AMAT 362 - PROBABILITY for Statistics

Recap
Lecture 10 : LINEARITY OF E(-)

$$E(aX+bY) = aE(X) + bE(Y)$$

Lecture 11 → Continue the study of Joint PMFs for 2 or more RVs

Ex

Draw 2 cards from a 52 card deck

X = # Queens ~~in~~ among 2 cards

Y = # Kings among 2 cards

Write down $P_{X,Y}(x,y) = P(X=x \text{ AND } Y=y)$ across all possible values for x, y

- $P(0,0) = ?$

$52 - 8 = 44$ non K or Q cards

$$\boxed{p(0,0) = \frac{\binom{44}{2}}{\binom{52}{2}}}$$

$$\boxed{P(2,0) = \binom{4}{2} / \binom{52}{2}}$$

$\overset{\shortparallel}{P}(0,2)$ By symmetry

$$\boxed{P(1,0) = \frac{\binom{4}{1}\binom{44}{1}}{\binom{52}{2}}}$$

$\overset{\shortparallel}{P}(0,1)$ By symmetry!

$$\boxed{P(1,1) = \frac{\binom{4}{1}\binom{4}{1}}{\binom{52}{2}}}$$

Joint Table

P_Y	Y=#Kings			
1%	2	<1%	0	0
14%	1	~13%	>1%	0
85%	0	~71%	~13%	<1%
		0	1	2
	P_X	85%	14%	1%

X=#Queens

N.B. X & Y ARE NOT INDEP!

OBSERVE:

1) NORMALIZATION AXIOM

$$\sum_{x,y} P(x,y) = 1$$

ex) 71% + 26% + 3% = 100% ✓

2) MARGINALIZATION AXIOM (Law of Total Probability)

$$\boxed{P_X(x) = \sum_y P(x,y)}$$

PMF for X = restricted sum of Joint PMF

Ex) $\underset{\text{NO Q's}}{P(X=0)} = \dfrac{\binom{48}{2}}{\binom{52}{2}} \approx 85\%$ ✓

$$\boxed{P_Y(y) = \sum_x P(x,y)}$$

INDEPENDENCE

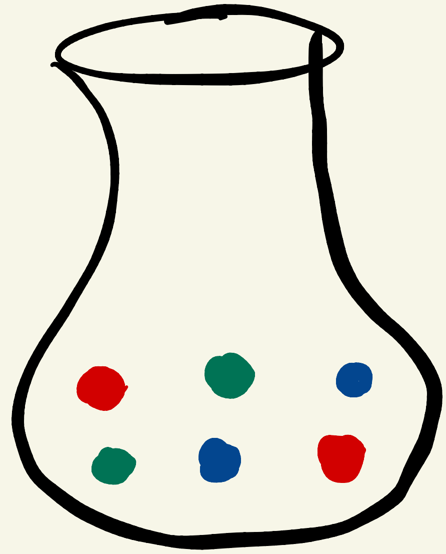
Def X & Y are independent RVs if

$$\boxed{P_{X,Y}(x,y) = P_X(x)P_Y(y)}$$

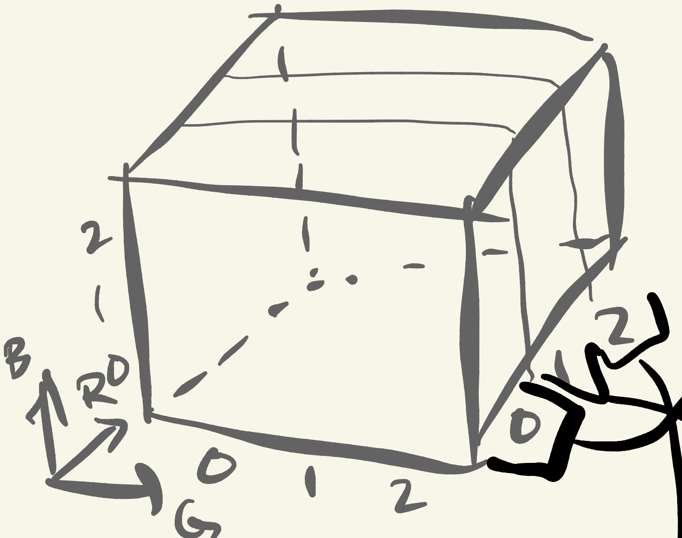
NOT ALWAYS TRUE!

Example of 3 RV's!

URN w/ 2 Red Balls, 2 Green Balls, 2 Blue Balls → Draw 2



Q: How do we visualize $P_{R,B,G}(r,b,g)$?



Fix R=0

B \ G	0	1	2
2	$\frac{\binom{2}{2}}{\binom{6}{2}}$	0	0
1	0	$\frac{\binom{2}{1}\binom{2}{1}}{\binom{6}{2}}$	0
0	0	0	$\frac{\binom{2}{2}}{\binom{6}{2}}$

1/15 4/15

$\frac{1}{\binom{6}{2}} = \frac{2}{6\cdot 5} = \frac{1}{15}$

Fix R=1

$\frac{\binom{2}{1}\binom{2}{1}}{\binom{6}{2}}$

B \ G	0	1	2
2	0	0	0
1	$\boxed{\frac{4}{15}}$	0	0
0	0	$\frac{4}{15}$	0

Fix R=2

B \ G	0	1	2
2	0	0	0
1	0	0	0
0	$\frac{\binom{2}{2}}{\binom{4}{2}}$	0	0

1/15

MULTIPLICATION RULE

$$P(X=x, Y=y) = P(X=x \mid Y=y)\,P(Y=y)$$

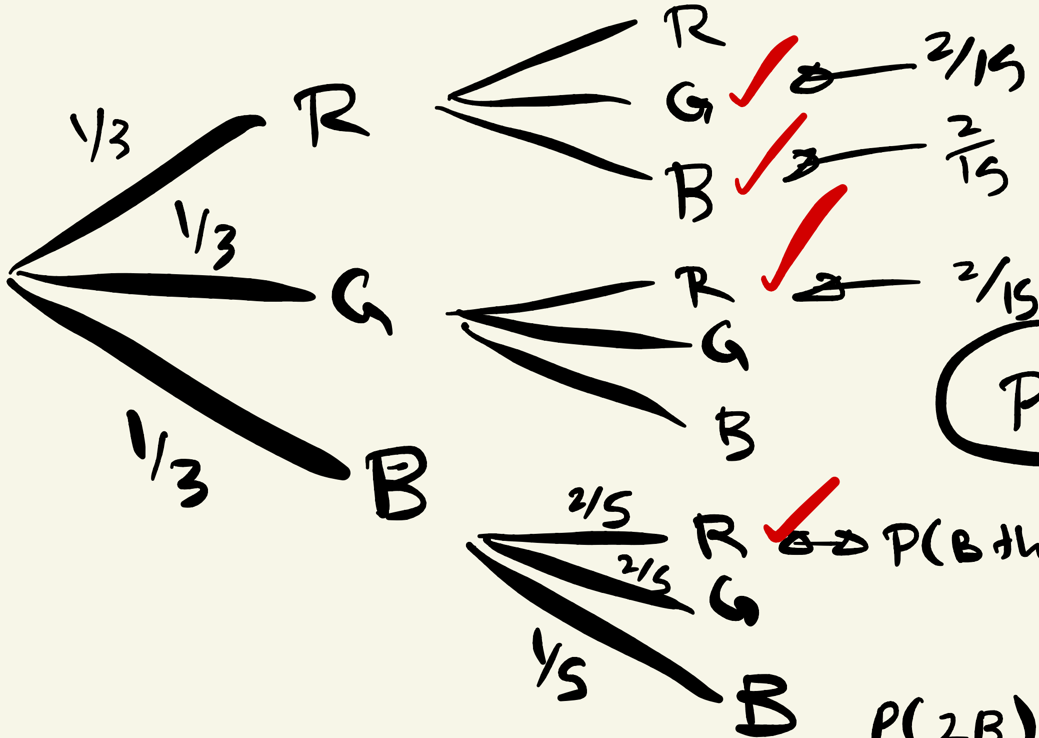
Ex: $P(R=1, G=0, B=1)$

$$= \underbrace{P(G=0, B=1 \mid R=1)}_{\text{Remove red balls, 4 left. Prob 1B}}\ \underbrace{P(R=1)}$$

$$\frac{\binom{2}{1}}{\cancel{\binom{4}{1}}} \quad \frac{\binom{2}{1}\cancel{\binom{4}{1}}}{\binom{6}{2}} = \frac{\binom{2}{1}\binom{2}{1}}{\binom{6}{2}}$$

3rd Way

TREE OF POSSIBLE DRAWS



$$\frac{\binom{2}{1}\binom{4}{1}}{\binom{6}{2}} = \frac{8}{15} = P(R=1)$$

$P(B \text{ then } R) = \frac{2}{15}$

$P(2B) = \frac{1}{15} = \frac{1}{\binom{6}{2}}$

INDEPENDENT VERSION of URN

↳ Draws w/ replacement

⇒ MULTINOMIAL DISTRIBUTION

Sps I have n indep. trials w/ m possible outcomes ("categories"/"colors") at each step

Define N_i = # trials whose outcome was i $(1 \leq i \leq m)$

$$\boxed{P(N_1 = n_1, N_2 = n_2, \ldots, N_m = n_m) = \frac{n!}{n_1! \cdots n_m!} p_1^{n_1} \cdots p_m^{n_m} \quad \text{where } p_i = \text{prob. of outcome } i}$$

Ex Urn w/ 2R, 2B, 2G
2 draws w/ replacement

$p_1 = p_2 = p_3 = \frac{1}{3}$

Use multinomial formula

R "N_1"	B "N_2"	G "N_3"	
2	0	0	$\to \frac{2!}{2!0!0!}\left(\frac{1}{3}\right)^2\left(\frac{1}{3}\right)^0\left(\frac{1}{3}\right)^0 = 1/9$
1	1	0	$\to \frac{2!}{1!1!}\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = 2/9$
0	2	0	$\to 1/9$
1	0	1	$\to 2/9$
0	0	2	$\to 1/9$
0	1	1	$\to 2/9$

$3/9$

$6/9$

$+$

$9/9$

Normalization ✓

Marginalization

$$P(R=r) = \sum_{(g,b)} P(r,g,b)$$

$P(R=2) = 1/9 \qquad P(R=1) = 4/9 \qquad P(R=0) = 4/9$

Check this w/ BINOMIAL DISTRIBUTION

p = prob of success = prob of red ball $= \frac{2}{6} = \frac{1}{3}$

q = prop of not red = prob of g or b $= 4/6 = 2/3$

$\Big\} \to 1$ ✓

$$P(R=r) = \binom{n}{r} p^r q^{n-r} \to$$

$P(R=2) = \binom{2}{2}\left(\frac{1}{3}\right)^2 = 1/9$ ✓

$P(R=1) = \binom{2}{1}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = 4/9$ ✓

$P(R=0) = \binom{2}{0}\left(\frac{2}{3}\right)^2 = 4/9$ ✓

Q: What is the expected # of Red balls?
(w/ replacement)

$$E(R) = 2 \cdot P(R=2) + 1 \cdot P(R=1) + 0 \cdot P(R=0)$$

$$= 2 \cdot \frac{1}{9} + 1 \cdot \frac{4}{9} = \frac{6}{9} = 2/3$$ ✓

DIRECT BY DEF. METHOD

INDICATOR METHOD

Def $R_1 = \begin{cases} 1 & \text{if 1st Ball Red} \quad p = 1/3 \\ 0 & \text{if 1st Ball} \neg \text{Red} \quad q = 2/3 \end{cases}$

$R = R_1 + R_2$ where

$R_2 = \begin{cases} 1 & \text{if 2nd Ball Red} \quad p = 1/3 \\ 0 & \text{o.w.} \quad q = 2/3 \end{cases}$

$$E(R) = E(R_1) + E(R_2)$$
$$= 1/3 + 1/3 = 2/3$$ ✓

GO BACK TO DRAWS W/O REPLACEMENT

$$E(R) = 2 \cdot \frac{1}{15} + 1 \cdot \frac{8}{15} + 0 \cdot \frac{6}{15}$$
$$= \frac{10}{15} = 2/3$$ WAAAAAT!

Used indicator method

$R = R_1 + R_2$ $\quad R_1 = \begin{cases} 1 & \text{if 1st Red} \quad 1/3 \\ 0 & \text{o.w.} \end{cases}$

$E(R) = E(R_1) + E(R_2)$ $\quad R_2 = \begin{cases} 1 & \text{if 2nd Red} \quad 1/3 \end{cases}$ ← Check this!

$= 2/3$ ✓

Two Upshots/Methods

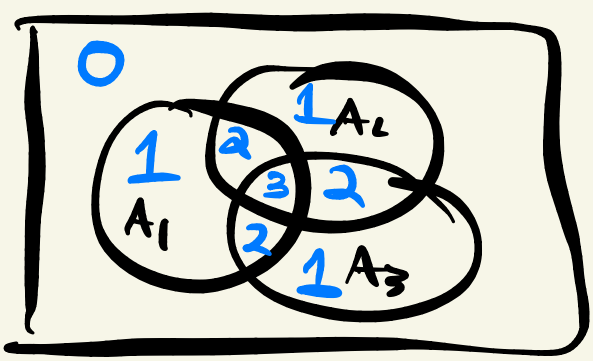
1) Given an event $A \subseteq \Omega$

define $I_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{o.w.} \end{cases}$

then $\boxed{E(I_A) = P(A)}$

2) If X counts the # of occurrences of an event among a collection of events $A_1, \dots, A_n$

$$\begin{aligned} E(X) &= E(I_{A_1} + \cdots + I_{A_n}) \\ &= E(I_{A_1}) + \cdots + E(I_{A_n}) \\ &= P(A_1) + \cdots + P(A_n) \end{aligned}$$



Ex $X = \#$ of 6's out of three rolls

A_1 = 1st roll is a 6

A_2 = 2nd roll is a 6

A_3 = 3rd roll is a 6

$E(X) = \frac{3}{6} = \frac{1}{2}$ = Expected # of 6's !