Math 362 Midterm 2 Practice

April 7, 2021

Name: SOLUTIONS

- You may use a calculator and notes from our class as well as worksheet solutions on Blackboard, but nothing else. If you don't understand a word, you can use an online dictionary. In general, please show your work, simplify as completely as possible, and *box your final answer*.
- *Collaboration is strictly prohibited* on this exam. Use of Chegg or similar sites carries steep penalties that are described in the syllabus.

Problem	Points	Score
1	12	
2	10	
3	15	
4	10	
5	5	
6	8	
Total:	60	

1. (12 points) (*Expectation and Variance*)

Consider the following discrete random variable with the following PMF:

X	0	1	2	3
p_X	1/4	1/4	1/4	1/4

(a) (2 points) Compute $E(X)$.

$$E[X] = \sum x\, P(X=x) = \frac{1}{4}(0+1+2+3) = \frac{6}{4} = 1.5$$
$$= \frac{3}{2}$$

(b) (2 points) Compute $V(X)$. $= E(X^2) - E(X)^2$

$$E(X^2) = \frac{1}{4}(0^2 + 1^2 + 2^2 + 3^2) = \frac{14}{4}$$

$$V(X) = \frac{14}{4} - \frac{9}{4} = \frac{5}{4}$$

(c) (2 points) Consider a new random variable Y, which is a linear function of X, whose PMF is below

Y	2	9	16	23
p_Y	1/4	1/4	1/4	1/4

$Y = aX + b$

$b = 2$

$9 = a + 2 \quad a = 7$

$\Rightarrow Y = 7X + 2$

Now, compute $E(Y)$.

$$E(Y) = 7E(X) + 2$$
$$= \frac{21}{2} + 2 = \frac{25}{2}$$

(d) (2 points) Compute $V(Y)$.

$$V(Y) = V(7X+2) = V(7X)$$
$$= 49 \cdot V(X)$$
$$\boxed{= 49 \cdot \frac{25}{2}}$$

(e) (2 points) You can think of X as describing the roll of a four-sided die. Suppose you roll two four sided die and sum their values, so $S_2 = X_1 + X_2$ where X_1 and X_2 are IID with the same PMF as X.
Now, compute $E(S_2)$.

$$E(S_2) = E(X_1 + X_2)$$
$$= E(X_1) + E(X_2)$$
$$= 2E(X)$$
$$\boxed{= 3}$$

(f) (2 points) Compute $V(S_2)$.

$$V(X_1 + X_2) = V(X_1) + V(X_2)$$
$$= 2V(X)$$
$$= 5/2$$

2. (10 points) Suppose I interview 8 people and I ask for their height H and their weight W, measure in inches and pounds respectively. Their answers are as follows:

H	62	62	64	65	69	69	72	72
W	115	135	135	165	170	180	180	210

(a) (2 points) Fill out the following joint probability table, whose entries $p_{H,W}(i,j)$ equals the probability of someone having height i and weight j, based on the data above.

	62 in	64 in	65 in	69 in	72 in
115 lbs	1/8				
135 lbs	1/8	1/8			
165 lbs			1/8		
170 lbs				1/8	
180 lbs				1/8	1/8
210 lbs					1/8

(b) (2 points) Compute the PMFs for height $p_H(i)$ and $p_W(j)$.

Marginalize

P_W

115	135	165	170	180	210
1/8	2/8	1/8	1/8	2/8	1/8

P_H

62	64	65	69	72
2/8	1/8	1/8	2/8	2/8

(c) (2 points) Compute $\mu_H = E(H)$ and $\mu_W = E(W)$

$\mu_W = 161.25$ lbs

$\mu_H = 66.875$ inches

(d) (2 points) Compute $E[(H - \mu_H)(W - \mu_W)]$. This is called the *covariance* of H and W.

Could do directly, but... $E[H \cdot W - H \cdot \mu_W - W \cdot \mu_W + \mu_H \mu_W]$

$= E[HW] - \mu_H E[W] - \mu_W E[H] - \mu_H \mu_W$

$= E[HW] - \mu_H \mu_W$

$10{,}866.875 - 161.25 \times 66.875 = 103.2$

(e) (2 points) Are H and W independent? Explain how you know this.

2 ways of knowing this

1) $P_{H,W} \neq P_H \times P_W$ & 2) cov $\neq 0$

3. (15 points) (*Important Distributions and Related Attributes*)

Kyra goes to a conference, where there are 730 other people. Kyra discovers that 6 other people that have her same birthday.

NOT including Kyra!

(a) (2 points) What's the expected number of people that will share Kyra's birthday?

$$\frac{730}{365} = 2$$

(b) (3 points) What's the *exact* probability that 6 other people will have Kyra's birthday?

Prob of 6 successes out of 730 trials

$$\binom{730}{6}\left(\frac{1}{365}\right)^6\left(\frac{364}{365}\right)^{724}$$

(c) (3 points) Using the Poisson approximation, compute the probability that 6 other people have the same birthday as Kyra.

$$\lambda = \frac{1}{365} \cdot 730 = 2$$

$$P(X=k) \approx \frac{\lambda^k}{k!}e^{-\lambda} \quad \text{so} \quad \frac{2^6}{6!}e^{-2}$$

$$\approx 1.2\%$$

(d) (2 points) Use Markov's inequality to give an upper bound on the probability that Kyra will find 6 or more people in a randomly selected group of 730 that have her same birthday.

$$P(X \geq 6) \leq \frac{E(X)}{6} = \frac{2}{6} = 1/3 = 33.\bar{3}\%$$

(e) (3 points) The variance of a Poisson random variable is λ. Use Chebyshev's inequality applied to the Poisson approximation to give an upper bound on the probability that Kyra will find 6 or more people in a randomly selected group of 730 that have her same birthday.

$$P(|X-2| \geq 4) \leq \frac{\lambda}{16} = \frac{2}{16} = \frac{1}{8} \approx 12.5\%$$

(f) (2 points) Explain why this problem is different from the usual Birthday problem, which asks for the probability that no two people share the same birthday.

The difference is that here we're trying to not repeat a specific birthday, which has prob 1/365

In The Bday problem we're trying to NOT repeat a growing list of bdays which is proportional to the size of the group.

4. (10 points) (*Important Distributions and Related Attributes*)

A die is rolled until the first time a 6 comes up. Denote this random variable by T.

(a) (2 points) Write below a formula for the probability that the first six occurs on the kth roll.

$$P(T = k) = \left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right)$$

(b) (2 points) Compute the expectation of T.

$$E(T) = 1/\tfrac{1}{6} = 6$$

(c) (2 points) Compute the variance of T.

$$V(T) = q/p^2 = 5/6 \cdot 6^2 = 30$$

(d) (2 points) Compute the probability that T is between 5 and 7 rolls, inclusive.

$$P(5 \le T \le 7) = \left[\left(\frac{5}{6}\right)^4 + \left(\frac{5}{6}\right)^5 + \left(\frac{5}{6}\right)^6\right] \cdot \frac{1}{6}$$

(e) (2 points) Compute the probability $P(T \ge 9)$

$$P(T \ge 9) = P(\text{fail } \underset{\text{at least}}{8} \text{ times}) = \left(\frac{5}{6}\right)^8$$

5. (5 points) (*Law of Large Numbers*) The average face value of an 8-sided die is $\mu = 9/2 = 4.5$. The standard deviation is $\sigma = \sqrt{65/12} \approx 2.32$. 63?

Suppose I roll an 8-sided die n times and compute the average face value A_n. How many times do I need to roll an 8-sided die to be 75% sure that I am within .5 of the true mean?

$$P\left(|A_n - \mu| \geq \frac{1}{2}\right) \leq \frac{\sigma_n^2}{(1/2)^2}$$

$$\sigma_n^2 = Var(A_n) = Var\left(\frac{X_1 + \cdots + X_n}{n}\right)$$

$$= \frac{1}{n^2} n \cdot Var(X_i) = \frac{65}{12 \cdot n}$$

So $$\frac{\sigma_n^2}{(1/2)^2} \leq .25 = \frac{1}{4}$$

$$\frac{65}{12 \cdot n} \leq \frac{1}{4} \cdot \frac{1}{4} \Rightarrow \frac{65}{12} \cdot 16 \leq n$$

$$\boxed{85.\overline{3} \leq n}$$

What about CLT?

$$P(a < S_n < b) = P(\underbrace{n\mu - n\tfrac{1}{2}}_{a} < S_n < \underbrace{n\mu + \tfrac{n}{2}}_{b})$$

$$P\left(\mu - \frac{1}{2} < A_n < \mu + \frac{1}{2}\right)$$ divide by n

$$\Phi\left(\frac{b - n\mu}{\sigma\sqrt{n}}\right) - \Phi\left(\frac{a - n\mu}{\sigma\sqrt{n}}\right) > .75$$

$$2\Phi\left(\frac{n}{2\sigma\sqrt{n}}\right) - 1 > .75 \quad \rightsquigarrow \Phi\left(\frac{\sqrt{n}}{2\sigma}\right) > \frac{1.75}{2} = .875$$

Reverse z-table lookup

$$\frac{\sqrt{n}}{2\sigma} > 1.15 \Rightarrow n > 28.11$$

6. (8 points) The daily high temperature in Albany in May is normally distributed with mean 71 degrees (Fahrenheit) and standard deviation of 10 degrees.

   (a) (2 points) Using the conversion formula $C = \frac{5}{9}(F-32)$, determine the mean and standard deviation for daily high temperature measured in degrees Centigrade.

$$E(C) = E\left(\frac{5}{9}(F-32)\right) = \frac{5}{9}\left[E(F) - 32\right]$$

$$= 5/9 \cdot (71-32) = \frac{5 \cdot 39}{9}$$

$$\mu_C = 21.\overline{6}$$

$$SD(C) = \frac{5}{9} SD(F)$$

$$\sigma_C = \frac{50}{9} = 5.\overline{5}$$

   (b) (3 points) Using the attached Z-table, determine the probability that the high is below 51 degrees Fahrenheit.

$$P(F < 51) = \Phi\left(\frac{51-71}{10}\right) = \Phi(-2)$$

$$= 1 - \Phi(2) = 2.28\%$$

   (c) (3 points) Using the attached Z-table, determine the probability that the high temperature is between 46 and 86 degrees Fahrenheit.

$$P(46 < F < 86) \approx \Phi\left(\frac{86-71}{10}\right) - \Phi\left(\frac{46-71}{10}\right)$$

$$= \Phi(1.5) - \Phi(-2.5)$$

$$= .9332 - (1 - .9938)$$

$$= .9332 - (.0062)$$

$$= 92.7\%$$

Standard Normal Probabilities

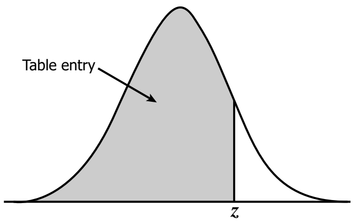

Table entry for z is the area under the standard normal curve to the left of z.

z	**.00**	**.01**	**.02**	**.03**	**.04**	**.05**	**.06**	**.07**	**.08**	**.09**
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998