PART III

Mathematical harmonics

CHAPTER 10

Pythagorean harmonics in the fifth century: Philolaus

With the work of Aristoxenus this phase of the empirical tradition of harmonic theory has run its course. We now turn the clock back a hundred years or so to consider our earliest evidence about the mathematical form of the discipline, which originated with the Pythagoreans of the fifth century or conceivably with Pythagoras himself in the sixth. Evidence from later writers gives us a fair general picture of the approach to harmonics which they adopted and whose outlines I sketched in Chapter 1; I shall not repeat the programmatic points I made there. But on matters of detail and on the work of individual Pythagoreans of the early period the late sources are often unreliable. For the most part I follow Burkert in treating Aristotle as the most authoritative of our sources on the subject, together with a few fragments from the work of the late fifth-century Pythagorean Philolaus which most modern commentators take to be genuine. If they are, Philolaus is the only Pythagorean harmonic theorist before the fourth century of whose work we have solid and significant details.[1]

The prize exhibit is a short passage from Philolaus' essay *On Nature* quoted by Nicomachus and Stobaeus. In modern editions it is sometimes printed as a continuation of Philolaus fragment 6, sometimes as a separate item; I shall refer to it as frag. 6a. It has been much discussed,[2] but I have a serious purpose in re-examining it at some length here. Although much of what is regularly said about it is true, one important aspect of it seems to have been overlooked, or at any rate underplayed, and it has a crucial bearing on the relation between Philolaus' approach to harmonics and those of other theorists in both the main traditions. In order to unearth this feature and

1 For the most thorough modern investigations of the fragments' claims to authenticity and of Philolaus' work as a whole see Burkert 1972, especially 218–98 and 386–400, Huffman 1993, and more briefly Kahn 2001 ch. 3. There is a good deal to be said for Burkert's view (broadly shared by Huffman and Kahn) that much of the information retailed by Aristotle also came from Philolaus' treatise.

2 It appears as the second paragraph of DK 44B6. Detailed discussions of it will be found in Burkert 1972: 389–94 and Huffman 1993: 145–65.

to explore its implications we need to look closely at the whole fragment, even though this will involve repetitions of points made often by others before. The translation below is as nearly literal as I can make it, and with its thicket of transliterations and other barbarous coinages it may be barely intelligible as it stands. But it will be best to have it in front of us as a point of reference, in a version which does not smooth out the peculiarities of its diction. The first part of my discussion will try to elucidate them.

FRAGMENT 6A: PRELIMINARY ANALYSIS

The size of *harmonia* is *syllaba* and *di' oxeian*, and *di' oxeian* is greater than *syllaba* by an epogdoic. For from *hypata* to *messa* is a *syllaba*, from *messa* to *neata* is a *di' oxeian*, from *neata* to *trita* is a *di' oxeian*; and what lies between *trita* and *messa* is an epogdoic. *Syllaba* is epitritic, *di' oxeian* is hemiolic, and *dia pasan* is duple. Thus *harmonia* is five epogdoics and two *diesies*, *di' oxeian* is three epogdoics and a *diesis*, and *syllaba* is two epogdoics and a *diesis*.

The expressions *harmonia*, *di' oxeian* and *syllaba* refer to the octave, the perfect fifth and the perfect fourth. Hence at the most straightforward level the first sentence means simply 'an octave is a fourth plus a fifth', as indeed it is. But Philolaus' language needs closer inspection. His terms do not carry any reference to numbers, as do 'octave', 'fifth' and 'fourth' (which indicate that these intervals span the ranges between any given note of a scale and the eighth, fifth and fourth notes in order from it respectively; rather similarly, the regular Greek term for the fourth is *dia tessarōn*, 'through four [notes or strings]', the fifth is *dia pente*, 'through five', and the octave is *dia pasōn*, 'through all'). Philolaus' expressions *syllaba* and *di' oxeian* seem to come from the language of musicians, rather than philosophers or scientists, and *harmonia* inhabits both spheres. Both these facts are important. *Syllaba* means 'grasp'. According to the likeliest explanation, it referred originally to the group of strings which lie under a lyre-player's fingers in what we might call their 'starting position', poised over the four lowest strings on the instrument. *Di' oxeian* means 'through the high-pitched [strings]', that is, those that complete the span from the fourth string to the top of the octave. Thus in the musical scheme that Philolaus' description reflects, the most basic arrangement is one in which the pivot between the octave's main structural components is a fourth from the bottom and a fifth from the top.[3] The point that needs emphasis here is that when the terms are interpreted in this way, their direct reference is to components of the attunement which

[3] For an explanation of these terms along the lines I have sketched see Porph. *In Ptol. Harm.* 97.2–8.

lie in specific positions, and not to the sizes of intervals regardless of where they are placed.

The noun *harmonia* is one we have already met in several contexts. It is cognate with the verb *harmozein*, 'to fit together', 'to join', as in the work of a carpenter. It is the 'fitting-together' of diverse elements into a unity, and as such it plays a significant role in Presocratic cosmology, notably in Heraclitus and Empedocles[4] and in a crucial fragment of Philolaus himself, frag. 6, which we shall consider later. But it also figures prominently in Greek musical vocabulary, as we have seen, and in Philolaus it seems to form a bridge between the musical and cosmological domains. In one of its musical applications, and the most relevant here, a *harmonia* is an attunement, an integrated pattern of relations into which a collection of notes is 'fitted together' by a musician when he tunes his strings. Philolaus' attunement spans an octave, the range whose bounding notes are most perfectly coordinated with one another; and he knows the term regularly used by later writers to refer to that interval, *dia pasōn* (*dia pasan* in his Doric dialect). But he uses that expression only once in the fragment, at a point where he is simply identifying this interval's ratio. What his more prominent deployment of the word *harmonia* emphasises is the coherence and unity of the relation, not its dimensions. (Even his alternative expression, *dia pasan*, does not point directly to the interval's size or ratio, any more than do *syllaba* or *di' oxeian*, but only to its comprising the whole range covered by the attunement's notes.)

Hypata, *messa*, *trita* and *neata* are the Doric forms of the note-names which appear as *hypatē*, *mesē*, *tritē* and *nētē* in the Attic Greek of most theorists; from here onwards I shall use the words' more familiar forms. In the system which Philolaus describes, *hypatē* is the lowest note of the octave and *nētē* the highest; *mesē* is a fourth above *hypatē* and a fifth below *nētē*; *tritē* is a fourth below *nētē* and a fifth above *hypatē*; and *tritē* is higher than *mesē* by a whole tone (though Philolaus does not put it in that way, and his language poses problems to which we shall shortly return). These relations are set out in Figure 8. We have already become familiar with the division of an octave into two subsystems, each spanning a fourth and separated by a tone; it is normal in Greek theory, and other writers give the same names as Philolaus to the notes which form their boundaries, with one exception. The note in the position of Philolaus' *tritē* is usually called *paramesē*,

[4] See especially Heraclitus frag. 51, Empedocles frags. 26.11–14, where the cosmic principle of Love is called *Harmoniē* (cf. frags. 22, 35, and references elsewhere to Love under various names), and frags. 71, 96, 107.

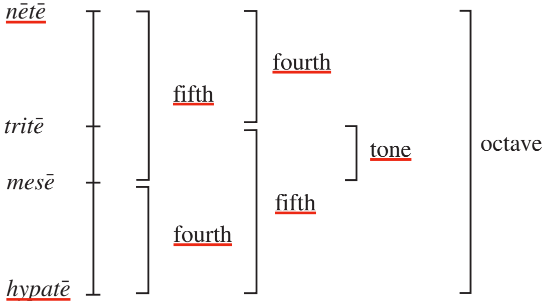

Figure 8 The *harmonia* of Philolaus

and *tritē* is differently placed. This oddity in Philolaus' usage has implications which will also be considered in due course.

I must explain one other group of expressions before we go on. The uncouth terms 'hemiolic', 'epitritic' and 'epogdoic' are crudely Englished versions of Greek adjectives referring to ratios. *Hēmiolios* means 'half-and-whole', and specifies the relation in which one item is one-and-a-half times the other, in the ratio 3:2. Expressions formed by adding the prefix *epi-* to an ordinal adjective (e.g. to *tritos*, 'third', or *tetartos*, 'fourth') mean 'a third in addition', 'a quarter in addition', and so on. Thus when the ratio between two terms is *epitritos*, 'epitritic', the larger amounts to the smaller 'and a third of the smaller in addition', and the ratio is 4:3. When it is *epogdoos*, 'epogdoic', the larger exceeds the smaller by one eighth of the smaller, in the ratio 9:8. The remaining ratio which Philolaus mentions, the 'duple' or 'double' (*diplasios*), is of course the ratio 2:1.

With these preliminaries behind us we can move on to consider the passage's overall structure.[5] It appears to be conceived as an argument, or perhaps as two separate arguments; the second sentence (given my punctuation) is linked to the first by the connective 'for' (*gar*), and the final sentence is introduced by the word 'thus' (*houtōs*). Any adequate interpretation must account for this impression of reasoning from premises to conclusions.

The position of the connective 'for' shows that if the first two statements amount to an argument, the first states the conclusion, the second the grounds on which it is based. Now every significant term in these sentences, with one exception, belongs to the vocabulary of musicians; only 'epogdoic' is a mathematician's expression, alien to the language of musicians and musically educated amateurs. If we set questions about this mathematical

[5] I shall postpone discussion of Philolaus' special way of using the word *diesis*.

intruder aside for a moment, Philolaus seems to be trying to convince anyone with an ordinary person's knowledge of music of the truth of his initial assertion. Because we all know that 'from *hypatē* to *mesē* is a *syllaba*, from *mesē* to *netē* is a *di' oxeian*', and so on, we can see that 'the size of *harmonia* is *syllaba* and *di' oxeian*, and *di' oxeian* is greater than *syllaba* by an epogdoic'. The argument is perfectly adequate. The pattern of relations set out in the second sentence, and apparently taken to be common knowledge, guarantees that a *syllaba* plus a *di' oxeian* is an octave; and a glance at Figure 8 will make it obvious that *di' oxeian* exceeds *syllaba* by an epogdoic.

All that is plain and simple, but there are two features of the argument that call for comment. The first is that despite its use of a mathematical expression referring to a ratio, nothing in it depends on the identity of this ratio or any other. No mathematical calculations are involved at all, and any arbitrary label for the interval between *mesē* and *tritē* would have served Philolaus' purpose equally well. His reasoning in these sentences would have been unaffected if he had called it the 'tone', *tonos* or *toniaion diastēma*, as other theorists do. I do not know why he chooses the more recondite expression, which is unlikely to have been used by musicians themselves. It is just possible that the more familiar *tonos* had not yet been adopted as this interval's name, though we would expect musicians to have had some word by which they could refer to it, and we know of no other.[6] It may be relevant that when the term *tonos* does acquire this meaning, it is normally treated as identifying an interval by reference to its size, and not (except in special contexts) by its position in the system; and the practice of assigning sizes to intervals, as I have already said, was itself a product of the theorists' work, not something inherited from traditional musical language. Whatever the explanation of Philolaus' choice may be, he must at some earlier stage in his essay have specified the ratios of *di' oxeian* and *syllaba* as 3:2 and 4:3 respectively, and shown that 3:2 exceeds 4:3 in the 'epogdoic' ratio, 9:8. From that moment on, he could use the word *epogdoos* either to refer to the relevant interval's ratio, or simply as a convenient stand-in for the (perhaps non-existent) musical name of the interval itself; and its function in the first two sentences is plainly the latter.

My second point about the first two sentences is this. If Philolaus' conclusion is obvious to anyone familiar with the pattern of concords formed by the four fundamental notes, it seems strange that he should need any sort of argument to prove it, especially one such as this, which would be incomprehensible to readers who lacked that elementary piece of musical

[6] For a brief and lucid discussion of the history of the word *tonos*, see Rocconi 2003: 23–4.

knowledge. I suggest that the reason is to be found in the fact which I mentioned above, that the familiar musical designations of the concords do not identify them by their sizes but by their locations in the system, and that the practice of assigning sizes to intervals was a theorist's innovation. Part of what Philolaus is doing is to persuade his audience that this practice makes sense, and that once we have accepted it we shall see that the interval between *tritē* and *nētē*, for instance, can intelligibly be described as the 'same' interval as the one between *hypatē* and *mesē*. In the language of ordinary musical discourse, *syllaba* is simply the name of the lower of these intervals. But as soon as we have introduced the notion of an interval's size, as Philolaus does in the first sentence, the name can be applied equally, as it is in the second sentence, to any interval with the same compass.

We shall return to this suggestion later, but first we must review the rest of the fragment. The next sentence is straightforward; it states, without argument, the ratios of the fourth, fifth and octave (it is here, where nothing but the ratios is in question, that we meet the standard term for the octave, *dia pasōn*). As I have pointed out, Philolaus' use of the term *epogdoos* presupposes that the ratios of the fourth and the fifth (and if theirs, then surely that of the octave) had been established in an earlier passage. This sentence, then, looks like a reminder, perhaps prompted by the immediately preceding reference to the epogdoic. If it has a function in the reasoning of the passage we have, it does not leap to the eye. This too is an issue we shall revisit later.

The final sentence, with its introductory 'thus', has the air of expressing a set of conclusions for which the necessary premises have just been provided. The conclusions are that *harmonia*, the whole structured complex of an attunement spanning an octave, amounts to five epogdoics and two *dieseis*,[7] the fifth or *di' oxeian* to three epogdoics and a *diesis*, and the fourth or *syllaba* to two epogdoics and a *diesis*. Plainly, however, these propositions cannot be inferred from the earlier part of the fragment without addition, if only because the *diesis* has not previously been mentioned. Three questions must be answered if we are to reconstruct Philolaus' train of thought. First, what, in his usage, is a *diesis*? Secondly, how can the surviving statements of the fragment most economically and plausibly be supplemented, so as to provide premises which will underpin his conclusions? Thirdly and crucially, in which of its roles does the term *epogdoos* appear here? Is its specification of a ratio relevant to the argument, or is it functioning (as

7 In Philolaus' Doric, the plural of *diesis* is *diesies*, whereas in the Attic of most other theorists it is *dieseis*. I shall use the Attic form, as I do for the names of the notes.

in the first sentence) merely as a label for a musical interval, in a piece of reasoning to which ratios make no contribution?

The term *diesis* is common in musicological writings and we have met it before. It always refers to a very small interval, but its scope is not restricted to intervals of any one size. When a writer uses it to pick out one size of interval in particular, it may be anywhere between (roughly) a quarter-tone and a semitone, and only the context or an adjective qualifying the noun will reveal which it is. Sometimes the author makes it clear that the intervals he designates as *dieseis* come in several sizes.[8] In Philolaus' fragment, however, the *diesis* must plainly be an interval of just one size, since it is the interval by which a perfect fourth exceeds two whole tones (two 'epogdoics'). Given that the ratio of the fourth is 4:3 and that of the tone is 9:8, it turns out to be the so-called 'minor semitone', slightly less than exactly half a tone, whose ratio is 256:243.[9]

Did Philolaus know this cumbersome ratio? Archytas and Plato in the next century certainly did, and it has sometimes been argued that Archytas' very *recherché* handling of it points to an earlier and more straightforward context for its use, which is assumed to be that of Philolaus.[10] I am not altogether convinced by this argument, though for reasons I shall give later I think its conclusion is true. Let us suppose that it is, that Philolaus had established it earlier in the treatise, and that when he mentions the *diesis* here, its ratio is relevant to the argument. In that case it must also be relevant that the ratio of the tone is 9:8, as Philolaus' term for it indicates, and the argument must be of a mathematical sort.

Such an argument could be reconstructed, but it would be moderately complex, and in order to give Philolaus appropriate premises we would need to make quite substantial additions to the statements we have. There is a simpler solution. The word *diesis* is almost certainly drawn, like others in this passage, from the language of musicians. Outside musical contexts it means a 'putting through' or 'letting through', and its musical use may be based on the image of a pipe-player 'letting the melody through' almost imperceptibly, perhaps without any detectable movement of his fingers, from one note to another that is very close by.[11] We can interpret Philolaus' usage as referring most directly to the passage through the tiny 'space' remaining between the penultimate note of the system and the lowest,

8 See e.g. Aristotle, *Metaph*. 1053a12–17, discussed on pp. 350–3 below.
9 In order to distinguish Philolaus' rather unusual use of the term from that of other theorists, I shall continue to italicise *diesis* when it occurs in a Philolaan context, and not otherwise.
10 See e.g. Burkert 1972: 388–9, Huffman 2005: 420–2.
11 Cf e.g. West 1992a: 255 n. 42, Rocconi 2003: 11 n. 35.

hypatē, after we have moved down the scale from *mesē* through two steps of a whole tone each.[12] In that case, in order to justify the conclusion, we need make only one addition to the statements made in the first two sentences: '*syllaba* is greater than two epogdoics by a *diesis*'.[13] We can then construe the mathematical meaning of *epogdoos* as irrelevant to the present stretch of reasoning, as it is (so I have argued) in the opening sentences; and taken together with the supplement I have suggested, the contents of those sentences are sufficient to justify the conclusion. From this perspective the third sentence, where the ratios of the concords are identified, can be no more than a parenthesis. We can elide it, and the passage becomes a single, coherent argument which makes no essential reference to ratios at all. But we shall shortly have to consider it again.

There is another small but significant pointer to the conclusion I have drawn. When Philolaus mentions epogdoics in his final sentence, he puts the adjective in the neuter plural, *epogdoa*. If there is a noun to which the adjective is implicitly attached, it cannot be *logoi*, 'ratios'; the only plausible candidate is *diastēmata*, 'intervals'. *Diastēma* is a musical rather than a mathematical term, and we have a report (whose credentials are respectable though not unchallengeable) about Philolaus' use of it. According to Porphyry, he used it to refer to the 'excess' (*hyperochē*) of one item over another;[14] and in context this means that when one interval differs from another by a certain *diastēma*, we reach the one from the other by a process of addition. We get from a fourth to a fifth by 'adding' a certain amount, a whole tone, and the tone is the 'excess' of the fifth over the fourth. Ratios do not behave like this; we do not add 9:8 to 4:3 to reach 3:2 – the notion is nonsensical, and this is part of Porphyry's point. Hence once again, if we infer from Philolaus' grammar that he is implicitly referring to *diastēmata*, we must conclude that he is not relying at this point on the measurement of intervals in terms of ratios.

Given the Pythagoreans' well-documented championship of ratios in musical analysis and this fragment's seminal place in the mythology of orthodox scholarship in Greek harmonics, this is a remarkable result. It is clear enough that Philolaus was well acquainted with a ratio-based method and used it himself, but in the reasoning of this passage ratios take a back

[12] It cannot be demonstrated from this fragment alone that the *diesis* is the lowest of the three intervals between *mesē* and *hypatē*. But every other theorist who considers systems of this type in their basic form locates it in this position.

[13] I postpone consideration of the grounds on which Philolaus could have justified this assertion. Given that he had previously done so, such a statement would fit perfectly at the end of the second sentence. It is tempting to conjecture that Philolaus put it there, and that it somehow got lost in the course of the passage's transmission, but I shall not commit myself to so impertinent a guess.

[14] Porph. *In Ptol. Harm.* 91.11–13.

seat. His arguments require us to do nothing more complex than to recognise that talk of the 'sizes' of musical intervals makes sense, that some fundamental intervals in the system are equal to others, and that larger intervals can be constructed by adding smaller ones together. This looks much more like the 'empirical' theorists' procedure, and like theirs it is the direct offspring of musical experience, not of mathematics. In that case Philolaus' approach is a hybrid between two perspectives which were later treated as incompatible. But that fact, if it is one, is not after all so very startling. We have no indication that they were construed in that light until the last decades of the fourth century (see further pp. 362–3, 390–1 below).

So far as I am aware, this reading of the fragment is new, and no doubt it will be controversial. It also leads me to recant a position I have adopted in the past, that ideas attributed to Philolaus in two passages of Boethius should be rejected as spurious.[15] The principal reason for dismissing them is precisely that they mix a ratio-based analysis of intervals with one which represents intervals as having 'sizes', where these sizes are not expressed as ratios but as amounts to which simple magnitudes are assigned; and they imply that we can construct larger intervals from smaller ones simply by adding these magnitudes together. Although I would still maintain that Burkert's reasons for accepting Boethius' testimony are inadequate by themselves,[16] the reading of frag. 6a which I have offered suggests that the mixture of approaches involved in Boethius' accounts may have a genuinely Philolaan pedigree, despite its mathematical confusions. We must therefore take a look at this material; and this will lead back to some further thoughts on the passage we have been examining.[17]

[15] My only published comment on this issue (so far as I can recall), Barker 1989a: 38 n. 36, is non-committal, but that was mere cowardice. Until very recently I was convinced that Boethius should not be believed.

[16] Burkert 1972: 394–400. His main argument depends on the assumption that Nicomachus, on one of whose lost works Boethius' account was based, found these aspects of Philolaus' theories in the same source as frag. 6a, which he quotes in a surviving treatise (Nicom. *Harm.* ch. 9). But that assumption cannot be guaranteed. Cf. Huffman 1993: 370, and n. 17 below.

[17] Huffman 1993: 364–74, gives a careful assessment of the two passages' claims to authenticity. He recognises the attractions of the hypothesis that one of them (Boeth. *Inst. mus.* 3.8) reflects early Pythagorean ideas, but contends, rightly in my view, that it is closely related to the other (*Inst. mus.* 3.5), which he takes to be spurious (366). But though his discussion of 3.5 succeeds in casting suspicion on it, I do not think it conclusive. The argument he regards as clinching the case (373–4, cf. 362–3) focuses on the passage's association of the unit with the point, the number 3 with the 'first odd line', and the number 9 with the 'first odd square'; and it depends on Burkert's attempt to show that the sequence 'point-line-plane-solid' belongs to a Platonist and not to a Pythagorean repertoire (Burkert 1972: 66–70, cf. 23–4, referring especially to the evidence of Aristotle, *De caelo* 299a2 ff.). The issues are too intricate to pursue here; I merely record the fact that I am not persuaded that they compel us to reject Boethius' report. Burkert himself did not deploy his arguments in an assault on its authenticity, as Huffman does; as I noted above, he regarded its evidence as reliable.

THE EVIDENCE OF BOETHIUS

The first of Boethius' two reports[18] alleges that Philolaus identified the whole tone with a specific number of 'units', 27, and that he divided this into two unequal parts, one called the *diesis*, comprising 13 units, the other called the *apotomē*, which is 14. He credits Philolaus with two arguments, of which the second seems to explain how he arrived at these numbers, while the first is designed to show that the numbers 13 and 27 have special significance. Only the second concerns us at this stage. In outline, it is this. The ratio of the *diesis*, conceived as the remainder of a perfect fourth after two whole tones (each in the ratio 9:8), is 256:243. The difference between these numbers is 13. We now construct a note a whole tone above the upper boundary of the *diesis*; this upper boundary is represented by the number 243, since smaller numbers are here assigned to higher pitches. The number to which 243 stands in the ratio 9:8 is 216, and the difference between 243 and 216 is 27. If we attach this number to the whole tone and 13 to the *diesis*, there will be 14 left for the larger part of the tone, the *apotomē*.

According to Boethius, Philolaus produced this remarkable piece of reasoning when considering how the whole tone can be divided in two, and it strongly suggests that he could find no way – in terms of ratios – of dividing it into equal parts. That is to be expected, since the ratio 9:8 cannot be divided into equal sub-ratios of integers, as Archytas later demonstrated (see pp. 303–5 below). But we need not suppose that Philolaus was equipped with a proof. Nor need we imagine that he computed the ratio of the *apotomē*, which in fact is 2187:2048. Instead, he used the ratio of the *diesis* as a basis for representing the 'size' of each of these intervals by a single number, as though it were a simple quantity or a 'distance' between two notes and could be measured as so many units. This is of course mathematically absurd.[19] But it fits well with Philolaus' combination of the two systems of measurement in frag. 6a. It attempts, indeed, however incoherently, to integrate them even more closely, by deriving the single numbers which represent the intervals' sizes from the terms of the ratios themselves.

18 *Inst. mus.* 3.5, reproduced as DK 44A26.

19 Its absurdity becomes obvious when we recall that the *diesis* 'is' 13 units because its ratio is 256:243 and 13 is the difference between this ratio's terms, and that 27 'is' the whole tone because its ratio is 243:216 and 27 is the difference between these numbers. If we divide this tone into 13 parts for the *diesis* and 14 for the *apotomē*, the *diesis* will run either from 243 to 230, or from 229 to 216; and patently neither 243:230 nor 229:216 is equal to 256:243. (This way of illustrating the point comes from Burkert 1972: 396.)

Boethius' second report claims that Philolaus offered four definitions, which can be paraphrased as follows.[20] (i) A *diesis* is the interval by which the perfect fourth (ratio 4:3) exceeds two whole tones. (ii) A *komma* is the interval by which a whole tone (ratio 9:8) exceeds two *dieseis*. (iii) A *schisma* is half a *komma*. (iv) A *diaschisma* is half a *diesis*. It is again obvious that Philolaus is not thinking in terms of ratios alone. The ratio of the *komma* can be computed; it is 531441:524288, but this – in Burkert's phrase – is 'pure frivolity'.[21] Boethius has already told us, in fact (*Inst. mus.* 3.5), that Philolaus identified the *komma* with the unit, 1, as being the difference between a *diesis* (13) and an *apotomē* (14). No pairs of integers will specify the ratios of the *schisma* or the *diaschisma*. If these vanishingly small intervals had any place in Philolaus' thought, they must have been conceived simply as quantities or linear distances.

All this is very odd, but if Boethius' account is reliable, there is no great mystery about the roles in which Philolaus deployed these peculiar calculations and definitions. They need have nothing to do with the analysis of different varieties of musical scale, the chromatic and enharmonic systems, as Burkert (following Tannery) suggested;[22] Boethius says nothing to point in that direction, and if the conclusions I shall sketch later about Philolaus' overall agenda are correct, he had no reason to be interested in scales of those sorts. What the passage at *Inst. mus.* 3.8 indicates is that they are wholly focused on the division into equal halves of the elementary intervals of the system outlined in frag. 6a, the tone and the *diesis*. Half a *diesis* is a *diaschisma*, and half a tone is two *diaschismata* plus one *schisma*; and in order to describe the *schisma* we have first to define the *komma*. None of this can be understood in terms of ratios. All the material in Boethius seems, then, to be continuous with the contents of the fragment quoted by Nicomachus and Stobaeus. Its primary concern, like that of the fragment, is with the business of measuring intervals in a musical system against one another as simple quantities.

BOETHIUS AND FRAGMENT 6A

The question why Philolaus was concerned with the business of 'halving' the tone and the *diesis* is one I shall postpone; there are other matters that we must consider first. The simple quantities involved in Boethius' reports are derived, paradoxically, from ratios – specifically the ratio of the *diesis* –

[20] *Inst. mus.* 3.8, printed as an appendix to DK 44B6.
[21] Burkert 1972: 395. [22] Burkert 1972: 398; cf. Tannery 1904b: 224–5, Huffman 1993: 365.

and ratios are also mentioned in frag. 6a. Let us now return to that fragment and ask whether the ratios figure there, despite my earlier comments, in more than a merely parenthetical role. I have suggested that part of its purpose is to drive home the possibility of shifting from the old practice of referring to intervals by their positions in the system to one which identifies them by their 'sizes', in some sense of that expression. We have seen also that Philolaus must have offered some kind of demonstration, earlier in his treatise, of the ratios of the intervals he calls (in the relevant sentence) *syllaba*, *di' oxeian* and *dia pasōn*, and of the one that 'lies between *tritē* and *mesē*', the 'epogdoic'. If Boethius is to be trusted, he had also worked out the ratio of the *diesis*. If these intervals have determinate ratios, then there can be intervals elsewhere in the system which are relevantly 'the same'; they have the same ratios and therefore the same 'sizes'. The third sentence of the fragment, which alludes to the ratios, will then serve as a reminder that this approach has a solid basis, and reinforces the conceptual shift which the passage is negotiating.

The fragment's arguments, however, are couched mainly in terms familiar to a musically educated audience that is innocent of mathematics, and presses these terms into service in the new environment of sizes and measurement in ways that make no direct appeal to ratio-theory. The vocabulary of the Boethian passages, by contrast, with its *apotomē*, *komma*, *schisma* and *diaschisma*, goes far beyond the regular terminology of musical discourse, and the exposition contains strange manipulations of numbers. But the essential conceptual apparatus used there is a direct extension of that of frag. 6a. In both cases, despite the references to ratios, each interval is treated primarily as an item of such-and-such a size, and intervals can be added together, divided in half and so on, just like the linear 'distances' involved in the empiricists' system of measurement.

If this reading of the fragment and Boethius' reports is on the right lines, it smooths the path between them and strengthens the case for regarding the latter as reliable. It also opens up a new, though inevitably hypothetical perspective on Philolaus' bizarre approach to the quantification of intervals, which is obvious in the Boethian passages but which is also detectable – or so I have argued – in the fragment. Scholars have generally treated the curiosities expounded by Boethius either as displaying only mathematical incompetence, or as evidence of Philolaus' obsession with 'number-symbolism', or both. My suggestion, starting from the reinterpretation of frag. 6a, is that it represents an attempt to fuse two quite different ways of measuring intervals, both of which were current among theorists but still in their infancy in Philolaus' time, one set in terms of ratios and characteristic of 'Pythagorean' harmonics, the other staying closer to musical experience and used principally by theorists engaged professionally in musical practice. It undermines neither the accusation of mathematical incompetence nor the thesis, to which I shall return, that Philolaus credited certain numbers with symbolic meaning. It merely adds that a further explanation for his manoeuvres can be extracted from the history of harmonics itself; and it posits that the relation between the two approaches was less clear-cut and less adequately understood in this period than is usually supposed.

I have still not explained why (in my opinion) Philolaus undertook the project whose problematic remnants we find in the passages we have been considering, and why he thought it necessary to pursue these strange calculations. Any suggestion about the details must be premised on a view about the overall purposes of his foray into harmonics, and I shall say little about it at this stage. There is broad agreement between most scholars on the fundamental points; I think the consensus is correct, and I shall return to it later. To put it in its simplest terms, it is that Philolaus' harmonic investigations should be understood, at least primarily, as a contribution to cosmology and not to the study of music for its own sake. His analyses of musical relations are only means to the greater end of understanding reality as a whole; and if, as I have suggested, he takes a leaf out of the empiricists' book, that is only because he saw it as a useful addition to his cosmological tool-kit.

THE MUSICAL STRUCTURE OF PHILOLAUS' ATTUNEMENT

There is one intrinsically musical issue, however, that needs to be clarified first. The language of frag. 6a shows that Philolaus is drawing on his own and his readers' knowledge of features of a form of attunement used regularly in real music-making. Commentators from Nicomachus onwards have inferred that although it certainly spanned an octave it contained only seven notes; by the standards of the eight-note octaves treated as the norm by later theorists, one element is missing. The inference is based on Philolaus' use of the name *tritē* for the note a tone above *mesē* and a perfect fourth below *nētē* (elsewhere the note in this position is called *paramesē*). *Tritē* means 'third [note or string]', and in other sources invariably refers to the third note counting down from *nētē*. If it has the same sense here, there can be only one note instead of the usual two between the boundaries of the fourth at the top of the octave;[23] see Figure 9.

[23] Nicom. *Harm.* 9.253.3–254.2 Jan, cf. e.g. Huffman 1993: 153–6, 165.

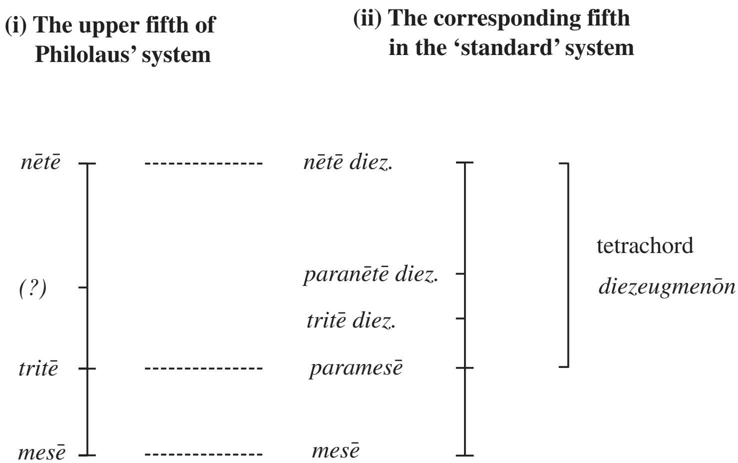

Figure 9 The 'third note' in Philolaus' *harmonia*

There is nothing very startling about this conclusion. Although instruments with eight strings (and sometimes more) were quite common among musicians of this period, we know from vase-paintings and allusions in poetry that there were only seven on the traditional tortoiseshell lyre, on which schoolboys still learned the rudiments of the art.[24] Anyone with enough education and leisure to take an interest in Philolaus would certainly have known the names of this instrument's notes and strings, even if they had only a concert-goer's more distant acquaintance with the elaborate instruments used by professionals.

Nicomachus also assumes – and most modern commentators agree – that the scale built into Philolaus' attunement proceeded, with one exception, through steps of a whole tone or a *diesis* each. The exception is the interval above *tritē*, where a note that would appear in the eight-note system is missing; and this, Nicomachus tells us, amounted to a *diesis* plus a tone, roughly a minor third.[25] This diagnosis, too, is plausible. Diatonic octave-systems in which the fourths at the top and the bottom are divided, reading downwards, into steps of tone, tone, *diesis* are well known elsewhere;[26] and

[24] Cf. e.g. the Homeric *Hymn to Hermes* 49–51, Pindar, *Nem.* 5.22–5, Ion frag. 32 West. Reputedly ancient seven-note systems are also mentioned in some theoretical and historical sources, e.g. [Arist.] *Probl.* 19.7, 32, 47, [Plut.] *De mus.* 1140f; Nicomachus himself describes another version of such a system in *Harm.* 3, cf. *Harm.* 5.

[25] Nicom. *Harm.* 9.253.6–7.

[26] Most relevantly in Plato, *Tim.* 35b–36b, [Eucl.] *Sect. can.* propositions 19–20.

Philolaus' scheme would differ from those only in so far as it omitted one note. Perhaps he gave a full description of this attunement's structure in a passage which has been lost. But we should note that there is nothing to prove it, either in frag. 6a or in Nicomachus. Nicomachus cites no authority for his statements apart from the fragment itself, and we may reasonably suspect that they represent only his attempt to interpret it. The fragment's last sentence quantifies the octave, fifth and fourth in terms of whole tones ('epogdoics') and *dieseis*, but it does not say that the system's scalar steps were of these sizes and no others. It cannot itself be construed as a list of these steps, if only because it quantifies the octave by reference to seven intervals and there can be only six steps in his seven-note system. The evidence leaves open the possibility – though we may think it far-fetched – that the scalar steps of the attunement which Philolaus had in mind were not all tones or *dieseis*; and in any case it does not guarantee that he analysed any system in all its details.

But let us suppose that he did, and that it followed the diatonic pattern I have specified. It remains true that the focus of the fragment is on the three concords, octave, fifth and fourth, which give the attunement its fundamental structure, on their sizes (whether conceived as ratios or as 'distances') and the quantitative relations between them, and on the way in which the fourths and fifths interlock to form the octave-attunement's integrated and symmetrical skeleton. We may pertinently add that if the attunement took the form that Nicomachus attributes to it, any substantial emphasis on its minutiae would have undermined the impression of complete symmetry conveyed by the fragment's second sentence, since the pattern of intervals into which the fourth between *hypatē* and *mesē* is divided is not perfectly replicated in the fourth between *tritē* and *nētē*. It seems to me that a good deal (not quite all) of the information retailed by Boethius fits neatly into the same agenda.

If the system's symmetry is an important consideration, we should be able to specify its mid-point, the fulcrum upon which equivalent structures at the top and the bottom are balanced. In its centre is the interval of a tone between *mesē* and *tritē*, in the epogdoic ratio 9:8; and 9:8 cannot be factorised into two equal ratios of integers (see p. 272 above). But there is an interval, the *diesis*, which is very close to half a tone, and which plays a part in the attunement's analysis. According to our present hypothesis it is a recognisable interval of a musical scale, and its ratio is known. On the basis of the terms of this ratio, 256:243, Philolaus converts the *diesis*, quite improperly, into a simple quantity represented by the number 13, and by a similar strategy he identifies the tone with the number 27. The larger remnant of the tone, amounting to 14 units, is then named as the *apotomē*, and the difference between the *diesis* and the *apotomē*, one unit, is called the *komma*. If we divide the whole tone in half, then, we are cutting it into segments amounting to a *diesis* plus half a *komma* each. Doggedly pursuing this line of thought, Philolaus assigns a name also to half a *komma*; it is the *schisma*. Hence we can identify the absolute centre of the system as lying at a point higher than *mesē* by a *diesis* and a *schisma*, and lower than *tritē* by the same amount.

This reconstruction of Philolaus' reasoning is obviously speculative, but it accounts for almost every ingredient of Boethius' reports. It falls short by failing to explain why he also provides a name, *diaschisma*, for half a *diesis*; and it says nothing about his reflections on the significance of the numbers 27 and 13. I shall comment briefly on the second of these points in due course, but I confess that I have no plausible explanation of the first. My interpretation has the substantial advantage, however, that it places these manoeuvres in precisely the context to which Boethius assigns them, that of the division of the tone. Rival interpretations, as I have said, have located them in another setting, the analysis of enharmonic and chromatic scales. In that project there will be a role for the *apotomē* (in the chromatic) and for the *diaschisma* (in the enharmonic).[27] But there will be none for the *komma* and the *schisma*; and there is no independent evidence in Boethius or anywhere else that Philolaus attempted an analysis of more than one variety of scale. Again, if it is agreed that his primary interests were cosmological rather than musical, it will not have been the special features of the various different types of scale that captured his attention, but their similarities, the shared features which gave all of them their structural coherence and distinguished them from arbitrary jumbles of notes and intervals. If we can extrapolate from the consensus of later writers, that coherence is created, in every case, by exactly the symmetrical interweaving of fourths and fifths, tying together the opposite ends of the octave, which Philolaus places at the heart of his enterprise.

HARMONICS AND COSMOLOGY

Let us now try to place all this in the context of Philolaus' cosmology. The only direct textual evidence that frag. 6a belongs with the more familiar and more explicitly cosmological frag. 6 (the first paragraph of DK 44B6) is very weak, consisting in nothing but the fact that they are quoted together

[27] For details see Burkert 1972: 398.

as a continuous passage by Stobaeus. But we are now in a position to see that they do indeed hang intelligibly together, whether or not frag. 6a followed frag. 6 immediately in Philolaus' treatise (which I am inclined to doubt), along with certain other Philolaan fragments (notably frags. 1–4), the material in Boethius and much of Aristotle's testimony. Students of Presocratic philosophy will be no strangers to the first half-dozen fragments of Philolaus; they have been discussed many times, and here I shall pick out only a few essentials.[28]

According to Diogenes Laertius, frag. 1 was the opening sentence of Philolaus' book. 'Nature in the *kosmos* was fitted together (*harmochthē*, from a verb cognate with *harmonia*) from things that are unlimited and things that impose limit, both the whole *kosmos* and all the things in it.' Frag. 2 offers an argument to show that things cannot all belong either to the class of 'unlimiteds' or to that of 'limiters'; and frag. 3 sketches another reason why they cannot all be 'unlimiteds'. Frag. 6 returns to the same ideas. It tells us that none of the things that there are, and of which we can have knowledge, could come into existence if the 'being' of the things from which the *kosmos* is constituted did not include both limiters and unlimiteds. Philolaus continues:

> But since these principles (*archai*) are not alike or of the same kind, it would have been impossible for them to be coherently integrated (*kosmēthēnai*) with one another if *harmonia* had not come upon them, in whatever way it came into being. Things that were alike and of the same kind had no need of *harmonia*, but it was necessary for things that were unlike, and not of the same kind or rank, to be held together by a *harmonia* of the sort that would hold them together in a *kosmos*.

Earlier Presocratics too had made use of the idea that material objects and the universe as a whole are made up of different or opposed ingredients fitted together or 'harmonised' in some special way; and medical writers more or less contemporary with Philolaus sometimes expressed this relation in explicitly musical terms, or assimilated musical *harmonia* to a universal principle through which mutually hostile elements are amicably integrated into a unity.[29] The language in which Philolaus expresses this thought is unusually abstract, and he does not clarify his remarks – at least, not in this passage – by mentioning examples of these limiters and unlimiteds. We can make some reasonable guesses about unlimiteds, however, and a strong

[28] For judicious brief discussions of these and other fragments see Kirk, Raven and Schofield 1983: 324–8, Kahn 2001: 23–38; see Burkert 1972 and Huffman 1993 for fuller treatments and bibliographies.

[29] See the passages from Presocratic philosophers cited in n. 4 above. For allusions in medical sources see the Hippocratic *De victu* (*On Regimen*) 1.8.2, a passage we shall look at briefly below, and the speech attributed to the doctor Eryximachus in Plato, *Symposium* 185e–188e, especially 187a–e.

candidate for the role of limiter is number. Length, for instance, conceived simply as such, is unlimited and inadequate to define any particular thing; it becomes something determinate only when pinned down by a number.[30]

I do not think we should hesitate to connect frag. 6a and the reports by Boethius closely with the ideas of frags. 1–6. The notion of an interval in the dimension of pitch picks out nothing determinate. It is unlimited in the sense that it has no definite boundaries; it can be as large or as small as you like. For any actual interval to come into being, this unlimited 'amount' must be limited by number, whether the numbering involves ratios or quantified linear distances. But the association of limit with the unlimited can give rise to a coherent whole (*kosmēthēnai*) only if '*harmonia* comes upon it'; and in the context of frag. 6a *harmonia* is musical attunement.[31] The fragment shows us how intervals determined by number are 'locked together', as frag. 6 has it, into a well-ordered and symmetrical whole.

Before glancing again at Boethius' reports in this connection, I want to add a word or two about a passage in a work of a very different sort, the Hippocratic treatise *On Regimen* mentioned in n. 29. It is generally assigned a date around 400 BC.[32] The writer is discussing the development of the human foetus.

> When it has moved to a different place, if it attains a correct *harmonia* containing three concords, *syllabē*, *di' oxeiōn* and *dia pasōn*, it lives and grows using the same nourishments as before. But if it does not attain *harmonia*, and the low-pitched (*barea*, lit. 'heavy') elements do not become concordant with the high-pitched (*oxea*, 'sharp') in the first concord or the second or that which runs through all (*dia pantos*), if just one of them is faulty the whole tuning (*tonos*) is useless. (*On Regimen* 1.8.2)[33]

The passage's resemblances to Philolaus leap to the eye. It echoes his musical terminology (transposed into the Attic dialect), which is uncommon elsewhere, and its use of the word *harmonia* (which recurs in 1.9) exactly parallels Philolaus' deployments of it in frags. 6 and 6a. *Harmonia* is simultaneously the principle which must govern relations between diverse elements in the developing foetus if it is to become a living whole, and a structure

[30] Compare frags. 3–4, where we are told first that nothing could be known if all things were unlimiteds, and secondly that all knowable things possess number and could not be known if they did not.

[31] This is a slightly different perspective from that of Eryximachus at Plato, *Symp*. 187a1–c5, where the hostile elements to be harmonised are high and low pitch, but the underlying thought is of the same order.

[32] See e.g. Joly 1960: 203–9.

[33] I follow here the text printed in the Budé edition of 1967. It involves several emendations, of which the most important is in the list of concords in the first sentence. The words '*syllabē, di' oxeiōn*' were suggested by Bernays in 1848 and independently by Delatte in 1930 to correct the MSS text *syllēbdēn diexiōn* (or a slight variant), which makes no good sense. Like most commentators I regard the emendation as certain.

spanning the compass of an octave, properly organised into substructures spanning a fourth and a fifth. It seems rather likely that the author had read Philolaus *On Nature*, and was drawing on it directly. What is particularly striking is that unlike many later non-musical authors who exploit notions in 'Pythagorean' harmonics, he makes no allusion to ratios; that aspect of frag. 6a is completely elided. For the writer of *On Regimen* it was apparently the musically described structures of an octave-attunement, not its interpretation in mathematical terms, that could provide a useful model for the organisation of a healthy embryo.[34] If Philolaus' priorities could be read in this way in his own time, this passage gives further support to my contention that the principal focus of his argument, in that part of his work, was on relations between intervals conceived in the manner of musicians rather than mathematicians, and that his approach had much in common with that of the empiricists.

We return now to Boethius' reports. I have explained how I interpret the bulk of them in relation to Philolaus' picture of a *harmonia* as a fully integrated structure, but there is one more aspect of the account at *Inst. mus*. 3.5 that calls for comment. Besides revealing how the numbers 27 (for the tone) and 13 (for the *diesis*) are derived from the intervals' ratios, Boethius tells us that Philolaus invested these numbers with significance in their own right. According to Philolaus,

> the tone had its origins in the number that constitutes the first cube of the first odd number, for that number was greatly revered among the Pythagoreans. Since 3 is the first odd number, if you multiply 3 by 3, and then this by 3, 27 necessarily arises; and this stands at the distance of a tone from the number 24, the same 3 being the difference. For 3 is an eighth part of the quantity 24, and when added to the same it gives the first cube of 3, 27.[35]

As for the *diesis*, its number is 13 partly because this is the difference between the terms of its ratio, 256 and 243, but also 'because the same number – that is, 13 – consists of 9, 3 and unity, of which unity holds the place of the point, 3 the first odd line, and 9 the first odd square'.[36]

Philolaus' main strategy here is clear enough; he aims to trace these musical numbers back to an origin in the number 3.[37] The Pythagoreans' interest in the symbolic meanings of numbers is well known, and I shall

[34] Hence it is misleading to use this passage, as Burkert does (1972: 262), to confirm that 'the numerical ratios' play a part in Hippocratic embryology.

[35] This translation is that of Bower 1989 with slight modifications.

[36] Translation from Bower 1989.

[37] Given that preoccupation and the use he makes of squares and cubes, he can hardly have failed to notice that one of his key numbers, 243, is the product of the square and the cube of 3. The other, 256, is based similarly on the numbers 2 and 4. It can be factorised as the square of 4 multiplied by itself, or as the cube of 4 multiplied by the square of 2, or in several other such ways.

not attempt any general exploration of that bewildering territory; but a few points deserve emphasis. First, the Greeks were finding non-mathematical significance in specific numbers long before the time of the Pythagoreans, whose speculations in this area probably involved more reflection on pre-existing ideas than autonomous invention.[38] Secondly, the number 3 seems to have been among the most important in this wider cultural context, and so too, in at least one rather striking setting, were its square and cube.[39] Finally, notions connected with these and other numbers in Pythagorean and other sources can be placed on a continuum from the less to the more abstract and mathematical. Thus we find a widespread association of 3 with the male and 4 with the female;[40] at a slightly more abstract level comes the link between the number 3 and completeness, since it has a beginning, a middle and an end;[41] while Philolaus' excavations of this number from 27 and 13, and his identification of it with the first odd number and the first odd line, are based in genuine arithmetic (and a species of geometry), even if his motivation has more impressionistic cultural roots.

The number 3 and the unit, in terms of which Philolaus analyses 27 and 13, also join 2 and 4 as elements in the celebrated *tetraktys* of the decad, the numbers 1, 2, 3, 4, adding up to the perfect number 10.[42] I need not harp on this theme, which is discussed by every modern writer on the Pythagoreans; and readers will hardly need to be reminded of Aristotle's sarcastic comments about the importance of this perfect number in Pythagorean astronomy.[43] But it is perhaps worth underlining Aristotle's remark in the same passage that the Pythagoreans linked to their account of the organisation of the universe 'everything they could find in numbers and in *harmonia* that agreed with the attributes and parts of the heaven'. This allusion to *harmonia* in the context of comments on the role of the number 10 points to a connection between this number and those involved in musical analysis; and it surely lies in the fact, repeatedly mentioned in later sources, that it is the numbers involved in the ratios of the primary concords, 2:1, 3:2 and 4:3, that come together to form the *tetraktys* of the decad.

Given that Aristotle drew much of his information about Pythagoreans from Philolaus, this would seem to be another instance of Philolaus' fusion

[38] See Burkert 1972: 474–9, with his references to the fascinating study of Homeric numerology in Germain 1954.

[39] Burkert 1972: 474–5. The numbers 3 and $3 \times 9 = 27$ appear in the context of the cult of the Erinyes at Soph. *OC* 479 ff.

[40] Burkert 1972: 475–6.

[41] See Aristotle, *De caelo* 268a10 ff.

[42] On ways of connecting 2 and 4 with Philolaus' numbers, see n. 37 above.

[43] Aristotle, *Metaph.* 986a3–12.

of the mathematics of ratio with thoughts about individual numbers. Very probably these ideas were widespread in the earlier Pythagorean tradition. The connections between the *tetraktys* and the harmonic ratios may be associated with the 'pebble-diagrams' attributed to Eurytus and others, in which the number 10 is represented by an array of ten dots or pebbles set out in rows to form an equilateral triangle, with one dot at the top, two in the row below it, three in the next and four in the last; and the ratios of the concords appear in the relations between the numbers of dots in adjacent rows.[44] This is a long way from anything we would consider soberly as 'mathematics'. The link between the ratios of *harmonia* and the hypothesis that there are ten principal bodies in the cosmos (which Aristotle derides) does not lie in the supposition that there are ten elements in a *harmonia*, integrated in determinate ratios; there are not. It has nothing to do with a ratio-based 'harmony of the spheres'.[45] Pride of place is given, once again, to numbers extracted individually from the ratios, not to the relations expressed by the ratios themselves.

But there is another twist in this curious maze. A passage in Nicomachus' treatise on arithmetic asserts that some people, 'following Philolaus', adopted the name 'the harmonic mean' (*mesotēs harmonikē*) for the mathematical mean otherwise known as 'subcontrary' (*hypenantia*), and offers a rather odd explanation for Philolaus' usage which we need not investigate.[46] The mean in question can be defined as follows. When we have three terms, A, B, C in descending order of magnitude, B is the harmonic or subcontrary mean between A and C if the fraction of C by which B exceeds it is the same as the fraction of A by which A exceeds B. In the example most commonly used by Greek writers, A is 12, B is 8 and C is 6. A second report, in a work of Iamblichus, asserts that Philolaus among others 'is found to have made use of musical proportion (*mousikē analogia*)'.[47] Musical proportion involves four terms, including between the two extremes both the harmonic mean and the arithmetic mean, which is such

[44] See e.g. Kahn 2001: 31–2.

[45] It should follow that when Aristotle wrote his famous account of just such a ratio-based 'harmony' at *De caelo* 290b12 ff., he was not thinking of a theory held by Philolaus or perhaps by any Pythagorean (he does not actually say who its champions were), though most commentators conclude that he was. But inferences of that sort are risky in this treacherous territory, and although we might find it hard to square the notion that there are ten celestial bodies with the thesis that they form a coordinated pattern of attunement, or to accommodate smoothly to one another the ways in which numbers and ratios are treated in the two contexts, it is perfectly possible that all these ingredients and numerical manipulations could co-exist comfortably in Philolaus' methodologically malleable mind.

[46] Nicom. *Arith.* 2.26.2.

[47] Iambl. *In Nicom.* 118.23; this and the passage of Nicomachus are printed together as DK 44A24.

that if we have three terms as above, A − B = B − C. The proportion is regularly illustrated by the series 12, 9, 8, 6, in which 9 is the arithmetic mean between 12 and 6, and 8 is their harmonic mean. The reliability of these reports has been carefully assessed by Carl Huffman, who concludes that there is nothing suspect or improbable about what Iamblichus says, but that Nicomachus' statement (which he takes to imply that Philolaus originated the use of the term 'harmonic mean') is open to serious doubt.[48] This scepticism about Nicomachus rests on foundations which I think can be undermined, but that issue need not concern us; if Philolaus understood the notion of musical proportion he must certainly have known of the subcontrary mean, whether or not he was the first to call it 'harmonic', since the latter is involved in the definition of the former.

In the case of Iamblichus too, Huffman's assessment is not conclusive, as I think he would agree. But its grounds are quite strong; let us assume that he is right. There is no difficulty in giving musical proportion a place in Philolaus' harmonic constructions, as we know them from frag. 6a. When the relations between *hypatē*, *mesē*, *tritē* and *nētē* in its octave attunement are specified as ratios, the ratio of *nētē* to *tritē* is 4:3, that of *tritē* to *mesē* is 9:8, and that of *mesē* to *hypatē* is 4:3. These relations are exemplified in the sequence 12, 9, 8, 6, and the whole construction is a specimen of musical proportion.

So far we have met no difficulties. But when we ask what role the concept of musical proportion might have played in Philolaus' scheme, we may hesitate about the answer. Means and proportions have serious work to do in the harmonic constructions of Archytas and Plato in the next century; their systems are grounded in the principle that notes (conceived as the terms of ratios) earn their places in a coherent pattern of attunement by intervening between other notes as means of appropriate sorts (see pp. 302–3 below). But the coherence of Philolaus' system, so far as our direct evidence takes us, rests on no such mathematically sophisticated foundations. It depends, as we have seen, on a conception of symmetry which makes no appeal to these means and proportions (of which we hear nothing in the other Philolaan fragments and testimonia), and in which the ratios themselves play only a secondary role. It appears to have been assembled partly from the culturally shared repertoire of musical knowledge (frag. 6a), and partly from piecemeal, unsystematic and mathematically naïve manipulations of numbers.

48 Huffman 1993: 168–71.

Perhaps, then, Huffman's doubts about Nicomachus' report are justified; but we can still accept what Iamblichus says, though not for Huffman's reasons. On what I take to be the likeliest interpretation of his statement, it is not only plausible but uncontroversially true. He does not say that Philolaus, or anyone else whom he names, understood the notion of musical proportion or even mentioned it, but only that it is something 'they are found to have used'. This need mean no more than that if you look at the system of ratios in Philolaus' *harmonia*, you will find that the terms are related in musical proportion, as indeed they are.[49] We even have a test case for this interpretation, since another writer in Iamblichus' list of those 'found to have used' musical proportion is 'Plato in the *Timaeus*'. Terms in the relevant proportion certainly figure in Plato's harmonic construction at *Tim*. 35b–36b, as they do in Philolaus, but the word *analogia*, 'proportion' does not occur in the passage; and though a 'proportion' involving four terms is mentioned earlier (32a–c), its description at 32b makes it clear that it is not 'musical proportion', and it is not called by that name. Iamblichus is merely identifying systems whose elements fit into this proportional pattern. He does not say that the conception was available to these systems' authors, still less that they understood and explicitly deployed it; and if that is what he meant to imply, the evidence of the *Timaeus* shows that he is not to be trusted.

My overall conclusion, then, is that the usual interpretations of Philolaus' work in harmonics are mistaken at least in their emphasis. I agree that his contributions to the science are to be placed in the context of his cosmology; and he was certainly familiar with the use of ratios in musical analysis and knew the ratios of certain intervals. But so far as our evidence goes, he did not treat ratios as the only acceptable representations of musical intervals, as later Pythagoreans and Platonists did, or even as the most illuminating ones. Nor did he require that a coherent system of attunement should be generated by any mathematical principle of proportion. He used ratios to underpin the (relatively new) idea that intervals can be identified by their sizes, and as a starting-point for identifying certain intervals with particular numbers. His approach combines calculations involving ratios with the 'linear' conceptions of practical musicians and empirical theorists,

[49] Cf. Huffman 1993: 169. 'That Philolaus knew of and used the "musical proportion" (12, 9, 8, 6) is very probable in the light of the musical theory found in F6a. In fact when Iamblichus says Philolaus is "found using" the musical proportion he may well be referring exactly to F6a.' So he may. But if frag. 6a is the basis of Iamblichus' remark, we have no reason to accept Huffman's inference that Philolaus 'knew of' this type of proportion and deliberately used it. The presence of musical proportion in the *harmonia* is determined independently, as it turns out, by the facts of musical practice.

and it is the latter that play the most important part in his account of the system's integration and symmetry.

None of this conflicts with the testimony of Aristotle. He explicitly mentions the Pythagoreans in connection with ratio-based harmonics only once, in the course of his long exposition of Pythagorean ideas at *Metaph.* 985b23 ff.[50] Here his reference to the 'ratios of the *harmoniai*' (985b31–2) appears in a context preoccupied with their emphasis on numbers, not on relations between numbers, as the basic elements in reality; and the discussion runs on seamlessly to their contention that there must be ten principal bodies in the universe, since 10 is the perfect number. A remark a little later in the *Metaphysics* (987a22–5) seems to reflect a slippage between ratios and individual numbers very similar to those we have noted in Philolaus. The Pythagoreans' definitions are superficial, he says, because 'they supposed that the first [item, perhaps number] to which a given definition belonged was the essence (*ousia*) of the thing, as if one supposed that double and the dyad are the same, on the grounds that the double always belongs first to the number 2'.[51] In general Aristotle is sharply critical of the Pythagoreans, especially of their misunderstandings about number; whereas he takes a positive view of the credentials of mathematical harmonics as such, as we shall see in Chapter 13, and exploits some of its propositions for his own purposes. It does not necessarily follow, of course, that what he knew of specifically Pythagorean (or Philolaan) harmonics was infected with their confusions about number in wider metaphysical or cosmological contexts, but I can see no reason why it should have been an exceptional case, somehow quarantined from the contagion.

50 The ratios of the concords play an essential role in the theory of the harmony of the spheres discussed by Aristotle at *De caelo* 290b12 ff., and when he mentions 'certain people' who hold this theory he may well have Pythagoreans in mind. But as I noted above, he does not say so.

51 This strategy is not identical with the one used by Philolaus in defining the *diesis* as the number 13, but their affinities are obvious.

CHAPTER II

Developments in Pythagorean harmonics: Archytas

Archytas of Tarentum, according to Aristoxenus, was the 'last of the Pythagoreans' in the continuous tradition stretching back to the founder. The precise dates of his birth and death are not known, but his life seems to have spanned almost exactly the same period as Plato's (427–347 BC), and if the seventh Platonic letter is genuine they were personal friends.[1] Certainly there are close connections between their writings on harmonics, though the evidence about the relation is not always easy to interpret.

Archytas was by all accounts a remarkable man, distinguished simultaneously as a philosopher, a mathematician, an inventor of ingenious gadgets, a statesman and a military commander, and admired also for his personal qualities, his kindness, resourcefulness, self-control and affection for children. He counts as a heroic figure in the early history of mathematical harmonics, which he raised to new levels of conceptual and technical sophistication and channelled in unprecedented directions. Only a few fragments of his writings survive, along with reports in various (and variously reliable) later sources, but they are enough to allow us to reconstruct a coherent general outline of his approaches and ideas, and to piece certain parts of his work together in some detail.

Half a millennium later, in the most accomplished of all Greek essays in the mathematical style of harmonics, Ptolemy speaks of Archytas with evident admiration. Though Ptolemy's general attitude to his predecessors is less contemptuous than Aristoxenus' and his comments on them less consistently vituperative, he rarely mentions them except to criticise; and though his approval of Archytas is also undercut by criticisms and reservations, he writes of him more warmly than of any other theorist. I have argued elsewhere that he borrows from Archytas rather more freely than he

[1] See Plato, *Ep*. 7, 338c–339d. For a thorough examination of everything to do with Archytas' life and work see Huffman 2005. I refer to him repeatedly in this chapter, and our differences over certain points should not disguise the fact that his is by far the best study of Archytas that we have. There is a useful shorter survey in Kahn 2001: 39–48.

admits, and may have regarded Archytas' work, in important respects, as the one true ancestor of his own.[2] It is from Ptolemy that we learn most of what we know about the details of Archytas' harmonic analyses, and about some of the principles which helped to determine the form they took. The evidence is therefore far removed in time from the original, and Ptolemy himself was probably dependent on an intermediate source.[3] There is nevertheless no good reason to doubt its authenticity, though we must of course be careful to distinguish, where we can, the data which Ptolemy had at his disposal from the inferences he drew from them and the opinions he expressed about them.

ARCHYTAS AND PTOLEMY'S 'PRINCIPLES OF REASON'

> Archytas of Tarentum, of all the Pythagoreans the most dedicated to music, attempted to preserve that which is in accordance with reason, not only in connection with the concords but also in the divisions of the tetrachords, on the grounds that commensurability between the differences is intrinsic to the nature of melodic intervals. (Ptol. *Harm*. 30.9–13)[4]

Earlier passages of the *Harmonics* allow us to give this initially obscure remark a clear interpretation. In saying that Archytas 'attempted to preserve that which is in accordance with reason', Ptolemy means, in the first place, that the ratios he assigned to the intervals of an attunement were not chosen merely as those which seemed to fit the musical data most closely, but were required to conform to the dictates of a 'rational' principle. In the language of this work, a 'rational' principle is one grounded in mathematics, and we can infer that in Ptolemy's opinion Archytas constructed his analyses, in part at least, on the basis of mathematical considerations.

That is a little vague; mathematical considerations are very various. But we can be more precise. Ptolemy means also, as the sequel shows, that the principle Archytas tried to apply approximated to his own notion of harmonic rationality, in short, that it was identical with or closely related to the central principle governing his own analyses of musical systems. A 'difference' (literally 'excess'), in this context, is the difference between the terms of a ratio expressing the size of a musical interval. The items with which such a difference must be commensurable are the terms of the

2 Barker 1994b, cf. Barker 2000a: 65–7, 120–8.

3 The most likely source is a musical theorist named Didymus, who can tentatively be dated to the middle of the first century AD, and who appears to have had direct access to a work by Archytas himself; see p. 438 below.

4 Where references to Ptolemy's *Harmonics* are given in this form they are cited by page and line of Düring's edition. References in the form 'II.13' are by the work's book and chapter numbers.

ratio themselves; and it is commensurable with them if it constitutes a unit by which each term can be exactly measured. It must therefore be an integral factor of each. This is precisely the condition which must be met, on Ptolemy's own view, by the ratio of every incomposite melodic interval, that is, every individual step of a well-formed scale.[5] Ptolemy explains earlier that the Pythagoreans agreed that a condition of this sort must apply to the ratios of concords.[6] Thus the difference between the terms of the ratio of the perfect fourth or the perfect fifth, for example (3:2 and 4:3), which is in each case 1, is an integral factor of both the ratio's terms. In ratios such as 5:3, 9:5 and so on, this condition is not satisfied. It entails, in fact, that every appropriate ratio, when expressed in its lowest terms, has the form $n + 1{:}n$, so that the two numbers are successive integers. Mathematicians call such ratios 'superparticular'; the corresponding Greek adjective is *epimorios*, and I shall call them 'epimoric' (for a more technical definition of 'epimoric' see n. 7 below).

Ptolemy's statement implies that in the opinion of theorists other than Archytas, this condition must be fulfilled by the ratio of any interval which is genuinely a concord, and that Archytas' originality lay in his extension of the principle to 'melodic' intervals as well. This implication needs to be qualified in at least three ways. First and straightforwardly, not all the concords recognised by the Pythagorean theorists Ptolemy mentions have epimoric ratios, as Ptolemy knew and had previously explained; the ratios of some of them are multiple.[7] But this is a quibble. Ptolemy is abbreviating the point in the interests of his immediate focus of attention, to which only epimoric ratios are relevant. Secondly, if Ptolemy means that Archytas' position was an advance on that of his predecessors, there is no independent evidence that the principle (in the form 'all concords must have ratios that are either multiple or epimoric') was current before the time of Archytas himself. There are passages in Plato and in Aristotle that may hint at it;[8] but

5 Ptol. *Harm.* 1.7, 1.15. 6 *Harm.* 1.5.

7 *Harm.* II.18–20, 12.1–5. The ratio of an octave plus a fifth is 3:1, and that of a double octave is 4:1. These ratios are called 'multiple' (in Greek *pollaplasios*), since the larger term is an exact multiple of the smaller; and here it is not the difference between the terms that constitutes the unit by which they are measured, but the lower term itself. The ratio of the octave itself, 2:1, is also multiple rather than epimoric, even though its terms are successive integers. An epimoric ratio is defined as one in which the larger term is equal to the smaller plus a unit-part (one half, or one third, and so on) of the smaller; whereas in the ratio 2:1 what has to be added to the smaller to produce the larger is not a part of the smaller but the whole of it. Ratios such as 9:5, which are neither multiple nor epimoric, are usually called *epimereis* in Greek, which I shall anglicise as 'epimeric' (in the Latin-based vocabulary of mathematicians they are 'superpartient'), or are described by the phrase 'number to number'.

8 Plato, *Tim.* 35b–36b (where the expressions 'multiple' and 'epimoric' do not occur, but the only ratio mentioned which does not have one of those forms is explicitly distinguished as a ratio of 'number to number'), Ar. *De sensu* 439b–440a, where the concords, like pleasant colours, are said to 'depend on the best-ratioed (*eulogistoi*) numbers'.

it first appears explicitly in surviving sources in the Euclidean *Sectio canonis*, which dates from the end of the fourth century at the earliest.[9] It is from this treatise, I believe, that Ptolemy drew much of his information about what he thought of as non-Archytan Pythagorean theory; and if he supposed that a direct enunciation of the principle predated Archytas (he does not explicitly say this) he may well have been wrong. Such general theoretical axioms are remote from what little we know of fifth-century Pythagorean thought, even from that of Philolaus, and it is at least as likely that it originated with Archytas himself. Thirdly, as Ptolemy goes on to emphasise, if Archytas extended the principle to melodic as well as concordant intervals, he did not do so consistently. This fact, which stands out obtrusively from the 'divisions of the tetrachord' to which we shall shortly turn, may reasonably make us hesitate to accept Ptolemy's contention that Archytas 'attempted' to assign epimoric ratios to all melodic intervals, and yet for some reason failed.

Whether it is applied only to concordant intervals or also to melodics, the principle can hardly stand without some argumentative backing. Purely arbitrary principles are obviously unacceptable, and this one needs justification urgently, since in either form it has awkward consequences. If it is restricted to concords, it requires that one interval which Greek ears recognised as concordant (as did the logic of some harmonic theorists) is in fact no such thing; the ratio of the octave plus a fourth is 8:3, which is neither multiple nor epimoric. If it is extended to melodic intervals a number of other difficulties will arise; most straightforwardly, one of the intervals involved in a simple structure repeatedly treated as fundamental by mathematical theorists, including Plato and the author of the *Sectio canonis* and perhaps Philolaus before them, must be rejected as improperly formed. The interval is the one which, together with two whole tones in the ratio 9:8, fills out the span of a fourth (ratio 4:3), and whose ratio is 256:243.[10] By the criterion which Ptolemy attributes to Archytas and adopts himself, it cannot be a 'rationally' formed melodic interval, and the structure in which it plays a part is flawed.[11]

9 [Eucl.] *Sect. can.* 149.12–24. In most respects the approach taken to harmonics in the *Sect. can.* differs substantially from that of Archytas, as it is presented by Ptolemy and in other sources to be considered below; but it certainly draws on his mathematical work in proposition 3, and may do so elsewhere. The treatise is discussed in Ch. 14 below.

10 We have already met this interval in connection with Philolaus, and its ratio is specified in Boethius' reports about him, which (so I have argued) deserve to be taken seriously. There is no doubt that Archytas knew the ratio, as we shall see below. But the evidence for Philolaus and Archytas comes from much later sources, and the first surviving text to quantify the interval as a ratio is Plato *Tim.* 36b2–5.

11 For Ptolemy's own very different treatment of a system of this sort see *Harm.* 39.14–40.20.

The requirement that musical relations should conform to mathematical principles of any sort reflects the idea that the quality we perceive in some pitch-relations and not in others, on the basis of which we call the former and not the latter 'musical', is the audible expression of a privileged variety of mathematical form. The notes of a melody or of the scale on which it is based, or those bounding a melodically acceptable interval, stand in relations to one another which, as we put it, make musical sense. They are not merely different from one another but are also in some way akin, coming together as elements in a coherent unity. It is the translation of this intuition into mathematical language that generates the requirement of 'commensurability'. The terms of a ratio, corresponding to the notes of an interval, can come together coherently only if the relation between them can be grasped as intelligible. This means that they must be capable, in Plato's phrase, of being 'measured against one another'; and this is possible only if the unit by which each term is measured is the same. This unit, furthermore, must exist as one of those elements in the ratio whose auditory counterparts are detected by the musical ear, since the perception of an interval's musicality need not depend on our comparing it, or its notes, with anything outside itself. Out of all this emerges the requirement that the ratio of a concordant or a melodic interval must be either multiple or epimoric. If it is multiple, the smaller term is the measure of the greater. If it is epimoric, the difference between the terms is the measure of both. If it is neither, the ratio contains no component which can serve as the unit of measurement for both terms, and the relation between them, from that perspective, is uncoordinated and unintelligible.

I have argued elsewhere that this account corresponds to Ptolemy's own justification for his mathematical principles.[12] I cannot prove that Archytas reasoned along the same lines; but if he did indeed adopt, in any form, the principle of 'commensurability' which Ptolemy attributes to him, it is hard to see how considerations of any other sort could have led him to it.[13] If Ptolemy's evidence is anywhere near the mark, however, it brings to light one very important point, regardless of the nature of Archytas' reasoning. An enquiry within which the forms available to the ratios of musical intervals are determined by mathematical axioms involves a much more sophisticated conception of science than does an unembroidered

[12] Barker 2000a: 74–87.

[13] No fourth-century writer preserves an argument of this sort, though there may be traces of it in Aristotle's treatment of ratios at *De sensu* 439b–440a, and cf. [Ar.] *Problems* 19.41. The argument offered at *Sect. can.* 149.17–24 for the principle as it applies to concords seems, on the face of it, trivial, and can in any case have no bearing on questions about melodic intervals; see pp. 375–8 below.

attempt to correlate recognised intervals with the ratios that are judged, by empirical means, to correspond most closely to them. This is a science which goes beyond a formal interpretation of the facts, to account for them on the basis of high-level abstract principles. In Aristotle's terms, it is a science which is not content simply to record 'the fact that . . .', but articulates also 'the reason why . . .'; and it is significant that Aristotle marks the distinction between 'empirical' and 'mathematical' harmonics in precisely this way.[14]

ARCHYTAS' DIVISIONS OF THE TETRACHORD: MATHEMATICAL PRINCIPLES AND MUSICAL OBSERVATIONS

In the passage from which our short excerpt was taken, Ptolemy goes on to set out, and to criticise, three 'divisions of the tetrachord' which, so he says, Archytas articulated. He names them as enharmonic, chromatic and diatonic, and they do indeed correspond, in their general outlines, to the schemata which are given these names in sources from Aristoxenus onwards. Whether Archytas himself designated them in this way we cannot tell, but that is unimportant. In either case, assuming that they are authentic, they are the earliest analyses we have which bring together three different types of system matching the three genera of Aristoxenian theory. If Philolaus' *harmonia* has affinities with any of the genera, it is (in Aristoxenus' terms) diatonic,[15] and so is that of Plato's *Timaeus* (pp. 319–21 below). Elsewhere in Plato, and in Aristotle, the various patterns of attunement are designated in a completely different way, as Dorian, Phrygian and so on, corresponding to distinctions drawn by the *harmonikoi* within the framework of structures most closely related to the enharmonic; and the *harmonikoi* discussed by Aristoxenus, as we have seen, dealt only with enharmonic systems. The remarks of the author of the Hibeh papyrus fragment, though he mentions all three names, betray the absence of any clear distinction between chromatic and diatonic. So far as we can tell from the surviving evidence, Archytas' detailed and fully quantified analysis of the three harmonic systems was unprecedented.

In Aristoxenian theory, an enharmonic tetrachord fills up the span of a perfect fourth with two very small intervals (quarter-tones) at the bottom, followed by a single step of a ditone. The two lowest intervals in a chromatic

[14] Ar. *An. post.* 78b34–79a6; see pp. 353–61 below.

[15] Or, just possibly, enharmonic; see Winnington-Ingram 1928. But I have argued (above pp. 276–8) that Philolaus' analysis may not presuppose any particular way of filling in the concordant intervals between the attunement's fundamental notes.

tetrachord are larger than those of the enharmonic, but when taken together they too occupy less than half the compass of the fourth. The upper interval is correspondingly smaller than it is in the enharmonic, but must occupy more than a tone and a quarter. In diatonic tetrachords the space occupied by the two lowest intervals together amounts to at least half the span of the fourth and commonly to more; hence the highest interval must always be a tone and a quarter or less. With certain qualifications to be noted below, Archytas' three schemata fit these criteria. Their intervals are of course expressed as ratios, rather than as Aristoxenian tonal distances. The ratios of intervals in his tetrachords, reading from the top downwards, are as follows:

Enharmonic: 5:4, 36:35, 28:27;
Chromatic: 32:27, 243:224, 28:27;
Diatonic: 9:8, 8:7, 28:27.[16]

Despite their general affinities with the Aristoxenian patterns, these divisions differ from his in several ways, and present other features too that seem puzzling. I do not claim that the list which follows is complete. (i) Most of Aristoxenus' divisions, though not all, include two intervals that are equal. None of Archytas' does. (ii) In the Archytan divisions, and in no others known to us, the lowest interval in each genus is the same. (iii) In those of the Aristoxenian divisions which gained widest currency, the two highest intervals in diatonic, taken together, are equal to the sum of the two highest intervals in chromatic, and also to the highest interval in enharmonic; all three magnitudes are ditones. In Archytas, the same equality holds between the overall spans of the two upper intervals in diatonic and chromatic (though the span in question is not exactly two whole tones), but the highest enharmonic interval is smaller. (iv) Aristoxenus enunciates the rule that the central interval of the three in a division can never be smaller than the lowest, a principle in which he is followed by Ptolemy, and to which the great majority of divisions known to us conform.[17] Archytas' enharmonic breaks this rule.[18] (v) Finally and most strangely, the ratios of the two highest chromatic intervals are strikingly anomalous, since unlike all the others in these systems, neither is epimoric. It is this peculiarity that Ptolemy has in mind when he comments that Archytas failed in his

16 For a meticulous study of these systems and of the passage of Ptolemy where they are set out and discussed (*Harm*. 30.3–32.23) see Huffman 2005: 402–28.

17 See Aristox. *El. harm*. 52.8–12, Ptol. *Harm*. 32.7–10.

18 So does the chromatic attributed by Ptolemy to Didymus in the tables of *Harm*. II.14 (see also Ptolemy's comment on the division in the preceding chapter, at *Harm*. 68.27–9). But such divisions are exceedingly rare.

attempt to achieve 'commensurability' in the ratios of melodic intervals (*Harm*. 30.13–14, 32.1–3).

One might set about explaining these features of the divisions in either of two ways, or through a combination of both. One approach would treat them as consequences of some mathematical operation through which the divisions were generated; the other would seek to account for them empirically and historically, as characteristics of genuine musical systems current in Archytas' time, ones that were later modified or abandoned. The latter strategy would presuppose that Archytas had set himself, like the *harmonikoi* but in a different way, to specify the structures of attunements actually used by contemporary musicians, rather than to derive a collection of purely theoretical systems from abstract mathematical principles. At least part of his intention, on this reading, would have been to represent the data of real musical practice in mathematically intelligible terms. Even if he conceived his results as significant also in a metaphysical or cosmological context, he would nevertheless have treated the perceptual data as essential evidence to which his conclusions must conform. This hypothesis is supported by the fact that despite the differences I have noted, there are very close correspondences between Archytas' divisions and some of those described, in different language, by Aristoxenus.[19] It is encouraged also by the various apparent anomalies in the divisions, which a completely theoretical scheme might be expected to eliminate, and again by the broadly approving attitude of Ptolemy, for whom the perceptual data constituted a crucial control on the work of mathematical construction. The same conclusion might be drawn from the mere fact that Archytas' divisions seem designed to accommodate all three of the main categories into which Aristoxenus and his successors divided the melodic systems of musical practice, including the hitherto vaguely conceived chromatic, and that his diatonic abandons the pattern (9:8×9:8×256:243) derivable from a simple manipulation of concords and their ratios, which was the central, if not the only point of reference for most metaphysically minded theorists.[20]

[19] They are examined in detail in Winnington-Ingram 1932, cf. Huffman 2005: 412–14.

[20] The interval of a tone (ratio 9:8) can be constructed in practice by moving from a given note through a perfect fifth upwards followed by a perfect fourth downwards, or the reverse. When two such tones have been constructed in succession, the *leimma* (ratio 256:243) will be left as the residue of a perfect fourth taken from the original note. The procedure is useful because, as Aristoxenus says, it is much easier to construct concords such as the fifth and the fourth accurately by ear than it is to construct discords such as the tone or the *leimma* (which Aristoxenus treats as a semitone); but it cannot be used to construct, for instance, the quarter-tones of Aristoxenus' enharmonic or the one-third tones of one of his forms of the chromatic, or any of Archytas' divisions. It is mentioned and explained in several theoretical sources (see especially Aristox. *El. harm*. 55.3–56.12, [Eucl.] *Sect. can*. prop. 17), and was very probably used by musicians themselves. The Greeks called the procedure *lēpsis dia symphōnias*; I shall refer to it as the 'method of concordance'.

I am tolerably confident that this diagnosis is part, at any rate, of the truth. If that is so, it is important, not just because these analyses, so construed, would allow us a glimpse of what early fourth-century melodies were like, but because it would mark a turning-point in the story of Pythagorean harmonics, a shift from a focus on exercises in mathematical cosmology to a direct engagement with the details of musical practice. At the same time one must not lose sight of the fact that mathematical considerations also had a part to play in these divisions' construction,[21] and hence that their peculiarities may not all arise straightforwardly from the attempt to provide a faithful representation of the 'facts'. The principle of 'commensurability' to which Ptolemy refers probably figured among these considerations (though in a slightly different form, as I shall try to explain); so almost certainly did another general mathematical thesis which will be outlined later (pp. 302–3).

Even if these divisions are designed to reflect patterns of tuning in contemporary musical currency, they cannot be supposed to replicate them exactly. For one thing, musicians' tuning-practices are variable, and the theorist's descriptions of them are not. For another, neither the instruments available to Archytas nor the discriminations of the human ear could be so finely calibrated as to guarantee that such-and-such an interval's ratio is exactly the one designated in his divisions, and not one marginally larger or smaller. It would be impossible, by purely observational techniques, to establish that the ratio of the middle interval in enharmonic, for instance, even in just one paradigmatic performance, was precisely 36:35 (and not e.g. 72:71 or 107:105). Archytas' figures must have been chosen, from among those within the range to which the perceptual evidence guided him, to fit a pattern determined by assumptions of a mathematical sort.

Let us turn now to the five features of the divisions which I listed above. (i) The fact that no two ratios in any one division are equal is almost inevitable if all the ratios are epimoric. There is only one way in which the ratio of the fourth, 4:3, can be divided into three sub-ratios all of which are epimoric and two of which are equal, and it is too remote from musical usage to be a candidate for serious consideration.[22] If we assume, then, that the non-epimoric ratios of Archytas' chromatic are exceptional cases calling for special explanation, this characteristic of his divisions emerges

[21] At least, I take it to be a fact, as I shall explain below, and I shall try to soften the impact of Huffman's criticisms of my position (2005: 416–17).

[22] It factorises 4:3 as $8:7 \times 8:7 \times 49:48$, which could only be construed as an impossible version of a diatonic division, containing two intervals rather larger than a tone, and a residue a good deal smaller than a quarter-tone. (The general formula for factorising $n+1:n$ into three epimorics two of which are equal is $(n + 1:n) = (b + 1:b) \times (b + 1:b) \times (b^2:b^2 - 1)$, where $b = (2n + 1)$. Thanks, here again, to Dr Jonathan Barker for lightening my mathematical darkness on these matters.)

directly from his otherwise consistent attribution of a privileged status to epimorics, supplemented by only the most rudimentary piece of musical knowledge. (ii) The equality of the lowest intervals in all three divisions cannot be explained on purely mathematical lines. Other theorists assign equal intervals or ratios to the lowest positions in certain forms of the chromatic and the diatonic, but the corresponding interval in enharmonic is always smaller; and those who represent intervals as ratios have no difficulty in constructing an enharmonic division in which this condition is met and all the ratios are epimoric, even if they agree with Archytas that its highest ratio is 5:4.[23] If there was a theoretical basis for Archytas' contention that the lowest interval is the same in all three genera, we have no idea what it was. It seems more likely that it reflected what he believed himself to have discovered, empirically, in the performing practices of contemporary musicians.[24]

(iii) Three quite different considerations can be brought to bear on the third issue, and all of them, I think, are sound. When Aristoxenus assigned the same span, a ditone, to the highest interval in enharmonic and to the sum of the two highest intervals in paradigmatic forms of each of the others, these equalities presumably represented his interpretation of aspects of familiar contemporary tuning-procedures (though this point will shortly be qualified). The fact that Archytas' highest enharmonic interval is smaller than the relevant composite intervals in diatonic and chromatic has one boringly obvious explanation. Given that the lowest interval in each of the divisions is the same, it is straightforwardly impossible to equate just one of the remaining intervals in enharmonic with the sum of the remaining two intervals in the others. It also seems easy enough to account for the fact that the highest enharmonic interval is not a ditone ($9:8 \times 9:8 = 81:64$) but slightly less ($5:4 = 80:64$). So small a difference looks marginal, and 5:4 is not only the epimoric ratio that approximates most closely to that of the ditone, but is satisfyingly simple. It is only to be expected that the next epimoric in order after those of the fifth and the fourth, 3:2 and 4:3, would find a significant role in a system grounded, at least partly, in mathematical conceptions.

[23] Ptolemy's enharmonic is $5:4 \times 24:23 \times 46:45$; that of Didymus is $5:4 \times 31:30 \times 32:31$. See the tables of Ptol. *Harm.* II.14.

[24] One might wonder whether conventions differed in his native Tarentum from those established elsewhere, in Athens, for example. But there is no evidence to support such a guess, and the fact that musicians were constantly on the move from one centre to another, and competed in the major festivals against others from quite different parts of the Greek world, points at least to a broad similarity between practices in different areas, even if there were minor regional variations. For discussion of the lives led by musicians see Bélis 1999, and more briefly Barker 2002a, ch. 8.

The third consideration is altogether more interesting. When Aristoxenus writes about the enharmonic, he recognises that the form he describes and treats as authentic is not acceptable to most contemporary listeners, and does not exactly match the tuning-pattern used for enharmonic melodies by most contemporary musicians. It is nevertheless, he asserts, the finest of all melodic systems, as is clear to those who have immersed themselves in the 'first and second of the ancient styles' (*El. harm.* 23.1–12). Aristoxenus' 'genuine enharmonic', then, is a historical reconstruction rather than a contemporary reality. The 'first and second ancient styles' in which he locates it are probably to be identified with those he attributes elsewhere to the aulete Olympus and his successors, who date (in so far as they are not merely figures of legend) from the seventh century and the early sixth.[25]

By these standards what passed for enharmonic music in Aristoxenus' time hardly deserved the name,[26] and the fact that Archytas' enharmonic does not tally completely with his may be a sign that it reflected fourth-century practice more faithfully. Certain additional details that Aristoxenus provides give firmer outlines to this initially vague suggestion. What musicians in his time found unacceptable, it turns out, was not the minuteness of its quarter-tones, as one might have expected,[27] but the size of its highest interval, the ditone.

It is not surprising that those who are familiar only with the currently prevailing type of melodic composition exclude the ditonal *lichanos*;[28] for the majority of people nowadays use higher *lichanoi*. The reason for this is their constant passion for sweetening (*glykainein*). An indication that this is their goal is that they spend most of their time working in the chromatic, and when they do, occasionally, approach the enharmonic they force it towards the chromatic, and so distort the melody. (*El. harm.* 23.12–22)

These people's 'distortions' of the enharmonic thus involve a reduction in the size of its upper interval, so that it sounds more like a chromatic system, and this practice arises from their pursuit of 'sweetness'.

These observations can be applied directly to the enharmonic of Archytas. Its upper interval is smaller than the ditone of the 'noble and ancient' enharmonic that Aristoxenus prefers. So far as quantitative descriptions reveal, whether they are Aristoxenian or Pythagorean, the difference is very slight. Aesthetically, however, it is not negligible. What makes one interval

[25] [Plut.] *De mus.* 1134f–1135b. [26] But cf. *El. harm.* 48.15–20.

[27] This is, however, the objection raised by unnamed musicians and theorists in a passage almost certainly derived from Aristoxenus at [Plut.] *De mus.* 1145a–c, and by a number of later writers.

[28] That is, the location of *lichanos*, the second-highest note from the top of the tetrachord under discussion, at a ditone below its highest note, *mesē*.

strike the ear as harsher, and another as smoother or 'sweeter', is the larger number of 'beats' or 'interferences' set up between the notes bounding the former. The number of beats generated in a given time from the interaction of pitches whose frequencies are related in epimoric ratios of small numbers is much less than that from those in more complex ratios involving larger terms. It is this phenomenon that allowed ancient musicians, and allows modern ones too, to distinguish such intervals as the perfect fifth and the perfect fourth, by ear, with great precision from their near neighbours. The relative infrequency of the beats they set up explains why Greek theorists described the notes involved in these concordant intervals as blending smoothly together, in a way that discordant pairings did not. By this criterion, an interval in the ratio 5:4, that of the upper interval of Archytas' enharmonic (and of a modern major third), will be noticeably 'sweeter' than a true ditone in the ratio 81:64. Tiny though the mathematical difference is, it takes only a moderately sensitive ear to appreciate its musical effect.[29]

(iv) Most Greek constructions are consistent with Aristoxenus' rule that the central interval of the three in the tetrachord is never smaller than the lowest. Archytas' enharmonic is not. This anomaly stems directly from the constraints imposed on the other two intervals in the division. Archytas was apparently convinced that the lowest interval in each genus was the same (see (ii) above). It must therefore appear in diatonic as well as in enharmonic and chromatic; and it is impossible to construct a plausible diatonic division in which the lowest interval is smaller than one in the ratio Archytas attributes to it, 28:27. His estimate of this interval's size in diatonic is in fact smaller than that of any other Greek theorist. Given that the lowest interval in his enharmonic must be no smaller than that, the second constraint, that its highest interval must come close to the compass of a ditone, ensures automatically that the middle interval will be smaller than the lowest. Hence even if there were already in Archytas' time a tendency towards the practice formalised in Aristoxenus' enunciation of the rule, it was apparently (in Archytas' opinion) over-ridden, in the case of the enharmonic, by these other considerations.

(v) We come, finally, to the peculiarities of Archytas' chromatic ratios, 32:27 × 243:224 × 28:27. Their oddity is accentuated by the fact that they approximate closely to those of an easily constructible chromatic division

[29] For some additional points on this matter see Huffman 2005: 412–13, with his references to Winnington-Ingram 1932.

which uses only epimoric ratios, 6:5 × 15:14 × 28:27.[30] An explanation of the strange-looking Archytan ratios is therefore urgently needed, and Ptolemy supplies one. If he and his source are to be trusted, the explanation is Archytas' own. 'He locates the second note from the highest in the chromatic genus from the one that has the same position in diatonic. For he says that the second note from the highest in chromatic stands to its counterpart in diatonic in the ratio of 256:243' (*Harm.* 32.2–6).

The explanation makes good sense. Let us assume that the tetrachord under consideration is the one focused on by most theorists, the one bounded by *mesē* at the top and by *hypatē mesōn* at the bottom. Immediately above *mesē*, in a typically constructed octave, lay a whole tone, followed by another identical tetrachord. The ratio of the highest interval in Archytas' diatonic tetrachord is that of a whole tone, 9:8; above it, then, lay another interval of the same size. The ratio 256:243, by which the second note of Archytas' chromatic is lower than its counterpart in diatonic, is that of the so-called *leimma*, the residue of a perfect fourth when two whole tones have been subtracted. Hence Archytas' second chromatic note could be reached by a very simple method, by tuning downwards through a perfect fourth from the note immediately above *mesē* (that is, *paramesē*). This is just the sort of method that a practical musician might be expected to adopt. Granted, then, that Ptolemy's explanation echoes that of Archytas, as his 'for he says' implies, we can infer that Archytas based his assessment of the highest chromatic ratio on his observation of musicians' tuning-procedures. The ratio is that of a perfect fourth less the whole tone lying above the tetrachord, i.e. 9:8 × 256:243 = 32:27. Once that is established, the ratio of the middle interval is determined by straightforward arithmetic, since that of the lowest interval is treated as a constant, 28:27. It is the ratio by which 4:3 exceeds 32:27 × 28:27, which is 243:224.

Both mathematical ideas and empirical observations seem therefore to be at work behind these features of Archytas' divisions, and in the most interesting cases the two coincide or combine. The mathematician's preference for epimoric ratios,[31] especially those whose terms are small numbers, comes

[30] This is the 'soft chromatic' described by Ptolemy (*Harm.* 35.4–6). Its highest interval exceeds that of Archytas' chromatic by an interval in the ratio 81:80, which is quantitatively marginal; and by the criteria outlined above the Ptolemaic version is perceptibly sweeter.

[31] I must insist that this *is* a mathematician's preference, and that purely musical considerations cannot explain all the major features of Archytas' analyses, though they are certainly involved as equal partners along with mathematical ones. What Huffman (2005: 416) describes as Burkert's '*reductio ad absurdum* of the thesis that Archytas was striving for superparticular ratios' is nothing of the sort (and Burkert himself did not regard it in that light), since it actually presupposes that Archytas deliberately adopted epimoric ('superparticular') ratios in most places; and that cannot be

together with contemporary musicians' predilection for 'smooth' or 'sweet' intervals such as the concords and the 5:4 major third, and with musically straightforward methods of tuning an instrument. The explanation of the highest chromatic interval that we have unearthed from Ptolemy's remarks suggests the possibility of accounting in a similar way for the ratio of the lowest interval, 28:27, which is common to all three attunements. Imagine, once again, that the specimen tetrachord is placed next to a whole tone in the ratio 9:8, but that this time the tone lies below the tetrachord instead of above it (as it does if the tetrachord is the one between *paramesē* and *nētē diezeugmenōn*). The interval formed by combining this tone with the interval at the bottom of an Archytan tetrachord is another epimoric whose terms are relatively small numbers; it is 7:6 ($9{:}8 \times 28{:}27 = 7{:}6$). This interval (the 'septimal' minor third, by contrast with the slightly larger and commoner version of this interval, in the ratio 6:5) is no doubt less easy to tune precisely than a perfect fourth or a major third. But in this case the musician has a second point of reference to help guide his ear. An interval in the ratio 7:6 falls short of a perfect fourth by an interval whose ratio is another simple epimoric, 8:7. Hence if he begins from the note a tone below the tetrachord and tunes upward through a perfect fourth, the upper boundary of the Archytan tetrachord's lowest interval will lie at a pitch which divides this fourth in the ratios 7:6 and 8:7. These are the two largest epimorics into which the fourth can be divided; and the musician can home in on the correct pitch for the note he is tuning by listening for the moment at which its relations with *both* the bounding notes of the fourth reach their maximum of 'smoothness'. There is scope for the possibility, then, that Archytas identified the ratio of the lowest interval of his tetrachords at least partly on the basis of his observation of performers' tuning-procedures, interpreted in the light of his mathematical assumptions.

At first sight, Archytas' divisions not only abandon the principle that all relevant intervals should have epimoric ratios (since two of the chromatic intervals do not), but also show no special preference for ratios whose terms are small numbers (since even if we leave the chromatic ratios on one side, those of the ratios 28:27 and 36:35 are hardly 'small' in this context). It

a mere accident. All it shows for Huffman's purposes is that it would have been theoretically possible for Archytas to find epimorics very close to the ratios he actually uses in chromatic, and that of course is true, as I have explained. If the reconstruction of his procedure I gave above is correct, his reasons for not making these adjustments were musical, not mathematical, and to that extent Huffman is right. But that does nothing to account for Archytas' exclusive choice of epimorics at every other point in his systems.

turns out, however, that if the scheme is expanded from a single tetrachord to a complete octave, every note in all three genera can be located from some other note in the same genus through an interval in a small-number epimoric ratio, one whose terms lie within the range of the numbers 1 to 9; and when the divisions are set out on that basis, every ratio of the form n + 1:n within that range, from 2:1 to 9:8, figures at least once in the pattern. By moving through a series of intervals in epimoric ratios, with every move beginning and ending on a note included in the system, one can reach every note of each system from any starting-point whatever; they are all linked by an unbroken chain of epimorics. By this method, in fact, one can reach all the notes of all three systems from any starting-point in any of them, since some of the moves will bring us to the 'fixed' notes which all of them share. I shall not go into the details here, which can readily be grasped from diagrams printed in two of my previous publications.[32]

I stand open to correction, but I do not see how the fact that all three systems fit so smoothly into this matrix can be regarded as a mere fluke. It suggests that Ptolemy's attribution to Archytas of a principle grounding acceptable divisions in epimoric ratios is well founded, but that his criticism is misplaced. Archytas need not have been trying, and inexplicably failing, to give epimoric ratios to every interval between adjacent notes in his divisions. His system is consistent with a different though related principle, that every note in each genus must be constructible, mathematically, as a term in an epimoric ratio with some other note in the same division. While there are connections, as we have seen, between small-number epimorics and efficient methods of tuning, it seems at least very likely that the mathematical requirement was an influential partner in Archytas' process of construction; what musicians actually did was interpreted in the light of

[32] Barker 1989a: 46–7, presented again in a slightly different way in Barker 2000a: 124. Huffman 2005: 417, argues that even so Archytas cannot have used the principle that all melodic intervals must be constructible by moves through superparticular (epimoric) or multiple ratios, since Ptolemy's evidence shows that he identified the position of one chromatic note by another method, which involves the non-epimoric ratio 256:243. So he does (p. 299 above); but that misses the point. What I meant (or since memory can mislead, at any rate what I mean now) is not that he used the 'epimorics only' principle as the basis of a method of discovery, but as a principle against which conclusions reached by any means whatever needed to be tested. In this particular instance, according to the account I gave earlier and whose outlines Huffman accepts (2005: 417–18), he 'discovered' the position of the chromatic note by inference from his observation of musicians' tuning-practices; but according to my present hypothesis he would have modified his conclusion if it had proved impossible, on reflection, to reach the position it assigned to that note through small-number epimorics. Of course this cannot be proved, and the 'epimoric map' given in my diagrams is my own construction, not that of Archytas. But I would argue that the pattern it shows is far too neat and comprehensive to be coincidental.

what they 'ought' to be doing, where this 'ought' reflects the perspective of philosophical and mathematical rationality.

THE THREE MATHEMATICAL MEANS

Another brief extract from Archytas' work points in a similar direction.[33] 'There are three means in music,' it begins. 'One is arithmetic, the second geometric and the third subcontrary, which people call "harmonic".' It goes on to define each of these mathematical means, and to pick out some of the properties of the relations generated when a mean of each sort is inserted between two other terms. Let me briefly recapitulate the basic facts about these means, which we have already glanced at in connection with Philolaus (pp. 283–4 above). The arithmetic mean, B, between two numbers A and C (where C is the smaller) is such that A−B = B−C. The geometric mean is such that A:B = B:C. The harmonic or subcontrary mean has a more complex definition. It is such that A–B is the same fraction of A as B−C is of C (e.g. if A−B is one quarter of A, then B−C is one quarter of C, as it is if A is 20, C is 12, and the harmonic mean between them, B, is 15).

As we noted earlier, the simplest musical application of this system of means is in the construction of concords. When an octave is divided in the familiar way into two perfect fourths separated by a whole tone, the structure is demarcated by four notes, of which the second note stands to the first in the ratio of the fourth, 4:3, the third note stands to the second in the ratio of a tone, 9:8, and the last note stands to the third in the ratio of a fourth, 4:3. (Hence the ratio of the third note to the first and of the last note to the second is that of the fifth, 3:2, and the ratio of the last to the first is that of the octave, 2:1.) The smallest whole numbers which capture this arrangement are 6, 8, 9, 12.[34] Of these four numbers, 9 is the arithmetic mean between 6 and 12 (since 9−6 = 12−9), and 8 is their harmonic mean (since 12−8 is one third of 12, and 8−6 is one third of 6). The geometric mean does not appear directly in this construction, but the terms of the ratios of a series of octaves (2:1 for the octave, 4:1 for the double octave, 8:1 for three octaves, and so on) form a geometric progression, 1, 2, 4, 8 . . ., so that 2, for example, is the geometric mean between 1 and 4 (since 2:1 = 4:2).

Intervals in the ratios 3:2, 4:3 and 9:8 can therefore be constructed mathematically by the insertion of the arithmetic and harmonic means between

[33] DK 47B2 (Archytas frag. 2 in other modern references), quoted at Porph. *In Ptol. Harm.* 93.6–17. For a full discussion see Huffman 2005: 162–81.

[34] This sequence of numbers appears repeatedly in later sources in connection with these calculations.

terms in the ratio defining the octave, these terms themselves being part of a geometric sequence. All four of the other epimoric ratios which map out Archytas' divisions, those in the series from 5:4 to 8:7, can be constructed in a similar way, in two easy steps, by introducing means between the terms of the ratios of the lesser concords, the fifth and the fourth. One can use means of either the arithmetic or the harmonic type; in each case the ratios between the middle term and the extremes will be the same, in reverse order (just as the insertion of an arithmetic mean between terms in the octave-ratio 2:1 gives 4:3 at the top and 3:2 at the bottom, while the insertion of the harmonic mean places them the other way round). Thus the ratio of the fifth, 3:2, can be represented equally well as 6:4. The arithmetic mean between its terms is 5, whose insertion generates the ratios 6:5 and 5:4. Inserting the harmonic mean between the terms of the same ratio (in this case most helpfully represented as 15:10) will give the same results in the opposite order.[35] An arithmetic mean can be placed between the terms of the ratio of the fourth, 4:3, if it is expressed as 8:6, where the arithmetic mean is 7, and this gives us the remaining ratios in the Archytan scheme, 8:7 and 7:6. To find an easy way of inserting the harmonic mean instead, first multiply the terms of the ratio 4:3 by 7, and the same results in the opposite order will again appear.[36] Hence every one of the ratios needed to define these divisions can be constructed by introducing arithmetic and harmonic means between the terms of the ratio of the octave, and then locating either arithmetic or harmonic means between the terms of the ratios of the smaller concords, into which the octave was divided in the first stage of the operation.[37]

ARCHYTAS' THEOREM ON THE DIVISION OF EPIMORIC RATIOS

These observations increase the probability that Archytas' analysis was driven in part by an impulse towards mathematical systematisation, and

[35] In the sequence 15, 12, 10, the number 12 is the harmonic mean between 15 and 10. 15:10 = 3:2, 15:12 = 5:4, and 12:10 = 6:5.

[36] In the sequence 28, 24, 21, the number 24 is the harmonic mean between 28 and 21, since 28–24 is one seventh of 28, and 24–21 is one seventh of 21. 28:21 = 4:3, 28:24 = 7:6, and 24:21 = 8:7.

[37] Huffman 2005: 169–70, rightly emphasises the fact that in frag. 2 Archytas does not only give the definitions of the 'three means used in music', but also explains that when the mean is arithmetical, the interval (*diastēma*) between the larger terms is smaller than that between the smaller terms; when it is geometrical the two intervals are the same size; and when it is harmonic the interval between the larger terms is the greater of the two. As Huffman points out, Archytas does not explain 'what we are to make of these comparisons'. But we might guess that one of his reasons for mentioning them was that he noticed and was intrigued by the fact that the insertion of arithmetic and harmonic means between terms in the same ratio always generates the same pair of ratios in the opposite order, as in the cases anatomised above.

there would be nothing surprising in that. He was renowned for his work in mathematics, which extended far beyond issues relevant to musical theory.[38] He also worked out a mathematical proof of a theorem that bears directly, and very importantly, on musical issues, the proposition that there is no mean proportional, 'neither one nor more than one', between terms in epimoric ratio.[39] Here a 'mean proportional' is a geometric mean. Between two terms in epimoric ratio, A and B, there is no intermediate term, X, such that A:X = X:B; nor is there a series of terms, X, Y, . . . , Z, such that A:X = X:Y = · · · = Z:B. The proof applies, of course, to epimoric ratios wherever they occur, not just in musical contexts. But it is most obviously relevant, and has the most striking consequences, in the field of harmonics,[40] and the fact that Archytas thought it significant enough to deserve formal proof is another sign of the privileged position he attributed to epimoric ratios in this domain. Its upshot, as Greek theorists understood it, is that no interval whose ratio is epimoric can be exactly halved, or divided into any number of sub-intervals all of which are equal;[41] and it was of major importance (though it was often misconstrued) both in the development of mathematical approaches to the division of intervals, and in later controversies between Aristoxenians and Pythagoreans. It was held to show, among other things, that it is impossible to divide the span of a tetrachord (a fourth, ratio 4:3) into equal sub-intervals, or into intervals all of which are multiples of the same unit; a tone (9:8), similarly, cannot be divided into any number of equal parts. In that case there must be something seriously wrong with the analyses of Aristoxenus, and indeed of the *harmonikoi* too, who talk blithely of quarter-tones, half-tones and the like, and suppose that a tetrachord in enharmonic, for example, contains two intervals of a quarter-tone each and another eight times that size, so that the whole fourth is divisible into ten equal segments.

38 See especially DK 47A14, quoted from Eutocius, which records a complex and sophisticated solution to the problem of constructing two mean proportionals between given terms, a question to which Archytas had reduced the famous 'Delian problem' of doubling the size of a cube. Other ancient references are given at DK 47A15. For a masterly discussion of this bewildering material see Huffman 2005: 342–401.

39 He is credited with the proof at Boethius, *Inst. mus.* III.11 (DK 47A19), and there is a closely related version at [Eucl.] *Sect. can.* proposition 3 (for some musical applications see propositions 10, 16, 18). For further discussion see pp. 351, 356, 380 below; and cf. Burkert 1972: 442–7, Knorr 1975 ch. 7, Huffman 2005: 451–70.

40 Both of the full statements of the proof preserved in ancient sources occur in musicological texts, not works of pure mathematics; so do most of the briefer references to it.

41 See e.g. *Sect. can.* props. 16 and 18, cf. Theo Smyrn. 53.1–16, 70.14–19 (in both of the passages in Theo Smyrn. the proof is misunderstood), Ptol. *Harm.* 24.10–11.

Others before Archytas had stumbled on some of the more obvious and awkward consequences of the proposition that Archytas proved; we have already seen Philolaus struggling manfully if incompetently to quantify a division of the tone (and, if my reconstruction is correct, the octave) into equal parts.[42] But there is no evidence and little likelihood that any of Archytas' predecessors had constructed a theorem to prove the proposition, or had even conceived or stated it as a truth about epimorics in general. Our reports about his theorem show beyond doubt, as Huffman says, 'that Archytas understood the demands of a rigorous deductive proof'.[43] It seems equally clear that he thought of harmonic theory as a discipline in which rigorous proofs of a mathematical sort should play a significant part.

HARMONICS, PHYSICAL ACOUSTICS AND MUSICAL PRACTICE

Archytas' harmonic constructions, in their mathematical guise, fit smoothly with his attempt to represent the pitch of sound as a quantitative variable;[44] what we perceive as a higher pitch is, in physical fact, a more rapid and vigorous movement (see pp. 27–9 above). The ratios assigned to musical intervals can therefore be straightforwardly understood as ratios between the speeds at which the notes forming the interval are transmitted through the air. The theory has obvious flaws and was adjusted in various ways by later writers. It nevertheless enables musical thought to cross a scientifically important boundary, that between the contents of our sensory and aesthetic awareness and an objectively real state of affairs outside us and independent of us. It provides the impressions we receive in our musical experience with an intelligible basis in the world accessible to the quantifications and measurements of a physicist; and the patterns we perceive as networks of musical relations find an objective counterpart in the dynamic interplay of aerial travellers, some faster, some slower, intertwining in precise schemes of mathematical order. It is their well-choreographed dances that are presented to our ears as attunements and melodies.

Pythagorean musical theory had been associated from the outset with cosmological and semi-scientific ideas. What mattered to Philolaus and his

[42] Huffman's comment (2005: 418) that Philolaus' procedure 'showed that the octave cannot be divided in half' is slightly misleading. The construction given in frag. 6a evidently *does* not so divide it, but if my reading of Boethius' evidence is correct he did not think the task impossible, and did his best to provide resources by which the outcome of such a division could be quantitatively expressed.

[43] Huffman 2005: 470.

[44] Archytas frag. 1, on which see Huffman 2005: 103–61, with the copious references to other discussions which he provides.

predecessors, so far as we can tell, was the mathematically specifiable system of order which certain fundamental musical relations exemplified and to which harmonic analysis gave access; and there are no good grounds for attributing to them an interest in the musical phenomena as such. Whatever may have been true of Hippasus and his other predecessors, Philolaus himself seems to have been uninterested even in the nature of the niche occupied by sounds in the world of matter and movement. Archytas' work opened the way for richer and more detailed explorations of such abstract patterns of order, most notably by turning the spot-light on the special status of epimoric ratios, by demonstrating techniques for manipulating them in the construction of harmonic divisions, by proving these ratios' resistance to equal division, by his classification and definition of the three 'musical means' and by his deployment of these means in his analyses of attunements.

But his studies in physical acoustics point also to a scientific interest in sound and pitch themselves, and reinforce the impression given by his tetrachordal divisions that he was concerned, much more directly than earlier Pythagoreans, with the domain of the audible for its own sake.[45] I have argued that the divisions were designed to reveal the mathematical organisation inherent in real musical practice; where I differ from other modern commentators, it is usually because they give even more weight than I do to the role of strictly musical considerations in Archytas' work. No matter whether their view or mine is nearer the mark, we are bound to conclude that Archytas pointed mathematical harmonics in an entirely new direction, and one, we must add, in which rather few of his successors seem to have followed him; most of them reverted to a more abstract approach, detached from the phenomena of musical experience, allied to the theories of Philolaus and heavily influenced by Plato.[46] It may be Archytas that Aristotle has in mind when he remarks that the role of mathematical harmonics is to explain the facts which empirical harmonics records; it could hardly do that if the system whose structures it explains, by grounding them in mathematical principles, were only the 'rational' constructions of abstract theory.

[45] A passage to be considered in Ch. 13 suggests that Archytas, like Philolaus and Plato, may also have thought of harmonics as relevant to issues in cosmology and metaphysics; see pp. 329–38. But the grounds of this hypothesis are not secure enough to carry much weight.

[46] The principal exception in later antiquity is Ptolemy (second century AD); there are traces of an interest in bringing mathematics into connection with musical realities in the work of Didymus (first century AD) and perhaps Eratosthenes (third century BC). I have discussed these people's approaches elsewhere: see Barker 2000a on Ptolemy, 1994a on Didymus, 2003 on Eratosthenes, on whom see also Barker and Creese 2001, Creese 2002: 126–56.

Archytas would also appear, in this respect, to be an appropriate target for Socrates' comment in the *Republic*, that the Pythagoreans concern themselves too much with things that are audible (531a1–3, cf. c1–2). Socrates' second and related criticism, however, seems wide of the mark where Archytas is concerned. 'They do not ascend to problems, to investigate which numbers are concordant and which are not, and in each case why' (531c2–4).[47] Whatever Socrates means by 'concordant' here,[48] the quest for principles which govern the ratios of well-formed intervals and complexes of intervals was as much a part of Archytas' enterprise as the analysis of contemporary tuning-systems. Plato was certainly aware of the fact, at least when he wrote the *Timaeus*, since in that dialogue he adopts one of Archytas' own central theoretical conceptions and redeploys it for his own purposes, and hints at his recognition of another (pp. 320–1 below). The *Republic*'s remarks about 'Pythagoreans' tell us, in the end, rather more about Plato's agenda than they do about the ideas of the people criticised; and about Archytas in particular, I think, they tell us no more than we knew already. They can indeed be misleading. We shall return to them in their own context in Chapter 12.

[47] Huffman 2005: 414 seems to conflate the two criticisms, and finds Archytas to be an appropriate target for both. But the distinction should be preserved; see further pp. 315–18 below.
[48] This poses a substantial problem; see p. 316 below.

CHAPTER 12

Plato

We have already called on Plato's help from time to time, and that is as it should be, even if we set aside his gigantic stature in the Western philosophical tradition. Once due allowance has been made for the fact that the conversations in his dialogues are fictional (though most of the characters are not), and for his own attitudes, prejudices and philosophical aims, the dialogues are an unparalleled source of information about the cultural milieu inhabited by elite Athenians and intellectually eminent visitors to the city in the late fifth and early fourth centuries, about the beliefs they held and the issues they discussed, and about the ways in which their ideas were expressed and debated. It is only to be expected, too, that music should figure prominently as a topic in these conversations, in view of the central place it held in every Greek's cultural experience. In fact, however, Plato shows rather little interest in it in his earlier work. There are various passing allusions, and a small handful of passages which from other perspectives have real theoretical interest; but in the context of a study of harmonic science, none of them has much to offer.[1]

MUSICAL ETHICS IN THE *REPUBLIC* AND THE *LAWS*

With the *Republic* the scene changes. Musical imagery becomes abundant, and sometimes does serious philosophical work; and there are two major set-pieces on musical topics. They treat the subject in quite different ways, and are segregated from one another in different parts of the dialogue. There is the passage on harmonics in Book VII, parts of which we have

[1] Leaving aside occasional, quite casual references to music (e.g. *Lysis* 209b4–7), there are only four passages in dialogues written (probably) before the *Republic* that have any musicological substance; they are *Lach*. 188c–d, *Prot*. 326a–b, *Symp*. 187a–e, *Phaedo* 85e–86d. It is worth noting that none of the statements in these passages is put into the mouth of Socrates himself, and that he discusses only one of them (very critically), at *Phaedo* 92a–95a. None of them provides much grist for our present mill, but for brief comments on the *Symposium* passage see pp. 72 n.4, 280 n.31 above. I examine them in a different context in Barker 2005a, chs. 3–4.

already reviewed and to which we shall shortly return. Much earlier, in Book III, in the context of a discussion of children's education, there is an elaborate examination of the ways in which different melodic and rhythmic styles reflect different dispositions of the human soul, and of the powerful influence they can exert, for good or ill, on the development of people's characters. The ethical significance of a melody is made to depend, in this passage, on the characteristics of the *harmonia*, the pattern of attunement (named here as Mixolydian, Lydian, Dorian, Phrygian and so on), which provides its framework of notes and intervals. It is these *harmoniai*, not the individual melodies as such, which are the bearers of ethical attributes, 'imitating' desirable or undesirable psychic dispositions and drawing their hearers' souls into their own likeness. Socrates therefore proposes a drastic purge of the *harmoniai* (and corresponding purges of musical instruments and of rhythmic structures too), banning from his ideal city all but the most edifying of them, Dorian and Phrygian. Plato's last work, the *Laws*, does not point an accusing finger at any specific *harmoniai* or varieties of scale; but it is if anything even more vehement in its condemnation of musical forms and practices that have a tendency to corrupt the souls of their executants and hearers. All the subtle complexities and aesthetically pleasing sophistications introduced into music by 'modern' composers are to be rejected, in favour of a noble simplicity that is supposed to have characterised the music of a lost golden age.[2]

Plato's Athenian spokesman in the *Laws* insists that musical practices in the city must be closely monitored and controlled, and that those who are to sit in judgement on them must be very thoroughly qualified for their role. Among other things, they must have a first-rate technical understanding of the elements of musical compositions and their inter-relations, at a much higher level than that of the citizen-amateurs who regularly performed in choruses, and even than that of most composers (*Laws* 670a–671a). But though the passage makes it clear that these people must master the kinds of analysis provided by harmonic scientists, it tells us nothing of their details. In Book III of the *Republic* there is no allusion to harmonic theory at all. In Socrates' ethically motivated witch-hunt among the *harmoniai*, the culprits are detected by their ethical or emotional resemblance to conditions of the soul. His arguments have sometimes been thought to presuppose an

[2] Plato projects his musical ideal onto the period before the Persian Wars, around 500 BC (*Laws* 700a–701b, with 698a–699e). But his golden age of music never really existed, of course, and certainly not at the time he imagines it. Far from being ruled by strict, stable and orderly norms, the musical world of the early fifth century was a ferment of innovation and controversy. For a brief and helpful survey see West 1992a: 341–55.

analysis of the structures of these *harmoniai*; but he gives none, and there is no reason to believe that Plato had such an analysis in the back of his mind and was implicitly relying on it. Socrates poses here as a musical ignoramus ('I do not know the *harmoniai*', *Rep*. 399a), deferring to the greater expertise of his companion Glaucon; and his arguments are based on Glaucon's identification of the *harmoniai* which match certain very impressionistic descriptions ('mournful', 'sympotic' and so on), without the least reference to structural considerations. No harmonic technicalities are involved, and the arguments should be understood in their own terms. The one point at which technical issues are raised and even Glaucon's knowledge proves inadequate has nothing to do with harmonics; and here Socrates shelves the matter as one on which they must consult the real expert, Damon.[3]

Plato's reticence about the details of harmonic theory in the *Laws* may be due simply to the understandable judgement that the context did not call for them; here he is not even attempting to identify precisely which musical attunement-systems should be approved of and which condemned. He needs only to point out that an adequate understanding of the ethical and socio-political significance of music in its various forms depends partly on a knowledge of musical 'theory', and specifically of harmonic analysis. The *Republic* is another matter, since the situation calls for the best arguments that can be mustered to support Socrates' purge of the *harmoniai*, and yet the arguments he puts forward against some and in favour of others do not depend at all on representations of the different structures which distinguish them. But if there is a reason for this over and above the contextual ones which might reasonably be adduced, I think it is straightforward. Of the two existing approaches to harmonic theory distinguished in *Republic* Book VII, Socrates and Glaucon regard that of the empiricists with undisguised contempt, and though the Pythagoreans are also criticised, Socrates seems to imply that their methods could be adapted to better effect. If either approach is to be preferred, then, it is theirs; and it has very close affinities with Plato's own major foray into harmonics in the *Timaeus*, which we shall examine below. But at the time the *Republic* was written, it was only the work of the *harmonikoi*, the despised empiricists, that could have had any bearing on Socrates' concerns in Book III. No pre-Archytan Pythagorean or other mathematical theorist, so far as we can tell, had even begun to dissect

[3] *Rep*. 400b–c. The topic here is rhythm, not *harmonia*; and though *Rep*. 424c and remarks by Nicias at *Lach*. 180c–d suggest that Socrates had a genuine (if slightly ironic) respect for Damon's intellectual, musical and educational attainments, I do not think he can be credited with any work in harmonic analysis at all, still less that the passage in *Rep*. III depended on it (see p. 47 above).

different forms of the scale, let alone to examine the differences between the *harmoniai* used by contemporary musicians. Philolaus' cosmologically oriented study of the basic outlines of an octave-attunement had nothing to offer in this context, and even Archytas' mathematical description of systems in the three genera, if Plato knew of it at this time, gave no purchase on the distinctions between the *harmoniai* which Socrates discusses. In that case there simply was no form of harmonic analysis which he could appropriately have used.

The purpose of harmonics in *Republic* Book VII and the *Timaeus* is to guide us towards an understanding of the principles governing the structure of reality as a whole, and to provide a form of understanding which will help us to restore our own distorted souls to their original perfection. The notion that the analysis of musical systems can and should lead to enlightenment in such mentally vertiginous and apparently non-musical areas as these was already current in the Pythagorean tradition, as we have seen, and that fact partly – but only partly – explains Plato's adoption of Pythagorean concepts and procedures in this second phase of his musical thought. Soberly considered, however, it is an extraordinary hypothesis. In its Platonic version it makes sense only in the context of his complex arguments and bold conjectures in metaphysics and epistemology, and it cannot be detached without serious loss from its settings in the dialogues in which it plays a part. I cannot do justice to these matters in a book of this sort. The brief sketch of the theory's philosophical environment that I shall offer is intended only as a rough guide for non-specialists, from which experienced students of Plato may prefer to avert their eyes. For anyone who wishes to pursue the issues accurately and in depth, enough has been written about them by scholars and philosophers down the centuries to occupy a reader for several lifetimes.[4]

THE PHILOSOPHICAL CONTEXT OF THE *REPUBLIC*'S DISCUSSION OF HARMONICS

The passage on harmonics in *Republic* Book VII, like the musical reflections in Book III, is part of a discussion of education; but the two educational programmes are very different. Book III was concerned with character-training, not with 'academic' learning, and those to be trained were children. Book VII turns to the intellectual education, in adult life, of a small, hand-picked elite

[4] For those who do not have a lifetime to spare, Annas 1981 is still in my view the best introduction available.

who – if their talents and application match up to the task – are to become philosophers, and rulers of the perfect city which Socrates and his friends are mentally constructing. In this city the authority of the philosopher-rulers or 'guardians' is absolute. But Plato was under no illusion that the total concentration of authority in a few hands is a panacea for political ills. He knew from personal experience as well as from reflective thought that a malign or misguided dictatorship is a catastrophe, worse even, in his opinion, than a democracy, for all the latter's light-headedness and lack of principle. It is essential, then, that the rule of the philosophers should be based on the firmest of foundations. The combination of Machiavellian skills, worthy intentions and plausible opinions which in the normal run of things can earn someone a reputation as a good statesman will by no means be enough to ensure that what the city gains from their government will genuinely benefit it. If they are to construct and preserve what is best, their grasp on which is best must have the unshakable authority of absolute knowledge.[5]

According to the *Republic*'s reasoning, anything whose nature can be known must, in the first place, be real; secondly, it must possess its attributes without qualification, independently of circumstances and of the enquirer's perspective; thirdly it must be eternal and eternally unchanging. Suppose I claim to know, absolutely and unshakably, that something has such and such a property, for instance that the cement-mixer in my barn is orange. If there is no such thing (the machine is a figment of my imagination or has been stolen), my claim is empty. If it does exist, its orangeness depends on the lighting conditions and the colour-sensitivity of the viewer; it is not an objective, knowable fact. Its paint will in any case flake off and it will cease to be orange. It will rust; one day, perhaps, it will have been recycled into dog-food cans, and no proposition about my cement-mixer will have any purchase on reality. But no proposition whose truth-value can change or evaporate can be known absolutely and without qualification. In this very strict sense there can be no knowledge of individual cement-mixers or apples or bottles of wine, or indeed of anything to which we gain access through our senses.[6]

Can there, then, be any genuine objects of knowledge? Two considerations in particular seem to have encouraged Plato to give this question an affirmative answer. One is the example provided by mathematics. Propositions such as '$2 + 2 = 4$', or 'the triangle's internal angles are equal to two

[5] For an account of Plato's relevant experiences, perhaps from his own hand, see Plato, *Ep.* 7, especially 324b–326b; cf. *Rep.* 555b–576b on democracy and tyranny and on the forms of human character which correspond to each of them, *Gorg.* 515b–519d on the failings of various famous Athenian statesmen.
[6] See particularly *Rep.* 476e–480a.

right angles' seem immune to the diseases infecting statements about the world presented to our eyes and ears. The number 2 and its attributes are independent of time, change and perspective, and the triangle discussed by mathematicians is not the imperfect and ephemeral diagram scribbled on a blackboard but something quite different, 'The Triangle', or possibly 'triangularity itself'. These 'objects' exist nowhere in the perceptible sphere; they are accessible only to the mind. Nevertheless they are real. The number 2 is not a figment of our imagination, and when we make true statements about it they are true objectively and eternally. Big Bangs and collapses of universes may come and go; the number 2 and its unchanging nature do not.

Secondly, Plato's dialogues are concerned above all with questions about values and virtues. 'What is courage?' asks Socrates in the *Laches*; 'What is piety?' in the *Euthyphro*; 'What is self-control (or moderation)?' in the *Charmides*; 'What is virtue?' in the *Meno*. In the *Republic* the core question (there are many others) is 'What is justice?' The entitlement of any particular person, disposition, action or mode of behaviour to be called 'just', 'virtuous', 'courageous' and so on is of course endlessly debatable. But Plato apparently thought that such debates make sense only if there is something which justice or courage or virtue really is, something to whose definition a person's actions or character may more or less imperfectly and temporarily correspond, or in whose nature they 'participate', as the *Republic* and other dialogues of that period express it. We must know what justice, for example, really is, before we can coherently make judgements about the extent to which this or that action exemplifies it. Such judgements can qualify as true or false only if there is something determinate which justice is, and though different people may have different views about its nature, as indeed they do, their views can be understood as competitors for the truth only if there is some one thing which each of them is attempting, felicitously or otherwise, to describe. Such entities as these, which Plato calls 'forms', are as objectively real, as knowable and as inaccessible to anything but the mind as are those discussed by mathematicians. Whatever their natures may be, they possess them absolutely, independently of us and our opinions, changelessly and eternally. They are fully qualified, then, as potential objects of knowledge. To find a way in which human minds can attain this knowledge is the task of a philosopher.

If a philosopher is to know what justice or virtue really is, it is not enough that he or she[7] should be able to grasp the truth about its nature; they must also understand why it is so, and why, demonstrably, it cannot be otherwise.

[7] For Socrates' defence of his thesis, outrageous by contemporary standards, that some women may have the abilities appropriate to the rulers or 'guardians' of his ideal city, and should therefore be educated and trained in the same way as the corresponding men, see *Rep.* 451b–457b.

It is by coming to grasp the reason why something is so that a person can 'tie down' a true opinion about its nature, as the *Meno* puts it, and convert it into knowledge (*Meno* 97d–98a). The philosophers must therefore pass beyond realities of the sorts I have mentioned to some higher principle or truth which determines that they are what they are, and explains why they must be so. In their dealings with mathematics they must also come to understand that when considered in isolation from higher levels of philosophical reasoning it is an insecure field of study, since it relies on axioms whose truth cannot be demonstrated or explained within mathematics itself. So long as its theses depend on undemonstrated postulates, which Plato calls '*hypotheseis*', it cannot carry the stamp of knowledge. What disqualifies it is not that the truth of its propositions is in serious doubt, but that they are not 'tied down'; and neither they nor propositions about values and virtues can be so tied until they are shown to be consequences of a higher principle which is grasped 'unhypothetically'. That is, when we understand it fully we shall see, without reference to anything else, that it cannot be otherwise; the act of understanding it provides its own guarantee or 'tether'.[8]

Since Plato represents reality as a single, rationally unified system, the natures of all its components are ultimately determined by that of a single nature which stands above them all. It is, so Socrates says, superior even to reality itself (*Rep.* 509b); it is that on which the nature of reality depends. There is thus only one unhypothetical principle, to which all genuine knowledge must be tied. Plato calls this highest of all beings 'the good', or 'the form of the good'. He nowhere attempts to provide a full account of its nature,[9] but its designation plainly implies that reality and goodness are indissolubly linked. The real is a manifestation of the good.[10] We know why reality is as it is if and only if we know why it is best that it should be so; and we can know that only if we understand fully what the good itself is. The *Republic*'s philosophers can therefore have no knowledge of values, or indeed of anything else, unless they have an unshakable grasp on this highest of all truths. But such understanding cannot be acquired easily or quickly. It can come only after a long and arduous programme of intellectual training, and even then only to a few gifted individuals,

8 See *Rep.* 509d–511e, particularly 510c ff.

9 Cf. *Rep.* 506b–e, and on the complexities and difficulties involved in an approach to an understanding of it, 531d–541b.

10 This is not to be construed as an expression of innocent optimism; Plato is under no illusion that human beings live in 'the best of all possible worlds'. The world we inhabit is unstable and imperfect; it is only the eternal reality of the forms that is wholly and immutably good.

exceptional in both mind and character. Before it advances to anything we would recognise as 'philosophy', this training demands the mastery of five mathematical disciplines, studied in a set order: arithmetic, plane geometry, solid geometry ('stereometry'), astronomy and finally harmonics.[11]

The main purpose of these studies is to accustom the trainee philosophers to thinking about realities accessible to the mind alone, wholly detached from any application to the unstable, mutable world presented to the senses. Arithmetic, as here conceived, is not about numerable pluralities of material objects but about numbers, simply as such. Geometry is not to do with the measurement of areas of land, but with entities such as the square and the circle in their own right. Platonic astronomy, more surprisingly, is not the study of the visible stars, but is – as we might put it – a purely theoretical enquiry concerned with the relations between abstract points in motion.[12] The task of the science of harmonics, correspondingly, is not to study sounds or to anatomise the music we experience through our ears, 'seeking the numbers in audible concords', but to examine relations between numbers; its central purpose, we are told in words I have quoted before, is to 'ascend to [mathematical] problems, to investigate which numbers are concordant [with one another] and which are not, and in each case why' (531c2–4).

HARMONICS IN THE *REPUBLIC*

Glaucon comments that this is an extraordinary (or perhaps 'superhuman') project, *daimonion pragma* (531c5). Like the preceding review of existing approaches to harmonics, both Pythagorean and empirical, his remark suggests that nothing comparable had hitherto been attempted. Socrates agrees, but insists that harmonics, so conceived, is 'useful in the quest for the beautiful and good, but pursued in any other way it is useless' (531c6–7). I have already outlined one reason for its usefulness which is repeatedly emphasised in the text; when the mathematical disciplines are treated as they should be, without reference to the objects of sense-perception (which are semi-real at best), they habituate the mind to the study of genuine realities, those that are changeless and eternal, and whose existence and nature can be grasped only in thought. But this is only part of the story. The other part is less explicit in Socrates' reflections, and it is in his comments on harmonics that it comes closest to the surface.

[11] The programme is introduced and described at *Rep.* 521c–531c.
[12] See e.g. Mourelatos 1980, 1981.

Harmonics, as described here, is not just the study of relations between numbers. Its enquiries turn on the distinction between numbers that are 'concordant' (*symphōnoi*) with one another and those that are not; and it is only at this point that it seems to have any plausible connection with the musically oriented harmonics of other theorists, or to acquire any right to the name. There is a difficulty here about the meaning of the adjective *symphōnos*. Socrates applies it to numbers, but this extended usage can be understood only in so far as we are clear about its meaning in the context from which it has been taken, that is, in its application to musical sounds. We need to know which attribute of sonorous relations it is that Socrates is transferring to relations between numbers. I do not think that in this passage it is the attribute which the word regularly designates in technical writings, that which marks off 'concords' from other classes of melodic relations. The sense on which Socrates' extended usage relies is more general, as it is in some other Platonic passages and often in non-technical literature of other sorts; it means something like 'sounding well together', and applies to notes in any relation that can occur as an element in a genuine musical melody.[13] Only on this interpretation, as it seems to me, can Platonic harmonics be reckoned a substantial science, let alone a *daimonion pragma*. If Socrates' 'concordant numbers' were merely the ratios corresponding to musically 'concordant' intervals, *symphōniai* in the technical sense, these intervals are very few and had long ago been assigned their mathematical counterparts. There was no mystery in Plato's time about 'which numbers are concordant' in that sense, though no doubt the question why they are so seemed a good deal thornier. But if the distinction, conceived musically, is more like that between 'melodic' and 'unmelodic' relations, the domain to be considered is vastly enlarged, and no one in the fifth century had even attempted to establish the mathematical identities of all the relations that fall into each category, or the criteria by which the matter is to be decided.[14]

But Socrates is not talking about music. He is talking about numbers, and about relations which hold between them simply as numbers. His distinction between relations that are *symphōnoi* and those that are not is

[13] See e.g. [Hom.] *Hymn to Hermes* 51, Aristoph. *Birds* 221, 659; the noun *symphōnia* has this broader sense at Plato, *Symp.* 187b, *Crat.* 405d.

[14] The studies of Archytas (Ch. 11 above) are evidently relevant here. The *Republic* cannot refer to them directly, since they postdate its dramatic scenario; and we do not even know whether the *Republic* was composed before or after them, or whether Plato knew about them when he wrote it. If he did, we might read the passage as a covert allusion to Archytas' work, and as expressing carefully qualified approval. Huffman 2005: 423–5 gives a very clear account of the import of the necessary 'qualifications', though this is too feeble a word for his picture of the distance between Archytas' studies and those that Plato's Socrates requires.

therefore to be understood as one that applies within mathematics itself, and as intrinsic to relations between numbers as such. This means that although the word used to convey the distinction is imported into the mathematical sphere from outside, from the language of musicians, the distinction itself is not (unlike, say, the distinction between numbers which are and those which are not the numbers of London buses). It belongs to mathematics as solidly and securely as the distinction between odd and even numbers.

This has a crucial implication. The judgement that certain relations are 'melodic', musically admissible, while others are not, is at least in significant part an evaluative one. Universally, in Greek discussions of these matters, musical relations are conceived as in some sense 'better' than unmusical ones, and it is because they are better that they can play a part in the construction of things of beauty. Relations of other sorts are excluded because they are in a corresponding sense defective. Plato's distinction between what is *symphōnos* and what is not carries with it the same evaluative loading; the point of distinguishing 'concordant' pairs of numbers from others (rather than classifying them along some other lines), and of grasping what it is that makes them so, is to establish a clear borderline between 'better' relations and 'worse'. It follows that mathematics itself, or at least the branch of it which Plato calls 'the study of *harmonia*', is not as one might have supposed a value-free zone. It distinguishes superior relations between numbers from inferior ones, and if pursued as it should be it will reveal also what it is about them that makes some of them 'musical' while others are not. Our perception that some audible relations belong properly in the domain of music, and that others are unsuited to it, is no more than a distant and derivative echo of this fundamental dichotomy in the realm of number itself.[15]

Plato's mathematical harmonics (perhaps better described as 'harmonic mathematics') therefore has something to teach us about value, that is, about what is and what is not objectively good. No doubt the direct contribution it can make to the philosophers' quest for the nature of the good itself is quite modest. At this stage they are still in the 'hypothetical' domain of mathematics, and they still have to tackle the business of high-level philosophical reasoning which Plato calls 'dialectic', from which alone a full understanding of the unhypothetical first principle can, in the end, emerge. But it is a step along the way, enabling the philosophers at least to grasp that goodness is not a construct of social or aesthetic convention, but

[15] For other indications of the evaluative strand in the mathematics of Plato's time see Barker 1994b, and on Plato in particular see Burnyeat 1987 (especially 238–40), and 2000.

inheres in formal, intelligible relations embedded in the structure of reality independently of human preferences, traditions and needs; and it enables them also to establish, in a preliminary, 'hypothetical' way, what some of these privileged relations are.

In the *Republic* Plato states none of the principles or propositions of his harmonics; we get no glimpse of the science in action. A justly famous and poetically delicious passage in Book x conjures up the picture of a musically ordered universe, in terms which are closely linked to the Pythagorean doctrine of the 'harmony of the spheres'. It is a literary *tour-de-force*, but its language is that of allegorical myth, not of science, and for all the ingenuity of devoted commentators it gives no purchase to detailed mathematical analysis, at least from the perspective of harmonics.[16] What the *Republic* lacks in this respect, however, is amply supplied in the *Timaeus*, in a passage which attracted more discussion in later antiquity than almost any other in the Platonic corpus, and which spawned an entire genre of writing in the field of harmonics.

HARMONICS IN THE *TIMAEUS*: THE SOUL OF THE UNIVERSE

The *Timaeus* begins with a recapitulation of the discussions of the *Republic* (*Tim.* 17a–19a), and so announces itself as some sort of continuation of the project initiated there. After a few pages it abandons the familiar Platonic dialogue form, and the bulk of it is a monologue spoken by the (real or fictional) Pythagorean, Timaeus of Locri, after whom it is named. What it offers is an elaborate account of the physical universe and its contents, premised on Plato's theory of forms and therefore presented as no more than a 'likely story' (29b–d), since according to that theory nothing in the perceptible and material realm can be certainly known. Its analyses and explanations of the origins and the structure of the cosmos and everything within it are grounded in mathematics, drawing on all five of the disciplines incorporated into the philosophers' education in the *Republic*. As a study of the workings of the physical universe it is continuous with the work of Presocratic cosmologists; and in its treatment of mathematics as the key to an understanding of the world it is a development of the Pythagorean tradition. But it is vastly more detailed and sophisticated than anything we know from those sources.

[16] *Rep*. 616b–617d. Something can indeed be made of it from an astronomical point of view, but that is another matter. It allows no secure inferences about the pattern of musical relations at which it hints.

Timaeus represents the universe as the product of a divine agent, not a creator in the Old Testament sense but a 'craftsman' (*dēmiourgos*), who fashions a pre-existing chaos into a system organised in the most perfect possible way. The universe he constructs is a living being, whose soul animates and sets in motion its bodily parts. This must be so, since the visible cosmos is perpetually in motion, most notably on the grand scale in the unfailing cycles of the stars and planets, and no mere body, in Plato's view, can move itself, or for ever sustain, through its own agency alone, a movement imparted to it from elsewhere. Only soul is a self-mover. The soul of the universe, furthermore, transmits to its bodily members not only movement but rationally intelligible order, detectable in its most spectacular form, once again, in the complex but regular and beautiful patterns woven by the movements of the celestial bodies. Hence the soul which the craftsman builds for the universe is itself a paradigm of rational order, whose self-movements are integrated within a perfectly, 'harmoniously' regulated structure.

The word 'harmoniously' is not chosen at random. It is most importantly in Timaeus' account of the construction of the soul of the universe (or the 'World Soul', as it is commonly known) that the conceptions of harmonic science come into play. The process by which the World Soul is built is described as if the craftsman were a skilled metal-worker.[17] He takes certain metaphysical stuffs, whose nature, fortunately, we need not examine, and fuses them into a compound as if blending metals into an alloy (35a1–b1). He then forms the compound into a strip, upon which he marks off lengths through a mathematically structured process of division.

> This is how he began to divide. First he took away one part from the whole, then another, double the size of the first, then a third, one and a half times the second and three times the first, then a fourth, double the second, then a fifth, three times the third, then a sixth, eight times the first, then a seventh, twenty- seven times the first. Next he filled out the double and triple intervals, once again cutting off parts from the material and placing them in the intervening gaps, so that in each interval there were two means, the one exceeding [one extreme] and exceeded [by the other] by the same part of the extremes themselves, the other exceeding and exceeded by an equal number. From these links within the previous intervals there arose hemiolic, epitritic and epogdoic intervals [i.e. intervals in the ratios 3:2, 4:3 and 9:8 respectively]; and he filled up all the epitritics with the epogdoic kind of interval, leaving a part of each of them, where the interval of the remaining part had as its boundaries, number to number, 256:243. And in this way he had now used up all the mixture from which he cut these portions. (*Tim.* 35b4–36b6)

[17] See particularly Zedda 2000.

All this is less complicated than it may at first appear. The process of division has three stages. In the first, the craftsman marks off lengths based on two geometrical progressions, a series of doubles, 1, 2, 4, 8, and a series of triples, 1, 3, 9, 27. Their combination gives the sequence 1, 2, 3, 4, 8, 9, 27. Secondly, he 'fills out the double and triple intervals (*diastēmata*)' by inserting two means between the terms bounding each such interval; these means are clearly defined, and are the harmonic and arithmetic means of Archytas' classification (pp. 302–3 above). If this phase of the division is to be represented in whole numbers its terms must be multiplied by 6. Thus the series of doubles becomes 6, 12, 24, 48, and the series of triples becomes 6, 18, 54, 162. The insertion of means into the series of doubles gives **6**, 8, 9, **12**, 16, 18, **24**, 32, 36, **48**, and their insertion into the series of triples gives **6**, 9, 12, **18**, 27, 36, **54**, 81, 108, **162**. Combining these two sequences into one we get **6**, 8, 9, **12**, 16, **18**, **24**, 27, 32, 36, **48**, **54**, 81, 108, **162**.

It is already clear that the division has something to do with music. This is shown not just by Plato's use of the term *diastēma*, 'interval' (which in any case has a non-musical sense that is perfectly appropriate in this context, meaning a 'gap' or 'distance' separating two things), and by his incorporation of Archytas' three kinds of mean into the structure.[18] It emerges from the shape of the division itself. The numbers from 6 to 24 represent the fundamental notes of an attunement spanning two octaves, each divided in the regular way to give perfect fourths at the top and the bottom, separated by the interval of a tone. Thus 12:6, for instance, is the ratio of an octave (2:1); 12:9 and 8:6 are each equivalent to 4:3, the ratio of a fourth; the intervening ratio, 9:8, is that of a tone. The sequence from 24 to 48 yields another octave with the same structure, except that as well as the two terms marking the inner boundaries of the fourths, 32 and 36, it includes also the term 27, which lies a tone from 24, in the ratio 9:8. Beyond the number 48 there lies another tone (54:48 = 9:8), which is best construed as a disjunction between two octaves, the next of which is bounded by 108 and 54. Only one term, 81, divides this fourth octave; 81:54 is the ratio of a perfect fifth, 3:2, and 108:81 is that of a fourth, 4:3 (the 'missing' term would be 72, since 72:54 = 4:3, 108:72 = 3:2 and 81:72 = 9:8). Finally, beyond this octave, we find 162:108, equivalent to 3:2, another perfect fifth. The entire structure spans four octaves, plus the tone separating the third octave from its successor and the concluding perfect fifth, in all four octaves and a major sixth.

[18] The geometric mean is not explicitly mentioned, but intermediate terms in the original series of doubles and of triples are the geometrical means between their neighbours.

The division's links with music might still, at a pinch, be reckoned coincidental, fortuitous consequences of a purely mathematical operation. Its third step, however, leaves no room for doubt. When an 'epitritic' interval (a perfect fourth in the ratio 4:3) is 'filled up with the epogdoic kind of interval' (these intervals are tones in the ratio 9:8) together with a residue in the ratio 256:243, what is generated is a tetrachord, divided into two tones and the small interval approximating to a semitone which Philolaus called the *diesis* and which later became known as the *leimma*.[19] Tetrachords of this form reappear times without number in post-Platonic sources as a representation of a diatonic division. We have seen also that such tetrachords can be accurately tuned in practice in the simplest possible way, through the 'method of concordance', and that this fact was apparently known to Archytas, though the ratios of his own diatonic are different (p. 299 above). There can be no motive for this division of each 'epitritic interval' into two epogdoics and a *leimma* except to capture the shape of a recognisably musical structure. From a purely mathematical perspective it looks both awkward and arbitrary, and the cumbersome 'number to number' ratio of 256:243 seems embarrassing in the company of this extensive array of ratios that are otherwise multiple or epimoric (cf. pp. 288–90 above). Another passage in the *Timaeus*, which we shall examine shortly, confirms not merely that the system has musical connotations, but also that for Plato's purposes in the dialogue it was essential that it should.

It is not, however, a description of any musical scale or attunement that was ever used in practice, if only because it has far too large a range. Most Greek analysis takes place within the compass of two octaves; Philolaus and the *harmonikoi* (and Archytas too, so far as our evidence takes us) were apparently content with one; and the range of any one instrument on which an attunement had to be formed can rarely have exceeded an octave by much. Even according to Aristoxenus' more generous estimate, the maximum compass of a single instrument or of the human voice is around two octaves and a fifth, and is certainly less than three octaves (*El. harm*. 23.22–32). Only one of the surviving musical scores has a range of more than an octave and a fifth, and few of them even approach that span.[20] The compass of the *Timaeus* 'scale' is not determined by musical

[19] The noun *leimma* is derived from the verb *leipein*, and means 'remainder', 'that which is left'. Plato himself does not use the noun, but the theorists who coined it probably did so on the basis of this passage, where it is described as the interval or distance (*diastasis*) that is 'left', *leiphtheisa* (this is the passive past participle of the same verb).

[20] For the (quite extraordinary) exception, a fragmentary score on a papyrus of the second century AD, with a compass of more than two octaves, see Pöhlmann and West 2001: 134–37. I am grateful to an anonymous reader for drawing it to my attention.

considerations at all, but by the metaphysically motivated requirement that its two primary sequences of terms should include both square and cubic numbers. It is to be considered musical in an abstract, mathematical sense, in virtue of its perfect proportionality, completeness and integration, to which human musical constructions can only distantly approximate; and the reasoning that generates it is the product of something that at least closely resembles the non-empirical, rationalistic harmonics adumbrated in the *Republic*.

These points do something to explain another feature of the division which would make it unhelpful in the context of an enterprise directed to the analysis of literally musical systems, that is, that from a musical point of view it is in certain respects underdetermined. There is at least one crucial general issue about which Plato leaves us entirely in the dark, since he does not tell us which way up we should read his scale, whether smaller numbers represent higher or lower notes. It is natural to assume that they are lower, so that they correspond to the slower speeds of movement with which lower pitches are linked in Archytas' acoustic theory, a variant of which reappears in the *Timaeus* itself (67b). But that interpretation generates an anomaly in the structure, since either the tone falling between the numbers 24 and 27 or that between the numbers 48 and 54 must lie within a tetrachord (the other can be construed as a disjunction); and whichever it is it will lie in the wrong place, at the bottom of a tetrachord, in the position occupied in a regular diatonic system by a *leimma*. We can resolve this problem by reading the scale in the other direction. But if we do so, the resulting correlation of larger numbers with lower notes is hard to justify except by reference to the relative sounding-lengths of strings or pipes; and this is precisely the sort of 'empirical' reference which Plato could be expected to avoid.

There are other uncertainties too. Timaeus has explained, in outline, how the open fourths in his system are to be 'filled up'. But he does not specify the order in which tones and *leimmata* are to be placed, and he says nothing explicitly about the subdivisions of the two open fifths (between 54 and 81 and between 108 and 162), or about that of the minor third between 27 and 32. These gaps in his account can be repaired, but the fact that he does not complete the task himself is significant. He is not interested in the task of making his construction correspond at every point to the shape of a system that could be used in practice. Nor does it matter, from the perspective of his project, whether high notes are associated with large numbers or with small, since the issue simply does not arise. There are no notes or pitches in his *harmonia*; there are only numbers.

When the craftsman has finished marking out his strip, he splits it lengthwise in two, and then, after fitting the two pieces together in the form of the letter X, he bends each of them round into a circle, and sets the two circles revolving. But the inner of the two circles, it now turns out, has itself been divided into seven concentric circles of different sizes, sizes which are once again determined by the two sequences of numbers involved in the first stage of the division, 1, 2, 4, 8 and 1, 3, 9, 27. These seven circles revolve in the same direction at various speeds, and their motion is also subject to that of the single outer circle (*Tim.* 36b–d). There are many problematic details here, but I shall ignore them. The general significance of the arrangement becomes clear in the immediate sequel, when the craftsman fastens bodily substance to these incorporeal circles, to be carried around by their revolutions. 'And it [the bodily substance] becomes the visible body of the heavens, while the other [the being whose construction has been described] is a soul, invisible but possessed of rationality and *harmonia*, the best of all the intelligible and eternal entities that have been brought into being by the best of agents' (36e–37a). More details are provided at 38c–e. The seven inner circles are those that carry the visible bodies we call moon, sun, Venus, Mercury, Mars, Jupiter and Saturn, while the undivided outer circle carries the fixed stars. As in Aristotle's account of a theory of the 'harmony of the spheres' in the *De caelo* and implicitly in *Republic* Book x, harmonics becomes fused with astronomy; and underlying both is the mathematics of 'concordant numbers'.

THE *TIMAEUS* ON HARMONICS AND HUMAN PSYCHOLOGY

A little later in the dialogue, Timaeus describes the way in which mortal creatures, including humans, were originally constructed. The craftsman himself builds their souls, on the same pattern as that of the universe though from an inferior alloy (41d4–7), but he entrusts to lesser gods the task of contriving bodies for them and fitting bodies to souls (41a7–d3). But when the mortal bodies are formed and the revolving circles are bound into them, the inrush of nourishment and sense-impressions throws the soul into confusion, so that the orderly structure on which its rationality and its grasp upon truth depend is distorted almost (but not quite) to breaking point.

The double and triple distances, three of each, and the means and linkages made up of hemiolic [3:2], epitritic [4:3] and epogdoic [9:8] ratios, while they cannot be totally destroyed except by the agent who bound them together, bend into twisted shapes of all sorts, and inflict every kind of breakage and ruin on the circles, so far as that is possible, so that they scarcely hold together, and move, but move irrationally, sometimes forwards, sometimes sideways, sometimes upside down. (43d4–e4)

It is like the condition of someone standing on his head and seeing things on his right as if they were on the left (43e4–8).

Musical mathematics thus enters human psychology. It is the structure of double and triple distances, and of the means which 'link' their terms and generate the ratios of the lesser concords and of the tones on which the divisions of tetrachords are based, that give the soul its rationality; and when they are disturbed the consequence is irrational confusion. It can be no coincidence that Timaeus' description of these distortions and their consequences evokes echoes of the ways in which fifth-century writers, especially the comic dramatists, depicted the corruptions and disfigurements inflicted on music by irresponsible modern composers. There are the same 'twists and turns' (*strophai*, elsewhere *kampai*) and 'ruinations' (*diaphthorai*), and the same image of seeing things the wrong way round.[21] Plato's implicit association of these musical malpractices with the psychological chaos into which the soul is initially plunged by its association with the body carries an incidental but clear message about music itself: to mistreat the structures proper to genuine melody and attunement is to abandon rationality.

The soul's confusion is at its most acute in the early years of mortal life, stirred up by the violent influxes and effluxes of nutriment and the onrush of bodily growth, and by the onslaughts of intense and unfamiliar sensations. Later, these disturbances lessen, and the soul's cycles can be restored to their proper condition. But complete recovery is possible only if the soul, at this stage, absorbs the 'true nourishment of education' (on all this see 43a–44c). The *Timaeus* does not revisit the detailed educational programmes of the *Republic*, but it does make it clear that music is among the most valuable resources on which we can draw when attempting to restore our souls to rationality and health.

In the relevant passage, Timaeus outlines the principal benefits we gain from our possession of sight and hearing. We have been equipped with sight not, primarily, for the humdrum purpose of finding our way about the world, but rather

> in order that we may see the revolutions of mind in the heavens, and apply them to the cycles of our own thought, which are akin to those others but disturbed while they are undisturbed, and by learning them thoroughly and engaging in reasonings true to what is naturally correct, and by imitating the altogether unwavering revolutions of the divine, we may establish soundly their wavering counterparts in ourselves. (47b6–c4)

[21] See particularly the famous passage of Pherecrates' *Chiron* quoted at [Plut.] *De mus.* 1141d–1142a (Pherecrates frag. 155); compare e.g. Aristoph. *Clouds* 968–72, *Thesm.* 49–69.

Voice and hearing have been given to us for a similar purpose. For one thing, they are the prerequisites of *logos* (which means both 'speech' and 'reason', as well as 'ratio' in appropriate contexts). But secondly,

> that part of music which can be deployed by the voice and directed to the hearing is given for the sake of *harmonia*. And *harmonia*, which has movements akin to the revolutions of the soul within us, is not reckoned useful, by anyone who treats the Muses intelligently, for the sake of irrational pleasure – as is nowadays generally supposed – but as having been given by the Muses as an ally in the attempt to bring the revolution of our soul, which has become ill-attuned (*anharmoston*), into proper order and concord with itself. (47c7–d7)

From a combination of two other passages of the *Timaeus* (70d7–72b5 and 79e10–80b9), it is possible to excavate quite a detailed account of the physiological and psychological processes which mediate the transition from the arrival of musical sounds at our ears to a rational appreciation and absorption of their divinely ordered patterns of movement. The extraction of this account from the text is a moderately intricate business; I have attempted it elsewhere and shall not now retrace all my steps.[22] Summarily, the upshot is this. The part of the soul upon which sounds, musical or otherwise, are initially registered is irrational, incapable of understanding. But it is not merely a recipient of physical stimuli. It is the locus of emotional reaction, and it could be no such thing unless the stimuli somehow presented themselves to it as meaningful. What Timaeus tells us is that 'messages' sent to it either through the channels of the senses or from the soul's rational part are transformed *en route* into 'images', quasi-pictorial likenesses of things that may be horrible and terrifying or sweet and delightful. (The agent of transformation is the liver, from whose surface the stimuli are reflected, as if from a mirror, but one which converts them from impulses into images. A rough modern analogue is the screen of a computer's monitor, which receives coded patterns of electronic impulses and presents them to our eyes as something entirely different, text and pictures.) Music, then, is received by the 'irrational soul' in the form of images or likenesses. At this level, as we are told in the *Republic* and the *Laws*, it is a fabric of *mimēseis*, 'imitations' of things other than itself.[23]

But the process does not end with the emotional responses of this part of the soul. The movement of sound through the body is circular, beginning in the head, travelling down to the liver and to the irrational soul which is housed nearby, and then back, transformed, to the head, the locus of the soul's rational part. The task of the rational soul is to interpret the

[22] Barker 2000b. [23] See e.g. *Rep.* 401a, *Laws* 668a–c.

imagistic 'phantasms' transmitted to it from below; it is like a *prophētēs* (one who 'speaks out clearly') who places comprehensible interpretations on the divinely inspired but inchoate and apparently crazy utterances of a *mantis*, the utterances, for example, of the priestess of the Delphic oracle. To ignorant people, says Timaeus at 80b5–8, musical combinations of sounds give pleasure (*hēdonē*); but to intelligent people they give delight, *euphrosynē*, 'because of their imitation, in mortal movements, of the divine *harmonia*'. They can give no such delight to people who fail to appreciate the connection of these sound-patterns with the *harmonia* of the World Soul. It follows that these 'intelligent' people are those who have mastered the science of harmonics in its Platonic form and have recognised its cosmological meaning.[24] The rational part of their souls, if of no one else's, is equipped to interpret the images conjured up by the impact of music on the irrational soul, and it is only their souls whose revolutions can be fully restored to their rational and harmonious order.

The *Timaeus* therefore implicitly assigns to mathematical harmonics, coupled with astronomy, a crucial role in the business of human life. It is the instrument through which we can regain the perfection which our souls lost when they were harnessed to the bewildering paraphernalia of bodily existence. In this light it no longer seems strange that later Platonists and Platonising 'Pythagoreans' (Nicomachus, Theon of Smyrna, Plutarch, Proclus and others) devoted so much study to the passage describing the musical structure of the World-Soul, or that harmonics played so large a part in their philosophical and mathematical reflections.

HARMONICS IN THE IVORY TOWER

The intellectual milieu into which Plato's harmonics inserted itself in its own time was quite different from that inhabited by the work of the *harmonikoi*. The ideas of the *harmonikoi* were propounded in the first instance for the benefit of practical musicians, and may also have been sketched, in outline, in public places for anyone to hear. Plato had views about the proper training of composers and performers, as we see in the *Laws*, but that is not the business in which he was engaged in the *Republic* or the *Timaeus*; and he did not teach in public. The Academy was a small, private institution for the intellectually ambitious. Some of the dialogues,

[24] At 47c Timaeus speaks of 'that part of music which can be deployed by the voice and directed to the hearing', implying that there is another 'part' of music inaccessible to these human faculties. This must be the music of the World-Soul and of its bodily organs, the stars and planets. We cannot hear or sing it, but a mind trained in mathematical harmonics can grasp its nature and significance.

particularly those written before the Academy's foundation, were certainly intended for a wider audience; but Plato's later writings, including the *Timaeus*, make no concessions to the uninitiated, and would have been unintelligible to all but a very few. Though Archytas' harmonic studies engage more closely and deliberately with the realities of human music-making, they too belong in the context of mathematical and philosophical enquiries relevant and accessible only to specialists. (There is no evidence, and no probability, that Archytas was a teacher in the mould of the sophists. His public persona was that of a statesman and a military commander.)

We are entering a new world here, one in which dedicated philosophers, scientists and mathematicians discourse with one another in language, and for purposes, beyond the imagining of outsiders. Such discourse is familiar enough in the modern world, where almost every advanced discipline is barricaded behind its own preconceptions, obsessions and jargon, where attempts in recent decades to open the frontiers between specialised university departments and research institutes have regularly failed, and where the guardians of academia typically discount efforts to communicate the experts' ideas to a non-specialist audience as negligible popularisations. But before Plato's time it had no precedent. Back in the fifth century even the most elevated intellectual enterprises were public property. Anyone could hear the sophists touting their wares, and buy more elaborate versions of them if they had some disposable funds; anyone could listen to Socrates' conversations and appreciate a caricature 'Socrates' on the comic stage; the reflections of sophists and Presocratic philosophers resurfaced repeatedly in publicly performed poetry and drama; and one could buy a copy of Anaxagoras' cosmological treatise for a few pence in the market-place. During the lifetimes of Archytas and Plato, and at least partly as a consequence of their work, mathematical harmonics joined the higher échelons of philosophy, mathematics and the natural sciences in withdrawing itself from public presentation and debate. Specialists talked to specialists, but to virtually nobody else. The work of Aristoxenus in the next generation, as we have already seen, took empirical harmonics a long way down the same path.

CHAPTER 13

Aristotle on the harmonic sciences

The only sustained discussion of musical issues in Aristotle's surviving works is in the last book of the *Politics*. Like the conversations in Book III of Plato's *Republic*, to which it is in part a response,[1] it focuses on the value of music in the life of a city and its citizens, and it says nothing about the musical sciences. It alludes several times, however, to the work of unnamed experts in musicology, at least some of whom were Aristotle's contemporaries.[2] This suggests that he had some acquaintance with up-to-date studies by musical specialists, perhaps including their work in harmonics; and references to harmonic science, and to concepts used by its exponents, are scattered here and there in his other writings. Almost all of them are brief. It seems fairly clear that Aristotle made no substantial contributions of his own to the subject, and that it was marginal to his main areas of interest. I shall argue, too, that despite the confidence of his various pronouncements, his grasp on some of its concepts and procedures was a little uncertain.

A study of his remarks pays dividends none the less, for three main reasons. First, slight though they are, they contribute rather more than has generally been recognised to our knowledge of mathematical harmonics in the fourth century, especially in its Pythagorean form. Secondly, they are relevant to an understanding of certain aspects of Aristotle's own thought, since he occasionally makes use of ideas drawn from mathematical harmonics in his studies of non-musical topics. Thirdly, we have seen that the ideas which he developed about scientific method in general, and about the structure of scientific knowledge, subsequently formed the backbone

[1] The views of Socrates in the *Republic* are explicitly mentioned and discussed at 1342a–b. It can plausibly be argued that other parts of the passage also address Plato's views.

[2] Aristotle attributes relevant ideas and investigations to 'those who have philosophised about this [musical] education' (1340b5–6), to 'some contemporary musical specialists (*mousikoi*) and those philosophers who are well versed in musical education' (1341b27–9), to 'some philosophers' (1341b33), to 'those who are involved in philosophical activity and in musical education' (1342a31–2), to 'people who work in this field', which is here evidently musical scholarship rather than philosophy (1342b8–9), and to 'some musical specialists' (1342b23–4).

of the influential approach to harmonics devised by Aristoxenus; and they can also be used to cast light on the procedures of another major treatise, the Euclidean *Sectio canonis*. I outlined the gist of Aristotle's ideas on these matters in Chapter 4 above, and shall not dwell further on their general characteristics; but his own view of their bearing on harmonic science is very different from that of Aristoxenus. His statements on this issue and on a handful of related concepts call for some comment. I shall therefore divide this chapter into three parts, corresponding to the three areas of interest I have identified.

AN ARISTOTELIAN FRAGMENT ON PYTHAGOREAN HARMONICS

I shall not reconsider here the passages on Pythagorean harmonics and philosophy which I touched on in Chapter 10 above, and which are in any case well known. Instead, I shall try to supplement what can be gleaned from them by mulling over the rather less familiar contents of a fragment from one of Aristotle's lost works, and attempting to reconstruct the outlines of the context in which it was originally set. It shows clearly that he was familiar with at least some of the work of Archytas, and with methods of analysis characteristic of the Pythagorean tradition, and I shall suggest that it gives us some reason to believe that he had studied them quite extensively. It is less obvious that he had absorbed their ideas and procedures with perfect accuracy. The passage, as we have it, begins with a direct quotation but continues as paraphrase, and its latter part contains vexing confusions. Our assessment of Aristotle's mastery of its topic will depend at least partly on whether we attribute these confusions to him or to the author of the paraphrase. In either case, however, we can extract valuable information from it about the preoccupations of mathematical theorists in his time or a little earlier.

The passage appears in the Plutarchan *De musica* at 1139b–f; the fragment itself is at 1139b, and runs as follows.

> *Harmonia* is celestial, and its nature is divine, beautiful and wonderful. In potential (*dynamis*) it is four-fold, and it has two means, the arithmetic and the harmonic; and its parts, magnitudes and excesses are revealed in accordance with number and equal measure; for melodies acquire their structure in two tetrachords.[3]

The compiler does not tell us which of Aristotle's works contained it, and commentators' views on the issue have differed. Rose assigned it to his *Eudemus* and Ross to his *De philosophia*; Gigon simply includes it among

[3] It is printed in the collections as frag. 47 Rose, frag. 25 Ross and frag. 908 Gigon.

the fragments whose work of origin is not named in our source, and does not commit himself to an opinion about the work's identity. Before considering its contents in detail I should like to make some comments on the question of its origins; and in due course I shall offer a suggestion of my own. If I am right, the nature of its Aristotelian context may have a substantial bearing on our interpretation of the fragment itself.

The fragment and the discussions associated with it occupy Chapter 23 of the *De musica*. It is the second in a run of four chapters (22–5) which stand out sharply from their surroundings. The bulk of the work is taken up with discussions of the history of musical styles and practices and of cultural issues connected with them; these draw occasionally on musical theory of an empirical or Aristoxenian sort, but never on mathematics. Chapters 22–5, unlike any other passages in the *De musica*, are concerned entirely with mathematical harmonics and its philosophical or cosmological applications. Chapter 22 discusses Plato's construction of the World-Soul in the *Timaeus*; Chapter 23 is the Aristotelian passage; and Chapters 24–5 give brief expositions of ideas apparently drawn from Pythagorean musical-cum-mathematical cosmology. It is true that the compiler makes an attempt to connect these passages with the preceding material. In Chapter 17 he argues that Plato's exclusion of all *harmoniai* except Dorian and Phrygian from his educational scheme in the *Republic* was not due to his ignorance of the others, or of the fact that Dorian and Phrygian were sometimes used in music unsuited to his purposes. Chapters 18–21 are a loose continuation of this theme, purporting to show that reputable 'ancient' composers who used only a limited number of rhythmic and melodic forms did so by deliberate choice, and not because they were unaware of the existence of other possibilities. The compiler then begins Chapter 22 by saying that he has shown that Plato did not reject other kinds of music out of ignorance, and will now demonstrate that he was also well versed in harmonics. But after the discussion of material from the *Timaeus*, Plato disappears from sight, never to re-emerge, and the exploration of themes in mathematical harmonics continues under its own momentum.

The compiler of the *De musica* used a number of sources, some of which he identifies by name, and all of which with a few very minor exceptions come from the fourth century or earlier. It seems clear that he adopted the practice of using the same source as the basis for quite long stretches of his own text, with only occasional brief intrusions from elsewhere; thus the bulk of Chapters 3–10, for instance, is derived (in my view) from Heraclides,[4]

[4] For a contrary opinion see Gottschalk 1980: 134 n. 22, and for an attempt to justify my own position see Barker (forthcoming).

and Chapters 31–9 (by scholarly consensus) from Aristoxenus. I would reckon it an odds-on bet that the self-contained discussion of mathematical harmonics in Chapters 22–5, so markedly different from anything else in the *De musica*, is also paraphrased from a single fourth-century treatise. Before presenting my suggestion about that treatise's author and identity I want to point to certain features of its four chapters which give further support to the hypothesis of a common origin.

In Chapter 22 the compiler sets off, as we have seen, by promising to show that Plato was well versed in harmonics. He continues:

> Thus in the passage on the creation of the soul in the *Timaeus*, he demonstrated his dedication to mathematics and music in the following words [from *Tim.* 35c–36a]: 'And after that he filled out the double and triple intervals by cutting off portions of his material and inserting them inside those intervals, in such a way that within each interval there were two means.'

He then lists and briefly defines the arithmetic, harmonic and geometric means, but he does not refer to them again in the chapter. Nor does he say anything about the in-filling of the triple intervals, or about any of the complexities in the remainder of the *Timaeus*' account. All he does is to explain in both musical and mathematical terms how the octave between *hypatē mesōn* and *nētē diezeugmenōn* can be divided in the familiar way by the insertion of two intermediate notes, *mesē* and *paramesē*. The arithmetical work is done by representing the octave-ratio as 12:6, and by assigning the numbers 8 and 9 to the intermediate notes. The rest of the discussion is an exploration of the symmetrical patterns formed within the octave by this elementary division, rather in the manner of Philolaus frag. 6a (p. 264 above). All the intricacies of the *Timaeus* passage have been elided, and we are apparently to be convinced of Plato's deep understanding of harmonics solely on the grounds that this rudimentary construction can be extracted from his account.

Chapter 23 turns to Aristotle, and it too is concerned only with the octave, which is named both in the opening quotation and at the end of the passage by the word *harmonia*; this again is reminiscent of Philolaus. Like its predecessor, this chapter identifies no notes or intervals within the octave between *hypatē mesōn* and *nētē diezeugmenōn* except those marked out by the 'fixed' notes *mesē* and *paramesē*, giving the boundaries of its upper and lower tetrachords; and it too makes use of the sequence of numbers 12, 9, 8, 6. But it supplements Chapter 22 by explaining (with some confusions which we shall consider below) how these results are reached by the introduction of the arithmetic and the harmonic mean between terms in the ratio of the octave, and by describing (again with some uncertainty of touch) the relations between the intervals constructed through the insertion of means of each type. Here we seem to move beyond the Philolaan model to observations set out by Archytas in frag. 2 (pp. 302–3 above) and exploited in the *Timaeus*.

The first sentence of Chapter 24 is evidently designed to run on directly from the last sentence of Chapter 23, since the latter provides its grammatical subject, which is again *harmonia*. It describes the 'parts' of *harmonia* in the unmistakably Pythagorean language of 'limit' and the 'unlimited' which is central to Philolaus frag. 6, and as even, odd and even-odd (that is, as the product of an even and an odd number). Here too *harmonia* is at least primarily exemplified in the structure of the musical octave, expressed in terms of number and ratio, since its even, odd and even-odd components are once again the numbers 12, 9, 8 and 6. As in the preceding chapters, the author refers only to the octave's division into its concordant substructures, and does not go on to analyse the divisions of its tetrachords in specific varieties of scale. Finally, the brief Chapter 25 touches on the idea that our senses are manifestations of *harmonia*, and that this is true above all of sight and hearing, which are 'celestial and divine', and which reveal *harmonia* to us with the help of light and sound. These notions may seem strange, but there are several parallel passages. The general thesis about the senses finds a partial echo in Aristotle's *De anima*; and very similar propositions about the 'celestial' status of sound and hearing and their value in revealing the secrets of *harmonia* appear in Plato's *Timaeus*, in another fragment of Aristotle, and later in a fascinating passage of Ptolemy's *Harmonics*; there are affinities, too, with remarks in the work of Archytas and in Plato's *Republic*.[5]

The focus on the basic mathematical structure of an octave-*harmonia* gives these chapters of the *De musica* a continuous theme, and they progress in an orderly way from one aspect of it to another. Thus an excerpt from the *Timaeus*, shorn of the complexities of its context, a quotation from Aristotle and recognisable ingredients of the work of Philolaus and Archytas have been brought coherently together. The result looks like an elaboration of the harmonic analysis of Philolaus frag. 6a, combined with the metaphysics and cosmology of frag. 6, given a deeper conceptual foundation through

[5] Aristotle, *De anima* 426a–b; Plato, *Tim*. 47b–d; Aristotle frag. 48 Rose, frag. 24 Ross (both editors assign the fragment, rightly in my view, to the same work as that to which they assign the fragment quoted in *De mus*. ch. 23, the *Eudemus* according to Rose, the *De philosophia* according to Ross); Ptolemy, *Harm*. 93.11–94.20. The ideas expressed in these passages seem also to be connected, though rather more distantly, with the representation of astronomy and harmonics as 'sister-sciences' in Archytas frag. 1, which is echoed by Plato at *Rep*. 530d and reappears again, ingeniously transformed, at the end of the passage in Ptolemy's *Harmonics*.

Archytas' account of the means, and extended into other areas touched on by Archytas and Plato. In short, it is an exploration of the way in which the octave's structure was represented and put to metaphysical use in a developed form of Pythagorean theory whose components were all in place by the second or third decade of the fourth century. There is nothing elsewhere in the *De musica* to encourage the thought that the compiler himself was capable of sifting and integrating passages and ideas from four or more original texts to produce this selective and well-constructed account; we can be confident, I suggest, that in line with his procedure elsewhere he was simply paraphrasing and summarising part of a single existing treatise. The chapters' contents gives us no reason to doubt that his source came from the same period as those he relied on in the bulk of the rest of the work, that is, from the fourth century, and it is a reasonable guess that it came from the same stable as his other major authorities, that is, from a writer in the Lyceum.

In that case the writer is likely to have been Aristotle himself, the latest of those to whom the passage explicitly or implicitly refers. The compiler regularly mentions his major sources (though he does not always tell us where their contributions begin and end), and none of the others named in the *De musica* is a plausible candidate. There is nothing intrinsically improbable about the hypothesis. We may even be able to identify the work on which the *De musica* draws, since there is one lost Aristotelian treatise into which a discussion of this sort would seem to fit perfectly. It was an essay which examined parts of the *Timaeus* in tandem with aspects of the work of Archytas. There is some doubt about the exact form and meaning of its title (recorded in slightly different versions by Diogenes Laertius and Hesychius) and we have very few clues about its contents, but it evidently made some attempt to relate aspects of Archytas' thought to ideas set out in the Platonic dialogue.[6] Since we know so little about it I cannot weigh up the merits of my suggestion by confronting it with detailed evidence from elsewhere, and to that extent it can be no more than a guess. I can only say that it strikes me as plausible; and if it were correct it would give us a glimpse of the strategy adopted in the treatise. It would suggest, for example, that Aristotle was not trying to bring out contrasts

[6] The evidence about this work and another Aristotelian essay on Archytas is assembled as testimonium A13 in Huffman 2005, and discussed on pp. 581–94. Huffman gives its title as *A Summary of the Timaeus and the Works of Archytas*; but we should be wary about the word 'summary', which is not guaranteed by the Greek, and also about the implications of the phrase 'the works', which might suggest a treatment of the whole body of Archytas' oeuvre. According to the ancient sources Aristotle's treatise was in only one book, and it seems unlikely to have had so compendious a scope.

between the two bodies of work he was considering but rather to establish common ground, and to extract from them a compendious representation of 'Pythagorean' philosophy in its most up-to-date form.

The hypothesis that these chapters are paraphrased from a work by Aristotle remains a strong one, in my view, whether or not it was the essay on the *Timaeus* and Archytas. In that case, since they are apparently designed to expound a central theme of Pythagorean mathematical harmonics and to sketch some of the metaphysical uses to which its exponents put it, it would be unsafe to assume that the direct quotation from Aristotle at the beginning of Chapter 23 is an expression of his own views. Most of it is soberly analytic and (given a little further clarification) could have been asserted by anyone who had mastered the relevant mathematics. But the rhapsodic opening sentence, '*Harmonia* is celestial, and its nature is divine, beautiful and wonderful', seems to hint at connections between harmonics and the study of the heavens which elsewhere Aristotle vehemently rejects. More probably, I think, the fragment is of a piece with all the other material in the passage surrounding it, and was part of his exposition of theses he found in his Pythagorean sources. He would hardly have taken the trouble to construct an account of their ideas if they had not struck him as interesting, but it does not follow that he shared them.

Of the four chapters in the *De musica* it is the one devoted to unravelling the meaning of the Aristotelian fragment that can contribute most for my purposes in this book. I shall now spend a little time on its details. After its opening sentence the fragment is densely compressed and bristles with technicalities. 'In potential it (*harmonia*) is four-fold, and it has two means, the arithmetic and the harmonic; and its parts, magnitudes and excesses are revealed in accordance with number and equal measure; for melodies acquire their structure in two tetrachords.' These statements must have served, in their enigmatic way, as an introductory summary of doctrine that would be explained in the sequel, and it seems probable that the discussion that follows in the *De musica* is a version of the original explanation.

The discussion establishes that what Aristotle calls the 'parts' of the octave-*harmonia* are its boundaries; the 'magnitudes' are the ratios 12:9, 9:6, 12:8 and 8:6; and the 'excesses' are the arithmetical differences between the ratios' terms. The 'magnitudes' 12:9 and 8:6 are 'equal in measure' (since each is equivalent to 4:3), as are 12:8 and 9:6 (equivalent to 3:2). 'Equality of measure' holds also, in a different but related sense, of the 'excesses' in the ratios 12:8 and 8:6, since 8 is the harmonic mean between 12 and 6; the difference between 12 and 8 is one third of 12, and that between 8 and 6 is one third of 6. The excesses in the ratios between the extremes and their arithmetic mean, 12:9 and 9:6, are 'equal in number'. The final remark seems designed to link the octave's mathematical structure to musical practice; the terms used in the mathematical analysis correspond to the boundaries of the tetrachords within which melodies are formed.[7]

Up to this point we are on firm and thoroughly Pythagorean ground, and there is nothing to suggest any misunderstanding on Aristotle's part; if the Plutarchan writer had not named the fragment's author, we would have had little hesitation in assigning it to a Pythagorean source. I shall not quote the whole of the subsequent paraphrase. It begins by attributing to Aristotle the statement that the 'body' of *harmonia* (which I take to be made up of the audible sounds in which its formal structure is instantiated) has parts which are unlike but concordant, and that its means are concordant with its parts (i.e. its boundaries) 'in accordance with numerical ratio'. It then identifies the various ratios and the intervals that correspond to them. There follows a statement whose cramped mode of expression owes more to a fondness for rhetorical counterpoint than to the pursuit of clarity, but whose content is elementary; it merely sets out basic features of the relations between the system's bounding terms and their arithmetic and harmonic means.

Difficulties start with the next pair of sentences. I translate them as they appear in the manuscripts.

> Aristotle reveals that their properties [those of the means and extremes] are such that *neatē* [the variant form of *nētē* used also by Philolaus] exceeds *mesē* by a third part of itself, and *hypatē* is exceeded by *paramesē* in the same way. Thus the excesses belong to the class of the relational; for they exceed and are exceeded by the same parts, since the extremes exceed and are exceeded by *mesē* and *paramesē* in the same ratio, the epitritic [4:3] and the hemiolic [3:2]. (1139d–e)

The initial problem is that one would naturally take the first sentence to refer to the harmonic mean between *nētē* (12) and *hypatē* (6); but their harmonic mean is *mesē* (8), and *paramesē* (9) has nothing to do with it. *Nētē* exceeds *mesē* by one third of itself, and *hypatē* is exceeded by *mesē*, not by *paramesē*, by the same fraction of itself. The obvious way of repairing the damage is to emend the MSS *paramesē* to *mesē*, as is done by Weil and Reinach. But we can adopt this manoeuvre only at the cost of making the latter part of the second sentence irrelevant; its reference to *paramesē* cannot be eliminated in the same way. If we follow Weil and Reinach, this remark must be struck out as an interpolation.

[7] What I have said so far about the fragment adds little to the notes *ad loc.* by Einarson and De Lacy 1967, with which I am in almost complete agreement.

But another and more interesting diagnosis is possible. If the closing statement is retained, its sense is that the ratio in which *nētē* exceeds *mesē* is the same as that in which *hypatē* is exceeded by *paramesē* (i.e. 3:2), and the ratio in which *nētē* exceeds *paramesē* is the same as that in which *hypatē* is exceeded by *mesē* (i.e. 4:3). This is of course true, and the point of stating it is presumably to bring out the system's structural symmetry, very much in the way that Philolaus had done (frag. 6a). What has happened in the first sentence, I suspect, is that a statement originally designed to specify the properties of the harmonic mean has been adapted to the purposes of the statement expressing the 'symmetry' thesis. The proposition lurking somewhere in the passage's history identifies *mesē* as the harmonic mean between *nētē* and *hypatē*, stating that *nētē* (12) exceeds *mesē* (8) by one third of itself, that is, of 12, and that *hypatē* (6) is exceeded by *mesē* by one third of itself, that is, of 6. As we have it in the MSS, by contrast, it must mean that *nētē* (12) exceeds *mesē* (8) by one third of 12, and that the excess of *paramesē* (9) over *hypatē* (6) is one third of 9. This amounts to the observation that the ratios of *nētē* to *mesē* and of *paramesē* to *hypatē* are equal. But the closeness of its language to standard ways of discussing the harmonic mean betrays its misunderstanding of the original statement underlying it.

The problems bedevilling the next part of the text (1139e–f) are even more acute and cannot be fully examined here; without emendation it is neither intelligible nor even grammatical. It is possible, however, that the distortions that have crept into the text arise once again from the superimposition of a new meaning on statements intended to convey another; and the passage has probably been further confused by later copyists' attempts to make sense of it. The underlying and superimposed senses are parallel to those in the earlier passage. This time the underlying statement identifies *paramesē* as the arithmetic mean between *nētē* and *hypatē*; *nētē* exceeds it, and *hypatē* is exceeded by it, by the same amount. The superimposed sense completes the evidence for the 'symmetry' thesis set out above; *nētē* exceeds *paramesē* in the same ratio as that in which *hypatē* is exceeded by *mesē*, a fact which the symmetry thesis presupposes, but which had not been expressed in the earlier passage's first sentence.

If this interpretation is on the right lines, the propositions expressed in both layers of the passage are true. From a mathematical perspective the relations identified in the superimposed layer are relatively trivial, but they are none the less striking and Philolaus had thought them important; they show that *harmonia* displays perfect symmetry of form. The underlying layer corresponds to Archytas' account of the harmonic and arithmetic means, and identifies their locations in the system; and its presence at some level of the text, as I have said, is unmistakably betrayed by its language. We are therefore faced with a choice. Commentators have regularly assumed that Aristotle's own version faithfully reproduced Archytan doctrine, and that the confusions (on my reading, the misinterpretations and superimpositions) are due to the author of the paraphrase. That may indeed be so. But since there can be no doubt that Aristotle, in his turn, was paraphrasing Archytas or a similar source (so that there are in fact three layers in the make-up of this passage rather than two), it is at least conceivable that the misunderstandings began with him.

The impulse to treat Aristotle as blameless and to fasten any confusions on the Plutarchan compiler is understandable. But there is one small piece of evidence that brings the case against Aristotle within the realm of possibility, though it does not amount to proof of guilt. In a passage of the *Politics* (1301b29–35) he distinguishes two kinds of equality. One is arithmetical, exemplified in the excesses of 3 over 2 and of 2 over 1. The other is equality 'in ratio', exemplified in the excesses of 4 over 2 and of 2 over 1, since the part (i.e. the fraction) of 4 by which 4 exceeds 2 is equal to the part of 2 by which 2 exceeds 1. In the first sentence quoted from the Plutarchan paraphrase, the 'superimposed' sense substitutes equality of ratio (12:8 = 9:6) for the relation created by the insertion of the harmonic mean (where 12–8 is the same fraction of 12 as 8–6 is of 6). In the *Politics*, apart from arithmetical equality, only equality of ratio is mentioned. The point is not that it neglects the 'harmonic' relation, but that the terms in which it describes equality of ratio are very similar to those in which Archytas described the harmonic mean, and which reappear in the Plutarchan discussion. The harmonic mean, according to Archytas, is such that 'the part of itself by which the first term exceeds the second is the same as the part of the third by which the middle term exceeds the third' (Archytas frag. 2). In the Plutarchan paraphrase, '*neatē* exceeds *mesē* by a third part of itself, and *hypatē* is exceeded by *paramesē* in the same way'. In the *Politics*, equality in ratio is exemplified by the excesses of 4 over 2 and 2 over 1, 'for 2 [the excess of 4 over 2] is the same part of 4 as 1 [the excess of 2 over 1] is of 2, since both are halves'.

The 'equality of ratio' which the *Politics* describes corresponds mathematically to the relation set up, in Archytas' classification, when a geometrical mean is introduced between two other terms. He describes this relation, however, in language completely different from Aristotle's: 'as is the first term to the second, so is the second to the third'. There is nothing wrong with Aristotle's formulation. But what we see now is that in discussing geometric equality and proportion, both he and the Plutarchan paraphrase use language reminiscent of Archytas' account of the harmonic, not the geometric mean. On this evidence, if Aristotle had wanted to convey the sense of the paraphrase's 'superimposed' thesis, he would have done so in much the same terms as those actually found in the text. The hypothesis that its origins there are with Aristotle himself must remain tentative, but it cannot be reckoned unthinkable.

ARISTOTLE'S OWN USES OF MATHEMATICAL HARMONICS

Let us turn now to territory where the ground is firmer, at least to the extent that we are dealing with the unadulterated Aristotle of the works that survive complete. Though he seldom engages closely with details of harmonic theory, he occasionally uses some of its concepts and conclusions to shed light on other topics. In the passage of the *Politics* where he discusses two types of equality, for instance, his topic is neither music nor mathematics. He is examining the different ways in which one might construe what people mean when they set up 'equality' as the goal of an attempt at political change,[8] and he uses the distinction between arithmetical equality and equality of ratio in order to rid their slogan of its ambiguities. But this passage does not allude explicitly to harmonics, and I shall put it aside. Several others might usefully be explored, but instead I shall select just one by way of example, and study it in a little detail.

In the third chapter of the *De sensu* Aristotle's topic is colour.[9] In the part we shall consider, he starts from the position that the fundamental colours are white and black; and claims that all other colours, red, green and so on, arise in one way or another from combinations of those two. One of Aristotle's main concerns is with the various kinds of 'combining' that can be involved. He looks at the case where the impression of a colour is produced by arrays of tiny black dots and white dots set side by side, the case where it is created by laying a thin film of white over a lower layer of black or vice versa, and the case in which white and black are fully blended together in something like a chemical fusion, where all parts of the mixture, however

8 Early in the passage (1301b6) he refers to changes that transform political institutions as *metabolai*, which is a commonplace word, but is also the one regularly used for musical 'modulations'. It is possible that Aristotle had that sense in mind, and that in this context the noun should be understood as a metaphor. Its musical associations will in any case have been obvious to his original readers.

9 A version of the discussion that follows was presented as a paper to the B Club in Cambridge in 2004, and I am grateful for comments made by members of my audience on that occasion.

small, are alike. The differences between these modes of combination need not concern us. Our starting point is Aristotle's supposition that in any of these scenarios, different colours are produced by combining white and black in different proportions; we get one colour when three units of white are put together with two units of black, for instance, and another when the ratio between the quantities is changed.

Aristotle's discussion is presented in three parts, each concerned with one of the sets of physical conditions I have sketched. In each of the three phases his comments on the relevant issues fall into two parts, one of which describes the kind of quantitative relation between white and black which produces colours of a superior sort, while the other contrasts that relation with one that is different and somehow inferior, and whose outcome, correspondingly, is a somehow inferior type of colour. These distinctions between better and worse relations do not depend on features of colours as such; they have nothing to do with the chemistry of pigments or the physics of light-refraction or the psychology of visual perception. They are of a mathematical sort, and can apparently be transferred, essentially unchanged, to any other domain in which relations between quantities play a significant role. Aristotle refers to mathematical relations between musical pitches as one parallel case; ratios of the same sorts are in play, and there is a similar distinction between better relations and worse.

Aristotle first introduces the ratios at 439b27 ff. Here he is discussing only the situation in which minuscule black dots and white dots are set side by side, in what we might call the pointillist scenario; but later he explicitly asserts that his remarks apply equally to the other kinds of case, those of superimposed colours and of genuine mixtures (440a12 ff., 440b18 ff.). The mathematical relations which generate each of the two categories of colour are the same in all three situations; and in that case we must suppose that the slightly varied descriptions he gives of each relation in different parts of the passage are intended to be equivalent. But the descriptions pose quite troublesome problems.

In connection with the first situation he says that the juxtaposition of particles of black and white colour will produce different colours when there are different ratios between the numbers of white and black particles, 'for they may lie beside one another in the ratio 3:2 or 4:3 or in accordance with other numbers'. This describes the 'superior' type of relation; of the other kind he says that they 'are in no ratio at all, but in an incommensurable (*asymmetron*) relation of excess and shortfall' (439b27–30). He then adds the following comment.

These colours [those of the better sort] are constituted in the same way as the concords. For the colours that are in the best-ratioed numbers, like the concords in their domain, are those of the colours that appear most pleasing, purple and red, for instance, and a few others of that kind; and they are few for the same reason that the concords too are few; while those that are not in numbers are the other colours. (439b31–440a3)[10]

In the second situation, where a black surface is seen through a film of white or vice versa, Aristotle says that many different colours will be produced 'in the same way as was previously stated; for there will be [in one kind of case] some ratio between the superimposed colours and the ones below, while others will be in no ratio whatever' (440a12–15). Finally, in the case where quantities of black and white are fused into a completely homogeneous mixture,[11] 'there will be many colours because the things that are mixed are capable of being mingled in many ratios; and some colours are in numbers, while in other cases there is only an excess [of black over white or the converse]' (440b18–20).

Thus where combinations of black and white produce the better kinds of colour, the colours are first described as being in some ratio of numbers, for instance 3:2 or 4:3, 'or in accordance with other numbers'. Secondly (in the same passage) we are told that they are 'in the best-ratioed (*eulogistoi*) numbers', like the concords, and that there are only a few such colours for the same reason that there are only a few concords. They are said, thirdly, to involve 'some ratio' between the amounts of white and black; and in their last appearance they are said to be 'in numbers'. We are given to understand that the sense of these various descriptions is in every case the same. Meanwhile the less attractive colours are said in the first passage to be 'in no ratio at all, but in an incommensurable relation of excess and shortfall', and are characterised shortly afterwards as 'not in numbers'. The formulations in the other passages simply repeat parts of these descriptions and introduce nothing new.

There is no great difficulty in construing 'in some ratio' and 'in numbers' (and their negative counterparts, 'in no ratio' and 'not in numbers') as equivalent expressions. At first sight the descriptions of the 'better' relations appear to apply to all ratios of integers without exception (all integers, and only they, can be described as *arithmoi*, 'numbers'), and if they did they

10 He follows this account of the difference between the better and worse relations with another, apparently offered as an alternative possibility; but since it does not reappear in the sequel I shall pass it by for present purposes. Aristotle seems not to have thought the possibility worth investigating further.

11 On 'complete mixture' of this sort see Aristotle, *De gen. et corr.* 1.10.

would give us an intelligible way of understanding the reference to cases where there is no ratio, but 'an incommensurable relation of excess and shortfall'. These would be cases where one of the quantities involved is greater than the other (it 'exceeds' the other and the other 'falls short' by some amount), but it is not possible to specify the relation between them as a ratio of integers; that would be true, for instance, of the relation between the lengths of the side and the diagonal of a square.

There are three interlocking complications, all arising from the first passage's comparison of the more attractive colours with the concords. First, we are told that there are only a few of these colours, just as there are only a few concordant intervals. But it is obviously not true that there are only a few ratios of integers; on the contrary, there are indefinitely many. If there are only a few relations of the relevant sort, they must correspond to some determinate small number of ratios of integers, and there must be some criterion by which they are marked off from the rest. Aristotle does not seem to provide a criterion which could play this role. Secondly, at this point in the text the musical concords and the attractive colours are not described simply as 'in ratios' or 'in numbers', but as 'in the best-ratioed numbers', which seems to mean something quite different. It also suggests that there is another class of relations which will still count as 'in numbers', but whose numbers come together in ratios of an inferior sort. They would be demarcated by their failure to meet the hypothetical criterion which I mentioned above; and the implication is puzzling for another reason too, since these relations would seem to be different from both the 'better' and the 'worse' ones described in the other formulations, and we hear nothing of a third, intermediate group of relations elsewhere in the discussion. Finally and rather similarly, if the cases where there is no ratio but an 'incommensurable excess' of one amount over the other are those where the relation cannot be expressed as a ratio of integers, it is not only the concords that will be excluded from this category, but also every one of the non-concordant intervals quantified by the mathematical theorists. The only intervals to which the description could apply are ones such as the exact half-tone; and in fact the theorists do not describe them in Aristotle's way, as intervals that can exist but are not in ratios of whole numbers. They simply deny that there can be such things. According to the regular interpretation of Archytas' theorem (pp. 303–5 above), a tone in the ratio 9:8 *cannot* be halved, since no interval whose ratio is epimoric can possibly be divided into equal parts.

We can reduce the number of major questions posed by these issues to two. What sort of relation is it that is 'in no ratio at all' but is constituted 'according to some incommensurate excess and shortfall'? And what is the criterion by which some determinate small number of ratios of intervals, those of the 'better' kind, can be distinguished from the rest? I think that the answers to these questions can be extracted from the resources of mathematical harmonics, and that there is indeed one answer that covers them both. From a historical perspective this answer is probably correct, and the fact that it will seem seriously inadequate when considered from a more abstract point of view should not lead us to reject it. Its shortcomings are symptomatic of anomalies in Greek mathematical harmonics itself.

Let us begin with my second question. By what licence are some ratios proclaimed as the best, and which are they? The clearest statement of an appropriate answer in writings on harmonics is in Ptolemy's account of ideas he attributes to 'the Pythagoreans', by which he certainly means Pythagoreans of the period we are considering here. Although one must be cautious in basing conclusions about fourth-century theories on so late (and in some respects so idiosyncratic) a source, there are good grounds for tracing these ideas back at least as far as Archytas, and for present purposes I shall take Ptolemy's evidence at face value.[12] What he says is that the Pythagoreans divided relations between musical pitches into two main classes, the concords and the discords, of which, so they said, the class of concords is the 'finer', *kalliōn*; and they divided ratios, correspondingly, into two primary categories. One of them 'is that of the so-called "epimeric" or "number to number" ratios, and the other is that of the epimorics and multiples; and of these the latter is better (*ameinōn*) than the former' (Ptol. *Harm*. 11.10–15). The sentence continues beyond this point, and I shall come back to it.

Epimoric and multiple ratios have already been mentioned frequently in this book, but a little recapitulation may be helpful. A ratio of the sort which Ptolemy identifies first is usually called 'epimeric' (*epimerēs*); his alternative expression for such ratios, 'number to number', is less common but was evidently current in the fourth century, since Plato uses it in the *Timaeus* (36b3). As conceived by Ptolemy and most other writers on harmonics, this category contains all the ratios that are neither multiple nor epimoric, ratios such as 7:4 or 15:11; Plato's example is the ratio of the *leimma*, 256:243.[13] We

[12] The issues are discussed in Barker 1994b; cf also Barker 2000a: 65–7.

[13] The Peripatetic philosopher Adrastus, around the end of the first century AD, is credited with a further classification of this motley crew of epimerics into subcategories with names that might almost have been coined by Aristophanes; we have the *pollaplasiepimereis* and the *pollaplasiepimorioi*, along with their ghostly Doppelgängers, the *hypopollaplasiepimereis* and the *hypopollaplasiepimorioi*, and Adrastus reserves the expression 'number to number' for ratios which fail to fit even under any of these descriptions (Theo Smyrn. 78–80). This is rollicking stuff, no doubt excellent sport for mathematicians, but we can ignore it and leave them to their fun.

are presented, then, with a pair of aristocrats in the kingdom of ratios, the multiples and the epimorics, and a plebeian gaggle including all the rest.

Ptolemy tells us that on the basis of the argument I have quoted, the Pythagoreans linked the 'better' class of ratios with the 'finer' class of musical intervals, the concords. At one level this association is obvious and by now very familiar; the ratios of the three primary concords are 2:1 for the octave, 3:2 for the fifth and 4:3 for the fourth, and of these 2:1 is multiple and the others are epimoric. But it soon emerges that Ptolemy means more than that; he means that these theorists adopted the principle that all concords *must* have multiple or epimoric ratios, and we have already seen that this principle was current in fourth-century mathematical harmonics. If the interpretation I offered in Chapter 11 is correct, Archytas extended its scope further, and insisted that not just the concords but all the intervals which form steps in any properly formed musical scale must be constructible through a procedure involving epimorics alone. Harmonic theory, then, has its elite ratios and its proletarians, and this obviously encourages the guess that Aristotle's 'best' ratios are precisely these elite ones, the multiples and epimorics. There is a very obvious objection to this hypothesis, but I shall leave it on one side until we have considered whether the same classification of ratios can give any help with our other question.

Aristotle speaks of relations that are constituted in no ratio whatever, 'but according to some incommensurate excess and shortfall (*kath' hyperochēn de tina kai elleipsin asymmetron*)'. What does he mean? Now in the course of Ptolemy's ruminations over notions he found in earlier Pythagorean forms of mathematical harmonics, his description of the feature which gives the aristocratic epimorics a higher status than their inferiors is almost an exact mirror-image of Aristotle's. The superior ratios are *en symmetrois hyperochais*, 'in commensurate excesses' (*Harm*. 16.13). A little later he credits Archytas with the thesis that 'melodic' intervals, that is, the individual steps of any scale, must be characterised by 'the commensurateness of the excesses', *to symmetron tōn hyperochōn* (*Harm*. 30.9–13); and it emerges that these expressions pick out precisely the class of the epimorics. If we take Ptolemy's expressions as a guide, we shall clearly be led to the conclusion that the relations which Aristotle describes as being in no ratio at all are in fact the epimerics.

In order to make further progress, we need to discover exactly what Ptolemy's expressions mean, and in what sense epimoric ratios have 'commensurate excesses'. Secondly, if these expressions specify the feature of epimorics which qualifies them for the ranks of the elite, we need to ask whether this feature has a counterpart in the case of the multiples, of such a sort that we can identify just one overarching consideration from which both classes of ratio derive their aristocratic credentials. The answers to both questions are most clearly and succinctly put, once again, by Ptolemy, in the later part of a sentence I quoted above. The Pythagoreans treated multiples and epimorics as the 'better' class of ratios, he says, 'because of the simplicity of the comparison, since in comparisons involving epimorics the excess is a simple part, and in the case of multiples the smaller term is a simple part of the greater' (*Harm.* 11.15–17). What he says here about epimorics reflects the fact that in such a ratio the 'excess', the difference between the two terms, is a 'simple part' (one half or one third and so on) of both the terms (see pp. 288–91 above). It can therefore serve as the unit by reference to which both terms are measured; and it is in this sense that it is *symmetros* with them. In ratios of the third, inferior sort, this is not the case; in the ratio 7:4, for instance, the difference, 3, is not a 'measure' of either 7 or 4. When we look at the multiples, it is obvious that the excess or difference cannot serve as a 'measure' except in the special case of 2:1; if we take the ratio 6:1, for example, the excess of the larger term over the smaller is 5, and this cannot be used as the measure of either 6 or 1. But in these cases the measure is the smaller term itself, of which, by definition, the larger is always a multiple.

These points lead to a more difficult question. In what sense does this feature of epimorics and multiples, the inclusion within them of an element by which the others can be measured, make them 'better' than the others, and why is it thought to underlie the superior kind of 'fineness' manifested audibly in the musical concords? Ptolemy's answer, which is hinted at in the phrase 'the simplicity of the comparison', is complex and hard to disentangle,[14] and is probably irrelevant to the Aristotelian context; I am reasonably confident that it is not one that he disinterred from fourth-century sources but a hypothesis of his own. The fourth-century answer is much more straightforward.

What gives the concords their special excellence, according to a whole series of writers from Plato onwards (*Tim.* 80a–b), including Aristotle (*De sensu* 7, 448a9 ff.), is the fact – as it is repeatedly said to be – that when the two notes of a concord are heard simultaneously, they are not perceived as two disparate items, isolated from one another. They blend together to form a single, unified sound which is identical with neither of them. When the two notes of a discord are heard at the same time, by contrast, no such well-integrated result is produced; they simply appear side by side as two separate sounds. To quote one clear definition from a later source,

[14] For discussion see Barker 2000a: 82–7.

intervals 'are concordant when the notes that bound them are different in magnitude [which in this author means "different in pitch"], but when struck or sounded simultaneously, mingle with one another in such a way that the sound they produce has a single form, and becomes as it were one sound. They are discordant when the sound from the two of them is heard as divided and unblended.'[15]

The emphasis of such accounts is always on the fusion and unification of a concord's elements to form a seamless whole. This is a special case of the same idea that gives us the 'harmonisation' of opposites through the influence of love in Eryximachus' speech in Plato's *Symposium* (185e–188e), and the synthesis of limiters with unlimiteds through *harmonia* in Philolaus frag. 6; more generally, it reflects the theme of diverse or even mutually hostile elements being integrated in a harmonious and admirable unity which runs through the Presocratic tradition from Heraclitus onwards. We can see how it links up with the ideas about multiple and epimoric ratios; the point is that they too are beautifully integrated complexes. The glue which holds together the greater and smaller terms is their 'common measure', the ingredient in the ratio that mediates between them, manifested in epimoric ratios by the 'commensurateness of the excesses' and in multiple ratios by the commensurateness of the terms themselves. In ratios of the third class, as in the case of the discords, there is no such ingredient to bring the terms into mutual agreement; as Aristotle says, the terms lie side by side 'according to some incommensurate excess and shortfall'.

Let us turn now to the obstacles in the way of this interpretation. The first is that Aristotle has described the substandard relations not only by reference to their incommensurate excesses, but as 'in no ratio at all'; and elsewhere in the passage the expression 'in no ratio' seems to be interchangeable with 'not in numbers'. The problem, obviously, is that neither description seems to fit the relations in the third class of ratios; whatever their deficiencies may be, they evidently are 'in ratios', and the ratios are defined by reference to whole numbers. The fact that Plato describes them precisely as relations 'of number to number' may well give us further cause for unease. As I said earlier, we would expect that 'in no ratio' and 'not in numbers' would designate genuinely irrational relations, like the relation between the diagonal of a square and its side; but we have seen that this cannot be what Aristotle meant.

This difficulty, however, may be illusory. There is at least one other relevant text which uses the 'no ratio' formulation, and spells out exactly

[15] Nicomachus, *Harm.* 12. 262.1–6.

the sense in which it is being used. A passage in the Aristotelian *Problems* discusses the intervals and the arithmetical relations created by doubling an octave, a fifth or a fourth.[16] The ratio of a double octave is 4:1; that of a double fifth is 9:4, and that of a double fourth is 16:9. All these are perfectly well-formed ratios of integers. But what the writer says is that the terms bounding the double fifth and the double fourth 'will have no ratio to one another (*pros allēlous oudena logon hexousin*)'. And he immediately explains what he means; *oute gar epimorioi oute pollaplasioi esontai*, 'for they will be neither epimoric nor multiple'. Here, then, we have the clearest possible evidence that the expression 'no ratio' could be used to refer to relations like 16:9 and 9:4, which by other standards would evidently count as ratios, ones belonging to the inferior 'number to number' class. (It may be worth noting that when Plato uses the 'number to number' formula at *Tim*. 36b3, he does not call the relation in question a *logos*, a ratio; but this by itself is not decisive, since in that passage not even ratios of the superior sorts are called by that name.)

In that case it is a reasonable hypothesis that Aristotle is using the expression 'in no ratio' in the same way as the writer of the Problem, to refer to these 'number to number' relations, all and any relations which are neither multiple nor epimoric. I do not know how this odd-looking usage arose; plainly, however, it existed. But there is still one glaring difficulty. When Aristotle describes the superior relations as 'in some ratio' or 'in numbers' he seems to be talking about relations in any genuine ratio whatever, that is, on our hypothesis, in any epimoric or multiple ratio. But when he describes them as those in the 'best-ratioed numbers', he seems to imply that there is a hierarchy even among the genuine ratios, and that the members of some sub-group of them are the best. That fits awkwardly with our impression that the contrasting group, the ones that are 'not in numbers', includes all and only the number-to-number relations; those of the genuine ratios which are not among the 'best' seem to be overlooked. Yet there is no doubt that those in the best-ratioed numbers amount only to a sub-set of the genuine ratios, not to all of them, since Aristotle tells us that there are only a few of them, just as there are only a few concords. We still have to locate the criterion by which some are marked off as the best, and we still have to account for the apparent absence of any reference to all the others.

At this point we come across a vulnerable spot in Greek mathematical harmonics. It was agreed on all sides that the concords are indeed few; within the span of the octave only three intervals are allowed to qualify for that

[16] [Aristotle] *Probl.* 19.41.

title, the fourth, the fifth and the octave itself. It is also common ground, with no dissenting voices whatever, that the distinction between concords and discords is determinate and absolute. Notes related in the fourth, fifth or octave blend together to form a unity in which neither note is heard individually, and notes combined in any other relation do not; writers of this period recognise no borderline cases. Aristotle holds, similarly, that the number of the most attractive colours is small; and I think he is making the same claim here as when he says later that the 'species' (*eidē*) of colour are limited, that is, as I read it, that there is only a determinate number of them. He states this at the end of Chapter 3 of the *De sensu*, and tries to demonstrate it in a very obscure argument in Chapter 6. In Chapter 4, at 442a20 ff., it turns out that if we exclude white and black (since it is colours produced by combining them that are in question here) there are exactly five, which seem to be red, purple, green, blue and yellow. All other hues are mixtures or variants of these colour-species.

We might expect the harmonic theorists to have worked out some mathematical counterpart of their clear-cut aesthetic distinction between concords and discords, one that would objectively distinguish the 'best' ratios, those of the concords, from all other epimorics and multiples. But they did not, and they can hardly be blamed for their failure to do so, for the very good reason that no such mathematical distinction exists. In the case of the epimorics, for instance, there is no mathematical justification whatever for locating a sharp borderline anywhere in the series 3:2, 4:3, 5:4, 6:5 and so on. The Pythagoreans, of course, were impressed by the fact that the terms involved in the ratios of the basic concords are 1, 2, 3 and 4, summing to the perfect number 10 and forming the *tetraktys* of the decad. But this is not the sort of consideration that would have appealed to Aristotle; and even though it is true that when we hear intervals whose bounding pitches have frequencies related to one another as ratios of small numbers, they strike our ears as 'smoother' than those where the terms of the ratios are larger, there are no arithmetical grounds for separating out 2:1, 3:2 and 4:3 or any other small group as uniquely privileged cases, definitively fenced off from the rest. The problem is apparently so distressing that the theorists never bring themselves to mention it; it is the Medusa's head of mathematical harmonics and will turn you to stone if you look at it directly. The only writer in the tradition who says things which when squarely faced would unveil this ghastly truth is Ptolemy, and even he quickly shuffles them under the carpet.[17] Nothing can be done with it except to pretend that it does not

[17] Ptol. *Harm*. 15.6–17, 16.12–21, cf. Barker 2000a: 74–5, 80–2.

exist; and in that case it would be foolish to imagine that Aristotle, in this passage, was in a position to do anything else. The awkward gap which my interpretation leaves in his account falls precisely where we ought to expect one.

My overall conclusion, then, is that the relevant passages of *De sensu* Chapter 3 should indeed be understood against the background of the harmonic theorists' classification of ratios, together with their attribution of privileged status to epimorics and multiples; and that once some apparent difficulties have been resolved the most obtrusive problem that remains is not of Aristotle's own making. The theorists were unanimous in contending that the ratios of the concords have special status, and Aristotle took over this idea without inspecting its credentials more closely than the theorists did themselves. One might argue that even so he has been a little careless, since even if there really were a sharp cut-off point between the 'best' ratios and the others, it would come in the wrong place. He apparently needs five ratios of the superior sort, one for each of his species of colour, and if the alleged boundary is located where the harmonic theorists put it there can be only three. But this problem can be evaded. It is at least very likely that by Aristotle's time the harmonic theorists had extended their studies from relations within the single octave to the two-octave span which was subsequently treated as containing all possible harmonic relations. That extension is clearly visible in Book XIX of the *Problems*, most of which, I believe, belongs to the fourth century, and again in the *Sectio canonis*; and the latter identifies just the right number of concords for Aristotle's purposes, adding the octave plus a fifth (ratio 3:1) and the double octave (4:1) to the original three.

One might have expected the octave plus a fourth to be included too, but it is not. Though no one disputed that the octave plus fourth sounds like a concord, its ratio is 8:3, which is the wrong sort, neither epimoric nor multiple. The octave plus fourth, one might say, is Medusa's little sister. If its ratio and the fact of its concordance are put together and confronted directly, they will explode the principle that all concords must be epimoric or multiple, to which the *Sectio* explicitly subscribes and which is a foundation-stone of its reasoning. The author deals with this embarrassing point by the simplest of all possible strategies; he does not mention the relevant interval at all, and neither do the *Problems*. I shall argue in the next chapter that the *Sectio* was probably written around 300 BC, and it seems to be original only in its systematic organisation of pre-existing ideas. There is then no difficulty in assuming that Aristotle's harmonic sources also claimed that the number of concords is five.

ARISTOTLE ON THE METHODOLOGY OF HARMONIC SCIENCE

The issue of measurement, which occupied us at the outset of this study, makes a convenient point of entry to Aristotle's reflections on the concepts and methods of harmonic science itself. In the course of a combative discussion of the notion of a 'one' or a 'unity', he writes as follows.

> It is obvious that the unit indicates a measure. In every domain what constitutes it is different, as for instance in *harmonia* it is a diesis, in length an inch [lit. 'a finger'] or a foot or something of that sort, and in rhythms a step or a syllable; similarly in weighing it is a weight of some determinate size.

He goes on to assert, *inter alia*, that if something is a 'one' or a 'unit' this implies that it is the measure of some plurality, and that the unit which is the measure for things of a given kind must itself be indivisible (*Metaph.* 1087b33–1088a4). The diesis is also mentioned in a very similar discussion earlier in the *Metaphysics*, as the indivisible unit and 'primary measure' in its domain (1016b17–24).

These references to a minimal and indivisible interval, the diesis, which functions as the measure of all others ('pluralities' of dieses), are bound to remind us of the 'smallest interval, by which measurement is to be made', mentioned in connection with empirical theorists at *Republic* 531a, and of the diesis or quarter-tone used for the same purpose in the work of the *harmonikoi* discussed by Aristoxenus. In the *Republic*, this interval's credentials as the unit of measurement depend on its being the smallest that the ear can detect, and in calling it 'indivisible' Aristotle seems to have the same thought in mind. At 1087b37–1088a3 he says that the unit of quantity in any domain is indivisible either 'in form' or 'in relation to perception', that is, so far as our senses can tell us. Of the examples he has given, only the syllable and perhaps the rhythmic step can be thought of as indivisible 'in form';[18] and it is hard to see how the units of measurement he has mentioned in connection with length and weight are indivisible even 'in relation to perception' (they are at best only indivisible within the system of measurement being employed). The notion that the musical diesis is indivisible in this latter sense reappears, however, in a passage of the *De sensu*, where Aristotle is trying to decide whether, and in what sense, a continuum can be conceived as having distinct parts. When we look at a millet seed, he says, our vision encounters the whole of it (and so, in a sense, all its parts); yet we cannot see a ten-thousandth part of it. Similarly, 'the

[18] An item is indivisible in form if it cannot be divided into other items of the same sort. A length is divisible into lengths, but a syllable is not divisible into syllables.

note inside the diesis escapes our perception, even though, when the *melos* is continuous, one hears the whole of it; and the interval between what is in the middle and the boundaries escapes perception' (445b32–446a4).[19] The diesis is indivisible in the sense that if a note were placed between its boundaries we could not distinguish its pitch from theirs, or perceive the interval by which it is separated from either of them.

There is no immediate difficulty, then, in relating these passages to the style of harmonic analysis practised by the *Republic*'s empiricists and by Aristoxenus' *harmonikoi*.[20] More complicated issues are introduced by another, closely related passage in the *Metaphysics* (1053a5–21). Here Aristotle says that in every context, people treat the smallest perceptible unit as the measure, and they suppose that they know something's size or quantity when they know it in relation to this unit. Once again he mentions the musical diesis as an example of such a unit, 'since it is smallest'. So far, everything is consistent with the other passages I have cited. But now he goes on: 'But the measure is not always numerically one; sometimes it is plural. The dieses, for instance, are two – not those assessed by hearing but those in ratios – and there are several vocal sounds by which we measure [in the science of phonetics], and the diagonal is measured by two units, as is the side, and so are all magnitudes' (1053a14–18).

Here, then, it is dieses 'in ratios', as conceived in mathematical harmonics, that are represented as units. Aristotle seems to mean that when larger intervals are expressed as ratios, their sizes need to be expressed in terms that refer to two different units of measurement. However this is understood,

19 This is the best translation I can offer for a difficult sentence. On the interpretation I shall follow here, one makes the *melos* between the boundaries of the diesis 'continuous' by 'sliding' from one of them to the other; on the probable origins of the word *diesis* in the performing techniques of aulos-players see n. 11 to Ch. 10 above. (For discussion of some of the problems see Barker 2004: 108–12, Sorabji 2004: 135–6; we shall revisit the passage briefly in Ch. 15.) In that case we hear the whole continuum of sound, but we cannot identify any particular pitch between the boundaries of the diesis, or decide what interval lies between any such pitch and either of the bounding notes. If this is the picture which Aristotle is presenting, it again recalls the procedures mentioned at *Rep.* 531a–b; Aristotle's allusion to 'the note in the middle' may well be an echo of the 'sound in the middle' at *Rep.* 531a6–7. A sceptical critic might argue that Aristotle knew little of these matters except what he could derive from this passage of Plato.

20 It may be objected that it is implausible to conceive the dieses of the *harmonikoi*, which are quarter-tones, as indivisible in precisely this sense, since a tolerably acute ear could indeed perceive a difference between a pitch lying within a quarter-tone's boundaries and the pitches of the boundaries themselves. The quarter-tone is not the smallest interval that the ear can detect, but the smallest that can be recognised as musically significant; the listener's ear will treat any smaller interval as a fudged attempt at the performance of a quarter-tone. The objection has some force. One might argue, however, that no smaller interval than the quarter-tone can be *identified* by perception; that is, that such intervals not only lack musical 'sense', but cannot be assigned any definite size, by the ear, in relation to other intervals. In that case they could neither be measured empirically nor be used as a unit of measurement for other intervals. To that extent at least, they 'escape perception'.

it need not undermine our interpretation of the previous passages, since his explicit distinction between the dieses 'assessed by hearing' and those 'in ratios' apparently implies that in the former case too the diesis is the measure, and that it does not fall into the special category where more than one unit of measurement is involved. But difficulties arise when we focus on mathematical harmonics itself, and ask what the two dieses are and how they serve as units of measurement.

The items to be measured must be intervals such as the octave, fifth, fourth and tone, and perhaps other melodic relations, all of them expressed as numerical ratios. Aristotle's statement that two units are required for their measurement suggests that he was aware of the proposition proved by Archytas (pp. 303–5 above), showing that no interval whose ratio has the form n + 1:n can be divided into any number of equal parts; and he apparently assumes that all the intervals with which harmonics is concerned either have ratios of that sort, or else fall into some other class of which the same proposition is true. Hence larger intervals, expressed as ratios, cannot be measured as multiples of any one smaller interval. If they are to be measured by reference to smaller ones, intervals in at least two different ratios will be needed to function as 'units'. Thus the ratio of the octave, 2:1, for instance, cannot be divided (factorised) into any number of equal ratios of integers; but it can be represented as the product of the ratios of two different smaller intervals, the fifth and the fourth (3:2 and 4:3), since 2:1 = 3:2×4:3.

But this only illustrates the general idea. Plainly the fifth and the fourth are not dieses, and are not the 'smallest perceptible units' to which Aristotle refers. He is apparently talking about a pair of small intervals whose ratios are such that those of *all* other relevant intervals can be expressed through combinations of some number of each. Thus – again only by way of imperfect illustration – the ratios of the fifth and the fourth can be combined to express not only that of the octave, but for instance those of the double octave or the octave plus a fifth, the former as 3:2 taken twice together with 4:3 taken twice (3:2×3:2×4:3×4:3 = 4:1), the latter as 3:2 taken twice and 4:3 taken once (3:2×3:2×4:3 =3:1). Which small intervals are they, then, that can plausibly be regarded as minimal in the required sense, and can also serve jointly to measure all other musical intervals?

If the intervals fundamental to harmonic organisation are the concords and the tone, clearly they must be among the intervals that are measured by the dieses. These dieses must therefore measure the tone; and they must also be capable of measuring the amount by which the fourth exceeds two tones and the fifth exceeds three, that is, the so-called *leimma* (or Philolaan *diesis*) in the ratio 256:243. These conditions can only be met if one of the dieses is the *leimma* itself, and if the other is the interval by which the tone exceeds the *leimma*, that is, the *apotomē* as defined in the Boethian report on Philolaus (p. 272 above), whose ratio is 2187:2048. The tone will then be measured as one *apotomē* and one *leimma* (that is, its ratio of 9:8 will be factorised as 2187:2048 × 256:243), the fourth as two *apotomai* and three *leimmata* (since it exceeds two tones by one *leimma*), and so on.

To the extent that the calculations work, the interpretation makes sense, and no other candidates for the roles of these dieses can put up nearly so good a case. But it faces three obvious objections. First, no other source mentions a harmonic methodology which treats the *leimma* and the *apotomē* in this manner, as the basis of a system of measurement. Arguments from silence are risky, of course, but not altogether negligible. Secondly, neither this pair of dieses nor any other could provide units capable of measuring intervals in all the ratios involved in fourth-century mathematical harmonics. These dieses, for instance, are helpless in the face of many of the ratios in the divisions constructed by Archytas. We would have to assume that they were used only in connection with a division of the sort found in the *Timaeus*, based on a tetrachord containing two whole tones and a *leimma* (9:8 × 9:8 × 256:243), and with a limited number of its variants. The third objection carries the greatest weight. The central purpose of measurement is to express quantities in terms of some fundamental unit (or units). The diesis of empirical harmonics is fundamental in the sense that it is the smallest interval that our ears can identify. The *leimma* and the *apotomē* cannot be accredited on precisely those grounds, since they cannot both be the smallest; and both are substantially larger than either the quarter-tones of the *harmonikoi* or several of the intervals used in the Archytan divisions. The limits of musical hearing are in any case largely irrelevant in the context of mathematical harmonics. But neither do they have any fundamental status when considered mathematically. Plainly they are not mathematical minima, and their rebarbative ratios disqualify them from consideration as intuitively acceptable mathematical starting-points. Nor could their ratios be established, independently of any others, through experiments with relative lengths of an instrument's string. They can be assigned their identities, that is, their ratios, only through a process of derivation from the ratios of larger intervals, the octave, fifth, fourth and tone; it is knowledge of the ratios of these larger intervals that allows us to measure the *leimma* and the *apotomē*, not the other way round.

That is the central point. Right from the beginning, mathematical harmonics took the ratios of the octave and the lesser concords as its point of departure, and based its assessment of the ratios of other intervals on them. It invariably proceeded by dividing larger intervals into smaller ones, not by building up the former from combinations of the latter. It had, in fact, no use for the notion of a 'unit of measurement' in the sense that Aristotle envisages. The passage of the *Metaphysics* which prompted all this puzzlement is premised on the assumption that all measurement works from a least unit upwards. When that assumption is coupled with the fact that in mathematical harmonics most of the important intervals cannot be divided into equal parts, it will inevitably lead to a conclusion of the sort that Aristotle draws. But the assumption is false, and mathematical harmonics is one area in which its writ does not run. There is no point in beating about the bush; so far as this aspect of the subject is concerned, Aristotle did not understand what he was talking about.

Let us turn now to a second issue to do with the methodology of harmonics. In Chapter 4 I outlined the account of a science's structure which Aristotle offers in the *Posterior Analytics*. Here I want to draw attention to what I called the 'same domain rule' which played a pivotal role in Aristoxenus' conception of his science; 'one cannot demonstrate what belongs to one science by means of another'. In Aristotle's formulation, however, the rule is qualified. One cannot do this 'except when they [the two sciences] are so related that one is subordinate to the other, as things in optics are to geometry and things in harmonics are to arithmetic' (75b14–17).

This tells us that there are exceptions to the 'same domain' rule, and that harmonics is one of the sciences to which it does not straightforwardly apply.[21] Aristotle returns to the point in several other passages, again using harmonics as one of his examples. Since the point is not always made or elaborated in quite the same way, it will be helpful to have all the significant variants in front of us before we examine them. Ignoring merely tangential allusions and repetitions, there are four instances to be considered, of which the one just quoted is the first; I shall repeat it for the sake of putting all the evidence in one basket.

(A) One cannot demonstrate what belongs to one science by means of another, except when they are so related that one is subordinate to the other, as optics is to geometry and harmonics is to arithmetic. (75b14–17)

(B) [In order to know something scientifically, Aristotle says at 76a4 ff., we must know it on the basis of principles which hold of it as such; and in that case the middle term of a scientifically demonstrative argument must belong to the same kind as the other terms. He continues:] Alternatively, it is like the way things in harmonics are demonstrated through arithmetic. For such things

[21] For two illuminating discussions of these exceptional sciences see Lennox 1986, Hankinson 2005.

as these, similarly, are demonstrated; but there is a difference. For the fact that something is so belongs to one science, since the underlying kind is different,[22] but the reason why belongs to the higher science, to which the attributes belong in themselves. Thus it is clear even from these cases that one cannot demonstrate anything absolutely except from its own principles; but the principles of these sciences have something in common. (76a9–15)[23]

(C) The reason why differs from the fact in another way, when each is studied by a different science. Such cases are those which are so related to one another that one falls under the other, as optics is related to geometry, mechanics to stereometry [i.e. solid geometry], harmonics to arithmetic and observation of the stars to astronomy. Some of these sciences have almost the same names, as for instance mathematical astronomy and nautical astronomy, or mathematical harmonics and harmonics based on hearing. In these cases it is the task of the empirical scientists to know the fact, and that of the mathematical scientists to know the reason why. For the latter possess the demonstrations which provide the explanations, and often do not know the fact, just as people who study universals often do not know some of the particular facts, since they have not examined them. (78b34–79a6)

(D) One science is more exact than another, and prior to it, if it is the science both of the fact and of the reason why, and not of the fact alone, separately from the science which deals with the reason why. Again, if a science deals with something without treating it as inhering in an underlying subject, it is more exact than one that deals with it while treating it as inhering in an underlying subject, as arithmetic is more exact than harmonics. (87a31–34)[24]

The general drift of these passages is that facts falling within the domain of harmonics (or of one variety of harmonics) can be demonstrated, and thus explained, on the basis of principles belonging to a higher science of a mathematical sort. But the terms in which Aristotle refers to the two sciences are not always the same. In three of the four passages, (A), (B) and (D), he speaks of harmonics as subordinate, in this sense, to arithmetic. Passage (C) contains the same formulation; but it goes on, almost in the same breath, to identify the relation as that between two species of harmonics, 'mathematical harmonics' and 'harmonics based on hearing'. These expressions can only refer, respectively, to harmonics in the Pythagorean style and harmonics of the sort practised by the people Aristoxenus calls *harmonikoi*;

22 English does not behave like Greek here, and no direct translation will quite capture the sense. Aristotle means that it is different from the kind that falls into the domain of the second science, which he has not yet mentioned in this sentence, and does belong in the province of the first.

23 A literal translation of the end of the last clause, as it appears in the MSS, would be '. . . have the common', or '. . . have that which is common'. A tiny emendation, reading *ti koinon* for *to koinon*, would straightforwardly produce the sense that seems to be called for, '. . . have something in common'; but in the face of the unanimous MSS tradition I doubt that the change is justified.

24 This second, very cumbersome sentence paraphrases the text rather than translating it. In Aristotle's Greek it contains just eleven words. I apologise, but can do no better.

and in that case, like some of Aristotle's statements about the diesis, they mark at least approximately the same distinction as that drawn in Book VII of the *Republic*.

Despite the close proximity of the references in (C), it is not clear whether Aristotle has the same distinction in mind when he speaks of 'harmonics' as subordinate to 'arithmetic'. It would be a strange way of expressing it. If he does mean that, 'harmonics' must be empirical harmonics, and mathematical harmonics is represented as coextensive with arithmetic, or as a part of it. Alternatively, 'harmonics' in these passages might refer to 'mathematical' or 'Pythagorean-style' harmonics, and his point will be that its propositions are demonstrated on the basis of principles proper to arithmetic, or rather *arithmētikē*, where this word has its regular sense, 'the science of number'. The evidence of (C) suggests that he may have confused or conflated these two lines of thought. Does he mean that empirical harmonics is subordinate to mathematical harmonics, or that mathematical harmonics is subordinate to arithmetic, or both? (I leave aside for a moment the more remote possibility that the relation holds between arithmetic and empirical harmonics. It will resurface shortly.)

If we are to do justice to Aristotle's various statements, the answer must be 'both'. Passage (C) guarantees that he is thinking at least of the relation between mathematical and empirical harmonics; the latter deals with the facts, the former with the explanations (78b39–79a3). Passage (D), by contrast, makes sense only if 'arithmetic' is given its ordinary meaning. It is more exact than harmonics because it deals with the items in its domain, numbers, 'without treating them as inhering in an underlying subject'. That is, it considers them simply as numbers, not as numbers 'of' or numbers 'inhering in' anything else. Harmonics, on the other hand, considers numbers only in so far as they characterise or inhere in pitched sounds (87a33–4). One cannot substitute 'mathematical harmonics' and 'empirical harmonics' for 'arithmetic' and 'harmonics' in this statement without making it unintelligible. Passage (D) does not show, by itself, that the relation between arithmetic and mathematical harmonics is precisely the same as that between mathematical and empirical harmonics in (C), where the higher science provides the explanations and the lower one the facts; but this is stated unambiguously in (B), and again in the first part of (C).

Aristotle's remarks can be combined into at least a superficially coherent picture if all three sciences are assimilated to a single hierarchy. The facts set out by propositions in empirical harmonics are demonstrated and explained by propositions in mathematical harmonics; and they in turn are demonstrated and explained by propositions in arithmetic. But since, on this interpretation, mathematical harmonics is not a fully autonomous science, and must call on arithmetic in the demonstration of its propositions, empirical harmonics will also depend, at one remove, on arithmetic for the explanation of its facts. In that case, when Aristotle speaks of harmonics as subordinate to arithmetic, 'harmonics' might after all be empirical harmonics, or simply harmonics in general, embracing both its variants. Passage (C) makes it clear, nevertheless, that Aristotle recognises the two harmonic sciences as distinct, representing one as concerned with the facts, the other as responsible (with the aid of arithmetic) for the explanations.

The relation that Aristotle apparently postulates between mathematical harmonics and arithmetic seems unproblematic, at least in outline. Harmonics of this sort depends fundamentally on operations with numerical ratios. It may call on principles which presuppose ways of classifying ratios by arithmetical criteria (as multiple, epimoric and so on), and upon classifications of means and proportions (geometric, arithmetic, harmonic). It sometimes appeals to quite sophisticated arithmetical theorems, such as the Archytan proof that no mean proportional lies between the terms of an epimoric ratio. The *Sectio canonis* in fact sets out a series of nine purely arithmetical theorems before proceeding to harmonic propositions, and draws on the former in all its proofs of the latter. Mathematical harmonics exploits principles proper to arithmetic at every turn, and there is an intelligible sense in which it is arithmetic that provides its explanations. Once it has been established, for instance, that the interval of a tone is in the ratio 9:8, it is argued that such an interval cannot be divided into equal parts *because* its ratio is epimoric, and *because* of the arithmetical theorem mentioned above.[25] At first sight all this seems clear, if a little uninteresting. There are in fact rather troublesome difficulties not far below the surface, as we shall see when we examine the *Sectio canonis* in its own right in Chapter 14; but we shall by-pass them here and turn to the supposedly parallel case of the relation between mathematical and empirical harmonics. Aristotle's account of this relation runs into choppy waters almost at once.

According to the account offered in passage (C), facts enunciated within empirical harmonics are explained by principles belonging to mathematical harmonics. Empirical harmonics, then, does not explain its own facts. Since these facts can nevertheless be explained, through demonstrations calling on principles of the higher science, and since all propositions that can be demonstrated are universal rather than particular, taking the form 'Every A is C', not 'This A is C', these facts must be generalisations grounded in

[25] See Proposition 16 in the *Sectio canonis*.

observation. Empirical harmonics, lacking the capacity to explain its own data, can amount to no more than a collection of such generalisations.[26] But it is hard to find any factual generalisation of that sort which could possibly be explained and confirmed on the basis of principles proper to the mathematical science.

Consider a candidate that may at first sight seem promising: 'The perfect fourth is concordant' (or 'Every interval that is a perfect fourth is concordant'). An appropriate demonstration of this fact, drawing on principles of mathematical harmonics, would take the following form:

(i) The perfect fourth is an interval whose ratio is of type f;
(ii) Every interval whose ratio is of type f is concordant; Therefore
(iii) The perfect fourth is concordant.

The problem here is straightforward. Mathematical harmonics, in any of its Greek versions, contains no principle corresponding to premise (ii). Theorists often adopted an axiom to the effect that every concordant interval has a ratio that is either epimoric or multiple, but that will not serve the purpose; many discordant intervals also have such ratios. There is no mathematically specifiable class of ratios ('type f') all of whose members are concordant.

Rather similar difficulties affect another plausible kind of example. It is a fact, adopted by mathematical and empirical theorists alike, that a perfect fourth added to (or compounded with) a perfect fifth makes an octave. One might suppose that this fact can be explained mathematically, roughly as follows (I ignore various intermediate steps that would be needed to plug all the logical gaps).

(i) The octave's ratio is 2:1, that of the fifth is 3:2, and that of the fourth is 4:3;
(ii) $3{:}2 \times 4{:}3 = 2{:}1$;
Therefore
(iii) A fourth compounded with a fifth makes up an octave.

The difficulty lies again in the first premise. Although it is fundamental to mathematical harmonics, it cannot be established by the principles of that science. The *Sectio canonis* does indeed contain arguments that purport to prove it, in Proposition 12. But Proposition 12 depends on Proposition 11, and Proposition 11 involves a logical mistake that cannot be repaired (see pp. 386–7 below). Even if it could, the first premise of our argument could not be established by Proposition 12 and still serve to demonstrate

[26] They may be stated in a different way, e.g. in the form 'The lowest interval in an enharmonic tetrachord is a quarter-tone.' But this is implicitly universal; it can be reformulated as 'Every interval which is the lowest in an enharmonic tetrachord is a quarter-tone.'

the conclusion we require, since Proposition 12 takes that conclusion as one of the assumptions from which it generates its proof of premise (i), and the putative demonstration would be circular.

The only way in which such a demonstration could persuasively be defended is by arguing that the first premise does not require demonstration or mathematical proof. Aristotle insists, after all, that the principles from which demonstrations are derived must be 'immediate', and do not call for demonstration in their turn. What will underpin the premise and allow it the status of a principle are repeated observations of the intervals produced from appropriately related lengths of a stretched string or a pipe.[27] As far as we can tell, it was in fact on observations of that sort, and not on mathematical reasoning, that theorists typically based their confidence in the relevant ratios; the *Sectio canonis*, probably at the end of the fourth century, offers the first (misguided) attempt at mathematical proof. This approach seems to avoid major difficulties; let us assume that any residual problems can be solved. But still it will not give Aristotle what he needs. No other propositions in mathematical harmonics, so far as I can see, could be established as principles in this manner, on the basis of observation alone. In that case the only demonstrations that could be offered would be ones that included premise (i), or a part of it, among their premises, and as a consequence the demonstrable part of empirical harmonics would be limited to a mere handful of elementary propositions. This example, then, exposes no logical flaws in Aristotle's position, but the position seems to be sustainable only at the cost of reducing the empirical science, in so far as the mathematical science can get any explanatory purchase on it, to triviality.

These examples have been hypothetical; perhaps we shall reach a better result by turning to propositions taken from an actual treatise in mathematical harmonics, with a view to discovering whether they are capable of explaining empirical facts. The *Sectio canonis* was composed, in my view, within a generation of Aristotle's death, and much of it seems to have been put together out of earlier sources.[28] It may therefore exemplify the kind of mathematical science that Aristotle had in mind, and it is the only continuous, full-length treatment of the subject that survives from the period. The bulk of it is set out as a connected series of propositions each of which is provided with a proof; and even if the proofs do not always match up to the strict requirements of Aristotelian demonstration, the work has plainly been

[27] On the route by which we can pass from empirical observations to principles, see pp. 110–12 above.
[28] The treatise and the problems surrounding its date are discussed in detail in Chapter 14 below.

conceived in the spirit of the *Posterior Analytics*, as an attempt to present a completed science (or a substantial part of one) in axiomatic form.

The first nine of the *Sectio*'s twenty propositions are purely arithmetical, and three others (which involve constructions rather than proofs) do not concern us here. These are the propositions proved in the remaining eight theorems.

Prop. 10: the interval of an octave is multiple [i.e. of multiple ratio].
Prop. 11: the intervals of a fourth and a fifth are both epimoric.
Prop. 12: the octave is 2:1; the fifth is 3:2; the fourth is 4:3; the octave plus a fifth is 3:1; the double octave is 4:1.
Prop. 13: the tone is 9:8.
Prop. 14: the octave is less than six tones.
Prop. 15: the fourth is less than two and a half tones, and the fifth is less than three and a half tones.
Prop. 16: the tone cannot be divided into two or more equal intervals.
Prop. 18: the *parhypatai* and *tritai* do not divide the *pyknon* into equal intervals.[29]

I have listed these propositions for one reason only; that is, to make it clear that not one of them can be understood as expressing any proposition in empirical harmonics (or 'harmonics based on hearing', as Aristotle describes it). None of the theorems proving them, therefore, can serve as a demonstration of a fact recorded by the empirical science. Propositions 10–13 specify the ratios of intervals, or the classes to which these ratios belong, and the ratio of an interval is not accessible to the ear. The remainder state conclusions which no empirical theorist of the period would accept. The claims made in Propositions 14 and 15 could, in principle, be tested aurally, but the only relevant fourth-century tests of which we know were taken, perhaps with some hesitation, to show that the first part of Proposition 15 is false.[30] In that case the second part of the proposition will be false too, since all parties accept that the fifth exceeds the fourth by a tone; and so is Proposition 14, since (again by general consensus) a fourth and a fifth together make up an octave. As to Propositions 16 and 18, there is plainly no way in which they can be construed as generalisations based on the evidence of the ear.

Nothing in the *Sectio canonis*, then, can support Aristotle's conception of the relation between the two forms of harmonics. Its mathematical

[29] The sense of this is that the two small intervals (according to empirical theorists, quarter-tones) at the bottom of an enharmonic tetrachord cannot be equal.

[30] Aristox. *El. harm.* 56.13–58.5.

reasoning neither proves nor explains anything in the domain of the empirical science. Statements made by Aristotle himself in passage (B) further undermine his position. It tells us that the 'underlying kind', that is, the subject of the conclusion demonstrated, belongs to the province of the empirical science, which deals with the fact; the attributes belong to the higher science which provides the explanation. The attribute assigned to the subject in the conclusion is therefore proper to the higher science, the subject itself to the lower. This seems to describe quite adequately the affiliations of subjects and attributes in at least Propositions 10–13 of the *Sectio*; the subjects are musical intervals, specified in terms familiar in empirical harmonics, and the attributes assigned to them are mathematical.[31] But despite the legitimacy of the subject-terms in the vocabulary of empirical harmonics, no such propositions will express facts that fall within its domain.

This result might be avoided if the term through which the demonstration links the subject and the predicate (the 'attribute') of the conclusion somehow belongs to both sciences. In the paradigmatic argument-form:

(i) Every A is B;
(ii) Every B is C;
Therefore
(iii) Every A is C,

the 'linking' or 'middle' term is B. This is perhaps what Aristotle has in mind at the end of Passage (B): 'the principles of these sciences have something in common'. But in that case, the premises which serve to prove and explain the (supposedly empirical) conclusion can belong to the higher, mathematical science only if A and C also belong in its domain, since both these terms figure in the premises. In that case the conclusion can also be read as a proposition in mathematical harmonics. Correspondingly, if the conclusion is an empirical proposition demonstrated from mathematical principles, not only its subject but its predicate too must fall within the empirical science's province. All three of the terms involved in the demonstration must in fact be 'common' to both sciences. But if every element in the demonstration has a foot in both camps, the whole argument can be understood either in mathematical or in empirical terms, and there seems to be no reason for introducing the slippery notion of 'kind-crossing' into any interpretation of it.

I think we must conclude that Aristotle's representation of the relation between mathematical and empirical harmonics not only fails to reflect

[31] In Propositions 12 and 13, where intervals are assigned their ratios, the ratio is expressed as an 'attribute' by means of an adjective; 'the octave is *diplasios*', 'the fifth is *hemiolios*' and so on.

the relation actually holding between them in the fourth century, but fails to make good sense. There is, however, at least one other way in which we might elucidate the thesis that mathematical harmonics explains facts catalogued in the science's empirical branch. Suppose that there is a set of mathematical principles from which the structure of a well-formed musical system such as a scale can be derived. The results of this derivation will be expressed mathematically, in the form of a sequence of ratios, as for instance those of the harmonic divisions of Archytas. So far we have nothing that could figure among the propositions of an empirical science. But suppose further that these sequences of ratios can be used to specify the relative lengths of a string or some comparable device from which notes can be produced, in such a way that (according to the postulates of mathematical harmonics) the intervals between them correspond to the ratios which the theoretical derivations produced. The resulting pattern of notes and intervals can then be assessed by ear, and perhaps some way might be found of describing it in empirical terms. If it strikes the musical ear as well formed, the thesis that it is well formed can become a proposition of empirical harmonics. But what explains its musical status will be the principles of mathematical harmonics from which its structure was derived.

This procedure is moderately close to one adopted, over four hundred years later, by Ptolemy. It is just possible that something like it was known in the fourth century, where its most probable locus is in the work of Archytas, though we have no clear evidence for it there or anywhere else. If it existed, the pattern of explanation it sets up might conceivably have prompted Aristotle's statements about the relation between the two forms of harmonics.

But this is hardly a promising hypothesis. Aristotle seems to regard the empirical science as going about its business of describing and cataloguing facts independently of its mathematical cousin, which intervenes only to provide explanations. Here, however, mathematical harmonics is the starting-point; the empirical scientist considers only constructions handed down to him from above, and the only constructions whose credentials can be mathematically explained are ones that have been drawn from that source. Further, as I have said, the notion that any such procedure existed in the contemporary repertoire is entirely speculative; and in any case the mode of explanation it offers seems resistant to representation in the form of an Aristotelian demonstration. I am no logician, and others may succeed where I have failed; but I see no way in which the trick can be done.

CONCLUSIONS

None of Aristotle's dealings with harmonics suggest that he thought of it as part of a musician's essential equipment, or as having any bearing on the wider concerns of musical critics and the educated public in general. It is a discipline pursued by specialists, partly for its own sake, and partly (at least in the mathematical form which captures most of Aristotle's attention) for any light that its concepts and conclusions may shed on problems in other scientific domains. His own scattered comments and discussions are unlikely to have interested anyone apart from dedicated scientists and students of metaphysics and logic. The passage containing the fragment which we examined at the beginning of this chapter shows that the Pythagoreans' version of the enterprise struck him as worth exploring in some detail. In other works he rejects their ways of exploiting its resources in the contexts of metaphysics and cosmology; but we see from his discussion of colours in the *De sensu* that he could treat relations between sounds as significantly parallel to relations between other objects of perception, and found mathematical harmonics itself, when shorn of peculiarly Pythagorean commitments, to be useful elsewhere as a source of illuminating analogies.

In that context his grasp on basic features of the harmonic theorists' work seems tolerably secure, and he is probably not responsible for the main problems generated by his treatment of it. But so far as his direct dealings with the intricacies of harmonics are concerned, my conclusions about Aristotle's understanding of the subject have been largely negative. In harping on his deficiencies I am not just indulging in the academic amusement of dancing on his grave. They are worth emphasising for a better reason, since they point to issues, connected for the most part with the relation between the two versions of harmonic science, that are both theoretically and historically problematic. In a broad sense, both sciences are attempting to establish the truth about the same subject. But they represent items fundamental to harmonic enquiry (notes, intervals and the like) in radically different ways; they differ in their methods of measurement, construction and analysis, and in the criteria by which they assess the credentials of putatively musical relations; and they reach incompatible conclusions. From the fourth century onwards they were regularly portrayed as rivals, and many theorists adopted positions entrenched in one camp or the other, explicitly advertising its merits and the defects of its competitor. But these belligerent gestures do not altogether conceal the fact that a great deal of diplomatic activity was going on behind the scenes, aimed at some sort of reconciliation between the warring factions; and both sides were routinely engaged also in piecemeal theoretical larceny, appropriating for their own purposes concepts, techniques and morsels of doctrine from the opposition's armoury. Aristotle's fumbling attempts in the *Metaphysics* to construe their methods of measurement in parallel ways, and in the *Posterior Analytics* to bring the two approaches into a coherent relationship with one another in the context of his theory of demonstration, are the first in a long and confusing line of bridge-building projects. Some are more impressive than others; none, in the end, is wholly successful.

CHAPTER 14

Systematising mathematical harmonics: the Sectio canonis

Up to the end of the fourth century, all our direct evidence about mathematical harmonics comes in the form of scraps. There are the fragments of Philolaus and Archytas, together with a few brief later reports on which we can reasonably rely; there are Plato's comments in the *Republic* and his psycho-musical construction in the *Timaeus*; there is a scattering of allusions and discussions in Aristotle and a couple of acid comments in Aristoxenus. Apart from the critique mounted by Theophrastus, which we shall consider in Chapter 15, there is very little else.

In earlier chapters I have tried to extract as much enlightenment from these bits and pieces as they can yield, and the amount is not negligible. But in some respects the absence of any complete treatise in the field leaves serious gaps in our knowledge. Quite apart from the loss of theories and arguments, we simply do not know what a 'complete treatise' of that sort would have looked like. We have nothing that unambiguously reveals the aims of such a work, the list of items that we might formulate as its table of contents, the way in which its propositions were combined with one another and integrated into a systematic whole (if indeed they were), or the style of presentation it adopted. It seems likely, in fact, that the various essays that once existed – those of Philolaus, Archytas and the mathematical theorists mentioned in the *Posterior Analytics*, for example – differed significantly in their aspirations and their modes of exposition; but we can say little about the overall structure and agenda of any of them.[1]

If its traditional dating is correct, the first example of a treatise of this sort that survives complete was composed around 300 BC. But this is a very substantial 'if'. Most scholars are prepared to accept that a continuous passage amounting to about two thirds of the whole can indeed be assigned to that period; but it has repeatedly been argued that the remainder cannot.

[1] For a well judged and intellectually satisfying (but inevitably hypothetical) attempt to reconstruct the programme of Archytas' work on harmonics, see Huffman 2005: 60–3.

These parts, it is said, must be later accretions, and the work was assembled in its existing form (that is, as the fullest MSS present it, and as it is printed in most modern editions) many centuries later. The difficulty is not one that we can responsibly ignore. We would be badly misled, both about the treatise itself and about the history of the science, if we took it to have been written, more or less as it stands, in Aristoxenus' lifetime or shortly afterwards, if in fact it emerged hundreds of years later from an entirely different intellectual environment.

Commentators' grounds for doubting that the whole text is an integrated product of the late fourth or early third century can conveniently be divided into two groups. One group fastens on details of the content of certain passages, which are held to be obtrusively anomalous. The other reaches similar conclusions by a different route, focusing primarily on aspects of the history of the text's transmission; the ways in which it is quoted and discussed by other authors are said to reveal that the version they knew was shorter, and lacked the parts whose origins are in dispute. This second group of contentions can be addressed without any close study of the treatise's contents, and after a few preliminary remarks I shall tackle it immediately. The first cannot be handled in the same way, and I shall deal piecemeal with the issues it raises, as they become relevant in the course of our section-by-section examination of the work. Although the grounds for my conclusions about these matters will not have been fully assembled until this chapter is complete, I shall resist the temptation to set the discussion out in detective-story mode, unveiling the solution (if such it be) only in the final scene. Such mystification might be mildly entertaining but would hardly conduce to clarity. I shall therefore come clean from the start; my view is that the reasons for denying the text's integrity as a document dating from about 300 BC are inadequate, and that though the positive reasons for taking the contrary view cannot be completely watertight they are strong and persuasive.

The Latin title by which the treatise is generally known is *Sectio canonis*; in Greek it is *Katatomē kanonos*, in English *Division of the Canon*. The Greek word *kanōn* refers, among other things, to a familiar device for constructing straight lines or measuring lengths, the ruler. In the present context it designates the ruler attached to or mounted on the sound-board of the instrument called the monochord (in later writings it is often used as the name of the instrument itself). As the name 'monochord' – which appears nowhere in this work – suggests, the instrument had a single string, stretched between two fixed bridges. A moveable bridge, placed at various positions on the sound-board underneath the string, determined the length of string that would be plucked and the pitch of the note it sounded. Marks on the ruler or *kanōn* indicated the positions at which the bridge should be placed successively to produce notes differing by predetermined intervals. Since the size of an interval depends on the ratio between the lengths of string that generate its bounding notes, the principles governing the procedure for marking out the *kanōn* are those of mathematical harmonics, in which each interval is correlated or identified with a ratio. To 'divide the *kanōn*' is to mark it out in this way, and a complete division of the *kanōn* is one that allows the instrument's user to play all the notes of a scale (or several scales), with all the intervals adjusted to the ratios deemed theoretically correct.

Only the last two paragraphs of the *Sectio* are directly concerned with such a division. They are preceded by a set of eighteen short arguments presented as theorems, and before the theorems comes a brief introduction. I shall call the theorems 'proposition 1', 'proposition 2', and so on; and though the two final paragraphs do not enunciate theorems, as their predecessors do,[2] I shall refer to them, for convenience, as propositions 19 and 20. The sequence of propositions falls clearly into two main parts (propositions 1–9, which are theorems in pure mathematics, and propositions 10–20, which introduce musical concepts and prove conclusions proper to harmonics); the second, explicitly musical group of propositions will need to be subdivided further, not always in quite the same way, as our examination of the work proceeds.[3]

THE *SECTIO CANONIS* IN PTOLEMY AND PORPHYRY

Questions about the history of the text's transmission, in both Greek and Latin versions, are far too complex to be treated fully here, and on these intricacies, along with many other matters, readers should consult the meticulous study by André Barbera.[4] I shall comment only on issues raised by the passages in later Greek writings which constitute the earliest evidence we have of the *Sectio*'s existence.[5] It will become clear that the writers in

2 From a formal perspective proposition 17 is also a little different from the others; like the passage dealing with the division of the *kanōn*, it offers a construction, not a proof.

3 The standard modern edition is Jan 1895: 148–66, reprinted with Italian introduction and notes in Zanoncelli 1990: 31–70. The most recent edition is Barbera 1991; see n. 4 below.

4 Barbera 1991. The book includes a careful examination of the MSS and of the citations of the *Sectio* in other ancient writings; texts and translations of the treatise in its various versions; and surveys of previous scholars' views and arguments about the origins of its parts. I am grateful to Professor Barbera for his comments, in correspondence, on an earlier draft of some of this chapter's ingredients. I am pleased to discover that our positions are not as far apart as I had supposed, but he should not be held responsible for any errors or confusions still lurking in these pages.

5 There are grounds for thinking that the *Sectio* was known to some writers who were at work rather earlier than those discussed here, specifically by Theon of Smyrna (early second century AD) and by his most important sources, Adrastus (late first century) and Thrasyllus (early first century). But the indications are too insecure to be relied on.

question assign it a date no later than the early third century BC, and I shall argue that despite some scholars' claims to the contrary, they provide no good reasons for believing that the work as they knew it lacked some parts of the text printed in modern editions. In fact they offer clear though not absolutely unchallengeable pointers to the opposite conclusion.

Porphyry, writing in the third century AD, quotes the first sixteen of the *Sectio*'s propositions as a single continuous passage which comes, he says, from a work by Euclid called *Division of the Canon*.[6] Some of the MSS of the treatise also ascribe it to Euclid, though others record it anonymously and others (quite implausibly) attribute it to a certain Cleonides.[7] If it could be shown that Euclid was indeed the author, the work would be firmly dated to around 300 BC; but doubts cannot be laid to rest so briskly. There is an independent tradition that Euclid wrote on music, but it is not a very solid one,[8] and the *Sectio*'s obvious similarities of language, presentation and method to Euclid's known writings might be products of later imitation rather than signs of genuinely Euclidean authorship. They might have been enough by themselves to persuade an editor or copyist to attach Euclid's name to manuscripts of the treatise, and to mislead Porphyry.

On the other side of the coin one might argue, first, that Porphyry's attributions of the many passages he quotes from earlier writers in this work are generally reliable.[9] Secondly, there is one passage in a pre-Porphyrian source, Ptolemy's *Harmonics* (12.8–27), which plainly draws on material from the *Sectio*. Ptolemy does not identify the work from which it is taken or name its author; he says only that the ideas and arguments it conveys are those of 'the Pythagoreans'. But this attribution is itself indicative, since most if not all of Ptolemy's material on the theorists to whom he gives this name seems to have come from sources preserving authentically fourth-century work.[10] Thirdly, the hypothesis that the *Sectio* was composed by an author other than Euclid who chose to imitate his manner is perfectly possible if he is held to have written at a date close to Euclid's own, but

[6] Porph. *In Ptol. Harm.* 99.1–103.25. The title and the ascription to Euclid are at 98.19.

[7] This is the same (but otherwise unknown) Cleonides to whom some of the MSS, and most modern commentators, attribute the Aristoxenian *Introduction to Harmonics* (*Eisagōgē harmonikē*) printed in Jan 1895: 179–207. On that work see Solomon 1980a, who discusses related issues about the *Sectio*'s authorship on pp. 368–73.

[8] The most significant source is Proclus, *In Eucl. Elem. I*, 69.3 Friedlein.

[9] The one case in which his ascription has repeatedly been disputed is that of the *De audibilibus*, which he attributes at *In Ptol. Harm.* 67.15 ff. to Aristotle. Here he may well be wrong (see especially Gottschalk 1968); but at least he has not assigned the work a date or an intellectual context very far removed from the true one.

[10] See Barker 1994b: 127–32. The fact that Ptolemy calls those responsible for these arguments 'Pythagoreans' should not be allowed to confuse the issue. It is merely his way of designating early exponents of a mathematical approach to harmonics, and need imply nothing about their other philosophical commitments.

much less persuasive if he is located in a later period, since Euclid's style is very different from that characteristically adopted by mathematical writers of later Hellenistic and early imperial times.[11]

No scholar, so far as I am aware, has found substantial grounds for doubting the integrity, as part or whole of a single work, of the segment of the *Sectio* attributed to Euclid by Porphyry, that is, the first sixteen propositions; nor have they found anything in this passage which conflicts with the hypothesis that it was written in the late fourth century or the early third.[12] The principal issue is whether the remaining parts of the text – the introduction, propositions 17–18, and the passage setting out the 'division of the *kanōn*' itself, propositions 19–20 – belong to the work in its original form. The fact that Porphyry presents us with a version from which these components are missing has been taken as evidence that they were not included in the text at his disposal, and the passages used by Ptolemy all fall within Porphyry's segment. It is arguable (though I shall dispute it) that such a text would make sense as it stands, without the rest, and when we come to consider the work's content and argumentation we shall see that it is precisely those remaining parts that contain the supposedly suspect features.

This reading of Porphyry does not convince me. When he quotes other people's writings, he does so to enhance his discussion of specific issues raised by Ptolemy's *Harmonics*, not to preserve whole works from the past for posterity. Take the case of the *Sectio*'s introduction. At the point where he quotes sixteen successive propositions from the treatise and attributes them to Euclid, Porphyry is commenting on the later part of Ptolemy *Harm*. 1.5 (12.8–27), whose own descent from the *Sectio* is beyond serious dispute. The introduction's exposition of theories in physical acoustics is quite irrelevant to this phase of Ptolemy's discussion, which is strictly mathematical in precisely the manner of those of the *Sectio*'s propositions which Porphyry

[11] These matters are discussed by F. E. Robbins in D'Ooge 1926: 28–34.

[12] Doubts of two kinds might be raised. First, if we are inclined to assume that the more ancient a text is, the more worthy it must be of our respect, the fact that the eleventh proposition contains a serious logical flaw (see pp. 386–7 below) might lead us to insist that it is the work of an inferior and therefore later writer; and since that proposition is pivotal to the whole, and the remainder cannot have been conceived without it, the entire treatise must be consigned to the intellectual swamps of later antiquity. Such snorts of outmoded prejudice should presumably be dismissed. Secondly, it is a fact that no echo of the treatise can confidently be identified in any writer before Ptolemy, in the second century AD, and this might constitute grounds for supposing it to have been written not much earlier than his time. But this point carries no weight; it merely puts the *Sectio* in the same condition as, for instance, the musicological works of Philolaus and Archytas, or the Peripatetic essay *De audibilibus*.

does quote. If he had been planning to present the introduction too, this was not the place to do it.

The fact is, in any case, that he does quote the introduction, virtually in full (a few phrases are missing), at the beginning of the same chapter of his commentary (90.7–22). Its purpose there is to introduce the notion of ratio, as it applies in musical contexts, and so to initiate a long discussion (90.24–95.23) about the relation between the concepts of *logos* (ratio) and *diastēma* (distance or interval). Evidently, then, the passage was one that Porphyry knew. A sceptical critic may object that he attributes it to no author, so that it remains possible that he regarded it as part of a different work. That is true. But the same consideration would count, to the same extent, against our attaching it to any author or text whatever; and the extent to which it should be allowed to count, I suggest, is zero. The anonymity of Porphyry's citation gives no positive reason for assigning the introduction to the *Sectio*, as he knew it, but equally gives none for denying the attribution. It is relevant, too, that the quotation is immediately followed in Porphyry's text by another which is also unattributed (90.24–91.1), but which we recognise as coming from the works of Euclid (*Elements* v, definition 3); and the fact that Euclid and the *Sectio* are much in Porphyry's mind at this moment is underlined by his quotation of two brief sentences from the *Sectio* itself shortly afterwards (92.29–93.2).

The absence of propositions 17–18 from Porphyry's long quotation provides, once again, no positive grounds for denying that they were part of the text on which he drew. The chapter of Ptolemy on which he is commenting, *Harmonics* 1.5, reviews certain basic theses, attributed by Ptolemy to the Pythagoreans, about the ratios of the concords and, incidentally, the interval of a tone. Porphyry's sixteen propositions are directly relevant to those topics, and to the treatment of them which Ptolemy is considering. Propositions 17–18 are not. They are concerned with relations specific to the enharmonic genus, with its incomposite ditones and so-called quarter-tones. These propositions have nothing to do with the issues of *Harmonics* 1.5, and it would have been quite inappropriate for Porphyry to include them; nor is there anywhere else in his commentary where they would naturally have found a place. Their absence tells us nothing about whether they were in his text of the *Sectio* or not.

The same point can be made about the 'division of the *kanōn*' in propositions 19–20, which is also missing from Porphyry's quotation. It is irrelevant to the immediate context in his commentary, and the method of construction it adopts is completely alien to Ptolemy's concerns, even when he sets about the task, later in the *Harmonics*, of devising his own divisions and criticising those of others. Porphyry had no good reason to include it. More positively, he himself provides a rather strong reason for believing that this material did indeed form part of the text he had in front of him. He cites the sixteen propositions as coming from 'Euclid's *Division of the Canon*', and it would be extraordinary if a work known by this title contained no division of the sort it announces. This way of referring to the treatise, which he uses also at 92.30, makes it virtually certain that whenever propositions 19–20 were composed, by Porphyry's time they were understood as an integral part of the work, and indeed as constituting its main agenda. The title indicates that the construction of the division is the treatise's primary purpose, what it is 'all about'.

I conclude, then, that it is a mistake to suppose that Porphyry and Ptolemy provide evidence for the existence of a 'short version' of the *Sectio*, which lacked the introduction and the last four propositions. On the contrary, these earliest allusions to the treatise give us good reasons for believing that it already contained the introduction and the division, and no solid grounds for doubting that propositions 17–18 were also included. Ptolemy's use of it suggests that he thought of it as originating in the fourth century, or very shortly thereafter, and Porphyry attributes it unambiguously to Euclid. Both of them, of course, might be wrong. We shall have to examine the work itself to find out whether or not the disputed parts of it fit comfortably within the intellectual context of that period, and whether, regardless of Porphyry's and Ptolemy's treatment, they can plausibly be regarded as elements of the text in its original form.

THE INTRODUCTION TO THE TREATISE

The introduction occupies only a page and a half of text, but by comparison with the spare, formal theorems it seems positively expansive. Its first part expounds a theory about the causes of sound and of differences in pitch. The writer then draws from that theory the conclusion that notes of different pitches are related to one another in numerical ratios, sketches a classification of ratios into three types, and finally offers grounds for the thesis that the ratios of all the concords fall under just two of the three headings. Some commentators have regarded this material as little more than a collection of miscellaneous jottings of a Pythagorean flavour, and as largely irrelevant to the subsequent theorems, which make explicit use of none of it except the closing contention about concords; and as a consequence they have argued that it cannot have been written by the same hand as the well-argued and systematically linked set of theorems that follows, or formed part of the same treatise.[13] This view strikes me as entirely mistaken. Despite its relative informality the introduction forms a tightly-knit, continuous argument. It is directed from start to finish to the task of establishing two quite specific conclusions; and these conclusions are indispensable to the work of the theorems, one as a necessary (though tacit) presupposition, the other as an explicit axiom. There is nothing haphazard or irrelevant about it, nor is there anything specifically Pythagorean.

The passage's argumentative structure will be seen at once if we set out its statements as a numbered sequence, together with indications of the logical relations between them. Numbering apart, what follows is a close paraphrase, almost a direct translation, with nothing substantial added or subtracted.

(i) If there were stillness and motionlessness, there would be silence; and
(ii) if there were silence and nothing moved, nothing would be heard. Then
(iii) if anything is going to be heard, impact and movement must first occur. Hence, since
(iv) (a) all notes occur when an impact has occurred; and
(b) it is impossible for an impact to occur unless movement has occurred previously; and
(c) of movements, some are more closely packed, some more widely spaced; and
(d) those that are more closely packed make the notes higher, and those that are more widely spaced make them lower; it is necessary that
(v) (a) some notes are higher, since they are composed of more numerous and more closely packed movements, and
(b) some are lower, since they are composed of fewer and more widely spaced movements. Hence
(vi) (a) those that are higher than what is required, when they are slackened [i.e. lowered in pitch], attain what is required by the subtraction of movement; and
(b) those that are lower than what is required, when they are tightened [i.e. raised in pitch], attain what is required by the addition of movement.[14] Hence it must be agreed that

[13] This was the opinion of Jan 1895: 115–20, and of Tannery 1904a.

[14] There is an ambiguity in the expression of this proposition and its predecessor. An alternative reading would be: '. . . those that are higher (lower) than what is required attain what is required through being slackened (tightened) by the subtraction (addition) of movement'. I cannot prove that the version adopted in my text is correct; but I think it is logically preferable.

(vii) notes are composed of parts, since they attain what is required by addition and subtraction. Now
(viii) all things composed of parts are related to one another in a ratio of number. Hence it is necessary that
(ix) notes, too, are related to one another in a ratio of number. Now
(x) of numbers, some are related in a multiple ratio, some in an epimoric, some in an epimeric. Hence it is necessary that
(xi) notes too are related to one another in these sorts of ratios. Now
(xii) of these [numbers], those in multiple and epimoric ratios are related to one another under a single name. And we know that
(xiii) (a) of notes, some are concordant, some discordant; and
(b) those that are concordant make a single blend out of the two notes, while
(c) those that are discordant do not. In that case it is reasonable (*eikos*) that
(xiv) since concordant notes make a single blend of sound out of the two, they are among those numbers which are related to one another under a single name, and so are either multiple or epimoric.

There are several points here at which one might question the logic of this argument or the credentials of its premises. We shall come to those in a moment. But the first and crucial point to notice is that it is nothing like a random miscellany. It is, unmistakably, an argument, a single, continuous and closely reasoned argument, directed throughout to the establishment of the conclusion given in (xiv). The author has evidently taken great pains to omit no necessary steps, though his expression of them is severely economical; and still more significantly, he has included nothing that is redundant – every statement and every inference has its part to play in the whole. Since the argument's conclusion, as we shall see, plays a major role in the reasoning of the main part of the treatise, the view that the introduction is irrelevant, like the view that it is haphazardly flung together, is not merely unproven but absurd.

One of the intermediate staging-posts in the argument, step (ix), serves two purposes. It contributes to the reasoning by which we reach step (xiv), but it also needs to be established in its own right before the theorems can proceed. There is nothing in our ordinary experience of sound or music to link intervals with mathematical ratios. If they are to be represented in that manner, as they are in the theorems, it must first be shown that regardless of the way we perceive them, the pitches of a higher and a lower note are in fact, as step (ix) puts it, 'related to one another in a ratio of number'. The first task of the *Sectio*'s introduction, then, is to demonstrate that this is so.

The route to this conclusion lies through a series of propositions in physical acoustics, treating sound as a movement or sequence of movements caused by impacts (presumably impacts made by some body on the air, though the writer does not say so). It is when pitched sound is conceived as physical movement, not as an auditory phenomenon, that point (ix) is taken to hold of it. In this respect the *Sectio*'s approach is similar to those of Archytas, Plato and Aristotle, and is as far as it could be from that of Aristoxenus; it would fall squarely into the class of those he considers irrelevant to harmonic science.[15] It is also worth reminding ourselves that although acoustic theories of this type probably originated in the context of Pythagorean thought, by the later fourth century they had been widely adopted by theorists and philosophers of several doctrinal persuasions. There is no mileage in the thesis that the *Sectio*'s introduction betrays Pythagorean affiliations of which the rest of the text is innocent.

The theory itself, however, differs from most of its known fourth-century predecessors. Plato seems to have modified Archytas' account, and Aristotle revised it further, but they share the view that sound is a movement transmitted through the air (or, if conceived as an audible phenomenon, is caused by such a movement), and that its pitch is dependent on the speed at which the movement is transmitted through the medium.[16] Many later writers adopt a similar position. Though the *Sectio* begins, like Archytas in frag. 1, from the thesis that a sound is a movement caused by an impact, it shifts at steps (iv)–(v) into talking of movements in the plural, and of each note as constituted by a sequence of movements, more or less closely packed together. When the movements follow one another in more rapid succession, the sound's pitch is higher.

The thesis that a sound which we hear as continuous is really constituted by a succession of impulses was probably inspired by observations of the behaviour of an instrument's string. When the string is plucked it oscillates to and fro; the writer conceives it as beating repeatedly on the surrounding air to make a sequence of separate impacts, and yet as long as its oscillations persist it emits an apparently continuous sound. Theories of this sort are found elsewhere too, though rather rarely, and if the *Sectio* dates from around 300 BC it is the earliest text to expound it unambiguously.[17]

15 See e.g. *El. harm*. 12.4–9, 32.19–28.

16 Archytas frag. 1, Plato, *Tim*. 80a–b, Aristotle, *De an*. 420a–b.

17 The clearest instances of such a theory, outside the *Sectio*, are at [Ar.] *De audibilibus* 803b–804a, and in a quotation from Heraclides by Porphyry at *In Ptol. Harm*. 30.1–31.21. The *De audib*. was probably written no later than the mid-third century (see Gottschalk 1968). It is uncertain whether the Heraclides cited by Porphyry is the well-known fourth-century philosopher or a later figure

Even fewer writers develop the idea, as the *Sectio* does, into a theory of pitch, where pitch depends on the rapidity with which impulses follow one another, not on the speed of the movements' transmission.[18] This theory too is likely to have been based on the observation of strings; the oscillations of a longer or slacker string are visibly less 'closely packed' than those of a shorter or tauter string, and the pitch it emits is lower.

As an explanation of differences in pitch, the *Sectio*'s hypothesis has advantages over ones based on speeds of transmission. It does not lead to the awkward consequence that differently pitched sounds originating at the same moment cannot arrive simultaneously at the ear; and it need not wrestle with the observation that objects in motion slow down as they move further from their source. It also seems well adapted to the purpose it will serve here, that of interpreting the thesis that the intervals between pitched sounds can be represented as ratios of whole numbers. A stretched string at rest is straight. We may suppose that it 'strikes' the air either at the instant when it departs from its position of rest, or when it reaches its position of maximum displacement. I suspect that the theory's proponents were impressed by the fact that in either case, the string will always have 'struck' the air a determinate number of times in a given period, a number that is in principle countable (though not of course in Greek practice). Hence there is always some ratio of whole numbers which corresponds to the difference between any two pitches. Each sound will have been constituted, during a given period of time, by a determinate number of 'parts' (cf. steps (vi)–(ix)).

There may seem to be a difficulty here. The pitch of a sound made up of 9 impulses per second, for instance, may be supposed to stand in the ratio 9:8, that of a tone, to one constituted by 8 impulses per second. Yet if they start simultaneously, at the moment when the latter's fourth impact occurs the former will not yet have made its fifth, so that both will have been constituted by four impacts and should apparently sound in unison. But the problem is soluble. Ratios become relevant when it is the relative pitches of sounds, not their absolute pitches, that are under consideration. If the first impulses of two different notes are simultaneous and their rates

who lived in the first century AD. Most commentators have taken the latter view (see Gottschalk 1980: 157), and I did so myself in Barker 1989a: 230. For indications that might point to the earlier Heraclides see Barker (forthcoming).

[18] The position of the *De audib.* on this matter is not entirely clear. In the passage cited in n. 17 it recognises that 'the impacts of air belonging to the higher notes occur more frequently', but this need not imply that higher pitch is *caused* by this greater frequency. At 803a the cause is the movement's greater speed. A passage of Theophrastus (frag. 716 Fortenbaugh) seems to direct some of its criticisms at theorists who take a view like that of the *Sectio* (the fragment will be discussed in Chapter 15); for other evidence of this position see the passages cited below in n. 19.

of impact are rationally related, there will follow a period of time during which their impacts do not coincide, and then a moment when they come back into phase. After that the pattern of non-coincidence will be repeated, and it is in that pattern that the difference between them consists. We may therefore stipulate that the numbers of impulses to be compared are those occurring between any two instants at which impulses of the two sounds coincide. In the case envisaged, there will always have been 9 of one and 8 of the other. There are indications that this approach was indeed sometimes adopted.[19] The supposition that there might be rates of impact such that after a first pair of simultaneous impacts the two would never again coincide goes well beyond the mathematical sophistication of these authors.

The last phase of the introduction's argument, steps (x)–(xiv), seems the weakest. It begins from a classification of ratios into three types, multiple (*pollaplasios*), epimoric or 'superparticular' (*epimorios*) and epimeric or 'superpartient' (*epimerēs*). Let us remind ourselves of what these terms mean. A multiple ratio has the form mn:n. In an epimoric ratio the greater term is equal to the smaller plus one integral part (one half, one third, and so on) of the smaller; when the ratios are expressed in their lowest terms this class includes all and only the ratios of the form n + 1:n, except 2:1, which is multiple. For present purposes an epimeric ratio is any ratio which is neither multiple nor epimoric. In the context of harmonics, the thesis that multiples and epimorics have a status that epimerics lack seems implicit in Archytas and Plato, and may go back to even earlier Pythagorean thought.[20] The conclusion with which the introduction ends, that all concords have ratios that are either epimoric or multiple, is taken as axiomatic by most later writers in the mathematical tradition of harmonics, and probably pre-dates the *Sectio*.[21]

It is obvious that the writer of the *Sectio* has set himself the task of persuading us that this conclusion is true, but the force of his argument seems questionable. It rests on two points. First, 'numbers in multiple and epimoric ratios are spoken of in relation to one another under a single name' (step (xii)); the implication is that those in epimeric ratios are not. Secondly, 'concordant notes make a single blend out of the two, while discordant

19 Porph. *In Ptol. Harm.* 107.15–108.21, [Aristotle] *Probl.* 19.39.

20 On Archytas see pp. 289–90 above, and on earlier Pythagoreans p. 272. Plato rather pointedly distinguishes the principal ratios involved in the construction of the World-Soul from that of the residual *leimma* of 256:243, describing the latter as a 'number to number' relation (*Tim.* 36b). In the surviving literature the term 'epimoric' itself occurs first at Ar. *Metaph.* 1021a2.

21 See e.g. [Ar.] *Probl.* 19.41, Thrasyllus at Theo Smyrn. 50.19–21, Ptol. *Harm.* 11.1–20. As we saw in the previous chapter, it is probably the epimorics and multiples that Aristotle has in mind when he connects concords and pleasant colours with 'well-ratioed' numbers at *De sensu* 439b–440a.

notes do not' (step (xiii b–c)). This second thesis need not detain us; the idea that the two notes of a concordant pair mingle together to form an integrated whole, and that it is this that distinguishes them acoustically from discords, was familiar in fourth-century and later thought.[22] The first, too, has a straightforward interpretation, though more recondite ones have sometimes been suggested.[23] Its sense, I think, is simply that each individual multiple and epimoric ratio is represented in the Greek language by a one-word expression.[24] There are no such words for epimeric ratios. Later mathematicians developed a terminology which in principle allowed each of these ratios, too, to be represented by a single word, but there is no reason to think that this cumbersome terminology was current in the fourth century.[25] At that period such a ratio was normally named by a composite expression which named each number involved in the ratio separately, in the form 'n to m', where the counterpart of 'to' is *pros*.[26]

The two premises, then, are tolerably unproblematic; it is the argumentative use to which they are put that may strike a reader as odd. It appears to rest on the assumption that if something is an integrated whole, 'blended' into one as concords are but discords are not, then language must be capable of representing it by a single word. This suggestion of a perfect fit between language (specifically, the Greek language) and reality seems remarkably optimistic.[27] Perhaps, however, it is a little less naïve than that. The thought might be that since language, and in particular the deliberately contrived, intellectually based language of mathematicians, marks a clear distinction between two classes of ratio, representing each item in

[22] See e.g. Plato, *Tim.* 80b, Ar. *De sensu* 447a–b, 448a, *De an.* 426b, and in later sources Aelianus at Porph. *In Ptol. Harm.* 35.26–36.3 (including an elaborate illustrative analogy), Nicomachus, *Harm.* 262.1–6, Cleonides 187.19–188.2.

[23] A reading of this sort was first proposed by Laloy 1900: 236–41. For interpretations other than the one I give here, see Barbera 1991: 56 with n. 149. None of them, I believe, will adequately fulfil the function for which the *Sectio* employs the statement in question.

[24] Terms for multiple ratios are formed with the suffix *-plasios*, giving *diplasios* for double, *triplasios* for triple, and so on. Among the epimorics there is a special word for 3:2, *hēmiolios* ('half-and-whole'). The remainder are named by terms in which the prefix *epi-* is attached to an expression meaning 'third', 'fifth' or the like, as in *epitritos* ('a third in addition') for 4:3, *epogdoos* ('an eighth in addition') for 9:8, etc.

[25] See Nicomachus, *Arithm.* 1.22–3 for a flurry of these terms, including coinages such as *tetraplasiepitetramerēs* for the ratio 24:5.

[26] Thus in Plato, as I have said, epimerics are called 'number to number' ratios, *Tim.* 36b3. Sometimes a good many words are needed to express such a ratio's size; in the same passage it takes Plato eleven words to say '256:243' (*hex kai pentēkonta kai diakosiōn pros tria kai tetterakonta kai diakosia*).

[27] It is nevertheless a view that might have been derived from an innocent-minded reading of Plato and Aristotle, both of whom (the latter very explicitly) regularly take 'what is said' as the starting-point for philosophical exploration. Their position seems to be that although linguistic distinctions may sometimes turn out to be misleading, if carefully interpreted and attentively related to others they typically reflect the contours of reality accurately enough to guide us towards the truth.

one class as a unity and those in the other merely as loose associations, it will have placed the ratios under consideration here, those of the concords, in the class appropriate to them. Even so, it is hard to see the argument as conclusive, and it is interesting that the writer does not state it as if it were. In inferences drawn earlier in the introduction he makes liberal use of expressions conveying logical necessity; given what has so far been established, the author asserts, the next proposition 'must' be true.[28] At the final step, however, these claims to iron-clad necessity evaporate, and are replaced by the altogether milder statement that it is 'reasonable' or 'to be expected' (*eikos*) that the conclusion holds.

Granted that the entire argument of the introduction has been designed to generate this conclusion, and that it is the only thesis stated here that is explicitly relied on in the theorems (crucially in propositions 10–11), the author's slippage at this critical moment from logical necessity to mere 'reasonableness' seems disappointingly feeble. But it should not really surprise us. Very few Greek writers attempt the task of proving by rigorous argument that all concords *must* have multiple or epimoric ratios; and those who do not are well advised, since the task is impossible. It cannot be demonstrated logically or mathematically that the proposition is true, and in fact, from a musician's perspective, it is not. The interval of an octave plus a fourth was widely recognised as a concord, and yet its ratio, 8:3, is neither epimoric nor multiple (this problem was mentioned on p. 348 above and will be discussed further below).[29]

From another point of view, too, a non-demonstrative argument is appropriate here. Its location in the introduction rather than in the theorems indicates that its conclusion is logically prior to the latter's reasoning. It is to be regarded as a principle on which the theorematic demonstrations of this science will draw, not as a proposition which they themselves can prove. We saw in earlier chapters that Aristotle, and following him Aristoxenus, insist that the principles of a science cannot be demonstrated; anything that can be demonstrated *must* be demonstrated if the science is to complete its task, and 'anything that requires demonstration does not have the nature of a principle'.[30] The strategy adopted in the *Sectio* is thus consistent with

[28] Cf. 'must' (*dei*) in step (iii), 'impossible' (*amēchanon*) in (iv), 'necessary' (*anangkaion*) in the transition to (v), 'it must be agreed' (*phateon*) in the transition to (vii), 'necessary' (*anangkaion*) again in the transitions to (ix) and (xi).

[29] The closest approximation to a general demonstration is probably one that can be extracted, though with some difficulty, from the text of Ptolemy's *Harmonics*. But it depends on some debatable assumptions, and Ptolemy's efforts to rid himself of the problem of the octave plus fourth are less than convincing. See Barker 2000a: 74–87.

[30] Aristox. *El. harm*. 44.14–15, cf. Aristotle, *An. post*. 71b26–9.

Peripatetic theories of scientific demonstration, and its author would have been misguided if he had looked for demonstrative proof.

Nor could the writer have based his principle securely on an inductive survey of cases, where observations of a range of instances in which a concord's ratio is epimoric or multiple lead us to the (probable) conclusion that all of them are so. For one thing, such a survey would inevitably stumble over the case of the octave plus fourth, which from an empirical or aesthetic perspective provides a compelling counter-example. Secondly, our confidence in a principle does not rest merely on the discovery that it holds good in all instances so far observed, but on an insight into the nature of the item under discussion; and this may indeed lead us to revise our view of what counts as an 'observed instance'. Here the insight focuses on the status of a concord as a blended unity. It will be this, if anything, that entitles a theorist to deny that the octave plus fourth can be a concord, despite the auditory impression it creates; and in the light of the other considerations expounded in the *Sectio*'s introduction, it is this intellectually enlightening conception of what a concord is that commends the principle to our understanding. The argument shows that when a concord is conceived in this way, the principle need no longer be regarded as an arbitrary postulate, and can be comfortably associated with other facets of what is known about sound and pitch, and with established mathematical classifications of ratios. It has done, in fact, about as much as any such argument could.

PROPOSITIONS 1–9: THE MATHEMATICAL GROUNDWORK

The propositions in this group say nothing explicit about music or sound. It is true that the word *diastēma* appears in them repeatedly, and that in musical contexts it means 'interval'. In later theorems the first nine propositions will be used to generate conclusions about musical relations, and there the word unambiguously acquires its musical sense. But this is done by treating musical intervals as cases of just one of the kinds that fall under the designation *diastēma*, a term whose range of application is much broader. A *diastēma* is literally that by which two items of any sort are separated, a gap or a distance or a quantitative difference. The reasoning of the first nine theorems is purely mathematical and will hold of *diastēmata* in general. We shall postpone consideration of the question whether it is to be understood in geometrical terms (as concerned with relations between magnitudes such as distances) or in arithmetical terms (as concerned with the relations between numbered pluralities).

But the situation is not quite as simple as that. Although a *diastēma*, in the first nine theorems, might be a geometrical or an arithmetical relation, it is not the distance between two points or the difference between two numbers, but a ratio. Interpreted geometrically, it is the ratio between two lengths; taken arithmetically, it is the ratio between two numbers. Proposition 1, for instance, reads: 'If a multiple *diastēma* taken twice makes some *diastēma*, the latter is also multiple.' As the phrasing implies and the working of the theorem confirms, this means (where A, B and C are numbers or lengths) that if the ratio A:B is multiple, and if A:B = B:C, then the ratio A:C is multiple too. This whole group of theorems, in short, proves propositions in the mathematics of ratio; and since musical *diastēmata* are conceived in the sequel as ratios, these propositions can be used there, without modification, as premises in theorems about them. It is in fact possible to read *diastēma* as if it meant 'ratio' throughout.[31] In another respect too these theorems are clearly tailored to their task in the musically oriented second half of the treatise. They do not amount to a complete exposition of ratio-theory, and if regarded from the perspective of that discipline alone would amount to a rather haphazard compilation. They have plainly been selected for their bearing on issues in harmonics; they include all and only those that will be needed to prove theorems about musical relations in propositions 10–20.[32] If the work is in any sense a 'complete' account of some subject, the subject is mathematical harmonics, not the mathematics of ratio in general.

I shall not examine the workings of each theorem in this group in detail; most of them are quite simple. But they nevertheless provide a good deal of food for thought. The first five deal with *diastēmata* conceived under general descriptions, as multiple or epimoric. The main focus is on the former. Four of them consider situations in which a *diastēma* is 'taken twice', to produce a sequence of three terms, A, B, C, such that A:B = B:C. We have noted proposition 1, that if a multiple *diastēma* is taken twice the *diastēma* formed from the two *diastēmata* together is also multiple. Proposition 2 states the complementary thesis: 'If a *diastēma* taken twice makes a whole that is multiple, that *diastēma* itself will also be multiple.' Proposition 4 is the negative counterpart of proposition 1. 'If an interval that is not multiple is taken twice, the whole will be neither multiple nor epimoric.' (The second part of the conclusion, introducing a reference to epimorics that is absent from propositions 1, 2 and 5, depends on proposition 3; see pp. 381–2 below.)

[31] This usage would not be unprecedented. Archytas frag. 2 may be a doubtful case, but Aristotle, *Phys.* 202a18 is not.

[32] This overstates the case very slightly. There is one exception, proposition 4; I discuss it on pp. 381–2 below.

Proposition 5 completes the set. 'If a *diastēma* taken twice makes a whole that is not multiple, that *diastēma* itself will not be multiple either.'

These four propositions form a tightly linked group. Inserted among them is one which at first sight is quite different, proposition 3. 'In an epimoric ratio there is no mean number, neither one nor more than one, which divides it proportionally.' In more familiar language, there are no mean proportionals between two terms in an epimoric ratio. That is, if A:C is epimoric, there is no term B such that A:B = B:C, nor can A:C be broken down into any larger number of sub-ratios all of which are equal (so that e.g. A:X = X:Y = Y:Z = Z:C).

The proposition differs from the four surrounding it in two principal ways. Its subject is epimoric ratios rather than multiples, and it is expressed in strikingly different language, speaking of mean numbers and proportionality rather than of *diastēmata* taken twice. This seems a little strange, since a broadly equivalent proposition could have been formulated in the latter way, for instance on a pattern derived from proposition 1: 'If a *diastēma* taken twice makes some *diastēma*, the latter *diastēma* will not be epimoric.' To put it more generally, the two modes of presentation seem to differ in that proposition 3 is clearly conceived in arithmetical terms, alluding to the insertion of numbers between other numbers, whereas the theses to be proved in the other four theorems might, on the face of it, be interpreted against the background of geometry, where the *diastēmata* are ratios between lengths.

But this apparent difference is illusory. When one looks at the reasoning of those four theorems, it turns out that though the argumentation of propositions 1, 4 and 5 might, at a stretch, be construed geometrically, that of proposition 2 cannot. It is explicit, like proposition 3, in referring to numbers and relations between numbers. So too, we may recall, is the latter part of the introduction, whose theory that pitch depends on the numerousness of a sound's constituent impulses points directly towards an arithmetical interpretation of the relations between notes. Notes, it says, are made up of parts, and all such things are related to one another 'in a ratio of number'. It is numbers, not merely quantities, whose ratios it classifies into three groups; and its final step states that concordant notes are 'among the numbers that are related to one another under a single name'.

There are strong indications, then, that the author of the *Sectio* thought of his science as based in arithmetic, not geometry. André Barbera has provided persuasive arguments of other sorts for the view that where expressions in the mathematical theorems, and the line-drawings included in some MSS, seem to hint at a geometrical treatment, these have been overlaid by later hands on an originally arithmetical text, and I think he is right.[33] The point is of some importance, and not just for any contribution it may make to the history of mathematics. A geometrical approach to harmonic relations is found in Plato and is common in later sources. It comes naturally to theorists who take as their starting point the relations between the lengths of string or pipe from which notes are produced, or the dimensions of other sounding bodies. It is also well adapted to acoustic theories which make pitch dependent on the speed of a sound's transmission, since speed can be specified in terms of the distance travelled in a given time.[34] The fact that the *Sectio* addresses these matters from an arithmetical angle shows that its author is not thinking, in the first instance, of the ratios as they are exhibited on the monochord's *kanōn*; that topic, when he reaches it, will involve a transition from an arithmetical to a geometrical view-point. He is dealing with ratios between numbers inherent in the physical constitution of pitched sounds themselves; and his non-Platonic, arithmetical approach in the theorems is well suited to his equally non-Platonic acoustic theory.

From this perspective proposition 3 fits comfortably with the others. We may still be puzzled, however, both by the fact that it enunciates that proposition in terms quite different from theirs, and by the intrusion of a theorem about epimoric ratios in the middle of a sequence dealing with multiples. The latter difficulty may seem easy to resolve; proposition 3 stands where it does because it is called upon in the proof of proposition 4. That is true, but proposition 4 itself has curious features. First, it is the only one of the propositions about multiple ratios that draws a conclusion referring also to epimorics: a non-multiple *diastēma* taken twice makes a whole that is 'neither multiple nor epimoric'. Secondly, it amounts in reality to two separate theorems, since the proof about multiples is separate from the one about epimorics and independent of it. Thirdly, it underplays its hand so far as epimorics are concerned; as proposition 3 shows, no *diastēma* whatever, when taken twice, makes a whole that is epimoric. Finally, and most strangely, it is unique and anomalous among the nine mathematical

[33] See Barbera 1991: 40–4. I accept the arithmetical interpretation with due humility, properly chided for previous errors by Barbera 1984a, and 1991: 41 n. 113.

[34] A later writer, Aelianus, makes this point lucidly in a passage quoted at length by Porphyry at *In Ptol. Harm.* 36.9–37.5; earlier parts of the passage illustrate an approach based on the dimensions of instruments (33.19–35.12). Among fourth-century authors, only Plato makes his position clear. As one might expect from his 'speed' theory of pitch (*Tim.* 80a), he treats harmonics, in the *Republic*, as the mathematical study of audible movements; it is the acoustic counterpart of astronomy, the mathematical study of visible movements, and both are conceived as investigations in what we might call 'kinetic geometry'. See *Rep.* 527c–530d, and cf. the explicitly geometrical construction of the musically organised World-Soul at *Tim.* 34b–36d.

propositions in that it does no work; nothing in the later theorems depends on it.

Any explanation of these oddities will be speculative, but I suggest the following. In one of his mathematical sources the author found propositions 1–2, the part of proposition 4 which deals only with multiples, and proposition 5; the source presented them as a coordinated group concerned with multiple ratios. Their context was purely arithmetical. The theorem proving proposition 4's statement about multiple *diastēmata* was included for the sake of completeness, and the fact that it has no bearing on the *Sectio*'s special agenda was in that context irrelevant. The *Sectio*'s author took them over as a package. He realised that proposition 3, which is an indispensable basis for the musical theorems (it is the logical pivot of proposition 10, on which all the others depend), shows that the 'whole' referred to in proposition 4 cannot be epimoric any more than it can be multiple. He therefore extended the thesis of proposition 4 to make it include this point, adding a brief argument alluding to proposition 3; and this made it essential to insert proposition 3 before proposition 4. He seems to have overlooked the fact that neither part of proposition 4 has anything to contribute to his project.

On this hypothesis, though proposition 3 is crucial to later phases of the *Sectio*'s programme, it has been levered awkwardly and for rather poor reasons into a sequence of theorems among which it did not originally belong. As it happens, we have independent evidence about its origins, since Boethius attributes an almost identical theorem to Archytas.[35] My hypothesis would not entail that the four propositions on multiples came from a different author, though that is quite possible; but if they too were borrowed from Archytas, they probably appeared in another part of his writings. If my earlier arguments hold water, their context was purely arithmetical and was not shaped by the requirements of harmonics. That is unlikely to be true of proposition 3 itself.

We can therefore glean from the first five theorems a few clues about the way in which the *Sectio*'s author went to work. They suggest that he may not have been an innovative mathematician, but that he was an enterprising and in most respects (though not quite all) a skilful synthesiser. We shall find other evidence shortly of his familiarity with aspects of fourth-century mathematics that lay at some distance from harmonic theory. The task he seems to have set himself is that of selecting appropriate arithmetical

[35] *Inst. mus.* III.11; see pp. 303–5 above. Boethius' authority throughout Book III is almost certainly Nicomachus; for a summary of the evidence for this view see Bower 1989: xxiv–xxvii. For discussion of the versions in Boethius and the *Sectio* see Knorr 1975 ch. 7, Barbera 1991: 58–60.

theorems from an existing repertoire, and of linking them together, in alliance with an unusual and well-judged hypothesis about the physics of sound and pitch, as a basis for the systematic demonstration of propositions in harmonics. Most – possibly all – of these propositions were already familiar, as we shall see, and previous theorists had commended them through various kinds of observation and reasoning. But at least as far as we know, there had been no previous attempt to organise and demonstrate them as a coordinated group.

Propositions 6–9 can be disposed of quickly. All of them consider ratios with specific values. Summarily, theorems 6–8 prove that the ratio 2:1 is the product of the two greatest epimorics, 3:2 and 4:3 (proposition 6); that the product of 2:1 and 3:2 is 3:1 (proposition 7); and that when the ratio 4:3 is 'taken away' from the ratio 3:2, the result is 9:8 (proposition 8). The MSS offer two distinct proofs for proposition 6, one for each of the others. They are cumbersome but essentially straightforward, and all of them (with the possible exception of the first proof of proposition 6[36]) are conceived in arithmetical rather than geometrical terms.

Proposition 9 states that the product of six *diastēmata* in the ratio 9:8 is greater than 2:1. It proves this by direct calculation, showing that when seven numbers are found such that each (apart from the first) stands to its predecessor in the ratio 9:8, the seventh number is more than double the first. To make all the terms whole numbers they have to be large (the first is 262144 and the last 531441); otherwise the argument is unproblematic. It is particularly obvious here that the theorem is chosen, and must indeed have originally been articulated, for its bearing on an issue in harmonics, by the author of the *Sectio* or (more probably) by some predecessor. It is of little or no intrinsic mathematical interest; its function is to serve as the basis of a proof that the octave, whose ratio is 2:1, is less than six whole tones in the ratio 9:8 (see proposition 14).

One other feature of propositions 1–9 calls for comment here. Theorems later in the series sometimes draw upon earlier conclusions in the course of their reasoning, as one would expect, and they have been carefully arranged to make this possible.[37] It is rather more surprising that they draw also on results which are not proved and procedures which are not expounded in the text itself at all, and that they sometimes do so explicitly. 'We have learned (*emathomen*)', says the author in proposition 2, 'that if there are numbers in proportion, however many of them, and the first measures [i.e.

[36] On this issue see Barbera 1991: 42–4, 134–9, 267.

[37] Thus proposition 4 calls on the conclusions of propositions 2 and 3, and proposition 5 on that of proposition 1.

is a factor of] the last, it also measures those in between.' In proposition 9, similarly, 'we have learned how to find seven numbers in the ratio 9:8 to one another'. The proof of the thesis in proposition 2 and the construction called on in proposition 9 both appear in Euclid's *Elements*, as does the proof of a thesis relied on but not proved in proposition 3.[38]

When the writer of the *Sectio* says 'we have learned', however, he need not mean 'we have learned from Euclid's *Elements*', though of course he may. The *Elements* bring together a great deal of material that was already known to fourth-century mathematicians, and the *Sectio* may be referring to pre-Euclidean sources. That is not a decidable issue. What is intriguing is that its readers are assumed to be already conversant with some moderately sophisticated mathematics. The author writes, in fact, as if he and they shared a common background in the discipline, and had mastered the same range of propositions. There is no way of telling, unfortunately, whether the 'we' is that of teacher and pupils ('we studied these points last term'), or the collegial 'we' of an intimate coterie of mathematicians, or the elitist 'we' which assumes that all readers are educated fellow-intellectuals, or whether the expression indicates that the *Sectio*, as a text, was conceived as a sequel to other written texts that covered the matters in question. It shows, at all events, that despite the air of self-contained completeness which its theorematic approach conveys, the work's cogency depends partly on its insertion into a specific disciplinary context, and that its arguments feed on the environment which its readers are taken to inhabit. Mathematical harmonics, as the *Sectio* presents it, is not an isolated, free-standing science, but belongs with others as part of a wider intellectual enterprise.

PROPOSITIONS 10–13: THE TRANSITION TO HARMONICS

The first two theorems in this group seek to prove that the octave *diastēma* is multiple (proposition 10), and that the *diastēmata* of the fourth and the fifth are both epimoric (proposition 11). Proposition 12 provides arguments to demonstrate the values of the *diastēmata* or ratios of the various concords, 2:1 for the octave, 3:2 for the fifth, 4:3 for the fourth, 3:1 for the octave plus fifth, 4:1 for the double octave. Proposition 13 shows that the ratio of the tone is 9:8.

The reasoning of all these theorems relies on conclusions established in the preceding arithmetical propositions. But it is obvious that they cannot

[38] The proofs relevant to propositions 2 and 3 are at Eucl. *El.* 8.7 and 8.8 respectively; for the construction in proposition 9 see *El.* 7.2.

prove anything about octaves, fifths and other musical relations from the resources of arithmetic alone. Pure *arithmētikē*, the 'science of number', says nothing about such things. Information about them must be fed in from other domains; and the procedure must also have access to principles which allow propositions about musical phenomena, such as the octave, to be brought into logical connection with propositions dealing with multiple ratios and other such mathematical abstractions.

To form an impression of the kind of musical information that is introduced, consider proposition 10.

> The octave *diastēma* is multiple. Let A be *nētē hyperbolaiōn*, let B be *mesē* and let C be *proslambanomenos*. Then the *diastēma* AC, being a double octave, is concordant. Hence it is either epimoric or multiple. It is not epimoric, since no mean falls proportionally in an epimoric *diastēma*. Therefore, since the two equal *diastēmata* AB and BC when put together make a whole that is multiple, AB is also multiple.

In order to make sense of this argument, we need to be familiar with the names of the notes, and to know the musical relations in which they stand; *nētē hyperbolaiōn* is an octave above *mesē*, and *mesē* is an octave above *proslambanomenos*. We must know also that the double octave is a concord. These are not very recondite pieces of knowledge; they belong to the most elementary level of harmonic theory in its empirical or descriptive guise. But they are plainly indispensable. The theorem calls, in addition, on two arithmetical results already proved, first that there is no mean proportional between terms in an epimoric ratio (proposition 3), and secondly that if the whole formed from two equal *diastēmata* is multiple, they are multiple too (proposition 2). The 'bridge' between the musical and arithmetical propositions is formed by the introduction's thesis that pitches are related to one another in numerical ratios, and by the principle it commends as 'reasonable', that the ratio of any concord must be either epimoric or multiple. They make it possible to cross the boundary between the two domains by tying a pair of musical conceptions, interval and concordance, to arithmetical categories. They themselves thus fall within the scope of neither of the two sciences at work elsewhere in the theorem, arithmetic and empirical harmonics. Neither of those sciences can demonstrate their truth, and nor can mathematical harmonics of the kind exemplified in the *Sectio*. Mathematical harmonics is constructed precisely through the formation of this bridge between the other two disciplines. Its theorems must therefore presuppose the bridging propositions and cannot be used to prove them.

Given the data taken from empirical harmonics, the arithmetical propositions and the bridging principles, the argument of proposition 10 is sound. Proposition 11 calls on resources of just the same sorts, and on the face of it its reasoning takes a course roughly parallel to that of proposition 10. But as commentators have repeatedly noted, it runs into trouble.

The *diastēma* of the fourth and that of the fifth are both epimoric. Let A be *nētē synēmmenōn*, let B be *mesē* and let C be *hypatē mesōn*. Then the *diastēma* AC, being a double fourth, is discordant. Hence it is not multiple. Therefore, since the two equal *diastēmata* AB and BC when put together make a whole that is not multiple, AB is not multiple either. But it is concordant; hence it is epimoric. The same demonstration applies also to the fifth.

This argument, like its predecessor, assumes familiarity with the named notes and the relations between them, and it assumes the knowledge that the double fourth (in modern terms, a seventh) is a discord. It relies on just one of the arithmetical theorems (proposition 5), and it seems to appeal twice to the bridging principle governing the ratios of concords. The second appeal, in the penultimate sentence, is legitimate; if the principle holds, and if AB is concordant but not multiple, it must be epimoric. The notoriously precarious move comes in the third and fourth sentences, where it is argued that since the double fourth is discordant it is not multiple. But the principle states only that all concords are either multiple or epimoric, not that all multiple *diastēmata* are concordant, and the inference from 'AC is discordant' to 'AC is not multiple' is unwarranted.

Scholars have sometimes tried to justify the move on the grounds that the writer, like most other harmonic theorists, concerns himself only with relations existing within the span of a double octave.[39] All the multiple *diastēmata* within that range are indeed concords (the octave, 2:1; the octave plus fifth, 3:1; the double octave, 4:1). It is only when we take the next step that we encounter the first interval in the series which by Greek standards is a discord; 5:1 is the ratio of an octave plus a major third. But the *Sectio* says nothing to indicate that the principle on which it relies is limited in this way, and its logic would be no less objectionable if it did. It could argue from the premise that all multiple *diastēmata* within the double octave are concords only if that fact were already established, and it can be established only by identifying the ratios belonging to the intervals involved. That is done in proposition 12. But the first step in proposition 12, showing that the ratio of the octave is 2:1, itself relies on the conclusion of proposition 11, and all its other steps depend on the first. The questionable move in

[39] See e.g. Barbera 1984a: 160–1.

proposition 11 would thus be 'legitimised' by reference to results which are derived from proposition 11's own conclusion, and the reasoning would be viciously circular.

Nor can one argue that the conclusions of proposition 12 can properly be relied on before being demonstratively established, since they were already thoroughly entrenched in the harmonic tradition and in that sense could be treated as known. That would involve using them like the data of empirical harmonics, which are introduced into the arguments as known facts which do not call for proof. Such a strategy would make no sense. Their role is quite unlike that of the empirical data. As proposition 12 shows, they are conclusions – arguably the most important conclusions – which this mathematical harmonics sets out to demonstrate, and they cannot simultaneously be factual assumptions on which those same demonstrations depend, even though both the writer and his readers will no doubt have been confident of their truth before any proofs were devised or presented.

The consequence is that the project of the *Sectio* is fatally flawed. From proposition 12 onwards, every conclusion depends directly or indirectly on proposition 11, and the latter's argumentation cannot be repaired. An analogous and acceptable theorem would have to include a premise of the form 'Every ratio of type T is the ratio of a concord.' The premise must avoid presupposing the conclusions of proposition 12; it must supply the necessary step in an argument to the conclusion that the ratios of the fourth and the fifth are epimoric; and there must be good reasons for thinking it true. The fact that no such premise was ever discovered by Greek theorists is no reflection on their ingenuity, since there is none to be found.[40]

Another difficulty arises from the summary statement with which proposition 12 ends: 'Thus it has been demonstrated, for each of the concords, in what ratio the bounding notes stand to one another.' This claim to comprehensiveness seems unjustified, since one interval regularly accepted as a concord, the octave plus fourth, has not been quantified or even mentioned. The reason for the omission is plain enough, since as I noted earlier, its ratio is 8:3 and is neither epimoric nor multiple. Hence the writer could not include it in his treatment without abandoning his principle about the ratios of concords. As I remarked in the previous chapter (p. 348 above), he deals with the problem by passing over it in silence.

But the problem is genuine and serious. When the *Sectio* treats the intervals listed in proposition 12 as concordant, it is relying on the evidence of musical perception, and the argument can only proceed on that basis. If

40 On this matter see also pp. 346–8 above.

perception finds the octave plus fourth concordant too, this evidence should arguably be given parity of treatment. It seems very clear that Greek ears did indeed perceive this relation as a concord, and that musicians and non-mathematical theorists were unanimous in accepting it.[41] A writer who uses the evidence of perception and practice to identify the concordant intervals, and who nevertheless commits himself to the thesis that only intervals with epimoric or multiple ratios can be concords, is effectively claiming that in the single case of the octave plus fourth, perception is unreliable. In that case good reasons must be given for regarding it as an exception. The *Sectio*'s author cannot have been unaware of the difficulty, and the fact that he gives no reasons of this sort makes it highly probable that he had none to offer. In later antiquity the problem became notorious, and various strategies were devised to cope with it.[42] None of them is very persuasive, and we have no grounds for reading any of them back into the *Sectio*.

PROPOSITIONS 14–18: INEQUALITIES AND CONTROVERSIAL CONCLUSIONS

We now reach a series of propositions whose role in the enterprise does not spring immediately to the eye. They all follow legitimately (given some further empirical input) from conclusions established or allegedly established previously. But they play no obvious part in the final stage of the work, the division of the *kanōn* in propositions 19–20, and we are entitled to wonder why the author has decided that they, rather than any others, merit demonstration in this treatise.

As a preliminary I shall review them briefly, partly in order to show that they have enough in common to justify my grouping them together. Proposition 14 shows that the octave is less than six tones, proposition 15 that the fourth is less than two and a half tones and the fifth less than three and a half, and proposition 16 that the tone cannot be divided into two or more equal parts. Proposition 17 is not a proof; it sets out a method of construction which is an adjunct to proposition 18. Proposition 18 itself shows that the lower moveable note in an enharmonic tetrachord, its *parhypatē* or *tritē*,

41 No harmonic scientist who takes the observational data seriously hesitates to accept it. Aristoxenus enunciates the general rule that when an octave is added to any concord the resulting interval is itself concordant (*El. harm*. 45.20–3). Ptolemy, whose approach is primarily mathematical, but who insists that 'rational' principles and conclusions must be consistent with the evidence of perception, elaborates Aristoxenus' rule and defends it vigorously, arguing in particular that it is absurd to deny the octave plus fourth the status of a concord (*Harm*. 13.1–23).

42 See e.g. Adrastus quoted at Theo Smyrn. 56.12–15, Ptolemy, *Harm*. 15.18–16.12, and cf. Ptolemaïs of Cyrene quoted at Porphyry, *In Ptol. Harm*. 23.25–31.

cannot divide the *pyknon* into equal parts. These conclusions fall neatly into two pairs; propositions 14 and 15 show that each of the principal concords amounts to less than some specified number of tones or half-tones, and propositions 16 and 18 show that certain *diastēmata* cannot be divided into equal sub-intervals or sub-ratios. More broadly, the group is united in its concern with inequalities; the octave is not equal to six tones, the ratios of the two intervals inside the enharmonic *pyknon* are not equal to one another, and so on.

The author's argumentative resources would have allowed him to demonstrate that none of the ratios that are most significant in harmonic analysis can be divided into equal sub-ratios of whole numbers. In the cases of the tone and the lesser concords – the fifth and the fourth – this follows directly from the fact that their ratios are epimoric, together with the Archytan theorem, proposition 3. Proposition 18 shows that the same consideration applies, through slightly more complex reasoning, to the *diastēma* comprising an enharmonic *pyknon*. Though the ratio of the octave is not epimoric but multiple, the argument supporting proposition 3 will cover this case too; all that is needed is a slight rephrasing of the proposition itself.[43] Taken together, these conclusions have an important methodological implication. It is that such intervals cannot be measured as multiples of any smaller interval, and that systems bounded by notes in these fundamental relations cannot be built up additively from units of any one size. There is no elementary unit of harmonic construction.[44]

It is therefore interesting that the *Sectio* does not adopt precisely this approach. It argues that the tone and the enharmonic *pyknon* cannot be divided equally, but in the case of the concords it concentrates on more specific inequalities. It does not claim or even imply that the octave, fifth and fourth cannot be divided into equal parts, only that the octave is less than six tones, that the fifth is less than three and a half tones and that the fourth is less than two and a half. (It leaves open, at this stage, the question whether there are such things as half-tones.)

No one who has studied later essays in this discipline will find this pattern strange; just the same group of contentions is routinely repeated in Platonist and neo-Pythagorean texts of the Roman era. But in the context of the period I am positing for the *Sectio*, we cannot appeal to scholarly routines of this sort, since there is no evidence that such semi-automatic

43 The reasoning of proposition 3 is sufficient to show that there is no mean proportional between terms in any ratio whatever of the form n + 1:n, and applies to the special case of the ratio 2:1 just as firmly as it does to those which can properly be described as epimoric.

44 On this point see also my comments on Aristotle's treatment of the diesis, pp. 350–3 above.

patterns of exposition had yet been set up. The likeliest diagnosis is that the propositions are set in this form because they will then respond most directly to current controversies. We saw in a previous chapter that Aristoxenus dedicated a surprising amount of space to a discussion of the 'size' of the perfect fourth (p. 190 above), and the thesis that it is in fact exactly two and a half tones seems to be one which he shared with his empirically minded predecessors. If they are right, it will follow at once (given the definition of a tone as the difference between a fifth and a fourth, and the agreement that a fifth plus a fourth make up an octave) that the fifth is three and a half tones and the octave six; conversely, if the fourth is not two and a half tones, those quantifications of the fifth and the octave will also fail. Aristoxenus and the empirical *harmonikoi* assume, in addition, that the tone and the enharmonic *pyknon* can be divided into equal parts. They take the same view about other intervals too, but these intervals have special importance. The 'parts' are regularly expressed as unit-fractions of a tone, and the *harmonikoi*, according to Aristoxenus, studied enharmonic systems and no others. These are also his own main focus of attention in Book III of the *El. harm.*

We may reasonably suppose, then, that propositions 14–18 are designed to consolidate a clear line of demarcation between the two harmonic traditions. Their methodological differences had been obvious since Plato's time if not before, and though Aristotle's grasp on the issues may have been uncertain, he plainly knew that they were at odds with one another over at least one of the substantive points set out in the *Sectio*.[45] It is a plausible guess that these points, or some of them, were set out in the work of Archytas. Since the *Sectio* expounds only a small selection of propositions in mathematical harmonics and leaves many topics untouched, and since those that it addresses are established from first principles, they are presumably to be reckoned as in some sense fundamental. The propositions establishing the ratios of the concords and the tone certainly have that status, and the inclusion of the controversial propositions 14–18 implicitly assigns them a similar rank.

The author seems to have thought, then, that mathematical harmonics lacks firm foundations until the theses over which it diverged from empirical harmonics had been brought together as a group and systematically demonstrated. It does not immediately follow that he construed this phase of his work as a full-dress refutation of the empiricist heresy. Aristoxenus, on the other side of the doctrinal fence, refrains from branding the physicists'

45 Plato, *Rep.* 530–1, Aristotle, *Metaph.* 1053a12–17; cf. p. 350 above.

and mathematical theorists' conclusions as false, a charge he scatters lavishly among his comments on the *harmonikoi*; physics and mathematical harmonics tackle questions quite different from his own, and simply have no bearing on his concerns (see pp. 166–8 above). Since the *Sectio* does not allude directly to the empiricists' procedures, we cannot be certain how its author or his fourth-century sources construed the relation between his conclusions and theirs, but such evidence as we have points to the more extreme interpretation. The *Sectio*'s unadorned statements suggest no scope for compromise or reconciliation; the tone, for example 'will not be divided into two or more equal parts' (proposition 16), and that is that. Since *tonos*, 'tone', is a term shared by theorists on both sides of the chasm, the statement as it stands flatly contradicts Aristoxenus and the *harmonikoi*, and later writers in the mathematical tradition seldom hesitate to conclude that the Aristoxenians and their allies must therefore be wrong. It is hard to imagine that the *Sectio*'s arguments were understood in a more generous sense in their own time. The treatise may indeed have played no small part in entrenching in Greek musicological thought the image of the two approaches as obdurate and irreconcilable rivals.

I have said that propositions 17–18 pose special problems of their own. The problems do not affect the interpretation of the passage's content or reasoning, but they are relevant to the other main issue that has been dogging us through the course of this chapter, since they have been thought to undermine the assumption that these propositions were included in the treatise in its original form. Apart from the fact, which I have already discussed, that they do not appear among Porphyry's quotations from the *Sectio*, two grounds for suspicion have been identified.

The first, however regarded, will not carry very much weight. It is that proposition 17 is presented in a different form from its predecessors. As Barbera notes,[46] all the earlier theorems have been cast as proofs, whereas proposition 17 offers, instead, a method of construction, explaining how to construct a ditone by moves through concordant intervals only. There is no doubt that this shift of approach takes place, but it does not strike me as significant enough to support the hypothesis of a change of author or date. For one thing, the division in propositions 19–20 also amounts to a constructional procedure, not a theorematic proof; but this point will cut no ice with those who reject propositions 17–18, since they typically treat the division, too, as an alien accretion. Barbera (p. 16) comments that proposition 17 marks a transition from mathematical demonstration

[46] Barbera 1991: 16, 24, 60.

to 'the style of a manual', meaning, I think, the style of a work which offers an introductory exposition of musical facts and procedures, designed to help students to find their way, in practice, around musical scales, intervals and so on. But this, I suggest, lays insufficient emphasis on the fact that propositions 17 and 18 belong together, and that the natural way of reading proposition 17 is simply as a preliminary to its sequel. Proposition 18 is, once again, a proof rather than an exposition of method, but we need the method described in proposition 17 if we are confidently to construct the intervals with which it is concerned. Furthermore, as Barbera points out (173, n. 61), even proposition 17 unquestionably displays 'aspects of the Euclidean model' on which its predecessors are patterned. The shift in presentation is not in fact very radical, nor, for the reasons I have given, is it unintelligible or inappropriate in its existing context.

The other objection to propositions 17–18 is more substantial, and has attracted the most discussion. They are concerned exclusively with intervals belonging to a scale in the enharmonic genus. Earlier propositions have examined intervals common to all the genera, and the system constructed in propositions 19–20 is diatonic. What reasons could there possibly be for a systematically minded author to move abruptly into an argument dealing only with the enharmonic, and without even announcing that the transition has occurred? (The name of the genus occurs nowhere in the passage or indeed anywhere else in the treatise.) I have argued above and more fully elsewhere that these propositions may be designed as part of a polemic against a rival school of theorists, the *harmonikoi* who according to Aristoxenus concerned themselves with enharmonic systems and no others, and very possibly also against Aristoxenus himself.[47] The proof with which they conclude purports to show that the small interval which is left in a perfect fourth after the subtraction of a ditone, and which constitutes the enharmonic *pyknon*, cannot be divided into equal intervals; and this would rule out of court any analysis in the manner of these rival theorists, since for them the *pyknon* was divided into equal quarter-tones. In its enunciation of a position incompatible with that of more empirically minded theorists, Aristoxenus included, this proof is of a piece with the *Sectio*'s demonstrations that the octave is not made up of six equal tones, and that the interval of a tone cannot be divided into equal portions (propositions 13 and 16). Further, the proof about the tone cannot be straightforwardly adapted to fit the case of the enharmonic *pyknon*, at least if that miniature structure is conceived as it is in propositions 17–18. The proof dealing with the tone

[47] See Barker 1978a, 1981b.

turns on the thesis that its ratio is epimoric; but the ratio of the enharmonic *pyknon*, as envisaged here, is not epimoric. It is the remainder of a perfect fourth, whose ratio is 4:3, after the subtraction of two whole tones, each in the ratio 9:8. Though the author does not say so, its ratio is 256:243; it is the Platonic or Pythagorean *leimma*. This ratio is epimeric or 'number to number', not epimoric, and nothing has been said to outlaw the equal division of epimeric ratios in general.[48] Hence a rather more elaborate approach is needed, and this is provided by the propositions in question.

I am now inclined to agree with Barbera (1991, 22), however, that the hypothesis of a polemical agenda is not enough by itself to allow us confidently to link propositions 17–18 with the preceding theorems. But there is another reason for believing that these propositions, whatever their relation to the remainder, were composed at an early date. Aristoxenus asserts that his predecessors had analysed systems in the enharmonic genus and no other (*El. harm*. 2.7–25). If that is true, it presumably reflects the currency, in those predecessors' time, of a mode of musical culture in which this genus had a central, privileged status. Support for this conclusion is provided both by the claim made by Aristoxenus – well known as a conservative in matters of musical taste – that it is the 'finest' of all genera, though falling into disuse in his day (*El. harm*. 23.3–23), and by the fact that it is not most commonly referred to as 'the enharmonic genus', but simply as *harmonia*, 'attunement'. There is also a tradition recurrent in later writings, and asserted in one document which probably dates from about 380 BC, that the music of the genre which had the highest status in classical Athens, that is, tragedy, was exclusively enharmonic, at least up to the closing years of the fifth century.[49] The 'ancient scales' described by Aristides Quintilianus (pp. 45–52 above) are based on an enharmonic structure; and if he is right to identify them with the *harmoniai* of Plato's *Republic*, they must also be those discussed by Aristotle in the *Politics*, since Aristotle makes it clear that the systems he is considering and those of the *Republic* are the same. Hence though neither Plato nor Aristotle specifies the *harmoniai* they review as being 'enharmonic', it is rather probable that they were. Neither in fact draws any distinctions recognisable as marking differences of genus. In tacitly assuming an enharmonic scheme, the *Sectio* would seem to be in good company.

48 Some of them can indeed be 'halved', in the sense intended here, those, that is, both of whose terms have rational square roots.

49 See Plut. *Quaest. conv*. 645d–e, [Plut.] *De mus*. 1137e–f, Psellus, *De trag*. 5. The probably fourth-century discussion appears in a papyrus fragment, *P.Hib*. 1.13; see West 1992a: 247–8, with the references in his n. 84; cf. also pp. 69–73 above.

With that in mind, we may judge that a treatise bringing together a corpus of significant fourth-century propositions about musical intervals could hardly fail to include some account of those peculiar to enharmonic systems. This rather impressionistic point can be reinforced. I have commented that the text does not actually state that the propositions refer to the enharmonic genus; it is the method of construction and the mathematics involved which assure us that they do.[50] The text names the note which is central to its argument simply as '*lichanos*', not as 'enharmonic *lichanos*', and the relevant structure simply as 'the *pyknon*', without distinguishing it from the various *pykna* of the chromatic genus. But this mode of reference would be unimaginable in a text originating in a period later than the fourth century. It makes sense only in a context where systems are presumed to be enharmonic unless explicitly described otherwise. If that context is assumed, the two propositions can stand, in a way precisely parallel to earlier ones, as orderly systematisations of pre-Euclidean material.

PROPOSITIONS 19–20: THE DIVISION OF THE *KANŌN*

This final stretch of the text describes a procedure for marking out the *kanōn* or 'ruler' under a monochord's string, in such a way that when a bridge is moved successively to each of the positions indicated, it will determine lengths of string that sound each successive note of a two-octave diatonic scale. As Barbera has noted, though the process of division is described in mathematical terms, and offers students no help with the practicalities of such operations as dividing a length into eight equal parts, the writer implies (realistically or otherwise) that the work will be carried out on an instrument, not merely in a diagram. The second sentence of proposition 19 reads: 'Let there be a length of the *kanōn* which is also the length AB of the string'.[51] The construction produces the results claimed for it, and the exposition is not hard to follow. But questions arise about certain details, and there is a problem of a more general sort. One would expect the sequence of steps followed in the division to be governed either by mathematical or by musical considerations, or perhaps by some intelligible combination of the two; and we shall find that the *Sectio*'s procedure has puzzling features on any of these interpretations.

The construction is in two parts. They correspond to the segments numbered as propositions 19 and 20 in modern editions, and though this

[50] The implication is obscured by Jan's emendations to the text of proposition 18, which are unnecessary and misleading.

[51] Barbera 1991: 60–1. The sentence quoted, with different and perhaps preferable punctuation, would read: 'Let there be a length of the *kanōn*, AB, which is also the length of the string.'

numbering has no manuscript authority, a line of demarcation is securely indicated by the text itself. Proposition 19 begins 'To mark out the *kanōn* according to what is called the immutable *systēma*', and ends 'Thus all the notes of the immutable *systēma* will have been found on the *kanōn*'; and proposition 20 begins 'It remains to find the moveable notes'. We can therefore start by considering proposition 19 by itself; discussion will be easier if we have a translation in front of us, along with an illustrative diagram.[52]

To mark out the *kanōn* according to what is called the immutable *systēma*. Let there be a length of the *kanōn* which is also the length AB of the string, and let it be divided into equal parts by C, D, E. Then AB, being the lowest in pitch, will be the bass note. This note AB is in the ratio 4:3 to CB, so that CB will be concordant with AB at the fourth above it. AB is *proslambanomenos*, and CB will therefore be *diatonos hypatōn*. Again, since AB is double BD, it will be in concord with it at the octave, and BD will be *mesē*. Again, since AB is four times EB, EB will be *nētē hyperbolaiōn*. I divided CB in half at F, and CB will be double FB, so that CB is concordant with FB at the octave, and FB will be *nētē synēmmenōn*. I cut off from DB one third, DG. DB will be in the ratio 3:2 to GB, so that DB will be concordant with GB at the fifth. Hence GB will be *nētē diezeugmenōn*. I set out GH, equal to GB, so that HB will be in concord with GB at the octave; thus HB is *hypatē mesōn*. From HB I took away one third, HK. HB will be in the ratio 3:2 to KB, so that KB is *paramesos*.[53] I marked off LK, equal to KB, and LB will become the low *hypatē*.[54] Thus all the notes of the immutable *systēma* will have been found on the *kanōn*.

Considered from a formal perspective, the route taken by this construction is straightforward but not strikingly systematic. After an initial quartering of the whole length, AB, producing four sounding-lengths (whole, three quarters, half and quarter), we halve the three-quarter length to construct a fifth length. We then take two thirds of half of AB to make the sixth length, double the sixth to produce the seventh, take two thirds of the seventh to produce the eighth, and finally double the eighth to construct the ninth.

The last four steps have a certain rhythm to them, with operations involving two thirds alternating with doublings. But by comparison with procedures adopted by some later theorists, the sequence is relatively haphazard. We may compare it, for example, with the one followed by Thrasyllus, some three centuries later. His division too falls into two distinct phases,

[52] A diagram broadly similar to the one given here appears in some MSS. Whether it or any other diagrams appeared in the original text is uncertain.

[53] *Paramesos* is found occasionally as a variant for *paramesē*.

[54] That is, the note usually named as *hypatē hypatōn*. Despite the unfamiliar expression used here, the author seems to have known the 'normal' way of naming it, since he designates the note above it as *parhypatē hypatōn* in proposition 20.

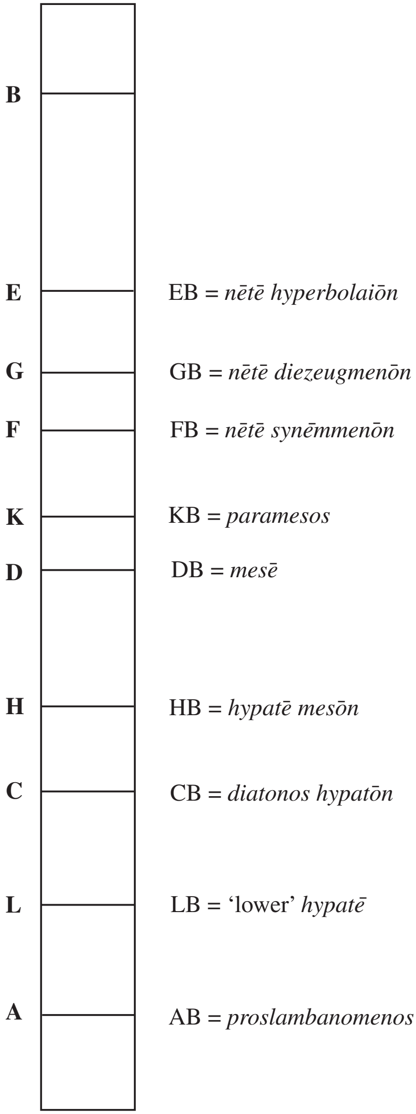

Figure 10 *Sectio canonis* proposition 19

corresponding roughly to the *Sectio*'s propositions 19 and 20; here we need consider only the first. It is remarkably orderly, beginning from the whole string, then halving it, then dividing it into thirds, and finally into quarters. By these means Thrasyllus constructs lengths for only six of the nine notes supplied by the *Sectio*'s proposition 19, leaving the remainder to be found in the second, more complex stage of the division. This seems anomalous, since the missing notes are among the fundamental, fixed notes of the system, and one might reasonably expect that fact to be reflected in the process by which they are mathematically constructed. Thrasyllus has evidently sacrificed musical considerations on the altar of formal orderliness, and has also enshrined in his procedure the symbolic significance of the numbers 1, 2, 3, 4, which make up the Pythagorean *tetraktys* of the decad.[55]

By Thrasyllus' standards the *Sectio*'s procedure is chaotic, and it shows no signs of allegiance to Pythagorean arithmological symbolism. It is less obvious, however, that we can form a sound judgement on it simply by reversing the verdict on Thrasyllus, arguing that at points where the two sets of requirements are in competition it has allowed musical principles free rein, and has made no concessions to systematic neatness for its own sake. That would suggest that it contains no significant musicological oddities, but in fact there are at least three. One of them, on the face of it, is merely terminological. The writer claims that he has constructed all the notes of the 'immutable' (*ametabolon*) *systēma*. If by this he means (in Aristoxenian language) 'all the fixed notes', the usage is intelligible; the system comprising all and only fixed notes is in an acceptable sense 'unchangeable'. But when the expression *ametabolon systēma* occurs in other writings it has a quite different sense, referring to the regular, 'unmodulated' ordering of all the notes within the standard two-octave compass.[56] Secondly, one of the notes constructed through the initial quartering of the string is not normally reckoned a fixed note at all. What the *Sectio* calls *diatonos hypatōn* is in other terminology the diatonic *lichanos hypatōn*, and is elsewhere usually assigned a status no different from that of any other moveable note, whose counterparts in the other genera are at different pitches. The *Sectio*'s treatment seems to award it a more fundamental role. Finally, another note constructed here, *nētē synēmmenōn*, is indeed treated by most theorists as

[55] That is, they sum to the perfect number 10, for whose numinous role in Pythagorean lore see e.g. Sextus Empiricus, *Adv. math*. 7.94–5. For Thrasyllus' construction see Theo Smyrn. 87.4–89.9. It is uncertain whether the introductory comment linking the procedure explicitly with the *tetraktys* comes from Thrasyllus or from Theon himself.

[56] For examples see Cleonides 201.14–18, Theo Smyrn. 92.26–7, Ptol. *Harm*. 52.11–12, 53.18, Arist. Quint. 14.24–5, and cf. Fig. 5 on p. 17 above.

a fixed note; but when the moveable notes of the system are located in proposition 20, those in the tetrachord whose upper boundary it is, the tetrachord *synēmmenōn*, are nowhere mentioned. *Nētē synēmmenōn* is left strangely isolated from the structure to which it belongs.

Let us take the second problem first. It is unlikely that *diatonos hypatōn* is included here merely because it is bound to emerge from a division of the length into quarters, since that move is itself unnecessary. The other notes constructed through this quartering are *mesē* and *nētē hyperbolaiōn*, and *mesē* could have been found by halving the string, *nētē hyperbolaiōn* by halving the half. The writer seems to have had a genuine reason for wishing to locate *diatonos hypatōn* as well; after *proslambanomenos* it is indeed the first note to be identified. In this respect the case is different from that of Thrasyllus' division, where this note is again constructed in the opening phase, since its appearance there is an inevitable consequence of his unwavering pursuit of a sequence through the terms of the Pythagorean *tetraktys*.

Thrasyllus, however, supplies a useful additional piece of evidence.[57] He refers to this note as '*hyperhypatē*, also known as *diatonos hypatōn*'. The term *hyperhypatē* reappears in the same role in a passage of Boethius derived from Nicomachus;[58] elsewhere it is almost unknown.[59] But the existence of a distinctive name for a note in this position, and one that does not specify it as diatonic, encourages the belief that in some Greek systems it had a role independent of genus, as part of the 'immutable' framework. This hypothesis is supported by Aristides Quintilianus' description of the Dorian *harmonia* he associates with Plato's *Republic*, in which a regularly formed enharmonic octave is supplemented, at the bottom, by the interval of a tone; the lowest note of the system would be this *hyperhypatē*.[60] It is encouraged also by the occurrence of a note in that position, in a non-diatonic context and immediately below a *pyknon*, in two of the surviving musical scores, those of a fragment from Euripides' *Orestes* and the Delphic paean of Limenius. In the *Orestes* the *pyknon* is enharmonic and in the

[57] In much of my discussion of this issue I am following the suggestions of Winnington-Ingram 1936: 25; cf. 28, 32, 35–6.

[58] Boeth. *Inst. mus.* 1.20. Here the mysteriously named Prophrastus of Pieria is credited with the addition of a ninth string, called *hyperhypatē* and tuned a tone below *hypatē*, to the eight allegedly first used by the equally mysterious Lycaon of Samos (possibly to be identified with Pythagoras). Cf. *Exc. ex Nicom.* 4, 274.3–4, and see Bower 1989: 33 n. 107, 34 nn. 109–10. These attributions are part of an almost wholly fictional reconstruction of progressive additions to the number of strings on the lyre or kithara; but they suggest that Nicomachus had some record of an ancient nine-note system, which included the note *hyperhypatē* lying at the interval of a tone below the regular octave.

[59] The usage at Arist. Quint. 8.12, which seems to imply that there are distinct diatonic, chromatic and enharmonic *hyperhypatai*, is probably based on a misunderstanding.

[60] Arist. Quint. 18.13–15; see p. 49 above.

paean chromatic, but in neither case does the note in question fit into the pattern standardly assigned to the relevant genus.[61]

Aristides attributes this form of the Dorian *harmonia* to the usage of 'very ancient times' (18.5–6). If the music of the *Orestes* fragment goes back to Euripides himself,[62] it originated in the fifth century; and the Delphic paeans of Athenaeus and Limenius, dating from 127 BC, adopt a deliberately 'antique' style. The various pieces of evidence are consistent, then, with the hypothesis that in representing *diatonos hypatōn* (or *hyperhypatē*) as one of an 'immutable' array of fundamental notes, the *Sectio* is in tune with musical realities, and in particular with ones prevalent in relatively early times.

The hypothesis has a bearing also on the fact that proposition 19 constructs *nētē synēmmenōn*, even though the tetrachord that includes it is ignored in proposition 20. The pertinent evidence I have found is thin, but points in a helpful direction. A passage of the Plutarchan *De musica*, certainly derived from Aristoxenus, reports that music in the very ancient 'libation-style' (*spondeiazōn tropos*) used certain notes in the accompaniment which it avoided in the melody. It also mentions the notes of the melody with which these accompanying notes formed concords or discords. The melodic notes are *parhypatē, lichanos, mesē, paramesē* and *paranētē diezeugmenōn*, and the accompanying notes are *tritē diezeugmenōn, nētē diezeugmenōn* and *nētē synēmmenōn*. (The system is to be construed throughout as enharmonic.) Taken together, these include all except the lowest of the notes of the tetrachord *mesōn*, and all the notes of the tetrachord *diezeugmenōn*; but the tetrachord *synēmmenōn* is represented only by its *nētē*.[63]

The absence of any reference to *paranētē* and *tritē synēmmenōn* might be an accident; but I rather think it is not. *Nētē synēmmenōn* lies a tone below *nētē diezeugmenōn*, and an octave above *diatonos hypatōn* or *hyperhypatē*, and is thus the analogue of the latter in the higher range of the system. (It is noteworthy that the only constructional use which the *Sectio* makes of *diatonos hypatōn* in proposition 19 is to locate *nētē synēmmenōn*, and that the latter is not employed in the construction of any other length.) In the Plutarchan account, *nētē synēmmenōn* appears in an enharmonic context, just as does its counterpart an octave below in Aristides' Dorian and in

61 For the *Orestes* fragment see Pöhlmann and West 2001: 12–13, where the note appears in bars 2, 8, 9 and 12 of their modern transcription; cf. West 1992a: 284–5. For the relevant part of Limenius' paean see Pöhlmann and West 2001: 80–1, where the note appears several times between figures 25 and 26 of their transcription; cf. West 1992a: 297.

62 I agree with West 1992a: 270, that there is no reason to doubt it. He provides a short bibliography on the piece on p. 278; see also the extensive bibliography on the scores in Pöhlmann and West 2001.

63 See [Plut.] *De mus.* 1137b–d.

the *Orestes* fragment; and the *Orestes* score itself includes the symbol for an instrumental (but not a melodic) note at the pitch of *nētē synēmmenōn*, repeated five times, an octave above the note which apparently plays the role of *hyperhypatē*. No other notes of the tetrachord *synēmmenōn* appear.[64]

Taken by themselves, these indications would not add up to solid evidence for the thesis that a note in this position was commonly used, irrespective of the genus of the prevailing *systēma*, as an item unattached to the tetrachord *synēmmenōn*. In the Plutarchan treatise (as in the *Sectio*), the name assigned to it seems to suggest the contrary; it identifies it precisely as the *nētē* of that tetrachord. But the source of that account is Aristoxenus, and he is using the terminology current in his own day; it is the only designation he has for a note at that pitch in an enharmonic context. When linked with the evidence about its counterpart an octave below, the argument for the thesis becomes a good deal stronger; and in both cases the practices in question apparently go back at least to the fifth century and probably further.

There may then be a simple explanation for the first difficulty raised by the *Sectio*'s nineteenth proposition, its anomalous use of the expression *ametabolon* ('immutable') *systēma*. If the hypotheses I have sketched are correct, it does indeed include only 'unchangeable' notes, but within a musical system which by the late fourth century was obsolescent if not extinct. The writer, then, is basing his account on an analysis that was already out-dated, and it would hardly be surprising if it brought with it elements of an earlier terminology. The expression *ametabolon systēma* is perfectly appropriate to the construction of proposition 19, just so long as theorists had not yet appropriated it for other purposes. It is likely, though not certain, that the person responsible for this appropriation was Aristoxenus.[65] We can therefore make sense of the *Sectio*'s usage so long as we attribute it either to a pre-Aristoxenian source, or to a writer for whom Aristoxenian terminology, even if he knew it, did not have canonical authority. The language and strategy of proposition 19 can thus be made musicologically intelligible; and there is something to be said, after all, for the suggestion that it has approached the matter from the opposite direction to Thrasyllus, allowing musical considerations rather than formal

64 Cf. West 1992a: 206–7.

65 The expression does not occur in the *El. harm.*, but the fact that it is used to mean 'unmodulated *systēma*' both by Cleonides (201.14–18) and by Aristides Quintilianus in the Aristoxenian phase of his treatise (14.24–5, explicating the phrase in a similar way to Cleonides) makes an Aristoxenian pedigree very likely. Cf. also Bacchius 74 (308.2–7). But by the Roman period the usage was widespread among theorists, regardless of their Aristoxenian or other allegiances. We may note that Thrasyllus signs off his entire division, including the moveable notes, with the comment that he has now 'filled out the entire *ametabolon systēma*', Theo Smyrn. 92.26–7.

or mathematical ones to guide the process of construction. We shall return to these issues shortly, but it will be helpful, first, to glance briefly at the final stage of the division, in proposition 20.

Sounding-lengths are constructed here for all the standardly recognised moveable notes in a two-octave diatonic system (omitting the tetrachord *synēmmenōn*), except two for which lengths were in effect established in proposition 19. (Diatonic *paranētē diezeugmenōn* stands at the same pitch as *nētē synēmmenōn* and need not be constructed independently; and the *lichanos* of the tetrachord *hypatōn* is identical with *diatonos hypatōn* or *hyperhypatē*.) Since there are eight moveable notes in all, two in each tetrachord, proposition 20 deals with six. A diagram of the completed *systēma* (Fig. 11 below) will give some help in following the reasoning.

(i) The first to be constructed is *diatonos hyperbolaiōn*,[66] a tone below the highest note of the system; the latter's sounding-length, EB, is one quarter of the whole length of string. This *diatonos* is found by dividing EB into eighths, and then adding to EB a distance equal to one of those eighths. The resulting length, MB, stands to EB in the ratio of the tone, 9:8. (ii) The lower moveable note in this tetrachord, *tritē hyperbolaiōn*, is a tone below *diatonos*, and is found by treating MB precisely as EB was treated in the previous step, to produce NB.

Once the highest tetrachord is completed, the procedure changes. (iii) Instead of constructing *tritē diezeugmenōn* by repeating the same manoeuvre with FB as its starting point (this is the length for *nētē synēmmenōn*, and therefore for diatonic *paranētē diezeugmenōn*), the author locates it by adding one third of itself to NB, *tritē hyperbolaiōn*. The new length, XB, is in the ratio 4:3 to NB, which corresponds to the fact that the two *tritai* are a perfect fourth apart. (iv) Next, XB is increased by half, giving OB, in the ratio 3:2 to XB. OB therefore sounds a fifth below XB, and is the length for *parhypatē mesōn*. (v) The fifth step amounts to doubling XB (*tritē diezeugmenōn*) to give PB (*parhypatē hypatōn*, an octave lower), though it seems to be conceived and expressed in a needlessly roundabout way.[67] (vi) Finally, CB (*diatonos hypatōn*) is decreased by a quarter to give RB, so that CB:RB = 4:3, and RB is *diatonos* (diatonic *lichanos*) *mesōn*, a fourth above *diatonos hypatōn*.

The procedure works, but there is no clear order in it, and it seems a little makeshift. Once again we can contrast the strategy of Thrasyllus,

66 The note is usually called diatonic *paranētē hyperbolaiōn*. The name *diatonos hyperbolaiōn* is uncommon, but reappears, for instance, in Thrasyllus' division.

67 It is done by adding to OX a length equal to itself, OP. Since OX is equal to half XB, this is equivalent to doubling XB to reach PB.

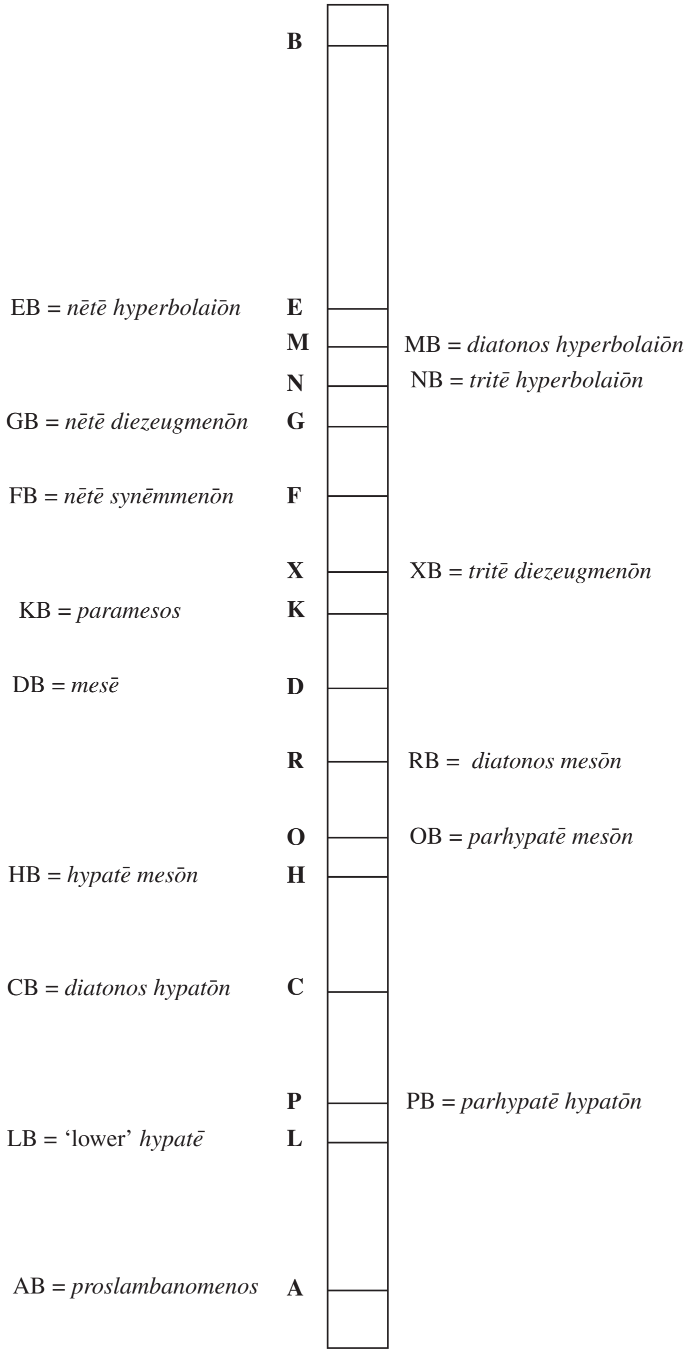

Figure 11 *Sectio canonis* proposition 20
[Notes constructed in prop. 19 are on the left, and those constructed in prop. 20 on the right]

which operates systematically, in every tetrachord, with repetitions of moves analogous to the first two in the *Sectio*.[68] Nor does the route taken in the *Sectio* seem to be governed by principles of a musical sort; the best we can say of it is that it takes the lower moveable notes of the tetrachords one after another in descending order (steps (ii)–(v)), and frames them with the construction of the two upper moveable notes that had not been put on the map in proposition 19. That gives the exposition a certain symmetry. But its basis is not very impressive, and it offers no explanation for the peculiarity of step (v). Since there are several other ways of presenting an equivalent construction in a systematic manner, one must conclude that the writer was unconcerned with such niceties, and perhaps also that his version belongs to a relatively undeveloped stage in the history of canonic division.

It is quite clear that the structure generated by the *Sectio*'s procedure is diatonic, or at least that it would have been so described by Aristoxenus and his successors. This creates a further difficulty for those who suppose, as I do, that the treatise as we have it is a single, fully integrated work, since propositions 17–18 presuppose an enharmonic structure, as we have seen, and nothing has been said to prepare us for the transition to a different system. The final part of my discussion of propositions 17–18 may indeed have made matters worse. If enharmonic is taken as the norm, the automatic point of reference unless we are told to the contrary, why is the division diatonic and not enharmonic, and why, given that it is, does the author not alert us to the change by naming the genus in which his construction is set?

The short answer to the first part of this question is that a complete division in the enharmonic genus is simply impossible on the basis of theorems propounded earlier in the work. There is nothing in them that would ratify any particular way of constructing the position of the middle note of the enharmonic *pyknon*, since the small intervals inside the *pyknon* have not been quantified. It is not merely, as Barbera puts it (1991: 24), that the author fails to provide 'the necessary mathematics to represent these intervals with ratios'. The fact is that there is no way of establishing them on the basis of the principles that the author had at his disposal, and any specific quantification would have been arbitrary. Among fourth-century theorists, only Archytas, so far as we know, offered a mathematical division of the enharmonic; and not only is it based on proportional principles of

[68] The presentation is complicated by Thrasyllus' construction, within the same sequence, of the chromatic *paranētai* and *lichanoi* (the *tritai* and *parhypatai* are identical with their diatonic counterparts). These too are found through repetitions of a single procedure.

which the *Sectio* is innocent, but it posits an enharmonic structure quite different from that of the present treatise. As late as the imperial period, writers who quantify the enharmonic *pyknon* in the same way as the *Sectio*, as the remainder of a perfect fourth after the subtraction of two tones in the ratio 9:8, do not attempt the task of dividing it mathematically.[69]

A diatonic system, by contrast, is easily constructed on the basis of the *Sectio*'s mathematical resources, since they provide a ratio for the tone, and allow us to treat the 'semitone' at the bottom of the tetrachord merely as the residue of the fourth after two tones; no separate quantification is needed. But none of this explains why the shift of genus from enharmonic to diatonic is unannounced, and why the author never explicitly refers to distinctions between genera at all.

The answer, I think, is that he recognised no such distinctions. In the light of the discussion so far this may seem a perverse opinion, but I believe that it both fits the facts and explains them. The first point to be made is terminological. Aristoxenus and later theorists regularly agree that the note with the most important function in determining genus is the higher of the two moveable notes in the tetrachord, which is usually named as *lichanos* in the tetrachords below *mesē*, and as *paranētē* in those above it. These primary names would, where necessary, be qualified by adjectives meaning 'enharmonic', 'chromatic' or 'diatonic'. Now it is these primary names, with no qualifying adjectives, that appear in propositions 17–18 of the *Sectio*. The adjectives' omission, in a context which in Aristoxenian terms is 'enharmonic', indicates that the names will automatically be understood as referring to notes lying roughly a ditone below the highest notes of their tetrachords.[70] When we come to propositions 19–20, where an Aristoxenian would call the upper moveable note of each tetrachord 'diatonic *lichanos*' or 'diatonic *paranētē*', the terms *lichanos* and *paranētē* disappear, and are replaced by the name *diatonos*, 'the note through a tone'.

This terminology is not unique. It is found occasionally in later sources (in Thrasyllus, for example); but I suspect that in those writers the usage is an archaism. Where it properly belongs is in a context where the notion of genus, as Aristoxenus presents it, had not yet taken firm root, and the special characteristic of what we call 'diatonic' systems was conceived less

[69] This is most clearly seen in Thrasyllus' sketchy treatment of the enharmonic at Theo Smyrn. 92.27–93.2.

[70] Traces of this convention survive in Aristoxenus, *El. harm.* Book III, notably at 65.25–68.1, in which (apart from a brief excursus on the diatonic at 66.17–25) the reference is exclusively but inexplicitly to enharmonic systems. In the immediate sequel (68.1–12), by contrast, where attention shifts to diatonic and chromatic, the genera are explicitly named.

as its transference of the notes *lichanos* and *paranētē* to a higher position in the tetrachord than as its omission of those notes, and its inclusion, instead, of a special note of its own, the 'note through a tone'. Structures that we would call 'enharmonic', conversely, are ones in which the 'note through a tone' is omitted, and the note marking off the interval which principally determines the system's musical character is given the special designation *lichanos* or *paranētē*, to distinguish it from the note which stands at the same pitch in a 'diatonic' tetrachord, its *parhypatē* or *tritē*. It was only with Aristoxenus' wholesale re-conceptualisation of musical systems and their relations that the notion of genus became fully articulated, and established the picture of a single note, *lichanos* or *paranētē*, moving up and down in pitch to determine the genus, while still retaining its own identity and hence the same name.[71]

In that case we can no longer insist on the point that propositions 17–18 do not announce themselves as concerned with the enharmonic genus, and propositions 19–20 do not indicate explicitly a shift to the diatonic. The language in which they are couched tells us quite clearly, in each case, which notes are being constructed or discussed. All that they presuppose is that the positions of the relevant notes are known, and that systems of two different sorts existed side by side. They are not distinguished from one another in the Aristoxenian manner, but by the fact that each of them uses a different selection of the notes available in the familiar repertoire.

Evidence of a different sort does a little to commend this hypothesis. The Plutarchan *De musica* records a report by Aristoxenus about the alleged 'invention' or 'discovery' of enharmonic music by the aulete Olympus (1134f–1135b). It is important to note that it is not represented as Aristoxenus' own analysis, but as his account of what other and presumably earlier 'musical experts' (*mousikoi*) have said about the matter (cf. pp. 98–100 above). Olympus is assumed to have been working, initially, within a diatonic structure; and he arrived at the basic outline of the enharmonic not by shifting the 'diatonic *lichanos*' to a lower position, but by omitting it altogether. The remaining note in the diatonic tetrachord, diatonic *parhypatē*,

[71] It is worth recalling that in his analyses of these structures and their relationships, the noun that Aristoxenus uses is *genos*, 'genus'. In musical contexts the usage is new, and is certainly drawn from the terminology of recent, Aristotelian science. If some other noun with the same reference had already been common currency among musicians and musical theorists, Aristoxenus would have been bound to make use of it, if only to connect his analyses with familiar descriptions of the 'facts', and perhaps to announce his intention of replacing this term, for scientific purposes, with another. No ancient text shows any trace of the existence of a pre-Aristoxenian noun of this sort. I conclude that there was none, and correspondingly no clearly articulated conception of the classes of system which Aristoxenus calls *genē*, 'genera'.

can hardly have retained the same name in this new context; in Aristoxenian terms it would have become enharmonic *lichanos*, but the writer does not tell us what it was originally called. This is disappointing but only to be expected, since Aristoxenus has evidently paraphrased the account given by the *mousikoi* into his own terminology; the passage is full of references to the enharmonic and diatonic genera, and it names the note that was 'omitted' as 'diatonic *lichanos*', not, like the *Sectio*, as *diatonos*. The crucial point is that the two systems are characterised by the presence or absence of specific notes, and not by the appearance of the same notes in different positions.

If these suggestions are on the right lines, they go some way towards an explanation of another puzzle in musicological history. There is evidence, as we have seen, that the core of the structures of the *harmoniai* of Plato's *Republic*, including those of which Socrates approves, was – in Aristoxenus' terms – enharmonic. That of the theoretically ideal system of the World-Soul in the *Timaeus*, by contrast, is diatonic, and yet it is to its perfection, not that of some enharmonic construction, that human music ought, we are told, to aspire. In this respect the transition from the *Republic* to the *Timaeus* parallels, on a much larger scale, the one between propositions 18 and 19 in the *Sectio*; and Plato, again like the author of the *Sectio*, nowhere makes any direct allusion to distinctions between the genera. On the hypothesis I am suggesting, Plato's strategy becomes at least a little less puzzling. There are not two radically different systems, nor is there a situation in which notes can move into different locations, of which – on Plato's view of things – only one can be correct. Instead there is a single reservoir of notes, each of which has its correct position, and musicians (or divine soul-makers) can form their harmonic patterns from them by selecting some and omitting others. I do not pretend that my reconstruction of this approach will resolve all the difficulties, either in Plato or in the *Sectio*; but I think it does a good deal to ease them.

CLOSING REFLECTIONS ON THE *SECTIO* AND ITS TARGET READERSHIP

Propositions 19–20 can be understood only by readers armed with a degree of musical knowledge, though it need not be great. They must be familiar with the names of all the notes in the system, and be aware of some (not necessarily all) of the relations that hold between them in a 'diatonic' structure. They must know that the interval between *nētē* and *diatonos hyperbolaiōn* is a tone, for instance, that the interval between *mesē* and *nētē diezeugmenōn* is a fifth, and so on. But the construction calls for knowledge only of intervals whose ratios have previously been established, that is, the concords and the tone, and in particular it neither asserts nor presupposes anything about the interval between a *tritē* or a *parhypatē* and the lowest note of its tetrachord, which will in fact, given the rest of the construction, be a *leimma* in the ratio 256:243. Nothing in the *Sectio* suggests that the construction it describes specifies just one system within the general category 'diatonic'; it points only to the conclusion that in one familiar pattern of attunement the two upper intervals of each tetrachord were commonly referred to as 'tones'. It assumes that this designation is accurate, and that readers of the *Sectio* will be familiar with it. Aside from that, they need know only the position of each named note in the tetrachord to which it belongs, and the (concordant) relations between the tetrachords themselves.

The exercise seems designed, in part, to show how the findings of propositions 12–13 (no others are needed) can be used to give a mathematical interpretation of a complete and well-known musical structure. But that could have been done, on the basis of the physical theory outlined in the introduction, without any reference to a string or to lengths on the *kanōn*. It would simply have specified the system's pattern of intervals as ratios, representing these, as in earlier propositions, as ratios between pitches rather than as ratios between lengths. To put the point another way, the writer could have continued with his original, arithmetical approach (see pp. 380–1 above), instead of shifting into a geometrical mode.

The way he actually goes to work has one obvious advantage. When the relations are constructed geometrically, on a monochord, the system of notes they define can be presented in its musical guise, to the ear as well as the mind.[72] Hence the claim that these mathematical manoeuvres produce the required musical results can be confirmed empirically, and their confirmation will give strong support to the thesis that these mathematical relations are responsible for the system's musical credentials. It would be misleading and anachronistic to suppose that the application of the mathematical formulae to the monochord was undertaken in an 'experimental' spirit, allowing mathematical hypotheses to be subjected to empirical tests. No harmonic theorist before Ptolemy seems to have envisaged the role of 'laboratory instruments' in this way. They were thought of rather as audiovisual aids, whose use would clarify the truths of mathematical harmonics

[72] There are of course practical difficulties to be overcome. Ptolemy argues that while the monochord is suitable for showing how concordant intervals correspond to ratios, it is unsatisfactory when used to display anything more complex, such as the mathematical anatomy of an entire *systēma*. See *Harm*. 66.5–69.12, especially 69.1–5.

and help learners to grasp them. The truths are presented, as it were, to the eyes on the *kanōn* and to the ears through their expression in sound, as well as abstractly to the intellect. There is no suggestion in any pre-Ptolemaic source that the audible results might put the mathematical conclusions at risk.[73]

The division is based, mathematically, on earlier theorems, but there is a logical gap between them which the *Sectio* does nothing to fill. The theorems' transition from pure arithmetic to harmonics is mediated by the physical acoustics of the introduction, which legitimises the treatment of pitch-relations as ratios of numbers. If a note at one pitch is made up of n impulses in a given time, and a note at another of m impulses; the relation between them is the ratio n:m. By itself, however, this does not justify the assumption, essential to propositions 19–20, that the same ratio will be reflected in the relation between the lengths of string from which the notes are produced. As later theorists often point out, the terms of the ratio have to be reversed when one shifts from ratios of pitches to ratios of lengths; the higher pitch, with its supposedly more rapid sequence of impacts or its greater speed of transmission, will attract the larger number, but it is produced by the smaller length. Thus n:m becomes m:n.[74] But this is not the main problem, which is that the *Sectio* has made no effort to show that ratios between pitches are correlated with ratios between lengths in any way at all.

The case for the correlation could evidently be argued on the basis of familiar observations of a kind apparently relied on in early Pythagorean musicology (see pp. 26–7 above). The fact that a string, stopped half-way along its length, produces a note an octave above that sounded by the whole string could also be linked intelligibly with the *Sectio*'s physics. The greater length oscillates more slowly and its impacts are more widely spaced in time; given the physical theory and the observations of a string's behaviour, one could infer that the half-length oscillates twice as rapidly as the whole, and similarly for the other relevant relations. But the *Sectio* is silent about such matters. It seems to take it for granted that its readers are as familiar with these observations as they are with the elementary musical data, and that they are capable of drawing the inference for themselves.

Readers of the treatise must therefore be musically literate, to the extent I have specified; they must have a background in mathematics, since it is assumed that they have mastered propositions not proved in the *Sectio* itself;

73 See for instance Theo Smyrn. 57.11–58.12, Nicomachus, *Harm.* ch. 10, and cf. Barker 2000a, chapters 10–11.

74 See for instance Theo Smyrn. 87.9–18, quoting Thrasyllus.

and they must be aware of the links that have been identified in the past between relative pitches and relative lengths of a string. The writer can also assume that they will recognise an implicit allusion to the monochord, and will understand what it is without being told; he mentions a string and the *kanōn*, but gives no description of the instrument to which they belong.[75] Rather few Greeks in 300 BC would have satisfied all these conditions. The *Sectio*, it seems, is appealing to a restricted and specialised audience.

In matching concordant intervals and the tone with the appropriate ratios (propositions 9–13), and in using the correlations to construct a musical *systēma* mathematically (propositions 19–20), the *Sectio* is doing nothing new. The fundamental ratios were known to the fifth-century Pythagoreans; the principles needed for a comparable construction procedure are at least embryonically present in Philolaus and are fully exploited in Plato's *Timaeus*. Some or all of the propositions about inequalities (propositions 14–16, 18) can persuasively be associated with Archytas and were known to Aristotle and Aristoxenus; the method of constructing a ditone which is used in proposition 17 was used also by Aristoxenus, and was probably common among practising musicians. What the *Sectio* contributes is its orderly, formal demonstration of these propositions. Its theorems about musical intervals are systematically arranged, and are dependent in their turn on a coordinated sequence of theorems in pure arithmetic; and they are also neatly integrated with a well-argued exposition of a theory about the physics of sound and pitch. Thus the harmonic propositions become assimilated into a nexus of interlinked explorations in several scientific domains. The *Sectio* is an attempt not only to establish propositions in mathematical harmonics on firm, rational foundations, but also to locate this science among others, fitting it smoothly into their multifaceted enterprise of investigating and understanding the world.

The fact that there are gaps and minor uncertainties in the *Sectio*'s exposition, and one crucial and irremediable logical error, scarcely detracts from its overall coherence or from the excellence of its aspirations. A more cogent general criticism would focus on its limitations. The harmonic theorems seek to demonstrate the ratios of only the most elementary musical relations; they mark off this science's findings from those of empirical harmonics at

[75] The earliest secure reference to the monochord dates from the late fourth century (Duris frag. 23). This passage, like the *Sectio*, mentions it without explaining what it is, and it implies that it had been in use for some time. See the discussion in Creese 2002: 22–5. But even if that is so, it had no public profile. There is no record of its use in musical performance until Roman times (Ptol. *Harm.* II.12); it served only as an accessory to scientific discussion, and only those exposed to such studies are likely to have known about it.

only the most obtrusive points of disagreement; and the concluding division describes only one pattern of attunement among many, from a mathematical perspective the simplest of all. In setting out the division it makes no attempt to ground its form in mathematical principles which might explain why it falls into this pattern rather than any other; it offers no counterpart of the theory of proportions which governs the analyses of Archytas and Plato. In these respects it seems primitive even by the standards of the earlier fourth century, and a comparison with the work of Aristoxenus shows immediately how large and complex a collection of musical relations and structures remains unexplored.

'Primitive', however, is probably the wrong word. There is no good reason to think that the writer was stuck in a time-warp, unaware of ways in which the science had moved on since the time of Philolaus; fairly clearly he was not. A better diagnosis is that he set himself only a determinate and restricted goal. His task was to secure firm foundations for mathematical harmonics, rather than to construct upon them an elaborate edifice to rival Aristoxenus'. That might indeed be a legitimate enterprise, but only when the masonry at the base of the building had been solidly cemented in place. It is notable, too, that the bed-rock upon which he sets it shows no trace of contamination with the deposits of Pythagorean or Platonist metaphysics. Certainly he has borrowed arguments and constructional procedures from those sources, directly or indirectly; but he has relocated them in the setting of a more hard-nosed, more nearly positivistic mathematics and science. In this he differs sharply from his most eminent predecessors,[76] as also from most of those successors who revived mathematical harmonics in the Roman period.

[76] It seems likely, however, though it cannot be proved, that in this respect his approach had precursors among the exponents of mathematical harmonics mentioned by Aristotle in the *Posterior Analytics*.

CHAPTER 15

Quantification under attack: Theophrastus' critique

Theophrastus was born in about 370 BC, succeeded Aristotle as head of the Lyceum in 322 BC and died, full of years, early in the second decade of the third century. Like several other Peripatetics of his generation he wrote copiously; Diogenes Laertius (v.42–50) lists the titles of 224 works, many of them in several books, amounting in all to 232,850 lines and addressing an astonishing range of topics. Later writers refer occasionally to his interest in music, and though he can hardly be regarded as a specialist in the field, the works in Diogenes' catalogue include an *On Music* in three books, and a *Harmonics* and an *On Musicians* in one book each. We know little of their contents. The great majority of his writings are lost, or survive only in fragments quoted by others, and from his publications on music we have only a handful of scraps.[1] Only one of them concerns us here. According to Porphyry, the source who quotes it, it comes from the second book of Theophrastus' *On Music*. It is by far the longest of the fragments on musical topics, running to 126 lines in Fortenbaugh's edition.[2]

The passage begins with a reference to a *kinēma melōidētikon*, a 'movement productive of melody' or a 'melody-making movement' which occurs in the soul. Right at the end of the fragment (130–1) Theophrastus asserts, more bluntly, that the 'nature' of music *is* a movement of the soul, a movement designed to release it from 'the evils caused by the emotions'. The precise relation between these statements is debatable, but the central point is that music and melody originate inside us. They arise from a movement intrinsic to the soul and consist, essentially, in such a movement; when they appear in the public domain as patterns of sounds, these must therefore be secondary manifestations of music, dependent on the first.

1 The fragments of Theophrastus' writings are collected in the two volumes of Fortenbaugh 1992; the passages on music are in the second volume, fragments 714–726C.

2 The work from which it is taken is named at Porph. *In Ptol. Harm.* 61.18–20, with the quotation itself at 61.22–65.15; frag. 716 Fortenbaugh. I shall cite it by the line numbers of Fortenbaugh's edition, which includes Porphyry's brief introductory remarks; the excerpt from Theophrastus begins at line 7.

Theophrastus' opening sentence draws attention to the remarkable accuracy with which the soul brings about the transition from psychic movement to audible sound, when it seeks to express its *kinēma melōidētikon* by means of the voice.[3] The soul 'turns' or perhaps 'steers' (*trepei*) the voice just as it wishes, 'to the extent that it is capable of steering something non-rational' (7–9).[4] Talk of the soul and its movement may be alien to modern minds, and a crude paraphrase of his assertion which strips it of its metaphysical associations will no doubt distort his meaning. But he has a point whose force we can easily recognise. A few unfortunates are 'tone dumb', as a friend of mine used to put it (or 'crows' in the patois of a teacher whose tyranny I endured over fifty years ago); but most of us, when we have a tune in our heads, can steer our voices around it with at least tolerable accuracy, singing just the notes we are imagining and no others. Soberly considered it is an extraordinary trick, and one that we perform without having the least idea how we do it.[5] It cries out for an explanation.

Some people, Theophrastus continues, tried to account for the soul's accuracy by reference to numbers, 'saying that the accuracy of the intervals arises in accordance with the ratios of the numbers' (10–12). He goes on to record their view that the ratio of the octave is 2:1, that of the fifth 3:2 and that of the fourth 4:3, and that 'for all the other intervals, in the same way, just as for the other numbers, there is a ratio peculiar to each. Hence they said that music consists in quantity, since the differences exist on the basis of quantity' (12–16).

Here we are on familiar ground, but the use to which, so Theophrastus implies, these theorists put their quantitative conceptions is one we have not previously met. There are, in fact, no other fourth-century allusions to the idea that the soul's ability to create an exact audible counterpart of its inner movements (or, more prosaically, our ability to sing the notes we intend) depends on the existence of precise, quantitative relations between the notes of a melody; and so far as I know there is only one in later literature.[6] The

[3] *Hermēneuein* means 'to express' or 'to interpret'. The related noun *hermēneia* is commonly used of a musician's 'performance' of a composition, and that sense is implicit here in the verb; the soul seeks to give voice to its silent melody in audible performance.

[4] 'Non-rational' translates *alogon*, 'without *logos*'. Sicking 1998: 107–8, is probably right to find a double meaning in it (I am less convinced by his third interpretation); vocal sound is 'something "irrational" and therefore "unmanageable"', and in Theophrastus' view, as we shall see, it is also *alogon* in the sense that it lacks any connection with the ratios (*logoi*) by reference to which some theorists account for the accuracy to which Theophrastus has alluded.

[5] The same point is made by Ptolemy, at the end of his brief account of the physical operations involved in singing. He is not given to exclamatory rhetoric; but he describes the capacity of the soul's ruling part to pick out and execute the appropriate physiological manoeuvres as 'astonishing' (*Harm.* 9.12).

[6] It appears in the same passage of Ptolemy as the remark mentioned in n. 5 above, *Harm.* 9.6–15. Ptolemy's theory is that the pitch of a vocal sound is determined by the distance between a point

thought seems to be that the soul identifies the relations between these notes through a kind of subconscious mental arithmetic, and reproduces them in perceptible form rather in the manner of a craftsman working from a set of measurements, transferring the results of its computations first to some quantifiable feature of the body's vocal apparatus, and from there to the medium of sound.

I do not know whether any fourth-century writer worked out any such hypothesis in detail, and from one point of view it hardly matters. Theophrastus will have none of it, and devotes the bulk of the passage to elaborate arguments designed to refute it. But he says very little more about its psychological aspects, or about the process through which inner movements or imagined melodies are made audible. All his criticisms in this part of the fragment are aimed at the familiar underlying theory that differences in pitch are essentially quantitative, and that each musical interval corresponds to a ratio of numbers. In the course of the discussion we learn a little about his own, strictly qualitative conception of the distinction between high pitch and low; and towards the end he turns his fire briefly on another quantitatively based approach, which seems – at least at first sight – to be that of theorists in the 'empirical' tradition which culminated in Aristoxenus. If that reading of the relevant lines is correct (which will turn out to be questionable), and if his arguments hold water, he will have undermined the basis of every variety of harmonic analysis known in his time.[7] His critique seems to have made little impact on later harmonic scientists. Porphyry, who was clearly impressed by its reasoning, is the only writer who mentions it; and when he says that though many others agree, 'I cannot list them all by name since I am not in possession of their writings,' and that 'Theophrastus will be adequate to stand for me in the place of them all' (1–3), one cannot help suspecting that the 'many others' are

in the windpipe from which an impulse of breath is thrown towards the mouth, and the point at which the impulse 'strikes' the outer air. The latter point is fixed; the former is shifted closer to it or further away through the agency of our 'ruling principles', which 'find and grasp astonishingly and easily, as though with a bridge [i.e. as on a monochord], the places on the windpipe from which the distances to the outer air will produce differences of sounds in proportion to the amounts by which the distances exceed one another'. It is possible that Ptolemy found this theory, like much else in *Harm.* 1.3, in a work of the fourth or third century BC; but there is nothing to prove it.

7 It is a great pity that his *Harmonics* is lost. Unless it consisted wholly of criticisms of existing approaches to the science, we might expect it to have sketched out a mode of analysis in which quantitative descriptions and comparisons of intervals play no part. The state of the evidence, unfortunately, offers little scope for informed speculation about its procedures or conclusions, or about the terms in which its analyses were expressed. I make a few tentative suggestions at the end of this chapter. For an interesting attempt to set Theophrastus' views on music within a wider philosophical framework see Lippman 1964: 157–66.

a mirage.[8] The passage is nevertheless important. It shows that the whole project of harmonic science, as it stood at the end of the fourth century, could be subjected to critical attack; its status as a reputable discipline was by no means assured. Even if Theophrastus' assault was unprecedented and waited six hundred years to find its first enthusiastic convert, it clearly deserves our attention.

Theophrastus' arguments are complex and often obscurely expressed; there are uncertainties both about their meaning and about the exact nature of the views which some of them are designed to refute. Along with several other commentators, I have tackled these and other minutiae elsewhere, and though we shall have to grapple with some of them in the course of this chapter, a good many will be elided.[9] Instead of following the passage through in precisely Theophrastus' order, I shall consider it under four headings, looking first at one group of arguments against mathematical theorists, then at another, thirdly at arguments apparently directed against an empirical, perhaps Aristoxenian approach, and finally at such hints as we have about Theophrastus' own views, most of which appear in the text just before the end of the second group of arguments against the *mathēmatikoi*.

ARGUMENTS AGAINST MATHEMATICAL THEORISTS: THE FIRST PHASE

The whole of Theophrastus' reasoning seems to be premised on the assumption that attributes fall into two types, quantitative and qualitative, and that these categories are mutually exclusive.[10] The position he initially attacks is encapsulated in the thesis attributed to 'some people' in lines 15–16, that 'music consists in quantity, since the differences exist on the basis of

[8] There are some affinities with Theophrastus' position, perhaps even some distant echoes of it, in a passage from an otherwise unknown mathematical writer, Panaetius the Younger, quoted by Porphyry immediately after the Theophrastan fragment (*In Ptol. Harm.* 65.21–67.10). But Panaetius' views seem a good deal less radical than Theophrastus'.

[9] See Barker 1978b and 1985, and the notes to Barker 1989a: 111–18. There is a valuable, more recent discussion (which includes trenchant comments on some of my earlier opinions) in Sicking 1998. For an older, more summary but still useful account see Lippman 1964: 157–61; cf. also Gottschalk 1998: 293–5.

[10] The terms regularly used in later writings to mark this contrast, *posotēs*, 'quantity', and *poiotēs*, 'quality', are apparently fourth-century philosophical coinages. In Aristotle the former occurs frequently, the latter less often; at *Theaetetus* 182a8–9 Plato indicates that *poiotēs* is an unfamiliar, 'uncouth' expression, perhaps one that he himself has just invented. In the present passage Theophrastus uses *posotēs* freely, but *poiotēs* does not appear; instead we find *idiotēs*, meaning something like 'characteristic property', and designating e.g. the redness of red things or the fluidity of liquids. This word too is rare before the third century (it occurs only once in Plato and not at all in Aristotle), and no earlier writer uses it as Theophrastus does, to contrast with *posotēs* and to refer specifically to non-quantitative attributes.

quantity'. These 'differences', as the context makes clear, are differences in pitch; and the issue on which Theophrastus focuses is whether, when one note is higher or lower in pitch than another, they differ quantitatively.[11]

The mathematical theorists' view leads, so he argues, to absurd consequences. Suppose first that the difference between a high note and a low one is a difference of quantity, just that and nothing else. In that case a difference of quantity, simply as such, must amount to 'a melody or part of a melody' (16–20). It will follow that any two things that differ in quantity, two different-sized patches of colour, for example, will differ in precisely the way in which two notes do, and they too will form 'a melody or part of a melody, if indeed melody and interval are number, and if melody and the [kind of] difference involved in it exist because of number' (20–3). In general, on this hypothesis, 'everything numerable would participate in melody to the extent that it does in number' (25–6). If, for example, I have twelve apples, nine peaches, eight pears and six bananas in my fruit-bowl, it would not merely be the case that the apples:peaches:pears:bananas ratio-pattern mirrors that of an octave divided into two fourths and a tone, but that I have a bowlful of music, humming along on the four fixed notes of a familiar attunement. Which is, alas, ridiculous.

Pitch-differences, then, cannot be merely differences of quantity. Theophrastus now shifts the angle of his attack, homing in on the more plausible idea that they might be quantitative differences attached to subjects of a particular sort, those that are sounds or notes. Suppose, he says, that 'plurality belongs to notes in the same way as it does to colour', in the way, that is, in which the redness of a set of red patches is one thing and anything quantitative about them (such as their number or size) is something quite different. On this approach 'a note is one thing and the plurality related to it is another' (26–7), and we are presented with two possibilities. One is that a high note and a low note differ 'as notes', that is, in some aspect of the feature that makes them notes rather than anything else, just as the colours of a geranium (red) and a delphinium (blue) differ in some

[11] He expresses the view he is criticising in various ways, representing it sometimes as saying that the difference 'depends upon' quantity, sometimes that it 'is' a quantity, sometimes that the items in question differ 'in quantity' (all these locutions can be found in lines 16–21). I do not think that the various forms of words mark significant differences in meaning. Nor do I agree with Sicking 1998: 112, that the shift of terminology in lines 22 ff., where he begins to speak of the thesis that an interval (*diastēma*) is a number (*arithmos*) or a plurality (*plēthos*) indicates that his attention is now directed to a different theory. Lines 17–26 form a single, continuous argument, and despite the variations in Theophrastus' formulations its target is the same throughout. Lines 26–49 pick up the point made in 17–26 and develop out of it a new line of criticism; but here again there is still (in my view) no change in the position under attack. Some reasons for adopting this interpretation will be offered below.

aspect of the feature that makes them colours. Alternatively, they differ 'in respect of their plurality'; their pitch-difference is a relation between quantitative attributes that they possess over and above the qualitative *idiotēs* which constitutes them as notes or sounds (27–9).[12]

These alternatives provide Theophrastus with the makings of a dilemma, on one horn or the other of which the theorists under attack must, so he argues, impale themselves. Suppose first that differently pitched notes differ 'in plurality, and the higher is as it is by being moved in respect of more numbers, the lower by being moved in respect of fewer'. But 'every sound is such as to be grasped either as high or as low, for every sound is higher than some other and lower than some other'. That is, the possession of pitch is essential to anything's being a sound; and if the characteristic of pitch is 'removed', by being treated as a quantitative attribute over and above the specific *idiotēs* of sounds and notes, 'what would be left over,' Theophrastus rhetorically demands, 'to make it a sound?' (29–36). We cannot abstract a sound's pitch from its character as a sound, in the way that we can abstract the size of a colour-patch from its character as a colour, for example; there will not be enough left of the subject to constitute it as a sound at all.

We move, then, to the second horn of the dilemma, and posit that high notes and low ones differ from one another 'as notes'. Here the argument is straightforward, since in that case 'we shall no longer have any need of plurality, for their own intrinsic difference will be sufficient by itself to bring melodies into being'. We are no longer supposing that pitch-differences depend on plurality or quantity; they depend, *ex hypothesi*, on the *idiotēs* of the sounds, and one note will differ from another in pitch just as one colour differs from another in hue, regardless of the quantities involved. If equal amounts of white and black are brought together, their 'numbers' do not differ, but their distinct characters as black and white remain. Just so, on this hypothesis, any quantitative attributes that notes may acquire on particular occasions are irrelevant to their intrinsic characters as high-pitched or low (36–46). Thus if we choose the first horn of the dilemma we reach an impossibility, while if we choose the second the quantitative account of pitch evaporates.

These intriguing and ingenious arguments are scarcely watertight, but it is not my business to criticise them here. It is more important to try

[12] Theophrastus shifts back and forth in this context between speaking of notes (*phthongoi*) and of sound (or 'vocal sound', *phōnē*). It is clear that he intends no distinction that is relevant to his argument, since he treats a note, for present purposes at least, simply as a pitched sound, and asserts at 32–3 that every sound (*phōnē*) is higher than some other and lower than some other; every sound, in short, or at least every sound that he is considering, is a pitched sound and hence a note.

to identify the thesis they are intended to demolish. They characterise it as one that makes pitch-differences depend on quantity or plurality or number; and these various, apparently interchangeable forms of expression, together with the very abstract character of the reasoning, suggest that the position under attack is conceived in rather broad and general terms. It is, perhaps, any kind of acoustic theory whatever that might serve to underpin the representation of pitch-relations as ratios of numbers. As he comes to the end of this stretch of argument, however, Theophrastus seems to indicate that he has either one quite specific version of such a theory in mind, or two, depending on how we read his remark. Looking back with evident satisfaction at the results of his reasoning, he concludes: 'thus the high-pitched sound does not consist of more numerous [parts] or move in accordance with more numbers, and neither does the low one' (46–7).

I have supplied the word 'parts' in the first half of this sentence, but it is hard to see how anything else could be meant. If it is correct, the thesis that is being denied must be that differently pitched notes are composed of different numbers of elements; and the only theory we know that carries such an implication is the one enunciated in the introduction to the *Sectio canonis*, where each note is formed from a number of impacts on the air, and a higher pitch arises from a more rapid and closely packed sequence of such impacts. We need therefore to ask whether Theophrastus is alluding to a different theory in the second half of the sentence, where he denies that a higher-pitched sound 'moves in accordance with more numbers', or is merely referring to the same one again in different terms. If Porphyry had broken off his quotation here, we could have been forgiven for supposing that the phrase refers, though rather obliquely, to the best-known theory of pitch current in the period, the theory adopted by Plato and (with some modifications) by Aristotle, that higher pitch is correlated with the greater speed of a sound's transmission through its medium. 'A higher-pitched sound moves in accordance with more numbers' might plausibly be taken as a roundabout way of saying 'a higher-pitched sound moves faster'. But in fact this cannot be so, since later in the fragment (lines 101–3) the thesis that a lower note 'moves in accordance with fewer numbers' (and by implication, that a higher note moves in accordance with more) is quite explicitly distinguished from the view that 'the high note is distinguished by its speed'; and the two are refuted separately and by different arguments.[13]

[13] Hence I cannot agree with Sicking 1998: 124 (cf. 115), when he asserts that the expression 'moves in accordance with more numbers' 'covers both the vibration theory and the speed theory', or at any rate not in the present context.

Since no third interpretation immediately suggests itself for the phrase 'moves in accordance with more numbers', we might conclude that like the phrase 'consists of more numerous [parts]', it must allude to a theory of the sort spelled out in the *Sectio*; and we might go on to revise our reading of the whole sequence of arguments we have just reviewed. Perhaps it is aimed exclusively at a thesis of that one particular type. But this interpretation also fails. If the next stretch of argument is directed against any one specific theory, it is certainly not that of the *Sectio*, as we shall see. Yet the formula 'moves in accordance with more numbers' reappears again during the course of it (line 69). Further, the way in which Theophrastus negotiates the transition from the arguments we have reviewed into the next, at line 50, clearly indicates that he does not mean us to think that the object of his criticism has changed. Lines 46–9 read as follows. 'Thus the high-pitched sound does not consist of more numerous [parts] or move in accordance with more numbers, and neither does the low one; for one can say this of the latter as well as the former, since there is a characteristic magnitude (*megethos*) that belongs to a low-pitched sound.' Line 50 introduces the new phase of the critique by saying: 'This is clear from the force exerted when people sing.' Hence the contention that a sound of any pitch has a 'characteristic magnitude' is used, in successive lines, first to help undermine the thesis that either high-pitched or low-pitched sounds possess more numerous parts or move in accordance with more numbers, and then to show the untenability of a theory which cannot be identical with the *Sectio*'s.

I shall argue that this new theory is not the 'speed' hypothesis either. But whether it is or not, the points I have raised suggest that the best interpretation of Theophrastus' strategy, both in the first phase of his polemic and (in view of the form of transition) in the second, is, after all, the one I initially sketched. Though at certain points in the discussion he is apparently thinking more of one type of theory than of another, he intends each of his arguments to cut against all of them equally.

ARGUMENTS AGAINST MATHEMATICAL THEORISTS: THE SECOND PHASE

There are arguments of two main sorts in this part of the text (50–80, together with a few lines combining criticism with positive theorising at 100–7). Both are essentially quite simple. In the first (50–64), Theophrastus draws attention to observable facts (or alleged facts) about the power or force required to produce high and low pitches by various different means. He argues that in singing and in playing a stringed instrument, the quantity of power needed to produce a note is the same regardless of its pitch, while in playing a wind instrument it is the lower pitch whose production calls for the greater power. (This last observation provides him with appropriate ammunition because all fourth-century theories of the type under examination associate greater numbers or quantities with higher pitches.) The second argument focuses on the phenomenon of concord. Summarily, if two notes form a concord they blend seamlessly together, and what we hear is the product of their fusion (see pp. 344–5 above); but if one were more forceful than the other it would overpower the weaker and would 'appropriate the perception for itself', so that there could be no experience of a fully blended, unified concordance (64–80).

In both of these arguments the larger numbers or pluralities or quantities of the earlier passage have been replaced by greater power (*dynamis*) or force (*bia*). If any specific theory is now under assault, it is therefore one that correlates higher pitch with the greater vigour of the action needed to produce it, and hence, we may infer, with greater vigour in the movement that constitutes the sound itself. This inference is supported by remarks in the second part of the argument. The theorists in question say, we are told, that 'the higher note is heard at a greater distance, through its travelling further because of the sharpness (*oxytēs*) of its movement, or because of its arising as the result of plurality' (69–72); and from the fact that it is capable of travelling further it is argued that it must be more vigorous (*sphodroteros*, 75–6).

We might expect Theophrastus to say that according to this theory, the higher note travels more swiftly, but in this phase of the discussion he does not. It is of course very probable – indeed virtually certain – that any theorist who associated higher pitch with greater force and power would link it also with greater speed, but we should not ignore the fact that Theophrastus has chosen, so far, to suppress that aspect of the matter, and so to obscure any connection between this theory and the views of Plato and Aristotle. (Plato says nothing about force or power in the relevant passage at *Timaeus* 79e–80b. Aristotle denies the existence of a direct correlation between higher pitch and greater force, in a complex argument at *De gen. an.* 786b–787a, and unless Theophrastus has misunderstood him badly, it cannot be his position that is under attack here.[14]) The only known fourth-century writer who correlates higher pitch directly and explicitly with the greater force exerted by an agent and with the strength of a sound's movement through the air is Archytas, in frag. 1 (see pp. 27–9 above); he also connects it with

[14] For the contrary view see Sicking 1998: 120. It may be my fault, but I fail to follow his reasoning.

greater speed, but speed figures much less prominently in his argument. In other respects too, Archytas' thesis fits smoothly into the target-area of Theophrastus' criticisms. He insists, for example, and Theophrastus denies, that a higher-pitched sound is heard at a greater distance because it travels more strongly and vigorously; and he offers an account of the forces involved in playing high and low notes on the aulos which Theophrastus confronts point by point.

If Theophrastus is deliberately criticising any one particular account of the phenomena, then, it must be that of Archytas (unless, of course, he found it in a source unknown to us), and I have no real doubt that the passage we know as Archytas fragment 1 is at the centre of his attention. But once again there are signs that he intends his comments to have wider application. We saw that the language of the first phase of his argument (lines 17–49) points the finger at a theory akin to that of the *Sectio*, if anywhere; and it certainly does not suggest the Archytan hypothesis. Yet two of its key expressions recur in the present passage, as if the criticisms developed here apply to the thesis or theses they designate just as much as they do to Archytas' position. Theophrastus' contention that two notes which jointly form a concord must be equal in power prompts his rhetorical query, 'for if the high note moved in accordance with more numbers, how could consonance come about?' (66–9, cf. 100–2); and the thesis that a higher note arises 'as a result of plurality' is represented as an alternative to the hypothesis that its movement is 'sharper', one that has relevantly identical implications and is refutable in the same way (69–78). It apparently makes no difference whether the position under bombardment is designated in ways that bring to mind Archytas' approach or some other; the same arguments will demolish it regardless of the language in which it is dressed.

If we suppose that Theophrastus was concerned to distinguish carefully between different types of mathematical theory and to refute them one by one, we shall be hard pressed to explain his treatment of the 'speed' hypothesis adopted, in one variant or another, by Plato, Aristotle and many later writers. It must have been very familiar to members of the Lyceum and associated intellectuals who took an interest in such matters, and it bore the *imprimatur* of the period's most respected philosophical heavyweights. As the dominant hypothesis in the field, it merited careful and extensive critical attention from any reputable opponent, certainly no less than the Archytan theory of force, and only secondarily speed, which it had largely displaced, or the theory of multiple impacts. Yet it surfaces explicitly in Theophrastus' polemic only once, and is dismissed in two short sentences. 'But neither can the high note be distinguished by its speed, for then it would occupy the hearing first, so that a concord would not arise. If it does arise, both [the high note and the low] are of equal speed' (103–5).

It is also relevant that at this point Theophrastus moves to an apparently much more general conclusion. In the preceding lines he has argued first that a lower note does not 'move in accordance with fewer numbers' (100–3), and then, in the sentences quoted above, that a higher note is not 'distinguished by its speed'. 'Hence,' he concludes, in a statement evidently designed to embrace both formulations, 'it is not some unequal numbers that give the explanation of the differences' (105–6). Similarly, as he reaches a peroration near the end of the fragment, he sums up his findings about mathematical theories by saying 'nor are the numbers causes, by the notes' differing from one another in quantity' (125–6). These generalised and capacious formulations seem designed to capture any theories whatever of the mathematical sort. The distinctions between them are unimportant, since each of the arguments that Theophrastus levels against them is intended to be sufficiently broad and flexible to count against them all. In that case he had no need to direct a special barrage of criticism against the 'speed' hypothesis, for all its prominence in contemporary thought. It had been eliminated along with the others, by the cumulative onslaught of his whole succession of arguments. The fact that with one brief exception he avoids formulations that would draw particular attention to Aristotle's theory might possibly be explained in personal or social terms, as prompted by *pietas* or prudence. Since his strategy did not require him to tackle head-on the views of his senior colleague and friend, even though he clearly disagreed with him, he chose to present his critique in a less obtrusively provocative guise. It would be pleasant to believe such a story, but we know almost nothing of the roles, if any, that were played by deference and tact in Aristotle's circle. It is only an agreeable speculation, though it might, after all, be true.

AN ARGUMENT AGAINST ARISTOXENUS?

The short passage we shall consider here (108–25) comes near the end of the fragment, when Theophrastus' critique of mathematical theories is complete, and immediately after an exposition of some positive views of his own which we shall examine in the next section of this chapter. It returns to the critical mode, but it is clearly directed at a new set of opponents. They too construe the relations between pitches in quantitative terms, but not in the manner of those who represent them as ratios. These people assert, Theophrastus says, that the causes of pitch-differences and of *emmeleia*, 'melodiousness', are intervals, *diastēmata*; and when the argument is over he sums up the conclusions of the whole passage preserved in the fragment by saying that the *diastēmata* are not the relevant causes, and 'neither are the numbers causes, by the notes' differing from one another in quantity'. Since this latter statement plainly encapsulates his conclusions about all mathematical theories, the thesis about *diastēmata* must come from a different stable.

Previous commentators, among whom I include myself, have generally identified the view criticised here as that of Aristoxenus, or of the empirically minded predecessors he calls *harmonikoi*, or perhaps of both. I shall argue, however, that unless Theophrastus' comments are very far from the mark indeed, this interpretation needs to be radically qualified. It is of course very tempting to assume that the theory comes from Aristoxenus or the earlier empiricist tradition, since we know of no other school of thought that could appropriately be contrasted with that of the number-theorists; and I accept that in a certain sense the assumption holds. But I shall try to show that Theophrastus' target cannot be Aristoxenus, and that although he may well have believed that he was engaging with the views of earlier empiricists, he relied on only one source for his information about them, one that presents a very peculiar and probably a misleading picture of their approach.

The passage begins as follows. 'Nor is it the intervals (*diastēmata*), as some people say, that are the causes of the differences and hence their principles (*archai*), since even when these are left out the differences remain' (108–9). The 'differences', here as in the earlier arguments, are those between differently pitched sounds or notes; and one might offer Theophrastus' statement a straightforward and moderately plausible interpretation. The distinct pitch of a note cannot be dependent on the intervals or 'distances' by which it is separated from other notes, since it retains that pitch even when it is sounded in complete isolation and all such *diastēmata* are 'left out'. But as the passage unfolds it becomes obvious that this cannot be what Theophrastus means.

For when something comes into being if certain things are left out, these are not the causes of its existence, not [that is] as productive causes, but [only] as not preventing it. For neither is the unmelodic (*ekmeleia*) a cause of the melodic (*emmeleia*), merely because the melodic would not come into being unless the unmelodic were rejected . . .[15] so that neither are the intervals the causes of melody as producing it, but as not preventing it. For if someone were to utter in addition

[15] The sense of the lines omitted here (112–15) is uncertain and the integrity of the text is debatable; see Sicking 1998: 130. On any reading, however, they will add little or nothing to the argument.

the continuous series of intervening positions, wouldn't he emit an unmelodic sound? So while the unmelodic would arise if these were not rejected, the melodic does not arise from their being left out, just because they would prevent it if they were not left out. (111–19)

The abstract theme running through this argument is that if something, X, cannot come into being unless certain other things, Y's, are left out or rejected, then the Y's are not the causes of X. That is, they are not its 'productive' causes, the items that actively bring X into existence; they are causes only in the very etiolated sense that so long as they are left out, they do not prevent its occurrence. The last sentence seems to make a rather stronger point. The fact that the omission of the Y's makes it possible for X to arise does not entail that their omission causes X's existence. Theophrastus apparently means that their omission is merely a necessary condition of X's coming into being, and that something else is needed as a 'productive' cause, to ensure that it actually arises.

In the case under scrutiny, the things that must be left out are the *diastēmata*. Since it is clear that Aristoxenus conceived intervals merely as empty spaces between differently pitched notes, and it seems likely that his predecessors did so too, the sense in which they must be omitted does not leap to the eye. Fortunately, Theophrastus explains what he means quite lucidly in the penultimate sentence. Melody and 'the melodic', *emmeleia*, can exist only when notes are distinctly pitched and the differences between them are clear. If a singer produced audibly not only the relevant notes but also 'all the intervening positions', the notes' pitches would be hopelessly obscured and the result would be unmelodic. What has to be left out between any two musical notes is not, then, the intervening *diastēma* conceived as an empty space, but what we might call its 'content' or 'potential content', that is, the continuous gradient of pitched sound by which the space could be filled.

But it is hard to find a theorist in the empirical tradition who identifies the *diastēmata* with their potentially audible content. Certainly Aristoxenus does not. On his account, 'a *diastēma* is that which is bounded by two notes which do not have the same pitch. To put it straightforwardly, an interval presents itself to perception as a difference between pitches, and a space capable of receiving notes higher than the lower of the pitches that bound the interval and lower than the higher' (*El. harm.* 15.24–31). It is a 'space capable of receiving notes', a *topos dektikos phthongōn*, and it is absolutely not a sound or a continuum of audible pitches. Further, in an elaborate passage at *El. harm.* 8.13–10.20 (see pp. 143–5 above), Aristoxenus argues at length for precisely the thesis that Theophrastus is asserting here, that in singing melodically the voice must sound only the notes bounding the intervals, and must pass silently across the intervals themselves. It therefore seems much more likely that Theophrastus is borrowing from Aristoxenus to fuel his polemic than that he is attempting to criticise him; and in due course we shall find other indications too that he is in Aristoxenus' debt. Nor does Aristoxenus suggest, as those attacked by Theophrastus apparently do, that the *diastēmata* are to be thought of as the 'causes' of differences in pitch. They are merely the spaces between pitches, and play no active role in bringing them into existence. Aristoxenus seems wholly uninterested, in fact, in causal issues of the sort that Theophrastus is addressing.

Nothing in Aristoxenus encourages the view that the predecessors he discusses adopted the strange position conjured up by Theophrastus, and neither does anything said by Aristotle about exponents of harmonics in its empirical form. Neither author tells us straightforwardly that these *harmonikoi* construed intervals precisely as Aristoxenus did, as gaps that could be filled with sound but considered simply as intervals are not; but equally they say nothing to suggest that their view was relevantly different. Even when Aristotle writes of those who treated certain tiny intervals as minimal, and regarded them as constituting units of which all other intervals are measurable as multiples, he does not say that they did so because they thought that there was no space left between their boundaries and that they were 'filled up' with pitched sound; so far as we can tell they were envisaged merely as the smallest 'spaces' or 'distances' that the ear could reliably identify.[16]

We need not suppose, however, that Theophrastus has arbitrarily fathered a ridiculous and imaginary theory on the unnamed objects of his criticism. There is one passage in earlier literature which does indeed seem to allude to a theory of this sort, and though it stands quite alone in this respect, it gains weight and apparent reliability from the high profile of the text and its author. It is in a part of Plato's *Republic* to which we have already returned more than once, in which Glaucon describes the activities of certain theorists to whom he mistakenly supposes that Socrates has just referred (this is the account subsequently embroidered by Socrates with his gruesome images of strings tortured on the rack, beaten with the plectrum, and so on). Glaucon describes these people as behaving absurdly,

[16] See e.g. Aristotle, *Metaph.* 1087b33–1088a7, where Aristotle says that such a *metron* may be indivisible either in form or 'in relation to perception'. There is no sense in which the small interval he calls the *diesis* is indivisible in form, and he must be treating it as an instance of the second sort; see pp. 349–50 above.

naming certain things *pyknōmata* and intently inclining their ears as if trying to detect a voice from next door, some of them saying that they can still just hear a sound in the middle and that this is the smallest interval, by which measurement should be made, while others disagree, claiming that the notes are already sounding at the same pitch; and both are putting their ears ahead of their mind. (*Rep*. 531a4–b1)

These people's quest is for 'the smallest interval, by which measurement is to be made', a project which we discussed in Chapters 1 and 2. Their procedure is to adjust the tensions of two strings until their pitches are so close together that any further adjustment would be heard as bringing them into unison; when that situation is reached, they suppose, the smallest *diastēma* has been found. It is thus to be identified by ear, not on the basis of any kind of argument; and it will therefore be the smallest that human hearing can detect, rather than being the smallest absolutely, in some theoretical sense. This is no doubt as it should be, since the crucial task is to locate an interval which harmonic scientists can pick out directly and use in practice as the unit of measurement.

All this makes sense of a sort, and it locates these theorists squarely in the empirical tradition to which Aristotle and Aristoxenus also refer. But there is one detail in Glaucon's depiction which marks it off from the others. Instead of representing the first group of these disputing investigators as saying that they can still just detect a gap or a space between the notes that the strings emit, he attributes to them the claim that they can still 'just hear a *sound* in the middle', where by 'in the middle' they must mean 'in between the notes in question'. Having said this, they add 'that this is the smallest interval, by which measurement should be made', where 'this' apparently refers to the sound just mentioned.[17]

It seems very strange that they should be claiming to detect a sound (*ēchē*) and not a gap, and if they are indeed identifying this sound itself as the 'smallest interval', that seems stranger still. But if Theophrastus had this passage and only this passage in his mind when he wrote his critique,[18]

[17] One can understand the statement in a slightly different sense, 'that this interval is the smallest . . .'; but on the syntactically most natural reading, 'this interval' will again refer back to the sound. The only way of escaping from this conclusion is to suppose that 'this' or 'this interval' refers to nothing explicitly mentioned before, and simply indicates the interval to which their manipulation of the strings has led them. That is possible, though more awkward. But whichever sense Plato intended, the puzzling reference to the sound remains; and if I am right in thinking that Theophrastus extracted the theory he attacks from this passage, he at least must have interpreted it as identifying the interval (*diastēma*) with the sound.

[18] There is one other passage which refers to a sound that lies between the boundaries of the minimal interval; according to Aristotle at *De sensu* 445a2, 'the note inside the diesis is undetectable'. But this does not carry the same implications as Plato's formulation; in fact it has quite the opposite sense, since Plato's theorists claim to be able to detect the 'sound in between'. The note (*phthongos*) in

he might well have taken it to mean that such theorists supposed the notes bounding an interval to be held apart, as it were, by the sound that occupies the space between them, and that it is this sound, and not an empty space, that constitutes the interval between them. If there is a sound between them, they are distinct notes, and if not, they must be identical in pitch. In that sense this 'sounding interval' can be thought of as the 'cause' of the notes' distinct pitches, as Theophrastus says. It is worth noticing that the word by which Glaucon refers to this sound, *ēchē*, is not the one normally used elsewhere for a musical sound (commonly *phōnē*) or a note (*phthongos*). Its range of application is rather more general than theirs, and can include any kind of audible noise. Theophrastus could plausibly have construed it here as indicating the 'unmelodic' continuum of pitch that makes up the 'content' of an interval.

In that case Theophrastus' criticisms will hit their target squarely enough. The target itself is extremely odd; but it would not really be surprising if, as I am suggesting, he extracted his picture of these theorists' conceptions from this passage and no other. Aristoxenus' discussions of them (assuming that they had already been written and had come to Theophrastus' attention[19]) offer no detailed analysis of their notion of a *diastēma*. Aside from Plato in the *Republic*, the only earlier writer to mention exponents of the empirical branch of harmonics is Aristotle,[20] and his allusions, again, are too slight and allusive to present a critic with a clear view of his quarry. One might expect such a critic to look for living examples of the approach he is criticising, not merely for other writers' reports about them. But it is entirely possible that no theorists of the empirical persuasion were at work in Theophrastus' own time, until their approach was revived and revolutionised by Aristoxenus.

question is not identified with the interval; the point, as Aristotle goes on to explain, is merely that we cannot identify the interval between any such note and either of the boundaries of the diesis; cf. pp. 349–50 above. There is nothing here to encourage an interpretation of the sort that Theophrastus indicates.

19 No firm conclusions can be drawn about the chronological relations between Theophrastus' *On Music*, from which the fragment is taken, and the various books of Aristoxenus' *Elementa harmonica*. It seems certain that Theophrastus knew something of Aristoxenus' work, both for the reason given on pp. 423–4 above, and because elements of his terminology and conceptual apparatus seem to echo those of Aristoxenus; I shall note a few of them below. It does not necessarily follow (though it seems likely) that he had read any part of the *El. harm.*, or even that it had been written; he may, for instance, have engaged in informal discussions with his colleague in the Lyceum or attended some of his lectures. Sicking 1998: 135–8, argues that Theophrastus' critique is directed against the Aristoxenus of *El. harm.* Book I, and that Book II, which was written later, incorporates an attempt to evade Theophrastus' objections. As I have explained, I do not think that the first part of this hypothesis holds water; and it seems to me, more generally, that Sicking's arguments cannot survive detailed exposure to Aristoxenus' text.

20 One should perhaps add the writer of the fragment preserved in *P. Hib.* I.13 (see pp. 69–73 above); but it says nothing to the point.

The fictional drama of the *Republic* projects its conversations back into the fifth century, and though the dates of Aristoxenus' *harmonikoi* cannot be conclusively fixed, most of them, too, seem to belong to that now distant period. In attempting to reconstruct their position, then, Theophrastus had precious little evidence to go on, and so far as the issues that interested him are concerned, much the fullest and most detailed account available was the *Republic*'s. His reliance on it is perfectly understandable.

His reconstruction hangs, however, on a strenuously literal interpretation of a single, seven-word phrase in Glaucon's speech. The thesis he extracts from it is justifiable to the extent that it does not distort the plain meaning of the text; but it is so bizarre that it is hard to believe that anyone really subscribed to it. We have absolutely no independent evidence that they did. There is plainly some cause for suspicion that when he was writing this colourful and rather over-excited passage, Plato's attention to detail wandered; or perhaps he inserted the slip deliberately, either to make the position he was satirising seem even more absurd, or to underline the muddle-headedness of his unfortunate brother Glaucon, Socrates' willing but confused respondent. However that may be, it seems to me overwhelmingly probable that the doctrine which Theophrastus attacks is a construct developed wholly out of Glaucon's speech, and has no more substantial claim to historical reality than that.

If I am even partly right in my reading of this passage, and even if I am wrong in connecting it so closely to the account in the *Republic*, there is no point in Theophrastus' polemic at which he takes issue with Aristoxenus. This seems remarkable, since Aristoxenus' picture of the relations between pitches is unambiguously quantitative in a sense to which Theophrastus might be expected to object. He discusses, for instance, the 'size' of the perfect fourth, and quantifies the intervals inside tetrachords in each genus in terms of tones, their multiples and their fractions. It makes no difference that in the second and third books of the *El. harm*. Aristoxenus emphasises the need to identify musical intervals, in the context of harmonic science, by reference to the *dynameis* of their bounding notes and not by their sizes. Every actual instance of such an interval still has a determinate size, and any two different pitches (as distinct from musical notes) are still separated by a measurable 'distance'.[21] If Theophrastus wishes to demonstrate that

[21] The Aristoxenian passages cited by Sicking 1998: 127, in support of his suggestion that *El. harm*. Book II is (in part) a response to Theophrastus' criticisms, are unquestionably important, but they do nothing to eliminate this difficulty. We may note, *inter alia*, that the two examples of Aristoxenian quantification which I mentioned above are both included in Book II.

any attempt whatever to quantify relations between pitches is inapposite and misguided, why does he not protest?

The answer, I think, is that he understood Aristoxenus well enough to grasp that a work like the *El. harm.* in fact offers no purchase to his criticisms. Theophrastus' arguments against theories underpinning mathematical harmonics focus on alleged defects in their accounts of the physical basis of pitch, and seek to show that pitch-differences cannot be quantitative when considered as attributes of movements in motion, as they really are 'out there'. His attack on the theory of *diastēmata* seems to be similarly motivated; its essential thesis is that these 'intervals' cannot be the *causes* of differences in pitch. But Aristoxenus, as we know, insists that he is not in the least concerned with such issues. His business is with melodic phenomena as they present themselves to perception, *kata tēn tēs aisthēseōs phantasian*, and it simply does not matter to him how questions about the underlying causes of perceived sounds and their attributes are to be answered.[22] Theophrastus recognised, I suggest, that this way of demarcating the scope of his discussion places Aristoxenus' theories firmly outside the battlefield upon which he himself is embroiled. Aristoxenus contends only that pitched sounds present themselves to the hearing as items separated by quantifiable distances. Underlying or causing the pitches of the sounds we perceive there is, no doubt, some variable attribute of objectively real events; its variations may be quantitative and the differences between its values measurable either as ratios or as quasi-linear distances, or again they may not. But about such matters Aristoxenus' thesis implies nothing at all, and his work is in this respect as irrelevant to those who debate them as theirs is to him.

Theophrastus keeps the spotlight on the doctrine about *diastēmata* for another six lines after the point we have reached (120–5). They introduce no fresh varieties of criticism, however, and the only new ideas they put forward will be more appropriately considered in the course of the next section, where certain ingredients of the preceding argument will also be briefly revisited.

THEOPHRASTUS' EXPOSITION OF HIS POSITIVE VIEWS

We pick up the text at the point where Theophrastus has just rebutted the thesis that it is because higher-pitched sounds move with more vigour and power that they can be heard at a greater distance from their origin than

[22] See particularly *El. harm.* 9.3–11, 12.4–19; cf. 32.20–8 and pp. 141–2, 166–8 above.

lower ones (69–80). He accepts that such sounds are indeed audible from further away, but he explains the phenomenon differently.

But since concordance exists, displaying the equality of the two notes, there is equality in their powers, differing in the specific quality (*idiotēs*) of each of them. For what is higher [lit. 'sharper'] is by its nature more conspicuous, not stronger, and that is why it is apprehended at a greater distance than the lower [lit. 'heavier'], just as white is by comparison with any other colour, or as is anything else that is more strongly apprehended not because the other is less what it naturally is, nor because it does not move in accordance with equal numbers, but because perception focuses more on the one than on the other on account of its unlikeness to its surroundings. Thus the low note penetrates too; but the hearing grasps the high note more readily because of its specific quality (*idiotēs*), not because of the plurality it contains. (78–87)

High notes, then, differ qualitatively from low ones; and the qualitative *idiotēs* of a high note makes it stand out in our perceptual field more vividly than a low one, just as a brightly coloured parrot shows up more conspicuously than a dull brown thrush. That, Theophrastus contends, is enough to account for the perceptibility of higher notes at greater distances without resort to quantitative considerations. But he now seems to make a concession to his opponents, allowing that it may after all be the case that a higher note travels further. If it really does so, however, 'it is not because it is moved in accordance with more numbers, but because of its shape, since a high-pitched sound travels more forwards and upwards, while a low one travels more equally all about' (87–90).

The concept of a movement's shape reappears in a few other writings on acoustics, where it plays various roles, not always in connection with differences of pitch;[23] but I have found no examples earlier than Theophrastus. The link that he makes between shapes and pitches may have been prompted, in part, by the root meanings and associations of the terms designating high pitch and low, *oxytēs* (sharpness) and *barytēs* (heaviness). These attributes, as they exist in sounds, seem to be analogous, says Aristotle, 'to the sharp and the blunt in the field of touch, for the sharp pierces, as it were, and the blunt pushes . . .' (*De an.* 420b1–3). Rather similarly, the writer of *Problem* XIX.8 asserts that a low-pitched, 'heavy' sound is larger and is like an obtuse angle, while a high-pitched, 'sharp' sound is like an

[23] According to Priscian, Theophrastus made a broader connection between sound and shape, asserting that hearing (and so, presumably, sound of any sort) occurs when the air is 'shaped' (*Metaphrasis* 1.30, Theophr. frag. 277A Fortenbaugh); see Gottschalk 1998: 294. (The miniature controversy between Gottschalk and myself mentioned in his n. 29 need not concern us here.) For later instances of connections between sound and shape see [Ar.] *Problemata* XI.20 and 23, XIX.8, [Ar.] *De audib.* 800a, Ptol. *Harm.* 7.8–15.

acute one. A good many other writers too seem to treat the meanings of these words in their tactile contexts as reliable guides to the nature of the acoustic properties to which they are transferred.[24]

But Theophrastus does not argue for a connection between the pitches of sounds and the shapes of their movements on the basis of these semantic associations, which he nowhere makes explicit. Instead, having stated his thesis in lines 87–90, he goes on to present evidence of its truth drawn from empirical observation. If you sing a high note and then a low one while touching your ribs with your hand, he says, you will feel the movement of the low note more in the periphery of your rib-cage than that of the high note. Similar results will be found if you touch the surface of the hollow parts of a lyre. 'For the low note travels everywhere all around, while the high note travels forwards, or in the direction in which the utterer forces it to go' (93–100).

Theophrastus uses his thesis as a weapon against the mathematical theorists, arguing that the greater forward motion of the higher note is balanced out by the wider spread of the lower, so that the latter, therefore, does not 'move in accordance with fewer numbers'. But it is presented as a fact of observation, not as an analysis of what high and low pitch are, and it is far from clear that he means to imply that the movement's shape determines or constitutes a sound's pitch. He need mean no more than that high-pitched sounds, in addition to their essential and constitutive *idiotēs*, have the characteristic of expending their impetus mainly in a narrow band of movement in one direction,[25] whereas low-pitched ones disperse it more evenly around their point of origin. Indeed, if he had intended to represent shape as constitutive of pitch he would have been in danger of falling into the snares of quantitativism himself. By his own account, a high note's forward motion and a low note's motion round about are (at least in principle) measurable, since they can be compared and reckoned to be quantitatively equal (100–1, cf. 126–9); and it would therefore have been tempting and presumably possible (again, in principle) to define any individual pitch by reference to the ratio between its forward and its peripheral movement.

Theophrastus could not afford to go down that road. He would have been better advised to stick firmly to his contention that high pitch and low pitch are qualitative *idiotētes* analogous to colours, and to treat the shapes of their movements merely as collateral attributes. I think it probable that

[24] The most elaborate development of such an approach is perhaps that at Ptol. *Harm.* 7.17–8.2. I discuss the behaviour of *oxys* and *barys* in writings on acoustics more fully in Barker 2002b.

[25] Or conceivably in two; line 89 says 'forwards and upwards', but lines 99 and 100 say only 'forwards'.

he intended to do so, and this interpretation of his position is entirely consistent with the text. It is positively encouraged, if not entailed absolutely, by the statement with which this part of the passage closes. 'Hence it is not some unequal numbers that account for the differences, but the sounds being naturally such as they are, naturally attuned together' (105–7). The regular sense of the word *toiaide*, which I have translated 'such as they are', invites the expanded translation 'such as they are *in quality*' (by contrast with 'in number' or 'in quantity'), and can hardly point to any aspect of the sounds but the attributes previously called *idiotētes* and made analogous to colours.

Theophrastus tells us no more about his notion of pitch. But the expression 'naturally attuned together' takes us into new territory. It has a recognisably Aristoxenian ring about it. Though the verb whose passive participle, *synhērmosmenai*, I have translated 'attuned together' does not occur in the *El. harm.*, we hear a good deal there about 'the nature of *to hērmosmenon* ("that which is attuned")' and similar conceptions. In his attack on the theory that the *diastēmata* are the causes of pitch-difference, which follows immediately, much of Theophrastus' discussion revolves around the relations between two other pivots of Aristoxenus' harmonics, *emmeleia* and *ekmeleia*, the melodic and the unmelodic; and in the remarks that conclude this passage he refers to the 'principles', *archai* of melodic sound. Taken together, these expressions suggest an interest in issues close to those addressed in the *El. harm*. What is it that constitutes the 'natural attunement together' of two pitches? What is the difference between the melodic and the unmelodic, and what are the principles that determine what is melodic and what is not? Another sentence (to which we shall return immediately) speaks of the conditions under which we are able to 'find the notes that are attuned to one another' (120–1). On what basis can we confidently identify them? All these questions are staples of Aristoxenus' enquiry, and we may reasonably conjecture that Theophrastus' lost *Harmonics* had an agenda and a conceptual apparatus not very far removed from his.

In the final phase of the fragment (120–32) Theophrastus sums up the conclusions of his various polemical arguments, of which I shall say no more, and adds a brief and enigmatic remark about the nature of music in general. Its connection with the main body of the passage seems tenuous; but the sentence at 120–1, part of which I quoted above, may do something to give it a context. We need therefore to look at this sentence more closely. Its exact wording, and with it its sense, are unfortunately a little uncertain.

As it stands in the manuscripts, the sentence must be translated roughly as follows. 'It is therefore a great help that *melōidia* revolves around [or "depends upon"] these, enabling us to find the notes that are attuned to one another.' For grammatical reasons, 'these' can only be *emmeleia* and *ekmeleia*, the melodic and the unmelodic, and the suggestion would be that we are helped to find the correctly attuned notes by our sense of what is melodic and what is not. That is intelligible, and could be elaborated from the resources of the *El. harm*. But it is open to a linguistic objection;[26] and even if that problem were ignored or resolved the thought conveyed would be completely isolated. Nothing in the text prepares us for it or explains it, and nothing in the sequel refers back to it. A one-letter emendation to the text[27] gives a different meaning, 'It is therefore a great help to *melōidia* that these are avoided . . .', where 'these' are now the *diastēmata*, conceived in this passage's peculiar way as the sonorous content of the spaces between notes.

The emended version is not problem-free,[28] but it has advantages; and I have come round to the view that it is probably preferable. Unlike the statement made by the MSS text, its thesis that in melody the *diastēmata* must be 'left out' or 'avoided' ties in firmly with the substance of the argument to which it is attached. Further, it gives a much clearer indication of the project in which 'help' is required, and allows us to place it in the context of the issues from which the entire passage began. Avoidance of the *diastēmata* is a great help, we are told, to *melōidia*. More exactly, it is a great help 'towards' *melōidia*; or, to paraphrase, it helps us greatly when *melōidia* is our goal. The word *melōidia*, I think, is carefully chosen. It is not the simple *melos*, 'melody', of lines 115 and 122. Though in some settings the two terms are practically interchangeable, and can refer either to a particular tune or to melody in general, *melōidia* may also be used in a way that connects much more closely with the related verb *melōidein*. In these cases it does not designate a 'thing' such as a tune, but an activity, singing or 'making melody'.[29] This interpretation makes good sense here. Avoidance

26 The verb *periistasthai* with a dative complement is used nowhere else in the sense required here, to 'revolve around' or 'depend upon'.

27 Reading *tauta eis* for *tautais*, as proposed by Alexanderson 1969 in his note *ad loc*.

28 There are clear examples elsewhere of the use of *periistasthai* to mean 'avoid', but they all occur in the Greek of a later period (the earliest is in Philodemus).

29 See for instance Aristox. *El. harm*. 27.19–20, where *melōidia* and *lexis* designate the activities of singing and speaking respectively; and especially 28.20–8, where it is said to be in the nature of *melōidia*, melodic singing, that the voice places its intervals in certain orders 'in accordance with *melos*', and the allusion to *melōidia* is developed by reference to the notes that the voice is capable of singing, *melōidēsai*. This rather intricate passage makes it clear that *melōidia* is an activity, the voice's actual movement through notes and intervals, whereas *melos* (in this instance) is an abstract 'essence' to whose principles *melōidia* naturally conforms. Cf. also 53.21–5.

of the *diastēmata* is a great help to us in our efforts to make melody with our voices, since it enables us to 'find the notes that are attuned to one another'. Theophrastus does not mean that we can identify the well-attuned notes theoretically, but that we can locate them in vocal practice, hitting the right pitches when we sing.

Clearly this takes us right back to the beginning of the fragment, with its comment on the accuracy with which the soul can bring into being an audible manifestation of its own internal 'melody-making movement'. It was this remarkable accomplishment that the ratio-based theory, with its underpinnings in physical acoustics, was allegedly designed to explain. Theophrastus has argued that its foundations are unsound, and it can therefore explain nothing. In the guise in which he presents it, the *diastēma* theory is equally flawed; and it is not, this sentence tells us, these content-filled *diastēmata* that help us to find the notes we need when translating our psychic movements into audible sound, but their omission. Though the passage is of course only an excerpt from a much longer work, this return to its opening suggests that Porphyry's choices of the places to begin and end his quotation are well judged. It seems to be a complete and well-rounded whole.

But it leaves large uncertainties about Theophrastus' own view of the matter. He may have explained elsewhere just how the omission of the *diastēmata* helps us to produce the right notes, how he construes the concept of 'the melodic', and what he means by saying that some notes are 'naturally attuned to one another', but he does not do so here. The closing lines of the passage offer instead the following compendious statements. 'The nature of music is one. It is the movement of the soul that arises for the sake of release from the evils due to the emotions;[30] and if it were not this, neither would it be the nature of music'[31] (130–2).

This tells us not only that music has its primary mode of existence within the soul, but that its role there is essentially therapeutic. Traces of the latter idea, perhaps initially prompted by some remarks of Aristotle in the *Politics*, can be found in several other reports about Theophrastus' opinions. No other Theophrastan view about music, in fact, is so frequently mentioned

[30] 'For the sake of' translates the preposition *kata* with an accusative. It might mean no more than 'in accordance with' or 'in correspondence with', but the purposive sense seems most appropriate here. Cf. the translation in Sicking 1998: 106, 'with a view to', and for the purposive usage in general see LSJ s.v. *kata* III.

[31] An alternative translation, equally possible from a linguistic perspective, is 'and if it did not exist, neither would the nature of music exist'.

in later antiquity, and it probably figured prominently in the work from which our fragment is taken.[32] Like Plato's views on the ethical influence and significance of music, the details of Theophrastus' ideas about its power to cure both psychic and bodily ills are beyond the scope of this study. But one aspect of them seems to have implications for his approach to the science of harmonics.

His thesis that music is 'a movement of the soul that arises for the sake of release from the evils due to the emotions' is presented as a statement of its essence, almost as a definition. 'If it were not this, neither would it be the nature of music'; that is, nothing amounts to music unless it is constituted by a psychic movement of this sort. The oracular remark prefacing these assertions, 'the nature of music is one', reinforces their definitional effect by assuring us that only one kind of activity or occurrence – the one specified in these lines – can genuinely qualify for the title 'music'. Exponents of mathematical harmonics had located the principles governing musical relations in the domain of numbers; they are mathematical principles first, to be discovered through abstract mathematical reflection, and apply only derivatively to the relations and structures with which harmonics is concerned. Aristoxenus represented *melos* or 'the attuned', *to hērmosmenon*, as having an autonomous nature of its own, and consequently held that the process through which the principles governing its modes of organisation and patterns of movement can be discovered must begin from a meticulous empirical examination of its behaviour. If, however, music's nature is that of a movement of the soul, and specifically that of one capable of releasing it from emotional stress and turmoil, the patterns of movement which constitute music will be marked off from those that do not by criteria of an essentially psychological sort. The principles governing it will be those that determine which inner movements are capable of generating emotional 'release'.

There are affinities between this view and the one set out by Plato in the *Timaeus*, where the human soul, when it is in its best condition, has a musical structure akin to that of the soul of the universe, and music can help us to regain that psychic perfection, and to restore 'the unattuned cycle of our soul to order and concord with itself' (*Tim.* 47d5–6). But the connection with Theophrastus is tenuous. None of our reports about him suggest that a 'healthy' soul is structured on a musical basis; and in fact he

[32] See Aristotle, *Pol.* 1342a4–15. The most relevant reports on Theophrastus appear in Fortenbaugh's edition as frags. 719A–B, 723, 726A–C. Cf. the comments of Sicking 1998: 140–2 (who questions, probably rightly, my earlier attempt to connect Theophrastus' ideas with Damon's), and Lippman 1964: 163–6. I discuss his 'musical therapeutics' more fully in Barker 2005a: 131–41.

could hardly have supposed that it was, given his thesis that music is psychic movement of an essentially therapeutic sort. It is difficult to see how its 'nature' could be identical with that of the movement of an emotionally afflicted soul through which its health is brought about, and at the same time constitute the structure of a soul that is already in a healthy condition. Again, the ideal musical structure of the soul in the *Timaeus* is grounded in principles of mathematical order and perfection which, according to Theophrastus, cannot be applied to musical relations at all, since these are not quantitative.

If Theophrastus' *Harmonics* attempted to follow up the implications of his statements in frag. 716, it must have differed from other known works in the field in at least two very striking respects. Musical relations between notes or pitches must have somehow been represented without recourse to quantification; and the principles governing them must have been derived from theories about the nature and operations of the soul. One cannot easily imagine the contents of such a work, and if it existed it seems to have left no trace in the writings of later theorists.[33] Perhaps, however, Theophrastus laid these inscrutable thoughts aside when composing the *Harmonics*. He cannot have followed the path laid down by mathematical theorists.[34] But I have argued that his non-quantitative view of musical relations in their 'real' or physical aspect (and presumably in the context of psychic movement too) is consistent with a broadly Aristoxenian approach to the phenomena as they present themselves in the field of human perception. In that case a Theophrastan *Harmonics* could engage with the musical 'appearances' in terms similar to those that Aristoxenus used, and could even adopt, at this level, a similar way of quantifying intervals. None of this would compromise his position on the real nature of musical relations; he would merely have to record his conviction that an account of this sort depicts them only as they appear and not as they really are. This, of course, is a distinction that Aristoxenus rejects; music, on his account, exists *only* as an audible phenomenon in the realm of 'appearances'. But Theophrastus can follow an Aristoxenian mode of analysis without subscribing to any

[33] The part of Theophrastus' theory that is not an attempt to define music, but simply credits it with psychotherapeutic effects, did of course leave such traces, and not only in the reports cited in n. 32 above; see for instance Arist. Quint. 57.31–58.32. But that is another matter.

[34] He could in principle have agreed that musical intervals are produced, as those theorists contended, from lengths of string or pipe in the ratios they specify. But a mathematical harmonics that restricts itself to enumerating ratios between the dimensions of sounding bodies is sadly limited. If it is to have a basis on which the results of these measurements can be explained, or to identify principles governing an attunement's mathematical form, it must posit that pitch-relations themselves are quantitative at the most fundamental ontological level; and here Theophrastus will certainly part company with them.

such view. He need only posit that the principles uncovered through such investigations, which govern relations and forms of organisation at the phenomenal level, are not autonomous but derivative, and that audible melodies are no more than external manifestations of the soul's silent dances on an internal stage, whose choreography would demand description in language of quite another sort.