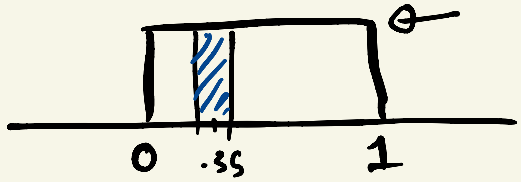
LECTURE 17 LIVE

Q1: $X \sim \text{Unif}(0,1)$



$$P(X \approx .35) = P(.345 \le X < .355)$$

after rounding the 3rd decimal place

$$= \frac{.355 - .345}{1 - 0}$$

$$= \frac{.01}{1} = .01 = 1\%$$

$b = 1$
$a = 0$

$$= \int_{.345}^{.355} \frac{1}{b-a}\,dx = x\Big|_{.345}^{.355}$$

$$= .355 - .345$$

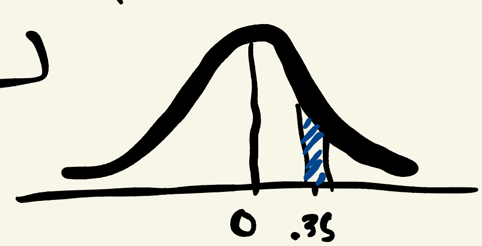
$$= .01$$

Q2 Redo Q1 w/ $X \sim \text{Norm}(0,1)$

PDF for Norm(0,1) is $\underbrace{\dfrac{e^{-x^2/2}}{\sqrt{2\pi}}} = \varphi(x)$

In general, $\text{Norm}(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/2\sigma^2}$$



$$P(.345 \le X < .355) = \int_{.345}^{.355} \varphi(x)\,dx$$

Q: What is the antiderivative of e^{-x^2}?

$$e^{-x} \xrightarrow{\int} -e^{-x} \xrightarrow{\frac{d}{dx}} -\frac{d}{dx}e^{-x} = -e^{-x}(-1)$$
$$= e^{-x} \checkmark$$

$$\frac{d}{dx}e^{u(x)} = e^{u}\cdot\frac{du}{dx}$$

CHAIN RULE

$$\frac{d}{dx}\left(e^{-x^2}\right) \stackrel{?}{=} e^{-x^2}\cdot(-2x)$$

PESKY!!!

$$\frac{d}{dx}F(x) = e^{-x^2}$$

FIND F(x)!!

$F = e^{-x^2+x^3}$ or $e^{-x^2}+e^{x^3}$ ← NOPE

$F = \ln \ldots ?$ NOPE

$$\frac{d}{dx}\left(\frac{1}{x}e^{-x^2}\right) = \frac{d}{dx}\left(\frac{1}{x}\right)\cdot e^{-x^2} + \frac{1}{x}\frac{d}{dx}\left(e^{-x^2}\right)$$

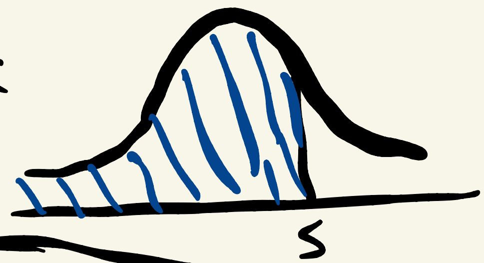
PESKY.... GOOD!

$$e^{z} = 1 + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

$$\Rightarrow e^{-x^2} = 1 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots$$

Integrate term by term! BUT STILL BAD

$$\Phi(s) = \int_{-\infty}^{s} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

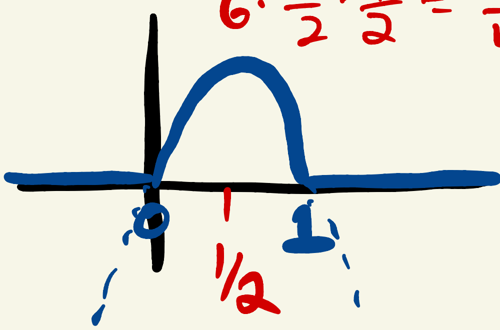


$$\boxed{\Phi(.355) - \Phi(.345)}$$

Answer NEED A BETTER Z-TABLE

③ X has PDF $f(x) = \begin{cases} cx(1-x) & 0<x<1 \\ 0 & \text{o.w.} \end{cases}$

$6 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{6}{4} = 1.5$



(P+a) Find c

Normalization Axiom

$$\int_{-\infty}^{\infty} f(x) = 1 = P(-\infty < X < -\infty)$$

$$\int_0^1 cx - cx^2 = c\int_0^1 x - x^2\,dx = c\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= c\left[\frac{1}{2} - \frac{1}{3}\right] = 1$$

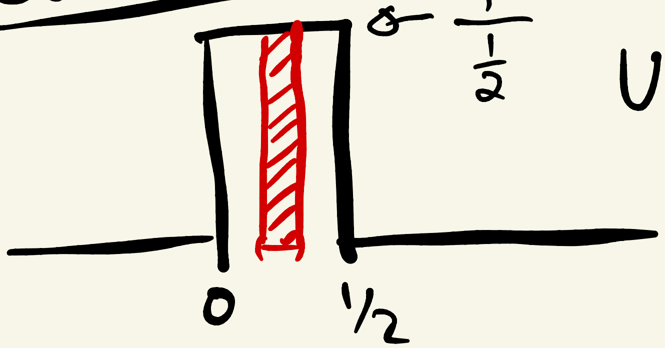
$$\frac{3}{6} - \frac{2}{6}$$

$$\Longrightarrow c \cdot \frac{1}{6} = 1$$

$$\Longrightarrow \boxed{c = 6}$$

N.B. $f_X(\frac{1}{2}) = 1.5 > 1$!! Why is this not a contradiction?

Consider



Unif$(0, \frac{1}{2})$

$f(x)$ = Probability Density

NOT PROBABILITY, WHICH IS

$$P(a < X < b) = \int_a^b f(x)\,dx = \boxed{\text{AREA UNDER CURVE}}$$

So $f(x)$ can be > 1

but $\int f(x) \leq 1$

Q3 Pt (b)

$$P(X \leq 1/2) = \int_0^{1/2} 6 \cdot x(1-x)\,dx$$

$$\int_0^{1/2} 6x - 6x^2 = 3x^2 - 2x^3 \Big|_0^{1/2}$$

$$= 3\left(\tfrac{1}{2}\right)^2 - 2\left(1/2\right)^3$$

Q3 Pt (c) ON YOUR OWN!

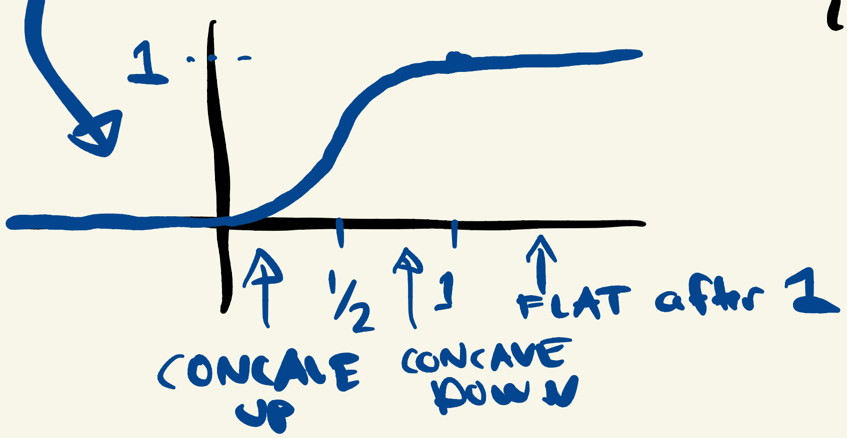
(d) Draw CDF of $f(x) = 6x(1-x)$ for $0<x<1$

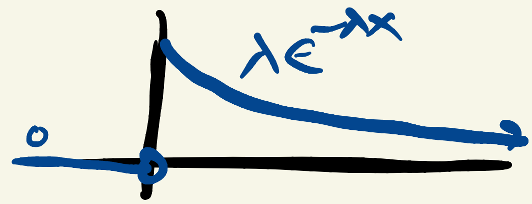
$$\int_{-\infty}^{s} f_X(x) = \int_0^s 6x(1-x) = 3x^2 - 2x^3 \Big|_0^s$$

$$F(s) = \begin{cases} = 3s^2 - 2s^3 & \text{for } 0 \le s \le 1 \\ = 0 & s<0 \\ = 1 & s \ge 1 \end{cases}$$

$3(1)^2 - 2(1)^3$
$= 3-2 = 1$ ✓

GRAPH





Q4) $X \sim \mathrm{Exp}(\lambda)$

(a) $E[X] \overset{\text{DEF}}{=} \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$

$$= \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\, dx$$

HOW TO INTEGRATE?

BY PARTS

$$\int_a^b u\, dv = uv \Big|_a^b - \int_a^b v\, du$$

$u = x \quad dv = \lambda e^{-\lambda x} \Rightarrow = -xe^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\, dx$

$$= -xe^{-\lambda x}\Big|_0^\infty + \underbrace{\int_0^\infty e^{-\lambda x}\,dx}_{= \frac{-e^{-\lambda x}}{\lambda}\Big|_0^\infty}$$

$$= \underline{\lim_{x\to\infty}\left(-xe^{-\lambda x}\right)} + 0$$

Convince yourself that this $\to 0$

$$\lim_{x\to\infty} \frac{-x}{e^{\lambda x}} \text{ "} = \frac{\infty}{\infty} \text{"}$$

Apply L'Hôpital

$$\boxed{E[X] = \frac{1}{\lambda}}$$

(b) $VAR[X] = E[X^2] - E[X]^2$

$$E[X^2] = \int_0^\infty x^2 \cdot \lambda e^{-\lambda x}\,dx$$

Integrate by parts 2x!

(HOMEWORK)

(c) $F(s) = \int_{-\infty}^{s} f(x)\,dx = \int_0^s \lambda e^{-\lambda x}\,dx$

$$= -e^{-\lambda x}\Big|_0^s = 1 - e^{-\lambda s}$$

for $s \geq 0$

GRAPH OF CDF

Q4 (d)

Recall median $= m$

$$P(X \geq m) \geq 1/2 \quad P(X \leq m) \geq 1/2$$

For monotonically increasing ($\S$ 1-1) CDF

$$P(X \geq m) = 1/2$$

$$= \int_m^\infty \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_m^\infty = \boxed{e^{-\lambda m} \overset{\text{set to}}{=} 1/2}$$

Solve for m $\quad -\lambda m = \ln(1/2)$

$$= \cancel{\ln 1}^{\,0} - \ln 2$$

$$-\lambda m = -\ln 2$$

Formula for

MEDIAN →

= Half-Life

$$\boxed{m = \frac{\ln 2}{\lambda}}$$

$M = 1$

Q5 Radioactive element has half-life of 1 year

Q: What is λ? $\quad \boxed{\lambda = \ln 2}$

$$P(T > 5) = \int_5^\infty \lambda e^{-\lambda t}\,dt = e^{-\lambda \cdot 5} = e^{(-\ln 2)\cdot 5} = \frac{1}{2^5} = \boxed{\frac{1}{32}}$$

(b) $$\int_{t_{10\%}}^{\infty} \lambda e^{-\lambda t}\,dt = 10\%$$

$$e^{-\lambda t_{10\%}} = .1 = \frac{1}{10}$$

$$-\lambda t_{10\%} = \ln\left(\frac{1}{10}\right) = -\ln 10$$

$$t_{10\%} = \frac{\ln 10}{\ln 2} \approx \underbrace{3.32}_{}$$

Amt of time in years for 90% to decay...