Advection Diffusion Equation

We are now interested in the advection diffusion equation in one-dimension, i.e. $\partial_t u = \eta \partial_x^2 u + \partial_x u$. We will rewrite the above equation as $\partial_t u = \mathcal{L} u$, with $\mathcal{L} u := \eta \partial_x^2 u + \partial_x u$. In particular we are interested in the case with homogeneous Dirichlet boundary conditions, i.e. $u(0) = u(1) = 0$. We consider the operator $T : H_0^1(0,1) \to H^{-1}(0,1)$ associated with the weak form of $\mathcal{L}$, i.e.

$$(Tu, v)_{L^2} = -(u', v')_{L^2} + (u', v)_{L^2} \quad \text{for any } u \in H_0^1(0,1)_{L^2}.$$

At first one might think that the operator T is NORMAL since integrating by parts we can show that $\hat{\mathcal{L}} = \eta \partial_x^2 u - \partial_x u$ and thus $\hat{\mathcal{L}}\mathcal{L} = \eta^2 \partial_x^4 u - \partial_x^2 u = \mathcal{L}\hat{\mathcal{L}}$. Yet once again the formal adjoint doesn't paint the full picture. In fact $D(TT') \neq D(T'T)$ which is the same issue we discussed for the advection equation in the first notes. In fact let us consider $u \in D(TT') \cap H^2(a,b)$ then we would have to sets of boundary conditions on u. In fact we first need $x \in H_0^1(0,1)$ but then we would also need $\mathcal{L} u \in H_0^1(0,1)$ which implies that trace $u'(x) + \eta u''(x)$ must vanish, while the same reasoning for $u \in D(TT') \cap H^2(\Omega)$ implies that $u \in H_0^1$ and $-u'(x) + \eta u''(x)$ must vanish. Thus we can clearly see why the advection diffusion is not NORMAL while the diffusion equation is self-adjoint and thus in particular normal.

Example In one dimension it is possible to compute analytically the eigenvalues and eigenfunction of the advection diffusion equation. To this aim we introduce the transformation $u(x) = v(x) e^{ax}$. Substituting this into the advection diffusion equation we get, $\eta v''(x) e^{ax} + (2\eta a + 1) v'(x) e^{ax} + (\eta a^2 + a) v(x)$. Hence selecting $a = -\frac{1}{2\eta}$ we are left with $\left(\eta v''(x) - \frac{1}{4\eta} v(x)\right) e^{ax}$. We can now look for the eigen. function of this form, i.e. $\eta v''(x) - \frac{1}{4\eta} v(x) = \mu v(x)$ which is equivalent to look for the eigenvalues of $\eta v''(x)$ and shift them by $-\frac{1}{4\eta}$. Hence we have that the eigenvalue and the eigenfunction of the advection diffusion equation are $\mu_n = -\frac{1}{4}\eta^{-1} - \eta n^2 \pi$ and $u_n = e^{-x/2\eta} \sin(n\pi x)$.

The question that arise quite naturally at this point is the following: "Given that we know the spectrum of the advection diffusion operator by analytical or numerical means, what can we say about the behaviour of the advection diffusion equation?"

We begin observing that the the presence of the exponential term in front of the first the eigenfunctions explains the apperance of a boundary layer, which gets more and more pronounced as $\eta \to 0$. Furthermore, as pointed out by [Davis 2004 London Math. Soc. J. Comp. Math.], the eigenvalue shift away from the origin suggesting that at $\eta \to 0$ we would have a faster decay to the equilibrium. Furthermore the eigenvalue gets closer and closer to each other as $\eta \to 0$ suggesting that the contribution of higher eigenfunctions remain significant for longer period of time as $\eta \to 0$. Furthermore it was first noted by [Trefethen, Rady 1994] that for any $\lambda \in \mathbb{C}$ the function $\phi(x) = \dfrac{e^{\alpha_+ x/\eta} - e^{\alpha_- x/\eta}}{(\alpha_+ - \alpha_-)\eta^{-1}}$ with $\alpha_\pm = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4\eta\mu}$ satisfy the equation $\mathcal{L}\phi = \lambda\phi$ and $\phi(0) = 0$. Furthermore if μ is inside the parabola $\mathrm{Re}(z) = -\eta \mathrm{Im}(z)^2$ then $\alpha_\pm$ both have negative real parts suggesting that ϕ decays exponentially and in particular $\phi(1) \leq C e^{\mu/\eta}$. This gives an intutive explanation to the following result,

Theorem Let γ be a fixed non-real number in the interior of the parabola defined by $\mathrm{Re}(z) = -\eta \mathrm{Im}(z)^2$, so that $\mu = \gamma/\eta$ so that μ lies inside the parabola $\mathrm{Re}(z) = -\eta \mathrm{Im}(z)^2$ and let $\alpha_\pm$ be defined as above. Introducing $\sigma, z \in \mathbb{R}$ such that $\alpha_+ - \alpha_- = \sqrt{1 + 4\gamma} = \sigma + iz$, then

$$\lim_{\eta \to 0} \|(T - \mu I)^{-1}\| \sim 2 e^{-\alpha^*/\eta} \frac{(\sigma^2 + z^2)^{1/2}}{(1 - \sigma^2)(1 + z^2)}, \quad \text{where } \mu = \max\{\mathrm{Re}\,\alpha_+, \mathrm{Re}\,\alpha_-\}.$$

Proof An in-depth discussion of this result can be found in [Trefethen-Embree] and a proof of this result can be found in [TREFETHEN-REDDY 1994].

It follows from the previous theorem that inside the parabola $Re(z) = -\eta\, Im(z)^2$ the norm of the resolvent set is exponentially large which implies that the Helmholtz equation loses physical significance for any value inside the parabola. In particular we notice that the Helmholtz equation associated with the advection diffusion operator loses physical significance outside the parabola $Re(z) = -\eta\, Im(z)^2$ which shrinks to the the real line as $\eta \to \infty$ and creeps toward the imaginary axis as $\eta \to 0$.