The stationary points of the distance function between a curve and a point give normals of the curve:

The distance between the point (x_1, y_1) and a curve given parametrically by $(x(t), y(t))$ is

$$l(t) = \sqrt{(x(t) - x_1)^2 + (y(t) - y_1)^2}. \tag{1}$$

Intuitively, the line between the point on a curve which is closest to (x_1, y_1) (the point with t such that $l(t)$ is a minimum) and (x_1, y_1) will be perpendicular to the tangent at that point. More formally, we can say that this point gives a global minimum of the distance function. We can prove this, and also show that all the stationary points of l give lines which are normals to the curve, by optimising the squared distance function, $l^2(t)$. We use the squared distance because its derivative is more convenient to calculate.

We start by calculating the derivative

$$\dot{l}^2(t) = 2\dot{x}(t)(x(t) - x_1) + 2\dot{y}(t)(y(t) - y_0),$$

and setting it to zero to give

$$\begin{aligned} \dot{x}(t)(x(t) - x_1) + \dot{y}(t)(y(t) - y_1) = 0 \qquad (2) \\ \Rightarrow \frac{y(t) - y_1}{x(t) - x_1} = -\frac{\dot{x}(t)}{\dot{y}(t)} \\ \Rightarrow \frac{y(t) - y_1}{x(t) - x_1} = -\frac{dx}{dy}. \end{aligned}$$

So the gradient of this line is the negative reciprocal of the derivative of the curve at t, which is the gradient of the tangent at t, meaning the line is a normal of the curve, passing through (x_1, y_1). Any value of t which satisfies Eq. (2) thus gives us a normal which passes through the desired point. In the case of an ellipse parameterised by $(a \cos t, b \sin t)$, this condition can be rewritten as

$$a \sin t(x_1 - a \cos t) = b \cos t(y_1 - b \sin t). \tag{3}$$