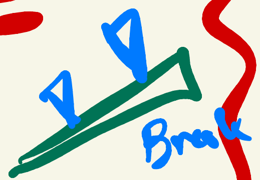
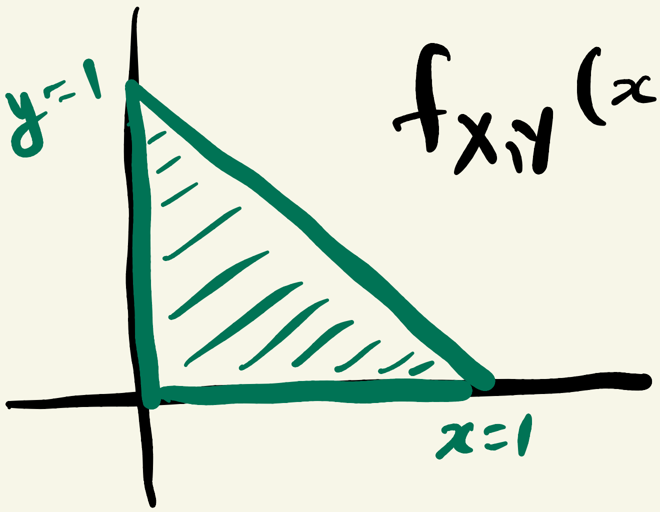
AMAT 362 - Probability for Statistics @ UAlbany

The Last Lecture #24

- Continuous Conditional Density
- Law of Iterated Expectations
- The Stick Problem
- Flipping a coin / inference





$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\text{Area}=1/2} & \text{for pts in supp} \\ 0 & \text{o.w.} \end{cases}$$

Practice Question

Pick a point at random according to this JDF. What is the expected area of the rectangle formed by the pt & (0,0)?

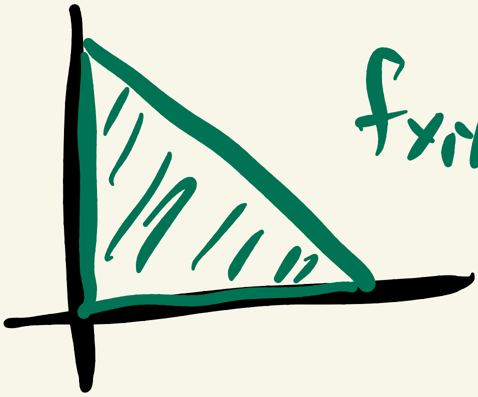
$E[XY]$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} xy\, f_{XY}(x,y)\, dy\, dx = 2\int_{x=0}^{x=1}\int_{y=0}^{y=1-x} xy\, dy\, dx$$

= THE REST IS ON YOU! ...

WHY DO WE CARE? $\text{cov}(X,Y) = E[XY] - E[X]E[Y]$

ARE X & Y INDEP IN THIS EXAMPLE?



$$f_{X,Y} = \begin{cases} 2 & x \,\&\, y \text{ in } \triangle \\ 0 & \text{o.w.} \end{cases}$$

NO! B/C $f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$

CONDITIONAL DENSITY

<u>Def</u> The conditional density function of X given $Y=y$ is

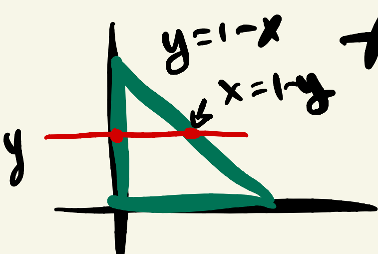
$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

← Assuming $f_Y(y) \neq 0$

<u>N.B.</u> If X & Y are independent then

$$f_{X|Y}(x|y) = \frac{f_X(x)\cancel{f_Y(y)}}{\cancel{f_Y(y)}} = f_X(x)$$

<u>STEP 1</u> Compute PDF for Y by marginalizing

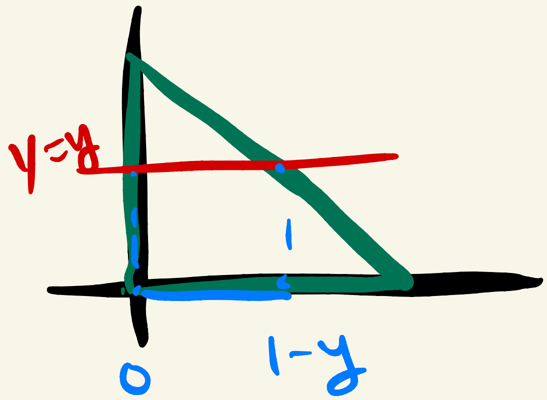


$$f_Y(y) = \int_{x=0}^{x=1-y} 2\,dx = 2x\Big|_{x=0}^{x=1-y} = 2(1-y) \quad 0 \leq y \leq 1$$

$$0 \quad \text{o.w.}$$

STEP 2: Plug in PDF for Y into the definition

of $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$



$$= \frac{2}{2(1-y)} = \frac{1}{1-y}$$

for $0 \le x \le 1-y$

STEP 3 Plug in a specific value if given one... $Y = 2/3$

$$f_{X|Y}(x|\,Y=2/3) = \begin{cases} \dfrac{1}{1-2/3} = 3 & 0 \le x \le 1-2/3 \\ 0 & \text{o.w.} \end{cases}$$

$\Rightarrow$ Uniform in $[0, 1/3]$

USUAL FORMULAS

$$P(a \le X \le b \,|\, Y=y) = \int_a^b f_{X|Y=y}(x|y)\,dx$$

$$E[g(X)\,|\,Y=y] = \int g(x) f_{X|Y=y}(x|y)\,dx$$

Ex From above

$$E[X|Y=y] = \int_0^{1-y} x f_{X|Y}(x|y)\,dx = \int_{x=0}^{x=1-y} \frac{x}{1-y}\,dx$$

$$= \frac{1}{1-y} \frac{x^2}{2}\Big|_{x=0}^{x=1-y} = \frac{1-y}{2} \checkmark$$

CONDITIONAL EXPECTATION IS A FUNCTION OF A R.V. → ITERATED EXPECTATION

Observe from last example

$E[X \mid Y=y]$ is a function of $y = Y$

⟹ Think of $E[X \mid Y]$ as a RV $g(Y)$

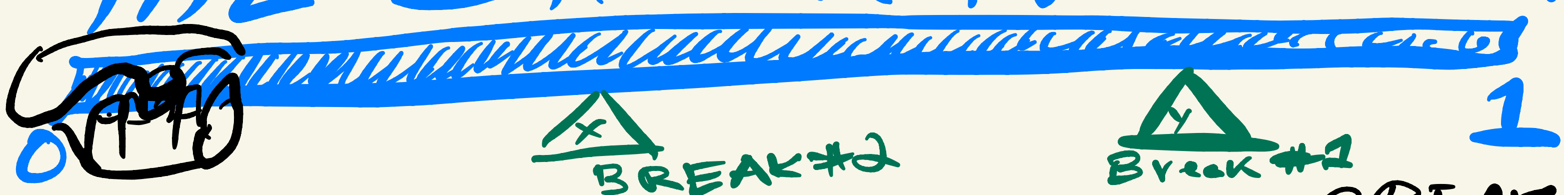
For Every RV can consider it's expectation

$$E\big[\underbrace{E[X \mid Y]}_{\text{A RV}}\big] = E[X]$$

¡AMAZING!

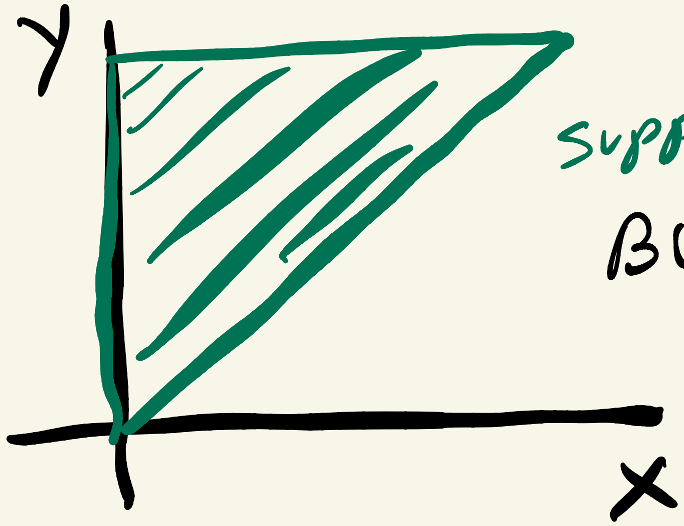
USEFUL when COMPUTING THE PDF OF X IS HARD

THE STICK PROBLEM



STEP 1 : PICK A LOCATION TO BREAK THE STICK $Y \sim \text{Unif}[0,1]$

STEP 2 : PICK ANOTHER LOCATION ON WHAT REMAINS TO BREAK AGAIN $X \mid Y=y \sim \text{Unif}[0,y]$



supp $= \{0 \le x \le y \le 1\}$

BUT $f_{X,Y}(x,y)$ is NOT UNIFORM

Q: How to determine $f_{X,Y}(x,y)$?

A: $f_{X,Y}(x,y) = f_{X|Y=y}(x|y) f_Y(y)$

Know $f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & 0 \le x \le y \\ 0 & \text{o.w.} \end{cases}$

$f_Y(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & \end{cases}$ i.e. $f_Y(y) = \mathbb{1}_{[0,1]}$ indicator function

Q: What is $E[X]$?

Apply the law of iterated expectations

$$E[E[X|Y]] = E[X]$$

$$E[X|Y=y] = \int x f_{X|Y}(x|y)\,dx$$

$$= \int_{x=0}^{x=y} x \frac{1}{y}\,dx = \frac{1}{y}\int_0^y x\,dx$$

$$\frac{1}{y}\left[\frac{x^2}{2}\right]_{x=0}^{x=y} = \boxed{\frac{y}{2}}$$

So...

$$E[\underbrace{E[X|Y]}] = E[Y/2] = \tfrac{1}{2}E[Y] = \tfrac{1}{4}$$

Longer way, apply definition $E[g(Y)] =$

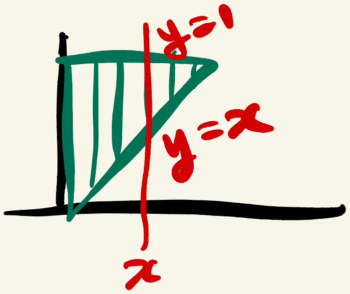
$$= \int_{-\infty}^{\infty} \frac{y}{2}\,\mathbb{1}_{[0,1]}\,dy = \int_0^1 \frac{y}{2}\,dy = \frac{y^2}{4}\Big|_0^1 = \frac{1}{4} \checkmark$$

THE REALLY LONG WAY...

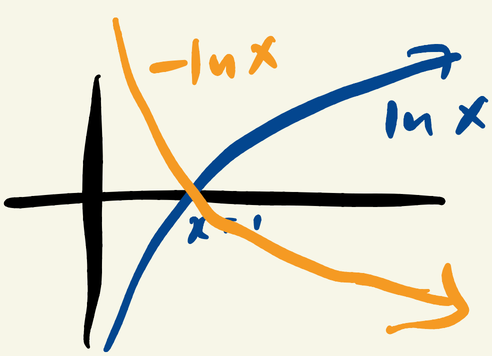
$$E[X] = \int_{-\infty}^{\infty} x\,\underbrace{f_X(x)}_{\text{DON'T KNOW!}}\,dx$$

Find $f_X(x)$ ...

$$f_X(x) = \int f_{X|Y}(x|y)\,f_Y(y)\,dy$$



$$= \int_{y=x}^{y=1} \frac{1}{y}\,\mathbb{1}_{[0,1]} = \int_{y=x}^{1} \frac{1}{y}\,dy$$



$$= \left[\ln y\right]_{y=x}^{y=1} = \boxed{-\ln x}$$

$0 < x \leq 1$

$$E[x] = \int_0^1 x(-\ln x)\,dx$$

$$= \ldots \text{Integrate by parts}$$

$$\int_0^1 u\,dv = \cancel{uv\Big|_0^1}^{\,0} - \int_0^1 v\,du$$

$u = \ln x$

$dv = -x\,dx$ ... FINISH FOR HW!

DICTIONARY for DISCRETE & CTS Conditional Probability

Conditioning Formulae: Discrete Case

Multiplication rule: The joint probability is the product of the marginal and the conditional

(JMF) Joint

$$P(X=x, Y=y) = P(X=x)P(Y=y \mid X=x)$$

Division rule: The conditional probability of $Y=y$ given $X=x$ is

$$P(Y=y \mid X=x) = \frac{P(X=x, Y=y)}{P(X=x)}$$

Bayes' rule:

$$\boxed{P(X=x \mid Y=y) = \frac{P(Y=y \mid X=x)P(X=x)}{P(Y=y)}}$$

BAYES

Conditional distribution of Y given $X=x$: Sum the conditional probabilities

$$P(Y \in B \mid X=x) = \sum_{y \in B} P(Y=y \mid X=x)$$

$\Sigma \longleftrightarrow \int$

Conditional expectation of $g(Y)$ given $X=x$: Sum g against the conditional probabilities

$$E(g(Y) \mid X=x) = \sum_{\text{all } y} g(y)P(Y=y \mid X=x)$$

Average conditional probability: LAW OF TOTAL PROBABILITY

$$P(B) = \sum_{\text{all } x} P(B \mid X=x)P(X=x)$$

$$P(Y=y) = \sum_{\text{all } x} P(Y=y \mid X=x)P(X=x)$$

Average conditional expectation: LAW OF ITERATED EXPECTATIONS

$$E(Y) = \sum_{\text{all } x} E(Y \mid X=x)P(X=x)$$

$$E[E[Y \mid X]] = E[Y]$$

Conditioning Formulae: Density Case

Multiplication rule: The joint density is the product of the marginal and the conditional

$$f(x,y) = f_X(x) f_Y(y \mid X=x)$$

Joint Density (JDF)

Division rule: The conditional density of Y at y given $X=x$ is

$$f_Y(y \mid X=x) = \frac{f(x,y)}{f_X(x)}$$

Bayes' rule:

$$\boxed{f_X(x \mid Y=y) = \frac{f_Y(y \mid X=x) f_X(x)}{f_Y(y)}}$$

Conditional distribution of Y given $X=x$: Integrate the conditional density

$$P(Y \in B \mid X=x) = \int_B f_Y(y \mid X=x)\,dy$$

Conditional expectation of $g(Y)$ given $X=x$: Integrate g against the conditional density:

$$E(g(Y) \mid X=x) = \int g(y) f_Y(y \mid X=x)\,dy$$

Average conditional probability:

$$P(B) = \int P(B \mid X=x) f_X(x)\,dx$$

$$f_Y(y) = \int f_Y(y \mid X=x) f_X(x)\,dx$$

Average conditional expectation:

$$E(Y) = \int E(Y \mid X=x) f_X(x)\,dx$$

WHAT DO YOU DO WHEN YOU DON'T KNOW IF A COIN IS FAIR?
(50-50)

Well... You flip it!

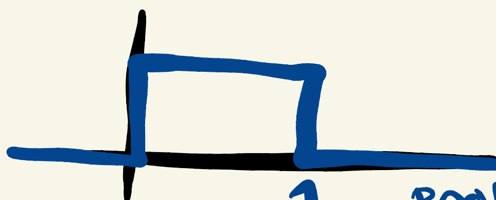
BUT what if this is an Alien coin & it behaves oddly ....?

Like $< 50\%$ time we get heads

If you truly don't know what the probability of a heads is, then a priori (PRIOR) the probability of a heads is some RV

$H \sim \text{Unif}[0,1]$

IS OUR PRIOR



So... Do an experiment...

$$P(S_n = k \mid H = p) = \binom{n}{k} p^k (1-p)^{n-k}$$

CONDITIONAL PMF = BINOMIAL RV

Suppose $n=20$ $k=7$ How do we update our beliefs about H?

BAYES RULE !!!

Instead of talking about $\mathbb{P}(H=p)$ (b/c $=0$) we need to talk about the conditional PDF for H, conditioned on $S_n=k$.

$$f_{H|S_n}(p \mid S_n=k) = \frac{f_{S_n|H}(S_n=k \mid H=p)\, f_H(p)}{f_{S_n}(S_n=k)}$$

BUT for Discrete RVs there's no difference b/w f & $\mathbb{P}$

$$= \frac{\overbrace{\boxed{P(S_n=k \mid H=p)}}^{\text{BINOMIAL}}\ \overbrace{\boxed{f_H(p)}}^{\text{PRIOR}}}{\boxed{P(S_n=k)}} \ ??$$

$$P(S_n=k) = \int_{-\infty}^{\infty} P(S_n=k \mid H=p)\, f_H(p)\, dp$$

$$= \int_{-\infty}^{\infty} \binom{n}{k} p^k (1-p)^{n-k}\, \mathbb{1}_{[0,1]}\, dp$$

$$= \int_0^1 \binom{n}{k} p^k (1-p)^{n-k}\, dp$$

$$= \binom{n}{k} \int_0^1 p^k (1-p)^{n-k}\, dp \leftarrow$$ How do we $\int dp$ that!??!

THE BETA INTEGRAL

$$B(r,s) = \int_0^1 u^{r-1}(1-u)^{s-1}\,du = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$$

For $r>0 \quad s>0$

Where for integers $r = n$

$$\boxed{\Gamma(n) = (n-1)! \text{ FACTORIAL}}$$

APPLIED TO COIN TOSSING

last ... page $= \binom{n}{k}\int_0^1 p^k(1-p)^{n-k}\,dp$

So... $k = r-1 \Rightarrow r = k+1$

AND... $n-k = s-1 \Longrightarrow s = n-k+1$

$$= \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} = \frac{\Gamma(k+1)\,\Gamma(n-k+1)}{\Gamma(\cancel{k}+1+n-\cancel{k}+1)}$$

$$= \binom{n}{k}\frac{k!\,(n-k)!}{(n+1)!} = \frac{1}{n+1}\binom{n}{k}\underbrace{\frac{k!\,(n-k)!}{n!}}_{\frac{1}{\binom{n}{k}}}$$

$$\frac{1}{n+1} \leftarrow = \frac{1}{n+1}\cancel{\binom{n}{k}}\frac{1}{\cancel{\binom{n}{k}}}$$

UPSHOT #1

$$f_{S_n}(k) = \int_0^1 f_{S_n|H}(k|p)\underbrace{f_H(p)}_{\text{Unif}[0,1]}\,dp$$

So $f_{S_n}(k) = \begin{cases} \frac{1}{n+1} & \text{for } 0 \le k \le n \text{ integers} \\ 0 & \text{o.w.} \end{cases}$

$\Rightarrow$ Uniform!

UPSHOT #2

BAYES RULE for Alien Coin

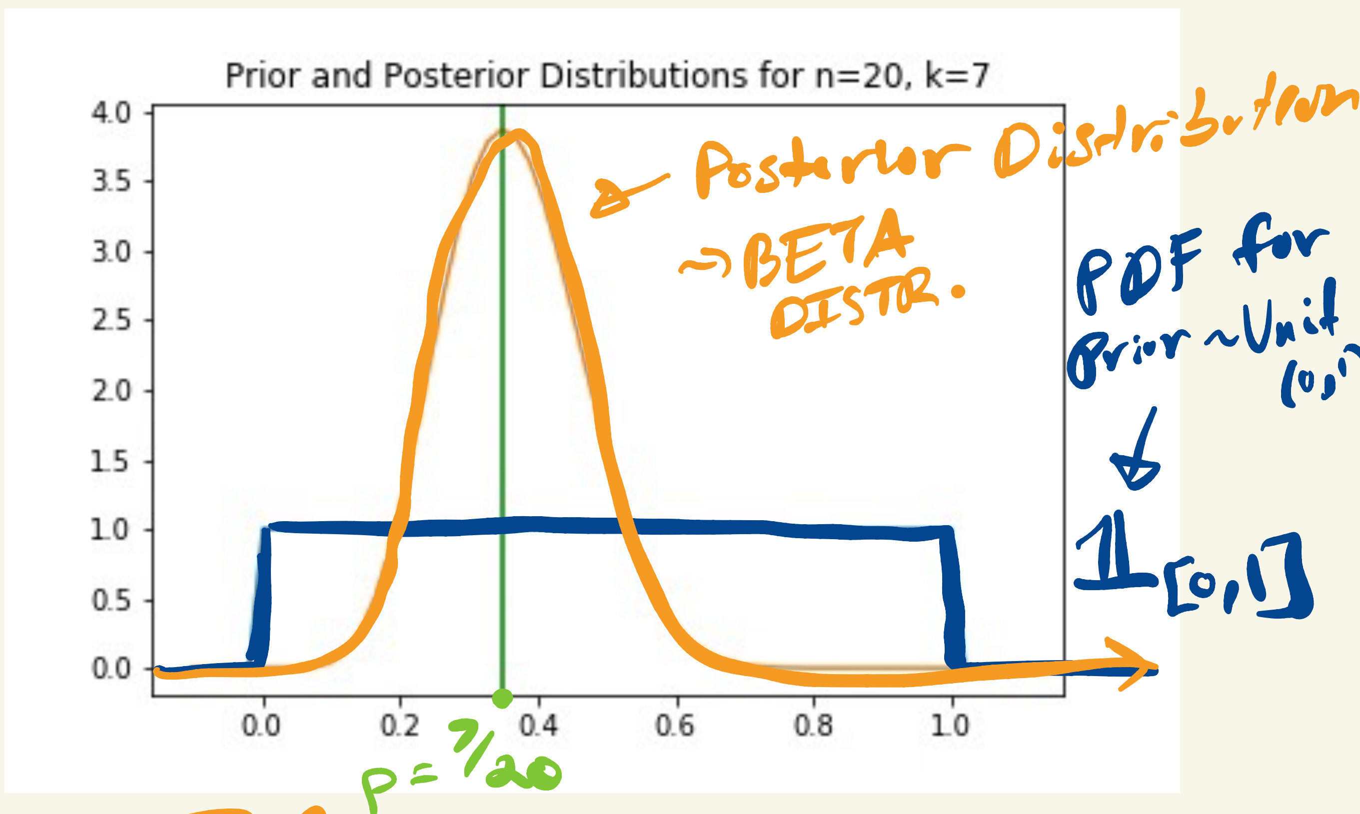
$$f_{H|S_n=k}(p|S_n=k) = \frac{\binom{n}{k} p^k (1-p)^{n-k} \cdot \mathbb{1}_{[0,1]}}{\frac{1}{n+1}}$$

CONDITIONAL PDF $= (n+1)\binom{n}{k} p^k (1-p)^{n-k} \quad 0 < p < 1$

FUNCTION OF P!

For $n=20$ flips & $k=7$ Alien Heads

Our beliefs about the probability of heads a posterior



BETA DISTRIBUTIONS

Beta (r,s) Distribution

For $r, s > 0$, the *beta* (r, s) distribution on $(0, 1)$ is defined by the density

$$\frac{1}{B(r,s)} x^{r-1}(1-x)^{s-1} \qquad (0 < x < 1)$$

where

$$B(r,s) = \int_0^1 x^{r-1}(1-x)^{s-1}dx$$

is the normalizing constant which makes the density integrate to 1. Viewed as a function of r and s, $B(r, s)$ is called the *beta function.*