Lecture 10 - LIVE -

WS10

(1) Toss a fair coin 3 times

↳ Heads ⟹ \$3 +

↳ Tails ⟹ Ø

\$5 Buy in

(a) Should you play?

Erick says no

$$P(X = \#\text{heads} \geq 2) = 50\%$$

Flip coin 3 times

↳ Ω HHH

$$3 \cdot \frac{1}{8} + \frac{1}{8} = \frac{4}{8}$$

$\cancel{3^2}$

$2 \times 2 \times 2 = 2^3 = 8$

$$P(X=2) = \binom{3}{2}\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)$$

$+$

$$P(X=3) = \binom{3}{3}\left(\tfrac{1}{2}\right)^3\cancel{\left(\tfrac{1}{2}\right)^0}\ 1$$

Payout table

$3X$

\$0	\$3	\$6	\$9
$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$E(3X) = 0\cdot\frac{1}{8} + 3\cdot\frac{3}{8} + 6\cdot\frac{3}{8} + 9\cdot\frac{1}{8}$$

$$= \frac{9 + 18 + 9}{8} = \frac{36}{8}$$

$$= 4.5 \$$$

Recognize

$$X = X_1 + X_2 + X_3$$

where $X_i = \begin{cases} 1 & \text{if } i\text{th throw heads} \quad p = 1/2 \\ 0 & \text{o.w.} \quad p = 1/2 \end{cases}$

Beautiful Fact

$$E(X) = E(X_1 + X_2 + X_3) = E(X_1) + E(X_2) + E(X_3)$$

$$= 3/2 = 1.5$$

$$E(3X-5) = 3E(X)-5$$
$$= 3\cdot\frac{3}{2} - 5$$
$$= \frac{9}{2}(\approx 4.5\$) - 5$$
$$= -50 \text{ cents!}$$

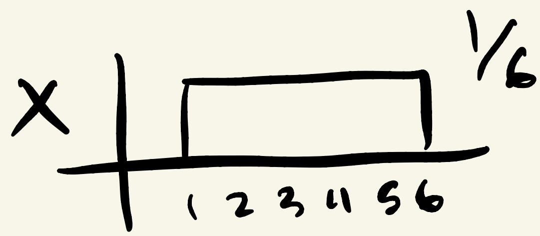
$$\boxed{E(aX+b) = aE(X)+b}$$
Pt 1 of L.o.E

Pt 2 of Linearity of Expectation (L.o.E)

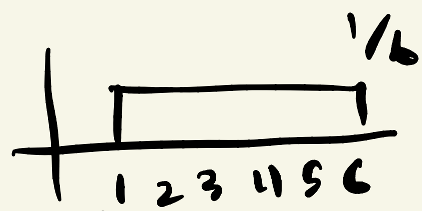
If X & Y then $E(X+Y) = E(X)+E(Y)$

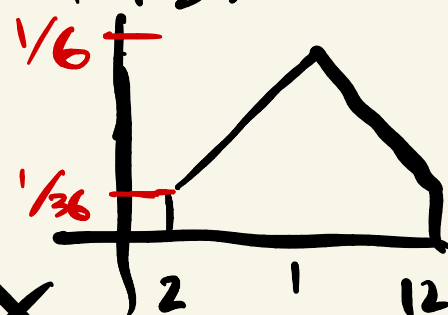
When do we encounter $X+Y (=Z)$

<u>Ex</u> Roll 2 dice



Y						
6	7	8	9	10	11	12
5	6	7	8	9	10	11
4	5	6	7	8	9	10
3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	2	3	4	5	6	7
	1	2	3	4	5	6
	X					





$P(X+Y=2) = \frac{1}{36}$ $P(X+Y=7) = \frac{6}{36} = \frac{1}{6}$

For $Z = X + Y$ $\quad X, Y \sim \text{Unif}\{1, \ldots, 6\}$

$$E(Z) = 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + \cdots + 7\left(\frac{6}{36}\right) + \cdots$$

$$\boxed{E(Z) = \sum_{z \text{ possible}} z \cdot P(Z = z)}$$

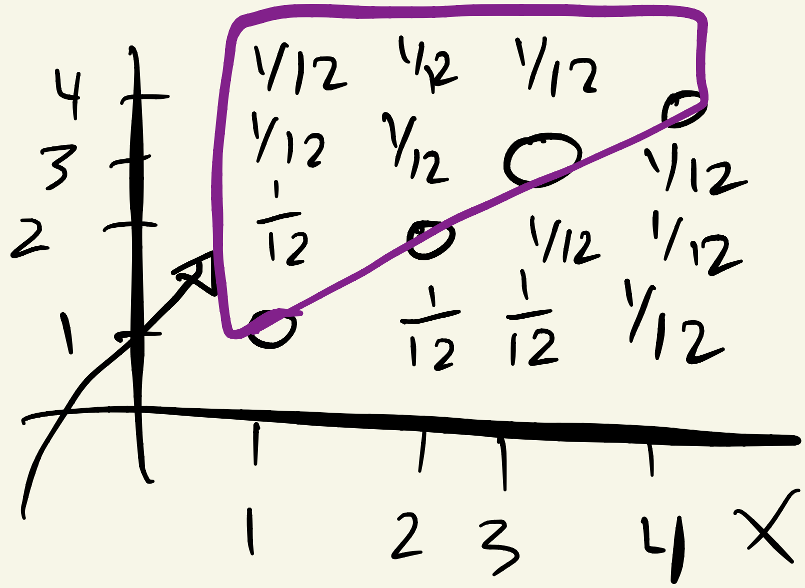
My L.O.E. Formula says

$$E(Z) = E(X+Y)$$
$$= E(X) + E(Y) = 2 \cdot \frac{7}{2} = 7$$

Recall $E(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \cdots + \frac{1}{6} \cdot 6$

$$= \frac{1}{6} \cdot (1 + 2 + \cdots + 6)$$
$$= \cancel{\frac{1}{6}} \frac{\cancel{6} \cdot 7}{2} = \frac{7}{2} = 3.5$$

(2)



$\binom{4}{2} = 6$

(a)

$\cancel{\binom{4}{2}}$ $\frac{4 \times 3}{2} = \frac{12}{2}$

$P(X \leq Y)$

$= \frac{6}{12} = \frac{1}{2}$

$P(1,2) = \left(\frac{1}{4}\right)\left(\frac{1}{3}\right)$

$Z = X + Y$

z	3	4	5	6	7
$P_Z(z)$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{4}{12}$	$\frac{2}{12}$	$\frac{2}{12}$

E(z)

$\frac{2}{12}\left(3+4+6+7\right) + 5 \cdot \frac{4}{12}$

$\frac{2 \cdot 20}{12} + \frac{20}{12} = \frac{60}{12} = 5$

w/ Replacement

$\Omega = 4 \times 4 = 16$ $\frac{1}{16}$

$\rightsquigarrow 7$

2	3	4	5	6	7	8

(c)

4	1/16	1/16	1/16	1/16
3	1/16	1/16	1/16	1/16
2	1/16	1/16	1/16	1/16
1	1/16	1/16	1/16	1/16
	1	2	3	4

How many (X, Y) possible pairs ? w/ replacement

$4 \times 4 = 16$

$$P(X \le Y) = \frac{16}{16}$$

(d) $E(X+Y) = E(X) + E(Y)$

First $E(X) = ?$ $= 2 \cdot \frac{5}{2}$

$= 5$

x	1,	2,	3,	4
$P(X=x)$	1/4	1/4	1/4	1/4

$E(x)$

$=2.5$

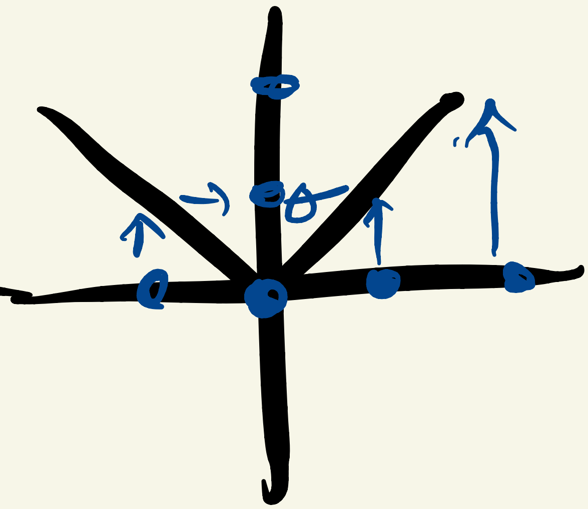
$E(X) = ? = \frac{1}{4}(1+2+3+4) = \frac{1}{\cancel{4}} \frac{\cancel{4} \cdot 5}{2}$

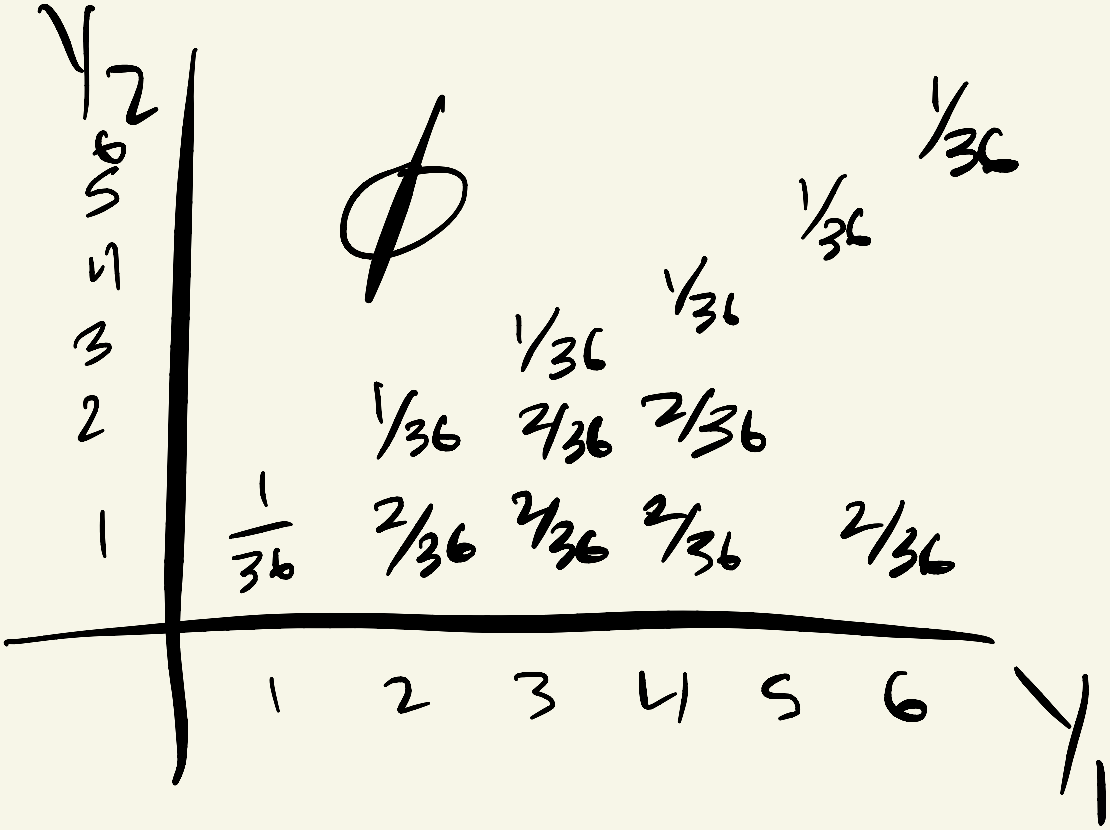
1

(c)

$X \sim$	0	1	2	3
	1/8	3/8	3/8	1/8
$X-1 \sim$	-1	0	1	2

$\|X-1\|$	0	1	2
	3/8	4/8	1/8





$$Y_1 = \max(X_1, X_2) \geq Y_2 = \min(X_1, X_2)$$

$$(i, j) \longmapsto (\max(i,j), \min(i,j))$$

$$(1, 6) \longmapsto (6, 1)$$
$$(6, 1) \longmapsto (6, 1)$$