LECTURE 16 LIVE

WS16 = NOT GRADED, NOT DUE

Q1

Draw w/ replacement until I get green ball

D = # draws needed

$\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) = P(D=2)$

(a) $P(D=1) = \frac{1}{4}$

(b) $P(D=3) = \left(\frac{3}{4}\right)^2 \frac{1}{4} = \underset{R}{\left(\frac{3}{4}\right)}\underset{R}{\left(\frac{3}{4}\right)}\underset{G}{\left(\frac{1}{4}\right)}$

(c) $P(D>3)$

Way 1

$$= 1 - P(D=1) - P(D=2) - P(D=3)$$

$$1 - \frac{1}{4} - \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) - \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)$$

$$1 - \frac{1}{4}\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2\right)$$

$$\frac{1}{4}\left(\frac{4^2}{4^2} + \frac{3\cdot 4}{4^2} + \frac{3^2}{4^2}\right)$$

$$16 + 12 + 9$$

...

Way 2 $P(D>3) = \sum_{k=4}^{\infty} \left(\frac{3}{4}\right)^{k-1}\left(\frac{1}{4}\right) = \frac{1}{4}\left(\left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \cdots\right.$

$= \frac{1}{4}\left(\frac{3}{4}\right)^3 \left[1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots\right] = \cancel{\frac{1}{4}} \cdot \left(\frac{3}{4}\right)^3 \cdot \cancel{4}$

Way 3 $P(D>3) = P(\text{failing } 3 \text{ times})$

$= \left(\frac{3}{4}\right)^3$

$$\sum_{k=0}^{\infty} a^k = \boxed{\underbrace{1 + a + a^2 + a^3 + \cdots}} = \frac{1}{1-a}$$

$= S \qquad 1 + a(\underbrace{1 + a + a^2 + \cdots}_{S})$

$S = 1 + aS$

$S(1-a) = 1 \quad \Rightarrow \quad S = \frac{1}{1-a}$

(d) $P(D>7 \mid D>4) = \frac{P(D>7 \text{ AND } D>4)}{P(D>4)}$

$= \frac{P(D>7)}{P(D>4)} = \frac{\left(\frac{3}{4}\right)^7}{\left(\frac{3}{4}\right)^4} = \left(\frac{3}{4}\right)^3$

$= P(D>3)$

"MEMORYLESS PROPERTY"

RULE $\boxed{P(T > a+b \mid T > a) = P(T > b)}$

(e) $E(D) = 1/p = 1/\tfrac{1}{4} = 4$

$$SD(D) = \sqrt{q/p^2} = \sqrt{3/4 \cdot 4^2} = \sqrt{12}$$

(f) NAMED DISTRIBUTION = GEOMETRIC

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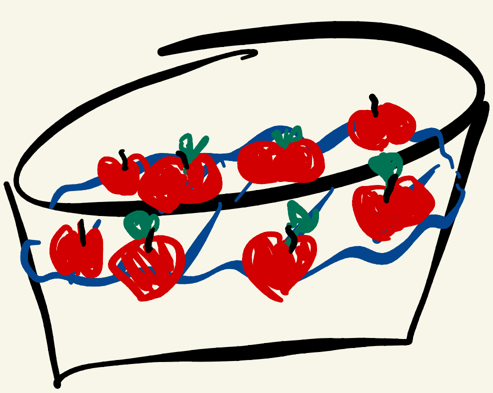


(a) NEGATIVE BINOMIAL

$p = 1/5$

B = RV that measures # Bobs unit 3rd Apple success

(b)
$$P(B=7) = \underbrace{\binom{6}{2}\left(\tfrac{1}{5}\right)^2\left(\tfrac{4}{5}\right)^4}_{B(n=6,\,p=1/5,\,k=2)} \times \underbrace{\tfrac{1}{5}}_{\text{Prob 7th attempt is a success}}$$

#needed successes

$$= \binom{6}{2}\left(\tfrac{1}{5}\right)^3\left(\tfrac{4}{5}\right)^4$$

(c) $E(B) = \dfrac{r}{p} = \dfrac{r=3}{p=1/5} = 15$

$$SD(B) = \frac{\sqrt{rq}}{p} = 5 \cdot \sqrt{3 \cdot 4/5}$$
$$= 7.5$$

③ $S_2 = X_1 + X_2$

when $X_i = \text{Unif}\{1, \dots, 6\}$

Q: What is the mode of S_2 ?

6	7	8	9	10		
5	6	7	8	9		
4	5	6	7	8		
3	4	5	6	7		
2	3	4	5	6		
1	2	3	4	5		
	1	2	3	4	5	6

$$P(S_2 = 7) = \frac{6}{36} = 1/6$$

Q: $E(S_2) = E(X_1 + X_2) = \underbrace{E(X_1)}_{7/2} + \underbrace{E(X_2)}_{7/2} = 7$

Here Mean = Mode!

Q $S_3 = X_1 + X_2 + X_3$

Mean ? $\frac{7}{2} \cdot 3 = 10.5$ Mode ? $\boxed{10 \text{ or } 11}$

$P(S_3 = 10.5)$? $= 0$

$$Var(S_2) = Var(X_1 + X_2) \overset{\text{INDEP ONLY}}{=} \underbrace{VAR(X_1)}_{35/12} + \underbrace{VAR(X_2)}_{35/12}$$

$$SD(S_2) = \sqrt{\frac{35}{12} + \frac{35}{12}}$$

$$= \sqrt{35/6}$$

$$SD(S_3) = \sqrt{\frac{35}{4}} = \sqrt{(n=3)\ \sigma^2} \qquad \sigma^2 = \frac{35}{12}$$

COLLECTOR'S PROBLEM

5 toys distributed in Apple Jacks Cereal

COLLECT THEM ALL!

Q: What is the expected # of Apple Jacks boxes you'll need to collect them all?

Guess = 5? GUESS = 25?

$E_1 = 1$ $\quad 5/4$

$p_1 = 1 \quad p_2 = \frac{4}{5} \quad p_3 = \frac{3}{5} \quad p_4 = \frac{2}{5} \quad p_5 = 1/5$

Boxes of Apple Jacks

$$N = T_1 + T_2 + T_3 + T_4 + T_5$$

$T_1 \sim \text{Geom}(1)$

$T_2 \sim \text{Geom}(4/5)$

$T_3 \sim \text{Geom}(3/5)$

$T_4 \sim \text{Geom}(2/5)$

$T_5 \sim \text{Geom}(1/5)$

$$E(N) = E(T_1) + E(T_2) + E(T_3) + E(T_4) + E(T_5)$$

$$= 1 + \frac{5}{4} + \frac{5}{3} + \frac{5}{2} + \frac{5}{1}$$

$$= 5\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right)$$

$$\approx 11.416$$

WS 14

Q1 $A_n = \dfrac{X_1 + \cdots + X_n}{n}$ Empirical Mean

$$P(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2} \leq .05$$

$c = 1 \quad X = A_n$

Important bit: Compute σ^2

12 sided die $\{1, \ldots, 12\}$

$$E(X_i) = \frac{12+1}{2} = \frac{13}{2}$$

$$D(X_i) = \sqrt{\frac{(12)^2-1}{12}} = \sqrt{\frac{143}{12}} \approx 3.45$$

$$SD(X) = \sqrt{\frac{(b-a+1)^2-1}{12}}$$

$\boxed{238.\overline{3}}$

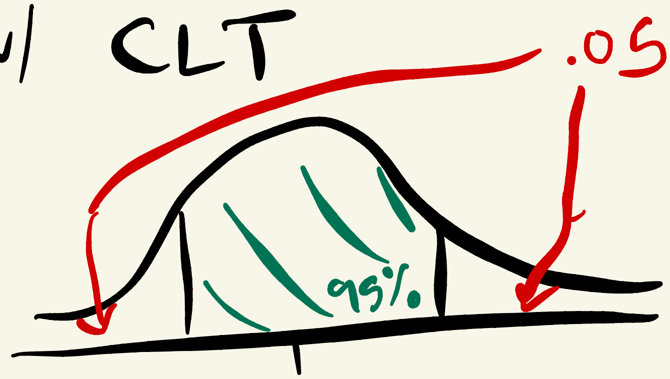
$$SD(A_n) = \sqrt{Var(A_n)}$$

$$VAR(A_n) = VAR\left(\frac{X_1 + \cdots + X_n}{n}\right) = \frac{1}{n^2} n \cdot Var(X_i) = \frac{1}{n}\underbrace{\left(\frac{143}{12}\right)}_{\sigma^2}$$

$$SD(A_n) = \sqrt{\frac{1}{n}\sigma^2} = \frac{\sigma}{\sqrt{n}}$$

$$P(|A_n - \mu| \geq 1) \leq \frac{VAR(A_n)}{1^2} = \frac{\sigma^2}{n} \leq .05 \Rightarrow \boxed{n \geq 20\sigma^2}$$

(b) w/ CLT



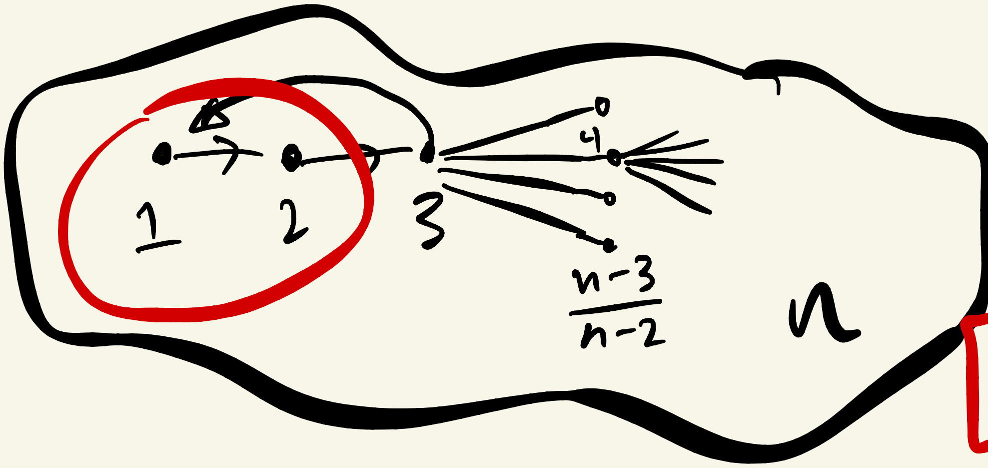
$$A_n \in \mu_{A_n} \pm \underbrace{2\sigma(A_n)}_{1} \approx$$

$$2\sigma(A_n) = 1$$

$$2\frac{\sigma}{\sqrt{n}} = 1$$

$$n > \left(\frac{1.96\,\sigma}{\varepsilon}\right)^2 = \left(1.96\sqrt{\frac{143}{12}}\right)^2 = 45.77$$

$$\varepsilon = 1 \qquad \sim 4 \cdot \frac{143}{12} = \frac{143}{3} \approx 47.6\ldots$$



$$p_4 = \left(1-\frac{1}{n-2}\right)\left(1-\frac{2}{n-2}\right)$$

$$\boxed{p_r = \left(1-\frac{1}{n-2}\right)\cdots\left(1-\frac{r-2}{n-2}\right)}$$

$$P_3 = \frac{n-3}{n-2}$$
$$= 1-\frac{1}{n-2}$$
$$= \frac{n-2-1}{n-2}$$

prob person 4 didn't repeat

$$P_4 = \left(\frac{n-4}{n-2}\right)\left(\frac{n-3}{n-2}\right)$$

person 4 can tell $\frac{n-4}{n-2}$ ~~people~~ people

Assuming person 3 didn't repeat

prob person 3 didn't repeat

$$P_r = \left(\frac{n-r}{n-2}\right)\left(\frac{n-r+1}{n-2}\right)\cdots\left(\frac{n-3}{n-2}\right)\underbrace{\left(\frac{n-2}{n-2}\right)}_{1}$$

$$= \left(\frac{n-2-r+2}{n-2}\right) \quad \left(\underbrace{\frac{\cancel{n-2}}{\cancel{n-2}}}_{1}-\frac{2}{n-2}\right)\left(\underbrace{\frac{\cancel{n-2}}{\cancel{n-2}}}_{1}-\frac{1}{n-2}\right) 1$$

$$= \left(1-\frac{r-2}{n-2}\right)\cdots\left(1-\frac{2}{n-2}\right)\left(1-\frac{1}{n-1}\right)$$

$$1-x \approx e^{-x}$$

$$p_r = \left(1-\frac{1}{n-2}\right)\left(1-\frac{2}{n-2}\right)\cdots\left(1-\frac{r-2}{n-2}\right) \approx e^{-\frac{1}{n-2}}e^{-\frac{2}{n-2}}\cdots e^{-\frac{r-2}{n-2}}$$

$$e^{-1/n-2} \quad e^{-2/n-2} \quad \cdots \quad e^{-r-2/n-2}$$

$$\boxed{e^{-\frac{(r-2)(r-1)}{2(n-2)}}}$$

$r=30$
$n=300$