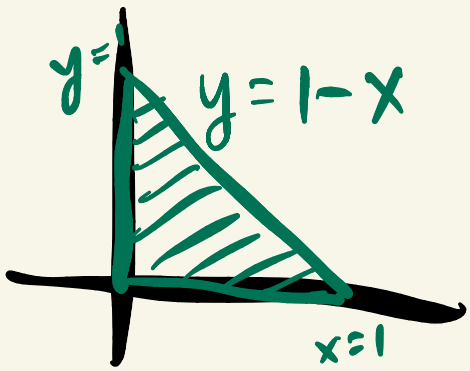
LECTURE 24 LIVE

Q1 $E[XY] = \cdots$

$$E[g(X,Y)] = \iint g(x,y)\, f(x,y)\,dx\,dy$$



$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} xy\,2\,dy\,dx$$

$$= 2\int_0^1 x \int_{y=0}^{y=1-x} y\,dy\,dx = 2\int_0^1 x\left[\frac{y^2}{2}\right]_{y=0}^{y=1-x} dx$$

$$= \cancel{2}\int_0^1 x\,\frac{(1-x)^2}{\cancel{2}}\,dx$$

$$= \cancel{2}\int_0^1 x\,(1-2x+x^2)\,dx$$

$$= \int_0^1 x - 2x^2 + x^3\,dx = \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\Bigg|_0^1$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$$

$$\frac{3}{4} - \frac{2}{3} = \frac{1}{12}$$

Q1(b) $\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$

WHY IS THIS IMPORTANT?

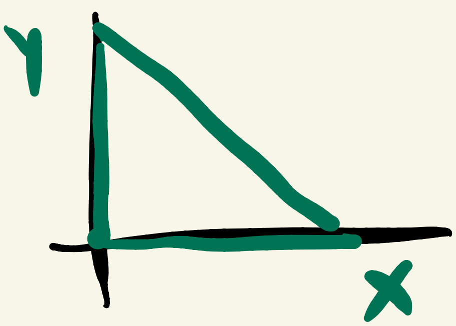
$$\boxed{VAR(X+Y) = VAR(X) + 2COV(X,Y) + VAR(Y)}$$

$$E[X] = \int x f_X(x)\,dx = \int_0^1 \int_{y=0}^{y=1-x} x\, 2\, dy\, dx$$

$$= 2\int_0^1 x\,[y]_{y=0}^{y=1-x}\,dx = 2\int_0^1 x(1-x)\,dx$$

$$= 2\int_0^1 x - x^2\,dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{2}{6} \checkmark$$



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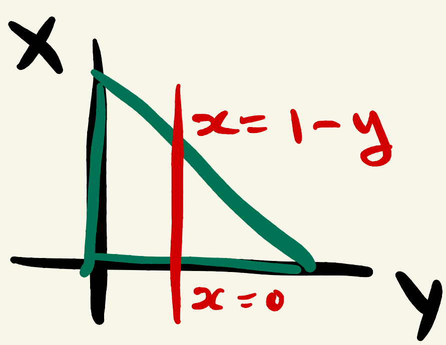


$$E[Y] = \int_{y=0}^{y=1} \int_{x=0}^{x=1-y} y\, 2\, dx\, dy = \frac{2}{6} \text{ By Symmetry}$$

$$= \frac{1}{3}$$

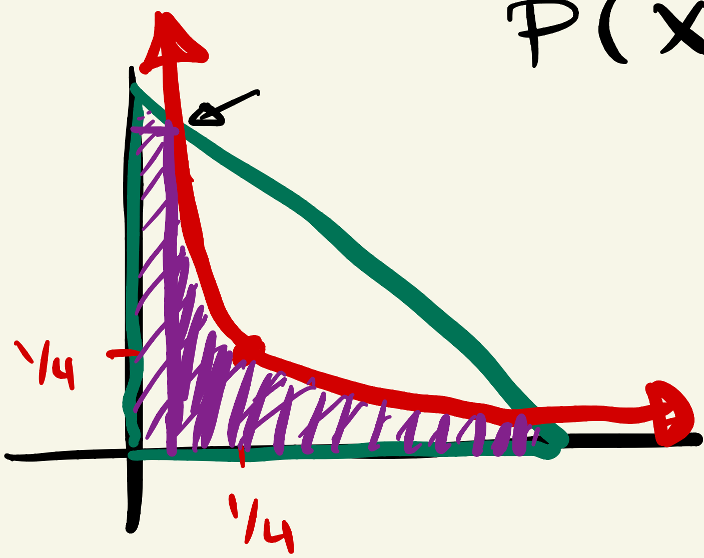
$$\mathrm{Cov}(X,Y) = \frac{1}{12} - \left(\frac{2}{6}\right)^2$$

$$-\frac{1}{36} \leftarrow = \frac{1}{12} - \frac{1}{9} < 0$$

Q1(c)

Compute $P(\text{Area} < 1/16)$

$$P(XY < 1/16)$$



$$y < \frac{1}{16x}$$

STEP 1: DETERMINE LIMITS of Integration

$$\left.\begin{aligned} xy &= \frac{1}{16} \\ y &= 1-x \end{aligned}\right] \longrightarrow x(1-x) = \frac{1}{16}$$

$$16x - 16x^2 = 1$$

$$16x^2 - 16x + 1 = 0$$

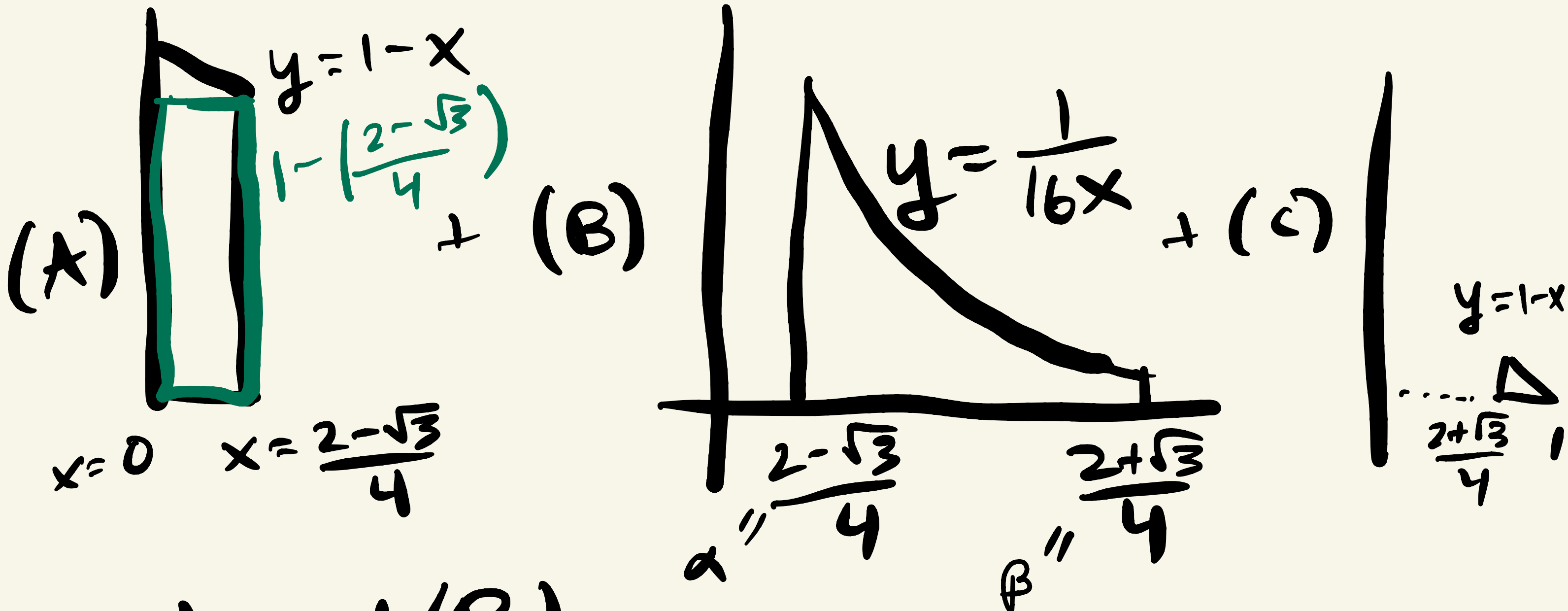
$$x^2 - x + \frac{1}{16} \qquad ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{1}{2} \pm \frac{\sqrt{1 - \frac{1}{4}}}{2} = \frac{\sqrt{3/4}}{2} = \frac{\sqrt{3}}{4}$$

$$x = \frac{2 \pm \sqrt{3}}{4}$$

3 Integrals to do



Integral (B)

$$\int_{\alpha}^{\beta} \frac{1}{16x}\,dx = \frac{1}{16} \ln x \Big|_{\alpha}^{\beta}$$

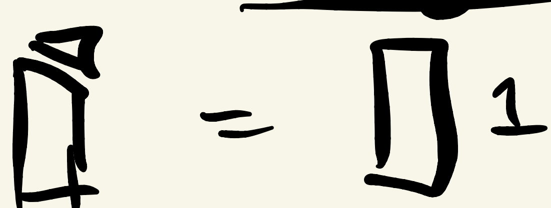
$$= \frac{1}{16}\left[\ln \beta - \ln \alpha\right]$$

$$\boxed{\frac{1}{16}\ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)} = \frac{1}{16}\ln\left(\frac{\beta}{\alpha}\right)$$

+ FINAL ANSWER

Integral (A) + Integral (C)

$= \frac{2-\sqrt{3}}{4}$



base $= \frac{2-\sqrt{3}}{4}$

② Show up at Nikes to work @ 10am

Customers show up $\sim \text{Exp}(\lambda = \frac{1}{10})$

X = service time $X \sim \text{Unif}(1, 6)$

Q: When do I expect to be done serving my first customer?

$$\rightarrow E[Y+X] = E[Y] + E[X]$$

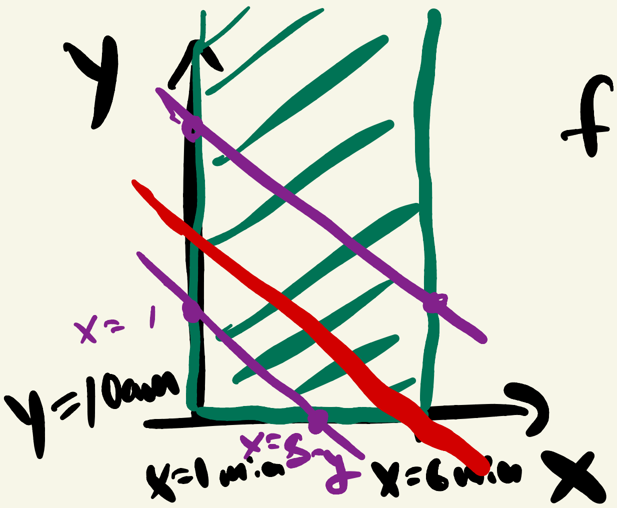
$$= 10{:}10\text{am} + \frac{6-1}{2}$$

$$\boxed{= 10{:}12\text{am } 30 \text{ secs}}$$

CAVEAT LECTOR ☆

I drew this diagram wrong in class. So skip this problem

OK Find Density of $Y + X = S$



$x + y = s$ $y = s - x$

$x = s - y$

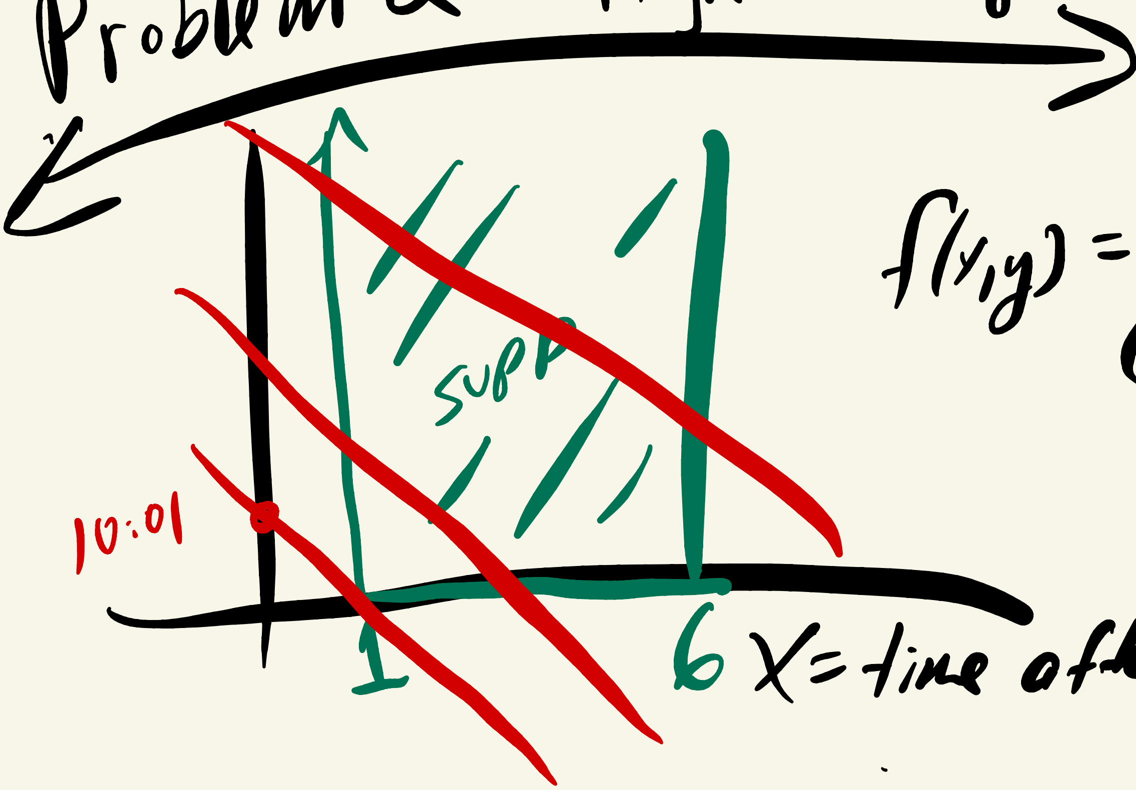
$$f_{X,Y} = \begin{cases} \frac{1}{10} e^{-y/10} & y \geq 10\text{ am} \\ & 1 \leq x \leq 6 \text{ mins} \\ 0 & \text{o.w.} \end{cases}$$

$$f_S(s) = \int_{x=1}^{x=s-y} \frac{1}{10} e^{-y/10} \, dx$$

DO THIS LATER

$$= \frac{1}{10} e^{-y/10} (s - y - 1)$$

Problem 2 - Right setup



$$f(x,y) = \begin{cases} \lambda e^{-\lambda y} & \text{for } y>0,\ 1 \le x \le 6 \\ 0 & \text{o.w.} \end{cases}$$

X = time after you start

"Don't worry about this problem
we'll do something similar for
the practice final on Mon. May 10"
- [illegible]

③ $X \sim \text{Unif}(0,1)$

$$P(A \mid X = x) = x^2$$

Q: What is $P(A)$?

(Rule of Averaged Conditionals aka Law of Total Probability)

$$P(A) = \int P(A \mid X = x) \underbrace{f_X(x)}_{= \mathbb{1}_{(0,1)}} \, dx$$

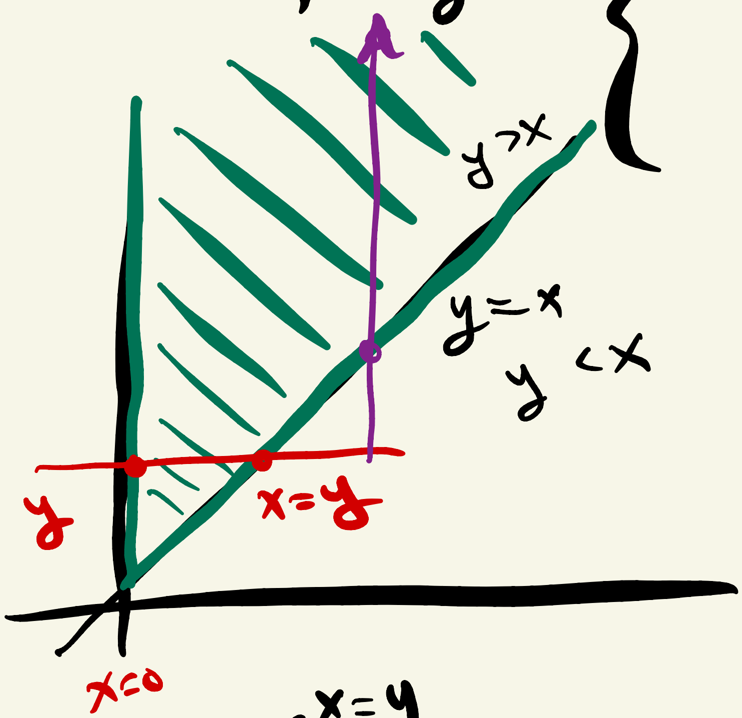
indicator function

$$= \int_0^1 P(A \mid X = x) \, dx$$

$$= \int_0^1 x^2 \, dx = \frac{x^3}{3} \Big|_0^1 = \boxed{1/3}$$

Q4

$$f(x,y) = \begin{cases} \lambda^3 x e^{-\lambda y} & 0<x<y \\ 0 & \text{o.w.} \end{cases}$$



(a) PDF for Y

$$f_Y(y) = \int_{x=-\infty}^{x=\infty} f(x,y)\,dx$$

$$f_Y(y) = \int_{x=0}^{x=y} \lambda^3 x e^{-\lambda y}\,dx = \lambda^3 e^{-\lambda y}\left[\frac{x^2}{2}\right]_{x=0}^{x=y}$$

$$\boxed{f_Y(y) = \frac{\lambda^3 y^2 e^{-\lambda y}}{2} \text{ for } 0<y}$$

Gamma(3, λ) Distribution

So $E = \frac{3}{\lambda}$

IGNORE THE REST

(b) $E[Y] = \int y f_Y(y)\,dy = \int_{y=0}^{y=\infty} \frac{\lambda^3 y^3}{2} e^{-\lambda y}\,dy$

JUST LOOK E[Gamma] up!

See Bottom of next page!

Integrate by Parts 3x! YUCK

ALTERNATIVELY

$$E[Y] = E[E[Y|X]]$$

$$E[Y|X=x] = \int y\, f_{Y|X=x}(y|x)\, dy$$

$$f_{Y|X=x} = \frac{f(x,y)}{f(x)} = \frac{\lambda^3 x e^{-\lambda y}}{\lambda^2 x e^{-\lambda x}} = \lambda e^{\lambda x - \lambda y}$$

$$f_X(x) = \int_{y=x}^{y=\infty} \lambda^3 x e^{-\lambda y}\, dy = \lambda^2 x \underbrace{\int_x^{\infty} \lambda e^{-\lambda y}}$$

$$\lambda^2 x e^{-\lambda x}$$

$$f_X(x) = \lambda^2 x e^{-\lambda x} \qquad 0 < x < \infty$$

SKIP THIS

$$E[Y|X=x] = \int y \lambda e^{\lambda x - \lambda y}\, dy$$

$$= e^{\lambda x} \int y \lambda e^{-\lambda y}\, dy$$

ERR... THIS ISN'T MUCH EASIER.

JUST USE THIS

$r=3$ so $\boxed{E = \frac{3}{\lambda}}$

Poisson Arrival Times (Gamma Distribution)

If T_r is the time of the rth arrival after time 0 in a Poisson process with rate λ, or if $T_r = W_1 + W_2 + \cdots + W_r$ where the W_i are independent with exponential (λ) distribution, then T_r has the *gamma* (r, λ) distribution defined by either (1) or (2) for all $t \geq 0$:

GOT TO RECOGNIZE

(1) **Density:** $P(T_r \in dt)/dt = P(N_t = r-1)\lambda = e^{-\lambda t}\dfrac{(\lambda t)^{r-1}}{(r-1)!}\lambda$

where N_t, the number of arrivals by time t in the Poisson process with rate λ, has Poisson (λt) distribution. In words, the probability per unit time that the rth arrival comes around time t is the probability of exactly $r-1$ arrivals by time t multiplied by the arrival rate.

(2) **Right tail probability:** $P(T_r > t) = P(N_t \leq r-1) = \displaystyle\sum_{k=0}^{r-1} e^{-\lambda t}\frac{(\lambda t)^k}{k!}$

because $T_r > t$ if and only if there are at most $r-1$ arrivals in the interval $(0, t]$.

(3) **Mean and SD:** $E(T_r) = r/\lambda \qquad SD(T_r) = \sqrt{r}/\lambda$

WS22

$$f(x,y) = f_X(x)\,f_Y(y)$$

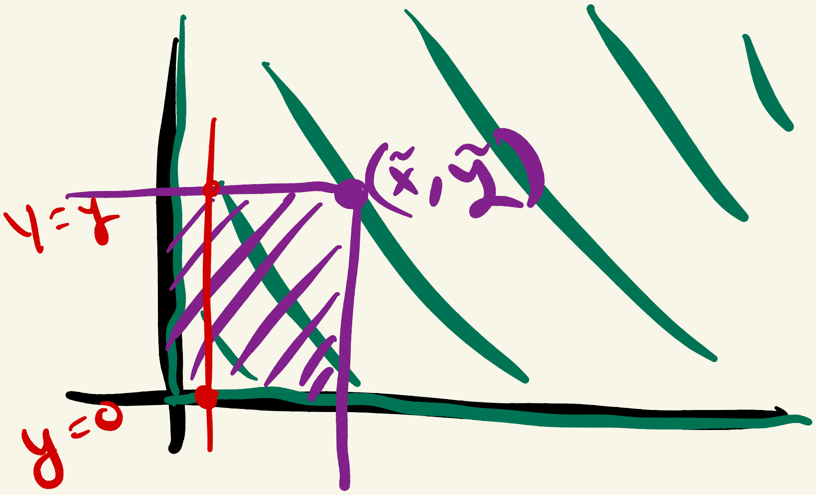
Q6

$$f(x,y) = 6e^{-2x-3y} \qquad x,y \geq 0$$

$$= \left(2e^{-2x}\right)\left(3e^{-3y}\right)$$

$$\rightsquigarrow \left(1-e^{-2\tilde{x}}\right)\left(1-e^{-3\tilde{y}}\right)$$

GUESS...



$$\int_{x=0}^{x=\tilde{x}} \int_{y=0}^{y=\tilde{y}} \left(2e^{-2x}\right)\left(3e^{-3y}\right) dy\, dx$$

$$= \int_{x=0}^{x=\tilde{x}} 2e^{-2x} \left[-e^{-3y}\right]_{y=0}^{y=\tilde{y}} dx = \left(1-e^{-3\tilde{y}}\right) \int^{\tilde{x}} 2e^{-2x}$$

$$\boxed{= \left(1-e^{-3\tilde{y}}\right)\left(1-e^{-2\tilde{x}}\right)}$$

$$P(X \leq \tilde{x}, Y \leq \tilde{y}) = F_X(\tilde{x})\,F_Y(\tilde{y})$$

BC Independent X & Y