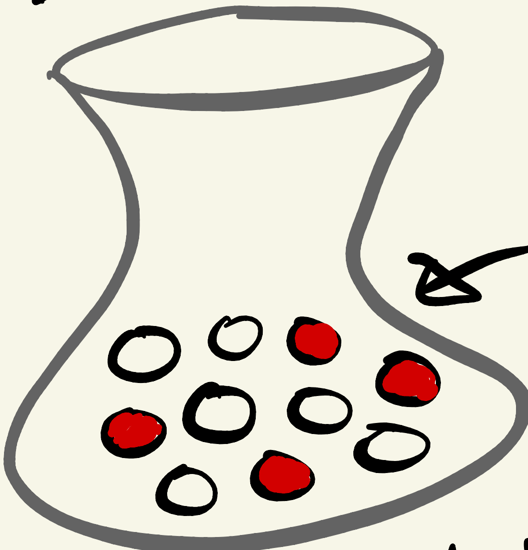
# AMAT 362 - PROBABILITY for STATISTICS

## LECTURE 5: CONDITIONAL PROBABILITY!



4 Red 6 White Balls in an urn

EXP: Draw balls out one at a time. w/o Replacement

Let $E_1$ = first ball is red

$E_5$ = fifth ball is red

Q $P(E_1) = ?$

A $P(E_1) = \frac{4}{10}$ ← # red balls / ← # balls in urn

Q $P(E_5) = ?$

A $\Omega = \{$all permutations of $\underbrace{RRRRWWWWWW}\}$

$$|\Omega| = \frac{10!}{4!\,6!} = \binom{10}{4} = \binom{10}{6}$$

$E_5 \subseteq \Omega$ → permutations of ← where R in the 5th position

x's → $\underset{1}{x}\ \underset{2}{x}\ \underset{3}{x}\ \underset{4}{x}\ \underset{5}{R}\ \underset{6}{x}\ \underset{7}{x}\ \underset{8}{x}\ \underset{9}{x}\ \underset{10}{x}$

9 remaining balls w/ 3R 6W balls

# perms of 6W's 3R's

$$= \frac{9!}{6!\,3!} = |E_5|$$

$$P(E_5) = \frac{|E_5|}{|\Omega|} = \frac{\frac{9!}{\cancel{6!}\,3!}}{\frac{\cancel{10!}}{\cancel{6!\,4!}}} = \frac{\cancel{6!}\,4!}{10!} \Big/ \frac{\cancel{6!\,4!}}{\cancel{10!}} = \frac{4}{10}$$

**N.B.!** $P(E_1) = P(E_5)$

Prob of 1st Ball Red = Prob of 5th ball Red !

**UPSHOT!** Unless you know what happened before, you can't include it in computing probability now!

## CONDITIONAL PROBABILITY

⟹ TOOL ALLOWS TO UPDATE OUR CALCULATIONS OF ODDS TO INCORPORATE INFORMATION

**Def** If $B \subseteq \Omega$ is an event w/ $P(B) > 0$

then $$\underbrace{P(A|B)}_{\text{"Probability of A GIVEN B"}} := \frac{P(A \cap B)}{P(B)}$$

Special case = Uniform Probability when $\Omega$ is finite

$$P(A|B) = \frac{|A \cap B|}{|B|}$$

EXAMPLE

$$P(E_5|E_1) = \frac{P(E_5 \cap E_1)}{P(E_1)}$$

6 white
2 Red

LONG WAY

| R | x | x | x | R | x | x | x | x | x |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

$P(E_1 \cap E_5)$
$= P(E_5 \cap E_1)$ $= \dfrac{\#\text{perms of } 6W\text{'s } 2R\text{'s}}{|\Omega|}$

$$\frac{\frac{8!}{6!\,2!}}{\binom{10}{4}} \qquad \frac{8!}{6!\,2!} = \binom{8}{6} = \frac{8\cdot 7}{2} = 28$$

$$\Rightarrow P(E_1 \cap E_5) = \frac{28}{\frac{10\cdot\cancel{9}_3\cdot\cancel{8}\cdot 7}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}} = \frac{4\cdot\cancel{7}}{10\cdot 3\cdot\cancel{7}} = \left(\frac{4}{10}\right)\left(\frac{1}{3}\right)$$

$$= \left(\frac{4}{10}\right)\left(\frac{3}{9}\right)$$

Prob 1st ball is red

probability 2nd observed ball is red Given one red is gone

EASY WAY

$$P(E_1 \cap E_5) = \left(\frac{4}{10}\right)\left(\frac{3}{9}\right)$$

$$P(E_s|E_1) = \frac{P(E_1 \cap E_s)}{P(E_1)} = \frac{\cancel{\left(\frac{4}{10}\right)}\left(\frac{3}{9}\right)}{\cancel{\left(\frac{4}{10}\right)}}$$

FINAL ANSWER $= \frac{3}{9}$

$33\%$

$.\overline{3}$

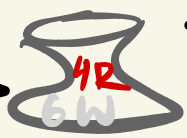
## MULTIPLICATION RULE

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow \boxed{P(A \cap B) = P(A|B)P(B)}$$

## ITERATED MULT. RULE

$A_1, A_2, \ldots, A_n$ w/ $P(A_1 \cap \cdots \cap A_n) > 0$

$$\Rightarrow \boxed{P(A_1 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap \cdots \cap A_{n-1})}$$

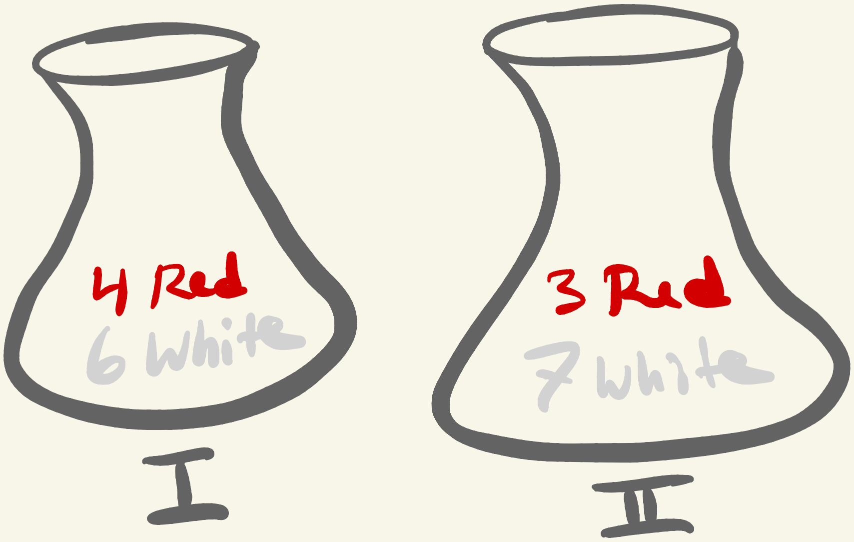
Ex (Same  )


$$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2)$$

1st, 2nd & 3rd Balls are red $= \left(\frac{4}{10}\right)\left(\frac{3}{9}\right)\left(\frac{2}{8}\right)$

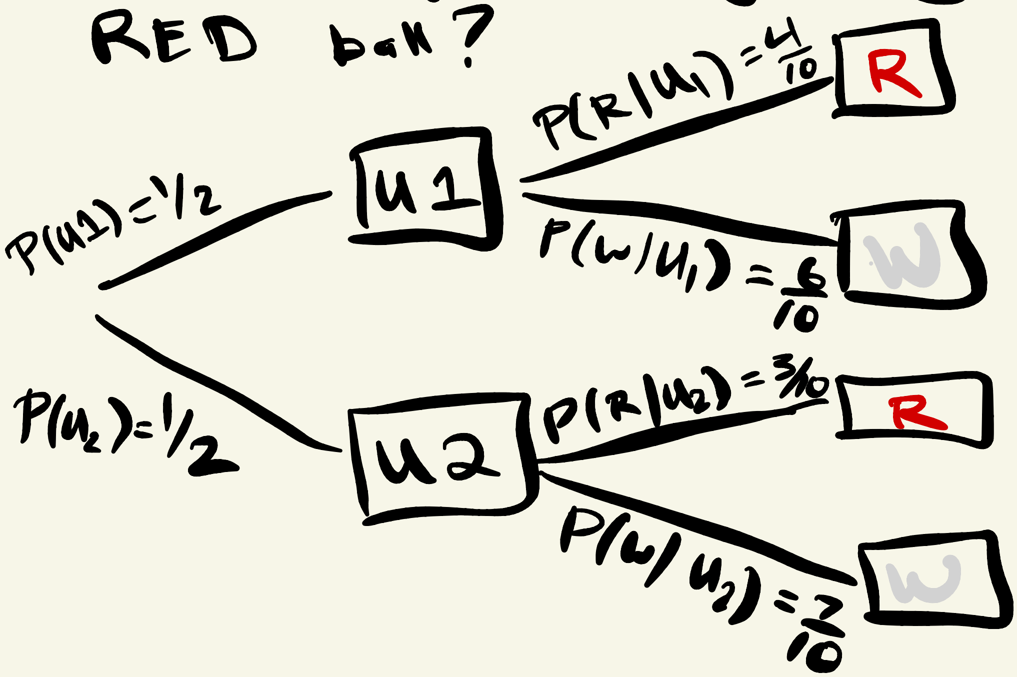
EASY!

# NEW SETUP: TWO POLYA URNS




**THE EXPERIMENT**: Randomly select URN I or URN II **then** pick out a ball

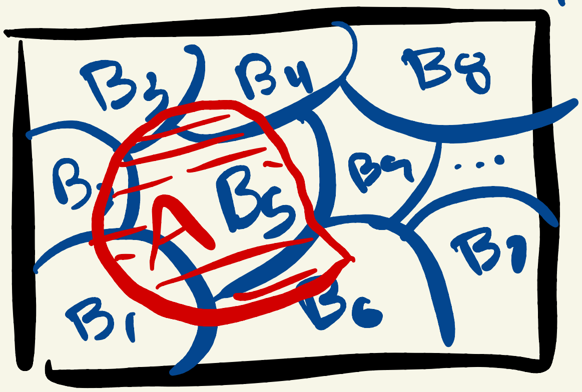
**Q**: What's the probability of getting a RED ball?




$$P(R) = P(R \cap U_1) + P(R \cap U_2)$$
$$= P(R|U_1)P(U_1) + P(R|U_2)P(U_2)$$
$$= \frac{4}{10}\cdot\frac{1}{2} + \frac{3}{10}\cdot\frac{1}{2} = \frac{7}{20}$$

# LAW OF TOTAL PROBABILITY

If $B_1, \dots, B_n$ partitions $\Omega$
and $P(B_i) > 0 \quad i = 1, \dots, n$




THEN

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i)$$

$$\Rightarrow P(A) = \sum_{i=1}^{n} P(A \mid B_i) P(B_i)$$