# AMAT 362 - Probability for Statistics
@UALBANY

# LECTURE 22:

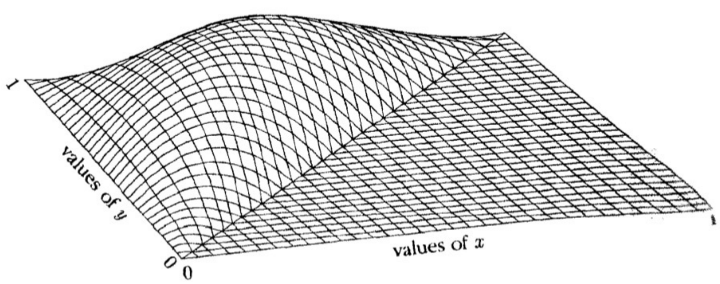
ONLY 2 MORE LECTURES!

- Joint Density Functions
- Marginalizations
- Independence
- Where $\sqrt{2\pi}$ comes from
  ↳ Two or More Normal RVs!

## What do more complicated JDFs look like?
(other than uniform)

The concept of a *joint probability density function* $f(x, y)$ for a pair of random variables $X$ and $Y$ is a natural extension of the idea of a one-dimensional probability density function studied in Chapter 4. The function $f(x, y)$ gives the density of probability per unit area for values of $(X, Y)$ near the point $(x, y)$.

FIGURE 1. **A joint density surface.** Here a particular joint density function given by the formula $f(x,y) = 5!\,x(y-x)(1-y) \quad (0 < x < y < 1)$, is viewed as the height of a surface over the unit square $0 \le x \le 1, 0 \le y \le 1$. As explained later in Example 3, two random variables $X$ and $Y$ with this joint density are the second and fourth smallest of five independent uniform $(0, 1)$ variables. But for now the source and special form of this density are not important. Just view it as a typical joint density surface.

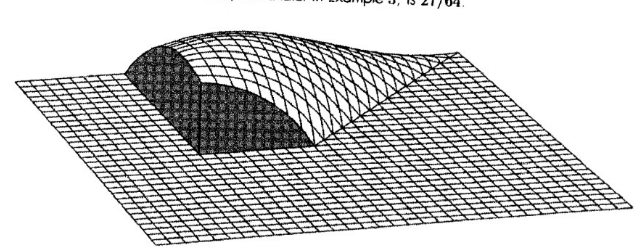



Examples in the previous section show how any event determined by two random variables $X$ and $Y$, like the event ($X > 0.25$ and $Y > 0.5$), corresponds to a region of the plane. Now instead of a uniform distribution defined by relative areas, the probability of region $B$ is defined by the volume under the density surface over $B$. This volume is an integral

$$P((X,Y) \in B) = \iint_B f(x,y)\,dx\,dy$$

This is the analog of the familiar area under the curve interpretation for probabilities obtained from densities on a line. Examples to follow show how such integrals can be computed by repeated integration, change of variables, or symmetry arguments. Uniform distribution over a region is now just the special case when $f(x, y)$ is constant over the region and zero elsewhere. As a general rule, formulae involving joint densities are analogous to corresponding formulae for discrete joint distributions described in Section 3.1. See pages 348 and 349 for a summary.

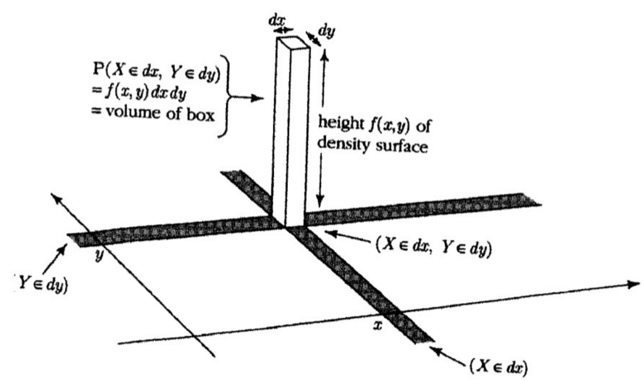
FIGURE 2. **Volume representing a probability.** The probability $P(X > 0.25 \text{ and } Y > 0.5)$, for random variables $X$ and $Y$ with the joint density of Figure 1. The set $B$ in this case is $\{(x, y) : x > 0.25 \text{ and } y > 0.5\}$. You can see the volume is about half the total volume under the surface. The exact value, found later in Example 3, is 27/64.



Informally, if $(X, Y)$ has joint density $f(x, y)$, then there is the *infinitesimal probability formula*

$$P(X \in dx, Y \in dy) = f(x,y)\,dx\,dy$$

This means that the probability that the pair $(X, Y)$ falls in an infinitesimal rectangle of width $dx$ and height $dy$ near the point $(x, y)$ is the probability density at $(x, y)$ multiplied by the area $dx\,dy$ of the rectangle.



# DICTIONARY B/W DISCRETE AND CONTINUOUS PROBABILITY (2 or more RVs)

## Discrete Joint Distribution

**Probability of a point:** $P(X=x \text{ AND } Y=y)$

$$P(X=x, Y=y) = P(x,y)$$

The joint probability $P(x,y)$ is the probability of the single point $(x,y)$.

**Probability of a set $B$:** The sum of probabilities of points in $B$

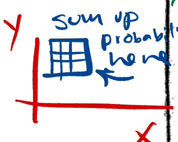
$$P((X,Y) \in B) = \sum_{(x,y)\in B} P(x,y)$$

Ex: 5 Red 5 Green balls, draw out 5 balls w/o replacement $X = \#$ Reds $Y = \#$ Greens

Q: $P(1 \le X \le 3 \text{ AND } 2 \le Y \le 4)$?

**Constraints:** Non-negative with total sum 1

$$P(x,y) \ge 0 \quad \text{and} \quad \sum_{\text{all } x}\sum_{\text{all } y} P(x,y) = 1$$




**Marginals:**

Important for today

$$P(X=x) = \sum_{\text{all } y} P(x,y)$$

$$P(Y=y) = \sum_{\text{all } x} P(x,y)$$

$$\sum \longrightarrow \int$$

**Independence:** $\boxed{P(x,y) = P(X=x)P(Y=y)}$ (for all $x$ and $y$)

**Expectation of a function $g$ of $(X,Y)$, e.g., $XY$,**

$$E(g(X,Y)) = \sum_{\text{all } x}\sum_{\text{all } y} g(x,y)P(x,y)$$

provided the sum converges absolutely.

## Joint Distribution Defined by a Density

**Infinitesimal probability:**

$$P(X \in dx, Y \in dy) = f(x,y)\,dx\,dy$$

Infinitesimal Event dx AND dy

The joint density $f(x,y)$ is the probability per unit area for values near $(x,y)$.

**Probability of a set B:** The volume under the density surface over $B$

$$P((X,Y) \in B) = \iint_B f(x,y)\,dx\,dy$$

Integrate

**Constraints:** Non-negative with total integral 1

Normalization Axiom

$$f(x,y) \ge 0 \quad \text{and} \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

Challenge: Check $\iint 5!\,x(y-x)(1-y) = 1$ $0<x<y<1$

**Marginals:**

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

**Independence:** $\boxed{f(x,y) = f_X(x)f_Y(y) \quad \text{(for all } x \text{ and } y)}$ Important

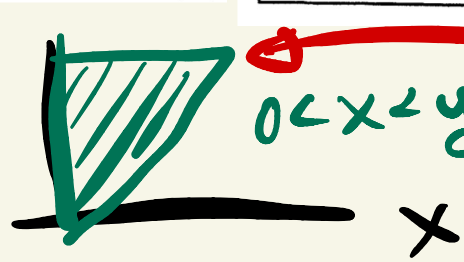
**Expectation of a function $g$ of $(X,Y)$, e.g., $XY$** ← $E[XY]$

$$E(g(X,Y)) = \iint g(x,y)f(x,y)dx\,dy$$

$xy\,f(x,y)$

provided the integral converges absolutely.

## Example




$0<x<y<1$

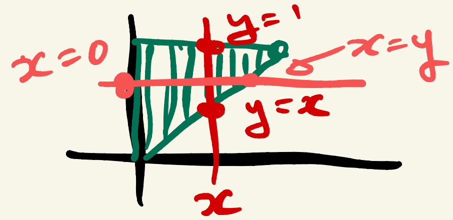
Area of Triangle $= \frac{1}{2} \cdot h \cdot b = 1/2$

$$f(x,y) = \begin{cases} 2 & \text{for } 0<x<y<1 \\ 0 & \text{o.w.} \end{cases}$$

**Q:** Where does 2 come from?

**A:** $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) = 1$ In order for Normalization Axiom to hold!

# Example Ct'd

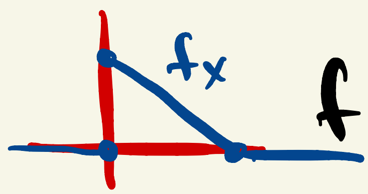



Find PDFs for X & Y

$$f_X(x) = \int_{y=-\infty}^{\infty} f(x,y)\,dy$$

What should the limits be?

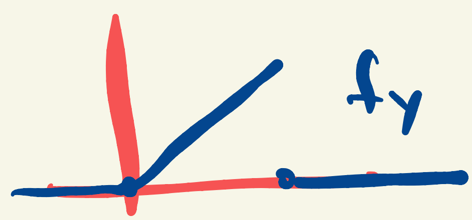
$$= \int_{y=x}^{y=1} 2\,dy = 2y\Big]_{y=x}^{y=1} = 2-2x = 2(1-x)$$




$$f_X(x) = \begin{cases} 2(1-x) & 0<x<1 \\ 0 & \text{o.w} \end{cases}$$

$$f_Y(y) = \int_{x=0}^{x=y} 2\,dx = 2x\Big]_{x=0}^{x=y} = 2y$$

$$f_Y(y) = \begin{cases} 2y & 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$




Q: Indep?

A: NO b/c $\text{supp}(f_{X,Y}) \neq (\text{supp} f_X) \times (\text{supp} f_Y)$

$$\boxed{\text{INDEP} \iff f_{X,Y}(x,y) = f_X(x)\,f_Y(y) \quad \forall x,y}$$

$2 \neq 2(1-x)2y$

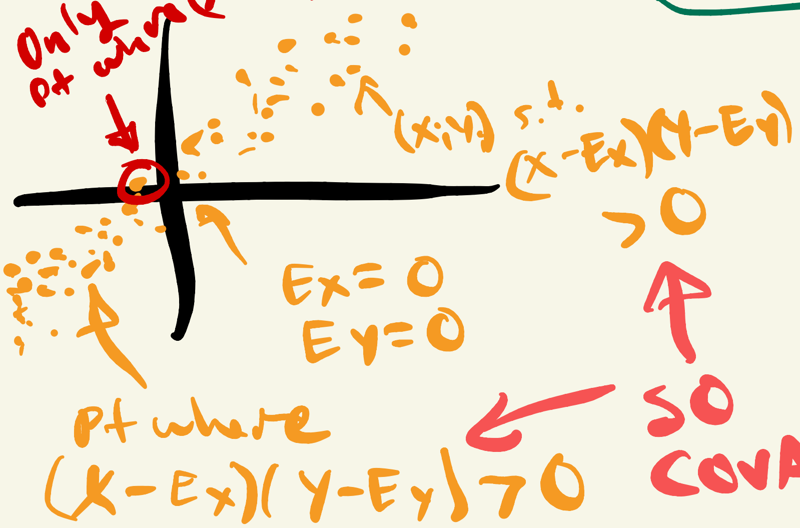
$\forall x$ & $y$

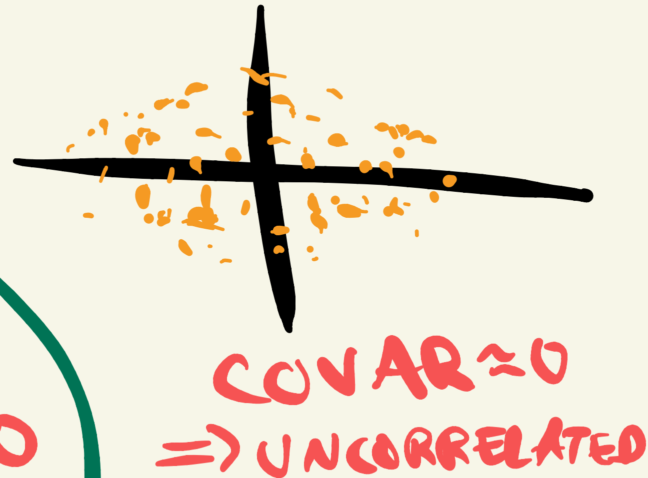
$2(1-\frac{1}{2})\,2\cdot\frac{3}{4} = \frac{3}{2}$

Q: $\mathrm{Cov}(X,Y) = ?$

$$\boxed{\mathrm{Cov}(X,Y) = 0 \iff X \,\&\, Y \text{ INDEP}}$$

$$\mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$




$$E[XY - X\mu_Y - \mu_X Y + \mu_X\mu_Y]$$

$$= E[XY] - \mu_Y E[X] - \mu_X E[Y] + \mu_X\mu_Y$$

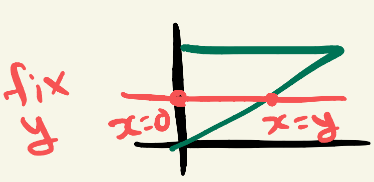
$$- \mu_Y\mu_X \quad \cancel{- \mu_X\mu_Y} \quad \cancel{+ \mu_X\mu_Y}$$

★★★

$$\boxed{\mathrm{COVAR} = E[XY] - E[X]E[Y]}$$

Ex

$$E[XY] = \iint xy\, f_{X,Y}(x,y)\,dx\,dy = 2\int_{y=0}^{y=1}\int_{x=0}^{x=y} xy\,dx\,dy = 2\int_{y=0}^{y=1} \frac{x^2 y}{2}\Big|_{x=0}^{x=y} dy$$




$$= 2\int_0^1 y^3/2\,dy = \frac{y^4}{4}\Big|_0^1 = 1/4$$

$$E(X) = \int_0^1 x f_X(x)\,dx$$

$$= \int_0^1 x\,[2(1-x)]\,dx$$

$$\int_0^1 2x - 2x^2 = x^2 - \frac{2x^3}{3}\Big|_0^1$$

$$\boxed{E[X] = 1 - \frac{2}{3} = 1/3}$$

$$E[Y] = \int_0^1 y f_Y(y)\,dy$$

$$\ldots = \frac{2}{3}$$

$$\rightarrow \mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$= \frac{1}{4} - \frac{1}{3}\cdot\frac{2}{3}$$

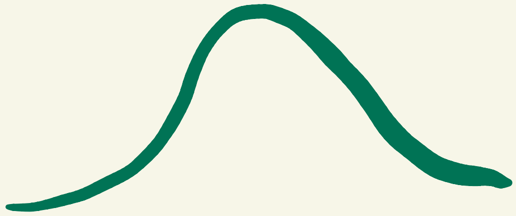
$$= \frac{9}{36} - \frac{8}{36} = \boxed{\frac{1}{36}}$$

**HW** If $X \sim \mathrm{Exp}(\lambda)$ and $Y \sim \mathrm{Exp}(\mu)$ ARE INDEP then $\rightarrow P(X>Y)$


$\leftarrow$ Volume under $f_{X,Y} = \lambda\mu e^{-\lambda x - \mu y}$ for this area

$$= P(X>Y)$$

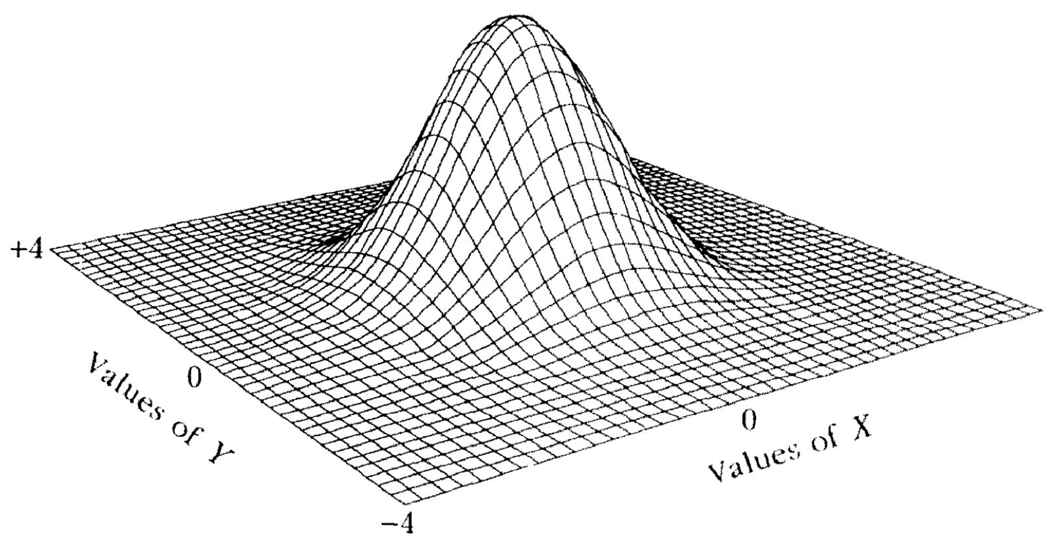
# INDEPENDENT NORMAL RVs



$X \sim \text{Norm}(\mu_X, \sigma_X^2)$ $\qquad$ $Y \sim \text{Norm}(\mu_Y, \sigma_Y^2)$

$\Rightarrow$ Result of $f_X(x)\, f_Y(y) \underset{\text{b/c indep}}{=} f_{X,Y}(x,y)$

FIGURE 1. Perspective plot of the joint density of $X$ and $Y$.




Let's Assume $\mu_X = \mu_Y = 0 \quad \sigma_X = \sigma_Y = 1$

i.e. both unit normal distributions

**Q**: What is $\displaystyle\int_{-\infty}^{\infty} e^{-x^2/2}\, dx$ ?

Approaching this directly is

IMPOSSIBLE!

B/c $e^{-x^2}$ has no elementary anti-deriv.

**Trick!**

Can compute $\left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)^2$ ★Magic★

$$= \left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2/2}\,e^{-y^2/2}\,dx\,dy$$

$$= \iint e^{-(x^2+y^2)/2}\,\underline{dx\,dy}$$

(More magic! CARTESIAN → POLAR)

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\,\underline{r\,dr\,d\theta} \qquad \boxed{x^2+y^2=r^2}$$

Try: $\frac{d}{dr}\left[e^{-r^2/2}\right] = e^{-r^2/2}\cdot\underbrace{\frac{d}{dr}\left(-r^2/2\right)}_{-\frac{\cancel{2}r}{\cancel{2}}}$

$$= -re^{-r^2/2}$$

$$2\pi\left[-e^{-r^2/2}\right]_0^{\infty}$$

$$\Rightarrow \boxed{\left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)^2 = 2\pi} \Rightarrow \int_{-\infty}^{\infty} e^{-x^2/2} = \sqrt{2\pi}$$

Takeaways

If $X$ & $Y$ are indep unit normals

then $f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$

in POLAR coords $f_{R,\theta}(r,\theta) = \frac{1}{2\pi} e^{-r^2/2}$

Marginalize w/rt $R$

$$f_R(r) = \int_0^{2\pi} \frac{1}{2\pi} e^{-r^2/2} \, r \cancel{\phantom{x}} \, d\theta$$

$r$ = const.

$$\boxed{f_R(r) = \begin{cases} r e^{-r^2/2} & r > 0 \\ 0 & \text{o.w.} \end{cases}}$$