Embedded Trefftz discontinuous Galerkin methods

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(polynomial) Trefftz-DG methods

$$\begin{cases} \mathcal{L}u = 0 \text{ in } \Omega, \\ + \text{ bndc. on } \partial\Omega, \end{cases}$$

$$\text{Find } u_h \in \mathbb{T}^p(\mathcal{T}_h), \quad \text{s.t. } a_h^{\mathsf{DG}}(u_h, v_h) = \ell_h^{\mathsf{DG}}(v_h) \qquad \forall v_h \in \mathbb{T}^p(\mathcal{T}_h) \text{ with}$$

$$\boxed{\mathbb{T}^p(\mathcal{T}_h) := \Pi_{K\in\mathcal{T}_h} \mathbb{T}^p(K), \quad \mathbb{T}^p(K) := \{v_h \in \mathbb{P}^p(K) \text{ s.t. } \mathcal{L}u = 0 \text{ on } K\}}$$

$$\dim \mathbb{T}^p(\mathcal{T}_h) = \mathcal{O}(p^{n-1}) \ll \dim \mathbb{P}^p(\mathcal{T}_h) = \mathcal{O}(p^n)$$

Assumption: Trefftz space is 'rich enough' ($\mathcal{L} = \sum_{l=1}^d \alpha_l \partial_{x_l}^{\beta_l}$ for $\alpha_l \in \mathbb{R}$ and $\beta_l \in \mathbb{N}$)

E. Trefftz, *Ein Gegenstück zum Ritzschen Verfahren*, Proc. 2nd Int. Cong. Appl. Mech., Zurich, 1926

Embedded Trefftz-DG method

- Goal: Represent Trefftz basis $\{\psi_j\}_M$ in the standard DG basis $\{\phi_i\}_N$:

$$\psi_j = \sum_{j=1}^{N} \mathbf{T}_{ij}\phi_i, \;\; j = 1, .., M, \;\; \text{for } \mathbf{T} \in \mathbb{R}^{N\times M}.$$

Then instead of solving $\mathbf{A}\mathbf{u}_{\mathbb{P}} = \mathbf{b}$ we solve $\mathbf{T}^T\mathbf{A}\mathbf{T}\;\mathbf{u}_{\mathbb{T}} = \mathbf{T}^T\mathbf{b}$.

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- Recipe: Find u for which $\|\mathcal{L}u\|_{0,h} = 0 \Leftrightarrow \langle \mathcal{L}u, \mathcal{L}v\rangle_{0,h} = 0, \;\; \forall v \in V_h$

$$\boxed{\mathbf{T} = \ker(\mathbf{W}) \text{ with } \mathbf{W} = \sum_{K\in\mathcal{T}_h} \langle \mathcal{L}\phi_i, \mathcal{L}\phi_j\rangle_K}$$

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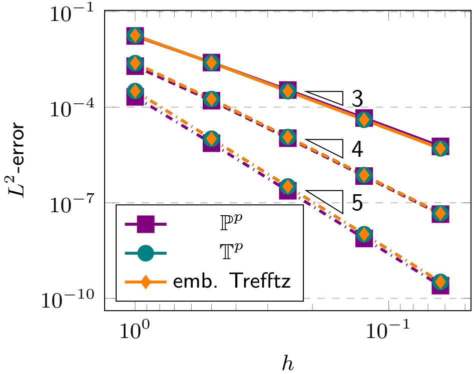
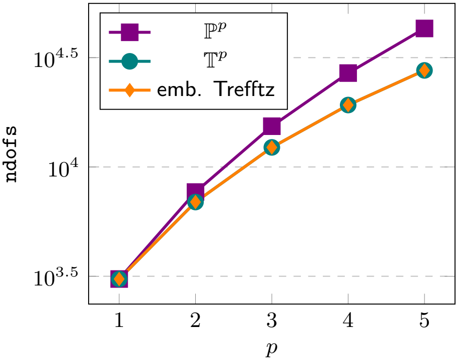
- Benefit: $\mathbf{T}^T\mathbf{A}\mathbf{T} \in \mathbb{R}^{M\times M}$ with

$$\dim \mathbb{T}^p(\mathcal{T}_h) = M \ll N = \dim \mathbb{P}^p(\mathcal{T}_h)$$

Example: Laplace equation

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$





Embedded Trefftz - no polynomial Trefftz space

What to do for operators like $-\Delta \pm \mathrm{id}$, $-\mathrm{div}(\alpha(\mathbf{x})\nabla\cdot)$, $\ldots$

- Idea: Instead of $\langle \mathcal{L}u, \mathcal{L}v\rangle = 0,\ \forall v \in V_h$ we use the relaxed condition

$$\langle \mathcal{L}u, w\rangle_{0,h} = 0,\ \forall w \in W_h \subset V_h$$

($W_h := \mathcal{L}\mathbb{P}^p(\mathcal{T}_h)$ recovers the previous embedding)

- Introduce a **weak Trefftz space** for the embedding

$$\boxed{\mathbb{T}^p(\mathcal{T}_h) = \{v \in \mathbb{P}^p(\mathcal{T}_h),\ \Pi_W \mathcal{L}v = 0 \text{ on each } K \in \mathcal{T}_h\}.}$$

- Proceed with the embedding

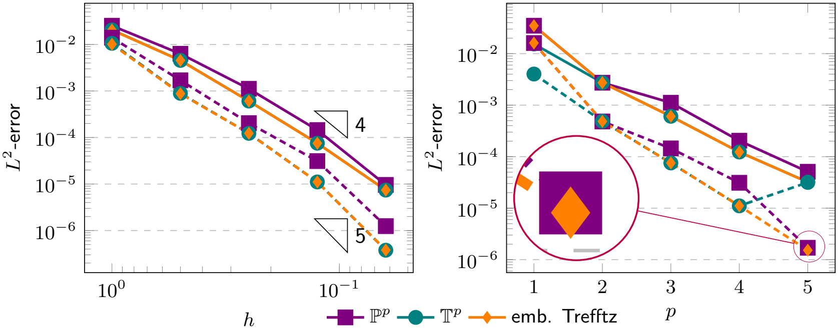
$$\mathbf{T} = \ker(\mathbf{W}) \text{ with } \mathbf{W} = \sum_{K\in\mathcal{T}_h} \langle \mathcal{L}\phi_i, \psi_j\rangle_K, \quad \forall \psi_j \in W_h$$

Example: Helmholtz

$$\begin{cases} -\Delta u - \omega^2 u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n_{\mathbf{x}}} + iu = g & \text{on } \partial\Omega. \end{cases}$$

$$\mathbb{T}^p = \{e^{-i\omega(d_j \cdot \mathbf{x})} \text{ s.t. } j = 0, \dots, k\}$$

$$\mathbb{ET}^p = \{v \in \mathbb{P}^p(\mathcal{T}_h),\ \Pi_{\mathbb{P}^{p-2}}(-\Delta - \omega^2)v = 0 \text{ on each } K \in \mathcal{T}_h\}$$



Conditioning

Lemma (Conditioning of the embedded Trefftz method)

$$\kappa_2(\mathbf{T}^T \mathbf{A} \mathbf{T}) \leq \kappa_2(\mathbf{A})$$

Proof.

By construction of $\mathbf{T}$ all its column vectors are orthogonal. □

Embedded Trefftz - inhomogeneous problem

What to do for inhomogeneous problems $\mathcal{L}u = f$

On each element we can construct a particular solution using the pseudo-inverse

$$\mathbf{W}^\dagger = \begin{pmatrix} | & & | & | & & | \\ \mathbf{v}_1 & \dots & \mathbf{v}_L & \mathbf{v}_{L+1} & \dots & \mathbf{v}_N \\ | & & | & | & & | \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sigma_1} & & & & \\ & \ddots & & & \\ & & \frac{1}{\sigma_L} & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{---} & \mathbf{u}_1^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{u}_L^T & \text{---} \\ \text{---} & \mathbf{u}_{L+1}^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{u}_N^T & \text{---} \end{pmatrix}$$

For $u_{h,f}$ a particular solution, we are looking for a solution $u_{\mathbb{T}} \in \mathbb{T}^p(\mathcal{T}_h)$ that (uniquely) solves

$$\boxed{a_h(u_{\mathbb{T}}, v_{\mathbb{T}}) = \ell(v_{\mathbb{T}}) - a_h(u_{h,f}, v_{\mathbb{T}}) \quad \forall\, v_{\mathbb{T}} \in \mathbb{T}^p(\mathcal{T}_h).}$$

Q. Hu, L. Yuan, *A plane wave method combined with local spectral elements for nonhomogeneous Helmholtz equation [...]* , Adv. Comput. Math., 2018,

A. Uściłowska-Gajda, et al., *Comparison of two types of Trefftz method for the solution of inhomogeneous elliptic problems*, Comput. Assist. Mech. Eng. Sci., 2003,

Example: Possion

$$\begin{cases} \Delta u = f \text{ in } \Omega, \\ u = g_D \qquad\qquad \text{on } \partial\Omega \end{cases}$$

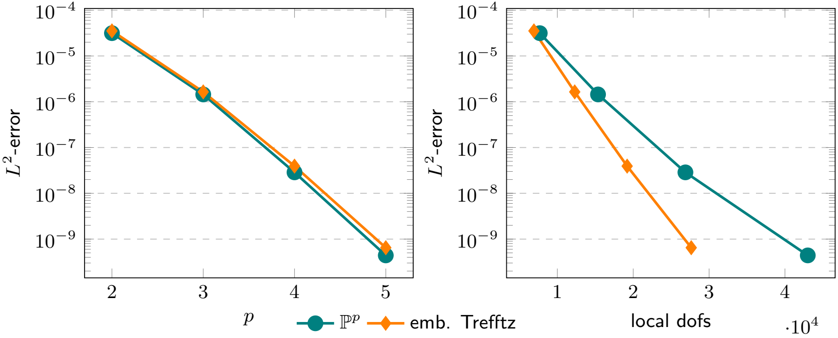


Figure: Numerical results for Poisson equation, 3D, mesh size $h = 0.25$.

Céa-type Lemma

Lemma (Céa)

Assume that problem

$$\text{Find } u_{hp} \in \mathbb{P}^p(\mathcal{T}_h), \quad \text{s.t. } a_h(u_{hp}, v_{hp}) = \ell(v_{hp}) \qquad \forall v_{hp} \in \mathbb{P}^p(\mathcal{T}_h).$$

is well-posed, specifically coercive with respect to $\|\cdot\|_h$*. Let* $u \in V(\Omega)$ *be a weak solution to the PDE problem under consideration and* $u_h \in \mathbb{T}^p(\mathcal{T}_h) + u_{h,f}$ *be the Trefftz-DG solution, then*

$$\|u - u_h\|_h \lesssim \inf_{\substack{v_h \in \mathbb{P}^p(\mathcal{T}_h) \\ \Pi\mathcal{L}v_h = \Pi f}} \|u - v_h\|_h$$

Conclusion

Summary

- reduce test/trial-spaces using a projection that infers structural properties
- construct an embedding of Trefftz (like) subspaces in a very generic way
- works for inhomogeneous PDEs and non-constant coefficients

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Error estimates for fully discrete generalized fems with locally optimal spectral approximations.
arXiv preprint, arXiv:2201.07041, 2021.

paulst.github.io/NGSTrefftz

Some number crunching

d	p	`ndofs` DG	HDG	TDG(1)	TDG(2)	`nnzes` DG	HDG	TDG(1)	TDG(2)
2	0	54	91	54	54	196	415	196	196
2	1	162	182	108	162	1,764	1,660	784	1,764
2	2	324	273	162	270	7,056	3,735	1,764	4,900
2	3	540	364	216	378	19,600	6,640	3,136	9,604
2	4	810	455	270	486	44,100	10,375	4,900	15,876
2	5	1,134	546	324	594	86,436	14,940	7,056	23,716
3	0	729	1,612	729	729	3,337	10,360	3,337	3,337
3	1	2,916	4,836	2,187	2,916	53,392	93,240	30,033	53,392
3	2	7,290	9,672	4,374	6,561	333,700	372,960	120,132	270,297
3	3	14,580	16,120	7,290	11,664	1,334,800	1,036,000	333,700	854,272
3	4	25,515	24,180	10,935	18,225	4,087,825	2,331,000	750,825	2,085,625
3	5	40,824	33,852	15,309	26,244	10,464,832	4,568,760	1,471,617	4,324,752

`ndofs`: globally coupled `ndofs`

Observations

- Trefftz DG always beats DG
- Trefftz DG shows no "low order overhead" as Hybrid DG
- Trefftz DG also beats Hybrid DG (unless "projected jumps" are possible!)
- For first order problems: Trefftz DG beats Hybrid DG by factor ≈ 2 (in `ndofs`)

Example: Laplace equation

$$\mathcal{L}u = -\Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega.$$

$$\boxed{\mathbb{T}^p(K) = \{1, x, y, xy, x^2 - y^2, x^3 - 3xy^2, \dots\}}$$

$$a_h(u,v) = \sum_K \int_K \nabla u \nabla v \, dx + \sum_{F \in \mathcal{F}_h^{\text{int}}} \int_F \overbrace{-\{\!\{\partial_{\mathbf{n}} u\}\!\}[\![v]\!]}^{\text{consistency}} \; \overbrace{-\{\!\{\partial_{\mathbf{n}} v\}\!\}[\![u]\!]}^{\text{symmetry}} \; \overbrace{+ \, \alpha p^2 h^{-1} [\![u]\!][\![v]\!]}^{\text{stability}} \, ds + \text{bnd}$$

$\{\!\{\cdot\}\!\}$: average across facets, $[\![\cdot]\!]$: jump across facets. $\rightsquigarrow$ communication between neighbors.

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R. Hiptmair, A. Moiola, I. Perugia, C. Schwab, *Approximation by harmonic polynomials [...] of Trefftz hp-dGFEM*, ESAIM Math. Model. Num. Anal., 2014

O. Cessenat, B. Despres *Application of an Ultra Weak Variational Formulation of Elliptic PDES to the Two-Dimensional Helmholtz Problem* SIAM J. Num. Anal. 2015

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$$\boxed{\mathbf{T} = \ker(\mathbf{W}) \text{ with } \mathbf{W} = \sum_{K \in \mathcal{T}_h} \langle \mathcal{L}\phi_i, \mathcal{L}\phi_j \rangle_K}$$

On each mesh element use SVD (or QR)

$$\mathbf{W}\big|_K = \begin{pmatrix} | & & | & | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_L & \mathbf{u}_{L+1} & \dots & \mathbf{u}_N \\ | & & | & | & & | \end{pmatrix} \cdot \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_L & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{---} & \mathbf{v}_1^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{v}_L^T & \text{---} \\ \text{---} & \mathbf{v}_{L+1}^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{v}_N^T & \text{---} \end{pmatrix}$$

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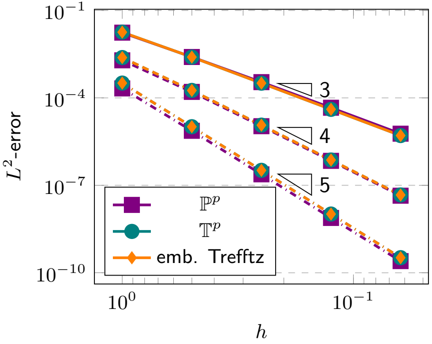

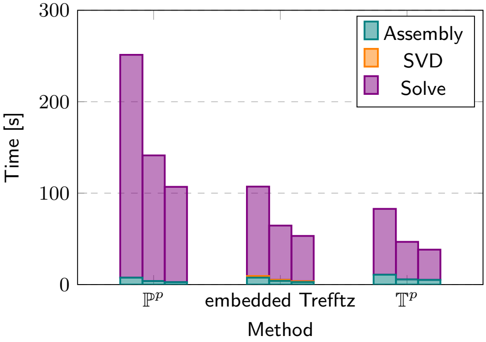

Benefits of Embedded Trefftz DG

So far:

Embedded Trefftz DG ...

1. ... facilitates implementation of existing polynomial Trefftz methods
2. ... is computationally (a bit) more expensive than "direct" Trefftz spaces
3. ... inherites conditioning properties from DG scheme

Next up:

Can we deal with ...

1. ... PDEs where no (suitable) polynomial Trefftz spaces exists?
2. ... inhomogeneous PDEs?

Non-polynomial Trefftz spaces

Many problems don't have suitable polynomial Trefftz spaces

Examples: $\mathcal{L} = -\Delta \pm \mathrm{id}$ $\qquad \mathcal{L} = -\operatorname{div}(\alpha \nabla \cdot)$, α not constant

R. Hiptmair and A. Moiola and I. Perugia, *A Survey of Trefftz Methods for the Helmholtz Equation*, Lect. Notes Comput. Sci. Eng., 2016

L.-M. Imbert-Gérard, A. Moiola, PS, *A space-time quasi-Trefftz DG method for the wave eq. with piecewise-smooth coefficients*, arXiv:2011.04617

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Trefftz DG based on plane waves

For Helmholtz $(-\Delta - \omega^2 \,\mathrm{id})$ Plane Wave DG (a Trefftz DG) spaces exist:

$$\mathbb{T}^p = \{e^{-i\omega(d_j \cdot \mathbf{x})} \text{ s.t. } j = 0, \ldots, k\}$$

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Quasi-Trefftz Methods

Let $\mathcal{L}_\alpha$ be diff. operator depending on a (element-wise) smooth α, define quasi-Trefftz space

$$\mathbb{QT}^p := \{v \in \mathbb{P}^p(\mathcal{T}_h) \mid T^{p-q}_{(\mathbf{x}_{\text{center}})}(\mathcal{L}_\alpha v) = 0\}$$

R. Hiptmair and A. Moiola and I. Perugia, *A Survey of Trefftz Methods for the Helmholtz Equation*, Lect. Notes Comput. Sci. Eng., 2016

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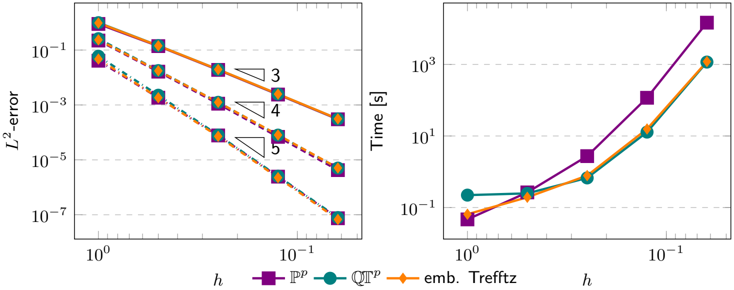
Example: Acoustic wave equation

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma} + c(\mathbf{x})^{-2} \frac{\partial v}{\partial t} = 0 & \text{in } \Omega \times [0,T], \\ \nabla v + \frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0} & \text{in } \Omega \times [0,T], \\ v(\cdot,0) = v_0, \ \boldsymbol{\sigma}(\cdot,0) = \boldsymbol{\sigma}_0 & \text{on } \Omega \times \{0\}, \\ v = g_D & \text{on } \partial\Omega \times [0,T], \end{cases}$$

With $c(x,y) = 1 + x + y$



L.-M. Imbert-Gérard, Andrea Moiola, PS, *A space-time quasi-Trefftz DG method for the wave equation [...]* , arXiv preprint, arXiv:2011.04617, 2021

Example code

Require: Basis functions $\{\phi_i\}_i$, DG formulation (a_h, l), operators $\mathcal{L}$, $\tilde{\mathcal{L}}$, truncation parameter ε, r.h.s. f

function DG MATRIX
- $(\mathbf{A})_{ij} = a_h(\phi_j, \phi_i)$
- $(\mathbf{l})_i = \ell(\phi_i)$

for $K \in \mathcal{T}_h$ **do**
- $(\mathbf{W}_K)_{ij} = \langle \mathcal{L}\phi_j, \tilde{\mathcal{L}}\phi_i \rangle_{0,h}$
- $\mathbf{T}_K = \ker_h(\varepsilon; \mathbf{W}_K)$
- **if** $f \neq 0$ **then**
  - $(\mathbf{w}_K)_i = \langle f, \tilde{\mathcal{L}}\phi_i \rangle_{0,h}$
  - $(\mathbf{u}_f)_K = \mathbf{W}_K^{\dagger} \mathbf{w}_K$

Solve $\mathbf{T}^T \mathbf{A} \mathbf{T}\ \mathbf{u}_{\mathbb{T}} = \mathbf{T}^T(\mathbf{l} - \mathbf{A}\mathbf{u}_f)$

$\mathbf{u}_h = \mathbf{T}\mathbf{u}_{\mathbb{T}} + \mathbf{u}_f$

output $\mathbf{u}_h$

```python
def Solve(mesh, order, dgscheme,
               L, Ltilde, eps,
               rhs):
  fes = L2(mesh,order=order,dgjumps=True)
  uh = GridFunction(fes)
  a,f = dgscheme(fes)
  u,v = fes.TnT()
  W = L(u)*Ltilde(v)*dx
  w = rhs*Ltilde(v)*dx
  T, uf = TrefftzEmbedding(W,fes,eps,w)
  Tt = T.CreateTranspose()
  TA = Tt@a.mat@T
  ut = TA.Inverse()*(Tt*(f.vec-a.mat*uf))
  uh.vec.data = T*ut + uf
  return uh
```

Algorithmic complexity: A rough comparison

- direct solver
- $N_{\text{el}} := \#\mathcal{T}_h \sim h^{-d}$
- p-scaling (no constants)

Costs:	Standard DG	Trefftz DG	Embedded Trefftz DG	Hybrid DG
Vector representation:				
total `ndofs` stored	$\sim N_{\text{el}} p^d$	$\sim N_{\text{el}} p^{d-1}$	$\sim N_{\text{el}} p^d$	$\sim N_{\text{el}} p^d$
globally coupled `ndofs`	$\sim N_{\text{el}} p^d$	$\sim N_{\text{el}} p^{d-1}$	$\sim N_{\text{el}} p^{d-1}$	$\sim N_{\text{el}} p^{d-1}$
Setup linear systems:				
`nnzes` $\mathbf{A}$	$\sim N_{\text{el}} p^{2d}$	$\sim N_{\text{el}} p^{2d-2}$	$\sim N_{\text{el}} p^{2d}$	$\sim N_{\text{el}} p^{2d}$
Additional costs:			Setup $\mathbf{T}$:	static cond.:
	—	—	$\sim N_{\text{el}} p^{3d}$	$\sim N_{\text{el}} p^{3d}$
Solving linear systems:				
global matrix	$\mathbf{A}$	$\mathbf{A}$	$\mathbf{T}^T \mathbf{A} \mathbf{T}$	$\mathbf{S}$
`nnzes`	$\sim N_{\text{el}} p^{2d}$	$\sim N_{\text{el}} p^{2d-2}$	$\sim N_{\text{el}} p^{2d-2}$	$\sim N_{\text{el}} p^{2d-2}$
arithmetic ops. ($\mathcal{O}(N^\alpha)$)	$\sim (N_{\text{el}} p^{2d})^\alpha$	$\sim (N_{\text{el}} p^{2d-2})^\alpha$	$\sim (N_{\text{el}} p^{2d-2})^\alpha$	$\sim (N_{\text{el}} p^{2d-2})^\alpha$