By the Name of Allah, the Most-Gracious, the Most-Merciful

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1 Constructing A Cutoff Function

Note: After writing this document, I (unsurprisingly) found some books that already have idea that I mention, here, as an exercise. For example, Lieb and Loss book, "Analysis", contains such a thing. I also edited this to add the case of the square, which I find important (and intriguing).

I have been trying to construct an example for a cutoff function, but I was stuck. There were multiple almost-there examples. But they weren't good enough (not smooth or not exactly cutoff). However, here is a way to create such a function using convolution of a Heaviside-like function and a standard mollifier.

Before proceeding, I should note that the constants below have been "tweaked" few times, to arrive at a desired structure. So, don't bother much about the boundaries.

Consider positive real numbers a and r, and the following functions from $\mathbb{R}$ to itself

$$H_{a+r/2}(x) = \begin{cases} 1 & \text{if } |x| < a + r/2 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\eta_r(x) = \begin{cases} c_r \exp\left(\frac{1}{x^2-(r/2)^2}\right) & \text{if } |x| < r/2 \\ 0 & \text{otherwise,} \end{cases}$$

where c_r is a positive constant that makes $\|\eta_r\|_{L^1} = 1$. Next, define

$$\xi_1^{a,r}(x) := (H_{a+r2} * \eta_r)(x).$$

Notice that $\xi_1^{a,r} \in [0,1]$ (which is clear by what follows, too). Now, a little of arithmetic gives us,

$$\xi_1^{a,r}(x) = \int_{-a-r/2}^{a+r/2} \eta_r(x-y)\,dy \; = \int_{x-a-r/2}^{x+a+r/2} \eta_r(z)\,dz\,,$$

and

$$\xi_1^{a,r}(x) = \int_{-r/2}^{r/2} H(x-y)\eta_r(y)\,dy\,.$$

Moreover, when $|x| \le a$, we have that $(x-a-r/2, x+a+r/2) \cap (-r/2, r/2) = (-r/2, r/2)$, and $\xi_1^{a,r}(x) = 1$. When $|x| \ge a + r$ and $|y| \le r/2$, we have that $|x-y| \ge a + r/2$, and $\xi_1^{a,r}(x) = 0$.

Therefore,

$$\begin{cases} \xi_1^{a,r}(x) = 1 & \text{when } x \in (-\infty, a] \\ 0 \leq \xi_1^{a,r}(x) \leq 1 & \text{when } x \in (a, a+r) \\ \xi_1^{a,r}(x) = 0 & \text{when } x \in [a+r, +\infty). \end{cases}$$

Since η_r is smooth, the convolution $\xi_1^{a,r}$ is smooth as well. Together with the above, this shows that $\xi_1^{a,r}$ is a 1-dimensional cutoff function. Now, we define the following *cutoff function* in $\mathbb{R}^d$ for $d \geq 1$,

$$\xi_d^{a,r}(x) := \xi_1^{a,r}(|x|)$$

2 An Almost-There Cutoff function

I have also tried to create a cutoff function using this "funny" method. I started with the mollifier η_r, and set

$$g_1(x) := e^{-1}\,\eta_r(x).$$

Then for natural numbers $m \geq 1$, I define the recursive relation

$$g_{m+1}(x) := g_m(x)\,\exp\left(1 - g_m(x)\right).$$

What's good about this is that $g_n(0) = 1$ always. And that if $g_n(0) = 0$ for some n, then $g_k(0) = 0$ for every k. Moreover, if $g_1(x) > 0$, then $g_n(x) \longrightarrow 1$ as $n \longrightarrow \infty$. This means that

$$g_n \longrightarrow H_{r/2} \quad \text{as } n \longrightarrow \infty.$$

This is when I got the idea of the construction in the previous section. I have plotted here, using Octave, an example for $r = 2$ with $m = 1, 3, 5$, and they look like cutoff function, except that they are not. Computer cannot detect the very small error.

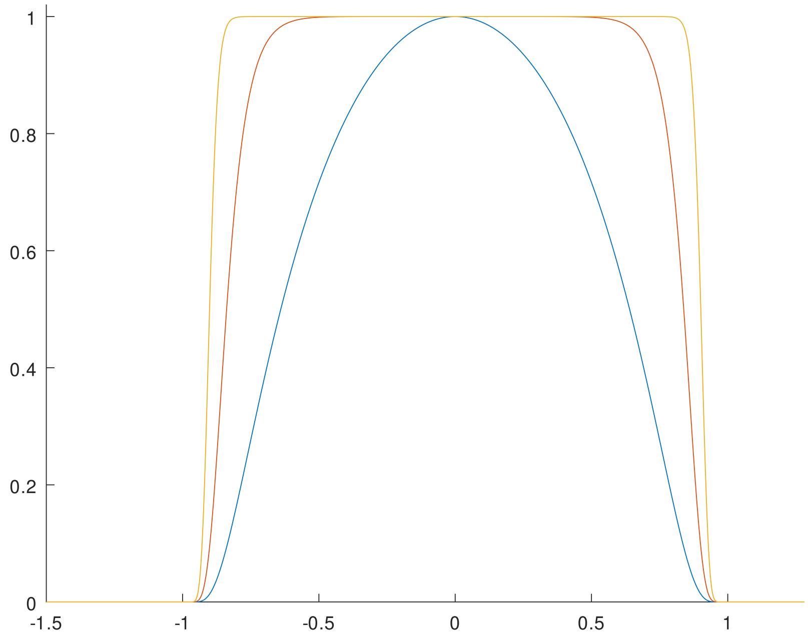

Figure 1: an almost-there cutoff function

3 An Cutoff function for the Square

Let $\xi_i : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a smooth cutoff function (perhaps something like the above), such that

$$\begin{cases} \xi_i(x) = 1 & \text{for } x \in [-\alpha_i, \alpha_i] \\ \xi_i(x) = 0 & \text{for } x \notin (-\beta_i, \beta_i), \end{cases}$$

where $\beta_i > \alpha_i > 0$. Then, define $\xi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$\xi(x) = \xi(x_1, ..., x_n) := \xi_1(x_1)\xi_2(x_2) \cdots \xi_n(x_n).$$

We can easily see that

$$\begin{cases} \xi(x) = 1 & \text{for } x \in [-\alpha_1, \alpha_1] \times \cdots \times [-\alpha_n, \alpha_n] \\ \xi_i(x) = 0 & \text{for } x \notin (-\beta_1, \beta_1) \times \cdots \times (-\beta_n, \beta_n). \end{cases}$$

Trivially ξ is continuous, as well. Thus, ξ is a cutoff function for the square $[-\alpha_1, \alpha_1] \times \cdots \times [-\alpha_n, \alpha_n]$.