LECTURE 21 - LIVE

3 Parts of this Course

1) Discrete Probability

2) Central Limit Theorem

3) Continuous Probability

BRIDGE B/W DISCRETE & Continuous

COMES FROM UNDERSTANDING

Probability of Sums $X+Y=S$

$$P(S=s) = \sum_{\substack{x,y \\ \text{s.t. } x+y=s}} P(x,y) = \sum_{x} \underbrace{P(x, s-x)}_{\substack{\shortparallel \\ y}}$$

Joint Probability Distribution of X, Y

If indep

Convolution of X & Y

$$\boxed{\sum_{x} P_X(x) P_Y(s-x)}$$

ALL OF THESE HAVE CTS ANALOGS

$$\sum \longleftrightarrow \int$$

PMFs $\qquad$ PDFs

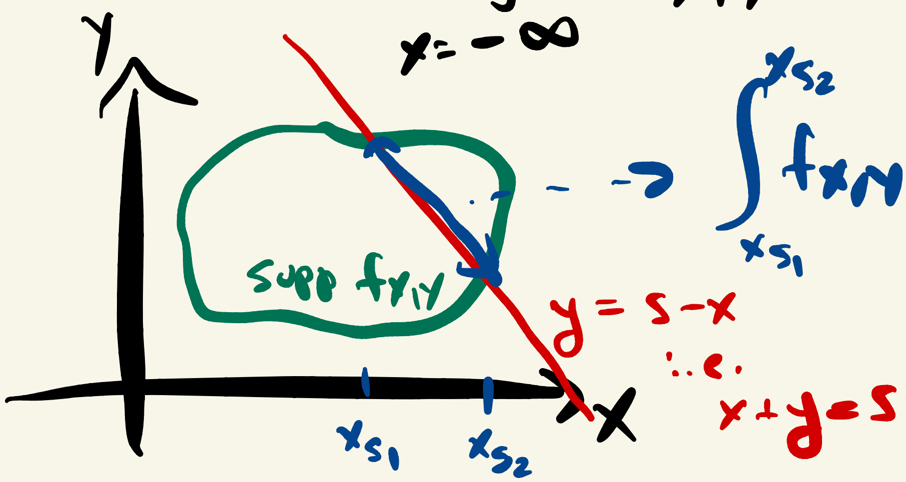
$$p_X \longleftrightarrow f_X$$

$$p_Y \qquad f_Y$$

So... If X & Y are cts RVs

$$f_{S=X+Y}(s) = \int_{x+y=s} f_{X,Y}(x,y)\,dx$$

$$= \int_{x=-\infty}^{x=\infty} f_{X,Y}(x, s-x)\,dx$$



$$\int_{x_{s_1}}^{x_{s_2}} f_{X,Y}(x, s-x)$$

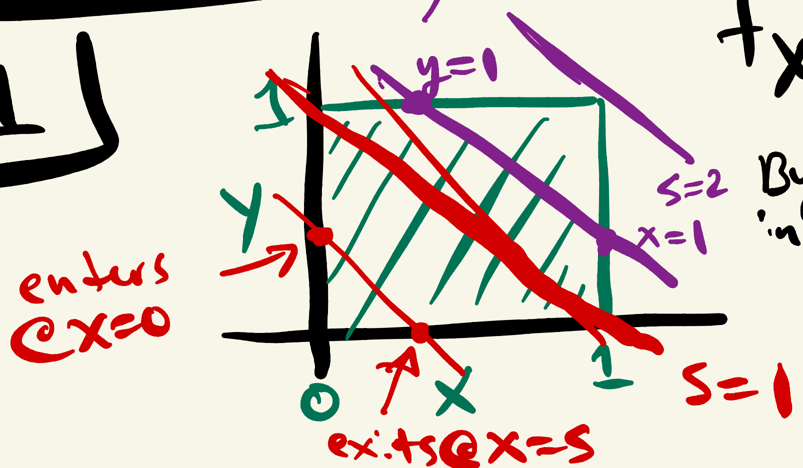
WS21

Q1

$x+y=s,\ y=1 \rightarrow \boxed{x=s-1}$

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\text{Area}} = 1 & 0\le x\le 1,\ 0\le y\le 1 \\ 0 & \text{o.w.} \end{cases}$$

By $\perp\!\!\!\perp$ indep $= f_X(x) f_Y(y)$



Find PDF for $S = X+Y$

Pick a little s, identify where $x+y=s$ enters the support of $f_{X,Y}(x,y)$

$\boxed{0 \le s \le 1}$

$x=0 \rightarrow y=0$

$y = s-x$

$\Rightarrow x = s$

$$f_S(s) = \int_{x=0}^{x=s} \underbrace{f(x', s-x')}_{=1} dx' = x \Big|_{x=0}^{x=s}$$

$\boxed{= s \quad \text{for } 0\le s\le 1}$

$\boxed{1 \le s \le 2}$

enters @ $y=1 = s-x$

$\Rightarrow x = s-1$

exits @ $x=1$

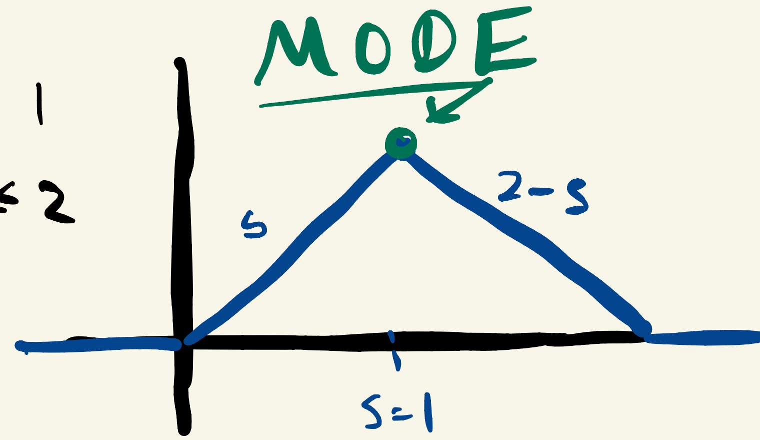
$$\rightarrow f_S(s) = \int_{x=s-1}^{x=1} \underbrace{f_{X,Y}(x', s-x')}_{=1} dx'$$

$= 1-(s-1)$

PDF for S for $1\le s\le 2 \Rightarrow \boxed{= 2-s}$

So... Write & Draw the PDF

$$f_S(s) = \begin{cases} s & \text{for } 0 \le s \le 1 \\ 2-s & \text{for } 1 \le s \le 2 \\ 0 & \text{o.w.} \end{cases}$$



So here

Mean = Median = Mode

$$E(X+Y) = E(X) + E(Y) \stackrel{\checkmark}{=} 1$$

$$= \frac{1}{2} + \frac{1}{2}$$

Q1 b

$$P(X+Y \le 1) = P(S \le 1)$$

Method 1

Use the PDF for S

$$= \int_0^1 f_S(s)\,ds$$

$$= \int_0^1 s\,ds = \frac{s^2}{2}\Big|_0^1$$

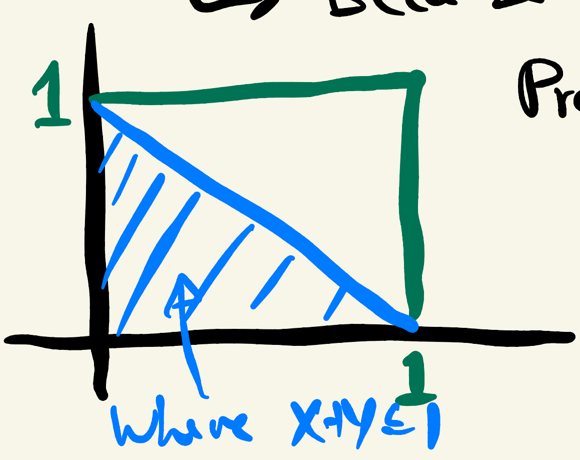
$$\boxed{= \frac{1}{2}}$$

Method 2

Use the JDF of X & Y

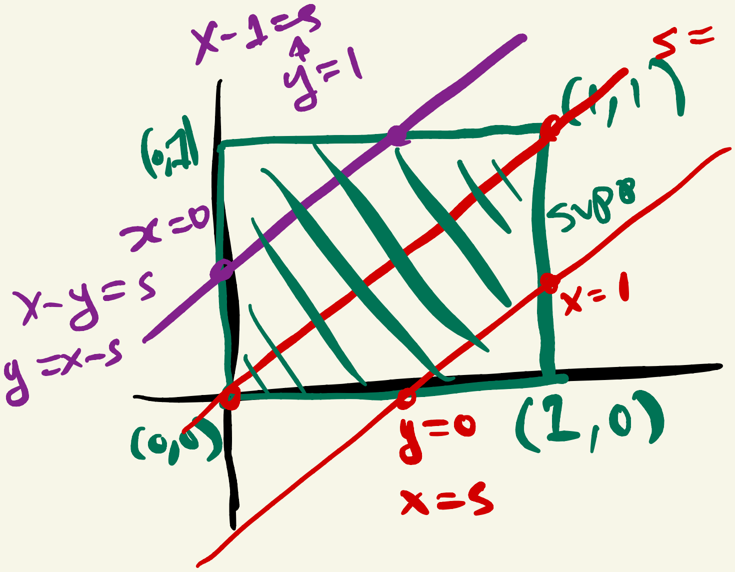
↳ Because $f_{X,Y}$ is uniform

Probability = Relative Area



$$\frac{\text{Area}(\triangle)}{\text{Area}(\square)} = 1/2 \checkmark$$

(c) Find PDF of $D = X - Y$

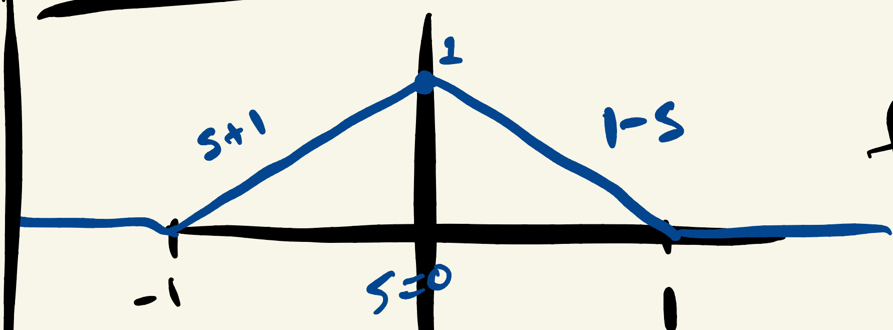


$x - y = s \longleftrightarrow y = x - s$

Here $s \geq 0$

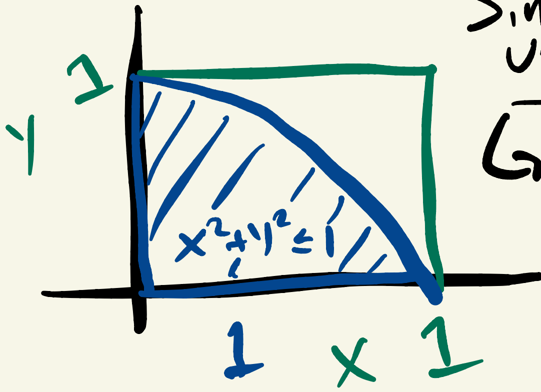
$\underline{0 \leq s \leq 1}$ $\Rightarrow f_D(s) = \int_{x=s}^{x=1} 1 \, dx' = 1 - s$

$\underline{-1 \leq s \leq 0}$ $f_D(s) = \int_{0}^{s+1} 1 \, dx = s + 1$



$$f_D(s) = \begin{cases} s+1 & -1 \leq s \leq 0 \\ 1-s & 0 \leq s \leq 1 \\ 0 & \text{o.w} \end{cases}$$

Q2



Since uniform...

$$\hookrightarrow \text{Prob} = \frac{\text{Area}(\quad)}{\text{Area}(\quad)} = \boxed{\frac{\pi}{4}}$$

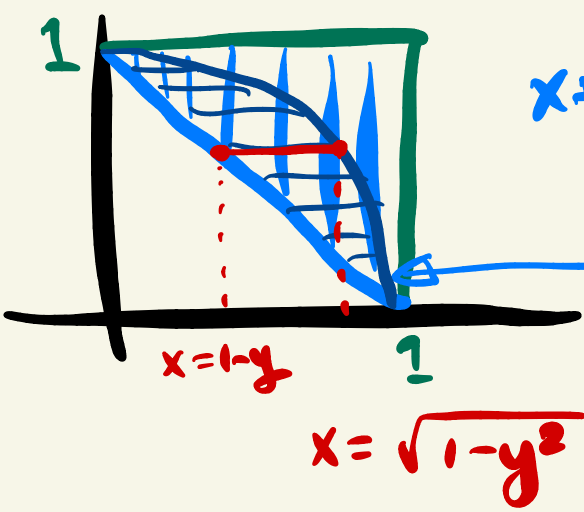
(a)

$\pi r^2 = \text{Area}$

$r = 1$

$$\frac{\pi r^2}{4}$$

(b) $$P(\underbrace{x^2+y^2 \le 1}_{A} \mid \underbrace{x+y \ge 1}_{B}) = \frac{P(A \cap B)}{P(B)}$$



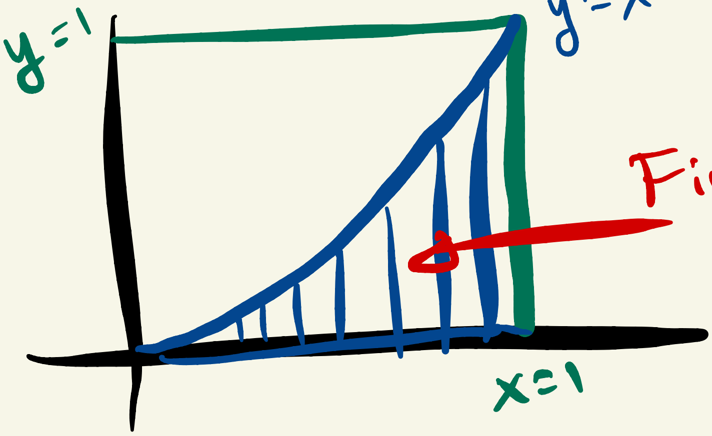
$x+y \ge 1$

$\Rightarrow y \ge 1-x$

$$= \boxed{\frac{\pi/4 - 1/2}{1/2}}$$

$$\int_{y=0}^{y=1} \int_{x=1-y}^{x=\sqrt{1-y^2}} 1 \, dx \, dy = \cdots$$

Q2 (c)



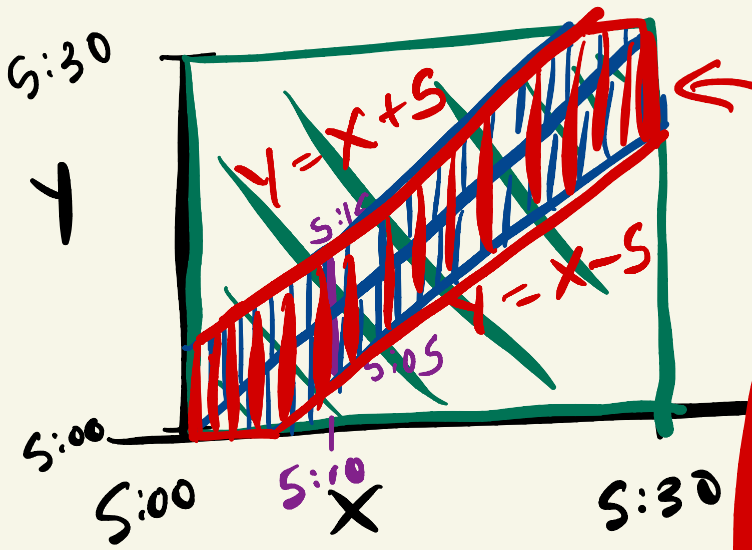
Find this area using calculus!

$$\text{Area} = \int_{x=0}^{x=1}\int_{y=0}^{x^2} f_{x,y}(x,y)\,dy\,dx = \int_{x=0}^{x=1}\int_{y=0}^{y=x^2} 1\,dy\,dx$$

$$\int_{x=0}^{x=1}\left[y\right]_{y=0}^{y=x^2} = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \boxed{\frac{1}{3}}$$

FINAL ANSWER

Q3 FAMOUS CTS Probability Problem



Q: What is our unit of time?

A: Minutes works best

$$f_X(x) = \begin{cases} \frac{1}{30} & \text{for } x \in [0, 30] \\ 0 & \text{o.w.} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{30} & y \in [0, 30] \\ 0 & \text{o.w.} \end{cases}$$

JDF

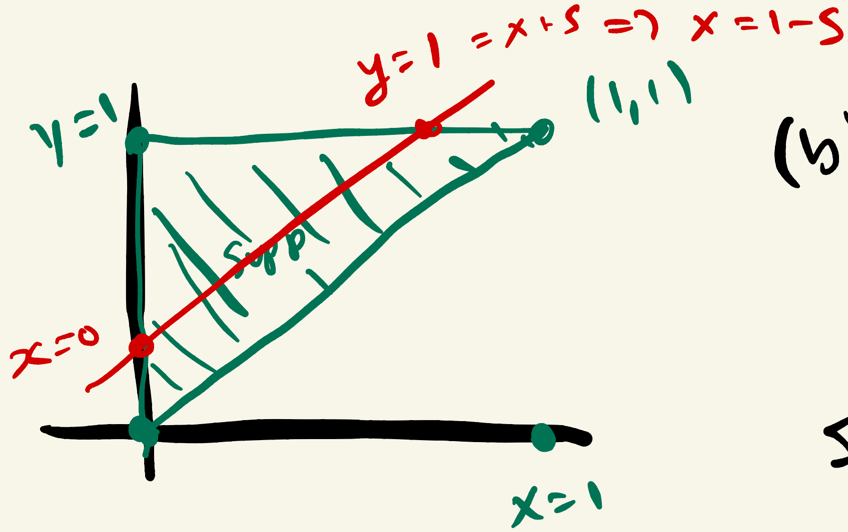
$$f_{X,Y} = \begin{cases} \frac{1}{900} & x \in [0, 30],\ y \in [0, 30] \\ 0 & \text{o.w.} \end{cases}$$

REL. AREA is the PROBABILITY OF THEM MEETING!

$$1 - \text{Area}\left(\quad\right) \qquad \boxed{= 1 - \left(\frac{5}{6}\right)^2} = 30.5\%$$

$$\text{Area} = \frac{2 \times \frac{1}{2} \times 25 \times 25}{900} = \left(\frac{25}{30}\right)^2 = \left(\frac{5}{6}\right)^2$$

Q4



(b) X & Y indep?

NO!

$\mathrm{supp}(f_{X,Y}) \neq \mathrm{supp}(f_X) \times \mathrm{supp}(f_Y)$

so $f_{X,Y} \neq f_X \cdot f_Y$

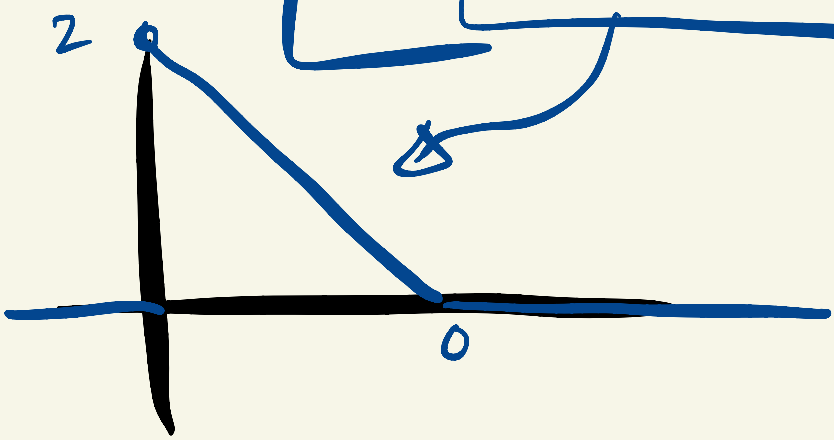
$$Y - X = S$$

$$y - x = s \Rightarrow y = x + s$$

For $0 \leq s \leq 1$

$$f(s) = \int_{x=0}^{x=1-s} \underbrace{f_{X,Y}(x, x+s)}_{=2 \text{ because } \frac{1}{\text{Area}} = \frac{1}{1/2}} \, dx$$

$$\boxed{= 2(1-s) \quad \text{for } 0 \leq s \leq 1}$$



(c) $E(Y-X) = E(Y) - E(X)$

(d) $\mathrm{cov}(X,Y) = E\left[(X-\mu_X)(Y-\mu_Y)\right]$