Lecture 11 - LIVE

① Roll a die 3 times

$X = \#\, 6\text{'s}$

$$E(X) \overset{?}{=} 3 \cdot \frac{1}{6^3} + \underbrace{2 \cdot \frac{1}{6^2}\left(\frac{5}{6}\right)}_{} + \underbrace{1 \cdot \frac{1}{6}\left(\frac{5}{6}\right)^2}_{}$$

✓

Missing $\binom{3}{2} = 3$

Missing $\binom{3}{1} = 3$

Prob of that sequence $\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)$

YNY	YNN
NYY	NYN
YYN	NNY

Intuitive Answer

$n = 3$

$p = \text{prob of } 6 = \frac{1}{6}$

$E(X) = np = \frac{3}{6}$

Indicator Method

Let $X_i = \begin{cases} 1 & \text{if 6 shows up on the } i\text{th roll} \\ 0 & \text{o.w.} \end{cases}$

$P(6) = \frac{1}{6}$

$E(X) = E(X_1 + X_2 + X_3) = E(X_1) + E(X_2) + E(X_3)$

Fundamental Facts

If X counts the number of successes out of n (independent) tries

For Binomial

1) $P(X=k) = \binom{n}{k} p^k q^{n-k}$ BINOMIAL DISTR.

2) $E(X) = E(\underbrace{X_1 + \cdots + X_n}_{S_n})$ where X_j indicates if the jth try was a success

(L.O.E.) $= E(X_1) + \cdots + E(X_n)$

$= P(X_1 = 1) + \cdots + P(X_n = 1)$

$= p + p + \cdots + p$

$= np$

Upshot

$$E(X) = \sum_{k=0}^{n} k \cdot P(X=k) = \sum_{k=1}^{n} k \cdot \frac{n!}{k!(n-k)!} p^k q^{n-k}$$

For Binomial

: LOT OF ALGEBRA

$= np$

① ctd

$Y = \#$ of 1, 3, 5 out of 3 rolls

$E(Y) = \ ? \ = \frac{3}{2} = 3 \cdot p$

p = prob of 1, 3 or 5

$= \frac{3}{6} = \frac{1}{2}$

$Y \sim \text{Binom}(p = \frac{1}{2}, n = 3)$

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② $X = \#$ of ♠ out of 7 cards dealt from 52

(There are 13 ♠'s)

$$\boxed{E(X) = 7 \cdot \frac{1}{4}}$$

Intuitive way!

$\frac{13}{52}$ = prob 1st card is a spade

$\frac{1}{4} =$

Let X_1 = indicator RV $= \begin{cases} 1 & \text{if 1st card is ♠} \\ 0 & \end{cases}$

$X_2 = \quad '' \quad = \begin{cases} 1 & \text{if 2nd card is a spade} \\ 0 & \text{o.w.} \end{cases}$

Are X_1 & X_2 are independent? No.

② cont'd

$X = \#$ of spades

$= X_1 + X_2 + \dots + X_7$

NOT INDEPENDENT!

But still $\Rightarrow E(X) = E(X_1) + E(X_2) \dots + E(X_7)$

$= \frac{1}{4} + \frac{1}{4} + \dots + \frac{1}{4}$

$= \frac{7}{4}$

$L = \#$ 1, 2, 3s

$M = \#$ 4s

$H = \#$ 5 or 6s

Roll 2x

L = 0

H \ M	0	1	2
2	$\left(\frac{2}{6}\right)^2$	0	0
1	0	$2\left(\frac{2}{6}\right)\left(\frac{1}{6}\right)$	0
0	0	0	$\left(\frac{1}{6}\right)^2$

$\frac{4}{6^2} + \frac{4}{6^2} + \frac{1}{6^2}$

$9/36$

L = 1

H \ M	0	1	2
2	0	0	0
1	$2\left(\frac{3}{6}\right)\left(\frac{2}{6}\right)$	0	0
0	0	$2\left(\frac{3}{6}\right)\left(\frac{1}{6}\right)$	0

$\frac{12 + 6}{36} = \frac{18}{36}$

L = 2

H \ M	0
0	$\left(\frac{3}{6}\right)^2$

$9/36$

(b) Distribution for L

Binomial $p = 1/2$ $n = 2$

k	0	1	2
$P(L=k)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$\| \quad \frac{9}{36} \quad \frac{18}{36} \quad \frac{9}{36}$

(c) $L + H = Z$ $P(L, M, H)$

$\underline{Z = 0}$ $P(M=2) = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$

$\underline{Z = 1}$ $P(L=1, M=1, H=0) = \frac{6}{36}$

$+P(L=0, M=1, H=1) = \frac{4}{36}$ $\Big] - \frac{10}{36}$

$\underline{Z = 2}$ $P(L=2, M=0, H=0) = 9/36$

$+P(L=1, M=0, H=1) = 12/36$

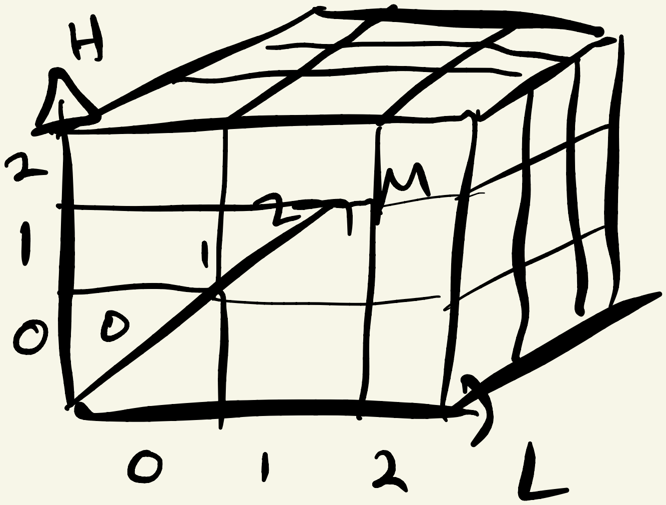
$+P(L=0, M=0, H=2) = 4/36$ $\Big] \frac{25}{36}$

③ 3 RVs L, M, H

$H \to 5, 6$

$M \to 4$

$L \to 1, 2, 3$



L	M	H
2	0	0
1	1	0
1	0	1

L = 0

#H's	0	1	2
2	$\left(\frac{2}{6}\right)\left(\frac{2}{6}\right) = \frac{4}{36}$	0	0
1	0	$2\cdot\left(\frac{1}{6}\right)\left(\frac{2}{6}\right)$	0
0	0	0	$\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$

#4's = M

$p(HM) = \frac{2}{6}\cdot\frac{1}{6}$

or

$p(MH) = \frac{1}{6}\cdot\frac{2}{6}$

L = 1

H	0	1	2
2	0	0	0
1	$2\left(\frac{2}{6}\right)\left(\frac{3}{6}\right)$	0	0
0	0	$2\left(\frac{3}{6}\right)\left(\frac{1}{6}\right)$	0

M

$p(LH) = \frac{3}{6}\cdot\frac{2}{6} = \frac{6}{36}$

$+ p(HL) = \frac{2}{6}\cdot\frac{3}{6} = \frac{6}{36}$

$\Big\} \to \frac{12}{36}$

L = 2

	0	1	2
2			
1			
0			

Multinomial Distribution

$n = 2$

$n_1 = \#L \quad n_2 = \#M \quad n_3 = \#H$

Each entry $\dfrac{n!}{n_1!\,n_2!\,n_3!} p_1^{n_1} p_2^{n_2} p_3^{n_3}$

WS 10 #4

Toss a coin 3 times

X = # heads in first 2 tosses

Y = # heads in last 2 tosses

	Y			
$2(\frac{1}{2})^3$	2	0	THH $(\frac{1}{2})^3$	HHH $(\frac{1}{2})^3$
$4(\frac{1}{2})^3$	1	TTH $(\frac{1}{2})^3$	HTH or THT $2(\frac{1}{2})^3$	HHT $(\frac{1}{2})^3$
$2(\frac{1}{2})^3$	0	$(\frac{1}{2})^3$	HTT $(\frac{1}{2})^3$	0
	X	0	1	2
		$\downarrow$ $2(\frac{1}{2})^3$	$\downarrow$ $4(\frac{1}{2})^3$	$\downarrow$ $2(\frac{1}{2})^3$