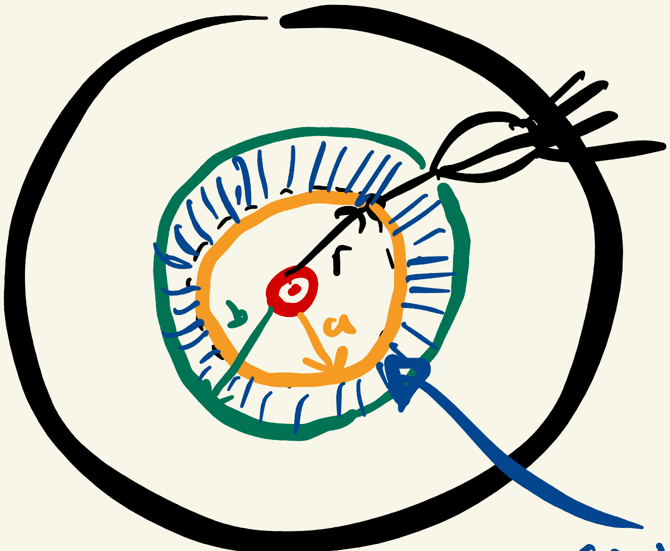
AMAT 362 PROBABILITY for STATISTICS

(@ UAlbany)

LECTURE 20: MORE CTS RVs

$$\frac{d}{dx}\text{CDF} = \text{PDF}$$

⟶ Rubber Pencil



strip b/w a&b

Q: Suppose I throw a dart uniformly at random at a dartboard, HOW is R = distance from the origin distributed?

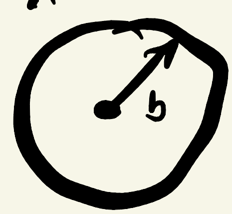
RECALL

$$P(a < R < b) = \int_a^b f_R(r)\,dr$$

Try to find f_R = PDF

$$\frac{\pi b^2 - \pi a^2}{\pi r_{\text{dartboard}}^2 \; (=1)} \leftarrow = \frac{\boxed{\text{Area of strip}}}{\text{Area of dartboard}}$$

$$P(a<R<b) = \frac{\pi b^2 - \pi a^2}{\pi 1^2} = b^2 - a^2$$

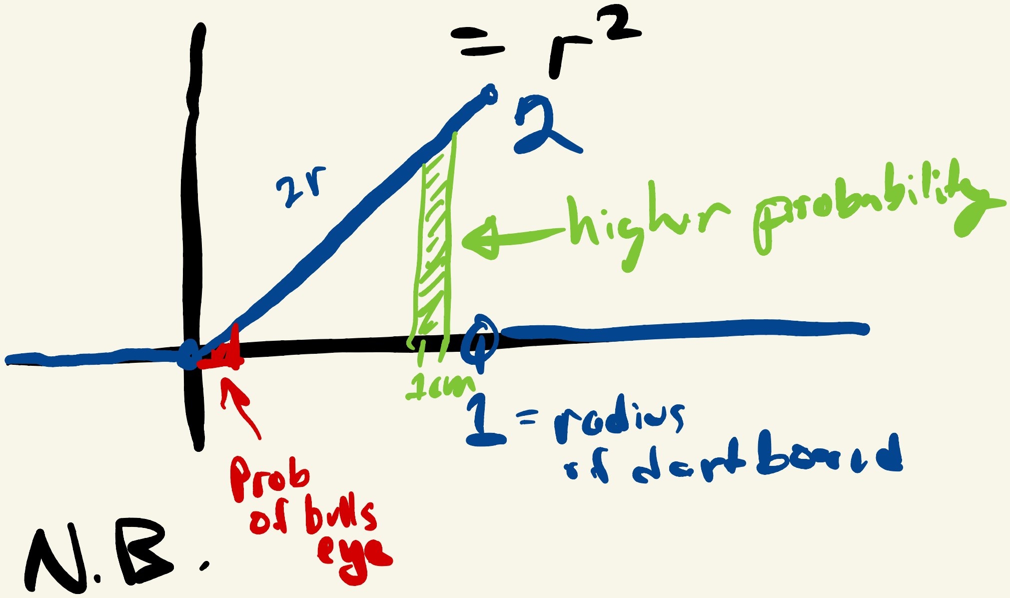
CDF for R $\quad F_R(b) = \int_0^b f_R(r)\,dr =$ Area 

$r=b$

$$\longrightarrow \quad F_R(r) = \int_0^r f_R(r')\,dr' \quad \overset{F.T.C}{\rightsquigarrow} \quad \frac{d}{dr}F_R(r) = f_R(r)$$

$$= r^2$$

$$\frac{d}{dr}(r^2) \boxed{= 2r = f_R(r)}$$



N.B.

1) $f_R(.75) = 2(.75) = 1.5 > 1$

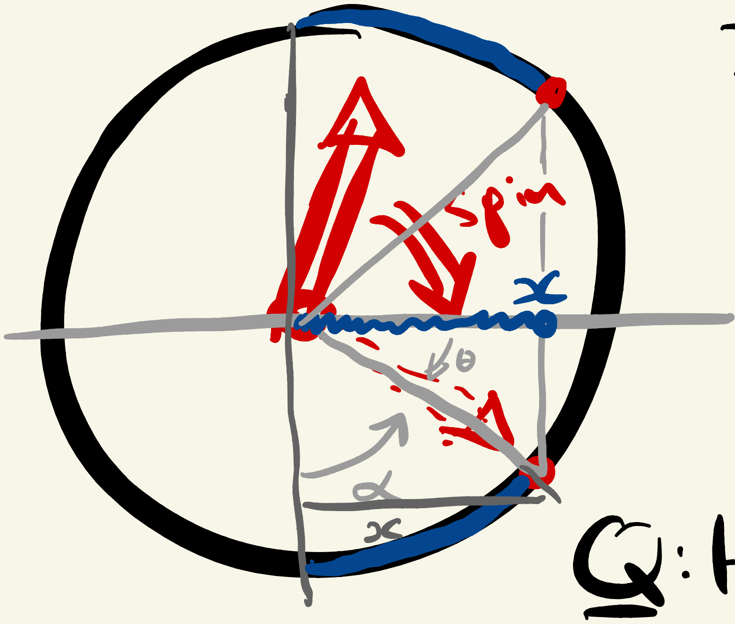
UPSHOT!

$\boxed{\text{Probability Density can be greater than 1} > 1 \text{ (possibly)}}$

2) In this example we see how probability density concentrates around the perimeter!

$\longrightarrow$ Makes sense

Another Circular EXAMPLE



I sample a pt uniformly at random from circle

$\rightarrow$ Read off x coord

$\Longrightarrow$ RV $= X$

Q: How is X distributed?

$$P(0 < X < x) = \frac{\text{Area of two blue arcs}}{\text{Circumference of the circle}}$$

B/C pt sampled uniformly

$$\left(= \int_0^x f_X(x')\, dx' \right)$$

$$= \frac{2 \text{ arclength of single blue arc}}{\text{circumference}}$$

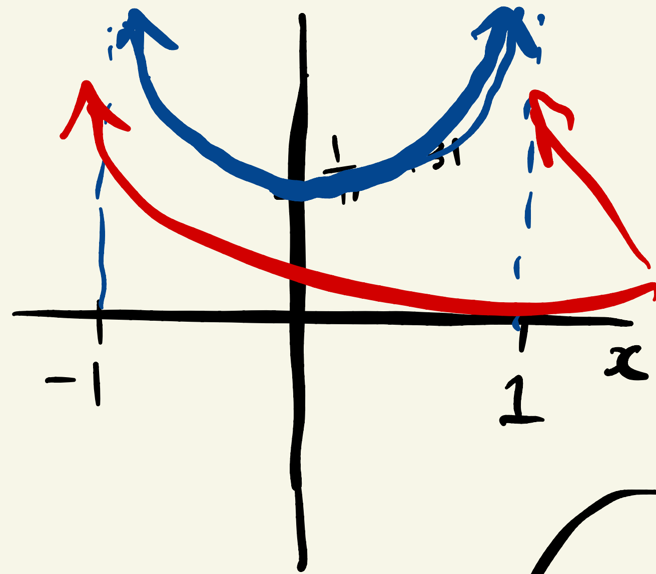
$$\text{CDF for } X = \frac{2\alpha}{2\pi} = \frac{\alpha}{\pi} = \frac{1}{\pi}\sin^{-1}(x)$$

$$\alpha = \text{arcsine}(x) = \sin^{-1}(x) \left(\neq \frac{1}{\sin x}\right)$$

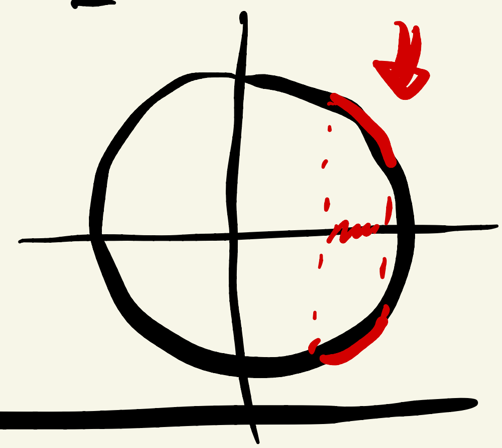
$$\frac{d}{dx}\left(\frac{1}{\pi}\sin^{-1}(x)\right) = \boxed{\frac{1}{\pi\sqrt{1-x^2}}} = \text{PDF!}$$

Graph of PDF for X

$$\frac{1}{\pi\sqrt{1-x^2}}$$



Why does $f_X \to \infty$ as $x \to \pm 1$

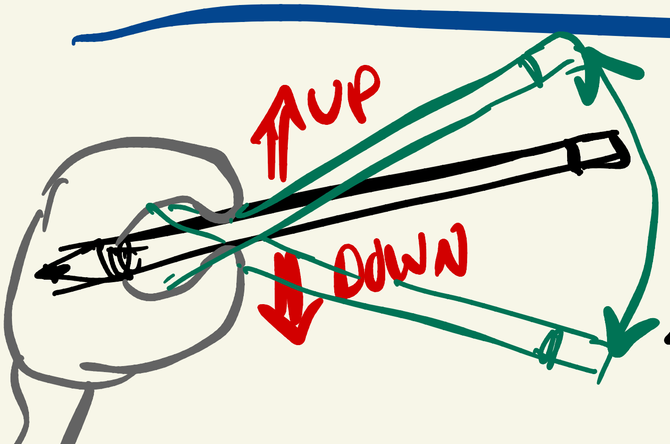


$x = \frac{1}{2}$

$$\frac{1}{\pi\sqrt{3/4}} = .367$$

ANOTHER EXAMPLE where this occurs...

THE RUBBER PENCIL



Observe the pencil mostly at these turnaround points

Model the eraser's location (= y position) as a function of time using a Simple Harmonic Oscillator $y(t) = y_0 \sin(t)$

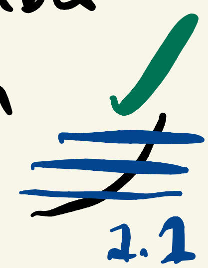
Imagine you're equally likely to observe the pencil at some time

$$T \sim \text{Unif}\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

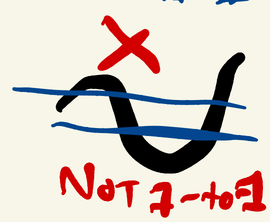
Q: What is the PDF for $Y = y_0 \sin(T)$?

USEFUL FORMULA

Let X be a R.V. and g a differentiable function that is 1-to-1 on its domain (unique inverse = horizontal line test passes the line test)



The domain = support ($f_X(x) \neq 0$)



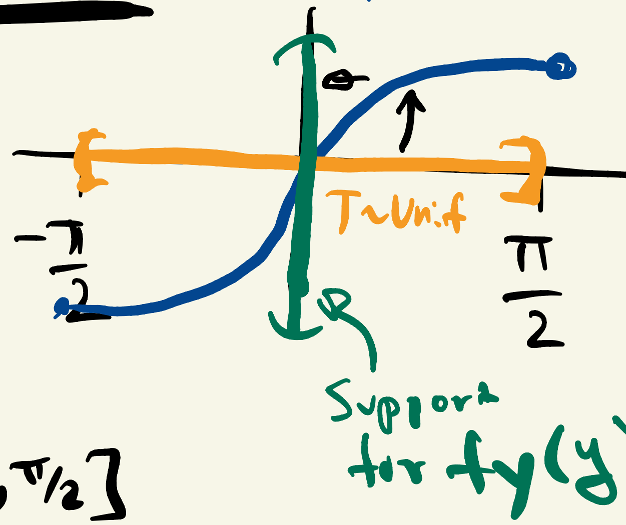
then PDF for Y is

$$f_Y(y) = f_X(\underbrace{g^{-1}(y)}_{\text{inverse of } y \text{ under } g}) \cdot \frac{1}{|g'(g^{-1}(y))|}$$

derivative of g

Returning to example

Passes (horizontal line test)?



$g(t) = y_0 \sin(t)$

$X = T \sim \text{Unif}\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$f_T(t) = \begin{cases} \frac{1}{\pi} & = \text{length of } [-\pi/2, \pi/2] \\ 0 & \text{o.w.} \end{cases}$

$g'(t) = y_0 \cos(t)$

$f_Y(y) = f_T(g^{-1}(y)) \cdot \frac{1}{|y_0 \cos(g^{-1}(y))|}$

$y = y_0 \sin t$

$\Rightarrow \frac{y}{y_0} = \sin t \longrightarrow t = \sin^{-1}\left(\frac{y}{y_0}\right)$

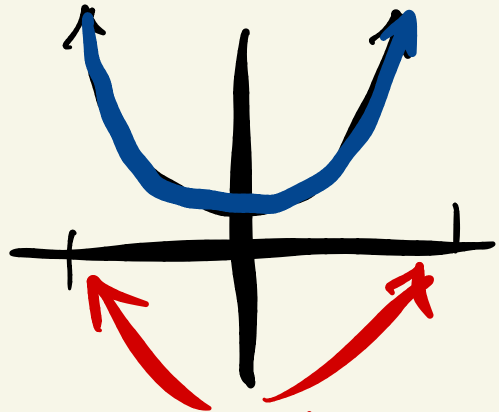
$f(t) = \frac{1}{\pi}$ for $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$f_Y(t) = \frac{1}{\pi} \frac{1}{|y_0 \cos(t)|} \Rightarrow$ need as a function of y?

$\cos(t) = \sqrt{1 - \sin^2 t} \Rightarrow f_Y(y) = \frac{1}{\pi} \frac{1}{|y_0 \sqrt{1 - \sin^2 t}|}$

$[\sin(t)]^2 = \left[\sin\left(\sin^{-1}\left(\frac{y}{y_0}\right)\right)\right]^2$

$$= \frac{1}{\pi} \frac{1}{\sqrt{1-\left(\frac{y^2}{y_0^2}\right)}}$$



Probability density concentrates here!