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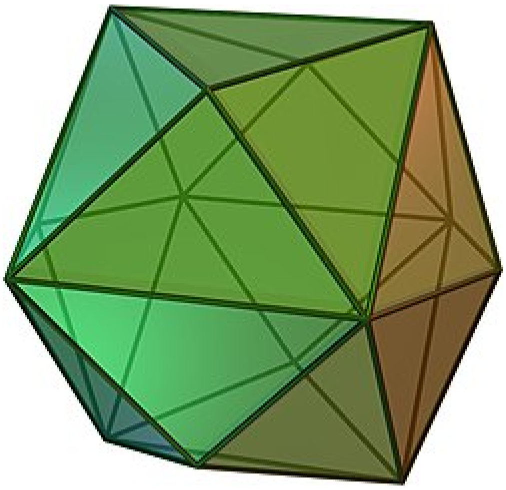
Tetrakis hexahedron

In geometry, a **tetrakis hexahedron** (also known as a **tetrahexahedron**, **hextetrahedron**, **tetrakis cube**, and **kiscube**[2]) is a Catalan solid. Its dual is the truncated octahedron, an Archimedean solid.

It can be called a **disdyakis hexahedron** or **hexakis tetrahedron** as the dual of an omnitruncated tetrahedron, and as the barycentric subdivision of a tetrahedron.[3]

Tetrakis hexahedron	

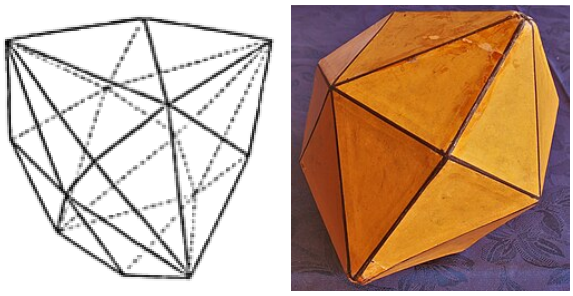


Type	Catalan solid Kleetope
Faces	24
Edges	36
Vertices	14
Dual polyhedron	truncated octahedron

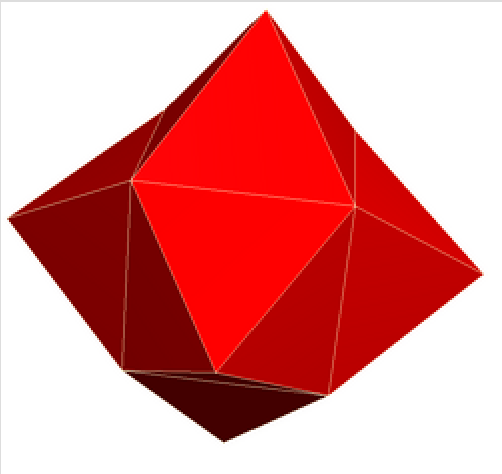
Die and crystal model

As a Kleetope

The name "tetrakis" is used for the Kleetopes of polyhedra with square faces.[4] Hence, the tetrakis hexahedron can be considered as a cube with square pyramids covering each square face, the Kleetope of the cube. The resulting construction can be either convex or non-convex, depending on the square pyramids' height. For the convex result, it comprises twenty-four isosceles triangles.[5] A non-convex form of this shape, with equilateral triangle faces, has the same surface geometry as the regular octahedron, and a paper octahedron model can be re-folded into this shape.[6] This form of the tetrakis hexahedron was illustrated by Leonardo da Vinci in Luca Pacioli's *Divina proportione* (1509).[7]



Drawing and crystal model of variant with tetrahedral symmetry called *hexakis tetrahedron* [1]

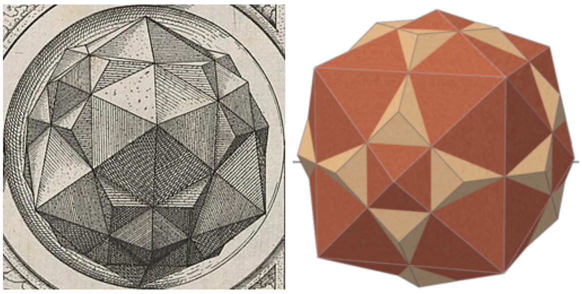


Non-convex tetrakis hexahedron with equilateral triangle faces

Denoting the edge length of the base cube by a, the height of each pyramid summit above the cube is $\frac{a}{4}$. The inclination of each triangular face of the pyramid versus the cube face is $\arctan\frac{1}{2} \approx 26.565°$ (sequence A073000 in the OEIS). One edge of the isosceles triangles has length a, the other two have length $\frac{3a}{4}$, which follows by applying the Pythagorean theorem to height and base length. This yields an altitude of $\frac{\sqrt{5}a}{4}$ in the triangle (OEIS: A204188). Its area is $\frac{\sqrt{5}a^2}{8}$, and the internal angles are $\arccos\frac{2}{3} \approx 48.1897°$ and the complementary $180° - 2\arccos\frac{2}{3} \approx 83.6206°$. The volume of the pyramid is $\frac{a^3}{12}$; so the total volume of the six pyramids and the cube in the hexahedron is $\frac{3a^3}{2}$.

This non-convex form of the tetrakis hexahedron can be folded along the square faces of the inner cube as a net for a four-dimensional cubic pyramid.

As a Catalan solid



Dual compound of truncated octahedron and tetrakis hexahedron. The woodcut on the left is from *Perspectiva Corporum Regularium* (1568) by Wenzel Jamnitzer.

The tetrakis hexahedron is a Catalan solid, the dual polyhedron of a truncated octahedron. The truncated octahedron is an Archimedean solid, constructed by cutting all of a regular octahedron's vertices, so the resulting polyhedron has six squares and eight hexagons.[8] The tetrakis hexahedron has the same symmetry as the truncated octahedron, the octahedral symmetry.[9]

Cartesian coordinates for the 14 vertices of a tetrakis hexahedron centered at the origin, are the points

$$\left(\pm\tfrac{3}{2},0,0\right),\ \left(0,\pm\tfrac{3}{2},0\right),\ \left(0,0,\pm\tfrac{3}{2}\right),\ \left(\pm1,\pm1,\pm1\right).$$

The length of the shorter edges of this tetrakis hexahedron equals 3/2 and that of the longer edges equals 2. The faces are acute isosceles triangles. The larger angle of these equals $\arccos\frac{1}{9}\approx 83.62^\circ$ and the two smaller ones equal $\arccos\frac{2}{3}\approx 48.19^\circ$.

Applications

Naturally occurring (crystal) formations of tetrahexahedra are observed in copper and fluorite systems.

Polyhedral dice shaped like the tetrakis hexahedron are occasionally used by gamers.

A 24-cell viewed under a vertex-first perspective projection has a surface topology of a tetrakis hexahedron and the geometric proportions of the rhombic dodecahedron, with the rhombic faces divided into two triangles.

The tetrakis hexahedron appears as one of the simplest examples in building theory. Consider the Riemannian symmetric space associated to the group $SL_4(\mathbf{R})$. Its Tits boundary has the structure of a spherical building whose apartments are 2-dimensional spheres. The partition of this sphere into spherical simplices (chambers) can be obtained by taking the radial projection of a tetrakis hexahedron.

Symmetry

With tetrahedral symmetry, the triangular faces represent the 24 fundamental domains of tetrahedral symmetry.[10] This polyhedron can be constructed from six great circles on a sphere. It can also be seen by a cube with its square faces triangulated by their vertices and face centers, and a tetrahedron with its faces divided by vertices, mid-edges, and a central point.

See also

- Disdyakis triacontahedron
- Disdyakis dodecahedron
- Kisrhombille tiling
- Compound of three octahedra
- Deltoidal icositetrahedron, another 24-face Catalan solid.

References

1. *Hexakistetraeder* in German, see e.g. *Meyers* page and *Brockhaus* page. The same drawing appears in *Brockhaus and Efron* as *преломленный пирамидальный тетраэдр* (*refracted pyramidal tetrahedron*).
2. Conway, *Symmetries of Things*, p.284

3. Langer, Joel C.; Singer, David A. (2010), "Reflections on the lemniscate of Bernoulli: the forty-eight faces of a mathematical gem", *Milan Journal of Mathematics*, **78** (2): 643–682, doi:10.1007/s00032-010-0124-5 (https://doi.org/10.1007%2Fs00032-010-0124-5), MR 2781856 (https://mathscinet.ams.org/mathscinet-getitem?mr=2781856)
4. Conway, John H.; Burgiel, Heidi; Goodman-Strauss, Chaim (2008), *The Symmetries of Things*, AK Peters, p. 284 (https://books.google.com/books?id=Drj1CwAAQBAJ&pg=PA284), ISBN 978-1-56881-220-5
5. Klein, Cornelis; Dutrow, Barbara (2007), *Manual of Mineral Science* (https://books.google.com/books?id=6XcPMx8rY58C&pg=PA202), John Wiley & Sons, p. 202, ISBN 978-0-471-72157-4
6. Rus, Jacob (2017), "Flowsnake Earth" (https://archive.bridgesmathart.org/2017/bridges2017-237.html), in Swart, David; Séquin, Carlo H.; Fenyvesi, Kristóf (eds.), *Proceedings of Bridges 2017: Mathematics, Art, Music, Architecture, Education, Culture*, Phoenix, Arizona: Tessellations Publishing, pp. 237–244, ISBN 978-1-938664-22-9
7. Pacioli, Luca (1509), "Plates 11 and 12" (https://archive.org/details/divinaproportion00paci/page/n205), *Divina proportione*
8. Williams, Robert (1979). *The Geometrical Foundation of Natural Structure: A Source Book of Design* (https://archive.org/details/geometricalfound00will/page/78). Dover Publications, Inc. p. 78–79. ISBN 978-0-486-23729-9.
9. McLean, K. Robin (1990), "Dungeons, dragons, and dice", *The Mathematical Gazette*, **74** (469): 243–256, doi:10.2307/3619822 (https://doi.org/10.2307%2F3619822), JSTOR 3619822 (https://www.jstor.org/stable/3619822), S2CID 195047512 (https://api.semanticscholar.org/CorpusID:195047512) See p. 247.
10. Raman, C. V.; Ramaseshan, S. (1946), "The crystal forms of diamond and their significance", *Proceedings of the Indian Academy of Sciences*, **24** (1)

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- Wenninger, Magnus (1983), *Dual Models*, Cambridge University Press, doi:10.1017/CBO9780511569371 (https://doi.org/10.1017%2FCBO9780511569371), ISBN 978-0-521-54325-5, MR 0730208 (https://mathscinet.ams.org/mathscinet-getitem?mr=0730208) (The thirteen semiregular convex polyhedra and their duals, Page 14, Tetrakishexahedron)
- *The Symmetries of Things* 2008, John H. Conway, Heidi Burgiel, Chaim Goodman-Strauss, ISBN 978-1-56881-220-5 [1] (https://web.archive.org/web/20100919143320/https://akpeters.com/product.asp?ProdCode=2205) (Chapter 21, Naming the Archimedean and Catalan polyhedra and tilings, page 284, Tetrakis hexahedron)

External links

- Weisstein, Eric W., "Tetrakis hexahedron (https://mathworld.wolfram.com/TetrakisHexahedron.html)" ("Catalan solid (http://mathworld.wolfram.com/CatalanSolid.html)") at *MathWorld*.
- Virtual Reality Polyhedra (http://www.georgehart.com/virtual-polyhedra/vp.html) www.georgehart.com: The Encyclopedia of Polyhedra
  - VRML model (http://www.georgehart.com/virtual-polyhedra/vrml/tetrakishexahedron.wrl)
  - Conway Notation for Polyhedra (http://www.georgehart.com/virtual-polyhedra/conway_notation.html) Try: "dtO" or "kC"
- Tetrakis Hexahedron (https://web.archive.org/web/20080828230230/http://polyhedra.org/poly/show/35/tetrakis_hexahedron) – Interactive Polyhedron model
- The Uniform Polyhedra (http://www.mathconsult.ch/showroom/unipoly/)