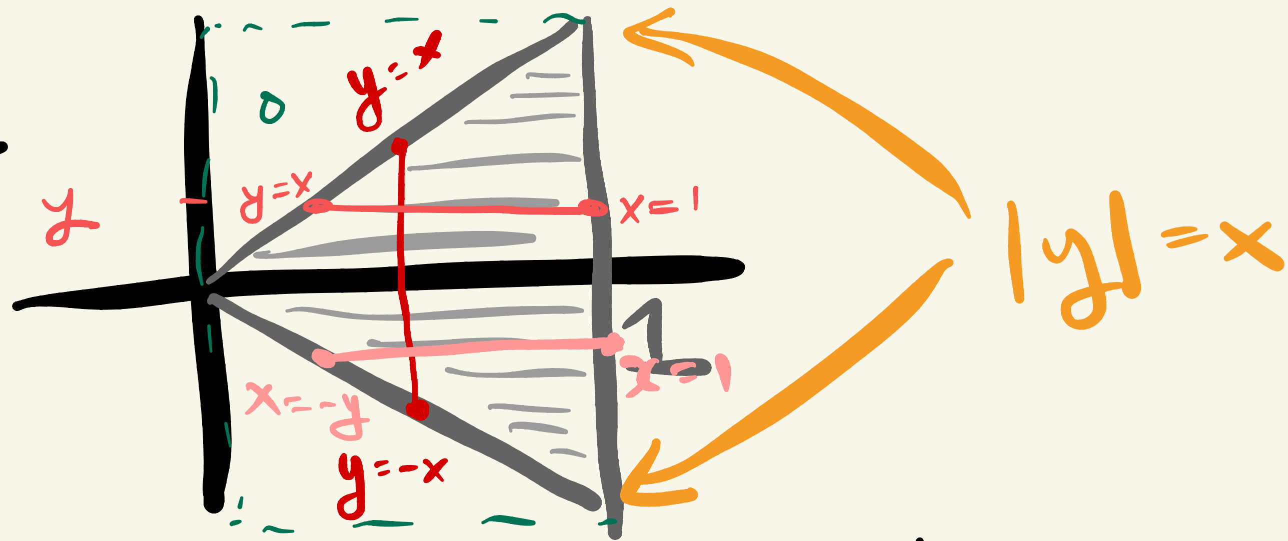
LECTURE 22 LIVE

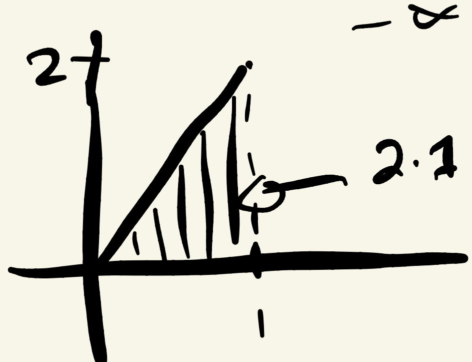
Q1



(a) Find JDF $f(x,y) = \begin{cases} \dfrac{1}{\text{Area}(\triangle)} \underset{=1}{} & 0<|y|<x<1 \\ 0 & \text{o.w.} \end{cases}$

(b) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{y=-x}^{y=x} 1\,dy = \boxed{2x}$

PDF for X



$2 \cdot 1 \cdot \frac{1}{2} = 1$ ✓

PDF for Y

$$\underset{y\geq 0}{f_Y(y)} = \int_{x=y}^{x=1} 1\,dx = \boxed{1-y \quad y\geq 0}$$

$$\underset{y\leq 0}{f_Y(y)} = \int_{x=-y}^{x=1} 1\,dx = \boxed{1+y \quad y\leq 0}$$

(c) NOPE.

(d) $E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 2x^2\,dx$

$= \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy$$

$$= \int_{-1}^{0} y(1+y)\,dy + \int_0^1 y(1-y)\,dy$$

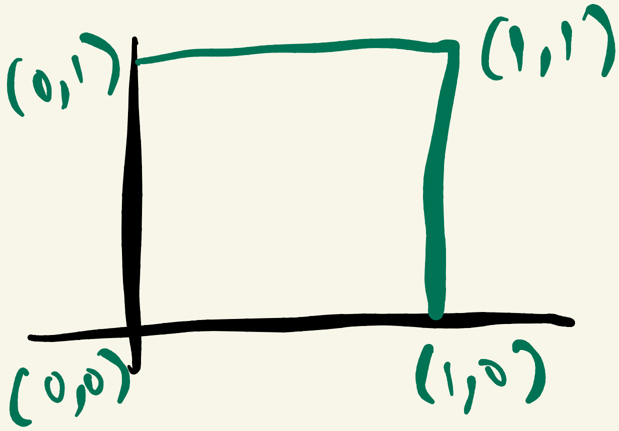
$$= \int_{-1}^{0} y+y^2\,dy + \int_0^1 y-y^2\,dy$$

$$= \frac{y^2}{2}+\frac{y^3}{3}\Big|_{-1}^{0} + \frac{y^2}{2}-\frac{y^3}{3}\Big|_0^1$$

$$= -\cancel{\frac{(-1)^2}{2}} - \cancel{\frac{(-1)^3}{3}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} = 0$$

PDF for Y

Q2)



$$f_{X,Y}(x,y) = \begin{cases} c(x^2+4xy) & 0<x<1 \\ & 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

(a) Find c.

By Normalization Axiom

$$\int_{y=0}^{y=1}\int_{x=0}^{x=1} c(x^2+4xy)\,dx\,dy = 1$$

FIND $\rightarrow$ $\displaystyle\int_{y=0}^{y=1}\left[\frac{x^3}{3}+2x^2y\right]_{x=0}^{x=1} dy$

$$= \int_{y=0}^{y=1} \frac{1}{3} + 2y\,dy = \left[\frac{1}{3}y + y^2\right]_{y=0}^{y=1}$$

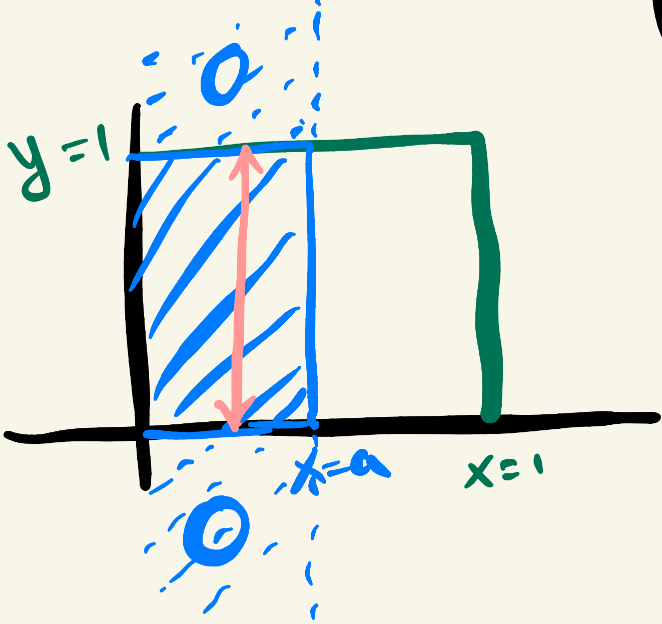
$$= \frac{1}{3} + 1$$

$$= \frac{4}{3}$$

$$\boxed{c = \frac{3}{4}}$$

So...

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{4}(x^2+4xy) & \text{for } \begin{matrix} 0<x<1 \\ 0<y<1 \end{matrix} \\ 0 & \text{o.w.} \end{cases}$$



$$P(X \le a)$$

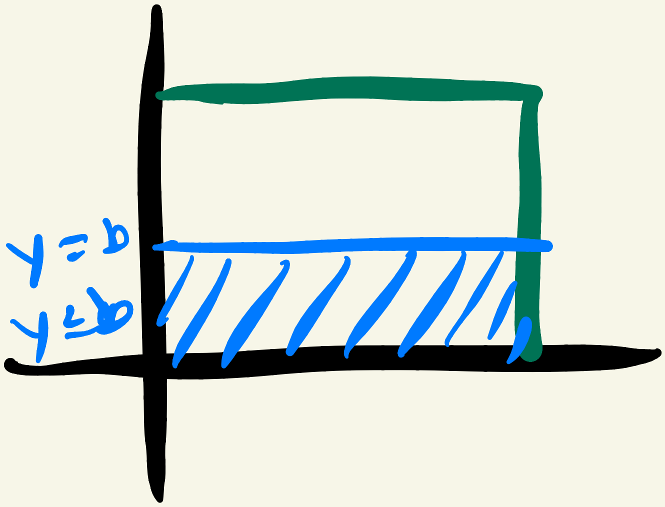
$$= P(X \le a, -\infty \le Y \le \infty)$$

$$= \int_{x=0}^{x=a} \int_{y=0}^{y=1} \frac{3}{4}(x^2+4xy)\, dy\, dx$$

$$= \frac{3}{4} \int_{x=0}^{x=a} \left[x^2 y + 2xy^2 \right]_{y=0}^{y=1} dx = \frac{3}{4} \int_{x=0}^{x=1} x^2 + 2x \, dx$$

$$= \frac{3}{4} \left[\frac{x^3}{3} + x^2 \right]_0^a = \boxed{\frac{3}{4} \left[\frac{a^3}{3} + a^2 \right]}$$

$P(X \le a)$



So...

$$P(Y \le b)$$

$$= \int_{y=0}^{y=b}\int_{x=0}^{x=1} \frac{3}{4}\left(x^2+4xy\right)\,dx\,dy$$

$$\frac{3}{4}\int_0^b \left[\frac{x^3}{3}+2x^2y\right]_{x=0}^{x=1} dy = \frac{3}{4}\int_0^b \frac{1}{3}+2y\,dy$$

$$= \frac{3}{4}\left[\frac{y}{3}+y^2\right]_{y=0}^{y=b} = \boxed{\frac{3}{4}\left(\frac{b}{3}+b^2\right)}$$

$P(Y \le b)$ "

Q3

$X \sim \text{Exp}(\lambda) \quad \lambda = \frac{1}{10}$

$E(X) = 10$

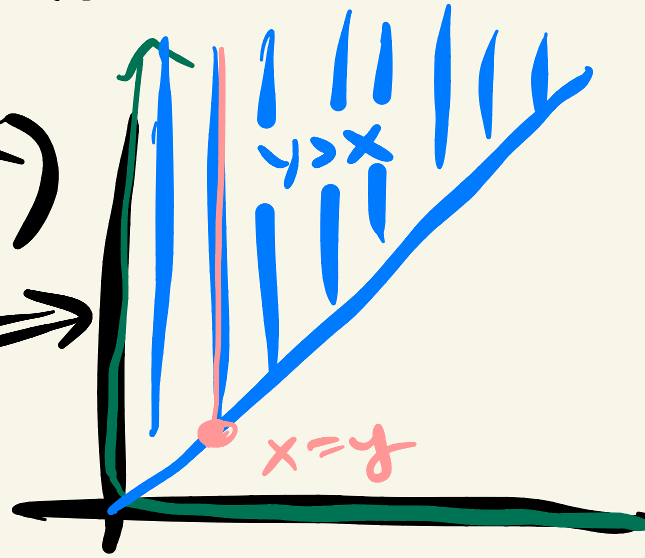
Washer →

Dryer → $Y \sim \text{Exp}(\mu) \quad \mu = \frac{1}{12}$

$E(Y) = 12$

(a) P(Dryer > Washer)

$\Leftrightarrow P(Y > X)$



$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} \lambda\mu \underbrace{e^{-\lambda x - \mu y}}_{e^{-\lambda x} e^{-\mu y}} \, dy \, dx$$

$\frac{1}{10} \cdot \frac{120}{22}$

$$\int_{x=0}^{\infty} \lambda e^{-\lambda x} \underbrace{\left(\int_{y=x}^{\infty} \mu e^{-\mu y} \, dy\right)}_{e^{-\mu x}} dx$$

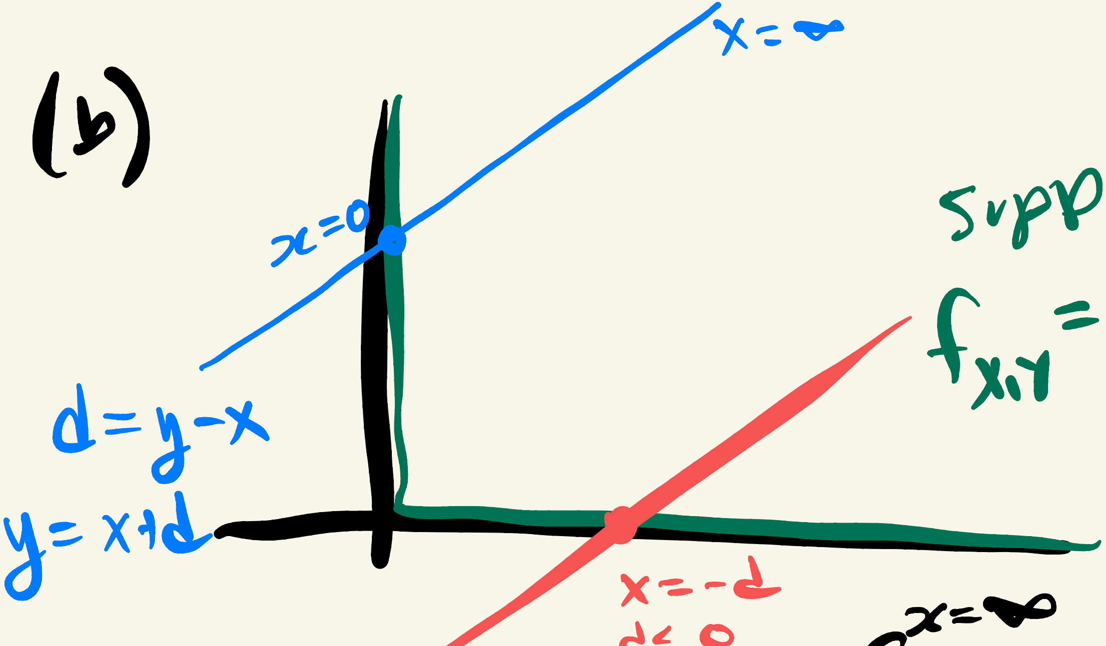
$$\frac{\frac{1}{10}}{\frac{1}{10} + \frac{1}{12}} = \frac{6}{10} = \frac{12}{22}$$

$$= \int_{x=0}^{\infty} \lambda e^{-(\lambda+\mu)x} \, dx$$

$$\frac{-\lambda}{\lambda+\mu} e^{-(\lambda+\mu)x} \Big|_0^{\infty}$$

$$\boxed{= \frac{\lambda}{\lambda+\mu}}$$

(b)



Supp $f_{X,Y} = \lambda\mu e^{-\lambda x} e^{-\mu y}$

For $d \geq 0$

$$f_D(d) = \int_{x=0}^{x=\infty} \lambda\mu e^{-\lambda x} e^{-\mu(\overset{y=x+d}{x+d})}\, dx$$

$$= \lambda\mu e^{-\mu d} \int_{x=0}^{\infty} e^{-(\lambda+\mu)x}\, dx$$

$$= \lambda\mu e^{-\mu d} \left[\frac{-e^{-(\lambda+\mu)x}}{\lambda+\mu} \right]_{x=0}^{x=\infty}$$

$$\boxed{f_D(d) = \frac{\lambda\mu e^{-\mu d}}{\lambda+\mu} \quad \text{for } d \geq 0}$$

FOR $d \le 0$

$x = -d$

$d \le 0$

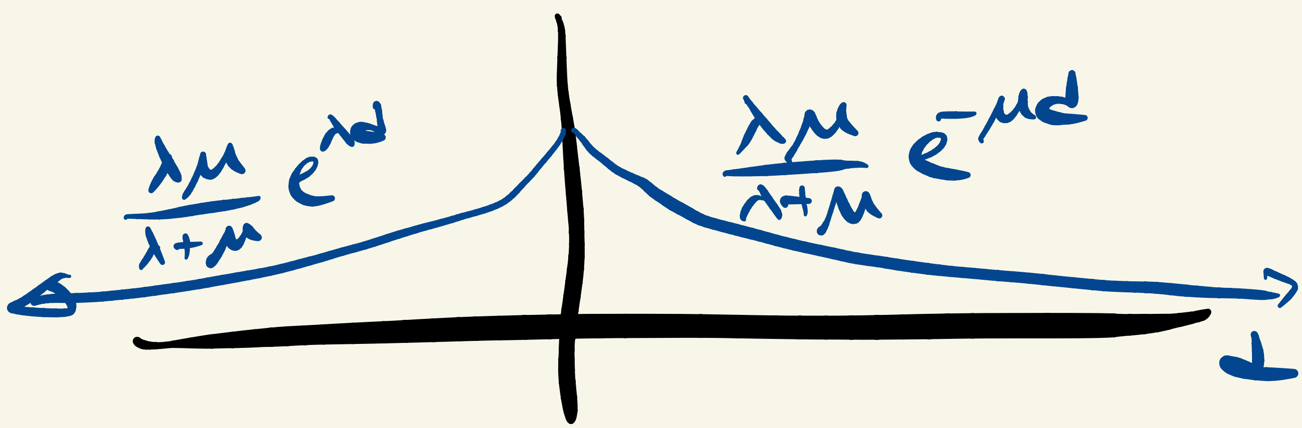
$y = x + d$

$$f_D(d) = \int_{x=-d}^{x=\infty} \lambda\mu e^{-\lambda x} e^{-\mu(x+d)}\,dx$$

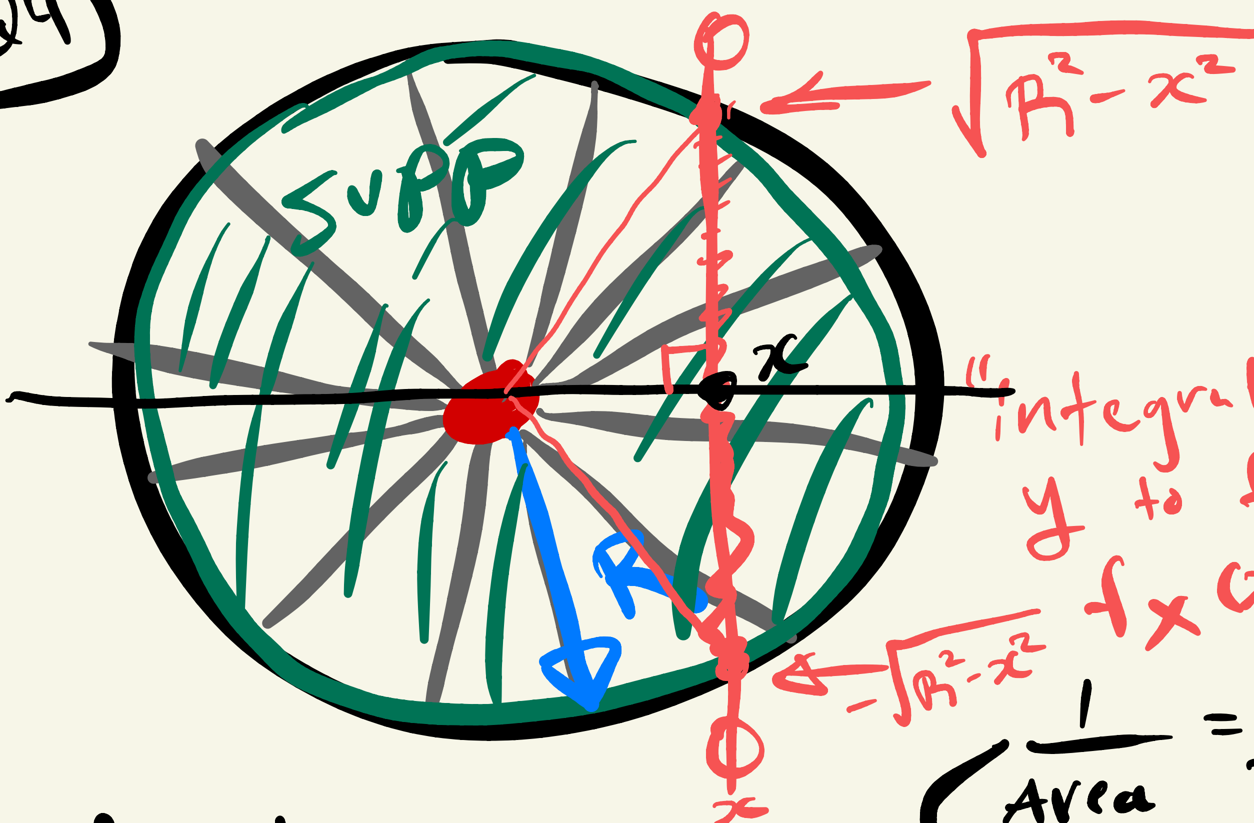
$$= \lambda\mu e^{-\mu d}\left[\frac{-e^{-(\lambda+\mu)x}}{\lambda+\mu}\right]_{x=-d}^{x=\infty}$$

$$= \frac{\lambda\mu e^{-\mu d}}{\lambda+\mu}\left[0 + e^{(\lambda+\mu)d}\right]$$

$$\boxed{f_D(d) = \frac{\lambda\mu e^{\lambda d}}{\lambda+\mu} \quad \text{for } d \le 0}$$



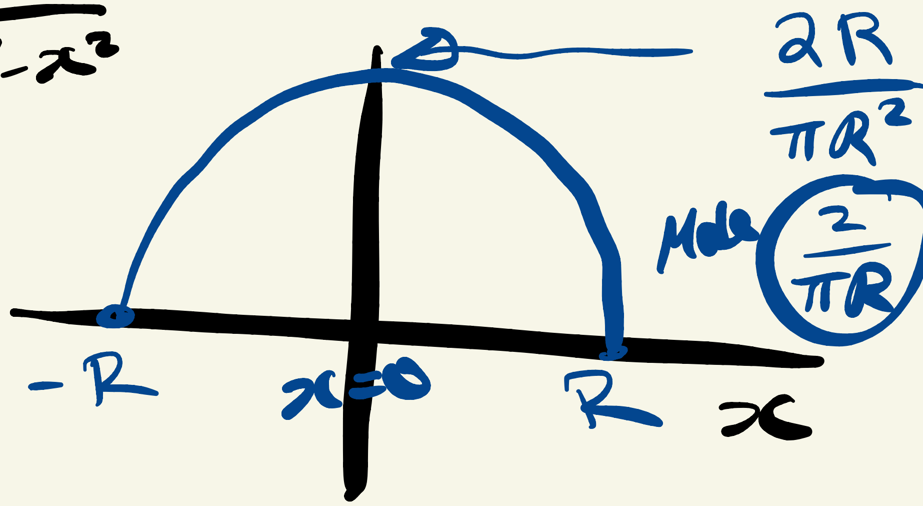
Q4)



"integrate out" y to find $f_X(x)$

Uniformly @random $\quad f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\text{Area}} = \dfrac{1}{\pi R^2} & \text{for } x^2+y^2 \le R^2 \\ 0 & \text{o.w.} \end{cases}$

$$f_X(x) = \int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}\, dy = \frac{2\sqrt{R^2-x^2}}{\pi R^2}$$



Q9

(a) Equiv to $\displaystyle\int_0^{R_{50\%}} r e^{-r^2/2}\, dr = 50\%$

$= 1/2$

$\rightarrow$ CDF for R

$$F_R(r) = 1 - e^{-r^2/2} \quad r \geq 0$$

$$= 1/2$$

$$\Rightarrow e^{-r^2/2} = 1/2 \qquad \frac{-R_{50}^2}{2} = -\ln 2$$

$$F_R(R_{50}) = 1/2 \Rightarrow \boxed{R_{50} = \sqrt{2\ln 2}}$$

$(R_{50} = 1.177)$