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S. J. C. INSTITUTE OF TECHNOLOGY, CHICKBALLAPUR

Department of Mathematics

LECTURE NOTES

Complex Analysis, Probability and Statistical Methods (18MAT41)

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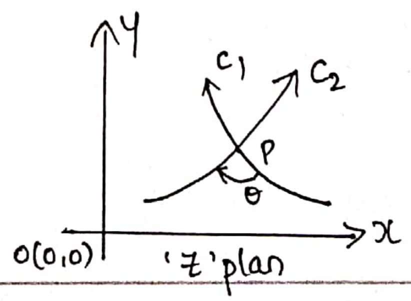
MODULE - 02

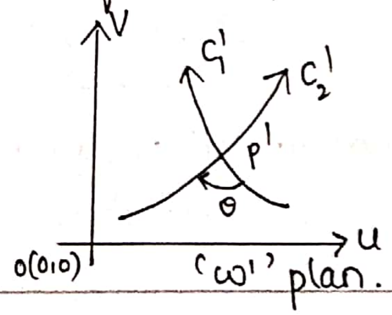
CONFORMAL MAPPING (OR) TRANSFORMATION.

Definition:

If the angle between any 2 curves in the Magnitude and its direction is called conformal transformation(or)

Suppose 2 curves c_1, c_2 in the 'z' plan intersect at the point 'p', and the corresponding curves c_1', c_2' in the 'w' plan intersect at 'p'', and if the angle of intersection of the curves 'p' is the same as the angle of intersection of the curves at 'p'' in Magnitude and direction, then this transformation is called the Conformal transformation.





Condition for $w=f(z)$ to represent a Conformal transformation:

1. **Necessary Condition**: If $w=f(z)$ represents a conformal transformation of a domain (D) in the 'z' plan into a domain (D') of the 'w' plan, then $f(z)$ is analytic function of z in D.

2. **Sufficient Condition**: let $w=f(z)$ be an analytic function of z in a region (D) of the 'z' plan and let $f'(z)\neq 0$ inside 'D' then the mapping $w=f(z)$ is Conformal at the points of D.

1. **Discussion of transformation $w=e^z$.**

Given, $w=f(z)=e^z$ is analytic

$\therefore f(z)$ is differentiable.

$\Rightarrow f'(z)=e^z, \quad z\neq 0$

we have, $w=f(z)=u+iv=e^z \rightarrow (1)$

let $z=x+iy$

$\therefore (1) \Rightarrow u+iv=e^{x+iy}$

$\Rightarrow u+iv=e^x.e^{iy}$

$\Rightarrow u+iv=e^x(\cos y+i\sin y)$

$\Rightarrow u+iv=e^x\cos y+ie^x.\sin y$

$\therefore u=e^x\cos y \rightarrow (2), \quad v=e^x.\sin y \rightarrow (3)$

By eliminating 'y', we have

$(2)^2+(3)^2 = u^2+v^2=(e^x\cos y)^2+(e^x.\sin y)^2$

$$\therefore u^2+v^2 = e^{2x}(\cos^2 y+\sin^2 y)$$

$$\Rightarrow u^2+v^2 = e^{2x} \rightarrow (4)$$

By eliminating 'x', we have

$$\frac{(3)}{(2)} \Rightarrow \frac{v}{u} = \frac{e^x \sin y}{e^x \cos y} = \frac{\sin y}{\cos y} = \tan y.$$

$$\Rightarrow v = \tan y \cdot u \rightarrow (5)$$

Case 1: let $x = c_1$ = constant, we have

$$\text{eq}^n (4) \Rightarrow u^2+v^2 = e^{2c_1} = r_1 \text{ (Say)}$$

$$\Rightarrow u^2+v^2 = r^2 \rightarrow (6)$$

'r' represents a circle having the centre as origin with radius 'r'.

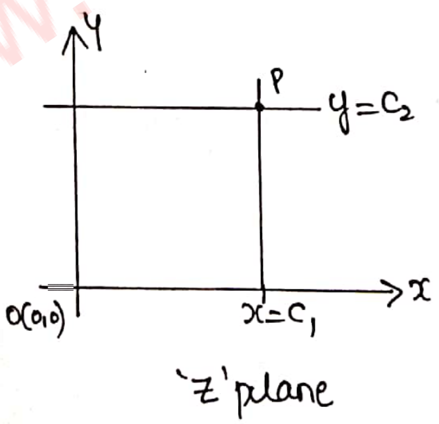
Case 2: let $y = c_2$ = constant, we have

$$\text{eq}^n (5) \Rightarrow v = \tan y \cdot u$$

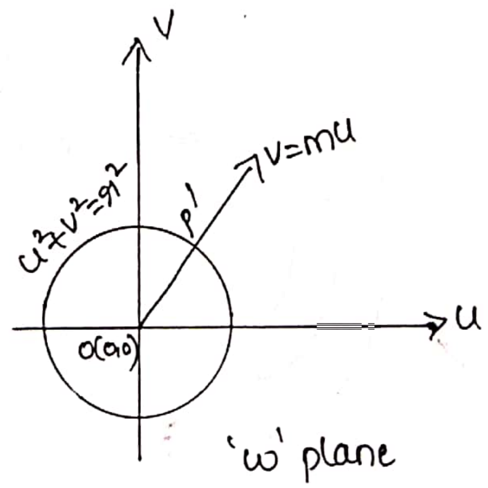
$$\Rightarrow v = \tan c_2 \cdot u$$

$$\therefore v = m.u \quad \text{where } m = \tan c_2 = \text{Slope} \rightarrow (7)$$

represents a straight line passing through the origin having the slope 'm'.



'z' plane



'w' plane

2\. Discussion of transformation $w = z^2$.

Given, $w = f(z) = z^2$ is analytic

$\therefore f(z)$ is differentiable.

$\Rightarrow f'(z) = 2z,\ z \neq 0.$

we have, $w = f(z) = u + iv = z^2 \rightarrow (1)$

let $z = x + iy$

$(1) \Rightarrow u + iv = z^2$

$\Rightarrow u + iv = (x+iy)^2 = x^2 + i^2y^2 + 2ixy$

$\Rightarrow u + iv = (x^2 - y^2) + i(2xy) \rightarrow *$

$\therefore u = x^2 - y^2 \rightarrow (2)$, $v = 2xy \rightarrow (3)$

By eliminating (u, v, x, y) continuously we have,

Case 1: let $u = c_1$ = Constant

$\therefore (2) \Rightarrow x^2 - y^2 = c_1 \rightarrow (4)$

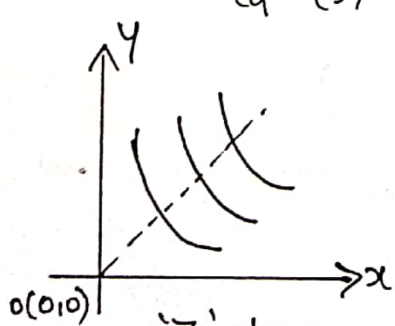
eqn (4) represents a hyperbola.

Case 2: let $v = c_2$

$\therefore (3) \Rightarrow 2xy = c_2$

$\Rightarrow xy = \frac{c_2}{2} \rightarrow (5)$

eqn (5) represents a rectangular hyperbola.



'z' plane

Case 3: let $x = c_3$

$\therefore (2) \Rightarrow u = c_3^2 - y^2$

$$y^2 = c_3^2 - u \rightarrow (6)$$

$\therefore (3) \Rightarrow v = 2xy$

$$2c_3 y = v$$

$$y = \frac{v}{2c_3} \Rightarrow y^2 = \frac{v^2}{4c_3^2} \rightarrow (7)$$

$\therefore$ From eqn (6) & (7)

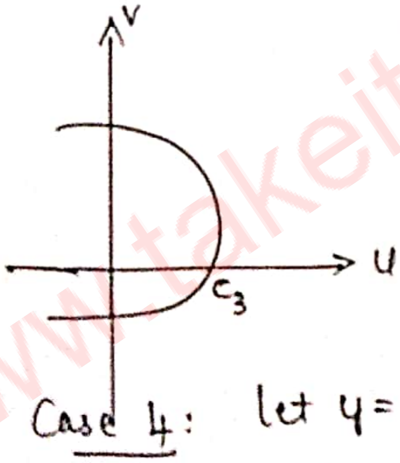
$$\frac{v^2}{4c_3^2} = c_3^2 - u$$

$$\Rightarrow v^2 = 4c_3^2 (c_3^2 - u)$$

$$\Rightarrow v^2 = -4c_3^2 (u - c_3^2)$$

$$(v-0)^2 = -4c_3^2 (u - c_3^2) \rightarrow (8)$$

$\therefore$ eqn (8) represents a parabola, symmetric about 'u' axis in the -ve direction having the vertex at $(c_3^2, 0)$



Case 4: let $y = c_4$

$\therefore (2) \Rightarrow u = x^2 - c_4^2$

$$x^2 = u + c_4^2 \rightarrow (9)$$

$\therefore (3) \Rightarrow v = 2xy$

$$v = 2x c_4$$

$$x = \frac{v}{2c_4} \Rightarrow x^2 = \frac{v^2}{4c_4^2} \rightarrow (10)$$

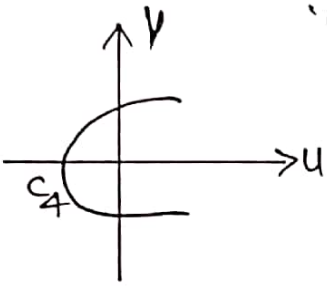
$\therefore$ From eqn (9) & (10)

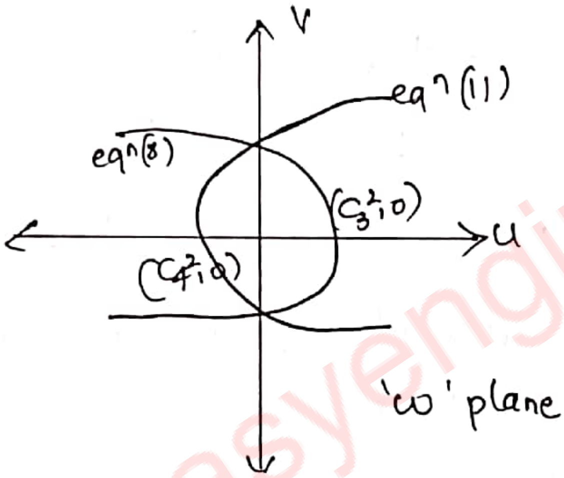
$$\frac{v^2}{4c_4^2} = u + c_4^2$$

$$v^2 = 4c_4^2\,(u + c_4^2)$$

$$(v-0)^2 = 4c_4^2\left(u - (-c_4^2)\right) \rightarrow (11)$$

$\therefore$ eqn (11) represents a parabola symmetric about 'u' axis in the +ve direction having the vertex at $(-c_4^2, 0)$.





3. Discussion of the transformation $\omega = z + \frac{1}{z}$, $z \neq 0$.

Given, $\omega = f(z) = z + \frac{1}{z}$ is analytic,

$\therefore f(z)$ is differentiable.

$$\Rightarrow f'(z) = z + \frac{1}{z}, \quad z \neq 0$$

$$= 1 - \frac{1}{z^2}, \quad z \neq 1$$

let $z = re^{i\theta}$

$\therefore$ WKT, $\omega = f(z) = u + iv = z + \frac{1}{z}$

$$\Rightarrow u + iv = re^{i\theta} + \frac{1}{re^{i\theta}}$$

$$\Rightarrow u + iv = re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$\Rightarrow u+iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= (r\cos\theta + ir\sin\theta) + \left(\frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta\right)$$

$$\Rightarrow u+iv = \left(r+\frac{1}{r}\right)\cos\theta + i\left(r-\frac{1}{r}\right)\sin\theta$$

$$\therefore u = \left(r+\frac{1}{r}\right)\cos\theta \rightarrow (1), \quad v = \left(r-\frac{1}{r}\right)\sin\theta \rightarrow (2)$$

$$\therefore (1) \Rightarrow \frac{u}{\left(r+\frac{1}{r}\right)} = \cos\theta \rightarrow (3) \quad \therefore (2) \Rightarrow \frac{v}{\left(r-\frac{1}{r}\right)} = \sin\theta \rightarrow (4)$$

$$\therefore (3)^2 + (4)^2 \quad \frac{u^2}{\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\left(r-\frac{1}{r}\right)^2} = \cos^2\theta + \sin^2\theta$$

$$\frac{u^2}{\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\left(r-\frac{1}{r}\right)^2} = 1 \rightarrow (5)$$

From eqn (1) $\Rightarrow \dfrac{u}{\cos\theta} = r+\dfrac{1}{r} \rightarrow (6)$ | $\dfrac{v}{\sin\theta} = r-\dfrac{1}{r} \rightarrow (7)$

$$\therefore (6)^2 - (7)^2$$

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r+\frac{1}{r}\right)^2 - \left(r-\frac{1}{r}\right)^2$$

$$\Rightarrow \frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4r\left(\frac{1}{r}\right) \qquad \Big| \; (a+b)^2-(a-b)^2 = 4ab$$

$$\Rightarrow \frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4$$

$$\Rightarrow \frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1$$

$$\Rightarrow \frac{u^2}{(2\cos\theta)^2} - \frac{v^2}{(2\sin\theta)^2} = 1 \rightarrow (8)$$

Case 1: when $r = c_1$, we can have eqn (5) $\Rightarrow \dfrac{u^2}{A^2} + \dfrac{v^2}{B^2} = 1$ represents Ellipse at Centre origin & vertex @ $(\pm 2, 0)$

Case 2: If $\theta = C_2$ is a constant, we have

$$eq^n\ (8) \Rightarrow \frac{u^2}{A^2} - \frac{v^2}{B^2} = 1$$

represents a hyperbola at the points $(\pm 2, 0)$

Also the Ellipse in the 'w' plane, we have $|z| = r$

$$= \sqrt{x^2+y^2} = r$$

$$\Rightarrow x^2+y^2 = r^2$$

represents a circle in the 'z' plane and for the hyperbola in the 'w' plane, we have θ = amplitude of z

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

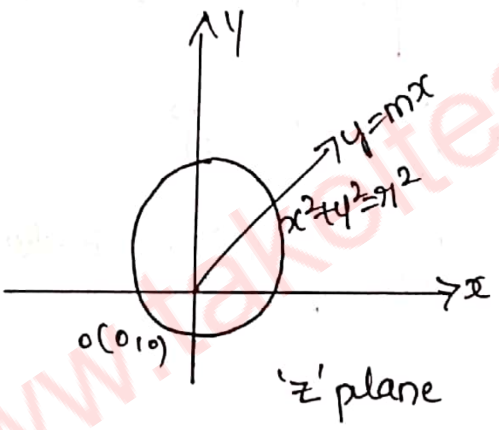
$\tan\theta = m =$ Slope

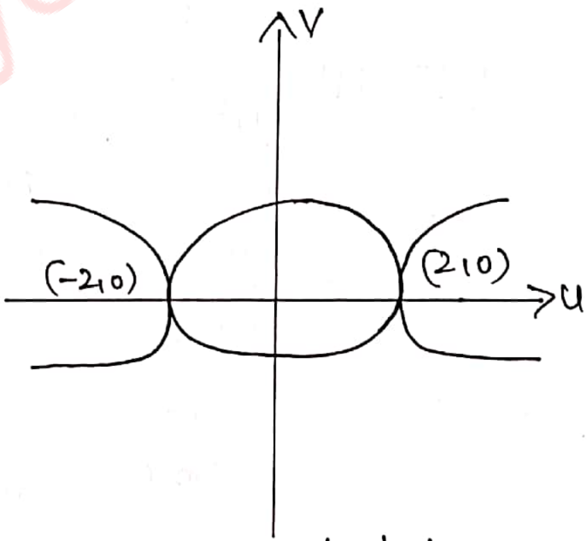
$$y/x = \tan\theta$$

$$y = x\tan\theta$$

$$\boxed{y = mx}$$

represents a straight line.





Belinear Transformation:

A transformation of the form $\omega = \dfrac{az+b}{cz+d}$, where a, b, c, d are real @ complex constants, such that $ad-bc \neq 0$ is called a Belinear transformation @ Linear fractional transformation @ Mobius transformation.

The ratio $\omega = \frac{az+b}{cz+d}$ can also be expressed as

$cwz + dw - az - b = 0$ is linear both in 'z' & 'ω', hence it is called a belinear transformation and if $\frac{dw}{dz} \neq 0$, hence it is the conformal.

properties of Belinear transformation:

(1) The transformation $\omega = \frac{az+b}{cz+d}$ sets of a one-two-one corresponds between the points of closed 'z'-plane and the closed ω-plane.

(2) If $ad - bc = 0$, then 'ω' is either a constant (or) meaningless.

(3) Invarient points (or) fixed points:

If a point 'z' maps into itself i.e., $\omega = z$ under the belinear transformation then the point is called invarient point (or) fixed points of belinear transformation.

Ex: Suppose invarient of $\omega = z^2$ are the solutions of equations

$\Rightarrow z^2 - z = 0$

$\Rightarrow z(z-1) = 0$

$\therefore z = 0, \ z = 1.$

(4). Cross ratio: If z_1, z_2, z_3, z_4 are four distinct points then the ratio is called the cross ratio of these points and it's denoted by $(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$

(5) A Bilinear transformation preserves the cross ratio of the 4 points (or) let w_1, w_2, w_3, w_4 be the images of 4 distinct points of z_1, z_2, z_3, z_4 in the z-plane under a Bilinear transformation, then

$$\Rightarrow (w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$$

$$\Rightarrow \frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

Problems:

1. Find the Bilinear transformation, which Map the points $z = 1, i, -1$ into $w = i, 0, -i$.

Given, $z_1 = 1$, $z_2 = i$, $z_3 = -1$ and $w_1 = i$, $w_2 = 0$, $w_3 = -i$

WKT, From the one-two-one bilinear transformation

$$\Rightarrow (w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

$$\Rightarrow \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(w-i)(0+i)}{(i-0)(-i-w)} = \frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$$\Rightarrow \frac{\cancel{i}(w-i)}{-\cancel{i}(w+i)} = \frac{(z-1)(1+i)}{-(z+1)(1-i)}$$

$$\Rightarrow \frac{w-i}{w+i} = \frac{(z+1)(1+i)}{(z+1)(1-i)}$$

$\Rightarrow (w-i)(z+1)(1-i) = (w+i)(z-1)(1+i)$

$\Rightarrow (w-i)(z-iz+1-i) = (w+i)(z+iz-1-i)$

$\Rightarrow \cancel{wz} - iwz + w - \cancel{iw} - iz - \cancel{z} - \cancel{i} - 1 = \cancel{wz} + iwz - w - \cancel{iw} + iz - \cancel{z} - \cancel{i} + 1$

$\Rightarrow -iwz + iwz + w + w = iz + iz + 1 + 1$

$\Rightarrow -2iwz + 2w = 2iz + 2$

$\Rightarrow \cancel{2}w(1-iz) = \cancel{2}(1+iz)$

$$\Rightarrow \boxed{w = \frac{(1+iz)}{(1-iz)}}$$

2. Find the Belinear transformation, which map the points $z = 1, i, -1$ into $w = 2, i, -2$. Also find the invariant points of transformation.

Given, $z = 1, i, -1$, $z_1 = 1$, $z_2 = i$, $z_3 = -1$ and

$w_1 = 2$, $w_2 = i$, $w_3 = -2$

WKT, Belinear transformation,

$$w = \frac{az+b}{cz+d} \rightarrow (1)$$

$\therefore$ when $z = 1$, $w = 2$

$$\therefore (1) \Rightarrow 2 = \frac{a(1)+b}{c(1)+d} = \frac{a+b}{d+c}$$

$\Rightarrow a+b = 2d+2c$

$\Rightarrow a+b-2c-2d = 0 \rightarrow (2)$

$\therefore$ when $z = i$, $w = i$

$$\therefore (1) \Rightarrow i = \frac{ai+b}{ci+d} \Rightarrow ai+b = i(ci+d)$$

$\Rightarrow ai+b = -c+id$

$\Rightarrow ai+b+c-id = 0 \rightarrow (3)$

$\therefore$ when $z=-1$, $\omega=-2$

$$\therefore (1) \Rightarrow -2 = \frac{a(-1)+b}{c(-1)+d}$$

$$\Rightarrow -2 = \frac{-a+b}{-c+d}$$

$$\Rightarrow -a+b = -2(-c+d)$$

$$\Rightarrow -a+b = +2c-2d$$

$$\Rightarrow -a+b-2c+2d=0 \rightarrow (4)$$

Now eq^n (2) $-$ (3)

$$\Rightarrow (a+\cancel{b}-2c-2d) - (ai+\cancel{b}+c-id) = 0$$

$$\Rightarrow a(1-i) - 3c + (i-2)d = 0$$

Now eq^n (3) $-$ (4)

$$\Rightarrow (ai+\cancel{b}+c-id) - (-a+\cancel{b}-2c+2d) = 0$$

$$\Rightarrow a(1+i) + 3c - d(i+2) = 0 \rightarrow (5)$$

$$\begin{array}{cccc} a & c & d & a \\ 1-i & -3 & i-2 & 1-i \\ 1+i & +3 & -(i+2) & 1+i \end{array}$$

$$\Rightarrow \frac{a}{3(i+2)-3(i-2)} = \frac{c}{(i-2)(1+i)+(1-i)(i+2)} = \frac{d}{3(1-i)+3(1+i)}$$

$$\Rightarrow \frac{a}{3(\cancel{i}+2-\cancel{i}+2)} = \frac{c}{+\cancel{i}-\cancel{2}-\cancel{1}-\cancel{2i}+\cancel{i}+\cancel{2}+\cancel{1}-2i} = \frac{d}{3(1-\cancel{i}+1+\cancel{i})}$$

$$\Rightarrow \frac{a}{12} = \frac{c}{-2i} = \frac{d}{6} = k$$

$\Rightarrow a = 12k, \quad c = -2ik, \quad d = 6k$

$\therefore eq^n(2) \Rightarrow 12\cancel{k} + b + 4ik - 1\cancel{2}k = 0$

$\Rightarrow b = -4ik$

$$\therefore (1) \Rightarrow w = \frac{12kz - 4ik}{-2ikz + 6k}$$

$$\Rightarrow w = \frac{12z - 4i}{-2iz + 6}$$

$$\Rightarrow w = \frac{6z - 2i}{-iz + 3} \rightarrow (6)$$

when $w = z$

$$(6) \Rightarrow z = \frac{6z - 2i}{-iz + 3}$$

$\Rightarrow z(-iz + 3) = 6z - 2i$

$\Rightarrow z(-iz + 3) - 6z + 2i = 0$

$\Rightarrow -iz^2 + 3z - 6z + 2i = 0$

$\Rightarrow -i^2 z^2 - 3zi + 2i^2 = 0$ (x/y by i)

$\Rightarrow z^2 - 3zi - 2 = 0$

$$\Rightarrow z = \frac{3i \pm \sqrt{9i^2 - 4(1)(2)}}{2(1)}$$

$$\Rightarrow z = \frac{3i \pm \sqrt{-9 + 8}}{2} = \frac{3i \pm \sqrt{-1}}{2} = \frac{3i \pm \sqrt{i^2}}{2}$$

$$\Rightarrow z = \frac{3i \pm i}{2}$$

Case 1 : $z = \frac{3i + i}{2} = \frac{4i}{2} = 2i$

Case 2 : $z = \frac{3i - i}{2} = \frac{2i}{2} = i$

$\therefore \boxed{z = 2i, i}$

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3. Find the Belinear transformation, which maps $z = \infty, i, 0$ into $w = -1, -i, 1$. Also find the fixed points of the transformation.

Soln For any real @ complex constants

WKT, Belinear transformation, $w = \dfrac{az+b}{cz+d} \rightarrow (1)$

$$\therefore (1) \Rightarrow w = \frac{a + b/z}{c + d/z} \rightarrow (2)$$

and given $z_1 = \infty, z_2 = i, z_3 = 0$ and $w_1 = -1, w_2 = -i, w_3 = 1$.

$\therefore$ when $z = \infty$, $\therefore w = -1$

$$\therefore (2) \Rightarrow -1 = \frac{a+0}{c+0} = \frac{a}{c}$$

$$a = -c$$

$$\underline{a+c=0} \rightarrow (3)$$

$\therefore$ when $z = i$, $w = -i$

$$\therefore (2) \Rightarrow -i = \frac{ai+b}{ci+d}, \Rightarrow -i(ci+d) = ai+b$$

$$\Rightarrow \underline{ai+b-c+id=0} \rightarrow (4)$$

$\therefore$ when $z = 0$, $w = 1$

$$\therefore (2) \Rightarrow 1 = \frac{b}{d}$$

$$\Rightarrow \underline{b-d=0} \rightarrow (5)$$

$\therefore$ (3) + (4)

$$a(1+i) + b + di = 0 \rightarrow (6)$$

$$\therefore (5) \Rightarrow 0.a + b - d = 0 \rightarrow (7)$$

a	b	c	a
$1+i$	1	i	$1+i$
0	1	-1	0

$$\frac{a}{(-1-i)} = \frac{b}{(1+i)} = \frac{d}{(1+i)}$$

$$\frac{a}{-(1+\cancel{i})} = \frac{b}{(1+\cancel{i})} = \frac{d}{(1+\cancel{i})} = K$$

$$\frac{a}{-1} = \frac{b}{1} = \frac{d}{1} = K$$

$$a = -K, \quad b = K, \quad d = K$$

and eqⁿ (3) $a + c = 0$

$$c = -a = -(-K) = \underline{\underline{K}}$$

WKT,

$$\omega = \frac{az+b}{cz+d} = \frac{-Kz+K}{Kz+K} = \frac{-z+1}{z+1}$$

$$\boxed{\omega = \frac{-z+1}{z+1}} \rightarrow (8)$$

To find invariant points, let $\omega = z$

$$(8) \Rightarrow z = \frac{-z+1}{z+1}$$

$$z(z+1) = (-z+1)$$

$$\Rightarrow (z^2+z)\,(+z-1) = 0$$

$$\Rightarrow z^2 + 2z - 1 = 0$$

$$z = \frac{-2 \pm \sqrt{4-4(1)(-1)}}{2(1)} = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm \sqrt{8}}{2}$$

$$\Rightarrow z = \frac{-2 \pm 2\sqrt{2}}{2} = \frac{-2\,(-1 \pm \sqrt{2})}{2} = -1 \pm \sqrt{2}$$

$$\Rightarrow z = -1 + \sqrt{2}, \quad z = -1 - \sqrt{2}$$

4. Find the Belinear transformation, which map the points $z = 1, i, -1$ into $w = 0, 1, \infty$.

Soln WKT, for any real or complex constants of a, b, c, d, the Belinear transformation is $w = \frac{az+b}{cz+d} \rightarrow (1)$

$$\therefore (1) \Rightarrow \frac{1}{\omega} = \frac{cz+d}{az+b} \rightarrow (2)$$

and given, $z_1 = 1$, $z_2 = i$, $z_3 = -1$ and $\omega_1 = 0$, $\omega_2 = +1$, $\omega_3 = \infty$

$\therefore$ when $z = 1$, $\omega = 0$ $\therefore \Rightarrow (1) \Rightarrow 0 = \frac{a+b}{c+d}$

$$\Rightarrow a + b = 0 \rightarrow (3)$$

$\therefore$ when $z=i,\ w=1$

$$\Rightarrow (1) \Rightarrow 1 = \frac{ai+b}{ci+d}$$

$$ai+b-ci-d=0 \rightarrow (4)$$

$\therefore$ when $z=-1,\ w=\infty$

$$\Rightarrow (2) \Rightarrow \frac{1}{\infty} = \frac{c(-1)+d}{d(-1)+b} = \frac{-c+d}{-d+b}$$

$$0=-c+d \rightarrow (5)$$

eqn (4) $-$ (3)

$$(i-1)a - ci - d = 0 \rightarrow (6)$$

$$\therefore (5) \Rightarrow 0.a - c + d = 0 \rightarrow (7)$$

$$\frac{a}{(i-1)} \quad \frac{c}{-i} \quad \frac{d}{-1} \quad \frac{a}{(i-1)}$$

$$0 \quad -1 \quad 1 \quad 0$$

$$\frac{a}{-i-1} = \frac{c}{-(i-1)} = \frac{d}{-(i-1)} = K$$

$$\Rightarrow a = -K(1+i),\quad c = -K(i-1),\quad d = -K(i-1)$$

$$c = K(1-i) \qquad d = K(1-i)$$

$\Rightarrow$ eqn (3) $b=-a$

$$b = -(-K(1+i))$$

$$b = K(1+i)$$

$$\therefore w = \frac{-K(1+i)z + K(1+i)}{K(1-i)z + K(1-i)}$$

$$w = \frac{-(1+i)z + (1+i)}{(1-i)z + K(1-i)}$$

Complex Integration:

Suppose $w=f(z)$ be a Continuous complex valued function over a region 'R'. for any Complex variable $z=x+iy$ in the Curve 'C'. then the Complex integration of $f(z)$ from the point 'p' to 'Q' can be defined as $\int_C f(z).dz$ and which will be evaluated by dividing the interval into 'n' no of parts.

Line Integral of a Complex valued function:

let $f(z)=u(x,y)+iv(x,y)$ be a continuous Complex valued function over a region 'R' of any Complex variable $z=x+iy$ in the Curve 'C', then the line of Integral of $f(z)$ Can be defined as $\int_C f(z).dz = \int_C (u+iv)(dx+idy)$

$$= \int_C (u.dx+iu.dy+ivdx-vdy)$$

$$= \int_C (udx-vdy)+i\int_C (vdx+udy)$$

Problems:

1. Evaluate $\int_C |z|^2.dz$, where 'c' is of square with the vertices $(0,0),(1,0),(1,1),(0,1)$.

Given, $\int_C |z|^2.dz$

$f(z)=|z|^2$, where $z=x+iy$

$\Rightarrow |z|=\sqrt{x^2+y^2}$

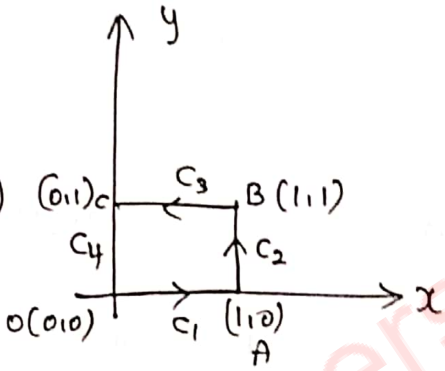
$\Rightarrow |z|^2=x^2+y^2$

and $dz=dx+idy$

$$\therefore \text{WKT} \int_C f(z).dz = \int_C (x^2+y^2)(dx+idy) \rightarrow (1)$$

$$= \int_{C_1} (x^2+y^2)(dx+idy) + \int_{C_2} (x^2+y^2)(dx+idy) + \int_{C_3} (x^2+y^2)(dx+idy) + \int_{C_4} (x^2+y^2)(dx+idy) \rightarrow (2)$$

(i) Along the curves, $C_1 \Rightarrow y=0$
$dy=0$



$$\therefore \int_C (x^2+y^2)(dx+idy) = \int_{x=0}^{1} (x^2)(dx+0)$$

$$= \int_0^1 (x^2)(dx)$$

$$= \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

(ii) Along the curves, $C_2 \Rightarrow x=1, dx=0$

$$\therefore \int_C (x^2+y^2)(dx+idy) = \int_{y=0}^{1} (1+y^2)(idy)$$

$$= i\int_0^1 (1+y^2)\,dy \Rightarrow i\,(y+y^3/3)\Big|_0^1$$

$$= i\left(1+\frac{1}{3}\right) = \frac{4i}{3}$$

(iii) Along the curves, $C_3 \Rightarrow y=1, dy=0$

$$\therefore \int_C (x^2+y^2)(dx+idy) = \int_{x=1}^{0} (x^2+1)\,dx$$

$$= \frac{x^3}{3}+x\Big|_1^0 \Rightarrow 0-\left(\frac{1}{3}+1\right) = \frac{-4}{3}$$

(v) Along the curves, $C_4 \Rightarrow x=0, dx=0$

$$\therefore \int_C (x^2+y^2)(dx+idy) = \int_{y=1}^{0} y^2\, idy \Rightarrow i\,\frac{y^3}{3}\Big|_1^0 \Rightarrow 0-\frac{i}{3} = \frac{-i}{3}$$

$$\therefore \int_C f(z)\,dz = \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3}$$
$$= -1 + 1i$$
$$= -1 + i$$

Cauchy's Integral Theorem (or) Fundamental Theorem.

Statement: If a function $f(z)$ is analytic at all the points within and on a closed contour 'c' then $\int_C f(z)dz = 0$.

Proof: Given, $\omega = f(z) = u(x,y) + iv(x,y)$ is analytic function over a region 'R' at any complex variable $z = x+iy$ of a curve 'c', we have $\int_C f(z).dz = \int_C (u+iv)(dx+idy)$.

$$\Rightarrow \int_C f(z).dz = \int_C (u\,dx - v\,dy) + i(v\,dx + u\,dy) \rightarrow (1)$$

By the Green's Theorem, WKT

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\therefore (1) \Rightarrow \int_C f(z).dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \rightarrow (2)$$

Given, $f(z) = u+iv$ is analytic and it can satisfy the C-R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\therefore (2) \Rightarrow \int_C f(z)dz = \iint_R \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy$$

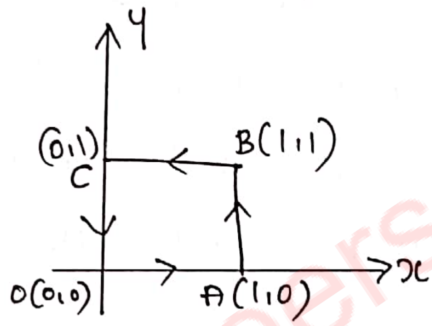
$$\Rightarrow \int_C f(z).dz = 0 + i0$$

$$\Rightarrow \int_C f(z)\,dz = 0$$

Problem:

1. Verifying Cauchy's Theorem for the function $f(z) = z^2$, where 'C' is the square having the vertices $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$.

Given, The Complex valued function $f(z) = z^2$, where 'C' is a closed square having the vertices $O(0,0)$, $A(1,0)$, $B(1,1)$ and $C(0,1)$.



WKT, $z = x+iy$

$\Rightarrow dz = dx + i\,dy$

$\therefore f(z)\,dz = z^2 . dz$

$\Rightarrow f(z) . dz = (x+iy)^2 (dx+i\,dy) \rightarrow (1)$

and $\int_C f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CO} f(z).dz \rightarrow (2)$

$\therefore$ Along OA $\Rightarrow y=0$
$dy=0$

$$\int_{OA} f(z)dz = \int_{OA} (x+i(0))^2 . [dx+i(0)]$$

$$= \int_{OA} x^2 . dx = \int_{x=0}^{1} x^2 . dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

$\therefore$ Along AB $\Rightarrow x = 1$
$dx=0$

$$\int_{AB} f(z)\,dz = \int_{AB} (1+iy)^2 . (0+i\,dy)$$

$$= i \int_{AB} (1+2iy-y^2)\,dy$$

$$= i \int_{y=0}^{1} (1+2iy-y^2)\,dy = i \left| y + iy^2 - \frac{y^3}{3} \right|_0^1$$

$$\int_{AB} f(z).dz = i \left| 1+i-\frac{1}{3} \right| = i \left| 1+\frac{2}{3} \right| = i^2 + \frac{2i}{3}$$

$$= -1 + \frac{2i}{3}$$

$$\therefore \int_{AB} f(z)dz = -1+\frac{2i}{3}$$

$\therefore$ Along BC $\Rightarrow y=1, dy=0$

$$\int_{BC} f(z).dz = \int_{BC} (x+i)^2 . (dx+i(0))$$

$$= \int_{BC} (x^2+2ix-1)\, dx$$

$$= \int_{x=1}^{0} (x^2+2ix-1)\, dx$$

$$= \left(\frac{x^3}{3} + 2i\frac{x^2}{2} - x \right|_1^0$$

$$= 0 - \left(\frac{1}{3} + i - 1 \right)$$

$$= -\left(i - \frac{2}{3} \right) = \frac{2}{3} - i$$

$\therefore$ Along CO $\Rightarrow x=0, dx=0$

$$\int_{CO} f(z).dz = \int_{CO} (0+iy)^2 . (0+idy)$$

$$= \int_{CO} i^2 y^2 . idy$$

$$= \int_{CO} -y^2 . idy \Rightarrow -i \int_{y=1}^{0} y^2 . dy$$

$$= -i \left| \frac{y^3}{3} \right|_1^0$$

$$= -i \left(0 - \frac{1}{3} \right) = \frac{i}{3}$$

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$\therefore (2) \Rightarrow \int_C f(z).\,dz = \frac{1}{3} - 1 + \frac{2i}{3} + \frac{2}{3} - i + \frac{i}{3}$

$$\int_C f(z)dz = 0$$

$\therefore$ Hence Cauchy's Theorem Verified.

Theorem : Cauchy's Integral formula.

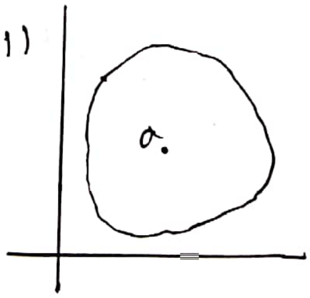
Statement: If $f(z)$ is analytic inside and on a Simple closed curve 'c' and if 'a' is anypoint within 'C', then $f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a}\,.\,dz$

Proof: Given $f(z)$ is an analytic function inside and on a Simple closed Curve 'c'.

Since 'a' is a point within 'C'. we shall enclose it by a circle 'C_1' and with $z=a$ as a Center and 'r' as a radius, Such that C_1 lies entirely within 'C'.

Therefore the function $\frac{f(z)}{z-a}$ is analytic inside and on the boundary of the region b/n 'c' and 'c_1'.

We have $\int_C \frac{f(z)}{(z-a)}\,dz = \int_{C_1} \frac{f(z)}{(z-a)}\,dz \rightarrow (1)$



WKT, $|z-a| = r$

$\Rightarrow z-a = re^{i\theta}$

$\Rightarrow z = a+re^{i\theta}$

$\Rightarrow \frac{dz}{d\theta} = (0+ri e^{i\theta})$

$\Rightarrow dz = ri.e^{i\theta}.d\theta$

$\therefore (1) \Rightarrow \int_C \frac{f(z)}{z-a}\,.dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\,.ri e^{i\theta}.d\theta$

$$\Rightarrow \int_C \frac{f(z)}{z-a} \cdot dz = i \int_0^{2\pi} f(a + re^{i\theta})\, d\theta \rightarrow (2)$$

when $r = 0$,

$$\therefore (2) \Rightarrow \int_C \frac{f(z)}{z-a}\, dz = i \int_{\theta=0}^{2\pi} f(a) \cdot d\theta .$$

$$\Rightarrow \int_C \frac{f(z)}{z-a} \cdot dz = i \cdot f(a) \cdot \int_0^{2\pi} 1 \cdot d\theta$$

$$= i \cdot f(a) \cdot \left|\theta\right|_0^{2\pi}$$

$$= i \cdot f(a)\, (2\pi)$$

$$\Rightarrow \int_C \frac{f(z)}{(z-a)}\, dz = i \cdot f(a) \cdot 2\pi$$

$$\therefore f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)}\, dz$$

Generalized Cauchy's Integral Formula:

If $f(z)$ is analytic inside and on a simple closed curve 'c' and if 'a' is any point within 'C', then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} \cdot dz$$

also $f^{n}(a) \cdot \frac{2\pi i}{n!} = \int_C \frac{f(z)}{(z-a)^{n+1}} \cdot dz$

Problems:

1. Evaluate $\int_C \frac{e^{2z}}{(z+1)(z+2)} \cdot dz$, where 'c' is a circle $|z| = 3$.

Sol let $I = \int_C \frac{e^{2z}}{(z+1)(z+2)}\, dz \longrightarrow (1)$

Since $c : |z| = 3$ is a circle with the center '0' (zero) and radius '3'.

$z=-1$ and $z=-2$ are inside of the circle.

$\therefore f(z)=e^{2z}$ is analytic at all points except $z=-1$ and $z=-2$.

$$\therefore \frac{1}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$\Rightarrow 1 = A(z+2) + B(z+1) \rightarrow (2)$$

when $z=-2$, when $z=-1$

$\therefore (2) \Rightarrow 1=-B$ $\quad$ $\therefore (2) \Rightarrow 1=A$

$\boxed{B=-1}$ $\quad$ $\boxed{A=1}$

$$\therefore \frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2}$$

$$\therefore \frac{e^{2z}}{(z+1)(z+2)} = \frac{e^{2z}}{z+1} - \frac{e^{2z}}{z+2}$$

$$\therefore \int_c \frac{e^{2z}}{(z+1)(z+2)} dz = \int_c \frac{e^{2z}}{z-(-1)} dz - \int_c \frac{e^{2z}}{z-(-2)} dz$$

$$= 2\pi i\, e^{2(-1)} - 2\pi i\, e^{2(-2)}$$

$$= 2\pi i\, e^{-2} - 2\pi i\, e^{-4}$$

$$= 2\pi i\left(e^{-2} - e^{-4}\right)$$

$$= 2\pi i\left(\frac{1}{e^2} - \frac{1}{e^4}\right) = 2\pi i\left(\frac{e^4 - e^2}{e^8}\right)$$

$$= 2\pi i\left(\frac{e^2-1}{e^4}\right)$$

2. Evaluate $\int_c \frac{e^{2z}}{(z+1)(z+2)} dz$, where 'c' is a circle $|z|=3$.

Sol: let $I = \int_c \frac{e^{2z}}{(z+1)(z+2)} . dz \longrightarrow (1)$

Since $c : |z|=3$ is a circle with the Center '0'(Zero) &

radius 3.

$z=-1$ and $z=2$ are inside of the circle.

$\therefore f(z)=e^{2z}$ is analytic at all points except $z=-1$ and $z=2$.

$$\therefore \frac{1}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2} \rightarrow (2)$$

$$1 = A(z-2) + B(z+1)$$

when $z=-1$, when $z=+2$

$1 = A(-3)$ $\quad$ $1 = B(3)$

$\boxed{A=-1/3}$ $\quad$ $\boxed{B=1/3}$

$$\therefore \frac{1}{(z+1)(z-2)} = \frac{1}{3(z-2)} - \frac{1}{3(z+1)}$$

$$\therefore \frac{e^{2z}}{(z+1)(z-2)} = \frac{e^{2z}}{3(z-2)} - \frac{e^{2z}}{3(z+1)}$$

$$\therefore \int_c \frac{e^{2z}}{(z+1)(z-2)}\,dz = \int_c \frac{e^{2z}}{3(z-2)}\,dz - \int_c \frac{e^{2z}}{3(z+1)}\,dz$$

$$= \int_c \frac{e^{2z}}{3(z-(2))}\,dz - \int_c \frac{e^{2z}}{3(z-(-1))}\,dz$$

$$= \frac{2\pi i}{3}\cdot e^{2(2)} - \frac{2\pi i}{3} e^{2(-1)}$$

$$= \frac{2\pi i}{3} e^{4} - \frac{2\pi i}{3} e^{-2}$$

$$= \frac{2\pi i}{3}\left(e^{4}-e^{-2}\right) = \frac{2\pi i}{3}\left(e^{4}-\frac{1}{e^{2}}\right)$$

3. Evaluate $\int_c \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}\,dz$ where 'c' is a circle $|z|=3$.

Sol: let $I = \int_c \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}\,dz \rightarrow (1)$

Since $C: |z| = 3$ is a circle with a circle '0' and radius 3'.

$\therefore z=1$ and $z=2$ are inside of the circle.

$\therefore f(z) = \sin(\pi z^2) + \cos(\pi z^2)$ is analytic at all points except $z=1$ and $z=2$.

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \longrightarrow (2)$$

$$1 = A(z-2) + B(z-1)$$

when $z=1$, when $z=2$

$1 = A(-1)$ $\boxed{A = -1}$

$1 = B(1)$ $\boxed{B = 1}$

From eqn (2)

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\therefore \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz = \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{z-2} . dz - \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{z-1} dz$$

where Substitute $z=2$ and $z=1$

$$= 2\pi i \left(\sin 4\pi + \cos 4\pi\right) - 2\pi i \left(\sin \pi + \cos \pi\right)$$

$$= 2\pi i (0+1) - 2\pi i (0-1)$$

$$= 2\pi i + 2\pi i$$

$$= \underline{4\pi i}$$

4. Evaluate $\int_C \frac{\sin(\pi z^2) + \cos \pi z^2}{(z-1)^2 (z-2)} dz$, where $C: |z| = 3$

Sol: let $I = \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)^2 (z-2)} dz \longrightarrow (1)$

Since $C: |z| = 3$ is a circle with Centre '0' and radius '3'

$z=2$ and $z=1$ are inside of the circle

$\therefore f(z) = \sin(\pi z^2) + \cos(\pi z^2)$ is a analytic function at all the points except $z=1$ and $z=2$.

$$\therefore \frac{1}{(z-1)^2(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{(z-2)}$$

$$1 = A(z-1)(z-2) + B(z-2) + C(z-1)^2 \longrightarrow (2)$$

when $z=1$, when $z=2$

$1 = B(-1)$ $\quad$ $1 = C$

$\boxed{B=-1}$ $\quad$ $\boxed{C=1}$

$\therefore A+C=0$

$A = -C = -1$

$\boxed{A=-1}$

$$\therefore \frac{1}{(z-1)^2(z-2)} = \frac{-1}{(z-1)} - \frac{1}{(z-1)^2} + \frac{1}{(z-2)}$$

$$\therefore \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)\,dz}{(z-1)^2(z-2)} = \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-2)}\,dz - \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-1}\,.dz - \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)^2}\,dz$$

$$= 2\pi i(\sin 4\pi + \cos 4\pi) - 2\pi i(\sin\pi + \cos\pi) - \int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^{1+1}}\,dz$$

$$= 2\pi i(0+1) - 2\pi i(0-1) - \frac{2\pi i}{1!} f'(1) \longrightarrow (3)$$

where $f(z) = \sin\pi z^2 + \cos\pi z^2$

$f'(z) = (\cos\pi z^2 - \sin\pi z^2)\,2\pi z$

$$\therefore f'(1) = (\cos\pi - \sin\pi)\, 2\pi$$
$$= (-1-0)\, 2\pi$$
$$= -2\pi$$

$\therefore$ from eqn (3)

$$\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)}\, dz = 2\pi i + 2\pi i - \frac{2\pi i}{1!}(-2\pi)$$
$$= 4\pi i + \frac{4\pi^2 i}{1}$$
$$= 4\pi i\,(1+\pi)$$

5. Evaluate $\int_c \frac{z^2 - z + 1}{z-1}\, dz$ where 'c' is a circle (i) $|z| = 1$
(ii) $|z| = 1/2$

Sol: let $I = \int_c \frac{z^2-z+1}{z-1}\, dz \rightarrow (1)$

(i) Since $c : |z| = 1$ is a circle with the centre 'O' and radius 1.
(or origin)

$z = 1$ is on the circle $|z| = 1$.

Hence $f(z) = z^2 - z + 1$ is analytic except $z = 1$.

$$\therefore \int_c \frac{z^2-z+1}{z-1}\, dz = 2\pi i\,(1^2 - 1 + 1)$$
$$= 2\pi i$$

(ii) Since $c : |z| = 1/2$ Can represent the circle with centre as origin and radius $1/2$.

$z = 1$ is out side of the circle $|z| = 1/2$

$$\therefore \int_c \frac{z^2-z+1}{z-1}\, dz = 0 \qquad (\because \text{outside No values})$$