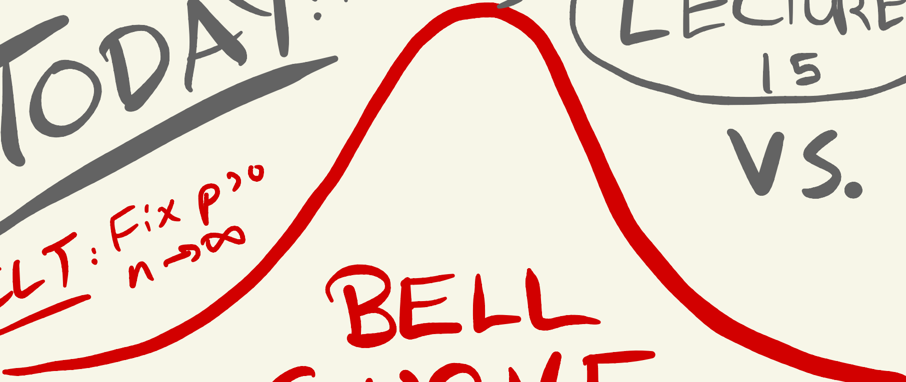
TODAY: LECTURE 15

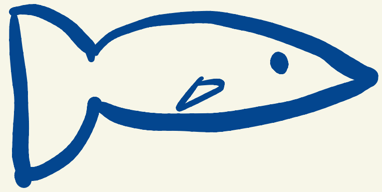


CLT: Fix $p > 0$, $n \to \infty$

BELL CURVE

VS.

Let $np \to \lambda$ as $n \to \infty$



POISSON

"FISH"

AMAT 362 - PROBABILITY for STATISTICS

Lecture 14 : CENTRAL LIMIT THEOREM

|| LAST LECTURE

→ Good for computing Binomial probabilities for large n

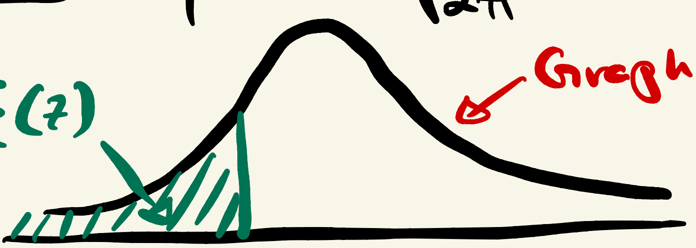
$$S_n = X_1 + \cdots + X_n$$

If I want to estimate for $S_n = X_1 + \cdots + X_n$ ← IID RVs w/ μ & σ

$$P(S_n = a) \approx \frac{\varphi\left(\frac{a - n\mu}{\sqrt{n\sigma^2}}\right)}{\sqrt{n\sigma^2}}$$

"Pointwise CLT"
assuming X_i integer valued

Rmk $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ PDF for Norm(0,1)



$\sqrt{n\sigma^2} = \sqrt{200 \cdot \frac{1}{2} \cdot \frac{1}{2}}$

Ex Flip fair coin 200 times

$$P(S_{200} = 100) \approx \frac{\varphi\left(\frac{100 - 200 \cdot \frac{1}{2}}{\sqrt{50}}\right)}{\sqrt{50}}$$

$$\approx \frac{1}{\sqrt{2\pi \cdot 50}} \approx 5.64\%$$

Ex $P(S_{200}=90) \approx \dfrac{\varphi\left(\frac{90-100}{\sqrt{50}}\right)}{\sqrt{50}} = \dfrac{1}{\sqrt{50}}\dfrac{1}{\sqrt{2\pi}} e^{-\left(\frac{-10}{\sqrt{50}}\right)^2 \frac{1}{2}}$

$= \dfrac{1}{10\sqrt{\pi}} e^{-1} \approx 2.07\%$

TODAY: Approximations that involve e

MARTIAN BIRTHDAY PROBLEM

A Martian year is 669 Martian Days

Q: What's the probability that in a room of 40 Martians, at least 2 share the same BDay?

$D = 669 \qquad n = 40$ # people / martians

A: Prob of no repeated bdays $= \left(\frac{D}{D}\right)\left(\frac{D-1}{D}\right)\left(\frac{D-2}{D}\right) \cdots \left(\frac{D-(n-1)}{D}\right)$

$= 1 \cdot \left(1-\frac{1}{D}\right)\left(1-\frac{2}{D}\right) \cdots \left(1-\frac{n-1}{D}\right)$

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

$\Rightarrow e^{-x} = 1 - x + \text{H.O.T.} \implies e^{-x} \approx 1 - x$ Good for small x

$\approx e^{-0} \cdot e^{-1/D} \cdot e^{-2/D} \cdots e^{-(n-1)/D}$

$= e^{-1/D - 2/D - 3/D \cdots - (n-1)/D}$

Use $1+2+3+\dots+n-1 = \frac{(n-1)n}{2}$

$\Rightarrow$ Prob of no repeats from n out of D possibilities $\approx e^{-\frac{(n-1)n}{2D}} \approx e^{-n^2/2D}$

So...

For $D=669$, $n=40$: $e^{-n^2/2D} \approx e^{-\frac{40^2}{2\cdot 669}} \approx 31.16\%$

$\Rightarrow$ At least 2 common Bdays $\boxed{68.8\%}$ For Mars

For Earthlings $D=365$, $n=40$: $e^{-40^2/2\cdot 365} \approx 11.8\%$

$\Rightarrow$ At least 2 common Bdays $\boxed{88.\%}$ For Earth

Q: How large does n need to be to get Prob$\left(\text{at least 2 common Bdays}\right) > 50\%$

A: $e^{-n^2/2D} < 1/2 \xrightarrow{\ln} -\frac{n^2}{2D} < -\ln 2$

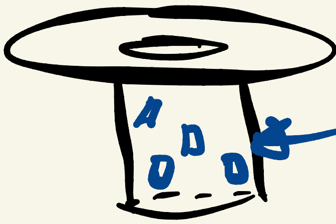
$n^2 > 2D\cdot\ln 2$

$\Rightarrow n > \sqrt{2D\cdot\ln 2}$

30.45

$n > 31$

Rmk You can apply this method



D tickets numbered $\{1, \dots, D\}$

Draw w/ replacement. What's the probability that by the n^{th} draw you've seen the same number come up at least twice

$$P(B_n) \approx 1 - e^{-\frac{n(n-1)}{2D}} \quad \underset{\substack{\text{exponent}\\ \text{approx}}}{\approx} \quad e^{-\frac{n^2}{2D}}$$

$$P(B_n) \approx 1 - e^{-n^2/2D}$$

POISSON APPROX. TO BINOMIAL

For even large n, if $p \approx 0$ then CLT is a poor approximation.

$$\boxed{P(S_n = k) = \underbrace{\binom{n}{k} p^k q^{n-k}}_{= b(n,p,k)} \approx \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{where } \lambda = np}$$

Aside on why this is true:

$$P(S_n = 0) = q^n = (1-p)^n \approx (e^{-p})^n = e^{-\lambda}$$

For small p $\quad \lambda = np$

Proof is by induction:

$$\text{N.B.} \quad \frac{b(n,p,k)}{b(n,p,k-1)} = \frac{n-k+1}{k} \cdot \frac{p}{1-p} = \frac{\lambda - (k-1)p}{k \cdot q}$$

$p \approx 0$, $q \approx 1$ $\Rightarrow \approx \frac{\lambda}{k}$

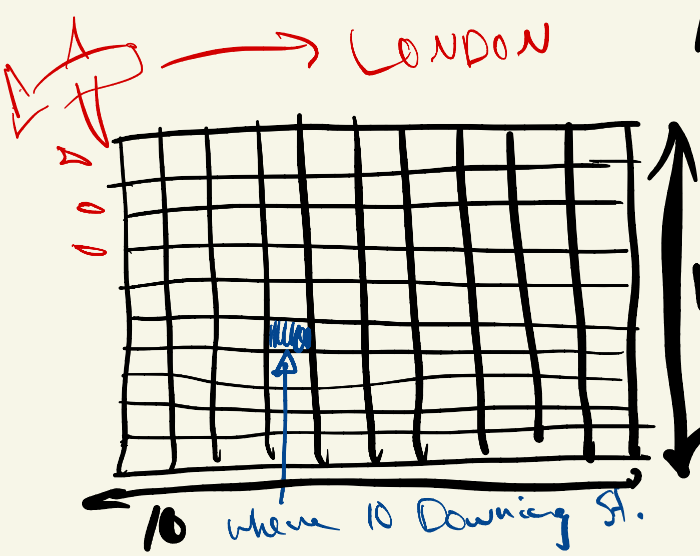
So if...

$$P(S_n=0) \approx e^{-\lambda}$$

then $P(S_n = 1) \approx \lambda e^{-\lambda}$ ($k=1$) and $P(S_n = 2) \approx \frac{\lambda}{2} \lambda e^{-\lambda} = \frac{\lambda^2}{2!} e^{-\lambda}$ ($k=2$)

APPLICATIONS

NAZIs BOMBING LONDON



Assume bombs dropped unif @ random

p = prob 10 Downing St block is bombed $= \frac{1}{100}$

1.8%

$$P(S_{400} = 0) \approx e^{-\lambda} = e^{-4}$$

400 bombs $= n$ $\quad \lambda = np = 4$

LEC 15, PHS

Exact Probability

$$b(400, p=\tfrac{1}{100}, k=0) = \left(1-\frac{1}{100}\right)^{400} = (.99)^{400}$$
$$= 1.7\%$$

vs 1.8% w/ e^{-4}

Q: What's prob 3 bombs hit?

$$\text{Exact Ans} = \binom{400}{3}(.01)^3(.99)^{397}$$

$$\text{Poisson Approx} \approx \frac{4^3}{3!}e^{-4} \approx 19.5\%$$

Typos in a Book

A typo occurs $\approx$ 1 out of 1000 words

$$\Longrightarrow p = \frac{1}{1000}$$

A page on a book has 200 words

Q: What's prob of at least one typo on a page?

$$\lambda = np = 200 \cdot \frac{1}{1000} = .2$$

$$1 - P(S_n = 0) \approx 1 - e^{-\lambda} = 1 - \underbrace{e^{-.2}}_{.818}$$

$n = 200$

$$\Longrightarrow 18.2\%$$

Prob of At least one typo!