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A blow-up criterion of strong solutions to the 2D compressible magnetohydrodynamic equations

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Abstract

This paper establishes a blow-up criterion of strong solutions to the two-dimensional compressible magnetohydrodynamic (MHD) flows. The criterion depends on the density, but is independent of the velocity and the magnetic field. More precisely, once the strong solutions blow up, the L^∞-norm for the density tends to infinity. In particular, the vacuum in the solutions is allowed.

Keywords: compressible magnetohydrodynamic equations; blow-up criterion; Cauchy problem; vacuum.

1 Introduction

We consider the system for the two-dimensional viscous, compressible magnetohydrodynamic (MHD) flows in the Eulerian coordinates as follows,

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \mu \Delta u + (\mu + \lambda)\nabla(\operatorname{div} u) + (\nabla \times H) \times H, \\ H_t - \nabla \times (u \times H) = \nu \Delta H, \quad \operatorname{div} H = 0, \end{cases} \tag{1.1}$$

where $\rho = \rho(x,t)$, $u = (u^1, u^2)(x,t)$, $H = (H^1, H^2)(x,t)$, and

$$P(\rho) = R\rho^\gamma \ (R > 0, \gamma > 1) \tag{1.2}$$

are the fluid density, velocity, magnetic field and pressure, respectively. R is a positive constant. Without loss of generality, we assume that $R = 1$. The constant viscosity coefficients μ and λ satisfy the physical restrictions:

$$\mu > 0, \quad \mu + \lambda \geq 0. \tag{1.3}$$

The constant $\nu > 0$ is the resistivity coefficient which is inversely proportional to the electrical conductivity constant and acts as the magnetic diffusivity of magnetic fields. We consider the Cauchy problem of (1.1) with the initial data

$$(\rho, u, H)|_{t=0} := (\rho_0, u_0, H_0), \quad x \in \mathbb{R}^2, \tag{1.4}$$

and the boundary condition at the far fields

$$(\rho, u, H) \to (0, 0, 0), \quad \text{as} \ \ |x| \to \infty. \tag{1.5}$$

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There have been large literature on the compressible MHD system (1.1) by many physicists and mathematicians due to its physical importance, complexity, rich phenomena and mathematical challenges, see [2–5, 8, 10, 11, 17, 18, 21, 24, 25, 29, 31, 32] and the references therein. When the initial density is uniformly positive, the local existence of strong solutions to the three-dimensional compressible MHD was proved by Vol'pert-Khudiaev [30]. Then, Kawashima [18] firstly obtained the global existence when the initial data are close to a non-vacuum equilibrium in H^3-norm. In particular, the theory requires that the solutions have small oscillations around a uniform non-vacuum state so that the density is strictly away from the vacuum.

For general large initial data, the global well-posedness of classical solutions to the compressible MHD system remains completely open. One of the main difficulties is that even the initial density is absence of vacuum, one could not know whether the vacuum states may occur or not within finite time. The system (1.1) may degenerate in the presence of vacuum, which produces new difficulty in mathematical analysis. Therefore, it is interesting to study the solutions with vacuum for the compressible MHD system. In fact, the local well-posedness of strong solutions of the system (1.1) in three dimensions was established by Fan-Yu [5] when the initial density may contain vacuum. Recently, Li-Xu-Zhang [21] proved the global existence of classical solutions with vacuum as far field condition to (1.1)-(1.5) in $\mathbb{R}^3$ for regular initial data with small energy but possibly large oscillations. However, the two dimensional case is quiet different from the three dimensional case when the far field condition is vacuum. Precisely speaking, the difference between 2D and 3D is that, if a function u satisfies $\nabla u \in L^2(\mathbb{R}^2)$, it is impossible to imply that $u \in L^p(\mathbb{R}^2)$, for any $p > 1$, while if $\nabla u \in L^2(\mathbb{R}^3)$, then $u \in L^6(\mathbb{R}^3)$. More recently, Lv-Huang [24] proved the local existence of strong solutions for the Cauchy problem of the two-dimensional compressible MHD equations with vacuum as far field density by weighted energy estimate and Lv et al. [25] generalized the previous work of Li-Xu-Zhang [21] on two-dimensional case.

Until now, all the global existence of solutions for the compressible MHD equations were obtained with some "smallness" assumptions on the initial data. However, there still remains a longstanding open problem: whether the strong (or classical) solutions to the compressible MHD system (1.1) can exist globally or not? Thus, it is important to study the mechanism of blow up and structure of possible singularities of strong solutions to the compressible MHD system (1.1). Along this direction, Xu-Zhang in [32] established the following Serrin's type criterion to the three-dimensional isentropic compressible MHD equations:

$$\lim_{T \to T^*} \left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^s(0,T;L^r_w)} \right) = \infty, \tag{1.6}$$

where $T^* \in (0, \infty)$ is the maximal time of existence for strong (or classical) solutions (ρ, u), L^r_w denotes the weak L^r-space and r, s satisfies

$$\frac{2}{s} + \frac{3}{r} \le 1, \quad 3 < r \le \infty. \tag{1.7}$$

Then Huang-Li in [16] extended the result of [32] to the non-isentropic case.

Although the Serrin type criterion for the Cauchy problem of three-dimensional MHD flows has been well established by Xu-Zhang in [32]. However, the two-dimensional case with the initial density containing vacuum becomes more difficult since the analysis of [32] for 3D case depends crucially on the L^6-bound on the velocity, while in 2D case, the velocity may not belong to $L^p(\mathbb{R}^2)$ for any $p > 1$. The main aim of this paper is to establish a blow-up criterion for the two-dimensional compressible MHD system.

Before stating the main results, we first explain the notations and conventions used throughout this paper. For $R > 0$, set

$$B_R \triangleq \left\{x \in \mathbb{R}^2 \big| \, |x| < R\right\}, \quad \int f dx \triangleq \int_{\mathbb{R}^2} f dx.$$

Moreover, for $1 \le r \le \infty$ and $k \ge 1$, the standard homogeneous and inhomogeneous Sobolev spaces are defined as follows:

$$\begin{cases} L^r = L^r(\mathbb{R}^2), \quad D^{k,r} = D^{k,r}(\mathbb{R}^2) = \{v \in L^1_{\text{loc}}(\mathbb{R}^2) | \nabla^k v \in L^r(\mathbb{R}^2)\}, \\ D^1 = D^{1,2}, \quad W^{k,r} = W^{k,r}(\mathbb{R}^2), \quad H^k = W^{k,2}. \end{cases}$$

Next, we give the definition of strong solution to (1.1) as follows:

Definition 1.1 (strong solution) *(ρ, u, H) is called a strong solution to* (1.1) *in* $\mathbb{R}^2 \times (0, T)$, *if for some $q_0 > 2$ and $a > 1$,*

$$\begin{cases} \rho \ge 0, \quad \rho \bar{x}^a \in C([0,T]; L^1 \cap H^1 \cap W^{1,q_0}), \quad \rho_t \in C([0,T]; L^2 \cap L^{q_0}), \\ (u, H) \in C([0,T]; D^1 \cap D^2) \cap L^2(0,T; D^{2,q_0}), \quad H \in C([0,T]; H^2), \\ (\rho^{1/2} u_t, H_t) \in L^\infty(0,T; L^2), \quad (u_t, H_t) \in L^2(0,T; D^1), \end{cases} \tag{1.8}$$

and (ρ, u, H) satisfies (1.1) *a.e. in $\mathbb{R}^2 \times (0,T)$, where*

$$\bar{x} \triangleq (e + |x|^2)^{1/2} \log^2(e + |x|^2). \tag{1.9}$$

Without loss of generality, assume that the initial density ρ_0 satisfies

$$\int_{\mathbb{R}^2} \rho_0 dx = 1, \tag{1.10}$$

which implies that there exists a positive constant N_0 such that

$$\int_{B_{N_0}} \rho_0 dx \ge \frac{1}{2} \int \rho_0 dx = \frac{1}{2}. \tag{1.11}$$

Our main result can be stated as follows:

Theorem 1.1 *Let $\Omega = \mathbb{R}^2$. In addition to* (1.10) *and* (1.11), *suppose that the initial data (ρ_0, u_0, H_0) satisfy, for any given numbers $a > 1$, $q > 2$,*

$$\begin{cases} \rho_0 \ge 0, \quad \bar{x}^a \rho_0 \in L^1 \cap H^1 \cap W^{1,q}, \quad \rho_0 u_0^2 + \rho_0^\gamma \in L^1, \\ (u_0, H_0) \in D^1 \cap D^2, \quad \bar{x}^{a/2} H_0 \in L^2, \quad \bar{x}^{a/2} \nabla H_0 \in L^2, \end{cases} \tag{1.12}$$

and the compatibility condition

$$-\mu \Delta u_0 - (\mu + \lambda) \nabla \text{div} u_0 + \nabla P(\rho_0) - (\nabla \times H_0) \times H_0 = \rho_0^{1/2} g, \tag{1.13}$$

holds for some $g \in L^2$.

Let (ρ, u, H) be a strong solution to the Cauchy problem (1.1), (1.5) *and* (1.4), *satisfying* (1.8). *If $T^* < \infty$ is the maximal time of existence, then*

$$\lim_{T \to T^*} \|\rho\|_{L^\infty(0,T; L^\infty)} = \infty. \tag{1.14}$$

Remark 1.1 *Theorem 1.1 means the blow-up criterion* (1.14) *is independent of the velocity u and the magnetic field H.*

We now make some comments on the analysis of this paper. As previously mentioned, the L^6-norm of velocity u plays crucial role in 3D MHD system, while for 2D, u may not belong to $L^p(\mathbb{R}^2)$, for any $p > 1$. The key observation of this paper is that, if we restriction the initial data in a smaller space, i.e. $\bar{x}^a\rho_0 \in L^1(\mathbb{R}^2)$ for some positive constant $a > 1$ (see (1.12)), we can show that $u\bar{x}^{-\eta}$ belongs to $L^{p_0}(\mathbb{R}^2)$, for some positive constant $p_0 > 1$, and $\eta \in (0, 1]$ (see (3.52)), here $\bar{x} = (e+|x|^2)^{1/2}\log^2(e+|x|^2)$. For this , we need to manipulate the weighted energy estimates throughout the Section 3 below.

Before finishing the introduction, we recall some related works. The global existence of weak solutions to the system (1.1) was established by Hu-Wang [10,11]. If there is no electromagnetic field effect, (1.1) turns to be the compressible Navier-Stokes equations. For the blow-up criterion of the compressible Navier-Stokes equations, we refer to [12, 13, 15, 27, 28] and the references therein.

The rest of the paper is organized as follows: In the next section, we collect some elementary facts and inequalities for the blow-up analysis. The main result, Theorem 1.1, is proved in Section 3.

2 Preliminaries

In this section, we will recall some known facts and elementary inequalities which will be used frequently later.

The local existence of strong solutions when the initial density may not be positive can be proved in a similar way as in [19] (cf. [24]).

Lemma 2.1 *Assume that the initial data (ρ_0, u_0, H_0) satisfy* (1.12). *Then there exist a small time $T_1 > 0$ and a unique strong solution (ρ, u, H) in the sense of Definition 1.1 to the Cauchy problem* (1.1)-(1.4) *in $\mathbb{R}^2 \times (0, T)$.*

Next, the following well-known Gagliardo-Nirenberg inequality (see [26]) will be used later.

Lemma 2.2 (Gagliardo-Nirenberg) *For $p \in [2, \infty), q \in (1, \infty)$, and $r \in (2, \infty)$, there exists some generic constant $C > 0$ which may depend on p, q, and r such that for $f \in H^1(\mathbb{R}^2)$ and $g \in L^q(\mathbb{R}^2) \cap D^{1,r}(\mathbb{R}^2)$, we have*

$$\|f\|^p_{L^p(\mathbb{R}^2)} \le C\|f\|^2_{L^2(\mathbb{R}^2)}\|\nabla f\|^{p-2}_{L^2(\mathbb{R}^2)}, \tag{2.1}$$

$$\|g\|_{C\left(\overline{\mathbb{R}^2}\right)} \le C\|g\|^{q(r-2)/(2r+q(r-2))}_{L^q(\mathbb{R}^2)}\|\nabla g\|^{2r/(2r+q(r-2))}_{L^r(\mathbb{R}^2)}. \tag{2.2}$$

The following weighted L^p bounds for elements of the Hilbert space $D^1(\mathbb{R}^2)$ can be found in [22, Theorem B.1].

Lemma 2.3 *For $m \in [2, \infty)$ and $\theta \in (1 + m/2, \infty)$, there exists a positive constant C such that we have for all $v \in D^{1,2}(\mathbb{R}^2)$,*

$$\left(\int_{\mathbb{R}^2} \frac{|v|^m}{e+|x|^2}(\log(e+|x|^2))^{-\theta}dx\right)^{1/m} \le C\|v\|_{L^2(B_1)} + C\|\nabla v\|_{L^2(\mathbb{R}^2)}. \tag{2.3}$$

The following lemma was deduced in [20], we only state it here without proof.

Lemma 2.4 *For $\bar{x}$ as in* (1.9), *suppose that $\rho \in L^\infty(\mathbb{R}^2)$ is a function such that*

$$0 \le \rho \le M_1, \quad M_2 \le \int_{B_{N_*}} \rho dx, \quad \rho \bar{x}^\alpha \in L^1(\mathbb{R}^2), \tag{2.4}$$

for $N_* \ge 1$ and positive constants M_1, M_2, and α. Then, for $r \in [2,\infty)$, there exists a positive constant C depending only on M_1, M_2, α, and r such that

$$\left(\int_{\mathbb{R}^2} \rho |v|^r dx\right)^{1/r} \le C N_*^3 (1 + \|\rho \bar{x}^\alpha\|_{L^1(\mathbb{R}^2)}) \left(\|\rho^{1/2} v\|_{L^2(\mathbb{R}^2)} + \|\nabla v\|_{L^2(\mathbb{R}^2)}\right), \tag{2.5}$$

for each $v \in \left\{ v \in D^1(\mathbb{R}^2) \middle| \, \rho^{1/2} v \in L^2(\mathbb{R}^2) \right\}$.

Next, for $\nabla^\perp \triangleq (-\partial_2, \partial_1)$, denoting the material derivative of f by $\dot{f} \triangleq f_t + u \cdot \nabla f$. We state some elementary estimates which follow from (2.1) and the standard L^p-estimate for the following elliptic system derived from the momentum equations in $(1.1)_2$:

$$\triangle F = \operatorname{div}\left(\rho \dot{u} - \operatorname{div}(H \otimes H - \frac{1}{2}|H|^2 \mathbb{I}_2)\right), \tag{2.6}$$

and

$$\mu \triangle \omega = \nabla^\perp \cdot \left(\rho \dot{u} - \operatorname{div}(H \otimes H - \frac{1}{2}|H|^2 \mathbb{I}_2)\right), \tag{2.7}$$

where

$$F \triangleq (2\mu + \lambda)\operatorname{div} u - P(\rho), \quad \omega = \partial_1 u^2 - \partial_2 u^1. \tag{2.8}$$

The symbol $\otimes$ denotes the Kronecker tensor product, i.e. $H \otimes H = (H^i H^j)_{2\times 2}$ and "$\mathbb{I}_2$" denotes the 2×2 unit matrix.

Lemma 2.5 *Let (ρ, u, H) be a smooth solution of (1.1). Then for $p \ge 2$ there exists a positive constant C depending only on p, μ, and λ such that*

$$\|\nabla F\|_{L^p(\mathbb{R}^2)} + \|\nabla \omega\|_{L^p(\mathbb{R}^2)} \le C(\|\rho \dot{u}\|_{L^p(\mathbb{R}^2)} + \||H||\nabla H|\|_{L^p(\mathbb{R}^2)}), \tag{2.9}$$

$$\begin{aligned} \|F\|_{L^p(\mathbb{R}^2)} + \|\omega\|_{L^p(\mathbb{R}^2)} &\le C \left(\|\rho \dot{u}\|_{L^2(\mathbb{R}^2)} + \||H||\nabla H|\|_{L^2(\mathbb{R}^2)}\right)^{1-2/p} \\ &\quad \cdot \left(\|\nabla u\|_{L^2(\mathbb{R}^2)} + \|P\|_{L^2(\mathbb{R}^2)}\right)^{2/p}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} \|\nabla u\|_{L^p(\mathbb{R}^2)} &\le C \left(\|\rho \dot{u}\|_{L^2(\mathbb{R}^2)} + \||H||\nabla H|\|_{L^2(\mathbb{R}^2)}\right)^{1-2/p} \\ &\quad \cdot \left(\|\nabla u\|_{L^2(\mathbb{R}^2)} + \|P\|_{L^2(\mathbb{R}^2)}\right)^{2/p} + C\|P\|_{L^p(\mathbb{R}^2)}. \end{aligned} \tag{2.11}$$

Proof: On the one hand, by the standard L^p-estimate for the elliptic systems (see [26]), (2.6) (2.7) yield (2.9) directly, which, together with (2.1), (2.6) and (2.7), gives (2.10). On the other hand, since $-\Delta u = -\nabla \operatorname{div} u - \nabla^\perp \omega$, we have

$$\nabla u = -\nabla(-\Delta)^{-1} \nabla \operatorname{div} u - \nabla(-\Delta)^{-1} \nabla^\perp \omega. \tag{2.12}$$

Thus applying the standard L^p-estimate to (2.12) shows

$$\begin{aligned} \|\nabla u\|_{L^p(\mathbb{R}^2)} &\le C(p)(\|\operatorname{div} u\|_{L^p(\mathbb{R}^2)} + \|\omega\|_{L^p(\mathbb{R}^2)}) \\ &\le C(p)\|F\|_{L^p(\mathbb{R}^2)} + C(p)\|\omega\|_{L^p(\mathbb{R}^2)} + C(p)\|P\|_{L^p(\mathbb{R}^2)}, \end{aligned}$$

which, along with (2.10), gives (2.11). Then, the proof of Lemma 2.5 is completed.

Finally, the following Beale-Kato-Majda type inequality, which was proved in [13], will be used later to estimate $\|\nabla u\|_{L^\infty}$ and $\|\nabla \rho\|_{L^2 \cap L^q} (q > 2)$.

Lemma 2.6 *For $2 < q < \infty$, there is a constant $C(q)$ such that the following estimate holds for all $\nabla u \in L^2(\mathbb{R}^2) \cap D^{1,q}(\mathbb{R}^2)$,*

$$\begin{aligned}\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \leq& C\left(\|\mathrm{div} u\|_{L^\infty(\mathbb{R}^2)} + \|\omega\|_{L^\infty(\mathbb{R}^2)}\right) \log(e + \|\nabla^2 u\|_{L^q(\mathbb{R}^2)}) \\ &+ C\|\nabla u\|_{L^2(\mathbb{R}^2)} + C.\end{aligned}$$

3 Proof of Theorem 1.1

Let (ρ, u, H) be a strong solution to the problem (1.1)-(1.4) as describe in Theorem 1.1. Suppose that (1.14) were false, that is,

$$\lim_{T\to T^*} \|\rho\|_{L^\infty(0,T;L^\infty)} = M_0 < \infty. \tag{3.1}$$

First, the standard energy estimate yields

$$\sup_{0\leq t\leq T} \int \left(\frac{1}{2}\rho|u|^2 + \frac{P}{\gamma-1} + \frac{|H|^2}{2}\right) dx + \int_0^T \int \left(\mu|\nabla u|^2 + \nu|\nabla H|^2\right) dxdt \leq C, \tag{3.2}$$

for $0 \leq T \leq T^*$. Throughout this paper, several positive generic constants are denoted by C and $C_i (i = 1, 2)$ depending only on $M_0,\ \mu,\ \lambda,\ \nu,\ T^*,\ q,\ a$ and the initial data.

The following lemma is based on (3.2).

Lemma 3.1 *It holds that for $q \in [2, \infty)$ and $0 \leq T \leq T^*$,*

$$\|H\|_{L^\infty(0,T;L^q)} + \int_0^T \int |H|^{q-2}|\nabla H|^2 dxdt \leq C. \tag{3.3}$$

Proof: We prove (3.3) as in He-Xin [8]. Multiplying $(1.1)_3$ by $q|H|^{q-2}H$ and integrating the resulting equation over $\mathbb{R}^2$ yield that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}} \int |H|^q dx + \nu \int \left(q|H|^{q-2}|\nabla H|^2 + q(q-2)|H|^{q-2}|\nabla|H||^2\right) dx \\ &= q\int H \cdot \nabla u \cdot H|H|^{q-2}dx - (q-1)\int \mathrm{div} u|H|^q dx \\ &\leq C\|\nabla u\|_{L^2}\||H|^{q/2}\|_{L^4}^2 \leq C\|\nabla u\|_{L^2}\||H|^{q/2}\|_{L^2}\|\nabla|H|^{q/2}\|_{L^2} \\ &\leq \delta\|\nabla|H|^{q/2}\|_{L^2}^2 + C(\delta)\|\nabla u\|_{L^2}^2\||H\|_{L^q}^q.\end{aligned} \tag{3.4}$$

Choosing δ suitable small in (3.4), we obtain (3.3) directly after using Gronwall's inequality and (3.2). Thus the proof of Lemma 3.1 is completed. □

Next, we give the key estimate on ∇u and ∇H in the following lemma.

Lemma 3.2 *Under the condition* (3.1)*, it holds that for $0 \leq T \leq T^*$,*

$$\sup_{0\leq t\leq T} \left(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\right) + \int_0^T \left(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2\right) dt \leq C. \tag{3.5}$$

$$\int_0^T \left(\|\nabla u\|_{L^4}^4 + \|\nabla H\|_{L^4}^4\right) dt \leq C. \tag{3.6}$$

Proof: First, multiplying the momentum equation $(1.1)_2$ by $\dot{u}$ and integrating the resulting equation over $\mathbb{R}^2$ gives

$$\begin{aligned}\int \rho|\dot{u}|^2 dx = & -\int \dot{u}\cdot\nabla P dx + \mu\int \dot{u}\cdot\triangle u dx + (\mu+\lambda)\int \dot{u}\cdot\nabla \mathrm{div} u\, dx \\ & -\frac{1}{2}\int \dot{u}\cdot\nabla|H|^2 dx + \int \dot{u}\cdot H\cdot\nabla H dx \triangleq \sum_{i=1}^{5} I_i,\end{aligned} \tag{3.7}$$

where we have used the fact that

$$(\nabla\times H)\times H = \mathrm{div}(H\otimes H) - \frac{1}{2}\nabla|H|^2 = H\cdot\nabla H - \frac{1}{2}\nabla|H|^2.$$

Since P satisfies

$$P_t + \mathrm{div}(uP) + (\gamma-1)P\mathrm{div}u = 0, \tag{3.8}$$

integrating by parts yields that

$$\begin{aligned}-\int \dot{u}\cdot\nabla P dx &= \int \Big((\mathrm{div}u)_t P - (u\cdot\nabla u)\cdot\nabla P\Big)dx \\ &= \left(\int \mathrm{div}u P dx\right)_t + \int \Big((\gamma-1)P(\mathrm{div}u)^2 + P\partial_i u\cdot\nabla u_i\Big)dx \\ &\le \left(\int \mathrm{div}u P dx\right)_t + C\|\nabla u\|_{L^2}^2.\end{aligned} \tag{3.9}$$

Integrating by parts also leads to

$$\begin{aligned}\mu\int \dot{u}\cdot\Delta u dx &= -\frac{\mu}{2}\left(\|\nabla u\|_{L^2}^2\right)_t - \mu\int \partial_i u^j\partial_i(u\cdot\nabla u^j)dx \\ &\le -\frac{\mu}{2}\left(\|\nabla u\|_{L^2}^2\right)_t + C\|\nabla u\|_{L^3}^3\end{aligned} \tag{3.10}$$

and that

$$\begin{aligned}(\mu+\lambda)\int \dot{u}\cdot\nabla\mathrm{div}u\, dx &= -\frac{\mu+\lambda}{2}\left(\|\mathrm{div}u\|_{L^2}^2\right)_t - (\mu+\lambda)\int \mathrm{div}u\ \mathrm{div}(u\cdot\nabla u)dx \\ &\le -\frac{\mu+\lambda}{2}\left(\|\mathrm{div}u\|_{L^2}^2\right)_t + C\|\nabla u\|_{L^3}^3.\end{aligned} \tag{3.11}$$

Using $(1.1)_3$ and (3.2), we get

$$
\begin{aligned}
I_4 =& \frac{1}{2}\int |H|^2 \mathrm{div} u_t dx + \frac{1}{2}\int |H|^2 \mathrm{div}(u\cdot\nabla u) dx \\
=& \left(\int \frac{|H|^2}{2}\mathrm{div} u dx\right)_t + \frac{1}{2}\int u\cdot\nabla |H|^2 \mathrm{div} u dx + \frac{1}{2}\int |H|^2\mathrm{div}(u\cdot\nabla u)dx \\
& - \int (H\cdot\nabla u + \nu\Delta H - H\mathrm{div} u)\cdot H \mathrm{div} u dx \\
=& \left(\int \frac{|H|^2}{2}\mathrm{div} u dx\right)_t - \frac{1}{2}\int |H|^2(\mathrm{div} u)^2 dx + \frac{1}{2}\int |H|^2 \partial_i u\cdot\nabla u^i dx \\
& - \int (H\cdot\nabla u + \nu\Delta H - H\mathrm{div} u)\cdot H \mathrm{div} u dx \\
\le& \left(\int \frac{|H|^2}{2}\mathrm{div} u dx\right)_t + C(\varepsilon)\int |H|^2|\nabla u|^2 dx + \varepsilon\|\nabla^2 H\|_{L^2}^2 \\
\le& \left(\int \frac{|H|^2}{2}\mathrm{div} u dx\right)_t + C(\varepsilon)\|\nabla u\|_{L^3}^2\|H\|_{L^2}^{4/3}\|\nabla^2 H\|_{L^2}^{2/3} + \varepsilon\|\nabla^2 H\|_{L^2}^2 \\
\le& \left(\int \frac{|H|^2}{2}\mathrm{div} u dx\right)_t + C(\varepsilon)\|\nabla u\|_{L^3}^3 + 2\varepsilon\|\nabla^2 H\|_{L^2}^2.
\end{aligned}
\tag{3.12}
$$

Similarly, we have

$$
I_5 \le -\left(\int H\cdot\nabla u\cdot H dx\right)_t + C(\varepsilon)\|\nabla u\|_{L^3}^3 + 2\varepsilon\|\nabla^2 H\|_{L^2}^2. \tag{3.13}
$$

Putting (3.9)-(3.13) into (3.7), and recalling (3.1), (2.11) yields

$$
\begin{aligned}
& B'(t) + \int \rho|\dot{u}|^2 dx \\
& \le C\|\nabla u\|_{L^2}^2 + C(\varepsilon)\|\nabla u\|_{L^3}^3 + 4\varepsilon\|\nabla^2 H\|_{L^2}^2 \\
& \le C(\varepsilon)\left(\|\nabla u\|_{L^2}^4 + 1\right) + C\||H||\nabla H|\|_{L^2}^2 + \varepsilon\|\rho^{1/2}\dot{u}\|_{L^2}^2 + 4\varepsilon\|\nabla^2 H\|_{L^2}^2.
\end{aligned}
\tag{3.14}
$$

where

$$
\begin{aligned}
B(t) \triangleq & \frac{\mu}{2}\|\nabla u\|_{L^2}^2 + \frac{\lambda+\mu}{2}\|\mathrm{div} u\|_{L^2}^2 - \int \mathrm{div} u P dx \\
& - \frac{1}{2}\int \mathrm{div} u |H|^2 dx + \int H\cdot\nabla u\cdot H dx.
\end{aligned}
\tag{3.15}
$$

Next, multiplying $(1.1)_3$ by $\triangle H$, and integrating by parts over $\mathbb{R}^2$, we have

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla H|^2 dx + 2\nu\int |\nabla^2 H|^2 dx \\
& \le C\int |\nabla u||\nabla H|^2 dx + C\int |\nabla u||H||\Delta H| dx \\
& \le C\|\nabla u\|_{L^2}\|\nabla H\|_{L^4}^2 + C\|\nabla u\|_{L^2}\|H\|_{L^\infty}\|\nabla^2 H\|_{L^2} \\
& \le C\|\nabla u\|_{L^2}\|\nabla H\|_{L^2}\|\nabla^2 H\|_{L^2} + C\|\nabla u\|_{L^2}\|\nabla^2 H\|_{L^2}^{3/2} \\
& \le \frac{\nu}{2}\|\nabla^2 H\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 + C\|\nabla u\|_{L^2}^2\|\nabla H\|_{L^2}^2.
\end{aligned}
\tag{3.16}
$$

Choosing C_1 suitably large such that

$$
\begin{aligned}
& \frac{\mu}{4}\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 - C\|P\|_{L^2}^2 \\
& \le B(t) + C_1\|\nabla H\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^2 + C\|\nabla H\|_{L^2}^2 + C\|P\|_{L^2}^2,
\end{aligned}
\tag{3.17}
$$

adding (3.16) multiplied by C_1 to (3.14), and choosing ε suitably small lead to

$$\begin{aligned}&(B(t)+C_1\|\nabla H\|_{L^2}^2)'+\frac{1}{2}\int\left(\rho|\dot{u}|^2+\nu C_1|\nabla^2 H|^2\right)dx\\&\le C+C\||H||\nabla H|\|_{L^2}^2+C(\|\nabla u\|_{L^2}^4+\|\nabla H\|_{L^2}^4).\end{aligned}\tag{3.18}$$

Integrating (3.18) over $(0,T)$, choosing $q=4$ in (3.3), and using (3.2) and Gronwall's inequality, we obtain (3.5). We can get (3.6) immediately by (2.11) and (3.5).

□

Next, we get some basic energy estimates on the magnetic field H.

Lemma 3.3 *Under the condition* (3.1)*, it holds that for* $0\le T\le T^*$,

$$\sup_{0\le t\le T}\||H||\nabla H|\|_{L^2}^2+\int_0^T\left(\|\Delta|H|^2\|_{L^2}^2+\||\Delta H||H|\|_{L^2}^2\right)dt\le C.\tag{3.19}$$

Proof: We will follow an idea in [25]. For $a_1,a_2\in\{-1,0,1\}$, denote

$$\tilde H(a_1,a_2)=a_1H^1+a_2H^2,\quad \tilde u(a_1,a_2)=a_1u^1+a_2u^2.\tag{3.20}$$

It thus follows from $(1.1)_3$ that

$$\tilde H_t-\nu\Delta\tilde H=H\cdot\nabla\tilde u-u\cdot\nabla\tilde H+\tilde H\mathrm{div}u.\tag{3.21}$$

Integrating (3.21) multiplied by $4\nu^{-1}\tilde H\triangle|\tilde H|^2$ over $\mathbb{R}^2$ leads to

$$\begin{aligned}&\nu^{-1}\left(\|\nabla|\tilde H|^2\|_{L^2}^2\right)_t+2\|\Delta|\tilde H|^2\|_{L^2}^2\\&=4\int|\nabla\tilde H|^2\Delta|\tilde H|^2dx-4\nu^{-1}\int H\cdot\nabla\tilde u\cdot\tilde H\Delta|\tilde H|^2dx\\&\quad+4\nu^{-1}\int\mathrm{div}u|\tilde H|^2\Delta|\tilde H|^2dx+2\nu^{-1}\int u\cdot\nabla|\tilde H|^2\Delta|\tilde H|^2dx\\&\le C\|\nabla u\|_{L^4}^4+C\|\nabla H\|_{L^4}^4+C\||H|^2\|_{L^4}^4+\|\Delta|\tilde H|^2\|_{L^2}^2,\end{aligned}\tag{3.22}$$

where we have used the following simple fact that

$$\begin{aligned}2\int u\cdot\nabla|\tilde H|^2\Delta|\tilde H|^2dx&=-2\int\partial_iu\cdot\nabla|\tilde H|^2\partial_i|\tilde H|^2dx+\int\mathrm{div}u|\nabla|\tilde H|^2|^2dx\\&\le C\|\nabla u\|_{L^4}^4+C\|\nabla H\|_{L^4}^4+C\||H|^2\|_{L^4}^4.\end{aligned}$$

Integrating (3.22) over $(0,T)$, and using (3.3) and (3.6), we obtain

$$\sup_{0\le t\le T}\|\nabla|\tilde H|^2\|_{L^2}^2+\int_0^T\|\Delta|\tilde H|^2\|_{L^2}^2dt\le C.\tag{3.23}$$

Noticing that

$$\begin{aligned}\||\nabla H||H|\|_{L^2}^2\le&\|\nabla|\tilde H(1,0)|^2\|_{L^2}^2+\|\nabla|\tilde H(0,1)|^2\|_{L^2}^2\\&+\|\nabla|\tilde H(1,1)|^2\|_{L^2}^2+\|\nabla|\tilde H(1,-1)|^2\|_{L^2}^2,\end{aligned}\tag{3.24}$$

and that

$$\begin{aligned}\||\Delta H||H|\|_{L^2}^2\le&C\|\nabla H\|_{L^4}^4+\|\Delta|\tilde H(1,0)|^2\|_{L^2}^2+\|\Delta|\tilde H(0,1)|^2\|_{L^2}^2\\&+\|\Delta|\tilde H(1,1)|^2\|_{L^2}^2+\|\Delta|\tilde H(1,-1)|^2\|_{L^2}^2,\end{aligned}\tag{3.25}$$

then we can get (3.19) from (3.23)-(3.25). □

In order to improve the regularity estimates on ρ, u and H, we start some basic energy estimates on the material derivatives of u.

Lemma 3.4 *Under the condition* (3.1), *it holds that for* $0 \le T \le T^*$,

$$\sup_{0\le t\le T}\left(\|\rho^{1/2}\dot u\|_{L^2}^2+\|\nabla u\|_{L^4}^4\right)+\int_0^T\|\nabla\dot u\|_{L^2}^2dt\le C. \tag{3.26}$$

Proof: We will follow an idea due to Hoff [9]. Operating $\partial/\partial_t+\mathrm{div}(u\cdot\ )$ to $(1.1)_2^j$ and multiplying the resulting equation by $\dot u^j$, one gets by some simple calculations that

$$\begin{aligned}\frac12\left(\int\rho|\dot u^j|^2dx\right)_t=&\ \mu\int\dot u^j(\Delta u_t^j+\mathrm{div}(u\Delta u^j))dx\\&+(\mu+\lambda)\int\dot u^j(\partial_t\partial_j(\mathrm{div}u)+\mathrm{div}(u\partial_j(\mathrm{div}u)))dx\\&-\int\dot u^j(\partial_jP_t+\mathrm{div}(u\partial_jP))dx\\&-\frac12\int\dot u^j(\partial_t\partial_j|H|^2+\mathrm{div}(u\partial_j|H|^2))dx\\&+\int\dot u^j(\partial_t(H\cdot\nabla H^j)+\mathrm{div}(u(H\cdot\nabla H^j)))dx\triangleq\sum_{i=1}^5J_i.\end{aligned} \tag{3.27}$$

First, integration by parts gives

$$\begin{aligned}J_1&=\mu\int\dot u^j(\Delta u_t^j+\mathrm{div}(u\Delta u^j))dx=-\mu\int(\partial_i\dot u^j\partial_iu_t^j+\Delta u^ju\cdot\nabla\dot u^j)dx\\&=-\mu\int(|\nabla\dot u|^2-\partial_i\dot u^ju^k\partial_k\partial_iu^j-\partial_i\dot u^j\partial_iu^k\partial_ku^j+\Delta u^ju\cdot\nabla\dot u^j)dx\\&=-\mu\int(|\nabla\dot u|^2+\partial_i\dot u^j\partial_ku^k\partial_iu^j-\partial_i\dot u^j\partial_iu^k\partial_ku^j-\partial_iu^j\partial_iu^k\partial_k\dot u^j)dx\\&\le-\frac{3\mu}{4}\|\nabla\dot u\|_{L^2}^2+C\|\nabla u\|_{L^4}^4.\end{aligned} \tag{3.28}$$

Similarly,

$$J_2\le-\frac{\mu+\lambda}{2}\|\mathrm{div}\dot u\|_{L^2}^2+C\|\nabla u\|_{L^4}^4. \tag{3.29}$$

It follows from integration by parts, and (3.8), (3.5) that

$$\begin{aligned}J_3&=-\int\dot u^j(\partial_jP_t+\mathrm{div}(u\partial_jP))dx=\int(\partial_j\dot u^jP_t+\partial_jPu\cdot\nabla\dot u^j)dx\\&=-\int((\gamma-1)\partial_j\dot u^jP\mathrm{div}u+\partial_j\dot u^j\mathrm{div}(Pu)+P\partial_j(u\cdot\nabla\dot u^j))dx\\&=-\int((\gamma-1)\partial_j\dot u^jP\mathrm{div}u+\partial_j\dot u^j\mathrm{div}(Pu)+Pu\cdot\nabla\partial_j\dot u^j+P\partial_ju\cdot\nabla\dot u^j)dx\\&=-\int((\gamma-1)\partial_j\dot u^jP\mathrm{div}u+P\partial_ju\cdot\nabla\dot u^j)dx\\&\le\frac{\mu}{4}\|\nabla\dot u\|_{L^2}^2+C.\end{aligned} \tag{3.30}$$

Next, it follows from $(1.1)_3$ and (2.1) that

$$
\begin{aligned}
J_4 &= \int \partial_j \dot{u}^j H \cdot H_t dx + \frac{1}{2}\int u \cdot \nabla \dot{u}^j \partial_j |H|^2 dx \\
&= \frac{1}{2}\int \partial_j \dot{u}^j \mathrm{div} u |H|^2 dx - \frac{1}{2}\int \partial_j u \cdot \nabla \dot{u}^j |H|^2 dx \\
&\quad + \int \partial_j \dot{u}^j H \cdot (H \cdot \nabla u + \nu \Delta H - H \mathrm{div} u) dx \\
&\le C \int |\nabla \dot{u}||\nabla u||H|^2 dx + C\int |\nabla \dot{u}||\Delta H \cdot H| dx \\
&\le \frac{\mu}{8}\|\nabla \dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4 + C\|H\|_{L^8}^8 + C\|\Delta H \cdot H\|_{L^2}^2.
\end{aligned} \tag{3.31}
$$

Similar to (3.31), we estimate J_5 as follows

$$
J_5 \le \frac{\mu}{8}\|\nabla \dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4 + C\|H\|_{L^8}^8 + C\|\Delta H \cdot H\|_{L^2}^2. \tag{3.32}
$$

Putting (3.28)-(3.32) into (3.27), which together with (3.3) gives

$$
\frac{\mathrm{d}}{\mathrm{dt}}\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2 dx \le C\|\nabla u\|_{L^4}^4 + C\||\Delta H||H|\|_{L^2}^2 + C. \tag{3.33}
$$

Integrating (3.33) over $(0,T)$, using (3.6) and (3.19), one has

$$
\sup_{0\le t\le T}\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \int_0^T \|\nabla \dot{u}\|_{L^2}^2 dt \le C. \tag{3.34}
$$

Then (3.26) can be obtained directly from (2.11), (3.1), (3.2), (3.19) and (3.34).

□

Next, the following Lemma 3.5 combined with Lemma 2.4 will be useful to estimate the L^p-norm of $\rho\dot{u}$ and obtain the regularity estimates on ρ.

Lemma 3.5 *Under the condition* (3.1)*, then there exists a positive constant N_1 depending only on N_0, T and the initial data such that for $0 \le T \le T^*$,*

$$
\int_{B_{N_1}} \rho(x,t) dx \ge \frac{1}{4}. \tag{3.35}
$$

Proof: First, multiplying $(1.1)_1$ by $(1+|x|^2)^{1/2}$ and integrating the resulting equality over $\mathbb{R}^2$, we obtain after integration by parts and using both (3.2) and the fact that

$$
\int \rho \, dx = \int \rho_0 \, dx = 1. \tag{3.36}
$$

This leads to

$$
\sup_{0\le t\le T}\int \rho(1+|x|^2)^{1/2} dx \le C. \tag{3.37}
$$

Next, for $N > 1$, let $\varphi_N(x)$ be a $C_0^\infty(\mathbb{R}^2)$ function such that

$$
0 \le \varphi_N(x) \le 1, \quad \varphi_N(x) = \begin{cases} 0 & \text{if } |x| \le N, \\ 1 & \text{if } |x| \ge 2N, \end{cases} \qquad |\nabla \varphi_N| \le 2N^{-1}.
$$

It follows from $(1.1)_1$, (3.2) and (3.36) that

$$\frac{\mathrm{d}}{\mathrm{dt}}\int \rho\varphi_N dx = \int \rho u\cdot\nabla\varphi_N dx \geq -2N^{-1}\left(\int \rho dx\right)^{1/2}\left(\int \rho|u|^2dx\right)^{1/2} \geq -CN^{-1}$$

which implies

$$\int \rho\varphi_N dx \geq \int \rho_0\varphi_N dx - CN^{-1}T. \tag{3.38}$$

It thus follows from (1.11) and (3.38) that for $N_1 \triangleq 2(2+N_0+4CT)$,

$$\int_{B_{N_1}} \rho dx \geq \int \rho\varphi_{\frac{N_1}{2}} dx \geq \frac{1}{4},$$

which gives (3.35). The proof of Lemma 3.5 is completed. □

The next key lemma is used to bound the density gradient and $L^1(0,T;L^\infty)$-norm of ∇u.

Lemma 3.6 *Under the condition* (3.1)*, for any* $q>2$*, it holds that*

$$\sup_{0\le t\le T}\left(\|\rho\|_{H^1\cap W^{1,q}} + \|\nabla u\|_{H^1}\right) + \int_0^T \|\nabla^2 u\|_{L^q}^2 dt \leq C. \tag{3.39}$$

Proof: In fact, for $p\in[2,q]$, $|\nabla\rho|^p$ satisfies

$$\begin{aligned}&(|\nabla\rho|^p)_t + \mathrm{div}(|\nabla\rho|^p u) + (p-1)|\nabla\rho|^p\mathrm{div}u\\ &+ p|\nabla\rho|^{p-2}(\nabla\rho)^t\nabla u(\nabla\rho) + p\rho|\nabla\rho|^{p-2}\nabla\rho\cdot\nabla\mathrm{div}u = 0.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{d}{dt}\|\nabla\rho\|_{L^p} &\leq C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^p} + C\|\nabla^2 u\|_{L^p}\\ &\leq C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^p} + C\|\rho\dot{u}\|_{L^p} + C\||H||\nabla H|\|_{L^p},\end{aligned} \tag{3.40}$$

due to

$$\|\nabla^2 u\|_{L^p} \leq C\left(\|\rho\dot{u}\|_{L^p} + \|\nabla P\|_{L^p} + \||H||\nabla H|\|_{L^p}\right), \tag{3.41}$$

which follows from the standard L^p-estimate for the following elliptic system:

$$\mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}u = \rho\dot{u} + \nabla P + \frac{1}{2}\nabla|H|^2 - H\cdot\nabla H, \quad u\to 0 \text{ as } |x|\to\infty.$$

Next, the Gagliardo-Nirenberg inequality, (2.8) and (2.9) implies

$$\begin{aligned}&\|\mathrm{div}u\|_{L^\infty} + \|\omega\|_{L^\infty}\\ &\leq C\|F\|_{L^\infty} + C\|P\|_{L^\infty} + C\|\omega\|_{L^\infty}\\ &\leq C(q) + C(q)\|\nabla F\|_{L^q}^{q/(2(q-1))} + C(q)\|\nabla\omega\|_{L^q}^{q/(2(q-1))}\\ &\leq C(q) + C(q)\left(\|\rho\dot{u}\|_{L^q} + \||H||\nabla H|\|_{L^q}\right)^{q/(2(q-1))},\end{aligned} \tag{3.42}$$

which, together with Lemma 2.6, (3.41) and (3.5), yields that

$$\begin{aligned}
\|\nabla u\|_{L^\infty} \leq & C\left(\|\operatorname{div} u\|_{L^\infty}+\|\omega\|_{L^\infty}\right) \log(e+\|\nabla^2 u\|_{L^q})+C\|\nabla u\|_{L^2}+C \\
\leq & C\left(1+\|\rho \dot{u}\|_{L^q}^{q/(2(q-1))}+\||H||\nabla H|\|_{L^q}^{q/(2(q-1))}\right) \\
& \cdot \log(e+\|\rho \dot{u}\|_{L^q}+\||H||\nabla H|\|_{L^q}+\|\nabla \rho\|_{L^q}) \\
\leq & C\left(1+\|\rho \dot{u}\|_{L^q}+\||H||\nabla H|\|_{L^q}\right) \log(e+\|\nabla \rho\|_{L^q}).
\end{aligned} \tag{3.43}$$

It follows from Lemma 2.4, Lemma 3.5, (3.37) and (3.26) that,

$$\int_0^T \|\rho \dot{u}\|_{L^q}^2 dt \leq C \int_0^T (\|\rho^{1/2} \dot{u}\|_{L^2}^2+\|\nabla \dot{u}\|_{L^2}^2) \leq C. \tag{3.44}$$

Moreover, we have by Hölder's inequality, Gagliardo-Nirenberg inequality that,

$$\begin{aligned}
\||H||\nabla H|\|_{L^q} & \leq C\|H\|_{L^{2q}}\|\nabla H\|_{L^{2q}} \leq C\|H\|_{L^2}^{1/q}\|\nabla H\|_{L^2}\|\nabla^2 H\|_{L^2}^{(q-1)/q} \\
& \leq C\|\nabla H\|_{L^2}(\|H\|_{L^2}+\|\nabla^2 H\|_{L^2}),
\end{aligned} \tag{3.45}$$

integrating this over $(0,T)$, which together with (3.2) and (3.5), yields

$$\int_0^T \||H||\nabla H|\|_{L^q}^2 dt \leq C. \tag{3.46}$$

Then, substituting (3.43) into (3.40) where $p=q$, we deduce from Gronwall's inequality, (3.44) and (3.46) that

$$\sup_{0\leq t\leq T} \|\nabla \rho\|_{L^q} \leq C, \tag{3.47}$$

which, along with (3.43), and (3.41), shows

$$\int_0^T \left(\|\nabla u\|_{L^\infty}+\|\nabla^2 u\|_{L^q}^2\right) dt \leq C. \tag{3.48}$$

Finally, taking $p=2$ in (3.40), one gets by using (3.3), (3.5), (3.48), and Gronwall's inequality that

$$\sup_{0\leq t\leq T} \|\nabla \rho\|_{L^2} \leq C. \tag{3.49}$$

The standard L^2-estimate for the elliptic system, (3.19), (3.26) and (3.49) lead to

$$\sup_{0\leq t\leq T} \|\nabla^2 u\|_{L^2}^2 \leq C \sup_{0\leq t\leq T} \left(\|\rho \dot{u}\|_{L^2}^2+\|\nabla P\|_{L^2}^2+\||H||\nabla H|\|_{L^2}^2\right) \leq C, \tag{3.50}$$

which together with (3.36), (3.1), (3.5) and (3.47)-(3.49) finishes the proof of Lemma 3.6. □

Next, it follows from (2.3), (3.35), and the Poincaré-type inequality [6, Lemma 3.2] that for $s>2$, $\eta\in(0,1]$, and $t\in[0,T]$,

$$\|u\bar{x}^{-1}\|_{L^2}+\|u\bar{x}^{-\eta}\|_{L^{s/\eta}} \leq C(s,\eta)\|\rho^{1/2}u\|_{L^2}+C(s,\eta)\|\nabla u\|_{L^2}, \tag{3.51}$$

which together with (3.2) and (3.5) gives

$$\|u\bar{x}^{-1}\|_{L^2}+\|u\bar{x}^{-\eta}\|_{L^{s/\eta}} \leq C(\eta,s). \tag{3.52}$$

With the help of (3.52), we can get the following spatial weighted mean estimate of the density, which has been proved in [20, Lemma 4.2].

Lemma 3.7 *Under the condition* (3.1)*, it holds that for* $a > 1$*,* $q > 2$ *and* $0 \le T \le T^*$*,*

$$\sup_{0 \le t \le T} \|\bar{x}^a \rho\|_{L^1 \cap H^1 \cap W^{1,q}} \le C. \tag{3.53}$$

Lemma 3.8 *Under the condition* (3.1)*, it holds that for* $a > 1$ *and* $0 \le T \le T^*$*,*

$$\sup_{0 \le t \le T} \|H \bar{x}^{a/2}\|_{L^2}^2 + \int_0^T \|\nabla H \bar{x}^{a/2}\|_{L^2}^2 dt \le C, \tag{3.54}$$

$$\sup_{0 \le t \le T} \|\nabla H \bar{x}^{a/2}\|_{L^2}^2 + \int_0^T \|\nabla^2 H \bar{x}^{a/2}\|_{L^2}^2 dt \le C. \tag{3.55}$$

Proof: First, multiplying $(1.1)_3$ by $H\bar{x}^a$ and integrating by parts yields

$$\begin{aligned} &\frac{1}{2}\left(\int |H|^2 \bar{x}^a dx\right)_t + \nu \int |\nabla H|^2 \bar{x}^a dx = \frac{\nu}{2} \int |H|^2 \Delta \bar{x}^a dx \\ &\quad + \int H \cdot \nabla u \cdot H \bar{x}^a dx - \frac{1}{2} \int \mathrm{div} u |H|^2 \bar{x}^a dx + \frac{1}{2} \int |H|^2 u \cdot \nabla \bar{x}^a dx \triangleq \sum_{i=1}^4 K_i. \end{aligned} \tag{3.56}$$

Direct calculations and (3.5) yield that

$$|K_1| \le C \int |H|^2 \bar{x}^a \bar{x}^{-2} \log^4(e + |x|^2) dx \le C \int |H|^2 \bar{x}^a dx, \tag{3.57}$$

and that

$$\begin{aligned} |K_2| + |K_3| &\le C \int |\nabla u| |H|^2 \bar{x}^a dx \le C \|\nabla u\|_{L^2} \|H \bar{x}^{a/2}\|_{L^4}^2 \\ &\le C \|\nabla u\|_{L^2} \|H \bar{x}^{a/2}\|_{L^2} (\|\nabla H \bar{x}^{a/2}\|_{L^2} + \|H \nabla \bar{x}^{a/2}\|_{L^2}) \\ &\le C \|H \bar{x}^{a/2}\|_{L^2}^2 + \frac{\nu}{4} \|\nabla H \bar{x}^{a/2}\|_{L^2}^2. \end{aligned} \tag{3.58}$$

Then, it follows from Hölder's inequality, (2.1) and (3.52) that

$$\begin{aligned} |K_4| &\le C \|H \bar{x}^{a/2}\|_{L^4} \|H \bar{x}^{a/2}\|_{L^2} \|u \bar{x}^{-3/4}\|_{L^4} \\ &\le C \|H \bar{x}^{a/2}\|_{L^4}^2 + C \|H \bar{x}^{a/2}\|_{L^2}^2 \|u \bar{x}^{-3/4}\|_{L^4}^2 \\ &\le C \|H \bar{x}^{a/2}\|_{L^2}^2 + \frac{\nu}{4} \|\nabla H \bar{x}^{a/2}\|_{L^2}^2. \end{aligned} \tag{3.59}$$

Putting (3.57)-(3.59) into (3.56), after using Gronwall's inequality, we have

$$\sup_{0 \le t \le T} \int \bar{x}^a |H|^2 dx + \int_0^T \int \bar{x}^a |\nabla H|^2 dx dt \le C. \tag{3.60}$$

Next, multiplying $(1.1)_3$ by $\Delta H \bar{x}^a$, integrating the resultant equation by parts over $\mathbb{R}^2$, it follows from the similar arguments as (3.16) that

$$\begin{aligned} &\frac{1}{2}\left(\int |\nabla H|^2 \bar{x}^a dx\right)_t + \nu \int |\Delta H|^2 \bar{x}^a dx \\ \le& C \int |\nabla H| |H| |\nabla u| |\nabla \bar{x}^a| dx + C \int |\nabla H|^2 |u| |\nabla \bar{x}^a| dx + C \int |\nabla H| |\Delta H| |\nabla \bar{x}^a| dx \\ &+ C \int |H| |\nabla u| |\Delta H| \bar{x}^a dx + C \int |\nabla u| |\nabla H|^2 \bar{x}^a dx \triangleq \sum_{i=1}^5 L_i. \end{aligned} \tag{3.61}$$

Using Gagliardo-Nirenberg inequality, (3.26) and (3.54), it holds that

$$\begin{aligned}
L_1 \le & C\int |\nabla H||H||\nabla u|\bar{x}^a(\bar{x}^{-1}|\nabla \bar{x}|)dx \\
\le & C\|H\bar{x}^{a/2}\|_{L^4}^4 + C\|\nabla u\|_{L^4}^4 + C\|\nabla H\bar{x}^{a/2}\|_{L^2}^2 \\
\le & C\|H\bar{x}^{a/2}\|_{L^2}^2\left(\|\nabla H\bar{x}^{a/2}\|_{L^2}^2 + \|H\bar{x}^{a/2}\|_{L^2}^2\right) + C + C\|\nabla H\bar{x}^{a/2}\|_{L^2}^2 \\
\le & C + C\|\nabla H\bar{x}^{a/2}\|_{L^2}^2,
\end{aligned} \tag{3.62}$$

$$\begin{aligned}
L_2 \le & C\int |\nabla H|^{(4a-1)/(2a)}\bar{x}^{(4a-1)/4}|\nabla H|^{1/(2a)}|u|\bar{x}^{-1/2}\bar{x}^{-1/4}|\nabla \bar{x}|dx \\
\le & C\||\nabla H|^{(4a-1)/(2a)}\bar{x}^{(4a-1)/4}\|_{L^{\frac{4a}{4a-1}}}\||\nabla H|^{1/(2a)}\|_{L^{8a}}\|u\bar{x}^{-1/2}\|_{L^{8a}} \\
\le & C\|\nabla H\bar{x}^{a/2}\|_{L^2}^2 + C\|\nabla H\|_{L^4}^2 \\
\le & C\|\nabla H\bar{x}^{a/2}\|_{L^2}^2 + C\|\nabla^2 H\|_{L^2}^2 + C,
\end{aligned} \tag{3.63}$$

$$\begin{aligned}
L_3 + L_4 \le & \|\Delta H\bar{x}^{a/2}\|_{L^2}\|\nabla H\bar{x}^{a/2}\|_{L^2} + \|\Delta H\bar{x}^{a/2}\|_{L^2}\|H\bar{x}^{a/2}\|_{L^4}\|\nabla u\|_{L^4} \\
\le & \frac{\nu}{4}\|\Delta H\bar{x}^{a/2}\|_{L^2}^2 + C\|\nabla H\bar{x}^{a/2}\|_{L^2}^2 + C,
\end{aligned} \tag{3.64}$$

$$\begin{aligned}
L_5 \le & C\|\nabla u\|_{L^\infty}\|\nabla H\bar{x}^{a/2}\|_{L^2}^2 \\
\le & C\|\nabla u\|_{L^2}^{(q-2)/(2q-2)}\|\nabla^2 u\|_{L^q}^{q/(2q-2)}\|\nabla H\bar{x}^{a/2}\|_{L^2}^2 \\
\le & C(1+\|\nabla^2 u\|_{L^q}^2)\|\nabla H\bar{x}^{a/2}\|_{L^2}^2.
\end{aligned} \tag{3.65}$$

Noticing the fact that

$$\begin{aligned}
\int |\nabla^2 H|^2\bar{x}^a dx = & \int |\Delta H|^2\bar{x}^a dx - \int \partial_i\partial_k H\cdot\partial_k H\partial_i\bar{x}^a dx \\
& + \int \partial_i\partial_i H\cdot\partial_k H\partial_k\bar{x}^a dx \\
\le & \int |\Delta H|^2\bar{x}^a dx + \frac{1}{2}\int |\nabla^2 H|^2\bar{x}^a dx + C\int |\nabla H|^2\bar{x}^a dx.
\end{aligned} \tag{3.66}$$

Submitting (3.62)-(3.65) into (3.61), and using (3.66), we obtain

$$\begin{aligned}
& \left(\int |\nabla H|^2\bar{x}^a dx\right)_t + \int |\nabla^2 H|^2\bar{x}^a dx \\
& \le C(1+\|\nabla^2 u\|_{L^q}^2)\|\nabla H\bar{x}^{a/2}\|_{L^2}^2 + C(\|\nabla^2 H\|_{L^2}^2 + 1),
\end{aligned} \tag{3.67}$$

which together with Gronwall's inequality, (3.5) and (3.39) yields (3.55). The proof of Lemma 3.8 is finished. □

Lemma 3.9 *Under the condition* (3.1)*, it holds that for* $0 \le T \le T^*$,

$$\begin{aligned}
& \sup_{0\le t\le T}\left(\|\rho^{1/2}u_t\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\nabla H\|_{H^1}^2\right) \\
& + \int_0^T \left(\|\nabla u_t\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 + \|\nabla^2 H\|_{L^q}^2\right)dt \le C.
\end{aligned} \tag{3.68}$$

Proof: First, the combination of (3.53) with (3.52) gives that for any $\eta \in (0,1]$ and any $s > 2$,

$$\|\rho^\eta u\|_{L^{s/\eta}} + \|u\bar{x}^{-\eta}\|_{L^{s/\eta}} \le C(\eta, s). \tag{3.69}$$

Differentiating $(1.1)_2$ with respect to t gives

$$\begin{aligned}
&\rho u_{tt} + \rho u \cdot \nabla u_t - \mu \Delta u_t - (\mu+\lambda)\nabla \mathrm{div} u_t \\
&= -\rho_t (u_t + u\cdot\nabla u) - \rho u_t \cdot \nabla u - \nabla P_t + \left(H\cdot\nabla H - \frac{1}{2}\nabla |H|^2 \right)_t .
\end{aligned} \tag{3.70}$$

Multiplying (3.70) by u_t, then integrating over $\mathbb{R}^2$, we obtain after using $(1.1)_1$ that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int \rho |u_t|^2 dx + \int \left(\mu |\nabla u_t|^2 + (\mu+\lambda)(\mathrm{div} u_t)^2\right) dx \\
&= -2\int \rho u \cdot \nabla u_t \cdot u_t dx - \int \rho u \cdot \nabla (u\cdot\nabla u \cdot u_t) dx \\
&\quad - \int \rho u_t \cdot \nabla u \cdot u_t dx + \int P_t \mathrm{div} u_t dx + \int \left(H\cdot\nabla H - \frac{1}{2}\nabla|H|^2\right)_t u_t dx \\
&\triangleq \sum_{i=1}^{5} \bar{J}_i.
\end{aligned} \tag{3.71}$$

Similar to the proof of [20, Lemma 4.3], and using (3.39), we have

$$\begin{aligned}
\sum_{i=1}^{4} \bar{J}_i &\le \frac{\mu}{4}\|\nabla u_t\|_{L^2}^2 + C\left(\|\nabla^2 u\|_{L^2}^2 + \|\rho^{1/2} u_t\|_{L^2}^2 + 1 \right) \\
&\le \frac{\mu}{4}\|\nabla u_t\|_{L^2}^2 + C\left(\|\rho^{1/2} u_t\|_{L^2}^2 + 1 \right).
\end{aligned} \tag{3.72}$$

For the term $\bar{J}_5$, we obtain after integration by parts that

$$\begin{aligned}
\bar{J}_5 &= -\int H_t \cdot \nabla u_t \cdot H dx - \int H \cdot \nabla u_t \cdot H_t dx + \int H \cdot H_t \mathrm{div} u_t dx \\
&\le C\|H\|_{L^4}\|H_t\|_{L^4}\|\nabla u_t\|_{L^2} \\
&\le \frac{\mu}{4}\|\nabla u_t\|_{L^2}^2 + \delta \|\nabla H_t\|_{L^2}^2 + C(\delta)\|H_t\|_{L^2}^2 .
\end{aligned} \tag{3.73}$$

Substituting (3.72) and (3.73) into (3.71) leads to

$$\frac{\mathrm{d}}{\mathrm{dt}}\|\rho^{1/2}u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \le C\delta \|\nabla H_t\|_{L^2}^2 + C(\delta)\left(\|\rho^{1/2}u_t\|_{L^2}^2 + \|H_t\|_{L^2}^2 + 1 \right). \tag{3.74}$$

Next, differentiating $(1.1)_3$ with respect to t shows

$$H_{tt} - H_t \cdot \nabla u - H\cdot \nabla u_t + u_t \cdot \nabla H + u\cdot\nabla H_t + H_t \mathrm{div} u + H \mathrm{div} u_t = \nu \Delta H_t. \tag{3.75}$$

Multiplying (3.75) by H_t, then integrating the resulting equation over $\mathbb{R}^2$, and using (3.3), (3.5), (3.51) and (3.54), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int |H_t|^2 dx + \nu \int |\nabla H_t|^2 dx = \int H_t \cdot \nabla u \cdot H_t dx \\
&\quad - \frac{1}{2}\int \mathrm{div} u |H_t|^2 dx + \int H\cdot\nabla u_t \cdot H_t dx + \int u_t \cdot \nabla H_t \cdot H dx \\
&\le C\|H_t\|_{L^4}^2 \|\nabla u\|_{L^2} + C\|H\|_{L^4}\|H_t\|_{L^4}\|\nabla u_t\|_{L^2} \\
&\quad + C\||H|^{\frac{1}{2a}}\|_{L^{8a}} \|(H\bar{x}^{\frac{a}{2}})^{\frac{2a-1}{2a}}\|_{L^{\frac{4a}{2a-1}}} \|u_t \bar{x}^{-\frac{2a-1}{4}}\|_{L^{8a}} \|\nabla H_t\|_{L^2} \\
&\le \delta \|\nabla H_t\|_{L^2}^2 + C(\delta)\left(\|H_t\|_{L^2}^2 + \|\rho^{1/2}u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \right),
\end{aligned} \tag{3.76}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{dt}}\|H_t\|_{L^2}^2+\|\nabla H_t\|_{L^2}^2\le C\|H_t\|_{L^2}^2+C\|\rho^{1/2}u_t\|_{L^2}^2+C_2\|\nabla u_t\|_{L^2}^2. \tag{3.77}$$

Then adding (3.74) multiplied by C_2+2 to (3.77), choosing δ suitable small, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\left((C_2+2)\|\rho^{1/2}u_t\|_{L^2}^2+\|H_t\|_{L^2}^2\right)+\|\nabla u_t\|_{L^2}^2+\frac{1}{2}\|\nabla H_t\|_{L^2}^2\\&\qquad\le C\left(\|\rho^{1/2}u_t\|_{L^2}^2+\|H_t\|_{L^2}^2+1\right),\end{aligned} \tag{3.78}$$

which together with Gronwall's inequality yields

$$\sup_{0\le t\le T}\left(\|\rho^{1/2}u_t\|_{L^2}^2+\|H_t\|_{L^2}^2\right)+\int_0^T\left(\|\nabla u_t\|_{L^2}^2+\|\nabla H_t\|_{L^2}^2\right)dt\le C. \tag{3.79}$$

Finally, it follows from (3.79), (3.3), (3.52), (3.26), (3.55) and Gagliardo-Nirenberg inequality that

$$\begin{aligned}\|\nabla^2 H\|_{L^2}^2&\le C\||u||\nabla H|\|_{L^2}^2+C\||H||\nabla u|\|_{L^2}^2+C\|H_t\|_{L^2}^2\\&\le C\|u\bar{x}^{-a/4}\|_{L^8}^2\||\nabla H|^{1/2}\bar{x}^{a/4}\|_{L^4}^2\||\nabla H|^{1/2}\|_{L^8}^2\\&\quad+C\|H\|_{L^4}^2\|\nabla u\|_{L^4}^2+C\\&\le C\|\nabla H\bar{x}^{a/2}\|_{L^2}\|\nabla H\|_{L^4}+C\\&\le\frac{1}{2}\|\nabla^2 H\|_{L^2}^2+C,\end{aligned} \tag{3.80}$$

and

$$\begin{aligned}\int_0^T\|\nabla^2 H\|_{L^q}^2dt&\le C\int_0^T\left(\||u||\nabla H|\|_{L^q}^2+\||H||\nabla u|\|_{L^q}^2+\|H_t\|_{L^q}^2\right)dt\\&\le C\int_0^T\big(\|u\bar{x}^{-a/2}\|_{L^{2q}}^2\|\nabla H\bar{x}^{a/2}\|_{L^{2q}}^2+\|H\|_{L^{2q}}^2\|\nabla u\|_{L^{2q}}^2\\&\qquad+\|H_t\|_{L^2}^{4/q}\|\nabla H_t\|_{L^2}^{2(q-2)/q}\big)dt\\&\le C\int_0^T\big(\|\nabla H\bar{x}^{a/2}\|_{L^2}^{2/q}(\|\nabla^2 H\bar{x}^{a/2}\|_{L^2}+\|\nabla H\nabla\bar{x}^{a/2}\|_{L^2})^{2(q-1)/q}\\&\qquad+\|\nabla u\|_{H^1}^2+\|\nabla H_t\|_{L^2}^2+1\big)dt\\&\le C\int_0^T\big(\|\nabla^2 H\bar{x}^{a/2}\|_{L^2}^2+\|\nabla H_t\|_{L^2}^2+1\big)dt\le C,\end{aligned} \tag{3.81}$$

which combined with (3.80) leads to (3.68). Thus the proof of Lemma 3.9 is finished. □

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1: Suppose that (1.14) were false, that is, (3.1) holds. Note that the generic constant C in Lemma 3.2 and Lemma 3.6-3.9 remains uniformly bounded for all $T<T^*$, so the functions $(\rho,u,H)\triangleq\lim_{t\to T^*}(\rho,u,H)(x,t)$ satisfy the conditions imposed on the initial data (1.12) at the time $t=T^*$. Furthermore, standard arguments yield that $\rho\dot{u}\in C([0,T];L^2)$, which implies

$$(\rho\dot{u})(x,T^*)=\lim_{t\to T^*}(\rho\dot{u})\in L^2.$$

Hence,

$$-\mu\Delta u - (\mu+\lambda)\nabla \mathrm{div} u + \nabla P - (\nabla \times H) \times H|_{t=T^*} = \sqrt{\rho}(x,T^*)g(x),$$

with

$$g(x) = \begin{cases} \rho(x,T^*)^{-1/2}(\rho\dot{u})(x,T^*), & \text{for } x \in \{x|\rho(x,T^*) > 0\}, \\ 0, & \text{for } x \in \{x|\rho(x,T^*) = 0\}, \end{cases}$$

satisfying $g \in L^2$ due to (3.26). Thus, $(\rho, u)(x, T^*)$ satisfies (1.13) also. Therefore, one can take $(\rho, u, H)(x, T^*)$ as the initial data and apply Lemma 2.1 to extend the local strong solution beyond T^*. This contradicts the assumption on T^*. We thus finish the proof of Theorem 1.1.

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