MATH 362—Work Sheet 06

Dr. Justin M. Curry

Due on Monday February 22nd, 2021

Name: ☆ SOLUTIONS ☆

1. (5 points) Assume the following (overly simplified) statements are true:

   - In the US, 50% of the population are men and 50% of the population are women.
   - 10% of the population in the US practices yoga.
   - Of yoga practitioners, 72% are women and 28% are men.

   (a) (1 point) Suppose that I'm told a person practices yoga, what is the probability that they are a woman?

   $$P(W/Y) = 72\%$$

   (b) (1 point) Are the events of being a woman and practicing yoga independent?

   NO b/c $\underset{72\%}{P(W/Y)} \neq \underset{50\%}{P(W)}$

   (c) (2 points) Suppose I randomly select a woman from the US population. What is the probability that they practice yoga?

   $$P(Y/W) = \frac{P(W \cap Y)}{P(W)} = \frac{P(W/Y)P(Y)}{P(W)} = \frac{(.72)(.1)}{\frac{1}{2}} = 14.4\%$$

   BAYES RULE

   (d) (1 point) What's the probability of a yoga class with five people in it consisting of all men?

   $$P(5M/Y) = \underbrace{P(M/Y)\,P(M/Y) \cdots P(M/Y)}_{5 \text{ times}}$$

   By conditional independence

   $$\boxed{= (.28)^5} \rightarrow .0017$$

   $< .2\%$!

2. (4 points) *Conditional Independence and Testing Twice for a Disease:* For two events E_1 and E_2 we sometimes write

$$P(E_1 \cap E_2) = P(E_1 \text{ and } E_2) \qquad \text{as} \qquad P(E_1E_2)$$

in order to save space. E_1 and E_2 are independent if $P(E_1E_2) = P(E_1)P(E_2)$. Furthermore, we say E_1 and E_2 are *conditionally independent given* B if

$$P(E_1E_2 \mid B) = P(E_1 \mid B)P(E_2 \mid B).$$

An example of conditional independence occurs when we re-run a test for a disease D. The probability of two positive tests assuming you have the disease can be computed as

$$P(++ \mid D) = P(+ \mid D)P(+ \mid D) = P(+ \mid D)^2$$

because each test is performed independently, even assuming you have the disease D.

For this question, assume the following is true:

- The probability that a randomly selected person has disease D is 0.5%.
- $P(+ \mid D) = 96\%$
- $P(+ \mid D^c) = 2\%$

(a) (2 points) Assuming someone tests positive for the disease D, what's the probability that they actually have disease D?

$$P(D|+) = \frac{P(+|D)P(D)}{P(+)}$$

$$P(+|D)P(D) + P(+|D^c)P(D^c)$$

$$\underbrace{(.96)(.005)}_{.0048} + \underbrace{(.02)(.995)}_{.0199}$$

$$= \frac{.0048}{.0048 + .0199} = 19.4\%$$

(b) (2 points) Assuming someone tests positive *twice* for a disease, what's the probability that they actually have disease D?

$$P(D|++) = \frac{P(+|D)^2 P(D)}{P(+|D)^2 P(D) + P(+|D^c)^2 P(D^c)}$$

$$\approx 92\%$$

3. (2 points) Suppose I have two urns, U_1 has two red balls and one white ball and U_2 has two red balls and two white balls. I select an urn uniformly at random, and draw out a red ball. What's the probability that I selected U_1?

$$P(U_1|R) = \frac{P(R|U_1)P(U_1)}{P(R|U_1)P(U_1) + P(R|U_2)P(U_2)} = \frac{(\frac{2}{3})\cancel{(\frac{1}{2})}}{(\frac{2}{3})\cancel{(\frac{1}{2})}^2 + (\frac{1}{2})\cancel{(\frac{1}{2})}}$$

$$\frac{2/3}{4/6 + 3/6} = \frac{2/1}{7/2}$$

$$= 4/7$$

4. (1 point) Suppose that $P(A) = 1/3$ and $P(B) = 1/3$ and $P(AB^c) = 2/9$. Are A and B independent?

$P(A) = P(AB^c) + P(AB)$

$1/3 = 2/9 + P(AB) \Rightarrow P(AB) = 1/9$

$\checkmark = P(A)P(B)$

YES INDEP!

Figure 1: Two Components in Series

5. (2 points) Suppose that in an electrical device with two components in *series*, the device works only if both components C_1 and C_2 work. Say the probability of each component failing on a given day is 10% and 5% respectively. What's the probability that the device works on a given day?

$P(\text{Works}) = P(C_1 \text{ works AND } C_2 \text{ works})$

Assuming independence $= P(C_1 \text{ works}) P(C_2 \text{ works})$

$= (90\%)(95\%) = 85.5\%$

For parallel

$P(\text{works})$

$= 1 - P(C_1 \text{ fails} \cap C_2 \text{ fails})$

$= 1 - P(C_1 \text{ fails}) P(C_2 \text{ fails})$

Assuming Indep.

$1 - (.1)(.05)$

$= 99.5\%$

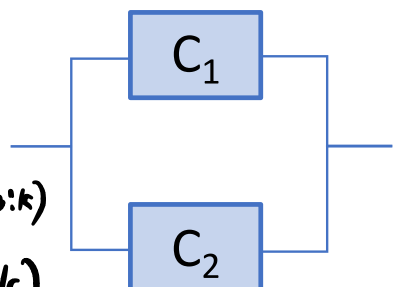

Figure 2: Two Components in Parallel

6. (2 points) Suppose that in an electrical device with two components in *parallel*, the device works only if either component C_1 or C_2 works. Say the probability of each component failing on a given day is 10% and 5% respectively. What's the probability that the device works on a given day?

MORAL: Designing systems in parallel is much more robust to failure!