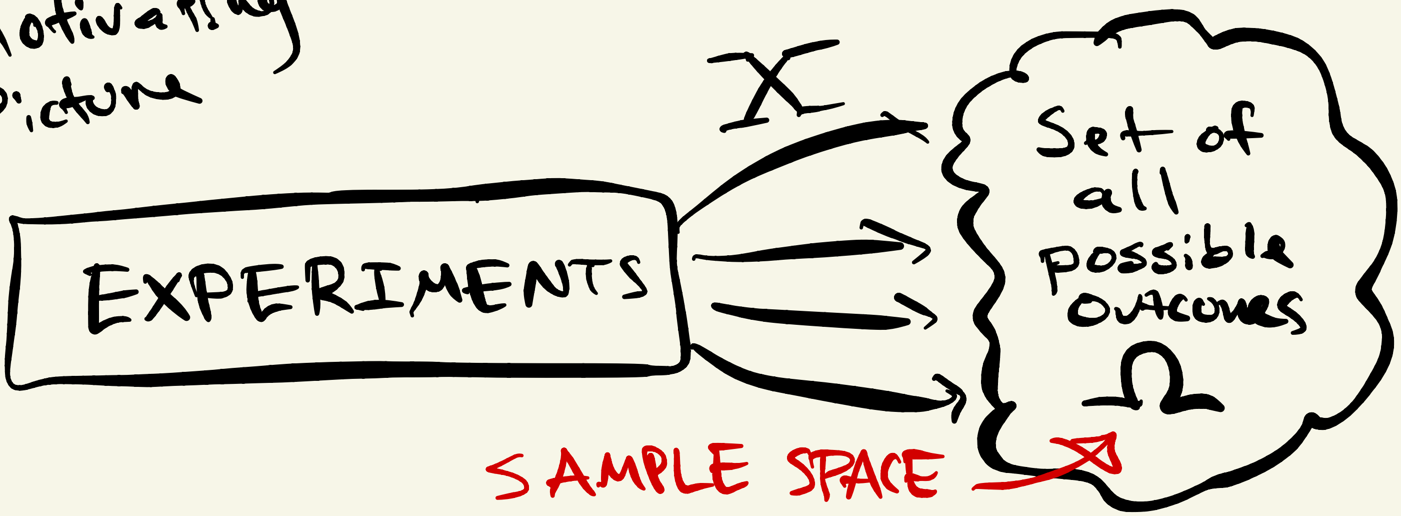
AMAT 362 – Probability for Statistics – Lec 2

(Lec 1)
Last Lecture: Set Theory – De Morgan's Law

This Lecture: Kolmogorov's Axioms and Introduction to Counting (combinatorics)

Motivating Picture



- Discrete Probability $\Longleftrightarrow$ Ω is a finite set (countable)

examples

$\Omega = \{H, T\}$

OR $= \{a, b, c, \dots, z\}$

$\Omega = \mathbb{N} = \{0, 1, 2, \dots\}$

contrast w/

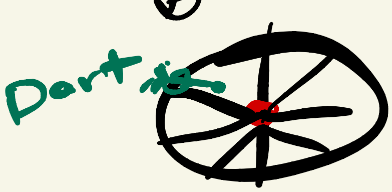
- Continuous Probability

$\Omega = \mathbb{R}$ $\longleftrightarrow$

ex where X = income

Problem! → Probability of someone making \$55,272.3165432159... cents $= 0$!

↓



Def Probability assigns chances to **events**, which is a subset of the set of possible outcomes Ω

Def The set of all subsets of Ω is

$$\mathcal{P}(\Omega) = \{A \mid A \subseteq \Omega\}$$

Q: If $|\Omega| = 5$ then how big is $\mathcal{P}(\Omega)$?

In general $|\Omega| = n \Rightarrow |\mathcal{P}(\Omega)| = 2^n$

A: $|\mathcal{P}(\Omega)| = 32 = 2^5$

Each subset $A \subseteq \Omega$ is encoded by a vector

$(x_1, x_2, x_3, x_4, x_5)$ where $x_i = \begin{cases} 1 & \text{if } x_i \in A \\ 0 & \text{if } x_i \notin A \end{cases}$

$$\Rightarrow |\mathcal{P}(\Omega)| = 2 \times 2 \times 2 \times 2 \times 2 = 2^5$$

N.B. $\phi \longleftrightarrow (0,0,0,0,0)$ no element is in ϕ

FACT If Ω is finite (discrete prob.) then every subset of Ω is an event.

AXIOMS FOR THE SET OF EVENTS $\mathcal{E}$

1) $\Omega \in \mathcal{E}$ 2) If $A \in \mathcal{E}$ then $A^c \in \mathcal{E}$

3) If $A_1, A_2, \dots, A_n, \dots$ is a countable collection of events, then $A_1 \cup A_2 \cup \cdots = \bigcup_{k=1}^{\infty} A_k \in \mathcal{E}$

Def (Kolmogorov) = Answer to "What is probability"?

A **probability measure** is a function

$$P : \mathcal{E} \longrightarrow [0,1]$$

where $A \longmapsto P(A)$ = Probability that A occurs

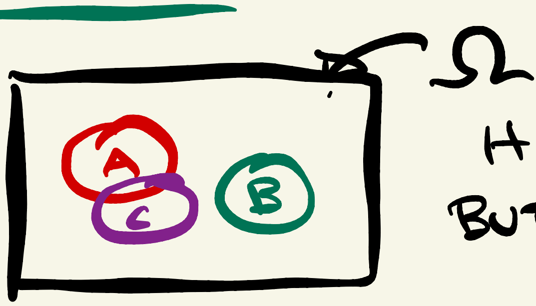
1) $0 \le P(A) \le 1$
2) $P(\Omega) = 1 \quad \Rightarrow P(\emptyset) = 0$
3) If $A_1, A_2, A_3, \ldots, A_n, \ldots$ is a countable collection of **disjoint** events, i.e. $A_i \cap A_j = \emptyset$, $i \ne j$

then $$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$$

N.B.

Easy 3) If $A \cap B = \emptyset$ (disjoint i.e. mutually exclusive) then $\boxed{P(A \cup B) = P(A) + P(B)}$

Intuition P measures percent area



$$P(A) = \frac{\text{Area}(A)}{\text{Area}(\Omega)}$$

Here $P(A \cup B) = P(A) + P(B)$

BUT $P(A \cup C) \le P(A) + P(C)$

Ex The uniform probability (measure) on a finite Ω is defined by $P(A) = \dfrac{|A|}{|\Omega|}$

Every outcome is equally likely

Ex Draw a card

$$P(\text{A King}) = \frac{\#\text{kings}}{\#\text{cards}} = \frac{4}{52} \checkmark$$

Computing uniform probabilities requires lots of tricks for counting, i.e.

COMBINATORICS

1) MULTIPLICATION RULE

Sps our experiment can be broken down into k steps, where each step i has n_i possible outcomes, then

$$\boxed{\underset{\overset{\shortparallel}{|\Omega|}}{\text{TOTAL \# OUTCOMES}} = n_1 \times n_2 \times \cdots \times n_k}$$

Ex 1 Flip a coin 3 times

$$\Rightarrow |\Omega| = 2 \times 2 \times 2 = 2^3 = 8$$

Ex 2 Yahtzee Roll 5 (distinguishable) die

= Rolling a single die 5 times

$$\Rightarrow |\Omega| = 6 \times 6 \times 6 \times 6 \times 6 = 6^5 = 7{,}776$$

Ex 3 Ordering from a Prix Fix Menu

1) SOUP OR SALAD
2) PASTA, FISH, STEAK
3) APPLE PIE OR BROWNIE

DECISION TREE

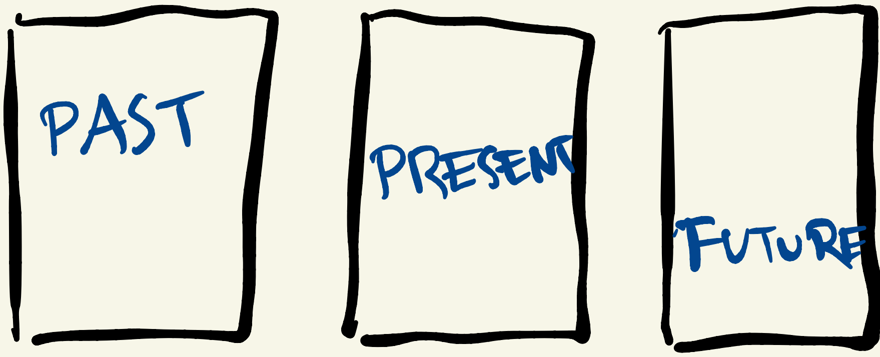
$$\#\text{LEAF NODES} = 2 \times 3 \times 2 = 12$$

Ex 4 TAROT CARDS

→ 22 Major Arcana

→ 56 minor Arcana

78 cards in a tarot deck



$$78 \times 77 \times 76 = \#\text{ Possible Fortunes}$$

Example of...

SAMPLING W/O REPLACEMENT (ORDER MATTERS)

URN

SAMPLE from n objects k times then

$$\#\text{possible outcomes} = n\cdot(n-1)\cdot\ \cdots\ (n-k+1)$$

$$|\Omega| = (n)_k = P_{n,k} = \boxed{\frac{n!}{(n-k)!}} = \frac{n\times(n-1)\times\cdots(n-k+1)\cdot\cancel{(n-k)}\cdots}{\cancel{(n-k)}\cancel{(n-k-1)}\cdots}$$

BIRTHDAY PARADOX

Suppose n people in a room

Q: What's the probability of two people sharing the same birthday?

$n=40 \rightsquigarrow$ 89% chance of at least two people have the same b-day.

event A = at least two people have the same b-day.

RULE OF COMPLEMENTS

$$P(A) = 1 - P(A^c)$$

A^c: A does NOT occur

Ex

$$P(A^c) = \left(\frac{365}{365}\right) \times \left(\frac{364}{365}\right) \times \left(\frac{363}{365}\right) \times \cdots \times \left(\frac{365-39}{365}\right)$$

$$= \frac{1}{365^{40}} \times \frac{365!}{325!} \approx .11 = 11\% \text{ chance}$$