AMAT362 → Probability for Statistics → LEC 7: INTRO to RANDOM VARIABLES

- BINOMIAL $\binom{n}{k}$
- Geometric $q^{n-1}p$

__Up to Now...__

Probability of events

$\rightsquigarrow P(\text{Rain Today} \mid \text{Rain Yesterday})$

__Today and going forward ...__

Studying #'s that vary according to some Random process / experiment

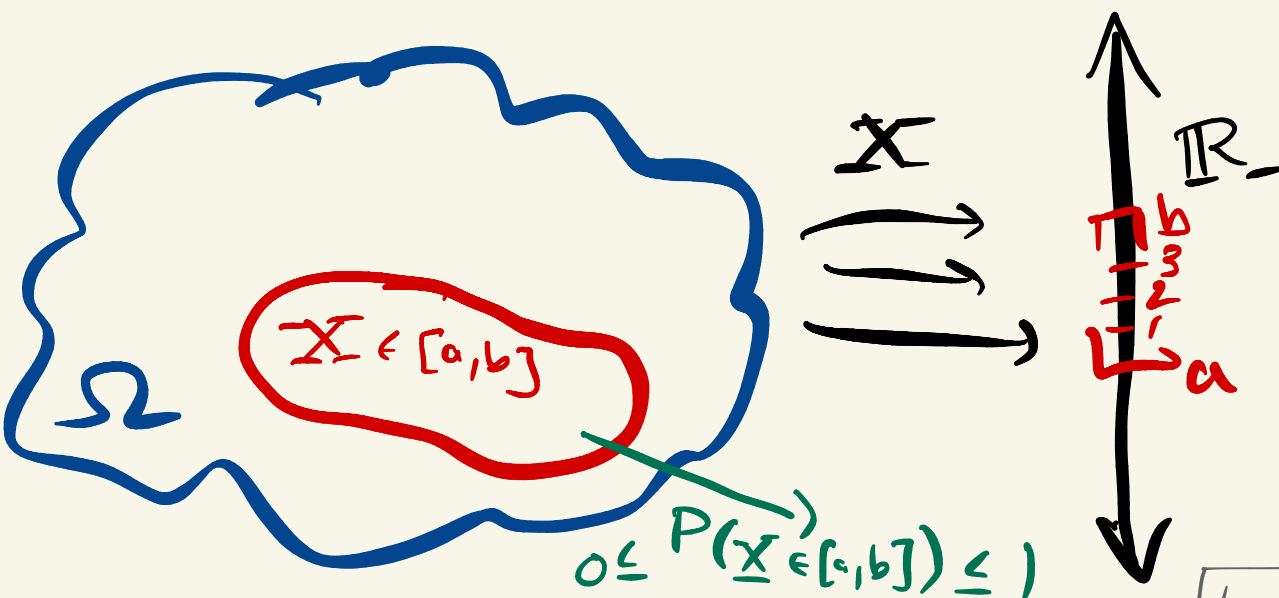
__Def__ A __RANDOM VARIABLE__ is a function $X : \Omega \to \mathbb{R}$

Ω: sample space

$\mathbb{R}$: #'s vary according to Ω

s.t.

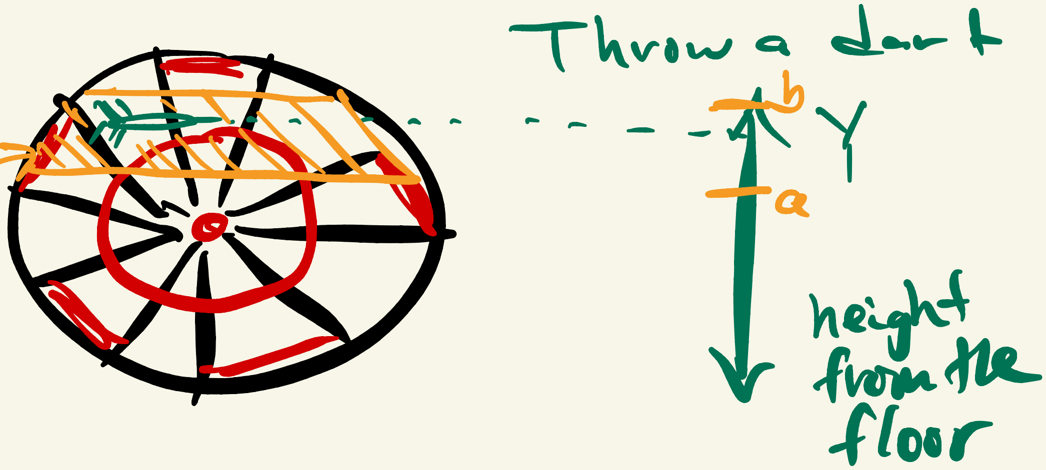
$P(a \le X \le b)$ is well-defined



Def A $\underline{\text{DISCRETE R.V.}}$ is one where range of X is finite or countable ($\mathbb{N} = 1, 2, 3, \ldots$)

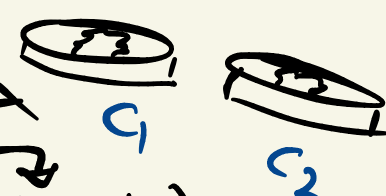
Def A $\underline{\text{CONTINUOUS R.V.}}$ is one where when range of X is uncountable

Ex.



$P(a \le Y \le b)$ $= \dfrac{\text{Area of this region}}{\text{Area of dart board}}$

2 COIN EXAMPLE



Sps we have 2 fair coins ($P(\text{heads}) = P(\text{tails})$)

Let H = # of heads

$\Rightarrow H \in \{0, 1, 2\}$

Event where c_1 is Tails

when c_2 is tails

$$P(H=0) = P(T_1 \cap T_2)$$

Prob of 0 heads

$$= P(T_1)P(T_2) \quad \text{by independence}$$

$$= (1/2)(1/2) = \boxed{1/4}$$

Dually

$$P(\mathcal{H}=2) = P(T_1^c \cap T_2^c)$$
$$= P(T_1^c)\,P(T_2^c)$$
$$= P(H_1)\,P(H_2) \boxed{= 1/4}$$

$$P(\mathcal{H}=1) = P(H_1 \cup H_2) - P(H_1 \cap H_2)$$

[Rmk: $P(\mathcal{H}_1 \geq 1) = P(H_1 \cup H_2) = P(H_1) + P(H_2) - P(H_1 \cap H_2)$]

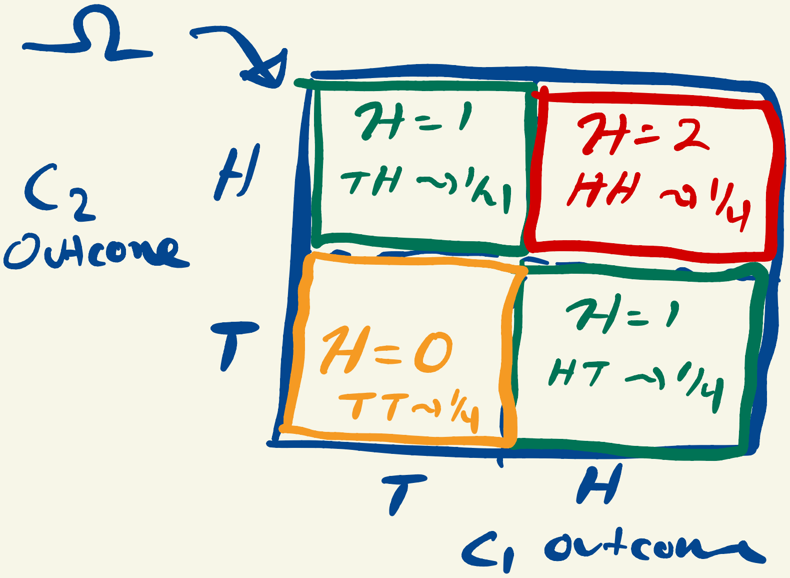
$$= P(H_1) + P(H_2) - 2P(H_1 \cap H_2)$$
$$= \frac{1}{2} + \frac{1}{2} - 2\left(\frac{1}{2}\cdot\frac{1}{2}\right)$$

$$\boxed{P(\mathcal{H}=1) = 1/2}$$

Alternatively...

$$\Omega = \{\underbrace{(T,T)}_{\mathcal{H}=0},\ \underbrace{(T,H),\ (H,T)}_{\mathcal{H}=1},\ \underbrace{(H,H)}_{\mathcal{H}=2}\}$$

(c_1 c_2)



$P(TH) + P(HT) = 1/4$

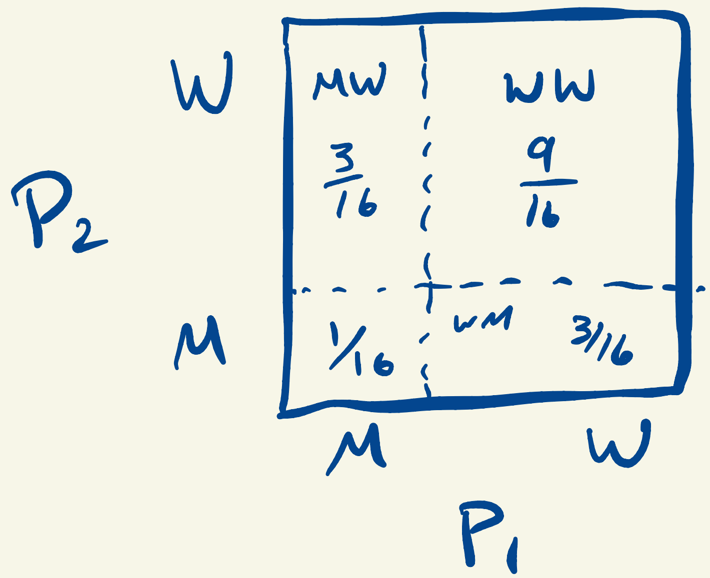
Example where counting is insufficient

Yoga Class

Man $\sim 25\% = \frac{1}{4}$

Women $\sim 75\% = \frac{3}{4}$

W = R.V. counts # of women in a 2 person yoga class



W	0	1	2
P_W	$\frac{1}{16}$	$\frac{6}{16}$	$\frac{9}{16}$

P — PROBABILITY

M — MASS

F — FUNCTION

BINOMIAL R.V.

Tracks the probability of k successes out of n identical independent tries $\to S_n \in \{0, \ldots, n\}$

Q: How to compute $P(S_n = k)$?

A: Two steps

1) Compute $P(S_1 S_2 \cdots S_k F_{k+1} \cdots F_n)$

By independence ... $= P(S_1)P(S_2) \cdots P(S_k) \cdot P(F_{k+1}) \cdots P(F_n)$

p = prob of success

q = prob of failure

$(p+q=1)$

$\to p^k q^{n-k}$

2) Count # ways of k successes out of n tries

$\to \binom{n}{k}$

— — . . . . . — . . . —

pick k successes

$\Rightarrow$ FINAL ANSWER $\boxed{P(S_n = k) = \binom{n}{k} p^k q^{n-k}}$

$B(n,k,p)$ BINOMIAL DISTRIBUTION

Example

3 Red + 1 Green
Draw w/ replacement 10x

Q: What is the probability of getting 5 greens out of 10 draws?

A: Drawing a green ball is a "success"

$p = 1/4$

$q = 3/4$

$$\boxed{P(S_{10}=5) = \binom{10}{5}\left(\tfrac{1}{4}\right)^5\left(\tfrac{3}{4}\right)^5}$$

(NOT SO?) SUBTLE VARIATION

Change the experiment: Draw balls w/ replacement until 1st Green Ball

Let T = Draw # where first green ball appears

Possible values for T: $1, 2, 3, 4, \ldots, 10^6, \ldots \ldots \infty$

$$P(T=1) = \left(\tfrac{1}{4}\right)$$

$$P(T=2) = \left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right)$$

$$\vdots$$

$$P(T=k) = \underbrace{\left(\tfrac{3}{4}\right)\cdots\left(\tfrac{3}{4}\right)}_{k-1 \text{ times}}\underbrace{\left(\tfrac{1}{4}\right)}_{k^{th} \text{ try}} = \left(\tfrac{3}{4}\right)^{k-1}\left(\tfrac{1}{4}\right)$$

GEOMETRIC R.V. $\boxed{P(T=k) = q^{k-1}p}$

Some final comments

- Draws w/ replacement have the effect of $P(E_1 \cdots E_n) = P(E_1) \cdots P(E_n)$

  AND $\rightsquigarrow$ PRODUCTS

- Draw w/o replacement means we have to compute things means we have to compute w/ conditional probability

Ex 3 Red + 1 Green balls in an urn

Draw <u>w/o</u> replacement

Let R = # red balls before green ball appears

$R \in \{0, 1, 2, 3\}$ $\qquad P(G / B_1 = R)$

$P(R=0) = \frac{1}{4} \qquad P(R=1) = \left(\frac{\cancel{3}}{4}\right)\left(\frac{1}{\cancel{3}}\right) = \frac{1}{4}$

$P(R=2) = \frac{\cancel{3}}{4} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{1}{\cancel{2}} = \frac{1}{4} \qquad P(R=3) = \frac{\cancel{3}}{4} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{1}}{\cancel{2}} \cdot \frac{1}{\cancel{1}}$

$\Rightarrow$ Uniformly Distributed R.V. $= \frac{1}{4}$