In this appendix we will study the eigenvalue problem associated with the Laplace operator in the domain depicted in Figure 1, using the technique of separation of variables. We will consider the following Dirichlet eigenvalue problem,

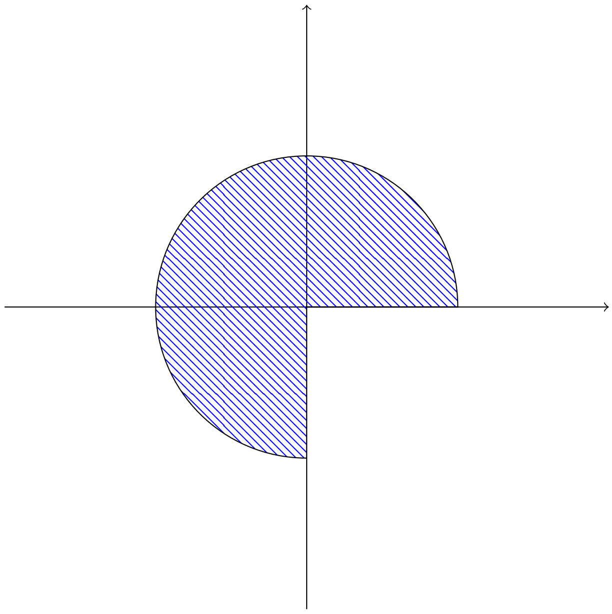
$$\begin{cases} \Delta\Phi(\rho,\theta) = -\lambda\Phi(\rho,\theta) \text{ in } \Omega, \\ \Phi(\rho,\theta) = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is the domain of the circular sector of radius 1 and angle $\frac{3}{2}\pi$, i.e.

$$\overline{\Omega} = \left\{ \rho e^{\theta} \in \mathbb{C} : \rho \in [0,1], \theta \in [0, \frac{3}{2}\pi] \right\}. \tag{2}$$

We will make one further assumption dictated by the physics of the problem, i.e $|\Phi(\rho,\theta)| < \infty$. As usual we assume Φ depends separately upon the radius

Figure 1: In the figure the domain where we solve the Dirichlet eigenvalue problem is drawn.



and the angle of the circular sector, $\Phi(\rho,\theta) = \Theta(\theta)R(\rho)$, expressing the Laplacian in polar coordinates we obtain

$$\Delta\Phi(\rho,\theta) = \partial_x^2\Phi(\rho,\theta) + \partial_y^2\Phi(\rho,\theta) = \frac{1}{\rho}\partial_\rho\Phi(\rho,\theta) + \frac{1}{\rho^2}\partial_\theta^2\Phi(\rho,\theta) + \partial_\rho^2\Phi(\rho,\theta)$$

$$\Delta\Phi(\rho,\theta) = \frac{1}{\rho}R'(\rho)\Theta(\theta) + \frac{1}{\rho^2}R(\rho)\Theta''(\theta) + R''(\rho)\Theta(\theta) \tag{3}$$

therefore imposing the eigenvalue problem we have the following expression,

$$\Delta\Phi(\rho,\theta) = -\lambda\Phi(\rho,\theta)$$

$$\frac{1}{\rho}R'(\rho)\Theta(\theta) + \frac{1}{\rho^2}R(\rho)\Theta''(\theta) + R''(\rho)\Theta(\theta) = -\lambda R(\rho)\Theta(\theta). \tag{4}$$

We perform some algebraic manipulations to obtain on one side an expression in θ and on the other side an expression in ρ.

$$\begin{aligned}
\Big(\frac{1}{R(\rho)\Theta(\theta)}\times\Big) \qquad \frac{1}{\rho}R'(\rho)\Theta(\theta)+\frac{1}{\rho^2}R(\rho)\Theta''(\theta)+R''(\rho)\Theta(\theta) &= -\lambda R(\rho)\Theta(\theta)\\
\Big(\rho^2\times\Big) \qquad \frac{1}{\rho}\frac{R'(\rho)}{R(\rho)}+\frac{1}{\rho^2}\frac{\Theta''(\theta)}{\Theta(\theta)}+\frac{R''(\rho)}{R(\rho)} &= -\lambda\\
\rho\frac{R'(\rho)}{R(\rho)}+\frac{\Theta''(\theta)}{\Theta(\theta)}+\rho^2\frac{R''(\rho)}{R(\rho)} &= -\lambda\rho^2\\
-\frac{\Theta''(\theta)}{\Theta(\theta)} &= \rho\frac{R'(\rho)}{R(\rho)}+\rho^2\frac{R''(\rho)}{R(\rho)}+\lambda\rho^2. \qquad (5)
\end{aligned}$$

Now since one side only depend on θ while the other only depend on ρ we obtain a well known one dimensional eigenvalue problem,

$$\begin{cases}\Theta''(\theta)=-\mu\Theta(\theta),\\ \Theta(0)=\Theta(\frac{3}{2}\pi)=0\end{cases}. \qquad (6)$$

It is well known that (6) has the following solutions,

$$\mu_n=\Big(\frac{2}{3}n\Big)^2 \qquad \Theta(\theta)=\sin(\frac{2}{3}n\theta)$$

which together with (5) gives an ODE for $R(\rho)$, i.e.

$$\rho\frac{R'(\rho)}{R(\rho)}+\rho^2\frac{R''(\rho)}{R(\rho)}+\lambda\rho^2-\mu_n=0, \qquad (7)$$

$$\rho^2R''(\rho)+\rho R'(\rho)+R(\rho)\Big(\lambda\rho^2-\mu_n\Big)=0. \qquad (8)$$

Performing the variable change $z=\sqrt{\lambda}\rho$ we obtain Bessel differential equation,

$$z^2R''(z)+zR'(z)+(z^2-\alpha_n^2)R(z)=0, \qquad \alpha_n^2=\mu_n. \qquad (9)$$

Bessel ODE has the following solution, $R(z)=AJ_{\alpha_n}(z)+BY_{\alpha_n}(z)$. Since we have the hypothesis $|R(\rho)|<\infty$ as $\rho\to 0$ then $B=0$, i.e.

$$R(r)=J_{\alpha_n}(\sqrt{\lambda}r), \qquad (10)$$

using homogeneous Dirichlet boundary condition we get $R(1)=J_{\alpha_n}(\sqrt{\lambda})=0$ and therefore λ must be equal to the square of the m-th zero of J_{α_n}, i.e. $\lambda=(z_{m,n})^2$. This yields the following solution to the eigenvalue problem associated with (1),

$$\Phi_{n,m}(\rho,\theta)=J_{\alpha_n}(z_{m,n}\rho)\sin(\frac{2}{3}\theta n), \qquad \lambda_{m,n}=z_{m,n}^2. \qquad (11)$$

Now we notice that using the fact that when $\rho \to 0$,

$$J_{\alpha_n}(z_{m,n}\rho) \approx \frac{1}{\Gamma(\alpha_n + 1)} \Big(\frac{z_{m,n}\rho}{2}\Big)^{\alpha_n}, \tag{12}$$

then we have the following approximation for $\Phi_{n,m}$:

$$\Phi_{n,m}(\rho,\theta) \approx C_{m,n}\rho^{\frac{2}{3}n} \sin(\frac{2}{3}n\theta). \tag{13}$$

Computing the norm in $\mathcal{W}^{1,2}\Big(B_\rho(\mathbf{0})\Big)$ and $\mathcal{W}^{2,2}\Big(B_\rho(\mathbf{0})\Big)$ we can show that $\Phi_{m,1}$ does live in $\mathcal{W}^{1,2}(\Omega)$ but not in $\mathcal{W}^{2,2}(\Omega)$. Furthermore if one uses approximation (13) to compute the Gagliardo-Slobodeckij semi-norm we can show that $\Phi_{m,1} \in \mathcal{W}^{\frac{5}{3}-\varepsilon}(\Omega)$. Expanding a generic solution of Laplace problem in the eigenspace we notice that the singular functions S_m are of the form $\Phi_{m,1}$ if $f \in \mathcal{L}^2(\Omega)$. In general the following result holds,

Theorem 1 *Let us consider a domain $\Omega \subset \mathbb{R}^2$ with a re-entrant corner of aperture ω. If $f \in \mathcal{W}^{0,p}(\Omega)$ then $u_0 \in \mathcal{W}_0^{1,p}(\Omega)$ is such that,*

$$u_0 - \sum_m C_m S_m \in \mathcal{W}^{2,p}(\Omega), \tag{14}$$

where the S_m are a particular set of singular functions belonging to the space $\mathcal{W}^{2-\frac{\pi}{\omega}-\varepsilon,p}(\Omega)$. Furthermore, $\Big(\mathcal{W}_0^{1,p}(\Omega), \mathcal{W}^{0,p}(\Omega), \mathcal{W}^{2,p}(\Omega)\Big)$ is a shift triplet for $u_0 - \sum_m C_m S_m$.

The reader interested in the proof of this result is referred to *Grisvard, P. (2011). Elliptic problems in nonsmooth domains. Society for Industrial and Applied Mathematics., Chapter 4.*