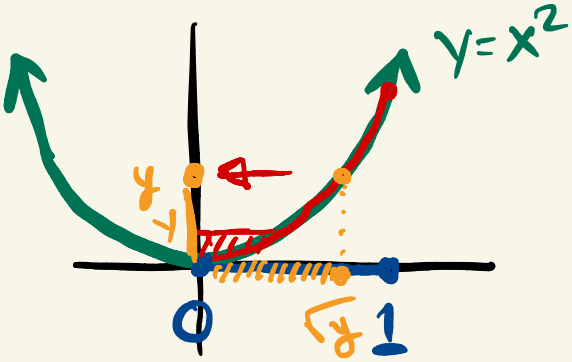
LECTURE 20 LIVE

→ WS19 First
→ WS20 (NOT DUE) 2 Questions
See Pitman §4.4 Ex 2 & 3

WS19 Q4

$X \sim \text{Unif}(0,1)$ Find PDF for $Y = X^2$



METHOD #1 = CDF

$$F_Y(y) = \int_{-\infty}^{y} f_Y(y')\,dy'$$

$$= P(Y \le y)$$

$$= P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

BUT $P(X<0) = 0$

$$= P(X \le \sqrt{y})$$

$$= \int_0^{\sqrt{y}} 1\,dx$$

$F_Y(y) = \sqrt{y}$

$$\boxed{\frac{d}{dy}\text{CDF} = \text{PDF}}$$

$$f_Y(y) = \frac{d}{dy}(\sqrt{y}) = \boxed{\frac{1}{2\sqrt{y}}} \quad \text{for } 0 < y < 1$$

0 o.w.

If $Y = g(X)$ w/ g 1-to-1 on supp(f) then

$$f_Y(y) = f_X(\underbrace{g^{-1}(y)}_{\text{inverse of } y \text{ under } g}) \cdot \frac{1}{|g'(g^{-1}(y))|}$$

derivative of g

Q4 METHOD #2

$y = g(x) = x^2$

$$f_X = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

So... $0 < y < 1$ $g^{-1}(y) < (-1, 1)$

BUT support of f_X is $(0,1)$

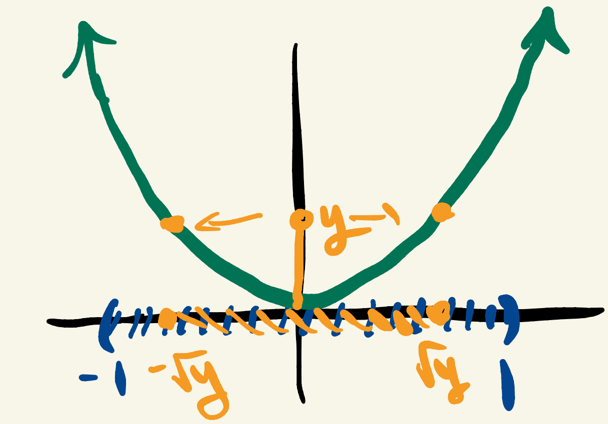
b/c Unif $\sim (0,1)$

$$f_Y(y) = 1 \cdot \frac{1}{|g'(\underbrace{g^{-1}(y)}_{= \sqrt{y}})|} = \boxed{1 \cdot \frac{1}{2\sqrt{y}}}$$

$g'(x) = \frac{d}{dx}(x^2)$

$= 2x$ $x = \sqrt{y}$ b/c $y = x^2$

Q5 $X \sim \text{Unif}(-1,1)$ $Y = X^2$



$$f_X(x) = \begin{cases} \frac{1}{2} & -1<y<1 \\ 0 & \text{o.w.} \end{cases}$$

METHOD #1 = CDF

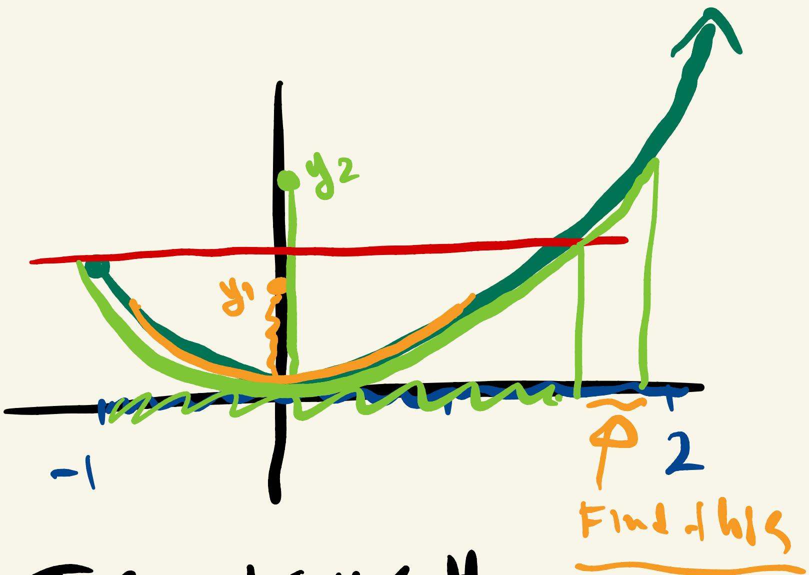
$$F_Y(y) = P(Y \le y) = P(X^2 \le y)$$
$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

$$\sqrt{y} = \frac{1}{2}\cdot\left(\sqrt{y} - (-\sqrt{y})\right) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}\,dx$$

$$\rightsquigarrow f_Y(y) = \frac{d}{\sqrt{y}}\left(\sqrt{y}\right)$$
$$= \begin{cases} \frac{1}{2\sqrt{y}} & 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

~~BONUS~~ Q6

$X \sim \text{Unif}[-1, 2] \qquad Y = X^2$



If $0 \le y \le 1$

$$P(Y \le y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{3}\,dx$$

$$P(Y \le y) = \frac{2\sqrt{y}}{3}$$

If $1 < y \le 4$

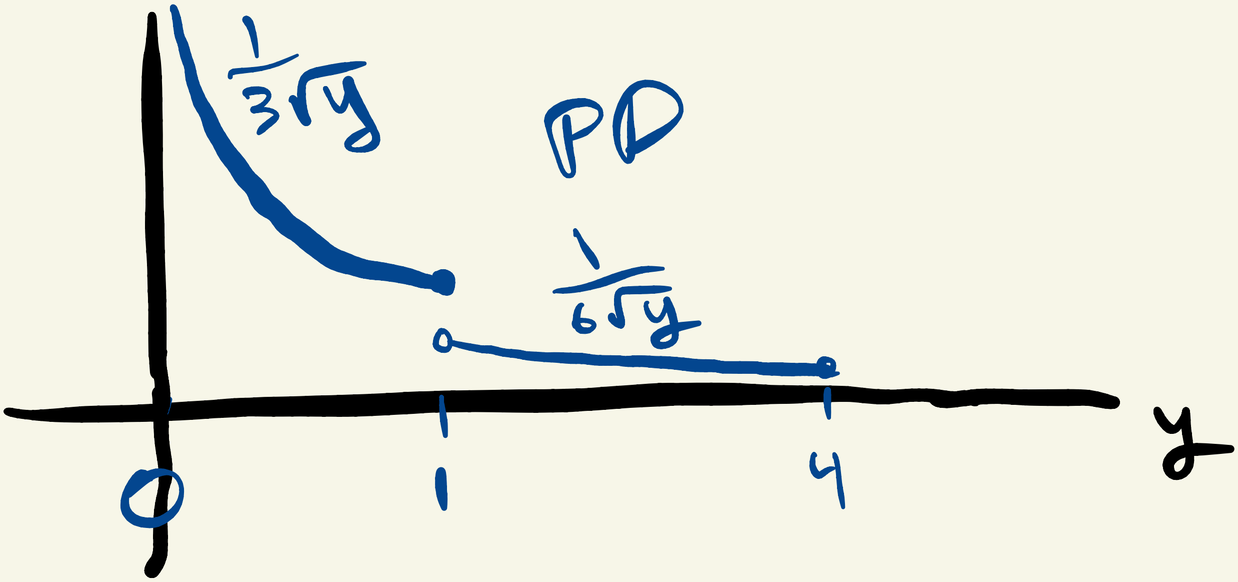
$$P(Y \le y) = \underbrace{P(Y \le 1)}_{= 2/3} + \underbrace{P(1 \le Y \le y)}_{= \int_{1}^{\sqrt{y}} \frac{1}{3}\,dx}$$

$$P(Y \le y) = \frac{2}{3} + (\sqrt{y} - 1)\frac{1}{3}$$

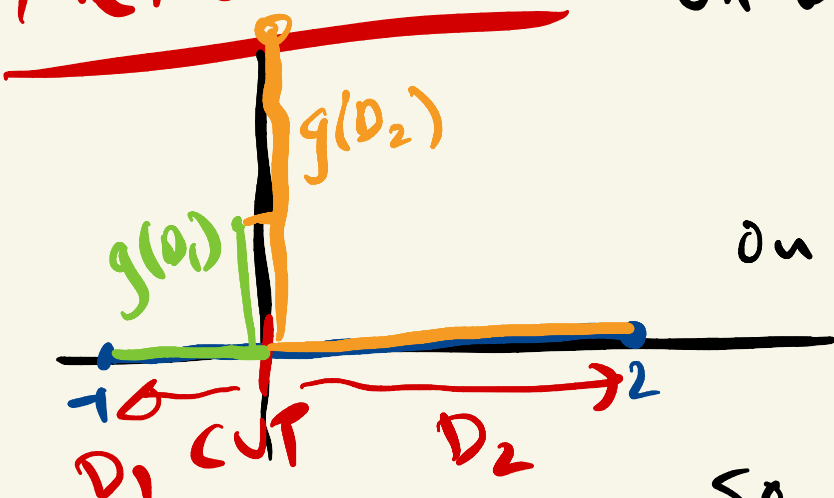
$\frac{d}{dy}\text{CDF} = \text{PDF}$ ... So... for $0 \le y \le 1$ $\quad \frac{d}{dy}\left(\frac{2}{3}\sqrt{y}\right) = \boxed{\frac{1}{3\sqrt{y}}}$

For $1 < y \le 4$

$$\frac{d}{dy}\left(\cancel{\frac{2}{3}} + \frac{1}{3}\left(\sqrt{y} - \cancel{1}\right)\right) = \frac{1}{6\sqrt{y}} \quad \text{for } y \in (1,4)$$



Method #2



On D_1 $f_Y(y) = f_X(g^{-1}(y))\Big|_{D_1} \frac{1}{|g'(\ldots)|}$

$= \frac{1}{3} \frac{1}{2\sqrt{y}}$

On D_2 $f_Y(y) = \frac{1}{3} \frac{1}{2\sqrt{y}}$

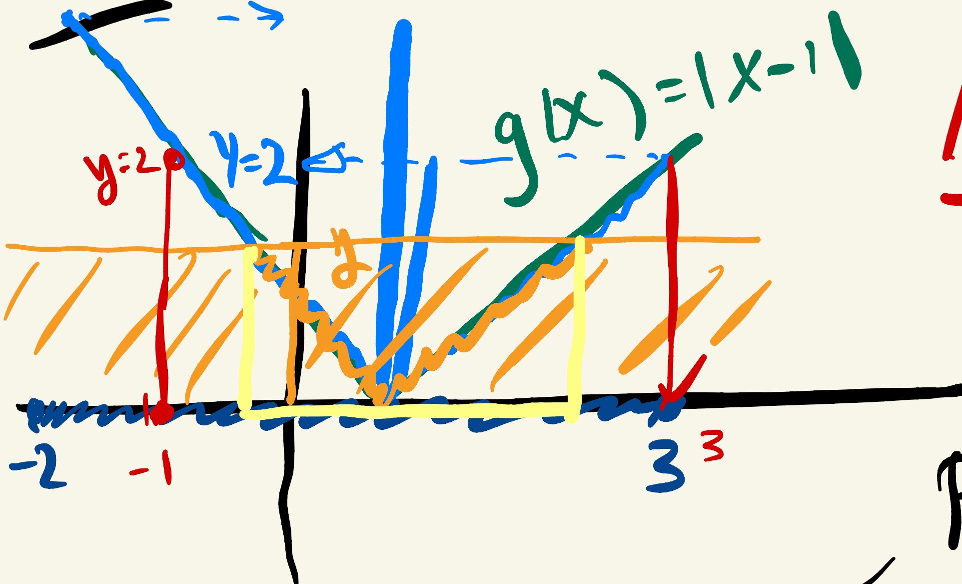
So add them together where defined for $\underline{y}$

$g(D_1) = [0,1]$

$g(D_2) = [0,4]$

$$= \begin{cases} \frac{1}{3\sqrt{y}} & 0 \le y \le 1 \\ \frac{1}{6\sqrt{y}} & 1 < y \le 4 \end{cases}$$

Q7 $X \sim Unif[-2,3]$ $Y = |X-1|$



METHOD 1 = CDF

$P(Y \le y)$ for $y \le 2$

$P(|X-1| \le y)$

$$P(-y \le X-1 \le y) = P(1-y \le X \le 1+y)$$

y = 2 ✓

$$= \int_{1-y}^{1+y} f_X(x)\,dx = \frac{1}{5}\left[(\cancel{1}+y) - (\cancel{1}-y)\right]$$

$$\boxed{F_Y(y) = \frac{2y}{5} \text{ for } 0 \le y \le 2}$$

Plugin y = 2

For $2 < y \le 3$

$$P(Y \le y) = \underbrace{P(Y \le 2)}_{\frac{4}{5}} + P(2 < Y < y)$$

$$= P(2 < |X-1| < y)$$

$$= P(1-y < X < -1)$$

$$\int_{1-y}^{-1} \frac{1}{5}\,dx = \frac{1}{5}\left[-1-(1-y)\right]$$

$$\boxed{= \frac{1}{5}[y-2]}$$

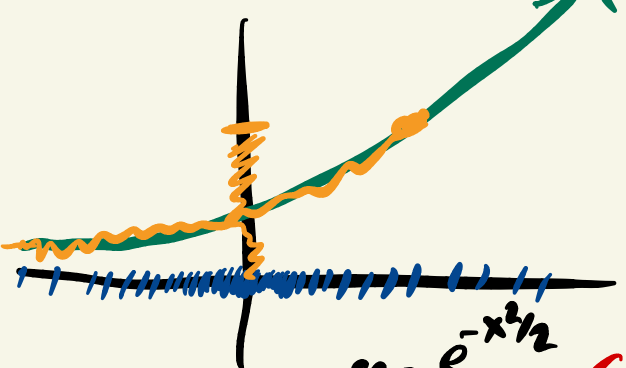
CDF for Y

$$F_Y(y) = \begin{cases} 0 & \text{for } y<0 \\ \frac{2y}{5} & \text{for } 0\le y\le 2 \\ \frac{4}{5} + \frac{1}{5}[y-2] & \text{for } 2\le y\le 3 \\ 1 & y\ge 3 \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{2}{5} & 0\le y\le 2 \\ \frac{1}{5} & 2< y\le 3 \\ 0 & \text{o.w.} \end{cases}$$

<u>Q8</u> $X \sim \text{Norm}(0,1)$ $\quad Y = e^X = g(X)$

$$P(Y\le y) = P(e^X \le y)$$
$$= P(X \le \ln y)$$
$$= \Phi(\ln y)$$



$X \sim \text{Norm}(0,1)$

$\varphi = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$ CHAIN RULE

$$\frac{d}{dy}\Phi(\ln y) = \varphi(\ln y)\cdot\frac{d}{dy}\ln(y)$$

$$F_Y(y) = \Phi(\ln y)$$

$$\frac{d}{dy}\left[\Phi(\ln y)\right] = \Phi'(\ln y)\frac{d}{dy}(\ln y)$$

$$= \boxed{\frac{e^{-\frac{(\ln y)^2}{2}}}{\sqrt{2\pi}} \cdot \frac{1}{y} \qquad 0 < y < \infty}$$

log-Normal Distribution