AMAT 362 - PROBABILITY for STATISTICS @UAlbany

LECTURE 21:

How to sum 2 cts RVs
$X, Y \rightsquigarrow Z = X+Y$

$f_{Z=X+Y} \neq f_X + f_Y$
$= f_X * f_Y$

Joint PDFs

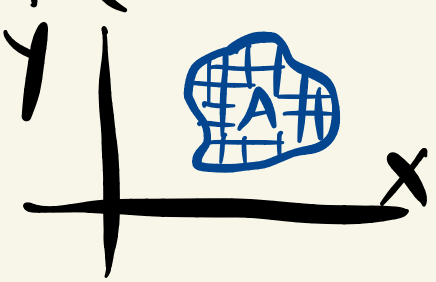
RECALL

The PDF for a cts R.V. X is a function

s.t. $P(a<X<b) = \int_a^b f_X(x)\,dx$

Analogously, the PDF for (X,Y) is a function $f_{X,Y}(x,y)$ s.t.

$$P((X,Y)\in A) = \iint_A \underbrace{f_{X,Y}(x,y)}_{\text{Joint PDF (JDF)}}\,dx\,dy$$



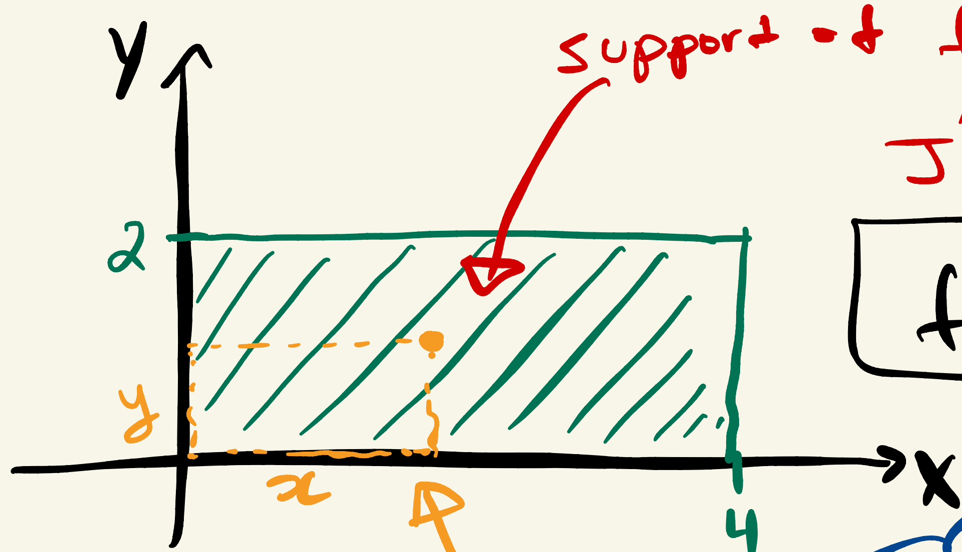
Motivating Question

Suppose I construct a rectangle randomly by choosing its width $X \sim \text{Unif}[0,4]$ and it's height from $Y \sim \text{Unif}[0,2]$ INDEPENDENTLY

→ How is its perimeter distributed?

→ What is the expected perimeter?

→ What is its variance?



How is perimeter distributed?

support of $f_{X,Y} = f_X(x) f_Y(y)$

"JDF"

B/C independent

$$\boxed{f_{X,Y}(x,y) = \begin{cases} \frac{1}{8} & 0 \le x \le 4,\ 0 \le y \le 2 \\ 0 & \text{o.w.} \end{cases}}$$

STEP 1: Find the JDF

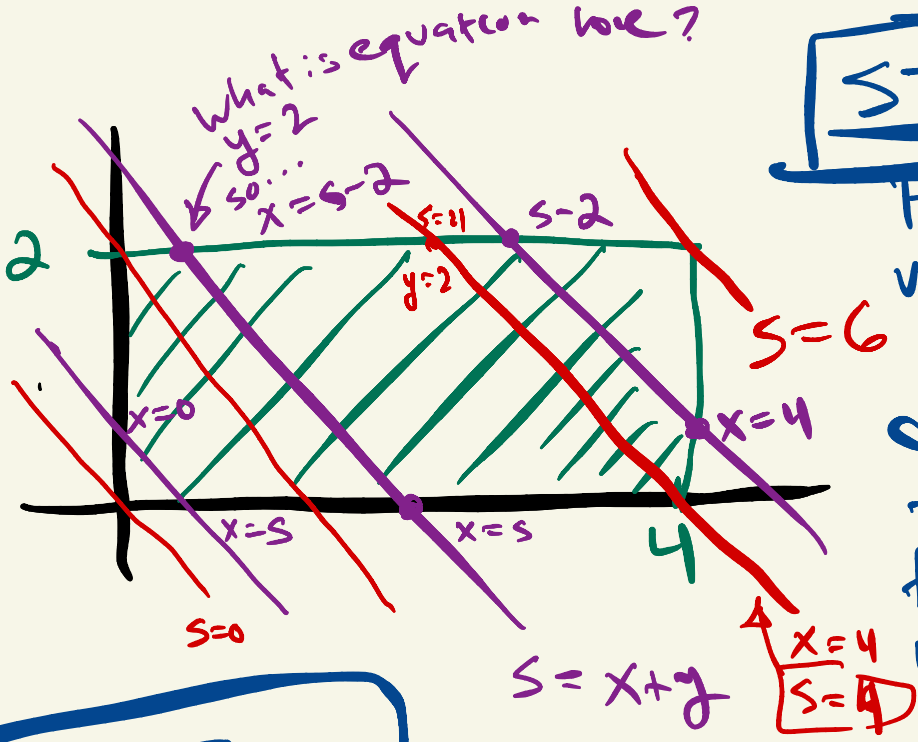
OBSERVE

$P = 2X + 2Y$ is perimeter

$= 2(X+Y)$

Let $S = X + Y$ is the sum ⟸ Core object of study

How is the sum distributed?



STEP 2

For varying values of $s = x + y$ determine where this line intersects the support of my JDF

|| points where $f_{x,y}(x,y) \neq 0$

$s = x + y$

$y = s - x$

check! $\Rightarrow \begin{cases} y = s - (s-2) \\ y = 2 \end{cases}$

STEP 3
INTEGRATE

$$P(X+Y=s) = \int_x f_{XY}(x, \overset{y}{\overset{\shortparallel}{s-x}})\,dx$$

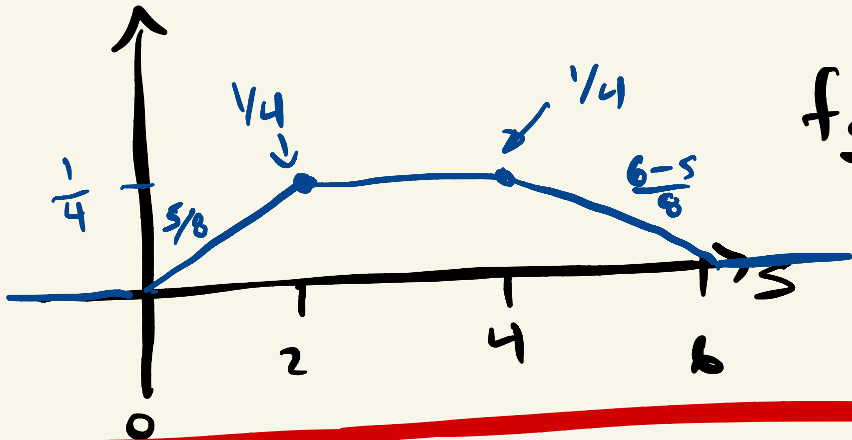
1) For $\underline{0 \le s \le 2}$ $\Rightarrow P(X+Y=s) = \int_{x=0}^{x=s} \frac{1}{8}\,dx = \boxed{\frac{s}{8}}$

2) For $\underline{2 \le s \le 4}$ $\Rightarrow \int_{s-2}^{s} \frac{1}{8}\,dx = \frac{1}{8}(s-(s-2)) = \boxed{\frac{1}{4}}$

3) $4 \le s \le 6 \Rightarrow \int_{s-2}^{4} \frac{1}{8}\,dx = \frac{1}{8}(4-(s-2))$

$$= \frac{1}{8}(6-s) = \boxed{\frac{6-s}{8}}$$

STEP 4 Write & Draw the PDF

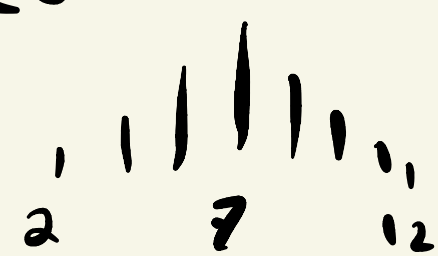


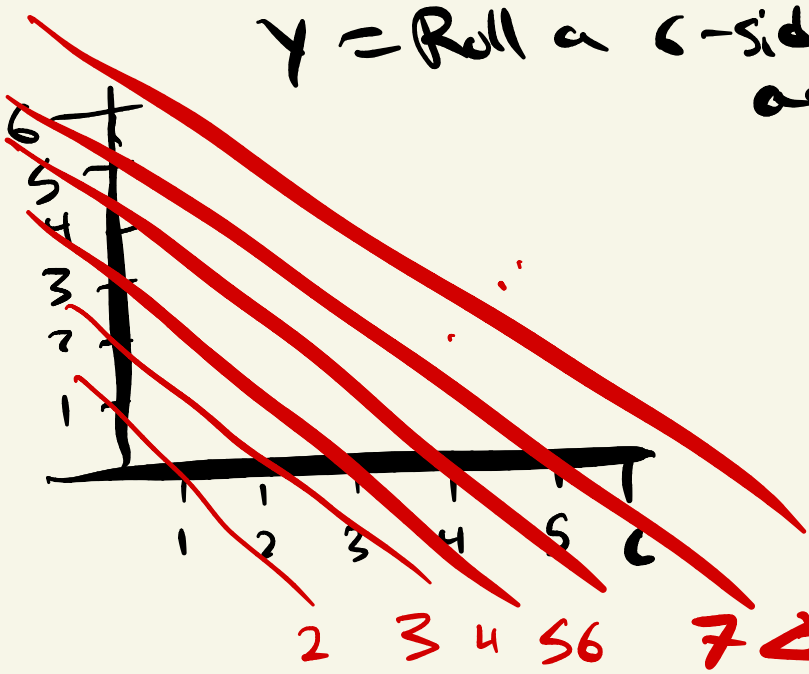
$$f_S(s) = \begin{cases} s/8 & 0 \le s \le 2 \\ 1/4 & 2 \le s \le 4 \\ \frac{6-s}{8} & 4 \le s \le 6 \\ 0 & \text{o.w.} \end{cases}$$

Note that the Probability of the sum $S = X + Y$ is NOT the sum of the probabilities!

This makes sense in the discrete case

X = Roll a 6-sided die
Y = Roll a 6-sided die again
→ $S = X + Y$





HIGH-LEVEL SUMMARY

The above steps for finding PDF for $S = X+Y$

1) Identify the joint PDF for X & Y
2) Determine the possible values for S and the limits of integration in terms of s
3) Integrate
4) Write & Draw

Are the steps for computing the <u>convolution</u> of X & Y.

$$f_{X*Y}(s) = \int_{x+y=s} f_{X,Y}(x, s-x)\,dx$$

is the PDF for S

Back to Question

Perimeter $= P = 2S$ ← Know PDF for S!

Apply the formula: If $Z = aW + b$

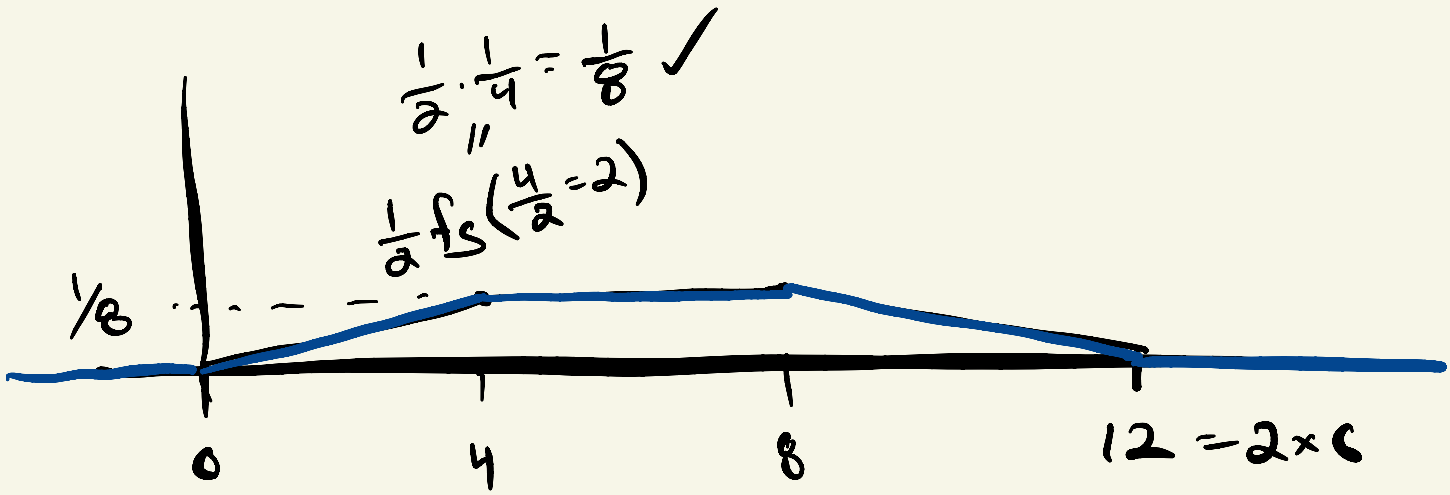
then $f_Z(z) = \frac{1}{|a|} f_W\left(\frac{z-b}{a}\right)$

PDF for Z PDF for W

$a = 2$ for us

$$f_P(p) = \frac{1}{2} f_S\left(\frac{p}{2}\right)$$

PDF for $P = 2S$



Expectation

$$E(P) = E(2S) = 2E(S)$$

$$E(S) = E(X+Y) = E(X) + E(Y)$$

$$\boxed{E[\text{Unif}[a,b]] = \frac{b-a}{2}}$$

$X \sim \text{Unif}[0,4]$
$Y \sim \text{Unif}[0,2]$

$= \frac{4-0}{2} + \frac{2-0}{2}$

$= 2 + 1$

$= 3$

$$\Rightarrow \boxed{E(P) = 6}$$

Variance

If X & Y are independent

then $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$

$$\boxed{\text{Var}[\text{Unif}[a,b]] = \frac{(b-a)^2}{12}}$$

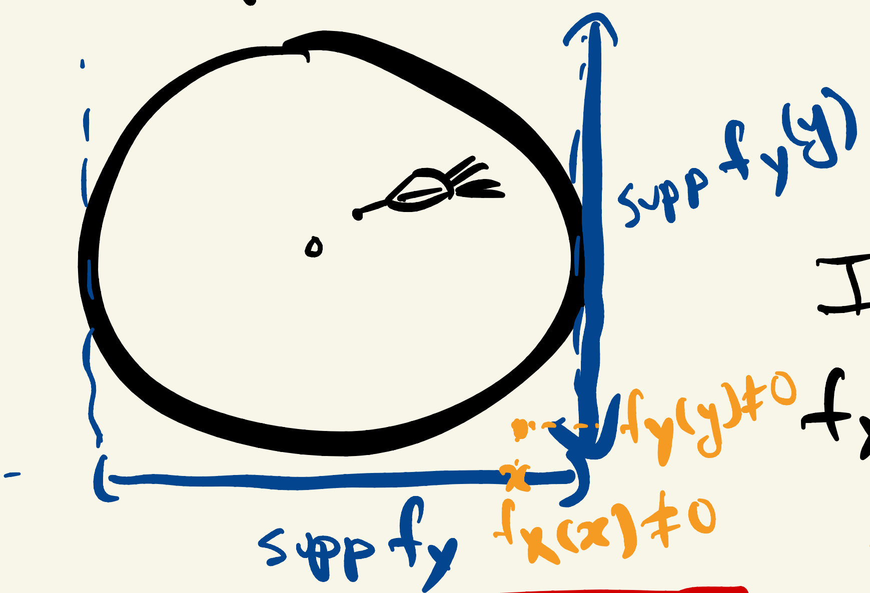
$\text{Var}(X+Y) = \frac{16}{12} + \frac{4}{12} = \frac{20}{12}$

$$\boxed{Var(aW) = a^2 Var(W)}$$

$$Var(2S) = 4 \cdot Var(S)$$
$$= 4 \cdot Var(X+Y)$$
$$= 4 \cdot [Var(X) + Var(Y)]$$
$$= 4 \cdot \frac{20}{12} = \frac{20}{3}$$

What happens if X & Y are NOT indep?

Example where this occurs: Throw dart at dartboard, assume I hit uniformly the board



If X & Y are indep

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

ZERO $\neq$ Not zero

Contradiction

True Always

$$\boxed{Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)}$$

where $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

(when X & Y are indep Cov = 0)