AMAT 362 - PROBABILITY for STATISTICS

LECTURE 18 : EXPONENTIAL RV's

↳ Where do these come from?

★ "POISSON POINT PROCESSES"

⤳ CDF TRICK

⟹ Gamma Distributions

RECALL

The Exponential RV ($f(t) = \lambda e^{-\lambda t}$ $t \geq 0$)

is the continuous (time) analog of the

Geometric RV (= try # of first success)

↳ "How many flips before first heads?"

ANALOGOUS QUESTIONS

1) HOW long do I need to wait at the DMV? — QUEUES

2) How long do I need remain on hold before getting through — CALLS (or emails)

3) How long do I need to wait before it is safe to touch something "SARS-COV2? — DETERIORATION or DECAY

So... YOU KNOW YOU NEED TO USE

$$T \sim \text{Exp}(\lambda) \quad \text{i.e. } P(a<T<b) = \int_a^b \lambda e^{-\lambda t}\,dt$$

(assuming $a, b \geq 0$)

HOW TO FIND λ?

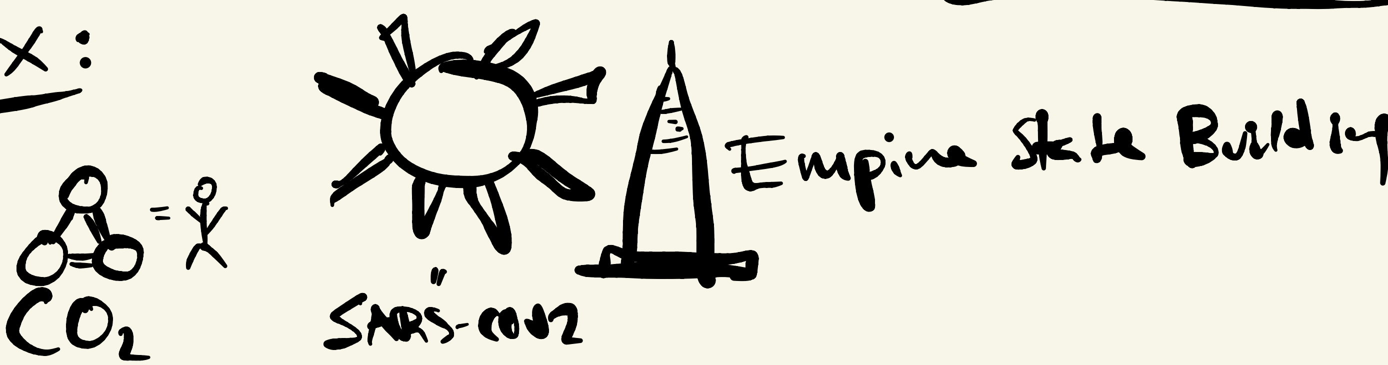
1) Avg Amount of time is given

$$E[T] = \frac{1}{\lambda} = \text{Avg} \quad \text{then} \quad \boxed{\lambda = \frac{1}{\text{Avg}}}$$

2) Half-life is given

$$\text{Median} = t_{1/2} = \frac{\ln(2)}{\lambda} \qquad \boxed{\lambda = \frac{\ln(2)}{t_{1/2}}}$$

Ex:



On a plastic surface SARS-COV2 has a half-life of 6.8 hours $\approx$ 7 hours

Q: A plastic bag has $1024 = 2^{10}$ SARS-COV2 on it

What's the amount of time required for all but 1 (on avg) coronavirus to decay?

Want $p = \text{Prob}(\text{A Coronavirus lives longer than } t)$

s.t. $np = 1$ where $n = 1024$

(Binomial expectation but w/ p a function of time)

$$p = \frac{1}{1024}$$

$$\frac{1}{1024} = 2^{-10} = P(T > t) = \int_t^\infty \lambda e^{-\lambda s}\,ds = e^{-\lambda t}$$

SURVIVAL FUNCTION

$$2^{-10} = e^{-\lambda t}$$

... Half-life is 7 hours $\Rightarrow \lambda = \frac{1}{7}\ln 2$

$$2^{-10} = \underbrace{e^{-\frac{t}{7}\ln 2}}_{\ln(-)}$$

$\ln(-) \rightarrow \quad \ln 2^{-10} = -10 \cdot \ln 2 = -\frac{t}{7}\ln 2$

$$\Rightarrow \boxed{t = 70}$$

EX

(b) What's the probability that after 70 hours No coronaviruses exist? (out of 1024)

$$\text{Prob}(\text{SARS-CoV2 surviving} > 70) = \frac{1}{1024}$$

$$\text{Prob not surviving} = 1 - \frac{1}{1024}$$

$\lambda = np = 1$

$$\text{Prob all 1024 viruses die} = \left(1 - \frac{1}{1024}\right)^{1024} \approx \left(e^{-\frac{1}{1024}}\right)^{1024} = e^{-1} \approx 36.79\%$$

$\text{Binom}\left(n = 1024,\ p = \frac{1}{1024},\ k = 0\right) \approx \text{Poisson}_{\lambda=1}(0)$

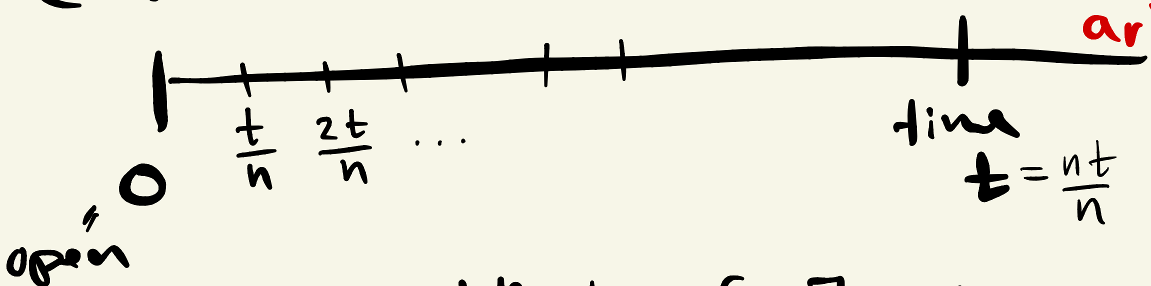
POISSON POINT PROCESSES

↳ Answer to "Why does $\text{Exp}(\lambda)$ model so many things?"

CONCRETE QUESTION

What is the probability distribution for the time of arrival of the first customer?

Consider



Avg # of arrivals in $[0,t]$ is λt

λ = Avg # of arrivals per unit time e.g. hrs, mins, etc.

Double check

so λt is right!

Let's subdivide $[0,t]$ into n little slices where n is s.t. only person could arrive in $[t \cdot \frac{i}{n}, t \cdot \frac{i+1}{n})$ window so e.g. n = # seconds in $[0,t]$

We can define $X_i = \begin{cases} 1 & \text{if arrival occurs in } [t \cdot \frac{i}{n}, t \frac{i+1}{n}) \\ 0 & \text{if no arrival occurs} \end{cases}$

Prob of arrival of a customer $= \boxed{\frac{\lambda t}{n} = p}$

→ Prob of no arrival $= 1 - \frac{\lambda t}{n}$

Prob of no arrivals in window $[0,t]$

Model as a geometric $P(T > n) = \left(1 - \frac{\lambda t}{n}\right)^n \approx e^{-\lambda t}$

$\rightarrow P(T \leq n) \approx 1 - e^{-\lambda t}$

Think of $P(T \le n) = 1 - e^{-\lambda t}$ as the CDF

"THE CDF TRICK" $\frac{d}{dt}(\text{CDF}(t)) = \text{PDF} @ t$

I.e. $F(t) = \int_{-\infty}^{t} f_X(x)\,dx$ then $F'(t) = f_X(t)$.

[Fundamental Theorem of Calculus!]

$$P(T \approx n) = \frac{d}{dt}\left[1 - e^{-\lambda t}\right] = \boxed{\lambda e^{-\lambda t}}$$

Prob of 1st arrival happens at $n \cdot \frac{t}{n} = t$

PDF for Exp(λ)

N.B. We just determined Probability of 1 arrival in $[0,t]$ $X_1 + \cdots + X_n = 1$

Q: How do we determine the probability of k arrivals?

$$P(S_n = k) \approx \text{Poisson}_{\lambda t}(k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

BINOMIAL b/c $np = n \cdot \frac{\lambda t}{n} = \lambda t$

Recall $|\text{Binom}(n,p,k) - \text{Poisson}_{np}(k)| \le np^2$ $n \cdot \left(\frac{\lambda t}{n}\right)^2 = \lambda^2 t^2 / n$

HOW GOOD POISSON APPROX.

POISSON POINT PROCESS APPLICATIONS

↳ Timeline of shark attacks at a beach

↳ Customer Arrivals in a shop

↳ Phone Calls to 911

→ "Point" in point process here refers to the point in time of the event

APPLICATION STEPS

1) Determine avg rate of arrivals per unit time (i.e. $t=1$) (hour or day) $= \lambda$

2) The # of arrivals in $[a,b]$ is $N([a,b]) \sim \text{Poisson}(\lambda(b-a))$

   DISCRETE RV

3) For Intervals $I_1, I_2, \ldots, I_k$ disjoint intervals

   $N(I_1), N(I_2) \ldots N(I_k)$ are indep RVs

Ex If on average 5 customers arrive per hour, what's the probability that 2 customers arrive between 10–11 am?

$$\text{Poisson}_{\lambda \cdot (11-10)}(2=k) = \frac{\lambda^2}{2!}e^{-\lambda} \quad \frac{5^2}{2!}e^{-5}$$

2nd TIME OF ARRIVAL

Know 1st time of Arrival $\sim Exp(\lambda)$

↳ Call 1st time of Arrival $T_1 \sim Exp(\lambda)$

What about T_2? HOW IS IT DISTRIBUTED

$P(T_2 > t)$ = Prob 2nd arrival after t

= Prob there were 1 or 0 arrivals in $[0,t]$

$$= P(\underbrace{N([0,t])}_{\text{Poisson w/ } \lambda \cdot t} \leq 1)$$

$= \text{Poisson}_{\lambda t}(0) + \text{Poisson}_{\lambda t}(1)$

$$P(T_2 > t) = e^{-\lambda t} + \lambda t e^{-\lambda t}$$

$$\Rightarrow P(T_2 \leq t) = 1 - e^{-\lambda t} - \lambda t e^{-\lambda t}$$

$\frac{d}{dt}$

$$f_{T_2}(t) = \frac{d}{dt}\left[1 - e^{-\lambda t} - \lambda t e^{-\lambda t}\right]$$

$$= \lambda e^{-\lambda t} - \lambda \underbrace{\frac{d}{dt}\left[t e^{-\lambda t}\right]}_{\text{Product Rule}}$$

$$= \cancel{\lambda e^{-\lambda t}} - \cancel{\lambda e^{-\lambda t}} - \lambda t \cdot (-\lambda e^{-\lambda t})$$

PDF for T_2 $\boxed{f_{T_2}(t) = \lambda^2 t e^{-\lambda t} \text{ for } t \geq 0}$

GAMMA DISTRIBUTION:

= Sum of n Exponential RVs

The PDF of the n^{th} arrival time is

$$\boxed{f_{T_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \text{ for } t \geq 0 \quad \Big| \; f_{T_n}(t) = 0 \text{ o.w.}}$$

Gamma distribution n & λ

Def The Gamma distribution w/ params r & λ say $X \sim \text{Gamma}(r, \lambda)$ is

$$f_X(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}$$

where $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x}\, dx$ is the continuous extension of $(n-1)!$ to non-integer n.