LEC 19 LIVE

Q1 $X_1, X_2, \ldots, X_{10k} \sim Exp(\lambda)$

$E(X_i) = 20$ days $\Rightarrow \lambda = \frac{1}{20}$

X_i = Time to failure of part i

$P(X_i > d) = \int_d^\infty \lambda e^{-\lambda t}\,dt = e^{-\lambda d}$ = prob of lasting d days

Define $S_{i,d} = \begin{cases} 1 & \text{if part } i \text{ is alive on day } d \\ 0 & \text{o.w.} \end{cases}$

$p = e^{-\lambda d}$

$q = 1-p$

$$\boxed{N_d = S_{1,d} + S_{2,d} + \cdots + S_{10k,d}}$$

Sum of Bernoullis $\Longrightarrow$ So... N_d is a BINOMIAL R.V.

$E(\text{Binomial}(n,p)) = np \Rightarrow$

$E[N_d] = np$
$= (10k\, e^{-\lambda d})$

Aside... Midterm 2 q4(d)
T~Geom $E[T] = 1/p \neq np$

(a) $d = 10$ days

$$p = e^{-\lambda \cdot 10} = e^{-\frac{10}{20}} = e^{-1/2}$$

$$E[N_{10}] = \underbrace{10{,}000}_{n} \cdot e^{-1/2} = \cancel{6{,}065} \quad \boxed{6{,}065}$$

plug in $d = 20 \rightarrow 10{,}000\, e^{-1} \sim 3{,}000$

$d = 30 \rightarrow 10{,}000\, e^{-3/2}$?

$$SD[\text{Binomial}] = \sqrt{npq}$$

$$n = 10{,}000$$

$$p = e^{-\lambda d} \qquad q = 1 - e^{-\lambda d}$$

Do the rest! Plug into a calculator

$d = 10 \quad SD(N_d) = 49$

$d = 20 \quad = 48.2$

$d = 30 \quad = 41.63$

(b) Expected time of first failure

$E[M]$ where $M = \min\{X_1, \dots, X_{10,000}\}$

$\Rightarrow M \sim Exp(10{,}000 \cdot \lambda)$

$\lambda = \frac{1}{20}$ $\quad Exp(500)$

$E[M] = \frac{1}{500}$ days

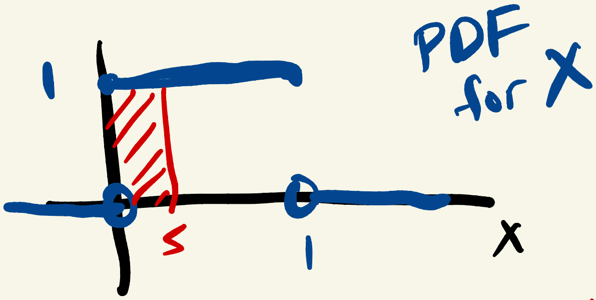
$$\frac{\overset{\text{mins/hr}}{60} \cdot \overset{\text{24 hours}}{24}}{500} = \boxed{2.88 \text{ minutes}}$$

$\rightarrow$ 2 mins

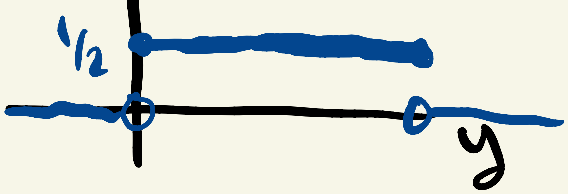
$\rightarrow$ 52.8 secs

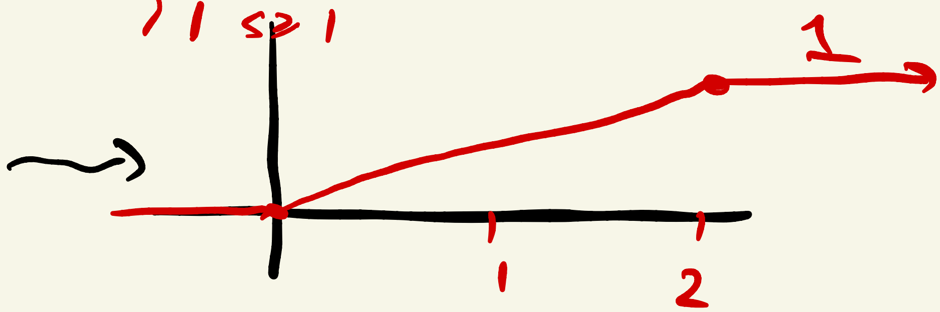
(1) $X \sim \text{Unif}[0,1]$ $\quad f_X(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$

$Y \sim \text{Unif}[0,2]$ $\quad f_Y(y) = \begin{cases} 1/2 & 0 \le y \le 2 \\ 0 & \text{o.w.} \end{cases}$



CDF for X $\Rightarrow F_X(s) = \begin{cases} 0 & s < 0 \\ s & 0 \le s \le 1 \\ 1 & s > 1 \end{cases}$





CDF for Y $\Rightarrow F_Y(s) = \begin{cases} 0 & s < 0 \\ \frac{1}{2}s & 0 \le s \le 2 \\ 1 & s > 2 \end{cases}$

(b) $M = \min\{X, Y\}$

Compute PDF for M? $\rightsquigarrow P(M > t)$

$= P(X > t \text{ AND } Y > t)$

$= P(X > t)P(Y > t)$

$= \boxed{(1-t)(1-\frac{t}{2})}$

$P(X \le t) = F_X(t)$

$\rightarrow P(X > t) = 1 - F_X(t)$

$P(M \le t) = 1 - P(M > t)$ $\quad$ For $0 \le t \le 1$

$$1-(1-t)\left(1-\frac{t}{2}\right) = P(M \le t)$$
$$= \text{CDF for } M$$

$$\cancel{1} - \left(\cancel{1} - \frac{t}{2} - t + \frac{t^2}{2}\right)$$

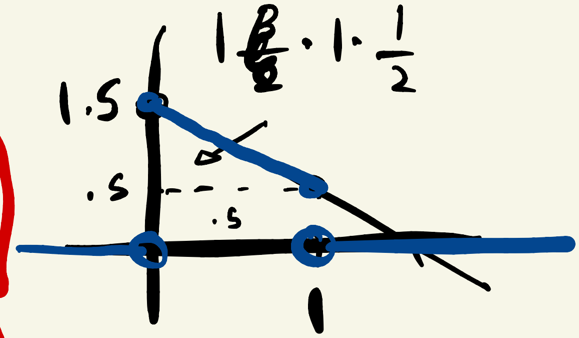
$$\frac{3}{2}t - \frac{t^2}{2} = F_M(t) = \text{CDF for } M$$

$$\frac{d}{dt}$$

$$\boxed{\text{PDF} = \frac{3}{2} - t}$$

$$\boxed{0 \le t \le 1}$$

o.o.w.



(b) $N = \max\{X, Y\}$

PDF for N? $\Rightarrow$ Consider the CDF!

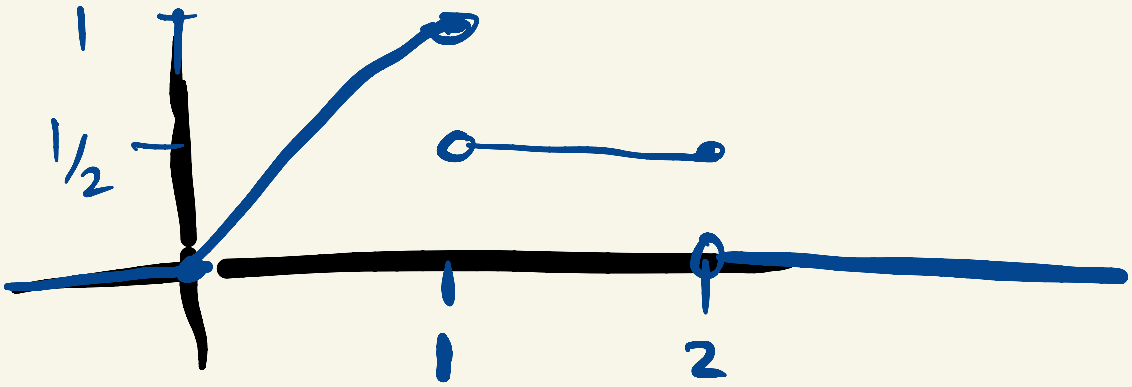
$$P(N \le t) = P(X \le t \text{ AND } Y \le t)$$
$$= P(X \le t)P(Y \le t)$$
$$= F_X(t)\, F_Y(t)$$

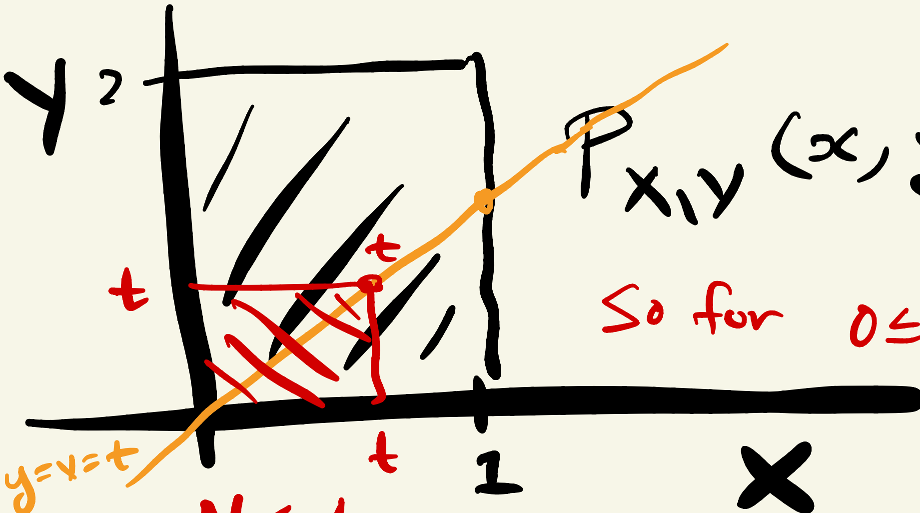
$t \quad 0 \le t \le 1$

$\frac{1}{2} \quad 1 \le t \le 2$

$\xleftarrow{\frac{d}{dt}}$

$$\begin{cases} = t \cdot \frac{t}{2} & \text{for } 0 \le t \le 1 \\ = 1 \cdot \frac{t}{2} & \text{for } 1 \le t \le 2 \end{cases}$$





$$P_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & 0 \le x \le 1 \\ & 0 \le y < 2 \\ 0 & \text{o.w.} \end{cases}$$

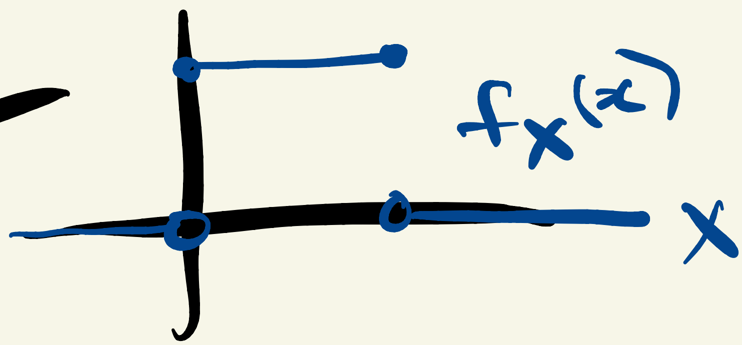
So for $0 \le t \le 1$

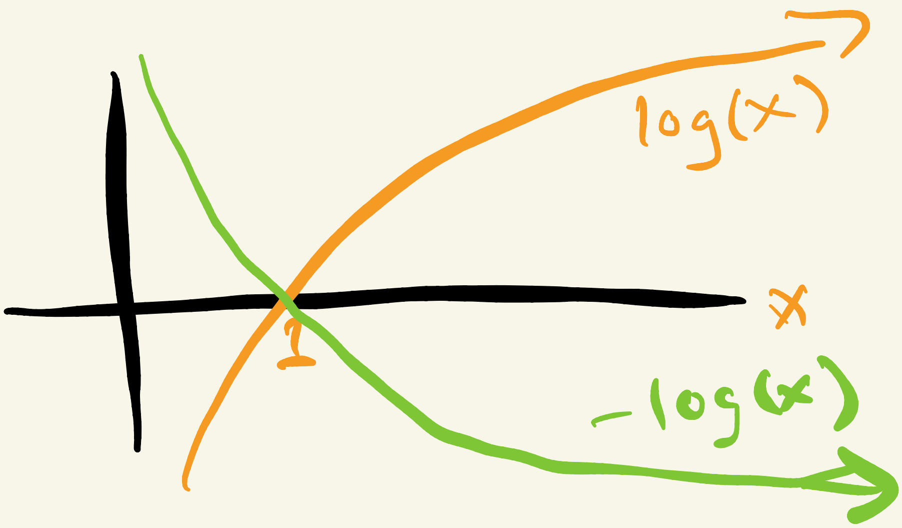
Accumulate area quadratically!

$N \le t$

$P(X \le t \text{ AND } Y \le t)$

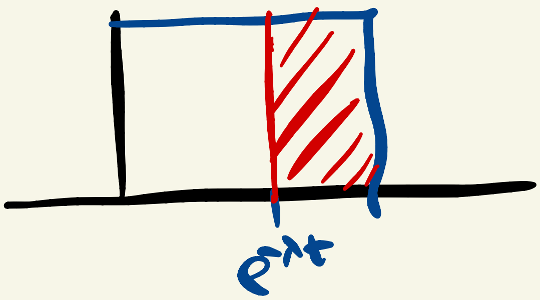
Q3





$$Y = -\frac{1}{\lambda}\log X$$

$$P(Y \le t) = P\left(-\frac{1}{\lambda}\log_e X \le t\right)$$



$$\log_e X \ge -\lambda t$$

$$P(X \ge e^{-\lambda t})$$

$$= 1 - e^{-\lambda t}$$

$$\frac{d}{dt}\left(1 - e^{-\lambda t}\right)$$

$$\boxed{f_Y(t) = \lambda e^{-\lambda t} \qquad t \ge 0}$$

so... $Y \sim Exp(\lambda)$!

WS18 Q5

(a) Call arrival rate $= 1 \frac{\text{call}}{\text{sec}}$

$= \lambda$

$W_i \sim Exp(\lambda = 1)$

$\rightarrow$ Find Probability 4th call occurs w/in 2 secs of 3rd call

W_1 = time till 1st call

W_2 = time till 2nd call AFTER 1st call

⋮

W_4 = time till 4th call AFTER 3rd call

$$P(W_4 \leq 2) = \int_0^2 \lambda e^{-\lambda t} dt$$

$$= 1 - e^{-2}$$

(b) $T_4 = W_1 + W_2 + W_3 + W_4$ = time to 4th call

$$P(T_4 \leq 5) = 1 - \underbrace{P(T_4 > 5)}$$

$$P(T_4 > 5) = P(N[0,5] \leq 4) = \text{Poisson}_5(0) + \text{Poisson}_5(1) + \text{Poisson}_5(2)$$

$$P(T_4 \geq 5) = \sum_{k=0}^{4} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

w/ $\lambda = 1 \quad t = 5$

$$\boxed{P(T_4 \geq 5) = \sum_{k=0}^{4} \frac{5^k}{k!} e^{-5}} \xrightarrow{\text{ANS}} \boxed{1 - P(T_4 \geq 5)}$$

(c) $E[T_4] = \frac{r}{\lambda} = \frac{4}{\lambda} = 4$

WS1B

Q6

PDF for X

$$f_X(t) = f_{X|\text{Machine 1}}(t) \cdot P(\text{Machine 1}) + f_{X|\text{Machine 2}}(t)\, P(\text{Mac. 2})$$

$$= \frac{4}{12} \lambda_1 e^{-\lambda_1 t} + \frac{8}{12} \lambda_2 e^{-\lambda_2 t}$$

where $\lambda_1 = \frac{1}{100}$ $\quad \lambda_2 = \frac{1}{200}$

(a) ... Do the integral ... $\boxed{\frac{4}{12} \cdot e^{-\lambda_1 200} + \frac{8}{12} e^{-\lambda_2 200}}$

(b) $E[X] = E[X \mid \text{Machine 1}] \cdot P(\text{Machine 1})$

$+ E[X \mid \text{Machine 2}]\, P(\text{Machine 2})$

$$= \boxed{\frac{4}{12} \cdot 100 + \frac{8}{12} \cdot 200}$$

$E[X^2] = E[X^2 \mid \text{Machine 1}]\, P(\text{Machine 1})$

$+ E[X^2 \mid \text{Machine 2}]\, P(\text{Machine 2})$

$$= \frac{2}{\lambda_1^2} \cdot \frac{4}{12} + \frac{2}{\lambda_2^2} \cdot \frac{8}{12}$$

$$= \boxed{2 \cdot 100^2 \cdot 4/12 + 2 \cdot 200^2 \cdot \frac{8}{12}}$$

$\rightarrow \text{VAR}[X] = E[X^2] - E[X]^2$