MATH 362—Work Sheet 18

Dr. Justin M. Curry

Due Monday, April 19, 2021

Name: SOLUTIONS

1. (1 point) If a part has a lifetime modeled by $T \sim \text{Exp}(\lambda)$, prove the **memoryless property**, which says that

$$P(T > a + b \mid T > a) = P(T > b)$$

According to Pitman (p. 281) this is like saying

"As long as a part is working, it's as good as new!"

$$P(T > a+b \mid T > a) = \frac{P(T > a+b)}{P(T > a)} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}} = e^{-\lambda b} = P(T > b)$$

2. (4 points) One of the reasons exponential RVs are important is that they model the time *between* earthquakes. Suppose the time to the next earthquake is exponentially distributed with rate 1 per year. Find the probability that the next earthquake happens

(a) (1 point) within one year;

$$P(T < 1\text{yr}) = 1 - e^{-1} = 63\%$$

(b) (1 point) within six months;

$$P(T < \tfrac{1}{2}\text{yr}) = 1 - e^{-1/2} \approx 39\%$$

(c) (1 point) after two years;

$$P(T > 2\text{yr}) = e^{-2} \approx 13.5\%$$

(d) (1 point) after two years, given that one year has already gone by without an earthquake.

$$P(T>2 \mid T>1) = P(T>1) = \boxed{37\% = \frac{1}{e}}$$

3. (5 points) Suppose component lifetimes are exponentially distributed with mean 10 hours. Find

(a) (1 point) the probability that a component survives 20 hours;

$\mu = 10 \Rightarrow \lambda = \frac{1}{10}$

$$P(T>20) = \boxed{e^{-2} = 13.5\%}$$

(b) (1 point) the median component lifetime;

$$\text{med} = \frac{\ln 2}{\lambda} = \boxed{10 \cdot \ln 2 = 6.9 \text{ hours}}$$

(c) (1 point) the SD of component lifetime;

$$SD = \frac{1}{\lambda} = 10 \text{ hours}$$

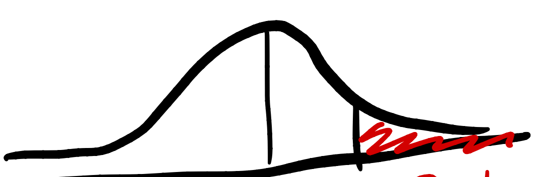
(d) (1 point) The probability that the average lifetime of 100 independent components exceeds 11 hours;

$$SD(A_n) = \frac{SD(X_i)}{\sqrt{n}} = \frac{1}{\lambda\sqrt{n}}$$

so $SD(A_{100}) = \frac{10}{10} = 1$

16%

$= 1 - \Phi(1)$



P Avg > 11 hrs

(e) (1 point) The probability that the average lifetime of 2 independent components exceeds 11 hours;

$$P\left(\frac{X_1+X_2}{2} > 11\right) = P(X_1+X_2 > 22)$$

Gamma Distribution

By Poisson Point Process $P(0 \text{ or } 1 \text{ failures in } 22 \text{hours})$

$$= \text{Poisson}_{22(\frac{1}{10})}(0) + \text{Poisson}_{22(1/10)}(1) = \boxed{e^{-2.2} + 2.2e^{-2.2} = 35.45\%}$$

4. (3 points) A store is open from 9am-6pm and averages 45 customers a day.

(a) (1 point) Compute the probability of no customers arriving between 9 and 10am. Call this event A_1.

9 hours 45 customers $\Rightarrow \lambda = 5 \frac{\text{customers}}{\text{hr}}$

$$\text{Poissin}_{5 \cdot 1hr}(0) = \boxed{e^{-5} = .67\%}$$

(b) (1 point) Compute the probability of 3 customers arriving between 10 and 10:30am. Call this event A_2.

$$\text{Poisson}_{5 \cdot (\frac{1}{2})}(3) = \boxed{\frac{(\frac{5}{2})^3}{3!} e^{-5/2} = 21.40\%}$$

(c) (1 point) Compute the probability $P(A_1 \cap A_2)$.

Since non-overlapping the processes are independent

$$P(A_1 \cap A_2) = P(A_1)P(A_2) = \boxed{\frac{(\frac{5}{2})^3}{3!} e^{-7.5}}$$

5. (3 points) For this problem you'll want to know that the probability distribution for T_r, which is the time of the r^{th} arrival in a Poisson Point Process with rate λ, or, alternatively, the distribution of $W_1 + \cdots + W_r$ the sum of r IID exponentials, has PDF

$$f_{T_r}(t) = \frac{\lambda^r t^{r-1}}{(r-1)!} e^{-\lambda t} \qquad \text{for } t \geq 0$$

and has mean r/λ and standard deviation $\sqrt{r}/\lambda$.

Suppose calls are arriving at a call center with an average rate of 1 call per second. Find:

$\lambda = 1$ call/sec

This is a trick question, b/c time between calls is just exponential.

(a) (1 point) the probability that the fourth call after $t = 0$ arrives within 2 seconds of the third call;

$$P(0<T<2) = 1 - e^{-\lambda 2} = 1 - e^{-2} = \boxed{86.5\%}$$

for $T \sim Exp(\lambda)$
$\lambda = 1$

(b) (1 point) the probability that the fourth call arrives by times $t = 5$ seconds;

$$P(T_4 < 5) = 1 - P(T_4 > 5) = 1 - P(0 \text{ or } 1 \text{ or } 2 \text{ or } 3 \text{ in } 5 \text{ secs})$$
$$= 1 - \left[e^{-5} + 5e^{-5} + \frac{5^2}{2!}e^{-5} + \frac{5^3}{3!}e^{-5}\right]$$

(c) (1 point) the expected time at which the fourth call arrives.

$$E[T_4] = \frac{4}{\lambda} = 4 \text{ secs}$$

6. (4 points) Transistors are produced by one machine have a lifetimes that is exponentially distributed with mean 100 hours. Transistors produced by a second machine have lifetime with exponential distribution and mean 200 hours. A package of 12 transistors has 4 produced by the first machine and 8 produced by the second machine. Let X be the lifetime of a randomly selected transistor from this package of 12. Find:

(a) (1 point) $P(X \geq 200 \text{ hours})$

$$\frac{4}{12} e^{-\frac{1}{100}\cdot 200} + \frac{8}{12} e^{-\frac{1}{200}\cdot 200} = 29\%$$

(b) (1 point) $E(X)$

$\frac{1}{\lambda}$

$$\frac{4}{12}\cdot 100 + \frac{8}{12}\cdot 200 = 166.\bar{6} \text{ hrs}$$

$$E[X^2] = E[X^2|M_1]P(M_1) + E[X^2|M_2]P(M_2)$$
$$= 2/\lambda_1^2 \cdot P(M_1) + \frac{2}{\lambda_2^2} P(M_2) = 2\cdot 100^2 \cdot \frac{4}{12} + 2\cdot 200^2 \cdot \frac{8}{12}$$

(c) (2 points) $Var(X)$

$$E[X^2] = 100^2\left(\frac{8}{12} + \frac{64}{12}\right) = 6\cdot 100^2$$

$$\Rightarrow VAR = 6\cdot 100^2 - (166.\bar{6})^2 \longrightarrow \boxed{32{,}221.\bar{T}} \Rightarrow \text{SDEV} = 179.5$$