

# Embedded Trefftz discontinuous Galerkin methods

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European Finite Element Fair 2023



# (polynomial) Trefftz-DG methods

$$\begin{cases} \mathcal{L}u = 0 \text{ in } \Omega, \\ + \text{ bndc. on } \partial\Omega, \end{cases}$$

Find  $u_h \in \mathbb{T}^p(\mathcal{T}_h)$ , s.t.  $a_h^{\text{DG}}(u_h, v_h) = \ell_h^{\text{DG}}(v_h) \quad \forall v_h \in \mathbb{T}^p(\mathcal{T}_h)$  with

$$\mathbb{T}^p(\mathcal{T}_h) := \Pi_{K \in \mathcal{T}_h} \mathbb{T}^p(K), \quad \mathbb{T}^p(K) := \{v_h \in \mathbb{P}^p(K) \text{ s.t. } \mathcal{L}u = 0 \text{ on } K\}$$

$$\dim \mathbb{T}^p(\mathcal{T}_h) = \mathcal{O}(p^{n-1}) \ll \dim \mathbb{P}^p(\mathcal{T}_h) = \mathcal{O}(p^n)$$

Assumption: Trefftz space is 'rich enough' ( $\mathcal{L} = \sum_{l=1}^d \alpha_l \partial_{x_l}^{\beta_l}$  for  $\alpha_l \in \mathbb{R}$  and  $\beta_l \in \mathbb{N}$ )

## Example: Laplace equation

$$\mathcal{L}u = -\Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega.$$

$$\mathbb{T}^p(K) = \{1, x, y, xy, x^2 - y^2, x^3 - 3xy^2, \dots\}$$

$$a_h(u, v) = \sum_K \int_K \nabla u \nabla v \, dx + \sum_{F \in \mathcal{F}_h^{\text{int}}} \int_F \overbrace{-\{\!\!\{\partial_{\mathbf{n}} u\}\!\!\} [v]}^{\text{consistency}} \overbrace{-\{\!\!\{\partial_{\mathbf{n}} v\}\!\!\} [u]}^{\text{symmetry}} \overbrace{+ \alpha p^2 h^{-1} [u] [v]}^{\text{stability}} \, ds + \text{bnd}$$

$\{\!\!\{\cdot\}\!\!\}$  : average across facets,  $[\![\cdot]\!]$  : jump across facets.  $\rightsquigarrow$  communication between neighbors.

# Embedded Trefftz-DG method

- Goal: Represent Trefftz basis  $\{\psi_j\}_M$  in the standard DG basis  $\{\phi_i\}_N$  :

$$\psi_j = \sum_{i=1}^N \mathbf{T}_{ij} \phi_i, \quad j = 1, \dots, M, \quad \text{for } \mathbf{T} \in \mathbb{R}^{N \times M}.$$

Then instead of solving  $\mathbf{A} \mathbf{u}_{\mathbb{P}} = \mathbf{b}$  we solve  $\mathbf{T}^T \mathbf{A} \mathbf{T} \mathbf{u}_{\mathbb{T}} = \mathbf{T}^T \mathbf{b}$ .

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- Recipe: Find  $u$  for which  $\|\mathcal{L}u\|_{0,h} = 0 \Leftrightarrow \langle \mathcal{L}u, \mathcal{L}v \rangle_{0,h} = 0, \quad \forall v \in V_h$

$$\mathbf{T} = \ker(\mathbf{W}) \quad \text{with} \quad \mathbf{W} = \sum_{K \in \mathcal{T}_h} \langle \mathcal{L}\phi_i, \mathcal{L}\phi_j \rangle_K$$

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- Benefit:  $\mathbf{T}^T \mathbf{A} \mathbf{T} \in \mathbb{R}^{M \times M}$  with

$$\dim \mathbb{T}^p(\mathcal{T}_h) = M \ll N = \dim \mathbb{P}^p(\mathcal{T}_h)$$

# Embedded Trefftz-DG method

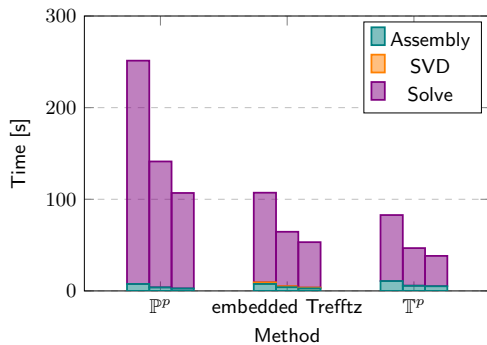
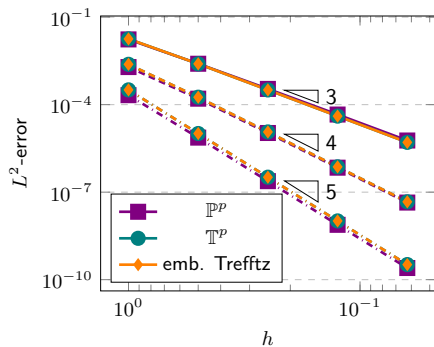
$$\mathbf{T} = \ker(\mathbf{W}) \text{ with } \mathbf{W} = \sum_{K \in \mathcal{T}_h} \langle \mathcal{L}\phi_i, \mathcal{L}\phi_j \rangle_K$$

On each mesh element use SVD (or QR)

$$\mathbf{W}|_K = \begin{pmatrix} | & & | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_L & \mathbf{u}_{L+1} & \dots & \mathbf{u}_N \\ | & & | & & | \end{pmatrix} \cdot \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_L & & \\ & & & 0 & \ddots \\ & & & & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{---} & \mathbf{v}_1^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{v}_L^T & \text{---} \\ \text{---} & \mathbf{v}_{L+1}^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{v}_N^T & \text{---} \end{pmatrix}$$

## Example: Laplace equation

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$





# Benefits of Embedded Trefftz DG

So far:

Embedded Trefftz DG ...

1. ... facilitates implementation of existing **polynomial** Trefftz methods
2. ... is computationally (a bit) more expensive than "direct" Trefftz spaces
3. ... inherits **conditioning** properties from DG scheme

Next up:

Can we deal with ...

1. ... PDEs where no (suitable) polynomial Trefftz spaces exists?
2. ... inhomogeneous PDEs?

# Non-polynomial Trefftz spaces

Many problems don't have suitable polynomial Trefftz spaces

Examples:  $\mathcal{L} = -\Delta \pm \text{id}$        $\mathcal{L} = -\text{div}(\alpha \nabla \cdot)$ ,  $\alpha$  not constant

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Trefftz DG based on plane waves

For Helmholtz  $(-\Delta - \omega^2 \text{id})$  Plane Wave DG (a Trefftz DG) spaces exist:

$$\mathbb{T}^P = \{e^{-i\omega(d_j \cdot \mathbf{x})} \text{ s.t. } j = 0, \dots, k\}$$

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R. Hiptmair and A. Moiola and I. Perugia, *A Survey of Trefftz Methods for the Helmholtz Equation*, Lect. Notes Comput. Sci. Eng., 2016

L.-M. Imbert-Gérard, A. Moiola, PS, *A space-time quasi-Trefftz DG method for the wave eq. with piecewise-smooth coefficients*, arXiv:2011.04617

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## Quasi-Trefftz Methods

Let  $\mathcal{L}_\alpha$  be diff. operator depending on a (element-wise) smooth  $\alpha$ , define quasi-Trefftz space

$$\mathbb{QT}^p := \{v \in \mathbb{P}^p(\mathcal{T}_h) \mid T_{(\mathbf{x}_{\text{center}})}^{p-q}(\mathcal{L}_\alpha v) = 0\}$$

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# Embedded Trefftz - no polynomial Trefftz space

What to do for operators like  $-\Delta \pm \text{id}$ ,  $-\text{div}(\alpha(\mathbf{x})\nabla \cdot)$ ,  $\dots$

- ▶ Idea: Instead of  $\langle \mathcal{L}u, \mathcal{L}v \rangle = 0$ ,  $\forall v \in V_h$  we use the relaxed condition

$$\langle \mathcal{L}u, w \rangle_{0,h} = 0, \quad \forall w \in W_h \subset V_h$$

( $W_h := \mathcal{L}\mathbb{P}^p(\mathcal{T}_h)$  recovers the previous embedding)

- ▶ Introduce a **weak Trefftz space** for the embedding

$$\mathbb{T}^p(\mathcal{T}_h) = \{v \in \mathbb{P}^p(\mathcal{T}_h), \Pi_W \mathcal{L}v = 0 \text{ on each } K \in \mathcal{T}_h\}.$$

- ▶ Proceed with the embedding

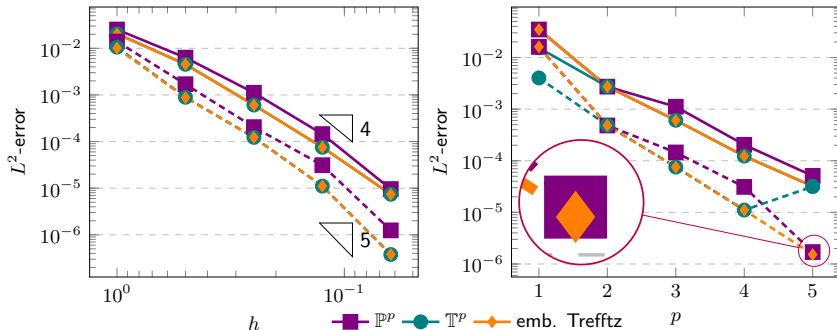
$$\mathbf{T} = \ker(\mathbf{W}) \text{ with } \mathbf{W} = \sum_{K \in \mathcal{T}_h} \langle \mathcal{L}\phi_i, \psi_j \rangle_K, \quad \forall \psi_j \in W_h$$

# Example: Helmholtz

$$\begin{cases} -\Delta u - \omega^2 u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n_{\mathbf{x}}} + iu = g & \text{on } \partial\Omega. \end{cases}$$

$$\mathbb{T}^p = \{e^{-i\omega(d_j \cdot \mathbf{x})} \text{ s.t. } j = 0, \dots, k\}$$

$$\mathbb{ET}^p = \{v \in \mathbb{P}^p(\mathcal{T}_h), \Pi_{\mathbb{P}^{p-2}}(-\Delta - \omega^2)v = 0 \text{ on each } K \in \mathcal{T}_h\}$$



Lemma (Conditioning of the embedded Trefftz method)

$$\kappa_2(\mathbf{T}^T \mathbf{A} \mathbf{T}) \leq \kappa_2(\mathbf{A})$$

Proof.

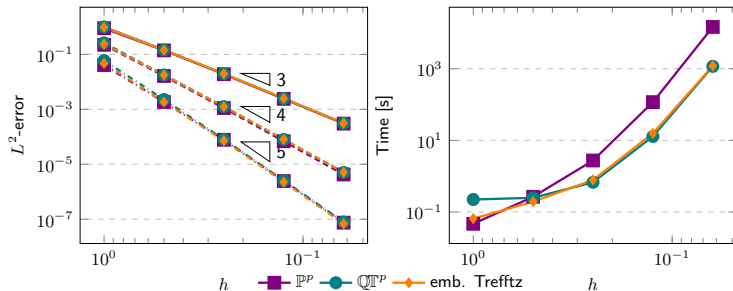
By construction of  $\mathbf{T}$  all its column vectors are orthogonal.



# Example: Acoustic wave equation

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma} + c(\mathbf{x})^{-2} \frac{\partial v}{\partial t} = 0 \\ \nabla v + \frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0} \\ v(\cdot, 0) = v_0, \boldsymbol{\sigma}(\cdot, 0) = \boldsymbol{\sigma}_0 \\ v = g_D \end{cases} \quad \begin{aligned} &\text{in } \Omega \times [0, T], \\ &\text{in } \Omega \times [0, T], \\ &\text{on } \Omega \times \{0\}, \\ &\text{on } \partial\Omega \times [0, T], \end{aligned}$$

With  $c(x, y) = 1 + x + y$





# Embedded Trefftz - inhomogeneous problem

What to do for inhomogeneous problems  $\mathcal{L}u = f$

On each element we can construct a particular solution using the pseudo-inverse

$$\mathbf{W}^\dagger = \begin{pmatrix} | & | & | & | & | \\ \mathbf{v}_1 & \dots & \mathbf{v}_L & \mathbf{v}_{L+1} & \dots & \mathbf{v}_N \\ | & | & | & | & | \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sigma_1} & & & & \\ & \ddots & & & \\ & & \frac{1}{\sigma_L} & & \\ & & & 0 & \ddots & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{---} & \mathbf{u}_1^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{u}_L^T & \text{---} \\ \text{---} & \mathbf{u}_{L+1}^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{u}_N^T & \text{---} \end{pmatrix}$$

For  $u_{h,f}$  a particular solution, we are looking for a solution  $u_{\mathbb{T}} \in \mathbb{T}^p(\mathcal{T}_h)$  that (uniquely) solves

$$a_h(u_{\mathbb{T}}, v_{\mathbb{T}}) = \ell(v_{\mathbb{T}}) - a_h(u_{h,f}, v_{\mathbb{T}}) \quad \forall v_{\mathbb{T}} \in \mathbb{T}^p(\mathcal{T}_h).$$

## Example: Poisson

$$\begin{cases} \Delta u = f \text{ in } \Omega, \\ u = g_D \end{cases} \quad \text{on } \partial\Omega$$

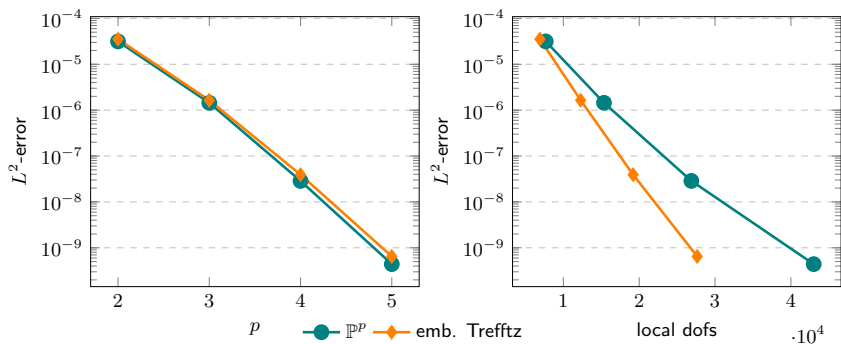


Figure: Numerical results for Poisson equation, 3D, mesh size  $h = 0.25$ .

# Céa-type Lemma

## Lemma (Céa)

*Assume that problem*

$$\text{Find } u_{hp} \in \mathbb{P}^p(\mathcal{T}_h), \quad \text{s.t. } a_h(u_{hp}, v_{hp}) = \ell(v_{hp}) \quad \forall v_{hp} \in \mathbb{P}^p(\mathcal{T}_h).$$

*is well-posed, specifically coercive with respect to  $\|\cdot\|_h$ . Let  $u \in V(\Omega)$  be a weak solution to the PDE problem under consideration and  $u_h \in \mathbb{T}^p(\mathcal{T}_h) + u_{h,f}$  be the Trefftz-DG solution, then*

$$\|u - u_h\|_h \lesssim \inf_{\substack{v_h \in \mathbb{P}^p(\mathcal{T}_h) \\ \Pi \mathcal{L} v_h = \Pi f}} \|u - v_h\|_h$$

# Example code

**Require:** Basis functions  $\{\phi_i\}_i$ , DG formulation  $(a_h, l)$ , operators  $\mathcal{L}, \tilde{\mathcal{L}}$ , truncation parameter  $\varepsilon$ , r.h.s.  $f$

```
1: function DG MATRIX
2:    $(\mathbf{A})_{ij} = a_h(\phi_j, \phi_i)$ 
3:    $(\mathbf{l})_i = \ell(\phi_i)$ 
4: for  $K \in \mathcal{T}_h$  do
5:    $(\mathbf{W}_K)_{ij} = \langle \mathcal{L}\phi_j, \tilde{\mathcal{L}}\phi_i \rangle_{0,h}$ 
6:    $\mathbf{T}_K = \ker_h(\varepsilon; \mathbf{W}_K)$ 
7:   if  $f \neq 0$  then
8:      $(\mathbf{w}_K)_i = \langle f, \tilde{\mathcal{L}}\phi_i \rangle_{0,h}$ 
9:      $(\mathbf{u}_f)_K = \mathbf{W}_K^\dagger \mathbf{w}_K$ 
10: Solve  $\mathbf{T}^T \mathbf{A} \mathbf{T} \mathbf{u}_\mathbb{T} = \mathbf{T}^T (\mathbf{l} - \mathbf{A} \mathbf{u}_f)$ 
11:  $\mathbf{u}_h = \mathbf{T} \mathbf{u}_\mathbb{T} + \mathbf{u}_f$ 
12: output  $\mathbf{u}_h$ 
```

```
1 def Solve(mesh, order, dgscheme,
2           L, Ltilde, eps,
3           rhs):
4     fes = L2(mesh, order=order, dgjumps=True)
5     uh = GridFunction(fes)
6     a, f = dgscheme(fes)
7     u, v = fes.TnT()
8     W = L(u)*Ltilde(v)*dx
9     w = rhs*Ltilde(v)*dx
10    T, uf = TrefftzEmbedding(W, fes, eps, w)
11    Tt = T.CreateTranspose()
12    TA = Tt@a.mat@T
13    ut = TA.Inverse()*(Tt*(f.vec-a.mat*uf))
14    uh.vec.data = T*ut + uf
15    return uh
```

# Algorithmic complexity: A rough comparison

► direct solver    ►  $N_{\text{el}} := \#\mathcal{T}_h \sim h^{-d}$     ►  $p$ -scaling (no constants)

<u>Costs:</u>	Standard DG	Trefftz DG	Embedded Trefftz DG	Hybrid DG
<u>Vector representation:</u>				
total ndofs stored	$\sim N_{\text{el}}p^d$	$\sim N_{\text{el}}p^{d-1}$	$\sim N_{\text{el}}p^d$	$\sim N_{\text{el}}p^d$
globally coupled ndofs	$\sim N_{\text{el}}p^d$	$\sim N_{\text{el}}p^{d-1}$	$\sim N_{\text{el}}p^{d-1}$	$\sim N_{\text{el}}p^{d-1}$
<u>Setup linear systems:</u>				
nnzes <b>A</b>	$\sim N_{\text{el}}p^{2d}$	$\sim N_{\text{el}}p^{2d-2}$	$\sim N_{\text{el}}p^{2d}$	$\sim N_{\text{el}}p^{2d}$
<u>Additional costs:</u>				
	—	—	Setup <b>T</b> : $\sim N_{\text{el}}p^{3d}$	static cond.: $\sim N_{\text{el}}p^{3d}$
<u>Solving linear systems:</u>				
global matrix	<b>A</b>	<b>A</b>	<b>T<sup>T</sup>AT</b>	<b>S</b>
nnzes	$\sim N_{\text{el}}p^{2d}$	$\sim N_{\text{el}}p^{2d-2}$	$\sim N_{\text{el}}p^{2d-2}$	$\sim N_{\text{el}}p^{2d-2}$
arithmetic ops. ( $\mathcal{O}(N^\alpha)$ )	$\sim (N_{\text{el}}p^{2d})^\alpha$	$\sim (N_{\text{el}}p^{2d-2})^\alpha$	$\sim (N_{\text{el}}p^{2d-2})^\alpha$	$\sim (N_{\text{el}}p^{2d-2})^\alpha$

# Conclusion

## Summary

- ▶ reduce test/trial-spaces using a projection that infers structural properties
- ▶ construct an embedding of Trefftz (like) subspaces in a very generic way
- ▶ works for inhomogeneous PDEs and non-constant coefficients

## References



F. Heimann, C. Lehrenfeld, P. Stocker, and H. von Wahl.  
Unfitted Trefftz discontinuous Galerkin methods for elliptic boundary value problems.  
*arXiv preprint arxiv:2212.12236*, 2022.



R. Hiptmair, A. Moiola, and I. Perugia.  
A survey of Trefftz methods for the Helmholtz equation.  
*Lect. Notes Comput. Sci. Eng.*, 2016.



E. Kita and N. Kamiya.  
Trefftz method: an overview.  
*Adv. Eng. Softw.*, 24(1–3):3–12, 1995.



C. Lehrenfeld and P. Stocker.  
Embedded Trefftz discontinuous Galerkin methods.  
*arXiv preprint, arXiv:2201.07041*, 2022.



C. Ma and R. Scheichl.  
Error estimates for fully discrete generalized fems with locally optimal spectral approximations.  
*arXiv preprint, arXiv:2201.07041*, 2021.



[paulst.github.io/NGSTrefftz](https://paulst.github.io/NGSTrefftz)

# Some number crunching

$d$	$p$	ndofs	DG	HDG	TDG(1)	TDG(2)	nnzes	DG	HDG	TDG(1)	TDG(2)
2	0		54	91	54	54		196	415	196	196
2	1		162	182	108	162		1,764	1,660	784	1,764
2	2		324	273	162	270		7,056	3,735	1,764	4,900
2	3		540	364	216	378		19,600	6,640	3,136	9,604
2	4		810	455	270	486		44,100	10,375	4,900	15,876
2	5		1,134	546	324	594		86,436	14,940	7,056	23,716
3	0		729	1,612	729	729		3,337	10,360	3,337	3,337
3	1		2,916	4,836	2,187	2,916		53,392	93,240	30,033	53,392
3	2		7,290	9,672	4,374	6,561		333,700	372,960	120,132	270,297
3	3		14,580	16,120	7,290	11,664		1,334,800	1,036,000	333,700	854,272
3	4		25,515	24,180	10,935	18,225		4,087,825	2,331,000	750,825	2,085,625
3	5		40,824	33,852	15,309	26,244		10,464,832	4,568,760	1,471,617	4,324,752

ndofs: globally coupled ndofs

## Observations

- ▶ Trefftz DG always beats DG
- ▶ Trefftz DG shows no "low order overhead" as Hybrid DG
- ▶ Trefftz DG also beats Hybrid DG (unless "projected jumps" are possible!)
- ▶ For first order problems: Trefftz DG beats Hybrid DG by factor  $\approx 2$  (in ndofs)

## Example: Laplace equation

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

