AMAT 362 - PROBABILITY for STATISTICS - LEC 8

STARS & BARS, GEOMETRIC & BINOMIAL

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GAMBLER'S RULE

RVs → P+ 2!

COMBINATORICS

1) How many distinct permutations are there of WACKAWACKA?

$n_W = 2 \quad n_C = 2$

$n_A = 4 \quad n_K = 2$

$$\frac{10!}{2!\,4!\,2!\,2!}$$

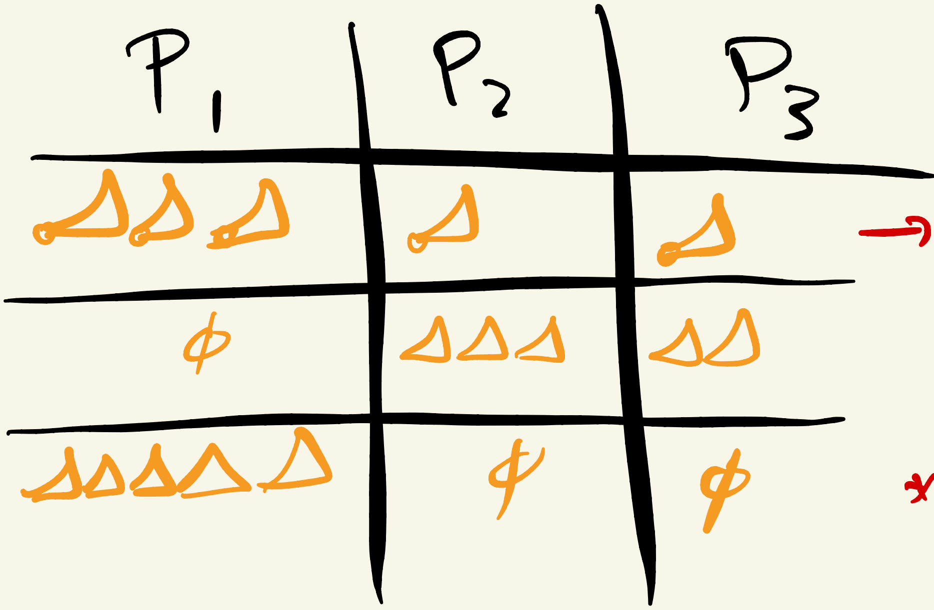
2) How many size 3 subsets are there of a set of size 7?

x_1	x_2	x_3	x_4	x_5	x_6	x_7
✓	✓	✓	×	×	×	×
✓	✓	×	✓	×	×	×
✓	✓	×	×	✓	×	×

perms of 3 ✓s & 4 ×s

$$\frac{(3+4)!}{3!\,4!} = \binom{7}{3} = \binom{7}{4}$$

3) How many ways can I give 5 bananas to 3 people?



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\# of banana → people assignments $= \dfrac{(5+2)!}{5!\,2!}$

$\binom{7}{2}$ or $\binom{7}{5}$

STARS & BARS

\# ways of distributing n identical objects to k distinct people is

$$\frac{(\overbrace{n}^{\#\text{ objects}} + \overbrace{k-1}^{\#\text{ bars}})!}{\underbrace{n}_{\#\text{ objects}}!\;\underbrace{(k-1)}_{\#\text{ bars}}!}$$

GEOMETRIC R.V.'s

Sps I apply for jobs one at a time each one I have a chance $p = \frac{1}{10}$ of getting.

Q: What's the probability that I have to apply for, strictly more than 4 jobs? (i.e. 5^{th} or 6^{th} or 7^{th} ... ✓ ☺)

A: V1] T = geometric R.V. that counts # tries until first success $\in \{1, 2, 3, 4, \dots\}$

KNOW THAT $P(T=k) = q^{k-1} p$, $q = 1-p$

$$P(T>4) = 1 - P(T \le 4)$$

$$= 1 - \underbrace{\left[\left(\tfrac{1}{10}\right) + \left(\tfrac{9}{10}\right)\left(\tfrac{1}{10}\right) + \left(\tfrac{9}{10}\right)^2\left(\tfrac{1}{10}\right) + \left(\tfrac{9}{10}\right)^3\left(\tfrac{1}{10}\right)\right]}_{34.39\%}$$

$$\boxed{= 65.61\%}$$

V2] LONG WAY, COMPUTE DIRECTLY

$$P(T>4) = P(T=5) + P(T=6) + P(T=7) + \dots$$

$$= \left(\tfrac{9}{10}\right)^4\left(\tfrac{1}{10}\right) + \left(\tfrac{9}{10}\right)^5\left(\tfrac{1}{10}\right) + \left(\tfrac{9}{10}\right)^6\left(\tfrac{1}{10}\right) + \dots$$

$$= \left(\tfrac{9}{10}\right)^4\left(\cancel{\tfrac{1}{10}}\right)\left[1 + \left(\tfrac{9}{10}\right) + \left(\tfrac{9}{10}\right)^2 + \dots\right]$$

$$\boxed{= \left(\tfrac{9}{10}\right)^4} \qquad \frac{1}{1-\frac{9}{10}} = \cancel{10}$$

RECALL

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

GEOMETRIC SERIES

V3] P(fail 4 times) $= \left(\frac{9}{10}\right)^4$

LEC 8 Pg 3

GENERAL PROPERTY of RVs

$X : \Omega \to \mathbb{R}$

$\Rightarrow$ Discrete Setting $\displaystyle\sum_{\substack{\text{all possible}\\ \text{values } k}} P(X=k) = 1$

The RANGE OF X

NORMALIZATION AXIOM

Check this...

GEOMETRIC

$$\sum_{k=1}^{\infty} P(T=k) = \sum_{k=1}^{\infty} q^{k-1} p$$

$$= p \sum_{k=1}^{\infty} q^{k-1}$$

$m = k-1$

$k=1 \Rightarrow m=0$

$$= p \underbrace{\sum_{m=0}^{\infty} q^m}_{\text{Geom. Series}}$$

$$= p \frac{1}{1-q} = \frac{\cancel{p}}{\cancel{p}} = 1 \checkmark$$

BINOMIAL R.V.

$$P(S_n=k) = \binom{n}{k} p^k q^{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \stackrel{?}{=} 1$$

Has to be True for Normalization to be true

BINOMIAL EXPANSION $\Big] \to (p+q)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$

$$(p+q)(p+q)(p+q)\cdots(p+q)$$

Coefficient of $p^k q^{n-k}$

pick k of these factors to multiply w/ p & $n-k$ w/ q

Q: Sps I play a game w/ $\frac{1}{N}$ odds of winning. How many times do I need to play to have better than 50% odds of winning at least once

A: Like the BDAY Problem..

$$P\left(\begin{matrix}\text{at least one win}\\ \text{out of } n \text{ attempts}\end{matrix}\right) = 1 - P\left(\begin{matrix}\text{no wins}\\ \text{out of } n \\ \text{attempts}\end{matrix}\right)$$

$$= 1 - \underbrace{\left(1 - \frac{1}{N}\right)}_{\text{Prob of not winning 1x}}^{n}$$

$\times n$ times for n tries

Equivalently Want n s.t.

$$\left(1-\frac{1}{N}\right)^n < \frac{1}{2} \rightsquigarrow \log\left(1-\frac{1}{N}\right)^n < \log\frac{1}{2}$$

$$n\log\left(1-\frac{1}{N}\right) < \log\frac{1}{2}$$

$$n\cdot\left(-\frac{1}{N}\right) < \log_e\left(\frac{1}{2}\right)$$

$$n > N\cdot\underbrace{\ln(2)}_{0.69}$$

RECALL

$$\log_e(1-z) = \ln(1-z)$$

$$\underset{\text{small } z}{\approx} -z - \frac{z^2}{2} - \frac{z^3}{3}, \ldots$$

$$\Rightarrow \log_e(1-z) \sim -z$$

"2/3rds RULE for GAMBLERS"

$\#\text{tries} > 69\% \frac{1}{P}$

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More BINOMIAL RV's

Q1 Flip a fair coin 6 times

What's $P(4 \text{ or more heads})$?

$$P(S_6=k) = B(\underset{\substack{\shortparallel \\ n}}{6}, \underset{\substack{\shortparallel \\ \#\text{successes}}}{k}, p=\tfrac{1}{2}) = \binom{6}{k}\cancel{\left(\tfrac{1}{2}\right)^{k}}\left(\tfrac{1}{2}\right)^{6-\cancel{k}}$$

$$= \frac{1}{2^6}\binom{6}{k}$$

$$P(S_6=4) + P(S_6=5) + P(S_6=6)$$

$$\frac{1}{2^6}\binom{6}{4} + \frac{1}{2^6}\binom{6}{5} + \frac{1}{2^6}\binom{6}{6}$$

$$\frac{15 + 6 + 1}{2^6} = \frac{22}{2^6} = \frac{11}{32}$$

Q2) 5 families each w/ 6 children

What's the probability of at least 3 families having 4 or more girls?

$n = 5$ families

$k = 3, 4 \text{ or } 5$ $\qquad p =$ prob that of 6 children at least 4 are girls

$= \text{by above } p = \frac{11}{32}$

$$\underbrace{\binom{5}{3}\left(\frac{11}{32}\right)^3\left(\frac{21}{32}\right)^2 + \binom{5}{4}\left(\frac{11}{32}\right)^4\left(\frac{21}{32}\right) + \binom{5}{5}\left(\frac{11}{32}\right)^5\left(\cancelto{1}{\frac{21}{32}}\right)^0}$$

$= 22.6\%$