2v \cdot v_x$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} = 2\left(u_x \cdot u_x + u \cdot u_{xx}\right) + 2\left(v_x \cdot v_x + v \cdot v_{xx}\right)$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} = 2\left(u_x^2 + u.u_{xx} + v_x^2 + v.v_{xx}\right) \rightarrow (3)$$

$$\therefore \frac{\partial \phi}{\partial y} = 2\,u.u_y + 2\,v.v_y$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial y^2} = 2.\left(u_y.u_y + u.u_{yy}\right) + 2\left(v_y.v_y + v.v_{yy}\right)$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial y^2} = 2\left(u_y^2 + u.u_{yy} + v_y^2 + v.v_{yy}\right) \rightarrow (4)$$

from (3) & (4) (or) LHS (consider)

$$LHS = 2\left(u_x^2 + u.u_{xx} + v_x^2 + v.v_{xx} + u_y^2 + u.u_{yy} + v_y^2 + v.v_{yy}\right)$$

$$= 2\left[u_x^2 + v_x^2 + u_y^2 + v_y^2 + u\left(u_{xx} + u_{yy}\right) + v\left(v_{xx} + v_{yy}\right)\right]$$

$$= 2\left[u_x^2 + v_x^2 + u_y^2 + v_y^2 + u(0) + v(0)\right]$$

$$= 2\left[u_x^2 + v_x^2 + (-v_x)^2 + (u_x)^2\right]$$

$$= 2\left(u_x^2 + v_x^2 + v_x^2 + u_x^2\right)$$

$$= 2\left[2\left(u_x^2 + v_x^2\right)\right]$$

$$= 4\left(u_x^2 + v_x^2\right)$$

$$= 4\left(f'|z|^2\right)$$

$$= RHS$$

Theorem: If $w = f(z)$ is regular, then show that

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2.$$

Proof: Given $w = f(z) = u + iv$ is a regular or analytic.

$\therefore f(z)$ is differentiable at any point of $z = x+iy$.

$\Rightarrow f'(z) = u_x + iv_x$

and WKT $|f(z)| = \sqrt{u^2+v^2} = \phi$

$\Rightarrow u^2+v^2 = \phi^2 \rightarrow (1)$

and $|f'(z)| = \sqrt{u_x^2+v_x^2}$

$\Rightarrow |f'(z)|^2 = u_x^2+v_x^2 \rightarrow (2)$

diff. (1) wrt 'x' partially,

$\therefore (1) \Rightarrow 2u.u_x + 2v.v_x = 2\phi\phi_x$

$\Rightarrow uu_x + v.v_x = \phi\phi_x \rightarrow (3)$

|||ly diff. (1) wrt 'y' partially,

$(1) \Rightarrow 2uu_y + 2vv_y = 2\phi\phi_y$

$\Rightarrow uu_y + vv_y = \phi\phi_y \rightarrow (4)$

$\therefore$ funⁿ $f(z)$ is analytic.

$\therefore$ WKT $u_x = v_y$, $v_x = -u_y$

$u_y = -v_x$.

$\therefore (4) \Rightarrow -u.v_x + v.u_x = \phi\phi_y$

$vu_x - u.v_x = \phi\phi_y \rightarrow (5)$

$\therefore (3)^2+(5)^2$

$$(uu_x+v.v_x)^2 + (vu_x - uv_x)^2 = \phi^2\phi_x^2 + \phi^2\phi_y^2$$

$$\Rightarrow u^2u_x^2 + v^2v_x^2 + \cancel{2uv.u_xv_x} + v^2u_x^2 + u^2v_x^2 - \cancel{2uv.u_xv_x} = \phi^2\phi_x^2 + \phi^2\phi_y^2$$

$$\Rightarrow u^2(u_x^2+v_x^2) + v^2(u_x^2+v_x^2) = \phi^2(\phi_x^2+\phi_y^2)$$

$$\Rightarrow (u^2+v^2)(u_x^2+v_x^2) = \phi^2(\phi_x^2+\phi_y^2)$$

$$\Rightarrow \cancel{\phi^2}(u_x^2+v_x^2) = \cancel{\phi^2}(\phi_x^2+\phi_y^2)$$

$$\Rightarrow \phi_x^2+\phi_y^2 = u_x^2+v_x^2$$

$$\Rightarrow \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 = |f'(z)|^2$$

$$\Rightarrow \left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$$

Theorem: If $f(z)$ is regular function with Constant Modulus, Show that $f(z)$ is also a Constant function.

Given, $w = f(z) = u + iv$ is regular function.

$\therefore f(z)$ is differentiable.

$\therefore f(z) = u_x + iv_x$

and $|f'(z)| = \sqrt{u_x^2 + v_x^2}$

$\Rightarrow |f'(z)|^2 = u_x^2 + v_x^2 \rightarrow (1)$

and given $|f(z)| = K$

$\Rightarrow \sqrt{u^2 + v^2} = K$

$\Rightarrow u^2 + v^2 = K^2 \rightarrow (2)$

diff. eqn (2) partially wrt 'x'.

$(2) \Rightarrow 2uu_x + 2vv_x = 0$

$\Rightarrow uu_x + vv_x = 0 \rightarrow (3)$

|||ly (2) $\Rightarrow$ wrt 'y' partially

$(2) \Rightarrow 2uu_y + 2vv_y = 0$

$\Rightarrow uu_y + vv_y = 0 \rightarrow (4)$

$\because$ But $f(z)$ is analytic

$\therefore u_x = v_y$ and $v_x = -u_y$

$\Rightarrow u_y = -v_x$

$\because (4) \Rightarrow u(-v_x) + vu_x = 0$

$\Rightarrow vu_x - uv_x = 0 \rightarrow (5)$

$(3)^2+(5)^2 \quad (uu_x+vv_x)^2+(vu_x-uv_x)^2=0$

$\Rightarrow u^2u_x^2+v^2v_x^2+2vu\cancel{v_xu_x}+v^2u_x^2+u^2v_x^2-2uv\cancel{u_xv_x}=0$

$\Rightarrow u^2(u_x^2+v_x^2)+v^2(u_x^2+v_x^2)=0$

$\Rightarrow (u^2+v^2)(u_x^2+v_x^2)=0$

$\Rightarrow K^2(u_x^2+v_x^2)=0$

$\Rightarrow K^2\neq 0,\ u_x^2+v_x^2=0$

$\Rightarrow |f'(z)|^2=0$

$\Rightarrow |f'(z)|=0$

$\therefore f(z)=c$

$\Rightarrow f(z)$ is Constant.

Problems:

1. Show that $f(z)=\cosh z$ is analytic and hence find its derivative.

Soln: $f(z)=\cosh z$

let $z=x+iy$

$\Rightarrow f(z)=\cosh(x+iy) \longrightarrow (1)$

WKT, $\cosh\theta=\cos i\theta$

$(1)\Rightarrow f(z)=\cosh i(x+iy)$

$=\cosh(ix+i^2y)$

$=\cosh(ix-y)$

$=\cos(ix).\cos y+\sin(ix).\sin y$

$=\cosh x.\cos y+i\sinh x.\sin y.$

$\Rightarrow f(z)=u(x,y)+iv(x,y)$

$\therefore u=\cosh x.\cos y \quad , v=\sinh x.\sin y$

$u_x=\sinh x.\cos y \quad , u_y=-\cosh x.\sin y$

$v_x=\cosh x.\sin y \quad , v_y=\sinh x.\cos y.$

$\therefore U_x = V_y, \quad V_x = -V_y$

$\therefore f(z)$ is analytic

$\therefore f'(z) = U_x + iV_x$

$\Rightarrow f'(z) = \sinh x \cdot \cos y + i\cosh x \cdot \sin y$

when $x = z, y = 0$

$\Rightarrow f'(z) = \sinh z$

2. Show that $f(z) = z + e^z$ is analytic, Hence find its derivative.

Soln: $f(z) = z + e^z$

let $z = x + iy$

$\Rightarrow f(z) = (x+iy) + e^{(x+iy)}$

$= (x+iy) + e^x \cdot e^{iy}$

$= (x+iy) + e^x \cdot (\cos y + i\sin y)$

$= x + e^x \cdot \cos y + iy + i\sin y \cdot e^x$

$= (x + e^x \cdot \cos y) + i(y + \sin y\, e^x)$

$\Rightarrow f(z) = u(x,y) + i\, v(x,y)$

$\Rightarrow u = x + e^x \cdot \cos y, \quad v = y + e^x \cdot \sin y$

$\therefore U_x = 1 + e^x \cdot \cos y, \quad U_y = -\sin y \cdot e^x$

$V_x = e^x \cdot \sin y, \quad V_y = 1 + e^x \cdot \cos y$

$\therefore U_x = V_y$ and $V_x = -U_y$.

$\therefore f(z)$ is analytic.

$\therefore f'(z) = U_x + iV_x$

$\Rightarrow f'(z) = 1 + e^x \cos y + i\, e^x \cdot \sin y$

$= 1 + e^x(\cos y + i\sin y)$

put $x = z, y = 0$ $\quad \Rightarrow f'(z) = 1 + e^z$

3. Find the analytic function, whose real part is $u = \dfrac{x^4 . y^4 - 2x}{x^2+y^2}$

Given, $u = \dfrac{x^4 . y^4 - 2x}{x^2+y^2} \rightarrow (1)$

diff. (1) partially wrt 'x'.

$\therefore (1) \Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{(x^2+y^2)(4x^3y^4 - 2) - (x^4 . y^4 - 2x)(2x)}{(x^2+y^2)^2}$

when $x = z, y = 0$

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{(z^2+0)(4z^3(0) - 2) - z^4.(0) - 2(z)(2z)}{(z^2)^2}$

$\Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{z^2(-2) + 4z^2}{z^4} = \dfrac{-2z^2 + 4z^2}{z^4}$

$\Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{2z^2}{z^4}$

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{2}{z^2}$

|||ly $\dfrac{\partial u}{\partial y} = \dfrac{(x^2+y^2)(4x^4y^3) - (x^4 . y^4 - 2x)(2y)}{(x^2+y^2)^2} = -\dfrac{\partial v}{\partial x}$

$\therefore \dfrac{\partial v}{\partial x} = \dfrac{2y\,(x^4 . y^4 - 2x) - (x^2+y^2)(4x^4y^3)}{(x^2+y^2)^2}$

when $x = z, y = 0$

$= \dfrac{2(0).(z^4(0) - 2(z)) - z^2(4z^4(0))}{z^4}$

$\therefore \dfrac{\partial v}{\partial x} = 0$

$\therefore$ WKT $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

$\Rightarrow f'(z) = \dfrac{2}{z^2} + i(0)$

$\Rightarrow f'(z) = \dfrac{2}{z^2}$

$$\Rightarrow \frac{df}{dz} = -\frac{2}{z^2}$$

$$\Rightarrow \int df = \int \frac{2}{z^2}\, dz$$

$$\Rightarrow f(z) = 2 \cdot -\frac{1}{z} + c$$

$$\Rightarrow \boxed{f(z) = -\frac{2}{z} + c}$$

4. Find the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$

Soln Given, $u = \frac{\sin 2x}{\cosh 2y - \cos 2x} \rightarrow (1)$

diff wrt 'x' (1) partially

$$\therefore \frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(0 + 2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{2\cos 2x\,(\cosh 2y - \cos 2x) - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

when $x = z, y = 0$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{2\cos 2z\,(1 - \cos 2z) - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1 - \cos 2z)^2} = \frac{-2\cancel{(1 - \cos 2z)}}{(1 - \cos 2z)^{\cancel{2}}}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{-2}{1 - \cos 2z}$$

Similarly $\frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x(2\sin h 2y - 0)}{(\cosh 2y - \cos 2x)^2}$

$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore \frac{\partial v}{\partial x} = \frac{\sin 2x(2\sin h 2y) - 0}{(\cosh 2y - \cos 2x)^2}$

when $z = x, y = 0$

$\frac{\partial v}{\partial x} = 0$

$\therefore$ WKT $f'(z) = u_x + iv_x$

$\Rightarrow f'(z) = \frac{-2}{1-\cos 2z} + i(0)$

$\Rightarrow \frac{df}{dz} = \frac{-2}{1-\cos 2z}$

$\Rightarrow df = \frac{-2}{1-\cos 2z} dz$

$\Rightarrow df = \frac{-\cancel{2}}{\cancel{2}\sin^2 z} dz = \frac{-1}{\sin^2 z} dz$

$\Rightarrow \int df = \int \frac{-1}{\sin^2 z} dz$

$\Rightarrow f(z) = \int -\operatorname{cosec}^2 z \cdot dz$

$\Rightarrow \boxed{f(z) = \cot z + c}$

5. Construct the Analytic function, whose real part is $\frac{\sin 2x}{\cosh 2y + \cos 2x}$.

Sol: Given, $u = \frac{\sin 2x}{\cosh 2y + \cos 2x} \rightarrow (1)$

$\frac{\partial u}{\partial x} = \frac{(\cosh 2y + \cos 2x)(2\cos 2x) - \sin 2x(0 + (-2\sin 2x))}{(\cosh 2y + \cos 2x)^2}$

$$\therefore \frac{\partial u}{\partial x} = \frac{(\cosh 2y + \cos 2x)\, 2\cos 2x + 2\sin^2 2x}{(\cosh 2y + \cos 2x)^2}$$

when $z = x, y = 0$

$$\frac{\partial u}{\partial x} = \frac{2\cos 2z\,(1 + \cos 2z) + 2\sin^2 2z}{(\cosh 2(0) + \cos 2z)^2}$$

$$= \frac{2\cos 2z\,(1 + \cos 2z) + 2\sin^2 2z}{(1 + \cos 2z)^2}$$

$$= \frac{2\cos 2z + 2\cos^2 2z + 2\sin^2 2z}{(1 + \cos 2z)^2}$$

$$= \frac{2\cos 2z + 2(1)}{(1 + \cos 2z)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2\,(1 + \cancel{\cos 2z})}{(1 + \cos 2z)^{\cancel{2}}} = \frac{2}{(1 + \cos 2z)}$$

lll^{ly} $\frac{\partial u}{\partial y} = \frac{(\cosh 2y + \cos 2x)(0) - \sin 2x\,(2\sinh 2y + 0)}{(\cosh 2y + \cos 2x)^2}$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial v}{\partial x} = \frac{\sin 2x\,(2\sinh 2y)}{(\cosh 2y + \cos 2x)^2}$$

when $x = z, y = 0$

$$\frac{\partial v}{\partial x} = \frac{\sin 2z\,(2\sinh 2(0))}{(\cosh(0) + \cos 2z)^2}$$

$$\therefore \frac{\partial v}{\partial x} = 0$$

$\therefore$ WKT, $f'(z) = u_x + i v_x$

$$\Rightarrow f'(z) = \frac{2}{1 + \cos 2z}$$

$$\Rightarrow \frac{df}{dz} = \frac{2}{1 + \cos 2z}, \quad \int df = \int \frac{\cancel{2}}{\cancel{2}\cos^2 z}\, dz$$

$$f(z) = \int \frac{1}{\cos^2 z}\, dz = \int \sec^2 z \, dz$$

$$\boxed{f(z) = \tan z + C}$$

6. Construct the Analytic function, whose real part is

$$x^2-y^2+\frac{x}{x^2+y^2}$$

Given, $u = x^2-y^2+\frac{x}{x^2+y^2}$

$$u = \frac{(x^2+y^2)(x^2-y^2)+x}{(x^2+y^2)} = \frac{(x^4-y^4+x)}{(x^2+y^2)} \rightarrow (1)$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(4x^3+1)-(x^4-y^4+x)(2x)}{(x^2+y^2)^2}$$

when $x=z, y=0$

$$\frac{\partial u}{\partial x} = \frac{z^2(4z^3+1)-(z^4+z)(2z)}{(z^2+0)^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{4z^5+z^2-2z^5-2z^2}{z^4}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{2z^5-z^2}{z^4}$$

$$= \frac{\cancel{z^2}(2z^3-1)}{z^4}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{2z^3-1}{z^2} = 2z-\frac{1}{z^2}$$

III^{ly} $$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)(-4y^3)-(x^4-y^4+x)(2y)}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}$$

when $z=x, y=0$

$$\boxed{v_x=0}$$

$$\therefore f'(z) = u_x+iv_x$$

$$\Rightarrow f'(z) = 2z-\frac{1}{z^2}+0$$

$$\Rightarrow \frac{df}{dz} = 2z-\frac{1}{z^2}$$

$$\int df = \int\left(2z-\frac{1}{z^2}\right)dz$$

$$\Rightarrow f(z) = \frac{2z^2}{2}-\left(\frac{-1}{z}\right)+c$$

$$\boxed{\therefore f(z) = z^2+\frac{1}{z}+c}$$

7. Find the analytic function $f(z) = u + iv$, whose imaginary part is $\dfrac{y}{x^2+y^2}$.

Given, $V = \dfrac{y}{x^2+y^2}$

$$\Rightarrow V_x = \frac{(x^2+y^2)(0) - y(2x)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

when $x = z$, $y = 0$

$$\therefore \boxed{V_x = 0}$$

$$\therefore V_y = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2}$$

$$\Rightarrow V_y = \frac{x^2-y^2}{(x^2+y^2)^2}$$ when $x = z$, $y = 0$

$$\therefore V_y = \frac{z^2}{z^4} = \frac{1}{z^2}$$

$\therefore$ WKT, $f'(z) = u_x + iV_x$

$$\Rightarrow f'(z) = V_y + iV_x$$

$$\Rightarrow f'(z) = \frac{1}{z^2}$$

$$\Rightarrow \frac{df}{dz} = \frac{1}{z^2}$$

$$\int df = \int \frac{1}{z^2}\, dz$$

$$\Rightarrow \boxed{f(z) = -\frac{1}{z} + c}$$

8. Construct analytic function, whose real part is $u = e^{2x}(x\cos 2y - y\sin 2y)$.

Sol Given, $u = e^{2x}(x\cos 2y - y\sin 2y) \rightarrow (1)$

$\therefore u_x = 2e^{2x}(x.\cos 2y - y\sin 2y) + e^{2x}(\cos 2y - 0)$

$u_x = 2e^{2x}(x.\cos 2y - y\sin 2y) + e^{2x}(\cos 2y)$

when $x = z, y = 0$.

$\Rightarrow u_x = 2e^{2z}(z-0) + e^{2z}(1)$

$u_x = 2e^{2z}(z) + e^{2z}$

$\therefore u_y = e^{2x}(-2x.\sin 2y - 1.\sin 2y - 2y\cos 2y)$

when $x = z, y = 0$

$u_y = e^{2z}(0)$

$\therefore u_y = 0$

$\therefore$ WKT, $f'(z) = u_x + iv_x$

$\Rightarrow f'(z) = u_x - iu_y$

$\Rightarrow f'(z) = 2ze^{2z} + e^{2z} - 0$

$\Rightarrow f'(z) = (2z+1)e^{2z}$

$\Rightarrow \dfrac{df}{dz} = (2z+1)e^{2z}$

$\Rightarrow \int df = \int (2z+1)e^{2z}.dz$

$\therefore f(z) = (2z+1)\int e^{2z}dz - \int(2.\int e^{2z}dz).dz$

$= (2z+1)\frac{1}{2}.e^{2z} - \frac{1}{2}e^{2z} + c$

$= ze^{2z} + \cancel{\frac{1}{2}e^{2z}} - \cancel{\frac{1}{2}e^{2z}} + c$

$\therefore f(z) = ze^{2z} + c$

9. Find the analytic function, whose imaginary part is

$V = e^{x}(x.\sin y + y\cos y)$

Given, $v = e^{x}(x\sin y + y\cos y)$

$\therefore v_x = e^{x}(x.\sin y + y\cos y) + e^{x}(\sin y)$

$\Rightarrow v_x = e^{x}(x.\sin y + y\cos y) + e^{x}(\sin y)$

when $x = z, y = 0$

$$\Rightarrow v_x = e^z (z(0)+0) + e^z(0)$$

$$\underline{v_x = 0}$$

$$\therefore v_y = e^x [x\cos y + \cos y + (-y\sin y)]$$

put $z = x, y = 0$

$$v_y = e^z (z+1+0)$$

$$= e^z (z+1)$$

$$\therefore f'(z) = u_x + i v_x$$

$$= e^z (z+1)$$

$$\therefore \frac{df}{dz} = e^z (z+1)$$

$$df = e^z (z+1) \cdot dz$$

$$\int df = \int e^z (z+1) \cdot dz$$

$$f(z) = (z+1) e^z - (1) e^z + c$$

$$= e^z ((z+1) - 1) + c$$

$$= z e^z + \cancel{e^z} - \cancel{e^z} + c$$

$$\therefore f(z) = \underline{z e^z} + c$$

10. Find the analytic function, $u+iv$ whose imaginary part is
$$V = \left(r - \frac{1}{r}\right)\sin\theta, \quad r \neq 0.$$

Given, $V = \left(r - \frac{1}{r}\right)\sin\theta, \quad r \neq 0$

$\left| \frac{d}{dr}\left(\frac{1}{r}\right) = \frac{-1}{r^2} \right.$

$\therefore V_r = \left(1 + \frac{1}{r^2}\right)\sin\theta \rightarrow (1)$

$\therefore V_\theta = \left(r - \frac{1}{r}\right)\cos\theta \rightarrow (2)$

$\therefore$ WKT, $f'(z) = e^{-i\theta}\left(u_r + iV_r\right)$

$\Rightarrow f'(z) = e^{-i\theta}\left(\frac{1}{r}V_\theta + iV_r\right)$

$\Rightarrow f'(z) = e^{-i\theta}\left(\frac{1}{r}\left(r - \frac{1}{r}\right)\cos\theta + i\left(1 + \frac{1}{r^2}\right)\sin\theta\right)$

when $z = r, \theta = 0$

$\Rightarrow f'(z) = \left[\frac{1}{z}\left(z - \frac{1}{z}\right)\right]$

$\Rightarrow f'(z) = 1 - \frac{1}{z^2}$

$\left| \because \int \frac{1}{z^2} = \frac{-1}{z} \right.$

$\Rightarrow \frac{df}{dz} = 1 - \frac{1}{z^2}$

$\Rightarrow \int df = \int\left(1 - \frac{1}{z^2}\right)dz$

$\Rightarrow \boxed{f(z) = z + \frac{1}{z} + c}$

11. Find the analytic function, $f(z) = u+iv$,
if $(u-v) = e^x[\cos y - \sin y]$

Given, $f(z) = u+iv \rightarrow (1)$

and given, $u - v = e^x(\cos y - \sin y) \rightarrow (2)$

different$^{\text{wrt}}$ 'x' eqn (2) partially

$(2) \Rightarrow u_x - V_x = e^x(\cos y - \sin y) \rightarrow (3)$

diff wrt 'y' partially,

$$(2) \Rightarrow u_y - v_y = -e^x(\cos y + \sin y) \rightarrow (4)$$

WKT, $u_y = -v_x$, $v_y = u_x$

$$\therefore (4) \Rightarrow -v_x - u_x = -e^x(\cos y + \sin y) \rightarrow (5)$$

(3)+(5),

$$u_x - v_x - v_x - u_x = e^x(\cos y - \sin y) - e^x(\cos y + \sin y)$$

$$-2v_x = e^x(\cos y - \cancel{\sin y} - \cos y + \cancel{\sin y})$$

$$+\cancel{2}v_x = +\cancel{2}e^x.\sin y$$

$$v_x = e^x.\sin y$$

when $x = z, y = 0$

$$\therefore v_x = 0.$$

(3)−(5), $2u_x = e^x(\cos y - \sin y) + e^x(\cos y + \sin y)$

$$2u_x = e^x(\cos y - \cancel{\sin y} + \cos y + \cancel{\sin y})$$

$$\cancel{2}u_x = \cancel{2}e^x.\cos y$$

$$u_x = e^x.\cos y$$

when $x = z, y = 0$

$$u_x = e^z$$

$\therefore$ WKT, $f'(z) = u_x + iv_x$

$$\Rightarrow f'(z) = e^z + 0 = e^z$$

$$\Rightarrow \frac{df}{dz} = e^z$$

$$\int df = \int e^z.dz$$

$$\Rightarrow f(z) = e^z + C$$

12. Find the analytic function $f(z) = u + iv$, if $(u - v) = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$ and $f\left(\frac{\pi}{2}\right) = 0$.

Given, $f(z) = u+iv \rightarrow (1)$

Given, $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)} \rightarrow (2)$

diff. (2) partially wrt 'x'.

$$\therefore u_x - v_x = \frac{1}{2}\left[\frac{(\cos x - \cosh y)(-\sin x + \cos x) + (\cos x + \sin x - e^{-y})\sin x}{(\cos x - \cosh y)^2}\right]$$

when $x=z, y=0$

$$\therefore u_x - v_x = \frac{1}{2}\left\{\frac{(\cos z - 1)(-\sin z + \cos z) + (\cos z + \sin z - 1)\sin z}{(\cos z - 1)^2}\right\}$$

$$\Rightarrow u_x - v_x = \frac{1}{2}\left\{\frac{\cos^2 z - \cancel{\cos z \sin z} - \cos z + \sin z + \cancel{\cos z \sin z} + \sin^2 z - \sin z}{(\cos z - 1)^2}\right\}$$

$$= \frac{-1}{2}\left|\frac{1 - \cancel{\cos z}}{(1-\cos z)^2}\right|$$

$$\therefore u_x - v_x = \frac{-1}{2}\frac{1}{(1-\cos z)} \rightarrow (3)$$

III^{ly} $$u_y - v_y = \frac{1}{2}\left\{\frac{(\cos x - \cosh y)e^{-y} + (\cos x + \sin x - e^{-y})\sinh y}{(\cos x - \cosh y)^2}\right\}$$

when $x=z, y=0$

$$= \frac{1}{2}\left(\frac{\cos z - 1}{(\cos z - 1)^2}\right)$$

$$\Rightarrow -v_x - u_x = \frac{1}{2}\frac{1}{\cos z - 1} = \frac{-1}{2}\frac{1}{(1-\cos z)} \rightarrow (4)$$

$(3)+(4) \Rightarrow -2v_x = 0$

$\Rightarrow v_x = 0$

$(3)-(4) \Rightarrow 2u_x = \dfrac{1}{1-\cos z}$

$u_x = 1/2(1-\cos z)$

$$\Rightarrow f'(z) = \frac{-1}{2}\left(\frac{1}{1-\cos z}\right)$$

$$= \frac{-1}{2}\left(\frac{1}{2\sin^2(z/2)}\right)$$

$$\Rightarrow \frac{df}{dz} = \frac{-1}{2}\left(\frac{1}{2}\frac{1}{\sin^2(z/2)}\right), \int df = \int \frac{-1}{2}\left(\frac{1}{2}\frac{1}{\sin^2(z/2)}\right)dz$$

$$\Rightarrow f(z) = \frac{1}{2}\int\left(-\operatorname{cosec}^2\left(\tfrac{z}{2}\right)\right)dz$$

$$\Rightarrow f(z) = +\frac{1}{2}\cot(z/2) + c \rightarrow (5)$$

By condition, $f(\pi/2) = +\frac{1}{2}\cot(\pi/\cancel{2}) + c$

$$0 = +\frac{1}{2}(1) + c$$

$\boxed{c = -1/2}$ $\therefore (5) = f(z) = +\frac{1}{2}\cot(\cancel{z/2}) - \frac{1}{2}$

13. Find the analytic function of $f(z) = u + iv$, if $u - v = (x-y)(x^2+4xy+y^2)$.

Given, $f(z) = u + iv$ and

Given, $u - v = (x-y)(x^2+4xy+y^2)$

$$\Rightarrow u - v = x^3 + 4x^2y + xy^2 - x^2y - 4xy^2 - y^3$$

$$\Rightarrow u - v = x^3 + 3x^2y - 3xy^2 - y^3 \rightarrow (1)$$

diff (1) w.r.t 'x' partially

$$\therefore (1) \Rightarrow u_x - v_x = 3x^2 + 6y\cancel{x} - 3y^2 \rightarrow (2)$$

when $x = z, y = 0$

$$\therefore (2) \Rightarrow u_x - v_x = 3z^2 \rightarrow (3)$$

|||^{ly}

$$u_y - v_y = 3x^2 - 6xy - 3y^2 \rightarrow (4)$$

when $x = z, y = 0$

$\therefore (3)\quad U_y - V_y = 3z^2 \rightarrow (5)$

WKT $U_y = -V_x,\ V_y = U_x$

$\therefore (4) \Rightarrow -V_x - U_x = 3z^2 \rightarrow (6)$

$\therefore (3)+(6) \Rightarrow -2V_x = 6z^2$

$\Rightarrow -V_x = 3z^2$

$\Rightarrow V_x = -3z^2$

$(3)-(6) \Rightarrow 2U_x = 0$

$\Rightarrow U_x = 0$

$\therefore$ WKT,

$f'(z) = U_x + iV_x$

$\Rightarrow f'(z) = 0 + i(-3z^2)$

$\Rightarrow f'(z) = -3iz^2$

$\Rightarrow \dfrac{df}{dz} = -3iz^2$

$\Rightarrow \int df = \int -3iz^2 \cdot dz$

$\Rightarrow \int df = -3i \int z^2 . dz$

$\Rightarrow f(z) = -\cancel{3}i \dfrac{z^3}{\cancel{3}} + C$

$\Rightarrow f(z) = -iz^3 + C$