Consider the following self-adjoint eigenvalue problem in variational form:

$$-u''(x) = \lambda u(x), \quad x \in (0,\pi), \quad u(0) = u(\pi) = 0 \tag{1}$$

where λ is the eigenvalue and $u(x)$ is the eigenfunction. The weak form of the problem is given by:

$$\int_0^\pi u'(x)v'(x)\,dx = \lambda \int_0^\pi u(x)v(x)\,dx \tag{2}$$

for all $v \in H_0^1(0,\pi)$, where $H_0^1(0,\pi)$ is the Sobolev space of functions that are square integrable and have square integrable weak derivatives, with boundary conditions $u(0) = u(\pi) = 0$.

1. Compute the first ten eigenvalues and eigenfunctions of the problem using the linear finite element method and the Firedrake library. [Hint: *the exact eigenvalues are given by* $\lambda_n = n^2$, $n = 1, 2, \ldots$].

2. What is the rate of convergence of the eigenvalues as the mesh is refined? [Hint: *try computing the eigenvalues for different mesh sizes, ex: 16, 32, 64, 128, 256 elements*].

3. Verify that all the eigenvalues are the discrete eigenvalue are positive and approximate the real eigenvalues from above. Can you explain why ? [Hint: *Use the Rayleigh quotient to show that the eigenvalues are positive and the min-max characterization of the eigenvalues to show that they are approximated from above. See Appendix B, for more details*].

4. What is the rate of convergence of the eigenfunctions in $H^1(0,\pi)$ as the mesh is refined? [Hint: *the exact eigenfunctions are given by* $u_n(x) = \sin(nx)$, $n = 1, 2, \ldots$].

Consider now the same eigenvalue problem, but in two dimensions, i.e. we look for $u \in H_0^1(\Omega)$ such that:

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\,dx = \lambda \int_\Omega u(x)v(x)\,dx \tag{3}$$

for all $v \in H_0^1(\Omega)$.

5. Compute the eigenvalues if $\Omega = (0,\pi) \times (0,\pi)$? How are eigenvalues with moltiplicity greater than one handled by the finite element method?

6. Let $\Omega = (0,\pi) \times (0,\pi)$. What is the order of convergence of the eigenvalues associated to the eigenvalue $\lambda_3 = 5$ as the mesh is refined? [Hint: *to compute this order of convergence, you need to find the best approximation in the eigenspace by a small Galerkin method.*]

7. What is the order of convergence of the first eigenvalue λ_1 as the mesh is refined, if Ω is given by

$$\Omega = \{\rho e^{i\theta} \in \mathbb{C} : 0 < \rho < 1, 0 < \theta < \frac{3}{2}\pi\}. \tag{4}$$

[Hint: *See appendix A, for the regularity of the first eigenfunctions and to see why the first eigenvalues should be given by* $\lambda_1 \approx (3.375610652)^2$].