AMAT 362 - Probability for Statistics - LEC 4

PASCAL's TRIANGLE

$$
\begin{array}{l}
0 \rightarrow \quad 1 \\
(x+y)^1 \quad 1 \quad 1 \\
(x+y)^2 \quad 1 \quad 2 \quad 1 \\
(x+y)^3 \quad 1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}
$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(x+y)(x+y) = x^2 + xy + yx + y^2$$
$$= x^2 + 2xy + y^2$$

Recall How many subsets of an n element set are there?

$\longrightarrow 2^n$

$$\underset{\substack{\shortparallel \\ x}}{(1} + \underset{\substack{\shortparallel \\ y}}{1)}^n = \underset{\binom{n}{0}}{1^n} + n \cdot \underbrace{1^{n-1} \cdot 1}_{1} + \binom{n}{2} \underbrace{1^{n-2} 1^2}_{1} + \cdots + \binom{n}{k} \underbrace{1^{n-k} 1^k}_{1} + \cdots + \underset{\substack{\shortparallel \\ 1}}{\binom{n}{n}}$$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \# \ k \text{ sized subsets of an } n \text{ elt set!}$$

DUALITY $\rightsquigarrow \binom{n}{k} = \binom{n}{n-k}$

BINOMIAL DISTRIBUTION

Q: Sps I flip a fair coin 6 times
What's the probability of getting **exactly** 2 heads out of 6 flips?

A: What is the sample space?

$$\Omega = \{ (f_1, f_2, f_3, f_4, f_5, f_6) \mid f_i \in \{H, T\} \}$$

$$|\Omega| = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$$

A = Exactly two of the f_i are H

$$|A| = \binom{6}{2} = \frac{6!}{4!\,2!} = \frac{6 \cdot 5}{2} = 15$$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{15}{64} \approx 23.4\%$$

PERMUTATIONS w/ REPEATING LETTERS

Q: HOW MANY DISTINCT WAYS CAN I REARRANGE THE LETTERS in 'TOOT'

A: TOOT OOTT

TOTO OTOT

TTOO OTTO

GENERAL ANSWER

If I have a word of length n w/ ℓ distinct letters which appear $n_1, n_2, \dots, n_\ell$ # times (i.e. $n_1+n_2+\dots+n_\ell=n$)

n_2 " # of times the 2nd distinct letter appears

THEN the # of rearrangements is

$$\frac{n!}{n_1!n_2!\dots n_\ell!} = \binom{n}{n_1, n_2, n_3, \dots, n_\ell} = \text{"Multinomial Coefficient"}$$

Ex OTTO $n=4$ $n_O=2$ $n_T=2$

$$\Rightarrow \frac{4!}{2!2!} = \frac{4\cdot 3}{2} = 6 \checkmark$$

Ex APPLE

$n_A=1$ $n_P=2$ $n_L=1$ $n_E=1$

$$\frac{5!}{1!2!1!1!} = 5\cdot 4\cdot 3 = 60$$

Ex FALAFEL

$n_F=2$ $n_L=2$

$n_A=2$ $n_E=1$

$$\boxed{\frac{7!}{2!2!2!}}$$

Why does this formula work?

Step 1: Make every letter distinct

$$OTTO \rightsquigarrow O_1 T_1 T_2 O_2$$

# of rearrangements I can distinguish is

$$n! \rightsquigarrow 4!$$

Step 2: Divide by the # of permutations that you can't really tell apart

$T_1 T_2 \overset{}{\underset{=}{\longleftrightarrow}} T_2 T_1$ 2! extra perms

$O_1 O_2 \underset{=}{\longleftrightarrow} O_2 O_1$ 2! extra perms

$$\Longrightarrow \frac{n!}{n_1! n_2! \cdots n_\ell!} \rightsquigarrow \frac{4!}{2!\,2!} = \binom{4}{2}$$

BACK TO THE COIN FLIPS

ways I can get EXACTLY 2 HEADS

= # of rearrangements of HHTTTT

$$\frac{n!}{n_H!\, n_T!} = \frac{6!}{2!\,4!} = \frac{6 \cdot 5}{2} = 15 \checkmark$$

OTHER APPLICATIONS of Multinomial Formula

★ Assigning n people to ℓ groups w/ each group getting $n_1, n_2, \ldots, n_\ell$ people.

Q: How many ways can I assign 2000 students to 3 schools where n_1 = # students in School 1 (capacity) $= 1000$

$$n_2 = 500$$

$$n_3 = 500$$

N.B.! $0! = 1$

A:

$$\underbrace{\binom{2000}{1000}}_{\substack{\text{\# ways of choosing} \\ \text{1000 students} \\ \text{from 2000}}} \underbrace{\binom{1000}{500}}_{\substack{\text{\# ways of choosing} \\ \text{500} \\ \text{remaining 1000}}} \binom{500}{500} = \frac{2000!}{1000!\,\cancel{1000!}} \cdot \frac{\cancel{1000!}}{500!\,500!} \cdot \frac{\cancel{500!}}{0!\,\cancel{500!}}$$

$$= \frac{2000!}{1000!\,500!\,500!}$$

$$= \binom{2000}{1000, 500, 500}$$

Line up students

S_1 S_2 S_3 S_4 .... S_{1500} S_{1501} ... S_{1998} S_{1999} S_{2000}

A B A A ... B A ... B A C

1000 A's → School 1

500 B's → School 2

500 C's → School 3

$$\frac{2000!}{1000!\,500!\,500!}$$ ← possible ways of assigning students to schools!

POKER

Q: How many ways are there of dealing 6 people 5 card hands?

A1: $\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5}\binom{32}{5}\binom{25}{5}$

ways of choosing 5 cards for P1

.... Another Way .. A2: $\dfrac{52!}{5!5!5!5!5!5!22!}$

CARTOON

C_1 C_2 C_3 C_4 ... C_{21} C_{22} ... C_{35} C_{36} ... C_{51} C_{52}

A B E A G C G F D G C F

5 A's 5 B's 5 C's 5 D's 5 E's 5 F's 22 G's DISCARD