Contents

1 What is space?

- When we think of space we think about space where we live but in mathematics it more then that In mathamatics any generic abstract collection of elements are called space

2 Function Space

- Space of all possible function F(x)
- Idea is come from linear algebra
- It is like vector space with infinit dimenstions

2.1 Hilbert space

Space all possible wave functions

3 Vectors

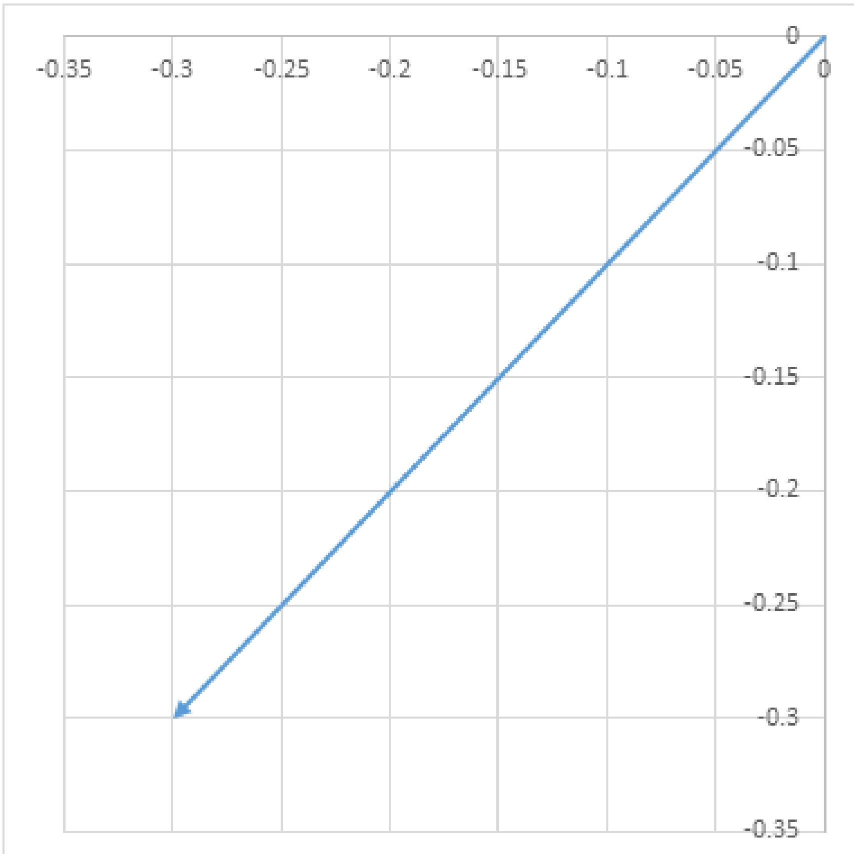
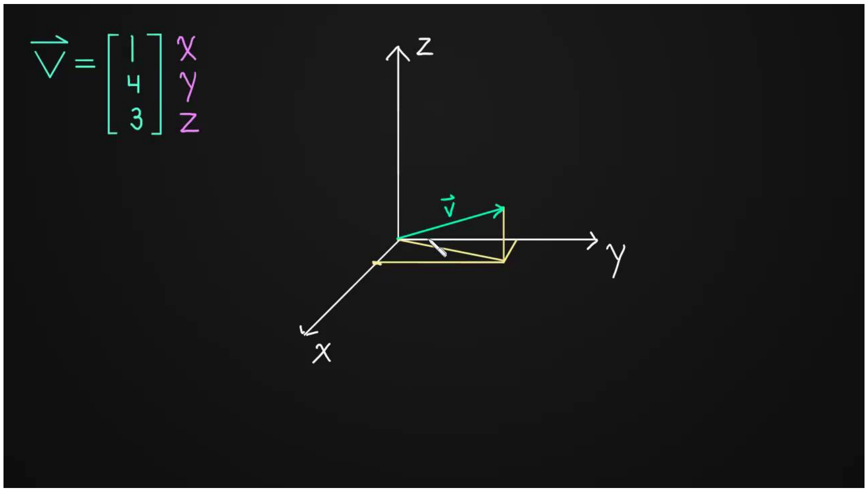


Figure 1: This is Vector

- It has both dirction and magnitude

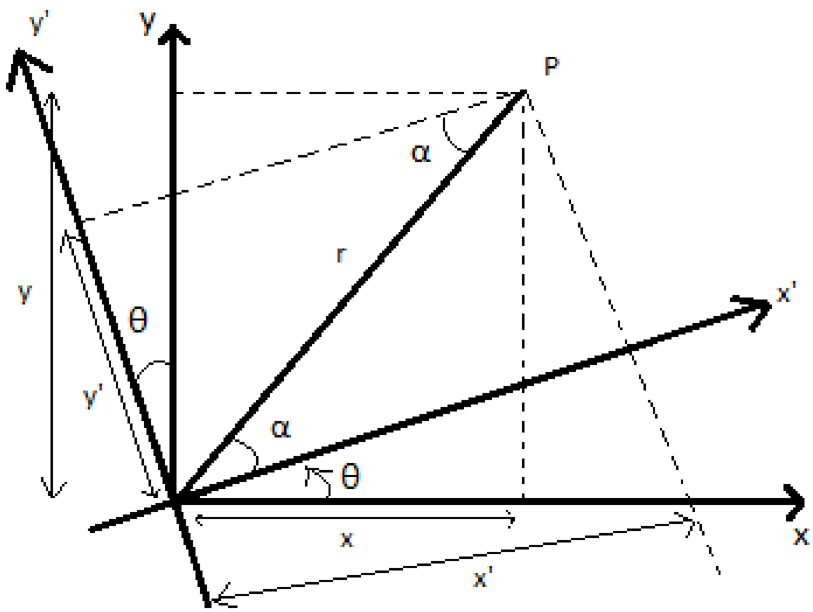
3.1 Vectors



3.2 Component reprasentation

- $\vec{a}$ = x$\hat{i}$ + y$\hat{j}$ + z $\hat{k}$
- Also i can write this way
- $\vec{a} = \frac{\vec{a}.\hat{i}}{||\hat{i}||^2} + \frac{\vec{a}.\hat{j}}{||\hat{j}||^2} + \frac{\vec{a}.\hat{k}}{||\hat{k}||^2}$

4 cordinate transformation



4.1 cordinate transformation

- I can write p interms of x and y
- $\vec{p} = \frac{\vec{p}.x}{||x||^2} + \frac{\vec{p}.y}{||y||^2}$
- I can alse write in terms of x' and y'
- $\vec{p} = \frac{\vec{p}.x'}{||x'||^2} + \frac{\vec{p}.y'}{||y'||^2}$
- here dot product give projection on to each axis and tell how much that vector in point in x dirction
- Also when dot product is zero the vectors are in orthogonal to each other

5 Inner product of functions

- It is just like dot product between two vectors
- A function can be thought as vector of infinit dimenstion

- For defining innerproduct just descritize our functions f and g in some intervel a and b
- We have $\begin{bmatrix} f_1 & f_2 & \dots & f_n \end{bmatrix}$ and $\begin{bmatrix} g_1 & g_2 & \dots & g_n \end{bmatrix}$ Now we can find innerproduct of this it just two vectors

5.1 Inner product continue

- Now the innerproduct is
- $$\sum_{k=0}^{n-1} f_k g_k$$
- This as one problem when n increses this changes by huge amound so we need to normalize this by Δ x

5.2 Inner product continue

- Now after normalization by Δ x the equation become
- $$\sum_{k=0}^{n-1} f_k g_k \Delta x$$
- This is Riemann approximation of inegral

5.3 Innerproduct

- Now the equation become
- <f(x),g(x)> = $\int_a^b$ f(x)g(x) dx

5.4 Inner product matlab

```
clear all, close all, clc

f = [0 0 .1 .2  .25  .2 .25 .3 .35 .43 .45 .5 .55  .5 .4 .425 .45 .425 .4 .35 .3 .25 .
g = [0 0 .025 .1  .2  .175 .2 .25 .25 .3 .32 .35 .375  .325 .3 .275 .275 .25 .225 .225

x = 0.1*(1:length(f));
```

```
xf = (.01:.01:x(end));
ff = interp1(x,f,xf,'cubic')

gf = interp1(x,g,xf,'cubic')


plot(xf(20:end-10),ff(20:end-10),'k','LineWidth',1.5)
hold on
plot(x(2:end-1),f(2:end-1),'bo','MarkerFace','b')
plot(xf(20:end-10),gf(20:end-10),'k','LineWidth',1.5)
plot(x(2:end-1),g(2:end-1),'ro','MarkerFace','r')


xlim([.1 2.7])
ylim([-.1 .6])
set(gca,'XTick',[.2:.1:2.6],'XTickLabels',{},'LineWidth',1.2)
set(gca,'YTick',[]);
box off

set(gcf,'Position',[100 100 550 250])

set(gcf,'PaperPositionMode','auto')
print('-depsc2', '-loose', '../figures/InnerProduct');

% %%
% xc = x;
% fc = f;
% n = length(x);
% hold on
% fapx = 0*ff;
% dx = xc(2)-xc(1);
% L = xc(end)-xc(1);
% L = 2.5
% A0 = (1/pi)*sum(fc.*ones(size(xc)))*dx*L;
% fapx = fapx + A0/2;
% for k=1:10
%     Ak = (1/pi)*sum(fc.*cos(2*pi*k*xc/L))*dx*L;
%     Bk = (1/pi)*sum(fc.*sin(2*pi*k*xc/L))*dx*L;
%
%     fapx = fapx + Ak*cos(2*k*pi*xf/L) + Bk*sin(2*k*pi*xf/L);
```

```
% end
%     plot(xf,fapx,'k')
```

6 Orthogonal Functions

- In vectors to check orthogonality we do dot product if dot product is zero then the vectors is orthogonal to each other
- $\vec{a}.\vec{b} = |\mathrm{a}||\mathrm{b}|\cos(\theta) = 0$
- mean $\theta = 90°$
- In functions we can do the same thing

6.1 Orthogonal Functions continue

- In function space if f and g are orthogonal to each other then inner-product is zero
- $\int_a^b \mathrm{f(x)g(x)\ dx} = 0$

6.2 Why Importent

- In vectorspace we represents vectors in terms of orthogonal basis
- Same can do in Function Space Represent any function interms of orthogonal functions
- One example of this is Fourier Transform
- It reprasent f(x) interms of orthogonal sins and cosins

7 Fourier Series

- It is a cordinate transformation
- It is made for solving heat equation in 1800s
- It decompose the signal f into sins and cosins
- sins and cosins are form a orthogonal basis for function space

7.1 Fourier Series

- Any periodic signals can be reprasent interms of sum of sins and cosins
- $$f(x) = \frac{A_0}{2} + \sum_{k=1}^{\infty}(A_k Cos(kx) + B_k Sin(kx))$$

7.2 FS continue

- It can be thought as ths
- $f(x) = \sum_{k=0}^{\infty} (<f(x),cos(kx)> \frac{cos(kx)}{||cos(kx)||^2} + <f(x),sin(kx)> \frac{sin(kx)}{||sin(kx)||^2})$

7.3 Fs

- $A_k = \frac{1}{||cos(kx)||^2} <f(x),cos(kx)>$
- $B_k = \frac{1}{||sin(kx)||^2} <f(x),sin(kx)>$
- $||f(x)||^2 = <f(x),f(x)>$

7.4 Complex Fourier Series

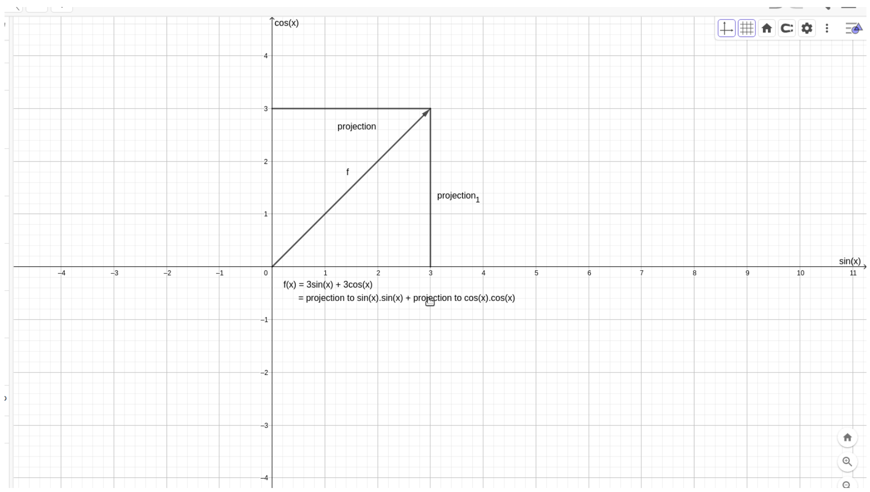
- it uses complex exponential to reprasent signal
- Coefficient can be found exactly same as that of fourier series
- project function into each complex exponential basis you get the coefficient c_k

7.5 Reprasentation

- $$f(x) = \sum_{k=-\infty}^{\infty} C_k e^{j\omega_0 kt}$$
- $C_k = \frac{1}{2\pi} <f(x),e^{jk\omega_0 t}>$

7.6 Example

- Assume we have a signal f(x) = 3sin(x) + 3cos(x) then it will look like this



7.7 Matlab

```
clear all, close all, clc

kmax = 7;

dx = 0.001;
L = pi;
x = (-1+dx:dx:1)*L;
f = 0*x;
n = length(f);
nquart = floor(n/4);
nhalf = floor(n/2);

f(nquart:nhalf) = 4*(1:nquart+1)/n;
f(nhalf+1:3*nquart) = 1-4*(0:nquart-1)/n;
subplot(3,1,1)
plot(x,f,'-','Color',[0 0 0],'LineWidth',1.5)
ylim([-.2 1.5])
xlim([-1.25*L 1.25*L])
set(gca,'LineWidth',1.2)
```

```
set(gca,'XTick',[-L 0 L],'XTickLabels',{});%{'-L','0','L','2L'})
set(gca,'YTick',[0 1],'YTickLabels',{});
box off

CC = colormap(jet(8));
% CCsparse = CC(5:5:end,:);
% CCsparse(end+1,:) = CCsparse(1,:);
CCsparse = CC(1:3:end,:);
%
subplot(3,1,2)
L = pi;
A0 = sum(f.*ones(size(x)))*dx;
plot(x,A0+0*f,'-','Color',CC(1,:)*.8,'LineWidth',1.2);
hold on
fFS = A0/2;
for k=1:kmax
    A(k) = sum(f.*cos(pi*k*x/L))*dx;
    B(k) = sum(f.*sin(pi*k*x/L))*dx;
    plot(x,A(k)*cos(k*pi*x/L),'-','Color',CC(k,:)*.8,'LineWidth',1.2);
%     plot(x,B(k)*sin(2*k*pi*x/L),'k-','LineWidth',1.2);
    fFS = fFS + A(k)*cos(k*pi*x/L) + 0*B(k)*sin(k*pi*x/L);
end
ylim([-.7 .7])
xlim([-1.25*L 1.25*L])
set(gca,'LineWidth',1.2)
set(gca,'XTick',[-L 0 L],'XTickLabels',{});%{'-L','0','L','2L'})
set(gca,'YTick',[-.5 0 .5],'YTickLabels',{});
box off
%
subplot(3,1,1)
hold on
plot(x,fFS,'-','Color',CC(7,:)*.8,'LineWidth',1.2)
l1=legend('     ','     ')
set(l1,'box','off');
l1.FontSize = 16;


subplot(3,1,3)
A0 = sum(f.*ones(size(x)))*dx;
plot(x,A0+0*f,'-','Color',CC(1,:),'LineWidth',1.2);
```

```
hold on
fFS = A0/2;
for k=1:7
    Ak = sum(f.*cos(pi*k*x/L))*dx;
    Bk = sum(f.*sin(pi*k*x/L))*dx;
    plot(x,Ak*cos(k*pi*x/L),'-','Color',CC(k,:)*.8,'LineWidth',1.2);
%     plot(x,Bk*sin(2*k*pi*x/L),'k-','LineWidth',1.2);
    fFS = fFS + Ak*cos(k*pi*x/L) + 0*Bk*sin(k*pi*x/L);
end
ylim([-.06 .06])
xlim([-1.25*L 1.25*L])
set(gca,'LineWidth',1.2)
set(gca,'XTick',[-L 0 L],'XTickLabels',{});%{'-L','0','L','2L'})
set(gca,'YTick',[-.05 0 .05],'YTickLabels',{});
box off

set(gcf,'Position',[100 100 550 400])
set(gcf,'PaperPositionMode','auto')
print('-depsc2', '-loose', '../figures/FourierTransformSines');

%% Plot amplitudes
clear ERR
clear A
fFS = A0/2;
A(1) = A0/2;
ERR(1) = norm(f-fFS);
kmax = 100;
for k=1:kmax
    A(k+1) = sum(f.*cos(2*pi*k*x/L))*dx*2/L;
    B(k+1) = sum(f.*sin(2*pi*k*x/L))*dx*2/L;
%     plot(x,B(k)*sin(2*k*pi*x/L),'k-','LineWidth',1.2);
    fFS = fFS + A(k+1)*cos(2*k*pi*x/L) + 0*B(k+1)*sin(2*k*pi*x/L);
    ERR(k+1) = norm(f-fFS)/norm(f);
end
thresh = median(ERR)*sqrt(kmax)*4/sqrt(3);
r = max(find(ERR>thresh));
r = 7;
subplot(2,1,1)
semilogy(0:1:kmax,A,'k','LineWidth',1.5)
hold on
```

```
semilogy(r,A(r+1),'bo','LineWidth',1.5)
xlim([0 kmax])
ylim([10^(-7) 1])
subplot(2,1,2)
semilogy(0:1:kmax,ERR,'k','LineWidth',1.5)
hold on
semilogy(r,ERR(r+1),'bo','LineWidth',1.5)
xlim([0 kmax])
ylim([3*10^(-4) 20])
set(gcf,'Position',[100 100 500 300])
set(gcf,'PaperPositionMode','auto')
% print('-depsc2', '-loose', '../figures/FourierTransformSinesERROR');
```

8 Fourier Transform

- Fourier series is for periodic signals
- If signal is not periodic then we can't use fourier series
- Fourier transform is limiting case of fourier series when L $\rightarrow \infty$

8.1 FT

- $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega x}dx$$
- $$F(\omega) = \int_{-\infty}^{infty} f(x)e^{-j\omega x}d\omega$$

8.2 Work in progress

9 Descrete Fourier Transform

- In real life the data sould be in measuremnts in some time
- We get time series insted of nice continues function
- So the descrete fourier transform is descritized version of fourier transform

9.1 DFT

- In dft the integration become summation
- DFT
- F(k) = $\sum_{n=0}^{N-1}$ $\mathrm{f_n}$ $\mathrm{e}^{-2\pi \ \mathrm{n} \ \frac{k}{N}}$
- k $\in$ 0 to N-1

9.2 Inverse DFT

- To come back to time series
- f(n) = $\sum_{k=0}^{N-1}$ $\mathrm{F_k}$ $\mathrm{e}^{2\pi \ \mathrm{k} \ \frac{n}{N}}$
- n $\in$ 0 to N-1

9.3 DFT

- let ω_{n} = $\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}}$
- Then we can reprasent DFT in matrix form

9.4 Matrics form

$$\begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_{n-1} \end{pmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & \omega & \dots & \omega^{N-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{n-1} & \dots & \omega^{(N-1)^2} \end{bmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix}$$

9.5 Beauty of matrices

- DFT matrix
- $$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & \omega & \dots & \omega^{N-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{n-1} & \dots & \omega^{(N-1)^2} \end{bmatrix}$$

9.6 Matlab code for DFT matrix

```
clear all, close all, clc
n = 256;
w = exp(-i*2*pi/n);

% Slow
for i=1:n
    for j=1:n
DFT(i,j) = w^((i-1)*(j-1));
    end
end

% Fast
[I,J] = meshgrid(1:n,1:n);
DFT = w.^((I-1).*(J-1));
imagesc(real(DFT))
```

9.7 Matlab Gibbs phenomena

```
clear all, close all, clc

kmax = 7;

dx = 0.001;
L = pi;
x = (-1+dx:dx:1)*L;
f = 0*x;
n = length(f);
nquart = floor(n/4);
nhalf = floor(n/2);

f(nquart:nhalf) = 4*(1:nquart+1)/n;
f(nhalf+1:3*nquart) = 1-4*(0:nquart-1)/n;
subplot(3,1,1)
plot(x,f,'-','Color',[0 0 0],'LineWidth',1.5)
ylim([-.2 1.5])
xlim([-1.25*L 1.25*L])
set(gca,'LineWidth',1.2)
```

```
set(gca,'XTick',[-L 0 L],'XTickLabels',{});%{'-L','0','L','2L'})
set(gca,'YTick',[0 1],'YTickLabels',{});
box off

CC = colormap(jet(8));
% CCsparse = CC(5:5:end,:);
% CCsparse(end+1,:) = CCsparse(1,:);
CCsparse = CC(1:3:end,:);
%
subplot(3,1,2)
L = pi;
A0 = sum(f.*ones(size(x)))*dx;
plot(x,A0+0*f,'-','Color',CC(1,:)*.8,'LineWidth',1.2);
hold on
fFS = A0/2;
for k=1:kmax
    A(k) = sum(f.*cos(pi*k*x/L))*dx;
    B(k) = sum(f.*sin(pi*k*x/L))*dx;
    plot(x,A(k)*cos(k*pi*x/L),'-','Color',CC(k,:)*.8,'LineWidth',1.2);
%     plot(x,B(k)*sin(2*k*pi*x/L),'k-','LineWidth',1.2);
    fFS = fFS + A(k)*cos(k*pi*x/L) + 0*B(k)*sin(k*pi*x/L);
end
ylim([-.7 .7])
xlim([-1.25*L 1.25*L])
set(gca,'LineWidth',1.2)
set(gca,'XTick',[-L 0 L],'XTickLabels',{});%{'-L','0','L','2L'})
set(gca,'YTick',[-.5 0 .5],'YTickLabels',{});
box off
%
subplot(3,1,1)
hold on
plot(x,fFS,'-','Color',CC(7,:)*.8,'LineWidth',1.2)
l1=legend('     ','     ')
set(l1,'box','off');
l1.FontSize = 16;


subplot(3,1,3)
A0 = sum(f.*ones(size(x)))*dx;
plot(x,A0+0*f,'-','Color',CC(1,:),'LineWidth',1.2);
```

```
hold on
fFS = A0/2;
for k=1:7
    Ak = sum(f.*cos(pi*k*x/L))*dx;
    Bk = sum(f.*sin(pi*k*x/L))*dx;
    plot(x,Ak*cos(k*pi*x/L),'-','Color',CC(k,:)*.8,'LineWidth',1.2);
%     plot(x,Bk*sin(2*k*pi*x/L),'k-','LineWidth',1.2);
    fFS = fFS + Ak*cos(k*pi*x/L) + 0*Bk*sin(k*pi*x/L);
end
ylim([-.06 .06])
xlim([-1.25*L 1.25*L])
set(gca,'LineWidth',1.2)
set(gca,'XTick',[-L 0 L],'XTickLabels',{});%{'-L','0','L','2L'})
set(gca,'YTick',[-.05 0 .05],'YTickLabels',{});
box off

set(gcf,'Position',[100 100 550 400])
set(gcf,'PaperPositionMode','auto')
print('-depsc2', '-loose', '../figures/FourierTransformSines');

%% Plot amplitudes
clear ERR
clear A
fFS = A0/2;
A(1) = A0/2;
ERR(1) = norm(f-fFS);
kmax = 100;
for k=1:kmax
    A(k+1) = sum(f.*cos(2*pi*k*x/L))*dx*2/L;
    B(k+1) = sum(f.*sin(2*pi*k*x/L))*dx*2/L;
%     plot(x,B(k)*sin(2*k*pi*x/L),'k-','LineWidth',1.2);
    fFS = fFS + A(k+1)*cos(2*k*pi*x/L) + 0*B(k+1)*sin(2*k*pi*x/L);
    ERR(k+1) = norm(f-fFS)/norm(f);
end
thresh = median(ERR)*sqrt(kmax)*4/sqrt(3);
r = max(find(ERR>thresh));
r = 7;
subplot(2,1,1)
semilogy(0:1:kmax,A,'k','LineWidth',1.5)
hold on
```

```
semilogy(r,A(r+1),'bo','LineWidth',1.5)
xlim([0 kmax])
ylim([10^(-7) 1])
subplot(2,1,2)
semilogy(0:1:kmax,ERR,'k','LineWidth',1.5)
hold on
semilogy(r,ERR(r+1),'bo','LineWidth',1.5)
xlim([0 kmax])
ylim([3*10^(-4) 20])
set(gcf,'Position',[100 100 500 300])
set(gcf,'PaperPositionMode','auto')
% print('-depsc2', '-loose', '../figures/FourierTransformSinesERROR');
```

9.8 Work in progres

10 FFT

- FFT is anlgorithm to compute DFT fast and efficiently
- It uses symetry in DFT
- To compute DFT Without FFT it require $\mathrm{O(n^2)}$ but FFT require only $\mathrm{O(nlog(n))}$

10.1

11 Gabor Transform

11.1 Limitations of Fourier transform

- FT is good for repreasenting smooth signal when there is sudden jump or discontinuity then it is not capture very well Gibbs phenomena
- FT is good for stationary signal
- Stationary means frequency of signal not change with time
- When we compute Fourier Transform we loss all of time information so we can't say when this frequency occured

- non stationary signals example is audio signal which frequency changes with time

11.2 Gabor transform

- it solve the problem of FT
- Gabor Transfom allow us to compute spectrogram a time frequency plot
- Also called windowed FT
- We take a window function multiply with the signal and translate the signal to get gabor transform

11.3 Gabor transform

- pull out both time and frequency content
- instead of computinf FT of entire signal we devide into several sections and compute FT of each section
- Mathamaticaly we can write
- $$G(f(t)) = \int_{\infty}^{\infty} f(\tau)e^{-i\omega\tau}g(\tau - t)d\tau$$
- g is the window function it can be gaussian or rectangular
- We can't know what frequency exist at what time instead but we can know what frequency band exist at what time

11.4 picture

- gabor grid
- `./gab.gif`

11.5 Problems of gabor transform

- Uncertainity principle
- It tells about when when you narrow the window you get better time resalution but you get poor frequency resalution
- when you stretch the window you get better frquency inforation but poor time information
- uncertainity principle tell us
- Δ t Δ f $\geq \frac{1}{4\pi}$

11.6 matlab code for spectrogram

```
clear all, close all, clc

n = 128;
L = 30;
dx = L/(n);
x = -L/2:dx:L/2-dx;
f = cos(x).*exp(-x.^2/25);                    % Function
df = -(sin(x).*exp(-x.^2/25) + (2/25)*x.*f);  % Derivative

%% Approximate derivative using finite Difference...
for kappa=1:length(df)-1
    dfFD(kappa) = (f(kappa+1)-f(kappa))/dx;
end
dfFD(end+1) = dfFD(end);

%% Derivative using FFT (spectral derivative)
fhat = fft(f);
kappa = (2*pi/L)*[-n/2:n/2-1];
kappa = fftshift(kappa);  % Re-order fft frequencies
dfhat = i*kappa.*fhat;
dfFFT = real(ifft(dfhat));

%% Plotting commands
plot(x,df,'k','LineWidth',1.5), hold on
plot(x,dfFD,'b--','LineWidth',1.2)
plot(x,dfFFT,'r--','LineWidth',1.2)
```

```
legend('True Derivative','Finite Diff.','FFT Derivative')
```

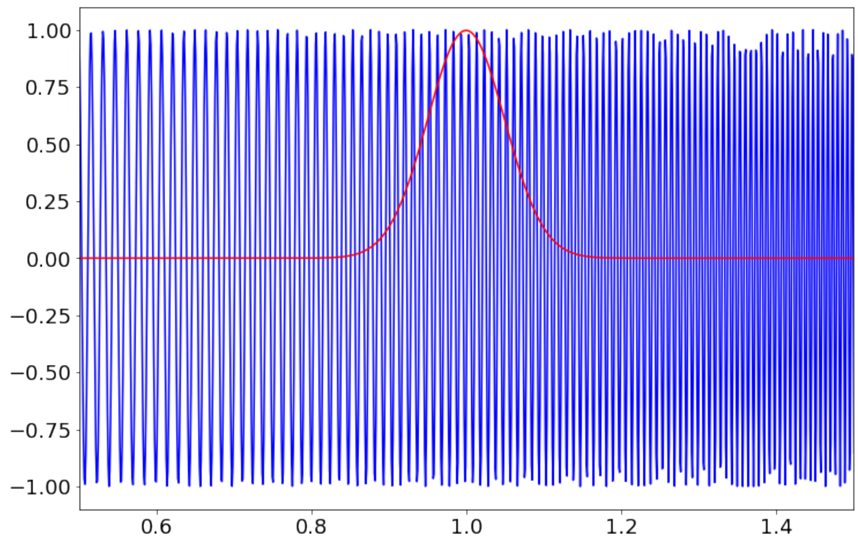
11.7 beethoven code matlab

```
clear all, close all, clc

% If you download mp3read, you can use this code
% also, need to download mp3read from
% http://www.mathworks.com/matlabcentral/fileexchange/13852-mp3read-and-mp3write
% [Y,FS,NBITS,OPTS] = mp3read('../../DATA/beethoven.mp3'); % add in your own song
% T = 40;            % 40 seconds
% y=Y(1:T*FS);       % First 40 seconds
load ../../DATA/beethoven_40sec.mat
%% Spectrogram
spectrogram(y,5000,400,24000,24000,'yaxis');

%% SPECTROGRAM
% uncomment remaining code and download stft code by M.Sc. Eng. Hristo Zhivomirov
% wlen = 5000;
% h=400;        % Overlap is wlen - h
% % perform time-frequency analysis and resynthesis of the original signal
% [S, f, t_stft] = stft(y, wlen, h, FS/4, FS);  % y axis range goes up to 4000 HZ
% imagesc(log10(abs(S)));
% load CC.mat
% colormap(ones(size(CC))-(CC))
%
% axis xy, hold on
% XTicks = [1 300 600 900 1200 1500 1800 2100];
% XTickLabels = {'0','5','10','15','20','25','30','35'};
% YTicks = [0 1000 2000 3000];
% YTickLabels = {'0','4000','8000','12000'};
% set(gca,'XTick',XTicks,'XTickLabels',XTickLabels);
% set(gca,'YTick',YTicks,'YTickLabels',YTickLabels);
%
% % plot a frequency
% freq = @(n)(((2^(1/12))^(n-49))*440);
% freq(40) % frequency of 40th key = C
```

11.8 Idea



12 Wavelet Transform

- supercharged Fourier transform
- Generalize Fourier transform
- Reprasent signals interms of other orthogonal functions
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12.1 Wavelet

- Wavelets are new basis functions also act as window function
- Wavelets are some wave like oscilationg functions in limited durations
- There are somany wavelets are avialable

12.2 Haar Wavelet