LECTURE 23 LIVE

BAYES RULE

$\boxed{Q1}$ A test is 95% accurate at detecting if disease 'D' is present. BUT! Only 1% of the population actually has disease D.

Q: What is the probability that a person who tests positive for disease D actually has the disease?

A: 95% accuracy $P(+ \mid D) = 95\%$

$\Rightarrow P(D \mid +) = ?$

know these

BAYES RULE

$$P(D \mid +) = \frac{P(+ \mid D)\,P(D)}{P(+)}$$

Need to find $\rightarrow P(+)$

"NOT D"

$$P(+) = P(+ \text{ AND have } D) + P(+ \text{ AND } \neg D)$$
$$= P(+ \mid D)P(D) + P(+ \mid \neg D)P(\neg D)$$

N.B. $P(+\mid \neg D)$ "false positive rate"

Say this happens only 5% of the time

MAKE ASSUMPTION

~~NOT TRUE IN GENERAL~~ $\Rightarrow$ " $P(-\mid D)$

TECHNICALLY, CAN'T INFER THIS FROM THE PROBLEM STATE

WHY?

$$P(-\mid D) = 1 - P(+\mid D)$$
$$= 100\% - 95\%$$
$$= 5\%$$

Assume $P(-\mid \neg D) = 95\%$

$\Rightarrow P(+\mid \neg D) = 5\%$

$$P(D\mid +) = \frac{(.95)(.01)}{(.95)(.01) + (.05)(.99)}$$

$$\boxed{= 16.1\%\,!}$$

Q2

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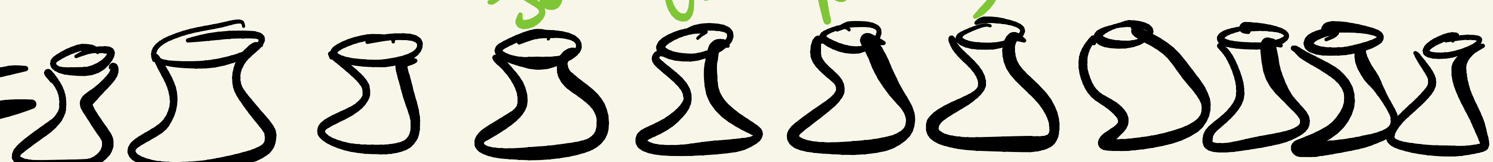


Ball / Brec kooken

U1	U2	U3	U4	U5	U6	U7	U8	U9	U10	U11
10R	9R	8R	7R	6R	5R	4R	3R	2R	1R	0R
0B	1B	2B	3B	4B	5B	6B	7B	8B	9B	10B

U = Urn # $U \sim \text{Unif}\{1, 2, \dots, 11\}$

B = # blue balls out of N draws

$$P(B=b \mid U=u) \sim \mathrm{Binom}\left(N, p=\frac{u-1}{10}\right)$$

$$= \binom{N}{b}\left(\frac{u-1}{10}\right)^b \left(1-\frac{u-1}{10}\right)^{N-b}$$

This is the conditional distribution for B given $U=u$

What is

$$P(B=b, U=u) \overset{?}{=} P(B=b \mid U=u)\,P(U=u)$$

$$= \begin{cases} \frac{1}{11}\binom{N}{b}\left(\frac{u-1}{10}\right)^b\left(1-\frac{u-1}{10}\right)^{N-b} & \text{for } 0 \le b \le N \text{ and } 1 \le u \le 11 \\ 0 & \text{o.w.} \end{cases}$$

(b) Suppose $N=10$ ~~&~~ # draws from mystery urn

DATA: $\boxed{B=b=3}$

GUESS Urn # right you win 10^6 \$
wrong nothing

Trying to maximize as a function of u

Binomial

$$P(U=u \mid \underset{=3}{B=b}, N=10) = \frac{\boxed{P(B=b \mid U=u)\,P(U=u)}}{P(B=b)}$$

Const

The Question boils down to ...
Where is the mode of a Binomial Distribution located?

→ one # above or below the mean = E

$$E(\text{Binom}(n,p)) = np$$

For our problem $= 10 \cdot \frac{u-1}{10}$

Want choose u s.t. $\frac{u-1}{10} \cdot 10 \sim B = \underline{3}$

→ $\boxed{u=4}$ seems most likely

But should do the math.

(c) Compute $E(B)$

Hint: DON'T ATTACK DIRECTLY

INSTEAD... USE LAW OF ITERATED EXPECTATION

$$E[B] = E[\underbrace{E[B \mid u]}]$$

Easy b/c we know $E[\text{Binom}] = np$ ✓

$$E[B \mid U=u] = N \cdot p = N\left(\frac{u-1}{10}\right)$$

$$\text{So}\ldots\quad E[B] = \sum_{u=1}^{11} \underbrace{N \cdot \left(\frac{u-1}{10}\right)}_{g(u)} \cdot \frac{1}{11}$$

$E[g(U)]$

$g(u) = E[B \mid u]$

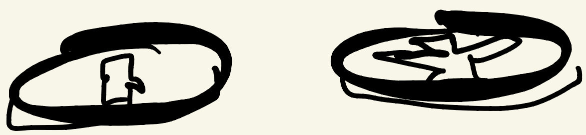
$$= \frac{N}{10 \cdot 11} \sum_{u=1}^{11} u-1$$

$$= \frac{N}{10 \cdot 11} \left(0 + 1 + 2 + 3 + \cdots + 10\right)$$

$$\frac{10 + 9 + 8 + \cdots + 1}{11 \quad 11 \quad 11 \quad\quad 11}$$

$$= \frac{N}{\cancel{10 \cdot 11}} \cdot \frac{\cancel{10 \cdot 11}}{2} = \boxed{\frac{N}{2}}$$

$\boxed{Q3}$



$\xrightarrow{\# \text{ heads}} X$

Flip Again

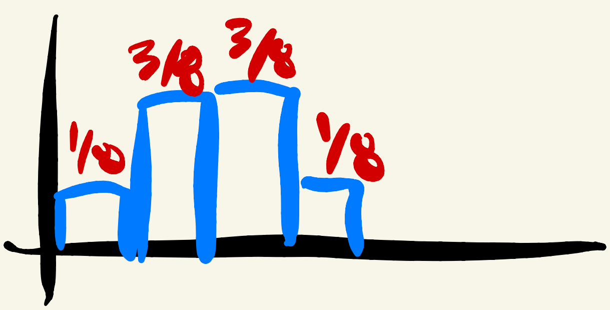
Let Y = denote # of Heads after you've re-flipped any tails in first toss

(a) BINOMIAL

$$P(X=x) = \binom{3}{x}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{3-x}$$

$$= \binom{3}{x} 2^{-3}$$

$x = \{0, 1, 2, 3\}$



1

1 1

1 2 1

1 3 3 1

(b) $P(Y=y \mid X=x)$ ... Err...

$P(Y=y \mid X=3)$ $\longrightarrow$

y	0	1	2	3
$P(Y=y \mid X=3)$	0	0	0	1

$$P(Y=y\mid X=0) = \begin{array}{c|cccc} y & 0 & 1 & 2 & 3 \\ \hline P(Y=y\mid X=0) & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array}$$

$$P(Y=y\mid X=1) = \begin{array}{c|cccc} y & 0 & 1 & 2 & 3 \\ \hline P(Y=y\mid X=1) & 0 & \frac{1}{4} & \frac{2}{4} & \frac{1}{4} \end{array}$$

heads out of 2 coins

$$\begin{array}{|ccc} 0^{+1} & 1^{+1} & 2^{+1} \\ \hline \frac{1}{4} & \frac{2}{4} & \frac{1}{4} \end{array}$$

$$P(Y=y\mid X=2) = \begin{array}{c|cccc} y & 0 & 1 & 2 & 3 \\ \hline & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$

$$\boxed{P(X=x, Y=y) = P(Y=y\mid X=x)\,P(X=x)}$$

Y \ X	0	1	2	3	(d)
3	$\frac{1}{8}\cdot\frac{1}{8}$	$\frac{1}{4}\cdot\frac{3}{8}$	$\frac{1}{2}\cdot\frac{3}{8}$	$1\cdot\frac{1}{8}$	$\frac{1+6+12+8}{64}=\frac{27}{64}$
2	$\frac{3}{8}\cdot\frac{1}{8}$	$\frac{2}{4}\cdot\frac{3}{8}$	$\frac{1}{2}\cdot\frac{3}{8}$	0	$\frac{3+12+12}{64}=\frac{27}{64}$
1	$\frac{3}{8}\cdot\frac{1}{8}$	$\frac{1}{4}\cdot\frac{3}{8}$	$0\cdot\frac{3}{8}$	0	$\frac{3+6}{64}=\frac{9}{64}$
0	$\frac{1}{8}\cdot\frac{1}{8}$	$0\cdot\frac{3}{8}$	$0\cdot\frac{3}{8}$	0	$\frac{1}{64}$ ✓

$P(Y=y)$

(e) $P(X=x \mid Y=y) = \dfrac{P(X=x, Y=y)}{P(Y=y)}$

From table

From part (d)

$P(X=x \mid Y=0)$

x	0	1	2	3
	$\frac{1}{8}\cdot\frac{1}{8}\cdot 64$	0	0	0

$P(X=x \mid Y=1)$

x	0	1	2	3
	$\frac{1}{8}\cdot\frac{3}{8}\cdot\frac{64}{9}$	$\frac{1}{4}\cdot\frac{3}{8}\cdot\frac{64}{9}$	0	0

$P(X=x \mid Y=2)$

x	0	1	2	3
$\frac{64}{27}\times$	$\frac{3}{8}\cdot\frac{1}{8}$	$\frac{2}{4}\cdot\frac{3}{8}$	$\frac{1}{2}\cdot 3/8$	0

$P(X=x \mid Y=3)$

x	0	1	2	3
$\frac{64}{27}\times$	$\frac{1}{8}\cdot\frac{1}{8}$	$\frac{1}{4}\cdot\frac{3}{8}$	$\frac{1}{2}\cdot 3/8$	$1\cdot 1/8$

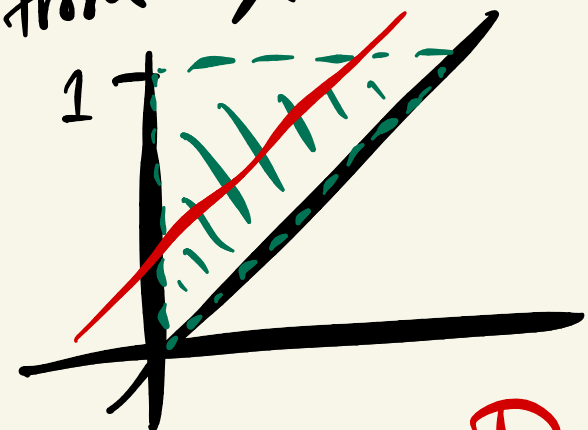
(f) Highlight largest entry in each table!

WS21

Q4 Can't distinguish $X \sim Unif(0,1)$

from $X \sim Unif[0,1]$

(a)



$Y - X = D$

$Y = X + D$

D is supported on $(0,1)$

(c) $E(Y-X) = E(Y) - E(X)$

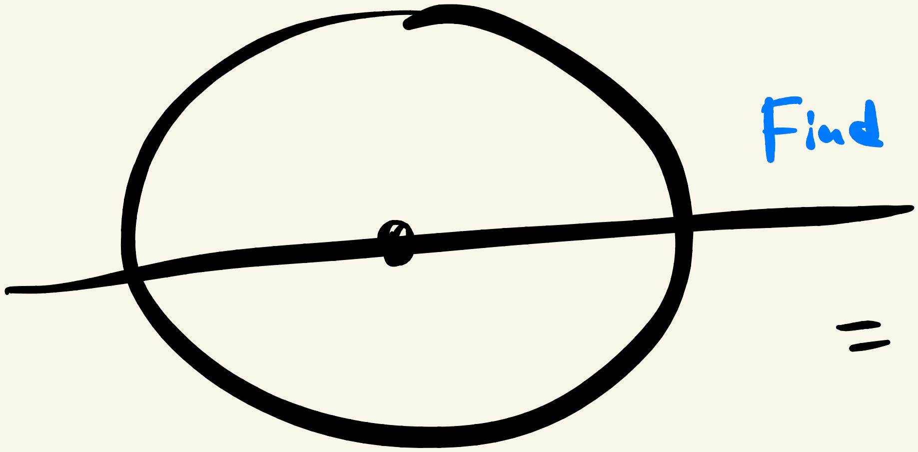
(d) $Cov(X,Y) = E[XY] - E[X]E[Y]$

Find this!

5(c) $X, Y \sim Norm(0,1)$

$\leadsto R = \sqrt{X^2 + Y^2}$ is Rayleigh

$re^{-r^2/2}$



Find $E[|Y|]$

$$= \int_{-\infty}^{\infty} |y| \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} y e^{-y^2/2} dy$$

$$\left[-e^{-y^2/2}\right]_0^{\infty}$$

$$= 2/\sqrt{2\pi}$$