

MATH 362—Work Sheet 07

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Due on Friday, February 26th, 2021

Name: _____

★ SOLUTIONS ★

1. (5 points) An urn contains 4 red balls and 3 green balls.

- (a) (2 points) Nine draws are made with replacement. Let X be the number of times a green ball appears. Identify by name the probability distribution of X . Find the probabilities $P(X \geq 1)$ and $P(X \leq 5)$.

$$X \sim \text{BINOMIAL DISTRIBUTION} \quad n=9 \quad p=3/7$$

$$P(X \geq 1) = 1 - P(X=0) = 1 - \left(\frac{4}{7}\right)^9 \sim 94.3\%$$

$$P(X \leq 5) = \left(\frac{4}{7}\right)^9 + \binom{9}{1}\left(\frac{3}{7}\right)\left(\frac{4}{7}\right)^8 + \binom{9}{2}\left(\frac{3}{7}\right)^2\left(\frac{4}{7}\right)^7 + \binom{9}{3}\left(\frac{3}{7}\right)^3\left(\frac{4}{7}\right)^6 + \binom{9}{4}\left(\frac{3}{7}\right)^4\left(\frac{4}{7}\right)^5 + \binom{9}{5}\left(\frac{3}{7}\right)^5\left(\frac{4}{7}\right)^4$$

$$\sim 86.5\%$$

- (b) (2 points) Draws with replacement are made until the first green ball appears. Let N be the number of draws that are needed. Identify by name the probability distribution of N . Find the probability $P(N \leq 5)$.

$$N \sim \text{Geometric Distribution} \quad p=3/7$$

$$P(N \leq 5) = \left(\frac{3}{7}\right) + \left(\frac{4}{7}\right)\left(\frac{3}{7}\right) + \left(\frac{4}{7}\right)^2\left(\frac{3}{7}\right) + \left(\frac{4}{7}\right)^3\left(\frac{3}{7}\right) + \left(\frac{4}{7}\right)^4\left(\frac{3}{7}\right)$$

OR $= 1 - P(N > 5) = 1 - P(1st+5 are Red balls)$

$$= 1 - \left(\frac{4}{7}\right)^5 = 93.9\%$$

- (c) (1 point) Compare $P(X \geq 1)$ and $P(N \leq 9)$. Is there a reason why these should be the same?

$$P(X \geq 1) = 1 - P(X=0) = 1 - P(9 \text{ Red balls in a row})$$

$$P(N \leq 9) = 1 - P(N > 9) = 1 - P(9 \text{ Red balls in a row})$$

Those are the same!

2. (2 points) The population of a small town is 500. 20% of the population has red hair. One morning I go to a diner and see 15 customers are there.

- (a) (1 point) What is the probability that 10 of the 15 customers have red hair?

$$\binom{15}{10} (.2)^{10} (.8)^5 \approx \frac{1}{10,100}$$

(b) (1 point) What is the probability that at most 2 of the 15 customers has red hair?

$$P(X \leq 2) = (.8)^{15} + \binom{15}{1} (.2) (.8)^{14} + \binom{15}{2} (.2)^2 (.8)^{13}$$

$\sim 39.8\%$

3. (2 points) An urn contains 4 red balls and 3 green balls. Two balls are sampled at random.

(a) (1 point) Let Z denote the number of green balls in the sample when the draws are done without replacement. Give the possible value of Z and its probability mass function (PMF).

$Z \in \{0, 1, 2\}$

Alternatives

$$P(Z=0) = \binom{4}{2} \binom{3}{0} = \frac{2}{7}$$

$$P(Z=1) = \binom{4}{1} \binom{3}{1} = \frac{4}{7}$$

$$P(Z=2) = \binom{3}{2} \binom{4}{0} = \frac{1}{7}$$

(b) (1 point) Let W denote the number of green balls in the sample when the draws are done with replacement. Give the possible values and the PMF of W .

W is Binomially distributed w $n=2$ $p=3/7$

$W=k$	0	1	2
$P_W(k)$	$\binom{4}{0} \left(\frac{3}{7}\right)^0 \left(\frac{4}{7}\right)^2$	$2 \binom{4}{1} \left(\frac{3}{7}\right)^1 \left(\frac{4}{7}\right)^1$	$\binom{3}{2} \left(\frac{3}{7}\right)^2 \left(\frac{4}{7}\right)^0$

4. (5 points) Choose one of the words in the following sentence uniformly at random and then choose one of the letters of that word, again uniformly at random:

SOME DOGS ARE BROWN

(a) (1 point) Find the probability that the chosen letter is R.

$$P(R) = P(R|ARE)P(ARE) + P(R|BROWN)P(BROWN)$$

$$= \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{4} = \frac{1}{4} \left(\frac{5}{15} + \frac{3}{15} \right) = \frac{2}{15}$$

(b) (1 point) Let X denote the length of the chosen word. Determine the PMF of X .

$X=k$	3	4	5
$P_X(k)$	$1/4$	$2/4$	$1/4$

- (c) (1 point) For each possible value k of X determine the conditional probability $P(X = k | X > 3)$.

3 WORDS w/ $X > 3$: SOME, DOGS, BROWN

$$P(X = 4 | X > 3) = \frac{2}{3} \quad P(X = 5 | X > 3) = \frac{1}{3} \quad \text{For all other } k$$

- (d) (1 point) Determine the conditional probability $P(\text{the chosen letter is R} | X > 3)$.

SOME
DOGS
 $\frac{1}{3}$ BROWN $\frac{1}{5}$ R

$$\frac{1}{15}$$

- (e) (1 point) Given that the chosen letter is R, what is the probability that the chosen word was BROWN?

$$P(\text{BROWN} | R) = \frac{P(\text{BROWN} \cap R)}{P(R)} = \frac{\frac{1}{5} \cdot \frac{1}{4}}{\frac{1}{3} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{4}} = \frac{\frac{1}{5} \cdot \frac{1}{4}}{\frac{1}{3} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{4}} = \frac{\frac{1}{5} \cdot \frac{1}{4}}{\frac{1}{3} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{4}} = \frac{1}{8}$$

5. (2 points) This is a challenge question and will not be on the exam. You are given a fair die. You must decide ahead of time how many times to roll. If you roll exactly 2 sixes, you win a new Tesla. How many rolls should you take to maximize your chances and what are the chances of winning? Note: there are two equally good choices for the best number of rolls.

Question asks to maximize

$$P(X=2) = \binom{n}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{n-2}$$

across all possible n .

Use a calculator to find out!