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# DYNAMICS OF DIRECTOR FIELDS*

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**Abstract.** An asymptotic equation for weakly nonlinear hyperbolic waves governed by variational principles is derived and analyzed. The equation is used to study a nonlinear instability in the director field of a nematic liquid crystal. It is shown that smooth solutions of the asymptotic equation break down in finite time. Also constructed are weak solutions of the equation that are continuous despite the fact that their spatial derivative blows up.

**Key words.** nonlinear hyperbolic partial differential equations, liquid crystals, geometrical optics

**AMS(MOS) subject classifications.** 35L70, 35C20, 76A99

## 1. Introduction.

In this paper, we study the nonlinear wave equation

$$\psi_{tt} = c(\psi)\left[c(\psi)\psi_x\right]_x. \tag{1.1}$$

This is a simplified equation for the director field of a nematic liquid crystal [7]. Liquid crystals are fluids made up of long rigid molecules. The orientation of the molecules is described macroscopically by a field of unit vectors $\mathbf{n}(\mathbf{x}, t) \in S^2$. Nematic liquid crystals are invariant under the inversion $\mathbf{n} \to -\mathbf{n}$, and then $\mathbf{n}$ is called a director field. The kinetic energy of the director field is usually neglected in studies of liquid crystals. Our aim is to investigate the effects of including the inertia of the director field in the simplest possible setting. We therefore consider the dynamics of the director field independently of any coupling with the fluid flow.

Equation (1.1) is the Euler–Lagrange equation of the variational principle

$$\delta \int_{t_1}^{t_2} \int_{-\infty}^{+\infty} \left[\psi_t^2 - c^2(\psi)\psi_x^2\right] dx\, dt = 0. \tag{1.2}$$

The potential energy in (1.2) is obtained by restricting the Oseen–Frank potential energy function for liquid crystals to directors that lie on a circle and depend on a single space variable $x$,

$$\mathbf{n} = \cos\psi(x, t)\mathbf{e}_x + \sin\psi(x, t)\mathbf{e}_y. \tag{1.3}$$

For the nondimensionalized kinetic energy, we use [5]

$$\tfrac{1}{2}\mathbf{n}_t \cdot \mathbf{n}_t = \tfrac{1}{2}\psi_t^2.$$

We then obtain (1.2) with

$$c^2(\psi) = \alpha\cos^2\psi + \beta\sin^2\psi. \tag{1.4}$$

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The term proportional to $\alpha$ describes the potential energy due to bending, and the term proportional to $\beta$ describes the potential energy due to splay. We present a detailed derivation of (1.1) and (1.2) in Appendix A of this paper.

In §2, we show that weakly nonlinear unidirectional waves satisfying (1.1) are described asymptotically by the equation

$$(u_t + uu_x)_x = \tfrac{1}{2}u_x^2. \tag{1.5}$$

Here, $u(x,t)$ is the perturbation in $\psi$ about some constant value $\psi = \psi_0$, and $x$ is a space variable in a reference frame moving with the linearized wave velocity. Equation (1.5) is the Euler–Lagrange equation of the variational principle

$$\delta \int_{t_1}^{t_2} \int_{-\infty}^{+\infty} \left(u_t u_x + uu_x^2\right) dx\, dt = 0. \tag{1.6}$$

Equation (1.5) follows from (1.1) when $c'(\psi_0) \neq 0$. We also derive corresponding asymptotic equations when $c'(\psi_0) = 0$. Extreme values of the wavespeed $c$ in (1.4) occur at $\psi_0 = 0$ or $\psi_0 = \pi/2$, corresponding to a director field that is aligned parallel to or perpendicular to the direction of propagation, respectively.

Equation (1.5) has several interesting features. In §3 we solve (1.5) using the method of characteristics and show that smooth solutions break down in finite time. In §4 we define admissible weak solutions of (1.5) and present a class of piecewise linear weak solutions of (1.5), which are continuous despite the fact that $u_x$ blows up in finite time. This differs from the behavior of weak solutions of the inviscid Burgers equation,

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0, \tag{1.7}$$

which are typically discontinuous due to the formation of shocks.

The inviscid Burgers equation is the canonical asymptotic equation for weakly nonlinear solutions of scale-invariant, hyperbolic conservation laws. In a similar way, (1.5) is the canonical asymptotic equation for weakly nonlinear solutions of scale-invariant, hyperbolic variational principles. We derive (1.5) from a general system of equations in Appendix B.

A second property of (1.5) is that the constant state $u = 0$ is nonlinearly unstable. As we show in §6, a small amplitude wave propagating through a constant director field (1.3) with $\psi = \psi_0$ and $c'(\psi_0) \neq 0$ will "knock" the field away from its unperturbed orientation. The change in the angle typically grows linearly in time and is in the direction of decreasing wavespeed. For (1.4), this is toward the minimum wavespeed, which is at $\psi = 0$ if $\alpha < \beta$, or at $\psi = \pi/2$ if $\alpha > \beta$. An asymptotic solution for the instability caused by a triangular pulse is shown in Fig. 5.

This instability can be explained heuristically from the variational principle (1.2). The potential energy density in (1.2) is $c^2(\psi)\psi_x^2$. If $\psi_x = 0$, then no value of $\psi$ is energetically preferred, and any uniform director field is in equilibrium. However, if $\psi_x \neq 0$, then decreasing $c(\psi)$ reduces the potential energy of the system. Suppose that the director field is perturbed by a wave which initially has no kinetic energy (i.e., $\psi_t(x,0) = 0$). At later times, the wave converts potential energy into kinetic energy, and the director field moves so as to reduce the wavespeed.

**2. Weakly nonlinear waves.** We look for a weakly nonlinear asymptotic solution of (1.1) of the form

$$\psi(x,t;\epsilon) = \psi_0 + \epsilon\psi_1(\theta,\tau) + O(\epsilon^2). \tag{2.1}$$

Here, the independent variables are

$$\theta = x - c_0 t, \qquad \tau = \epsilon t, \tag{2.2}$$

and $c_0 = c(\psi_0) > 0$ is the unperturbed wavespeed. The solution is formally valid in the limit $\epsilon \to 0$ with $\epsilon t = O(1)$. We use (2.1) and (2.2) in (1.1) and equate coefficients of $\epsilon^2$. This implies that

$$\left(\psi_{1\tau} + c_0'\psi_1\psi_{1\theta}\right)_\theta = \tfrac{1}{2}c_0'\psi_{1\theta}^2. \tag{2.3}$$

Here, $c_0' = c'(\psi_0)$, where $' = d/d\psi$. Assuming that $c_0' \neq 0$, the change of variables $u = c_0'\psi_1$, $x = \theta$, and $t = \tau$ transforms (2.3) into (1.5). In (1.5), $(x,t)$ are not the original space-time variables; instead $x$ is the space variable in a reference frame moving with the unperturbed wavespeed, and $t$ is a slow time.

Equation (2.3) is linear at a critical point of the wavespeed, where $c_0' = 0$. In that case, weakly nonlinear effects are significant over a longer timescale than $t = O(\epsilon^{-1})$. Suppose that $c^{(n)}(\psi_0) \neq 0$ and $c^{(k)}(\psi_0) = 0$ for $0 < k < n$ for some integer $n$. The appropriate weakly nonlinear expansion is then

$$\psi(x,t;\epsilon) = \psi_0 + \epsilon\psi_1(\theta,\tau) + O(\epsilon^{n+1}), \tag{2.4}$$

where

$$\theta = x - c_0 t, \qquad \tau = \epsilon^n t. \tag{2.5}$$

Using (2.4) and (2.5) in (1.1), equating coefficients of $\epsilon^{n+1}$, and rearranging the result, we obtain the following equation for $\psi_1$:

$$\left(\psi_{1\tau} + \Gamma\psi_1^n\psi_{1\theta}\right)_\theta = \frac{n}{2}\Gamma\psi_1^{n-1}\psi_{1\theta}^2. \tag{2.6}$$

In (2.6), $\Gamma = c^{(n)}(\psi_0)/n!$. To normalize (2.6), we let $u = |\Gamma|^{1/n}\psi_1$, $x = (\mathrm{sgn}\Gamma)\theta$, and $t = \tau$. This gives

$$\left(u_t + u^n u_x\right)_x = \frac{n}{2}u^{n-1}u_x^2. \tag{2.7}$$

The variational principle for (2.7) is

$$\delta\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}\left(u_t u_x + u^n u_x^2\right)dx\,dt = 0. \tag{2.8}$$

**3. Smooth solutions of the asymptotic equation.** The initial value problem for (1.5) is

$$\begin{aligned}\left(u_t + uu_x\right)_x &= \tfrac{1}{2}u_x^2,\\ u(x,0) &= F(x).\end{aligned} \tag{3.1}$$

In this section we consider only smooth (i.e., twice continuously differentiable) solutions of (3.1) with smooth initial data $F$. First, we use the method of characteristics to solve (3.1).

THEOREM 3.1. *Every smooth solution of* (3.1) *is given implicitly by*

$$\begin{aligned} u &= F(\xi) + tG(\xi) + H'(t), \\ x &= \xi + tF(\xi) + \tfrac{1}{2}t^2G(\xi) + H(t), \end{aligned} \tag{3.2}$$

*where $H$ is any function with $H(0) = H'(0) = 0$, and $G$ satisfies*

$$G'(\xi) = \tfrac{1}{2}F'^2(\xi). \tag{3.3}$$

*Proof.* Suppose that $u(x,t)$ is a smooth solution of (3.1). We introduce a characteristic coordinate $\xi$, where

$$\begin{aligned} x &= X(\xi,t), \\ U(\xi,t) &= u[X(\xi,t),t], \end{aligned} \tag{3.4}$$

and $X(\xi,t)$ satisfies

$$\begin{aligned} X_t &= U, \\ X(\xi,0) &= \xi. \end{aligned} \tag{3.5}$$

We also define

$$V(\xi,t) = X_\xi(\xi,t). \tag{3.6}$$

It then follows from (3.1) that $V$ satisfies

$$\begin{aligned} VV_{tt} &= \tfrac{1}{2}V_t^2, \\ V(\xi,0) &= 1, \\ V_t(\xi,0) &= F'(\xi). \end{aligned} \tag{3.7}$$

The solution of (3.7) is

$$V = X_\xi = \left[1 + \tfrac{1}{2}F'(\xi)t\right]^2. \tag{3.8}$$

Solving the equation $U_\xi = V_t$ for $U$ and then solving (3.5) for $X$ implies (3.2) and (3.3).

Conversely, it is easy to check that (3.2) and (3.3) define a solution of (3.1), provided that (3.2b) can be inverted to give $\xi = \Xi(x,t)$.

To determine the arbitrary function $H$ in this solution we need to add a boundary condition. For a wave propagating to the right into an undisturbed region, the appropriate boundary condition is

$$u(x,t) \to 0 \quad \text{as} \quad x \to +\infty. \tag{3.9}$$

Provided that the initial data satisfies $F(x) \to 0$ as $x \to +\infty$ and $\int_0^{+\infty} F'^2(x)\,dx < +\infty$, there is a unique smooth solution of (3.1) and (3.9) given by (3.2) with

$$G(\xi) = -\frac{1}{2}\int_\xi^{+\infty} F'^2(\zeta)\,d\zeta, \qquad H(t) = 0.$$

Theorem 3.1 implies that smooth solutions break down in finite time. The breakdown time is exactly twice the breakdown time for the inviscid Burgers equation (1.7) with the same initial data.

THEOREM 3.2. *Suppose that $F$ is not monotone increasing. Let $t_* = 2/\sup[-F'(x)]$. Then* (3.1) *has a smooth solution $u(x,t)$ in $0 < t < t_*$ and $\inf(u_x) \to -\infty$ as $t \uparrow t_*$.*

*Proof.* By the implicit function theorem, there is a smooth solution $\xi = \Xi(x,t)$ of (3.4a) if $X_\xi \neq 0$. From (3.8), this is true when $t < t_*$, with $t_*$ defined in the statement of the theorem. Moreover, since

$$u_x = \frac{U_\xi}{X_\xi} = \frac{F'}{1 + \frac{1}{2}F't},$$

it follows that $u_x$ blows up as $t \uparrow t_*$.

Next, we use Noether's theorem to obtain conservation laws for (1.5). Noether's theorem states that there is a conservation law associated with each variational symmetry $\tilde{x} = X_\epsilon(x,t,u)$, $\tilde{t} = T_\epsilon(x,t,u)$, $\tilde{u} = U_\epsilon(x,t,u)$ of the Lagrangian in (1.6). Calculating the possible infinitesimal generators of these symmetries [6] shows that there are only four: translations in space and time, scalings, and Galilean transformations. The associated conservation laws are as follows:

(1) translations in space ($\tilde{x} = x + \epsilon$, $\tilde{t} = t$, $\tilde{u} = u$),

$$\left(u_x^2\right)_t + \left(uu_x^2\right)_x = 0; \tag{3.10}$$

(2) translations in time ($\tilde{x} = x$, $\tilde{t} = t + \epsilon$, $\tilde{u} = u$),

$$\left(uu_x^2\right)_t - \left(2uu_xu_t + u_t^2\right)_x = 0; \tag{3.11}$$

(3) scale changes ($\tilde{x} = \epsilon x$, $\tilde{t} = \epsilon t$, $\tilde{u} = u$),

$$\left(tuu_x^2 - xu_x^2\right)_t - \left(xuu_x^2 + 2tuu_xu_t + tu_t^2\right)_x = 0; \tag{3.12}$$

(4) Galilean transformations ($\tilde{x} = x + \epsilon t$, $\tilde{t} = t$, $\tilde{u} = u + \epsilon$),

$$\left(u_x - tu_x^2\right)_t + \left(u_t + 2uu_x - tuu_x^2\right)_x = 0. \tag{3.13}$$

The conservation law (3.10) associated with translations in the phase variable $x$ is conservation of wave action. We make use of (3.10) in the next section; the remaining conservation laws (3.11)–(3.13) are not as useful because the conserved densities are not positive.

The last property of (1.5) discussed is the nonlinear instability of the constant solution $u = 0$. Consider a solution such that $u \to 0$ as $x \to +\infty$ and $u \to u_\infty(t)$ as $x \to -\infty$. Integrating (1.5) with respect to $x$ implies that

$$u'_\infty(t) = -\tfrac{1}{2}\|u_x\|^2(t), \tag{3.14}$$

where

$$\|u_x\|^2(t) = \int_{-\infty}^{+\infty} u_x^2(x,t)\,dx. \tag{3.15}$$

Since $u_x \to 0$ as $x \to -\infty$, integrating (3.10) over $\mathbf{R}$ shows that $\|u_x\|^2$ is constant. Therefore,

$$u_\infty(t) = u_\infty(0) - t\|u_x\|^2, \tag{3.16}$$

and $u_\infty$ grows linearly in time. Thus a compactly supported perturbation of $u = 0$ causes $|u(x,t)|$ to grow linearly in time for large negative $x$. This growth may change if singularities form in the solution, as we discuss at the end of the next section. The apparent nonlocal nature of the instability is due to the fact that the asymptotic equation (1.5) is unidirectional. We discuss the development of the instability for the two-way wave equation (1.1) in greater detail in §6.

**4. Weak solutions of the asymptotic equations.** Since (1.5) does not have global smooth solutions, we need to define weak solutions. The natural definition follows on multiplying (1.5) by a test function and integrating by parts. As we show by an explicit example below, weak solutions are not unique. We therefore also define an admissibility condition analagous to an entropy condition for hyperbolic conservation laws. The condition is motivated by the fact that smooth solutions of (1.5) satisfy the conservation law (3.10).

DEFINITION 4.1. A function $u(x,t)$ is a weak solution of (1.5) for $t > 0$ if $u \in H^1_{\text{loc}}(\mathbf{R} \times \mathbf{R}^+)$ and

$$\int \left[\phi_{xt} u + \tfrac{1}{2}\phi_{xx} u^2 - \tfrac{1}{2}\phi u_x^2\right] dxdt = 0 \tag{4.1}$$

for all test functions $\phi(x,t)$. The solution is weakly admissible if

$$\left(u_x^2\right)_t + \left(u u_x^2\right)_x \leq 0 \tag{4.2}$$

in the sense of distributions.

The admissibility condition (4.2) requires that $\|u_x\|(t)$ is decreasing; it restricts the appearance of corners in the solution.

The following explicit example shows that there are globally defined weak solutions. For $\alpha \geq 0$ and $t \neq 0$, we define $U(x,t;\alpha)$ by

$$U(x,t;\alpha) = \begin{cases} -\alpha t, & \text{if } -\infty < x \leq -\alpha t^2/2; \\ 2x/t, & \text{if } -\alpha t^2/2 < x < 0; \\ 0, & \text{if } 0 \leq x < +\infty. \end{cases} \tag{4.3}$$

We define $U(x,0;\alpha) = 0$. Then $U$ is a continuous function of $(x,t)$.

PROPOSITION 4.2. *The function*

$$u(x,t;\alpha,\beta) = \begin{cases} U(x,t;\alpha), & \textit{if } t \leq 0; \\ U(x,t;\beta), & \textit{if } t \geq 0. \end{cases} \tag{4.4}$$

*is a weak solution of* (1.5) *for any nonnegative constants $\alpha$ and $\beta$. It is a weakly admissible solution if and only if $\beta \leq \alpha$.*

*Proof.* Substituting the solution into (4.1) and using Fubini's theorem to evaluate the double integral directly shows that this is a weak solution. Moreover, a straightforward calculation shows that

$$\left(u_x^2\right)_t + \left(u u_x^2\right)_x = 2(\beta - \alpha)\delta(t)\delta(x).$$

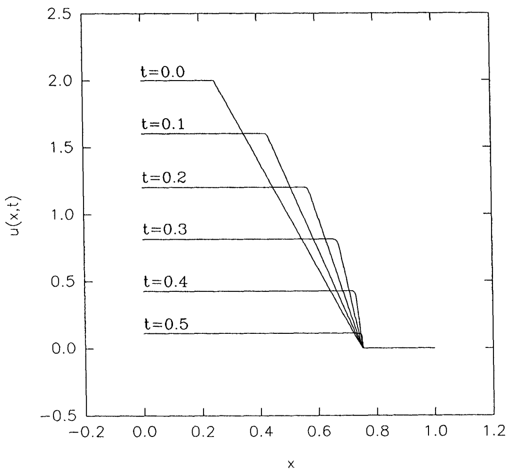


FIG. 1. *Numerical solution of* (5.1) *with initial data* (5.3).

Therefore the solution is weakly admissible when $\beta \leq \alpha$. We can also check that this solution satisfies (3.11) exactly in the sense of distributions. The reason is that $u = 0$ at the time $t = 0$ when $u_x$ blows up.

If $\alpha \neq 0$, the derivative of this solution blows up since $\inf u_x \to -\infty$ as $t \uparrow 0$. However, because the derivative is square integrable with respect to $x$, the length of the interval on which the derivative blows up shrinks to zero as $t \uparrow 0$. Moreover, the solution can be extended continuously past the blow-up time. A numerical computation of this solution is shown in Fig. 1.

Proposition 4.2 shows that weak solutions are not unique because corners can appear spontaneously in the solution. For example, choosing $\alpha = 0$ and $\beta > 0$ in (4.4) gives a nonzero weak solution which is identically zero for $t < 0$. Introducing new corners repeatedly leads to massive nonuniqueness. Even weakly admissible solutions are not unique, since weak admissiblity only requires $\beta \leq \alpha$ in (4.4). A stronger admissibility condition is therefore required. One possibility is to choose $\beta = 0$ in (4.4). This condition gives the smoothest solution possible for $t > 0$, and it is dissipative in the sense that strict inequality holds in (4.2) for nonsmooth solutions. We formulate an admissibility condition based on this idea in Definition 4.5, below. However, other admissiblity conditions are mathematically just as reasonable. For example, we can choose $\beta = \alpha$ in (4.4). This condition is conservative in the sense that the weak solutions satisfy (3.10) exactly. In this paper, we do not address the issue of which admissibility condition is most appropriate on physical grounds.

Before formulating a general admissibility condition, we show that patching together solutions like (4.4) gives global admissible weak solutions of (1.5) in a space $L_0$ of continuous piecewise linear functions.

DEFINITION 4.3. We say that $F \in L_0$ if and only if (a) $F : \mathbf{R} \to \mathbf{R}$ is continuous, piecewise linear, and consists of finitely many linear segments and (b) $F(x) = 0$ for $x$ sufficiently large and positive and $F(x) = \mu$ for some constant $\mu$ and for $x$ sufficiently large and negative.

Explicitly, $F \in L_0$ if and only if there are constants $\xi_1 > \cdots > \xi_N$, $\alpha_1, \cdots, \alpha_N$, and $\mu_1, \cdots, \mu_N$ such that $\mu_{n+1} - \mu_n = \alpha_n(\xi_{n+1} - \xi_n)$ and

$$F(x) = \begin{cases} 0, & \text{if } \xi_1 < x < +\infty, \\ \alpha_n(x - \xi_n) + \mu_n, & \text{if } \xi_{n+1} \le x \le \xi_n, \\ \mu_N, & \text{if } -\infty < x < \xi_N. \end{cases} \tag{4.5}$$

THEOREM 4.4. *For any initial data $F \in L_0$ there is a weakly admissible weak solution $u(x,t)$ of the initial-boundary value problem* (3.1), (3.9). *The solution is defined for all $t \ge 0$ and $u(\cdot, t) \in L_0$. Moreover,*

$$u_x(\cdot, t) : \mathbf{R}^+ \to L^1(\mathbf{R})$$

*is continuous.*

*Proof.* Consider the piecewise linear function

$$u(x,t) = \begin{cases} 0, & \text{if } x_1(t) < x < +\infty, \\ a_n(t)\left[x - x_n(t)\right] + u_n(t), & \text{if } x_{n+1}(t) \le x \le x_n(t), \\ u_N(t), & \text{if } -\infty < x < x_N(t). \end{cases} \tag{4.6}$$

This is a weak solution of (1.5) if the following conditions hold: (a) it is continuous; (b) it satisfies (1.5) pointwise for $x_{n+1}(t) < x < x_n(t)$; (c) the "corners" at $x = x_n(t)$ move at the local characteristic velocity $u_n(t)$. It follows that

$$u_{n+1} - u_n = a_n(x_{n+1} - x_n), \tag{4.7a}$$
$$\dot{a}_n + \tfrac{1}{2}a_n^2 = 0, \tag{4.7b}$$
$$\dot{x}_n = u_n. \tag{4.7c}$$

Solving these equations and imposing the initial condition (3.1b), (4.5), and the boundary condition (3.9) gives

$$\begin{aligned} a_n(t) &= \frac{2}{t - \tau_n}, \\ u_{n+1}(t) - u_n(t) &= -\frac{\delta_n}{a_n(t)}, \\ x_{n+1}(t) - x_n(t) &= -\frac{\delta_n}{a_n^2(t)}. \end{aligned} \tag{4.8}$$

Here, $u_1 = 0$, $x_1 = \xi_1$, and the constants of integration are defined by

$$\tau_n = -\frac{2}{\alpha_n}, \qquad \delta_n = \alpha_n(\mu_n - \mu_{n+1}) \ge 0.$$

Summing the telescoping series for $u_n$ and $x_n$ gives

$$\begin{aligned} u_n(t) &= -\sum_{k=1}^{n-1} \frac{\delta_k}{a_k(t)}, \\ x_n(t) &= -\sum_{k=1}^{n-1} \frac{\delta_k}{a_k^2(t)} + \xi_1. \end{aligned} \tag{4.9}$$

If $\tau_n < 0$ for all $1 \le n \le N$, then this solution is valid for all $t > 0$. Otherwise, the solution breaks down at $t = \tau_m$, where $\tau_m$ is a smallest positive element of $\{\tau_1, \cdots, \tau_N\}$.

It follows from (4.8) that $x_{m+1}(t) \uparrow x_m(t)$ as $t \uparrow \tau_m$. Thus as $t \uparrow \tau_m$ the solution approaches a piecewise linear function with one (or more) fewer linear segments than the initial data. We continue the solution past $t = \tau_m$ by using the limiting piecewise linear function as new initial data. Using a partition of unity, it is easy to check that this does give a weak solution.

The norm $\|u_x\|$ is conserved except at those times when a linear segment disappears from the solution. When segment $m$ disappears from the solution, the norm decreases by $\delta_m^{1/2}$ and (4.2) is satisfied. Thus the solution is admissible.

The continuity of the derivative $u_x(\cdot,t)$ into $L^1$ is immediate. It is worth noting that

$$u_x(\cdot,t) : \mathbf{R}^+ \to L^2(\mathbf{R})$$

is discontinuous at those times when a linear segment disappears from the solution.

This theorem suggests a way to analyze weak solutions with arbitrary initial data. Namely, approximate the initial data by piecewise linear functions, solve the resulting initial value problems, and show that the limit of a convergent subsequence of piecewise linear solutions is a weak solution of the original problem. We plan to carry out a detailed analysis in another paper [4].

Theorem 4.4 also suggests a stronger admissibility condition. The construction in the theorem picks out special solutions because we remove linear segments from the solutions whenever possible, but we never add segments. Not all solutions that are admissible in the sense of (4.2) are obtained in this way. For example, only the solution in Proposition 4.2 with $\beta = 0$ is so obtained. We also state a second admissibility criterion, based on a vanishing viscosity limit.

DEFINITION 4.5. A weak solution of (1.5) is *strongly admissible* if it is the limit in $H^1_{\text{loc}}(\mathbf{R} \times \mathbf{R}^+)$ of a sequence of piecewise linear solutions of the type constructed in Theorem 4.4. A solution is *viscously admissible* if it is the limit in $H^1_{\text{loc}}(\mathbf{R} \times \mathbf{R}^+)$ as $\epsilon \to 0+$ of solutions $u^\epsilon$ of the regularized equations

$$\left(u^\epsilon_t + u^\epsilon u^\epsilon_x - \epsilon u^\epsilon_{xx}\right)_x = \tfrac{1}{2}(u^\epsilon_x)^2. \tag{4.10}$$

Multiplying (4.10) by $u^\epsilon_x$ and rearranging gives

$$\left[(u^\epsilon_x)^2\right]_t + \left[u^\epsilon (u^\epsilon_x)^2\right]_x - \epsilon\left[(u^\epsilon_x)^2\right]_{xx} = -\epsilon(u^\epsilon_{xx})^2.$$

Taking the limit as $\epsilon \to 0+$ of the weak form of this equation implies (4.2). Thus a viscously admissible solution is weakly admissible. We conjecture that a viscously admissible solution is strongly admissible.

A piecewise linear solution (4.6), (4.8) approaches $U(x,t;\alpha)$ defined in (4.3) as $t \to +\infty$ for some value of $\alpha \geq 0$. This is because after a finite time all segments with negative slope, i.e., with $\tau_m > 0$, disappear from the solution, while if $\tau_m < 0$, the slope of the segment approaches $2/t$ for large times. We conjecture that (4.3) gives the large time behavior of any strongly admissible solution. This conjecture is supported by the numerical results shown in §5.

For nonsmooth solutions, $u_\infty(t) = u(-\infty,t)$ need not be given by (3.16). Integrating (3.14) gives

$$u_\infty(t) = u_\infty(0) - \int_0^t \|u_x\|^2(s)\,ds.$$

For an admissible solution, $\|u_x\|$ is nonincreasing, so this equation implies that

$$-t\|u_x\|^2(0) \leq u_\infty(t) - u_\infty(0) \leq 0.$$

If the solution approaches (4.3) as $t \to +\infty$, then since $\|U_x\|^2 = 2\alpha$,

$$u_\infty(t) \sim 2\alpha t \tag{4.11}$$

as $t \to +\infty$. Thus, for $\alpha \neq 0$, $u_\infty$ grows linearly in time.

Last, we give an explicit weak solution of (1.5), which has a singularity in the derivative,

$$u(x,t) = \begin{cases} \dot{s}_0(t) + \alpha\big(x - s_0(t)\big)^{2/3}, & \text{if } s_0(t) < x < +\infty, \\ \dot{s}_0(t) + \beta\big(s_0(t) - x\big)^{2/3}, & \text{if } -\infty < x < s_0(t). \end{cases} \tag{4.12}$$

Equation (4.12) defines a weak solution of (1.5) with an algebraic singularity at $x = s_0(t)$ for any constants $\alpha$ and $\beta$ and any continuously differentiable function $s_0(t)$. The singularity moves with the local characteristic velocity and $u_x$ is locally square integrable. More general solutions can be constructed by patching together solutions of this type with each other and with piecewise linear solutions.

## 5. Numerical solutions of the asymptotic equation.

The analytic solution (3.2) does not give an explicit solution of (1.5) for $u(x,t)$. Moreover, it breaks down once singularities form. We therefore compute some numerical solutions of (1.5). There is no evidence of shock formation. We write (1.5), with boundary condition (3.9) in the form

$$u_t + uu_x = -\frac{1}{2}\int_x^{+\infty} u_x^2\, dx. \tag{5.1}$$

We use the Enquist–Osher scheme to difference the left-hand side of (5.1), treating the right-hand side as an explicit source term. We denote the numerical approximation of $u(mh, nk)$ by $u_m^n$, where $h$ is the grid spacing, $k$ is the timestep, and $1 \leq m \leq M$. In the results shown below, we use $M = 2000$ grid points. The difference scheme is

$$u_m^{n+1} - u_m^n + r\left[f(u_{m+1}^n, u_m^n) - f(u_m^n, u_{m-1}^n)\right] = \frac{1}{2} r \sum_{k=m}^{M} \left(u_{k+1}^n - u_k^n\right)^2. \tag{5.2}$$

Here, $r = k/h$ and the numerical flux $f$ is

$$f(u,v) = \tfrac{1}{2}[\min(u,0)]^2 + \tfrac{1}{2}[\max(v,0)]^2.$$

For numerical boundary conditions we use $u_x = 0$; that is,

$$u_0^n = u_1^n, \qquad u_{M+1}^n = u_M^n.$$

In Fig. 1, we show a numerical solution with initial data corresponding to the solution in (4.4),

$$u(x,0) = \begin{cases} 2, & \text{if } -\infty < x \leq 0.25, \\ 3 - 4x, & \text{if } 0.25 < x < 0.75, \\ 0, & \text{if } 0.75 \leq x < +\infty. \end{cases} \tag{5.3}$$

The exact solution for $0 < t < 0.5$ is $u(x,0) = U(x - 0.75, t - 0.5; 4.0)$, where $U$ is defined in (4.4). The solution tends uniformly to zero as $t \uparrow 0.5$. Numerical dissipation

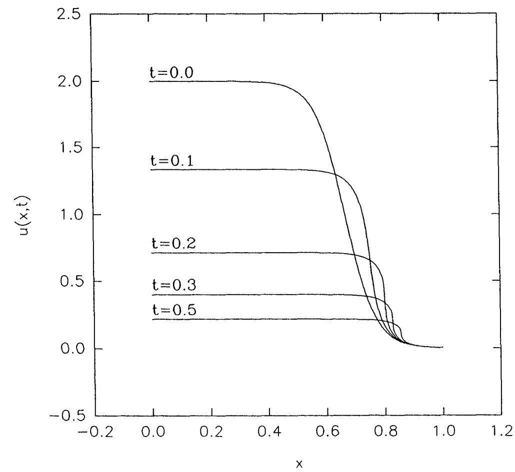


FIG. 2. *Numerical solution of* (5.1) *with initial data* (5.4).

smooths out the corners in the initial data. As a result, the numerical solution is not exactly zero at $t = 0.5$, although it does reproduce the general features of the analytical solution.

For $t > 0.5$, the numerical solution for (5.3) remains small and tends uniformly to 0 as $t \to +\infty$. The numerical scheme therefore approximates the solution (4.4) with $\beta = 0$. Numerical runs show that if we approximate given initial data by a piecewise linear function, then the numerical solution for the piecewise linear initial data approximates the numerical solution for the given initial data. This suggests that the finite-difference scheme (5.2) picks out strongly admissible solutions as $h \to 0$.

In Fig. 2, we show a numerical solution for monotone decreasing smooth initial data

$$u(x,0) = 1 - \tanh\left(\frac{x - \frac{2}{3}}{0.1}\right). \tag{5.4}$$

The numerical solution tends uniformly to zero. According to Theorem 3.2, the first derivative of the exact solution blows up at $t = 0.2$ for this initial data. The numerical solution is continuous, but it is consistent with there being a point $x_*(t)$ for $t \geq 0.2$, where $u_x\big(x_*(t), t\big) = -\infty$.

In Fig. 3, we show a numerical solution for Gaussian initial data,

$$u(x,0) = 2\exp\big(-y^2\big), \tag{5.5}$$

where $y = (x - 0.5)/0.15$. In this case, it appears that the solution approaches one of the weak solutions $U(x,t;\alpha)$ as $t \to +\infty$ with $\alpha > 0$. This behaviour is typical of other initial data that is not monotone decreasing. Another example is shown in Fig. 4 for initial data

$$u(x,0) = 2z\exp(-z^2) + 1 - \tanh z, \tag{5.6}$$

where $z = (x - 0.5)/0.1$.

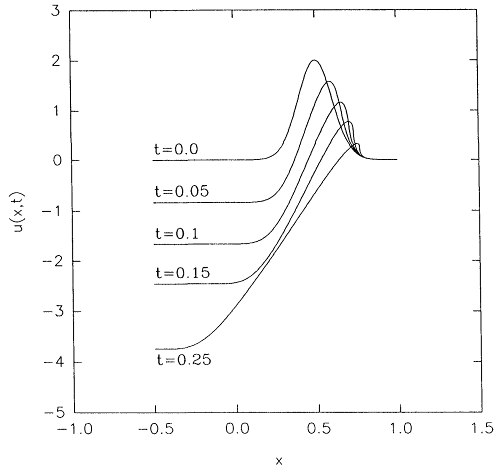


FIG. 3. *Numerical solution of* (5.1) *with initial data* (5.5).

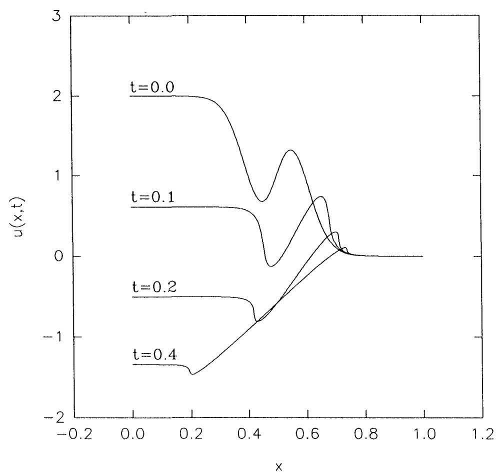


FIG. 4. *Numerical solution of* (5.1) *with initial data* (5.6).

**6. Weakly nonlinear development of instabilities in a director field.** In this section we derive an asymptotic approximation as $\epsilon \to 0$ of the solution of

$$\begin{aligned} &\psi_{tt} - c(\psi)\left[c(\psi)\psi_x\right]_x = 0, \\ &\psi(x,0;\epsilon) - \psi_0 = \epsilon f(x), \\ &\psi_t(x,0;\epsilon) = 0. \end{aligned} \tag{6.1}$$

Here, $f(x)$ is a compactly supported function that describes a small localized perturbation of a uniform director field $\psi = \psi_0$. We assume that the initial velocity is zero to simplify the presentation. Our objective is to show how the waves that propagate away from the perturbation "knock" the director field out of its unperturbed state. We use nondimensional variables in which the width of the perturbation and the unperturbed wavespeed $c_0 = c(\psi_0)$ are of the order one.

For $t = O(1)$, the leading-order approximation to $\psi$ is given by linearizing (6.1). Using d'Alembert's solution of the wave equation gives

$$\psi(x,t;\epsilon) - \psi_0 \sim \tfrac{1}{2}\epsilon\big[f(x - c_0 t) + f(x + c_0 t)\big]. \tag{6.2}$$

Thus the perturbation splits into two pulses moving to the left and right.

Over times $t = O(\epsilon^{-1})$, weakly nonlinear effects modify the pulses. By symmetry, it suffices to consider the right-moving pulse located near $x = c_0 t$. For $x - c_0 t = O(1)$ and $t = O(\epsilon^{-1})$, $\psi$ is described asymptotically by (2.1) and (2.3), where we assume that $c_0' \neq 0$. Matching with the linearized approximation (6.2) for $1 \ll t \ll \epsilon^{-1}$, gives an initial condition for (2.3),

$$\psi_1(\theta, 0) = \tfrac{1}{2} f(\theta). \tag{6.3}$$

Since the director field ahead of the pulse is undisturbed, we also require that

$$\psi_1(\theta,\tau) \to 0 \quad \text{as} \quad \theta \to +\infty. \tag{6.4}$$

Solving (2.3), (6.3), and (6.4) defines a function $\psi_\infty(\tau)$ where $\psi_1(\theta,\tau) \to \psi_\infty(\tau)$ as $\theta \to -\infty$. From (4.11), we expect that $\psi_\infty \sim \alpha\tau$ as $\tau \to +\infty$ for some constant $\alpha$.

Next, we consider the behavior of the solution for $t = O(\epsilon^{-1})$ and $x/t =$ constant. If $|x/t| > c_0$, then the director field is undisturbed and $\psi = \psi_0$. If $x/t = \pm c_0 + O(\epsilon)$, then the weakly nonlinear solution (2.1), (2.3) applies. Between the pulses, when $|x/t| < c_0$, we look for a solution of the form

$$\psi(x,t;\epsilon) = \psi_0 + \epsilon\tilde{\psi}(\xi,\tau) + O(\epsilon^2), \tag{6.5}$$

where $\xi = \epsilon x$ and $\tau = \epsilon t$. For $\tau = O(1)$, $\tilde{\psi}$ satisfies the linearized wave equation

$$\tilde{\psi}_{\tau\tau} = c_0^2 \tilde{\psi}_{\xi\xi}. \tag{6.6}$$

Matching with the weakly nonlinear solution implies that

$$\tilde{\psi}(c_0\tau,\tau) = \tilde{\psi}(-c_0\tau,\tau) = \psi_\infty(\tau). \tag{6.7}$$

Since $\psi_\infty(0) = 0$, the solution of (6.6) and (6.7) is

$$\tilde{\psi}(\xi,\tau) = \psi_\infty\big[\tfrac{1}{2}(\tau - c_0^{-1}\xi)\big] + \psi_\infty\big[\tfrac{1}{2}(\tau + c_0^{-1}\xi)\big]. \tag{6.8}$$

According to (6.5) and (6.8), the right-moving pulse emits a small, long left-moving wave whose amplitude grows with time. If the pulse has amplitude order $\epsilon$ and width $\lambda$, the wave has amplitude order $\epsilon$ and wavelength of the order $(\lambda c_0)/(\epsilon c_0')$. Similarly, the left-moving pulse emits a right-moving wave, and the director field between the pulses grows algebraically in time away from its unperturbed state.

An example where the asymptotic solution can be explicitly computed is shown in Fig. 5. The initial perturbation $f$ in (6.1) is

$$f(x;\lambda) = \begin{cases} 0, & \text{if } |x| \geq \lambda; \\ 2(1 - |x|/\lambda), & \text{if } |x| < \lambda. \end{cases} \tag{6.9}$$

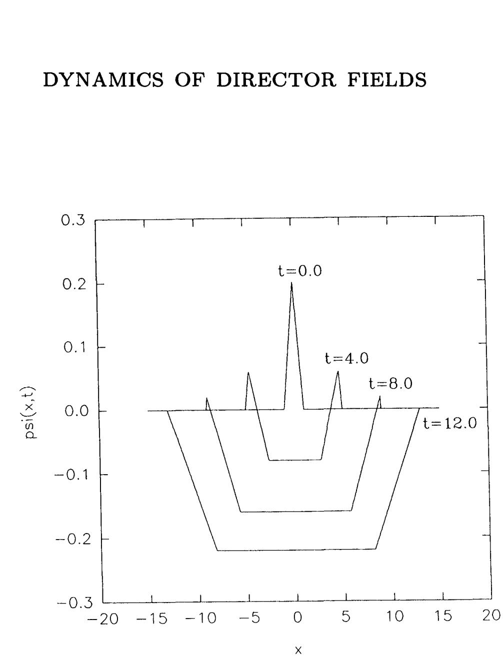


FIG. 5. *The asymptotic solution* (6.10)–(6.13) *of* (1.1). *The parameter values are* $\psi_0 = 0$, $c_0 = \lambda = 1$, $c_0' = 2$, *and* $\epsilon = 0.1$.

This initial data is piecewise linear, so the solution of (2.3), (6.3), (6.4), and (6.9) can be found by following the proof of Theorem 4.4. To write the result, let $\tau_* = 2\lambda/c_0'$ and $\sigma = \tau/\tau_*$. Then for $\tau < \tau_*$ the solution is

$$\psi_1(\theta,\tau) = \begin{cases} 0, & \text{if } \lambda \le \theta; \\ 2[\theta - \lambda]/[c_0'\tau_*(\sigma - 1)], & \text{if } \theta_2 \le \theta \le \lambda; \\ 2[\theta + \lambda - 2\lambda\sigma]/[c_0'\tau_*(\sigma + 1)], & \text{if } \theta_3 \le \theta \le \theta_2; \\ -2\sigma, & \text{if } \theta \le \theta_3. \end{cases}$$

The solution has "corners" at $\theta = \lambda$, $\theta = \theta_2(\tau)$, and $\theta = \theta_3(\tau)$, where $\theta_2 = \lambda\sigma(2-\sigma)$, $\theta_3 = -\lambda(1+2\sigma^2)$. The corners at $\theta = \lambda$ and $\theta = \theta_2(\tau)$ collide when $\tau = \tau_*$. For $\tau \ge \tau_*$ the solution is

$$\psi_1(\theta,\tau) = \begin{cases} 0, & \text{if } \lambda \le \theta; \\ 2[\theta - \lambda]/[c_0'\tau_*(\sigma + 1)], & \text{if } \hat\theta_3 \le \theta \le \lambda; \\ -(\sigma + 1), & \text{if } \theta \le \hat\theta_3. \end{cases}$$

Here, $\hat\theta_3 = -\lambda\sigma(2+\sigma)$.

Writing this solution in terms of the original space-time variables, we obtain an asymptotic solution of (6.1) with initial data (6.9). It is valid for $\epsilon c_0'/c_0 \ll 1$ and for times $t \sim t_*$, where $t_* = 2\lambda/(\epsilon c_0') \gg \lambda/c_0$ is the characteristic timescale of the instability. We assume that $\epsilon c_0' > 0$ when $t_* > 0$. By symmetry $\psi(-x,t) = \psi(x,t)$, so we only write out the solution for $x \ge 0$. We also introduce a dimensionless time variable $s = t/t_*$. For $t < t_*$,

$$\psi(x,t) - \psi_0 \sim \begin{cases} 0, & \text{if } x_1 \le x; \\ 2[x - c_0t - \lambda]/[c_0't_*(s-1)], & \text{if } x_2 \le x \le x_1; \\ 2[x - c_0t + \lambda - 2\lambda s]/[c_0't_*(s+1)], & \text{if } x_3 \le x \le x_2; \\ -2\epsilon s, & \text{if } 0 \le x \le x_3. \end{cases} \tag{6.10}$$

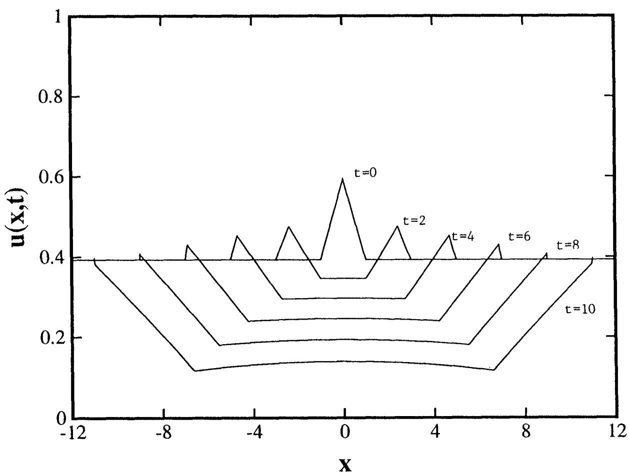


FIG. 6. *Numerical solution of* (1.1), (6.14), *and* (6.15). *Figure* 5 *shows the corresponding weakly nonlinear asymptotic solution.*

The corners are located at

$$\begin{aligned} x_1 &= c_0 t + \lambda, \\ x_2 &= c_0 t + \lambda s(2-s), \\ x_3 &= c_0 t - \lambda(1+2s^2). \end{aligned} \tag{6.11}$$

For $t > t_*$ the asymptotic solution is

$$\psi(x,t) - \psi_0 \sim \begin{cases} 0, & \text{if } x_1 \le x; \\ 2[x - c_0 t - \lambda]/[c_0' t_*(s+1)], & \text{if } \hat{x}_3 \le x \le x_1; \\ -\epsilon(s+1), & \text{if } 0 \le x \le \hat{x}_3. \end{cases} \tag{6.12}$$

Here,

$$\hat{x}_3 = c_0 t - \lambda s(2+s). \tag{6.13}$$

The main point of this solution is that in the region between the pulses, which is $|x| < \hat{x}_3(t)$ for $t > t_*$, the perturbation grows linearly in time in the direction of decreasing wavespeed like $\psi(x,t) - \psi_0 \sim -\epsilon\big(1 + t/t_*\big)$.

To verify the accuracy of this asymptotic solution, we show a numerical solution of the two-way wave equation (1.1) in Fig. 6. The initial data is

$$\begin{aligned} \psi(x,0) &= \frac{\pi}{8} + 0.1 f(x;1), \\ \psi_t(x,0) &= 0. \end{aligned} \tag{6.14}$$

Here, $f(x;\lambda)$ is defined in (6.9). The wavespeed in the numerical solution is

$$c^2(\psi) = (3 - 2\sqrt{2})\cos^2\psi + 2\sqrt{2}\sin^2\psi. \tag{6.15}$$

For this speed, $c(\pi/8) = 1.0$ and $c'(\pi/8) = 2.0$. These values are the same as the parameters used in the asymptotic solution plotted in Fig. 5. The agreement between Figs. 5 and 6 (with $\pi/8$ subtracted from $\psi$) is excellent.

**Appendix A. Derivation of the nonlinear liquid crystal wave equation.** We will begin by outlining the Ericksen–Leslie continuum description of liquid crystal theory, as given in [1], [2]. The description consists of a system of equations describing two independent vector fields $\nu(\mathbf{u},t)$ and $\mathbf{v}(\mathbf{u},t)$. Here the independent variable $\mathbf{u} \in \mathbf{R}^3$ is a spatial quantity, $\nu \in S^2$ defines the orientation of a director, and $\mathbf{v} \in \mathbf{R}^3$ is the local fluid velocity. We let $\mathbf{x} \in \mathbf{R}^3$ represent a material coordinate. Then $\mathbf{u}$ and $\mathbf{v}$ are related through

$$\begin{aligned} \dot{\mathbf{u}}(\mathbf{x},t) &= \mathbf{v}(\mathbf{u}(\mathbf{x},t),t), \\ \mathbf{u}(\mathbf{x},0) &= \mathbf{u}_0(\mathbf{x}). \end{aligned} \tag{A.1}$$

We also set

$$\mathbf{n}(\mathbf{x},t) = \nu(\mathbf{u}(\mathbf{x},t),t), \tag{A.2}$$

$$\dot{\mathbf{n}}(\mathbf{x},t) = \mathbf{w}(\mathbf{u}(\mathbf{x},t),t), \tag{A.3}$$

and $\mathbf{n}(\mathbf{x},0) = \mathbf{n}_0(\mathbf{x})$. Then

$$\mathbf{w}(\mathbf{u},t) = \nu_t + (\mathbf{v}\cdot\nabla)\nu. \tag{A.4}$$

The fluid component is conventionally regarded as incompressible, while the director field is a map into the unit sphere $S^2$. We therefore have the constraints

$$\nabla\cdot\mathbf{v} = 0, \tag{A.5}$$

$$|\nu| = |\mathbf{n}| = 1. \tag{A.6}$$

Setting the constant fluid density equal to unity, the equations of motion are [2]

$$\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla\cdot T = -\nabla p, \tag{A.7}$$

$$\rho[\mathbf{w}_t + (\mathbf{v}\cdot\nabla)\mathbf{w}] + \mathbf{H} = \lambda\nu. \tag{A.8}$$

Here, $\rho$ is a measure of the director inertia, which is often asumed to be negligable. Our interest is in examining the effects that this term can cause. We therefore set $\rho = 1$, subsequently. The fluid pressure $p$ and the scalar $\lambda$ are Lagrange multipliers corresponding to the constraints (A.5) and (A.6), respectively. The extra stress tensor $T$ and the force vector $\mathbf{H}$ are given by

$$T_{ij} = -\frac{\partial W}{\partial\,\partial_i\nu_k}\partial_j\nu_k + \frac{\partial\Delta}{\partial\,\partial_i v_j}, \tag{A.9}$$

$$H_i = \frac{\partial W}{\partial\nu_i} - \partial_k\frac{\partial W}{\partial\,\partial_k\nu_i} + \frac{\partial\Delta}{\partial w_i}. \tag{A.10}$$

Here

$$W = W(\nu,\nabla\nu), \qquad \Delta = \Delta(\nu,\mathbf{w},\nabla\mathbf{v}), \tag{A.11}$$

where $W$ is the internal stored energy of deformation of the director field, and $\Delta$ measures the energy dissipation. The functional form of $\Delta$ is restricted by the condition of positive dissipation [2],

$$\frac{\partial\Delta}{\partial\,\partial_i v_j}\partial_i v_j + \frac{\partial\Delta}{\partial w_i}w_i \geq 0. \tag{A.12}$$

Generally, equality in (A.12) is assumed to hold only for rigid body motions. In this paper, we neglect dissipative effects, so we set $\Delta \equiv 0$. Including dissipative effects will be helpful for picking out a unique solution after the formation of singularities.

When $\Delta = 0$, (A.7) and (A.8), with $\rho = 1$, are the Euler–Lagrange equations corresponding to the Lagrangian

$$\mathcal{L} = \int_{t_1}^{t_2} \int \left\{ \tfrac{1}{2}\left(|\mathbf{v}|^2 + |\mathbf{w}|^2\right) - W(\nu, \nabla\nu) + p\nabla\cdot\mathbf{v} + \tfrac{1}{2}\lambda|\nu|^2 \right\} d\mathbf{u}\, dt. \tag{A.13}$$

Equivalently, using (A.1)–(A.3) and (A.5),

$$\mathcal{L} = \int_{t_1}^{t_2} \int \left\{ \tfrac{1}{2}\left(|\dot{\mathbf{u}}|^2 + |\dot{\mathbf{n}}|^2\right) - W(\mathbf{n}, (\nabla\nu)\circ u) + p\circ u\cdot \det\nabla\mathbf{u} + \tfrac{1}{2}\lambda\circ u\cdot|\mathbf{n}|^2 \right\} d\mathbf{x}\, dt. \tag{A.14}$$

In (A.13), $\nu$ and $\mathbf{v}$ are independent vector fields. In (A.14) they are replaced by $\mathbf{n}$ and $\dot{\mathbf{u}}$ or by $\mathbf{n}$ and $\mathbf{u}$. It is simpler to take the derivatives of $\mathcal{L}$ with respect to $\mathbf{n}$ and $\mathbf{u}$ and transform to the spatial setting than to calculate $\delta_\nu\mathcal{L}$ and $\delta_\mathbf{v}\mathcal{L}$. The only slightly nontrivial calculation involves finding the variation with respect to $\mathbf{u}$ of $(\nabla\nu)\circ\mathbf{u}$, with $\mathbf{n}$ held constant.

We let

$$\mathbf{u}_\epsilon = \mathbf{u} + \epsilon\phi, \qquad \phi = \frac{d}{d\epsilon}\mathbf{u}_\epsilon\Big|_{\epsilon=0} = \frac{d^0}{d\epsilon}\mathbf{u}_\epsilon. \tag{A.15}$$

Then, using (A.15),

$$\begin{aligned}
\frac{d^0}{d\epsilon}\Big[(\nabla\nu)\circ u_\epsilon\Big]_{ij} &= \left(\frac{d^0}{d\epsilon}\frac{\partial\nu_i(u+\epsilon\phi)}{\partial(u_j+\epsilon\phi_j)}\right)\circ u \\
&= \left(\frac{d^0}{d\epsilon}\frac{\partial\nu_i}{\partial u_k}\frac{\partial u_k}{\partial(u_j+\epsilon\phi_j)}\right)\circ u \\
&= \left(\frac{d^0}{d\epsilon}\frac{\partial\nu_i}{\partial u_k}\left(\delta_{jk} - \epsilon\frac{\partial\phi_k}{\partial(u_j+\epsilon\phi_j)}\right)\right)\circ u \\
&= -\left(\frac{\partial\nu_i}{\partial u_k}\frac{\partial\phi_k}{\partial u_j}\right)\circ u.
\end{aligned} \tag{A.16}$$

So, for example in (A.14), since $\mathbf{n}_\epsilon = \mathbf{n}$ ($\mathbf{n}$ is held fixed for variations of $\mathbf{u}$), we have that

$$\begin{aligned}
\frac{d^0}{d\epsilon}\int_{t_1}^{t_2}\int W\left(\mathbf{n}_\epsilon, (\nabla\nu)\circ u_\epsilon\right) d\mathbf{x}dt &= \int_{t_1}^{t_2}\int \frac{\partial W}{\partial\,\partial_j\nu_i}\circ u\,\frac{d^0}{d\epsilon}\Big[(\nabla\nu)\circ u_\epsilon\Big]_{ij}\, d\mathbf{x}\, dt \\
&= \int_{t_1}^{t_2}\int \frac{\partial}{\partial u_j}\left(\frac{\partial W}{\partial\,\partial_j\nu_i}\frac{\partial\nu_i}{\partial u_k}\right)\phi_k\, d\mathbf{u}\, dt.
\end{aligned} \tag{A.17}$$

From (A.9), for $\Delta = 0$, this is

$$\int_{t_1}^{t_2}\int \left(\nabla\cdot T\right)_k \phi_k\, d\mathbf{u}\, dt, \tag{A.18}$$

which gives the third term in (A.7). The remaining terms in (A.7) and (A.8) are found more straightforwardly [1].

We now specialize to the case in which there is no fluid motion, when $\mathbf{u}(\mathbf{x},t) = \mathbf{u}_0(\mathbf{x})$. We take $\mathbf{u}_0(\mathbf{x}) = \mathbf{x}$ to be the identity for convenience. It follows from (A.1) and (A.2) that $\mathbf{v} = 0$ and $\nu = \mathbf{n}$. Equations (A.7) and (A.8) then reduce to

$$\partial_j \left( \frac{\partial W}{\partial\, \partial_j n_k} \partial_i n_k \right) = \partial_i p, \tag{A.19}$$

$$\ddot{n}_i + \frac{\partial W}{\partial n_i} - \partial_j \frac{\partial W}{\partial\, \partial_j n_i} = \lambda n_i. \tag{A.20}$$

We remark that if $\Delta \neq 0$ then the "viscosity" term $\partial \Delta / \partial \dot{n}_i$ that would appear in (A.20) is of the form $\gamma \dot{n}_i$. This is a lower-order damping term, which would not be expected to smooth out singularities.

We can use the constraint (A.6) to eliminate the Lagrange multiplier $\lambda$ in (A.20). This gives

$$\lambda = -\dot{n}_i \dot{n}_i + n_i \frac{\partial W}{\partial n_i} + \partial_j n_i \frac{\partial W}{\partial\, \partial_j n_i} - \partial_j \left( n_i \frac{\partial W}{\partial\, \partial_j n_i} \right). \tag{A.21}$$

Equations (A.19)–(A.21) are our equations for the director field.

Now we examine the form of the internal energy $W(\mathbf{n}, \nabla \mathbf{n})$. For nematic liquid crystals, $W$ is invariant under spatial rotations and reflections [2]. We use the Oseen–Frank form of $W$, which is given by

$$W(\mathbf{n}, \nabla \mathbf{n}) = \tfrac{1}{2} \left( k_1 (\nabla \cdot \mathbf{n})^2 + k_2 (\mathbf{n} \cdot \nabla \times \mathbf{n})^2 + k_3 |\mathbf{n} \times (\nabla \times \mathbf{n})|^2 \right), \tag{A.22}$$

where $k_1$, $k_2$, and $k_3$ are nonnegative constants. Sometimes, a null-Lagrangian is added to the right-hand side of (A.22). The system is said to undergo "splay," "twist," or "bend" if the first, second, or third term, respectively, on the right-hand side of (A.22) is the only nonzero term.

In this paper, we consider a simple class of director fields, $\mathbf{n}(x,y,z,t) = \mathbf{u}(x,t)$, $\mathbf{u} \cdot \mathbf{e}_z = 0$. Here, $\mathbf{e}_z$ is the unit vector in the z-direction and $\mathbf{u}$ is unrelated to the particle position earlier. Setting

$$\mathbf{u}(x,t) = \cos \psi(x,t) \mathbf{e}_x + \sin \psi(x,t) \mathbf{e}_y \tag{A.23}$$

gives

$$W(\mathbf{n}, \nabla \mathbf{n}) = \tfrac{1}{2} \left( k_1 \sin^2 \psi + k_3 \cos^2 \psi \right) \psi_x^2. \tag{A.24}$$

In particular, $\mathbf{u} \cdot (\nabla \times \mathbf{u}) = 0$, so there is no twist. If $\psi_x \neq 0$, the internal energy is minimized either at $\psi = 2n\pi$ (no splay) or at $\psi = (2n+1/2)\pi$ (no bending), depending on the relative magnitudes of $k_1$ and $k_3$.

It is simple to derive the equation of motion corresponding to (A.20) and (A.21) using the Lagrangian derived from (A.24). We set $k_1 = \beta$ and $k_3 = \alpha$ and define $c^2(\psi) = \alpha \cos^2 \psi + \beta \sin^2 \psi$. Observing that $|\dot{\mathbf{u}}|^2 = \dot{\psi}^2$, the corresponding Lagrangian is

$$\mathcal{L} = \int_{t_1}^{t_2} \int \tfrac{1}{2} \left( \dot{\psi}^2 - c^2(\psi) \psi_x^2 \right) dx dt.$$

The Euler–Lagrange equations are

$$\ddot{\psi} - c(\psi)\Big[c(\psi)\psi_x\Big]_x = 0. \tag{A.25}$$

**Appendix B. Nonlinear geometrical optics for variational principles with quadratic Lagrangians.** We consider a variational principle

$$\delta \int A^{ij}_{pq}(x,u)\frac{\partial u^p}{\partial x^i}\frac{\partial u^q}{\partial x^j}\,dx = 0, \tag{B.1}$$

where we use the summation convention. Here, $x \in \mathbf{R}^{n+1}$ are the space-time variables and $u : \mathbf{R}^{n+1} \to \mathbf{R}^m$ are the dependent variables. We assume that the coefficients $A^{ij}_{pq} : \mathbf{R}^{n+1} \times \mathbf{R}^m \to \mathbf{R}$ are smooth and, without loss of generality, that $A^{ij}_{pq} = A^{ji}_{pq} = A^{ij}_{qp}$.

It is straightforward to extend our analysis to include a constraint of the form $f(x^i, u^p) = 0$ (as we have in the variational problem for an arbitrary director field). Since this does not alter the form of the final result, we omit the details here.

The Euler–Lagrange equations for (B.1) are

$$\frac{\partial}{\partial x^i}\left(A^{ij}_{kp}\frac{\partial u^p}{\partial x^j}\right) = \frac{1}{2}\frac{\partial A^{ij}_{pq}}{\partial u^k}\frac{\partial u^p}{\partial x^i}\frac{\partial u^q}{\partial x^j}. \tag{B.2}$$

Equation (B.2) is scale invariant, meaning that it is invariant under the scale transformations $x \to \alpha x$. Thus the equation does not define any intrinsic length or timescales. This is a consequence of the fact that the Lagrangian density in (B.1) is a homogeneous quadratic function of $Du$. For example, the Oseen–Frank internal energy (A.22) is a quadratic function of $\nabla \mathbf{n}$, while the kinetic energy is a quadratic function of $\mathbf{n}_t$.

The characteristic equation of (B.2) is

$$\det\Big(\kappa_i\kappa_j A^{ij}_{pq}\Big) = 0. \tag{B.3}$$

Equation (B.2) is strictly hyperbolic, with $x^0$ as a time-like direction, if (B.3) has $2m$ distinct real roots for $\kappa_0$ for every $(\kappa_1, \cdots, \kappa_n) \in \mathbf{R}^n$. We assume that (B.3) is strictly hyperbolic at $u = 0$. This assumption is stronger than what we actually use below.

Weakly nonlinear geometrical optics [3] for first-order systems of quasilinear hyperbolic equations leads to an inviscid Burgers equation (1.7). Equation (B.3) is a second-order system, so these results do not directly apply to it. However, a similar expansion leads to (1.5) instead, as we now show. We look for small-amplitude, high-frequency asymptotic solutions of (B.2) of the form

$$u(x;\epsilon) = \epsilon u_1\big[x, \epsilon^{-1}\phi(x)\big] + \epsilon^2 u_2\big[x, \epsilon^{-1}\phi(x)\big] + O(\epsilon^3). \tag{B.4}$$

In (B.4), $\phi(x)$ is a phase variable. We define the wavenumber vector $\kappa = (\kappa_0, \cdots, \kappa_n) : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ by

$$\kappa_i = \frac{\partial \phi}{\partial x^i}. \tag{B.5}$$

We assume that $\kappa \neq 0$. Using (B.4) in (B.2), expanding the coefficients in Taylor series, and equating coefficients of $\epsilon$ and $\epsilon^2$ shows that $u_1(x,\theta)$ and $u_2(x,\theta)$ satisfy the equations

$$\kappa_i\kappa_j A^{ij}_{kp}\frac{\partial^2 u_1^p}{\partial \theta^2} = 0, \tag{B.6}$$

$$\kappa_i\kappa_j A^{ij}_{kp}\partial^2 u^p_2 \; over \partial\theta^2 + 2\kappa_i A^{ij}_{kp}\frac{\partial^2 u^p_1}{\partial\theta\partial x^j} + \frac{\partial}{\partial x^j}\left(\kappa_i A^{ij}_{kp}\right)\frac{\partial u^p_1}{\partial\theta} + \kappa_i\kappa_j\frac{\partial A^{ij}_{kp}}{\partial u^q}\frac{\partial}{\partial\theta}\left(u^q_1\frac{\partial u^p_1}{\partial\theta}\right) = \frac{1}{2}\kappa_i\kappa_j\frac{\partial A^{ij}_{pq}}{\partial u^k}\frac{\partial u^p}{\partial\theta}\frac{\partial u^q}{\partial\theta}. \tag{B.7}$$

Here and below, all coefficients and their derivatives are evaluated at $u = 0$. To simplify the notation, we do not indicate this explicitly.

Equation (B.6) implies that the wavenumber vector (B.5) satisfies the characteristic equation (B.3). This means that the phase $\phi(x)$ solves the linearized eikonal equation associated with (B.2), namely,

$$\det\left[\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^j}A^{ij}_{pq}(x,0)\right] = 0.$$

By strict hyperbolicity, there is a one-dimensional nullspace associated with $\kappa$, so the solution of (B.6) for $u_1$ is

$$u_1(x,\theta) = a(x,\theta)r(x). \tag{B.8}$$

In (B.8), $a : \mathbf{R}^{n+1}\times\mathbf{R} \to \mathbf{R}$ is an arbitrary scalar, which describes the wave amplitude and

$$r = (r^1,\cdots,r^m) : \mathbf{R}^{n+1} \to \mathbf{R}^m$$

is a null vector such that $\kappa_i\kappa_j A^{ij}_{kp}r^p = 0$.

The solvability condition for (B.7) is obtained by taking the inner product with $r$. After some algebra, this gives the following equation for $a(x,\theta)$:

$$\frac{\partial}{\partial\theta}\left(\frac{\partial a}{\partial s} + \Gamma a\frac{\partial a}{\partial\theta} + \Delta a\right) = \frac{1}{2}\Gamma\left(\frac{\partial a}{\partial\theta}\right)^2. \tag{B.9}$$

In (B.9), $\partial/\partial s = C^i(x)\partial/\partial x^i$ is a derivative along the rays associated with $\phi$, $\Gamma(x) \in \mathbf{R}$ is a coefficient that measures the strength of quadratically nonlinear effects, and $\Delta(x) \in \mathbf{R}$ describes the effects of focusing and nonuniformities on the wave. The coefficients are given explicitly by

$$\begin{aligned} C^i &= 2\kappa_j A^{ij}_{pq}r^p r^q,\\ \Gamma &= \kappa_i\kappa_j\frac{\partial A^{ij}_{pq}}{\partial u^k}r^k r^p r^q,\\ \Delta &= \frac{\partial}{\partial x^i}\left(\kappa_j A^{ij}_{pq}r^p r^q\right). \end{aligned} \tag{B.10}$$

We can regard (B.9) as a family of equations, with one equation for each ray. On a given ray, $a = a(\theta, s)$ is a function of the phase $\theta$ and the arclength variable $s$. The coefficients $\Gamma$ and $\Delta$ depend only on $s$. To normalize (B.9), we define $E(s) = \exp\int\Delta(s)\,ds$. Then the change of variables

$$u(x,t) = E(s)a(\theta,s), \quad t(s) = \int\frac{\Gamma(s)}{E(s)}\,ds, \quad x = \theta,$$

transforms (B.9) into (1.5).

**Appendix C. A family of nonlinear wave equations.** In this appendix we consider the equations

$$\left(u_t + u^n u_x\right)_x = \nu u^{n-1} u_x^2, \tag{C.1}$$

where $n > 0$. The initial condition is

$$u(x,0) = F(x). \tag{C.2}$$

Special cases of (C.1) are (a) $n = 1$, $\nu = 0$, the $x$-derivative of the inviscid Burgers equation (1.7); (b) $n = 1$, $\nu = \frac{1}{2}$, the variational equation (1.5); (c) $\nu = 0$, the $x$-derivative of the modified Burgers equation; (d) $\nu = n/2$, the higher-order variational equation (2.7).

As in the proof of Theorem 3.1, we introduce a characteristic variable $\xi$, where $x = X(\xi, t)$ and

$$X_t = U^n, \tag{C.3a}$$

$$X(\xi, 0) = \xi. \tag{C.3b}$$

We define

$$\begin{aligned} U(\xi,t) &= u\left[X(\xi,t), t\right], \\ V(\xi,t) &= X_\xi(\xi,t). \end{aligned} \tag{C.4}$$

PROPOSITION 3.1. *The characteristics of (C.1), (C.2) satisfy the equations*

$$\begin{aligned} X_{\xi t}^n &= n^n F'^n X_t^{n-1} X_\xi^\nu, \\ X(\xi,0) &= \xi. \end{aligned} \tag{C.5}$$

*Proof.* Writing (C.1) in characteristic form and eliminating $X$ from (C.3a) and (C.4b) gives

$$V U_{\xi t} = \nu U^{n-1} U_\xi^2, \tag{C.6a}$$

$$V_t = n U^{n-1} U_\xi. \tag{C.6b}$$

From (C.2)–(C.4), the initial conditions for (C.6) are

$$\begin{aligned} U(\xi,0) &= F(\xi), \\ V(\xi,0) &= 1. \end{aligned} \tag{C.7}$$

Using (C.6) shows that $\left(V^{-\nu/n} U_\xi\right)_t = 0$. Integrating and imposing the initial conditions (C.7) gives

$$U_\xi = F'(\xi) V^{\nu/n}. \tag{C.8}$$

Using (C.3a) and (C.4b) to eliminate $U$ and $V$ from (C.8) implies (C.5).

From (C.4) and (C.8),

$$u_x = F'(\xi) X_\xi^{-1+\nu/n}. \tag{C.9}$$

When $\nu < n$, smooth solutions of (C.1) break down if the solution of (C.5) satisfies $X_\xi(\xi_*, t_*) = 0$ for some $t_* > 0$. It is not obvious how to determine if this occurs for general values of $n$. However, we can answer this question when $n = 1$. For $\nu < 1$, the behavior is similar to that of the variational equation (1.5), but for $\nu > 1$ the solution itself, and not just the derivative, blows up in finite time.

PROPOSITION C.2. *Suppose that $n = 1$ and $F$ is real analytic and let $u(x,t)$ satisfy (C.1) and (C.2).*

(a) *If $\nu < 1$ and $F$ is not monotone increasing, then $\lim_{t\uparrow t_*} \inf_x u_x(x,t) = -\infty$, where*

$$t_* = -\frac{1}{1-\nu} \inf \frac{1}{F'(\xi)}.$$

(b) *If $\nu = 1$, then (C.1) and (C.2) have global smooth solutions.*

(c) *If $\nu > 1$ and $F$ is not monotone decreasing, then $\lim_{t\uparrow t_*} \sup_x |u(x,t)| = +\infty$, where*

$$t_* = \frac{1}{\nu - 1} \sup \frac{1}{F'(\xi)}.$$

*Proof.* (a) For $n = 1$, (C.8) and (C.6b) imply that

$$X_{\xi t} = F'(\xi) X_\xi^\nu. \tag{C.10}$$

Solving (C.10) for $\nu \neq 1$ gives

$$X_\xi = \left[1 + (1-\nu)F'(\xi)t\right]^{1/(1-\nu)}. \tag{C.11}$$

If $\nu < 1$, it follows from (C.9) and (C.11) that $u_x \to -\infty$ as $t \to t_*$, where $t_*$ is defined in the proposition.

(b) If $\nu = 1$, the solution of (C.10) is $X_\xi = \exp\left[F'(\xi)t\right]$. Since $X_\xi \neq 0$, we can solve for $\xi = \Xi(x,t)$ globally to obtain a smooth solution.

(c) From (C.8) and (C.11)

$$U_\xi = F'(\xi)\left[1 - \mu^{-1}F'(\xi)t\right]^{-(1+\mu)}, \tag{C.12}$$

where $\mu = 1/(\nu - 1) > 0$. Suppose that $(\nu - 1)F'(\xi_*) = t_*$. Taylor expanding $F'$ gives

$$F'(\xi) = F'_* + \frac{1}{m!}F_*^{(m+1)}(\xi - \xi_*)^m + \cdots.$$

Here, $m \geq 2$ is even since $F'$ has a maximum at $\xi_*$. Using this expansion in (C.11) and (C.12), integrating, and assuming $m\mu \neq 1$ gives

$$X - x_* \sim C\mathrm{sgn}(\xi - \xi_*)|\xi - \xi_*|^{1-m\mu},$$
$$U \sim K\mathrm{sgn}(\xi - \xi_*)|\xi - \xi_*|^{1-m-m\mu}.$$

Here $C$, $K$, and $x_*$ are constants. It follows that $U$ is unbounded as $\xi \to \xi_*$. If $m\mu < 1$, then $X \to x_*$ as $\xi \to \xi_*$. Nearby characteristics approach $(x_*, t_*)$, and $u(x,t)$ is unbounded in a neighborhood of this point. If $m\mu \geq 1$, then $X(\xi, t) \to \pm\infty$ as $t \to t_*$ when $\xi$ is sufficiently close to $\xi_*$, and the characteristics diverge to infinity.

Last, we consider (C.1) with special initial data for which an interval of characteristics intersect simultaneously at the same point. Our aim is to determine the size of the jump that forms in the solution. For the inviscid Burgers equation (1.7), the jump is finite. For the variational equation (1.5), the jump is zero (cf. Proposition 4.2).

PROPOSITION C.3. *Suppose that* $u(x,t)$ *is a solution of* (C.1) *such that the characteristics with* $a \le \xi \le b$ *intersect at* $x = x_*$ *when* $t = t_* > 0$ *and that* $F'$ *has one sign on* $(a,b)$. *Also suppose that* $u$ *is smooth for* $0 \le t < t_*$. *Let*

$$[u](t) = u\big(X(b,t),t\big) - u\big(X(a,t),t\big).$$

*Then*

$$\lim_{t\uparrow t_*}[u](t) = \begin{cases} 0, & \text{if } \nu > 0; \\ C, & \text{if } \nu = 0; \\ \infty, & \text{if } \nu < 0. \end{cases}$$

*Here,* $C$ *is a finite constant.*

*Proof.* From (C.8),

$$[u](t) = U(b,t) - U(a,t) = \int_a^b F'(\xi) X_\xi^{\nu/n}(\xi,t)\, d\xi. \tag{C.13}$$

By assumption, $X(\xi,t_*) = x_*$ for all $a \le \xi \le b$, so that $X_\xi(\xi,t_*) = 0$. Taking the limit as $t \uparrow t_*$ in (C.13) implies the result. In particular, when $\nu = 0$ we have the usual result that $[u] = F(b) - F(a)$.

We can explicitly construct piecewise linear solutions that illustrate Proposition C.3 when $n = 1$. They are

$$u(x,t) = \begin{cases} 0, & \text{if } 0 \le x; \\ x/(1-\nu)t, & \text{if } x_0 \le x \le 0; \\ u_0(t), & \text{if } x \le x_0, \end{cases}$$

where

$$x_0(t) = \alpha(\nu - 1)|t|^{1/(1-\nu)},$$
$$u_0(t) = \alpha|t|^{\nu/(1-\nu)},$$

and $\alpha$ is a constant. For $\nu < 1$ and $\alpha > 0$, the characteristics originating from $-\alpha(1-\nu) \le x \le 0$ at $t = -1$ intersect at $(x,t) = (0,0)$. If $0 < \nu < 1$, then $[u](t) = u_0(t) \to 0$ as $t \uparrow 0$, and if $\nu < 0$, then $[u](t) \to +\infty$, as required by Proposition C.3. For $\nu > 1$ and $\alpha < 0$, the characteristics starting in $x < 0$ at $t = -1$ all tend to $-\infty$ as $t \uparrow 0$ and $u(x,t) \downarrow -\infty$ as $t \uparrow 0$ for all $x < 0$.

The borderline case $\nu = 1$ is interesting. Equation (C.1) is then $u_{xt} + uu_{xx} = 0$. The piecewise linear solution is

$$u(x,t) = \begin{cases} 0, & \text{if } 0 \le x; \\ \alpha x, & \text{if } x_0 \le x \le 0; \\ u_0(t), & \text{if } x \le x_0. \end{cases}$$

Here, $\alpha$ is a constant and

$$x_0(t) = -\exp(\alpha t),$$
$$u_0(t) = -\alpha \exp(\alpha t).$$

Depending on the sign of $\alpha$, this solution tends exponentially to zero or to infinity as $t \to +\infty$. For $\nu < 1$ the corresponding solutions tend to zero in finite time or grow algebraically, whereas for $\nu > 1$ the solutions tend to zero algebraically or to infinity in finite time.

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