MATH 362—Work Sheet 17

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Due Saturday, April 17, 2021

Name: SOLUTIONS

1. (1 point) Suppose a number is generated uniformly at random from the unit interval $(0,1)$, i.e. $X \sim \text{Unif}(0,1)$. What's the probability of X being within 2 decimal places of .35, after rounding? For example, .349 would round up to .35 and .353 would round down.

$$P(.345 \le X < .355) = .355 - .345 = .01$$

2. (1 point) Repeat the question above except under the assumption that $X \sim \text{Norm}(0,1)$ distribution, i.e. normally distributed with mean 0 and variance 1.

$$P(.345 \le X < .355) = \Phi(.355) - \Phi(.345)$$

Use online calculator... $.0038$

3. (5 points) Suppose X is a random variable whose density is $f(x) = cx(1-x)$ for $0 < x < 1$ and $f(x) = 0$ otherwise.

   (a) (1 point) Find the value of c in order for this to be a valid PDF.

$$\int_{-\infty}^{\infty} f_X(x) = 1 \quad \leadsto \quad c\int_0^1 x - x^2\,dx = 1$$

by NORMALIZATION AXIOM

$$\Rightarrow \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{6}$$

$$\boxed{c = 6}$$

   (b) (1 point) $P(X \le 1/2)$

Let's find the CDF for part (d)

$$F_X(s) = \int_{-\infty}^{s} 6x(1-x) = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^s = \boxed{3s^2 - 2s^3}$$

$$= \frac{3}{4} - \frac{2}{8} = 1/2$$

   (c) (1 point) $P(X \le 1/3)$

$$F_X(1/3) = 3\left(\frac{1}{9}\right) - 2\left(\frac{1}{27}\right) = \frac{7}{27}$$

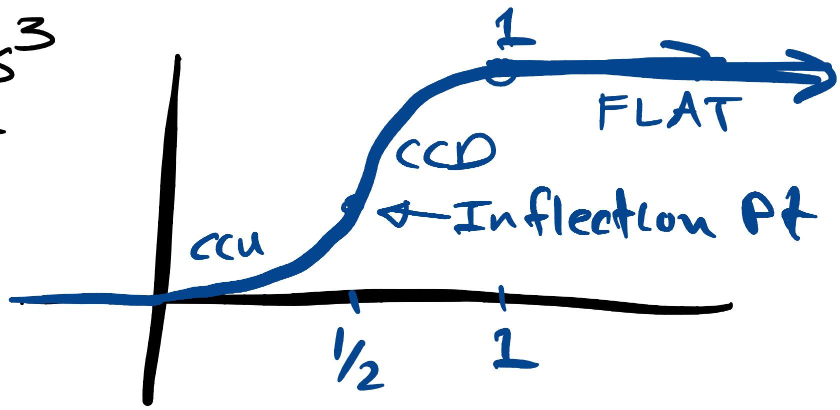
(d) (2 points) Make a drawing of the function $F(s) := \int_{-\infty}^{s} f(x)dx$, which is called the *cumulative distribution function (CDF)* of X.

$F(s) = 3s^2 - 2s^3$

$F'(s) = 6s - 6s^2$

$F''(s) = 6 - 12s$
$= 0$

$s = 1/2$



4. (4 points) *We'll do this in class, but please copy down the derivations and understand them!* Recall that X is exponentially distributed with parameter λ, written $X \sim \text{Exp}(\lambda)$, if its PDF is $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and is zero otherwise.

(a) (1 point) Derive a formula for $E(X)$ in terms of λ.

By L'Hôpital

$$\int_0^\infty x \cdot \lambda e^{-\lambda x} dx = \underbrace{-x e^{-\lambda x} \Big|_0^\infty}_{0} + \int_0^\infty e^{-\lambda x} dx$$

$$= -\frac{e^{-\lambda x}}{\lambda} \Big|_0^\infty = \boxed{\frac{1}{\lambda}} = E[\text{Exp}(\lambda)]$$

(b) (1 point) Derive a formula for $V(X)$ in terms of λ.

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = -x^2 e^{-\lambda x} \Big|_0^\infty + \underbrace{\int_0^\infty 2x e^{-\lambda x} dx}_{(\star)}$$

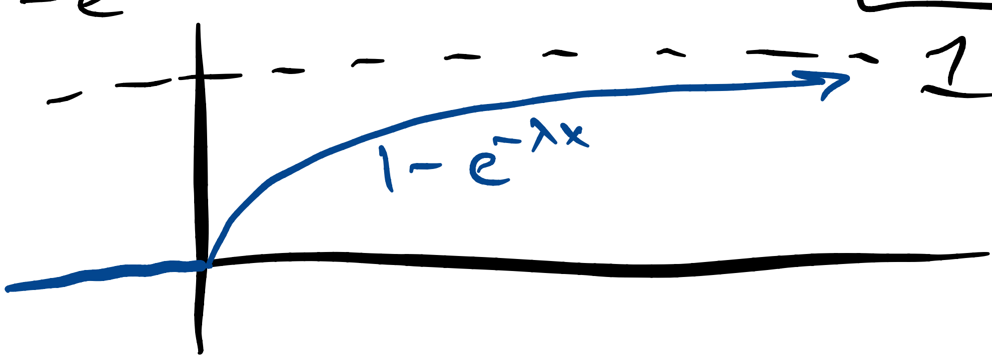
0 by L'Hôpital

Now from (a) know $(\star) = \frac{2}{\lambda^2} \Rightarrow V(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$

$$\boxed{= 1/\lambda^2}$$

(c) (1 point) Derive and draw the CDF for X.

$1 - e^{-\lambda x}$



(d) (1 point) Derive a formula for the median of X in terms of λ. This is also called the *half-life*, which measures the amount of time needed for a sample of radioactive material to degrade by half or for some chemical to degrade or be absorbed by half. Bonus question: what is the half-life of caffeine?

$$1 - e^{-\lambda m} = 1/2 \Rightarrow \boxed{m = \frac{\ln 2}{\lambda}}$$

5. (2 points) Suppose a particular kind of radioactive element has a half-life of 1 year. Find

(a) (1 point) The probability that an atom of this type survives for at least 5 years.

First need to find λ! $m = 1 \Rightarrow \lambda = \ln 2$

$$P(T > 5\text{ years}) = e^{-(\ln 2)5} = \frac{1}{2^5} = \frac{1}{32}$$

(b) (1 point) The time at which a sample of this element decays to 10% of its original purity.

Want $e^{-(\ln 2)t} = \frac{1}{10} \Rightarrow -t \ln 2 = -\ln 10$

$$t = \frac{\ln 10}{\ln 2} = 3.32 \text{ years}$$

6. (2 points) Suppose the length L of a phone call is exponentially distributed with $\mu = 10$ minutes.

(a) (1 point) Compute $P(L \geq 20)$

$\mu = 10 \text{ mins} = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{10}$

$$P(L \geq 20) = e^{-\lambda \cdot 20} \boxed{= e^{-2} = \left(\frac{1}{e}\right)^2}$$

(b) (1 point) Compute $P(8 \leq L \leq 22)$

$$\boxed{e^{-2.2} - e^{-.8}}$$

7. (3 points) Measurements on the mass of a metal part produced at a factory are IID with $\mu = 12$ grams and $\sigma = 1.1$ grams.

(a) (1 point) Find the chance that a single measurement is between 11.8 and 12.2 grams, assuming the mass varies according to a normal distribution.

$$\Phi\left(\frac{12.2-12}{1.1}\right) - \Phi\left(\frac{11.8-12}{1.1}\right)$$

$$2\Phi\left(\frac{.2}{1.1}\right) - 1 = .144 \text{ or } 14.4\%$$

(b) (1 point) Estimate the chance that the average of 100 measurements is between 11.8 and 12.2 grams. Answer this is as well: Is it necessary to assume that each measurement is normally distributed?

Same as asking

By Lec 14 CLT

$$P(100 \times 11.8 \le S_n \le 100 \times 12.2) \approx \Phi\left(\frac{100 \times 12.2 - 100 \times 12}{\sigma\sqrt{100}}\right) - \Phi\left(\frac{100 \times 11.8 - 100 \times 12}{\sigma\sqrt{100}}\right)$$

$$\approx 2\Phi\left(\frac{10}{1.1}(.2)\right) - 1$$

$$\approx 2\Phi(1.81) - 1 \approx 93.09\%$$

NO B/c CLT does need each X_i to be normal !