Embedded Trefftz discontinuous Galerkin methods

Christoph Lehrenfeld, Paul Stocker

Georg-August-Universität, Göttingen, Germany

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Trefftz-DG methods

$$\begin{cases} \mathcal{L}u = 0 \text{ in } \Omega, \\ + \text{ bndc. on } \partial\Omega, \end{cases}$$

$$\text{Find } u_h \in \mathbb{T}^p(\mathcal{T}_h), \quad \text{s.t. } a_h^{\mathsf{DG}}(u_h, v_h) = \ell_h^{\mathsf{DG}}(v_h) \qquad \forall v_h \in \mathbb{T}^p(\mathcal{T}_h) \text{ with}$$

$$\boxed{\mathbb{T}^p(\mathcal{T}_h) := \Pi_{K\in\mathcal{T}_h} \mathbb{T}^p(K), \quad \mathbb{T}^p(K) \subset \{u \text{ s.t. } \mathcal{L}u = 0 \text{ on } K\}}$$

$$\dim \mathbb{T}^p(\mathcal{T}_h) = \mathcal{O}(p^{n-1}) \ll \dim \mathbb{P}^p(\mathcal{T}_h) = \mathcal{O}(p^n)$$

with the same approximation properties as $\mathbb{P}^p(\mathcal{T}_h)$.

E. Trefftz, *Ein Gegenstück zum Ritzschen Verfahren*, Proc. 2nd Int. Cong. Appl. Mech., Zurich, 1926

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$\rightsquigarrow$ Thu 15h, 4B4-1, talk by Sergio Gomez!

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Example: Laplace equation

$$\mathcal{L}u = -\Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega.$$

$$\boxed{\mathbb{T}^p(K) = \{1, x, y, xy, x^2 - y^2, x^3 - 3xy^2, \dots\}}$$

$$a_h(u,v) = \sum_K \int_K \nabla u \nabla v \, dx + \sum_{F \in \mathcal{F}_h^{\text{int}}} \int_F \overbrace{-\{\!\{\partial_{\mathbf{n}} u\}\!\}[\![v]\!]}^{\text{consistency}} \; \overbrace{-\{\!\{\partial_{\mathbf{n}} v\}\!\}[\![u]\!]}^{\text{symmetry}} \; \overbrace{+\, \alpha p^2 h^{-1} [\![u]\!][\![v]\!]}^{\text{stability}} \, ds + \text{bnd}$$

$\{\!\{\cdot\}\!\}$: average across facets, $[\![\cdot]\!]$: jump across facets. $\rightsquigarrow$ communication between neighbors.

F. Li, C.-W. Shu, *A local-structure-preserving local discontinuous Galerkin method for the Laplace equation*, Methods Appl. Anal., 2006

R. Hiptmair, A. Moiola, I. Perugia, C. Schwab, *Approximation by harmonic polynomials [...] of Trefftz hp-dGFEM*, ESAIM Math. Model. Num. Anal., 2014

O. Cessenat, B. Despres *Application of an Ultra Weak Variational Formulation of Elliptic PDES to the Two-Dimensional Helmholtz Problem* SIAM J. Num. Anal. 2015

Embedded Trefftz-DG method

- Goal: Represent Trefftz basis $\{\psi_j\}_M$ in the standard DG basis $\{\phi_i\}_N$:

$$\psi_j = \sum_{j=1}^{N} \mathbf{T}_{ij} \phi_i, \;\; j = 1, .., M, \;\text{ for } \mathbf{T} \in \mathbb{R}^{N \times M}.$$

Then instead of solving $\mathbf{A}\mathbf{u}_{\mathbb{P}} = \mathbf{b}$ we solve

$$\boxed{\mathbf{T}^T \mathbf{A} \mathbf{T} \; \mathbf{u}_{\mathbb{T}} = \mathbf{T}^T \mathbf{b}.}$$

with $\dim \mathbb{T}^p(\mathcal{T}_h) = M \ll N = \dim \mathbb{P}^p(\mathcal{T}_h)$

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  with $\dim \mathbb{T}^p(\mathcal{T}_h) = M \ll N = \dim \mathbb{P}^p(\mathcal{T}_h)$
- Recipe: Find u for which $\|\mathcal{L}u\|_{0,h} = 0 \Leftrightarrow \langle \mathcal{L}u, \mathcal{L}v\rangle_{0,h} = 0, \;\forall v \in V_h$

$$\boxed{\mathbf{T} = \ker(\mathbf{W}) \text{ with } \mathbf{W} = \sum_{K\in\mathcal{T}_h} \langle \mathcal{L}\phi_i, \mathcal{L}\phi_j\rangle_K}$$

Embedded Trefftz-DG method

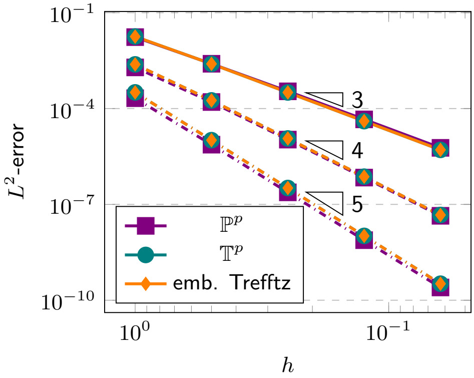
$$\mathbf{T} = \ker(\mathbf{W}) \text{ with } \mathbf{W} = \sum_{K\in\mathcal{T}_h} \langle \mathcal{L}\phi_i, \mathcal{L}\phi_j \rangle_K$$

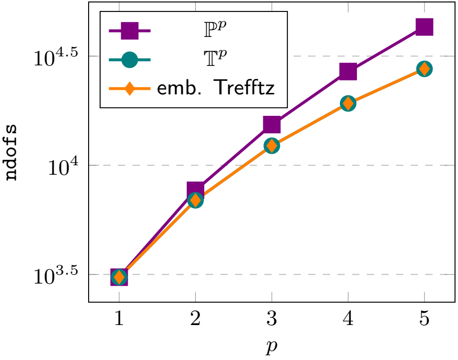
On each mesh element use SVD (or QR)

$$\mathbf{W}\Big|_K = \begin{pmatrix} | & & | & | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_L & \mathbf{u}_{L+1} & \dots & \mathbf{u}_N \\ | & & | & | & & | \end{pmatrix} \cdot \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_L & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{---} & \mathbf{v}_1^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{v}_L^T & \text{---} \\ \text{---} & \mathbf{v}_{L+1}^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{v}_N^T & \text{---} \end{pmatrix}$$

Example: Laplace equation

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$





Embedded Trefftz - inhomogeneous problem

What to do for inhomogeneous problems $\mathcal{L}u = f$

On each element we can construct a particular solution using the pseudo-inverse

$$\mathbf{W}^\dagger = \begin{pmatrix} | & & | & | & & | \\ \mathbf{v}_1 & \dots & \mathbf{v}_L & \mathbf{v}_{L+1} & \dots & \mathbf{v}_N \\ | & & | & | & & | \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sigma_1} & & & & \\ & \ddots & & & \\ & & \frac{1}{\sigma_L} & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{---} & \mathbf{u}_1^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{u}_L^T & \text{---} \\ \text{---} & \mathbf{u}_{L+1}^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{u}_N^T & \text{---} \end{pmatrix}$$

For $u_{h,f}$ a particular solution, we are looking for a solution $u_{\mathbb{T}} \in \mathbb{T}^p(\mathcal{T}_h)$ that (uniquely) solves

$$\boxed{a_h(u_{\mathbb{T}}, v_{\mathbb{T}}) = \ell(v_{\mathbb{T}}) - a_h(u_{h,f}, v_{\mathbb{T}}) \quad \forall\ v_{\mathbb{T}} \in \mathbb{T}^p(\mathcal{T}_h).}$$

Q. Hu, L. Yuan, *A plane wave method combined with local spectral elements for nonhomogeneous Helmholtz equation [...]* , Adv. Comput. Math., 2018,

A. Uściłowska-Gajda, et al., *Comparison of two types of Trefftz method for the solution of inhomogeneous elliptic problems*, Comput. Assist. Mech. Eng. Sci., 2003,

Example: Possion

$$\begin{cases} \Delta u = f \text{ in } \Omega, \\ u = g_D \qquad\qquad \text{on } \partial\Omega \end{cases}$$

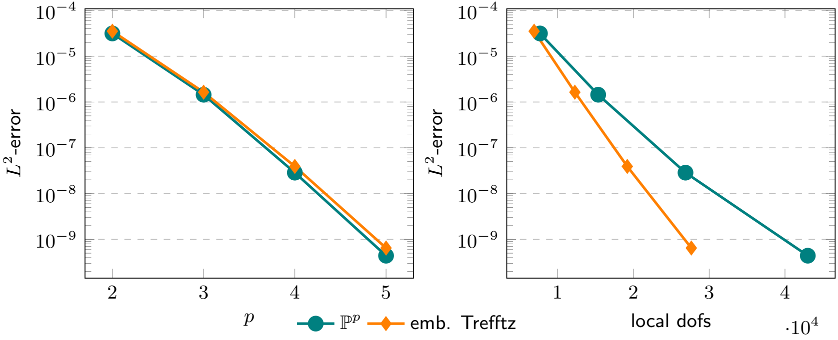


Figure: Numerical results for Poisson equation, 3D, mesh size $h = 0.25$.

Embedded Trefftz - no polynomial Trefftz space

What to do for operators like $\quad -\Delta \pm \mathrm{id}, \quad -\mathrm{div}(\alpha(\mathbf{x})\nabla\cdot), \quad \dots$

- Idea: Instead of $\langle \mathcal{L}u, \mathcal{L}v\rangle = 0,\ \forall v \in V_h$ we use the relaxed condition

$$\langle \mathcal{L}u, w\rangle_{0,h} = 0,\ \forall w \in W_h \subset V_h$$

($W_h := \mathcal{L}\mathbb{P}^p(\mathcal{T}_h)$ recovers the previous embedding)

- Introduce a **weak Trefftz space** for the embedding

$$\boxed{\mathbb{T}^p(\mathcal{T}_h) = \{v \in \mathbb{P}^p(\mathcal{T}_h),\ \Pi_W \mathcal{L}v = 0 \text{ on each } K \in \mathcal{T}_h\}.}$$

- Proceed with the embedding

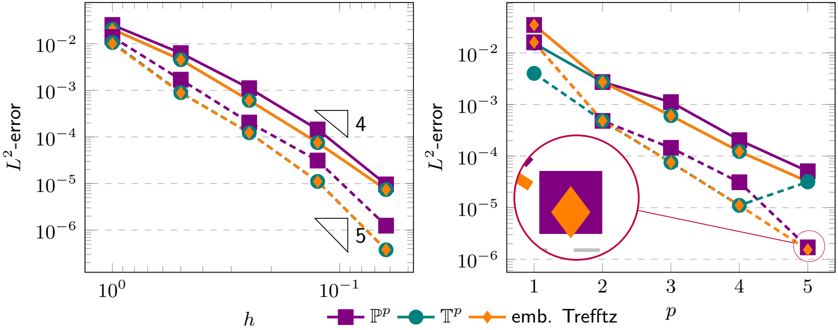
$$\mathbf{T} = \ker(\mathbf{W}) \text{ with } \mathbf{W} = \sum_{K\in\mathcal{T}_h} \langle \mathcal{L}\phi_i, \psi_j\rangle_K, \quad \forall \psi_j \in W_h$$

Example: Helmholtz

$$\begin{cases} -\Delta u - \omega^2 u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n_{\mathbf{x}}} + iu = g & \text{on } \partial\Omega. \end{cases}$$

$$\mathbb{T}^p = \{e^{-i\omega(d_j \cdot \mathbf{x})} \text{ s.t. } j = 0, \dots, k\}$$

$$\mathbb{ET}^p = \{v \in \mathbb{P}^p(\mathcal{T}_h),\ \Pi_{\mathbb{P}^{p-2}}(-\Delta - \omega^2)v = 0 \text{ on each } K \in \mathcal{T}_h\}$$



Conditioning

Lemma (Conditioning of the embedded Trefftz method)

$$\kappa_2(\mathbf{T}^T\mathbf{A}\mathbf{T}) \leq \kappa_2(\mathbf{A})$$

Proof.

By construction of $\mathbf{T}$ all its column vectors are orthogonal. $\square$

Example: Linear transport

We consider

$$\mathbf{b} \cdot \nabla u = f \quad \text{in } \Omega$$

and the weak Trefftz space

$$\mathbb{ET}^p = \{v \in \mathbb{P}^p(\mathcal{T}_h),\, \Pi_{\mathbb{P}^{p-1}}(\mathbf{b} \cdot \nabla)v = 0 \text{ on each } K \in \mathcal{T}_h\}$$

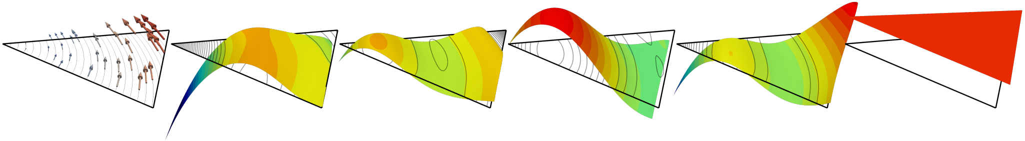


Figure: (Non-constant) Flow field (left) and shape functions of the weak Trefftz space for $p = 4$ for a triangle and $\mathcal{L} = \mathbf{b} \cdot \nabla$.

Céa-type Lemma

Lemma (Céa)

Assume that problem

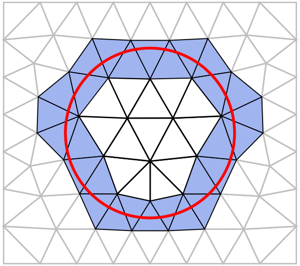
$$\textit{Find } u_{hp} \in \mathbb{P}^p(\mathcal{T}_h), \quad \textit{s.t. } a_h(u_{hp}, v_{hp}) = \ell(v_{hp}) \qquad \forall v_{hp} \in \mathbb{P}^p(\mathcal{T}_h).$$

is well-posed, specifically coercive with respect to $\|\cdot\|_h$. *Let* $u \in V(\Omega)$ *be a weak solution to the PDE problem under consideration and* $u_h \in \mathbb{T}^p(\mathcal{T}_h) + u_{h,f}$ *be the Trefftz-DG solution, then*

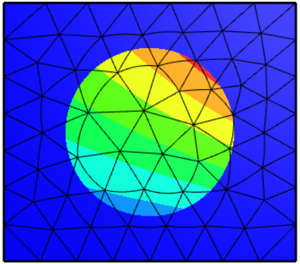
$$\|u - u_h\|_h \lesssim \inf_{\substack{v_h \in \mathbb{P}^p(\mathcal{T}_h) \\ \Pi \mathcal{L} v_h = \Pi f}} \|u - v_h\|_h$$

Outlook: Trefftz on unfitted geometries

- Trefftz methods can be used with unfitted DG method in a natural way
- cuts can lead to shape irregular elements require stabilization
- different stabilization techniques can be easily used with Trefftz methods



$\longrightarrow$



$\rightsquigarrow$ Thu 10h, 4A6-1, talk by Henry von Wahl!

Fabian Heimann, Christoph Lehrenfeld, PS, Henry von Wahl, *Unfitted Trefftz discontinuous Galerkin methods for elliptic boundary value problems*, arXiv:2212.12236, 2022

Conclusion

Summary

- reduce test/trial-spaces using a projection that infers structural properties
- construct an embedding of Trefftz (like) subspaces in a very generic way
- works for inhomogeneous PDEs and non-constant coefficients

References

S. Gómez, A. Moiola, I. Perugia, and P. Stocker.
On polynomial Trefftz spaces for the linear time-dependent Schrödinger equation.
arXiv preprint, 2023.

F. Heimann, C. Lehrenfeld, P. Stocker, and H. von Wahl.
Unfitted Trefftz discontinuous Galerkin methods for elliptic boundary value problems.
arXiv preprint arxiv:2212.12236, 2022.

P. L. Lederer, C. Lehrenfeld, and P. Stocker.
Trefftz discontinuous Galerkin discretization for the Stokes problem.
arXiv preprint, 2023.

C. Lehrenfeld and P. Stocker.
Embedded Trefftz discontinuous Galerkin methods.
Int. J. Numer. Methods Eng., 124(17):3637–3661, 2023.

C. Ma and R. Scheichl.
Error estimates for fully discrete generalized fems with locally optimal spectral approximations.
arXiv preprint, arXiv:2201.07041, 2021.

paulst.github.io/NGSTrefftz

Some number crunching

d	p	`ndofs` DG	HDG	TDG(1)	TDG(2)	`nnzes` DG	HDG	TDG(1)	TDG(2)
2	0	54	91	54	54	196	415	196	196
2	1	162	182	108	162	1,764	1,660	784	1,764
2	2	324	273	162	270	7,056	3,735	1,764	4,900
2	3	540	364	216	378	19,600	6,640	3,136	9,604
2	4	810	455	270	486	44,100	10,375	4,900	15,876
2	5	1,134	546	324	594	86,436	14,940	7,056	23,716
3	0	729	1,612	729	729	3,337	10,360	3,337	3,337
3	1	2,916	4,836	2,187	2,916	53,392	93,240	30,033	53,392
3	2	7,290	9,672	4,374	6,561	333,700	372,960	120,132	270,297
3	3	14,580	16,120	7,290	11,664	1,334,800	1,036,000	333,700	854,272
3	4	25,515	24,180	10,935	18,225	4,087,825	2,331,000	750,825	2,085,625
3	5	40,824	33,852	15,309	26,244	10,464,832	4,568,760	1,471,617	4,324,752

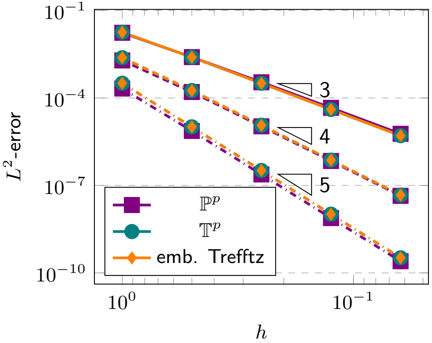
`ndofs`: globally coupled `ndofs`

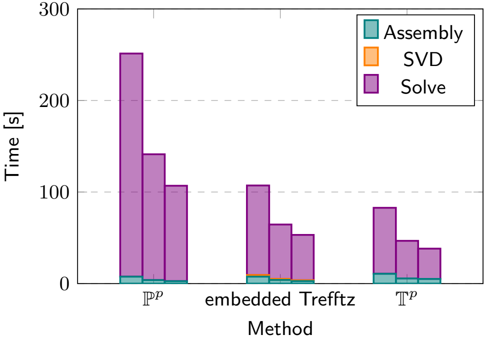
Observations

- Trefftz DG always beats DG
- Trefftz DG shows no "low order overhead" as Hybrid DG
- Trefftz DG also beats Hybrid DG (unless "projected jumps" are possible!)
- For first order problems: Trefftz DG beats Hybrid DG by factor ≈ 2 (in `ndofs`)

Example: Laplace equation

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

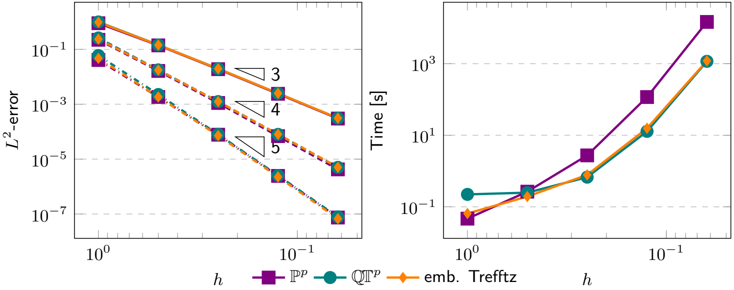




Example: Acoustic wave equation

$$
\begin{cases}
\nabla \cdot \boldsymbol{\sigma} + c(\mathbf{x})^{-2}\frac{\partial v}{\partial t} = 0 & \text{in } \Omega \times [0,T], \\
\nabla v + \frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0} & \text{in } \Omega \times [0,T], \\
v(\cdot,0) = v_0, \ \ \boldsymbol{\sigma}(\cdot,0) = \boldsymbol{\sigma}_0 & \text{on } \Omega \times \{0\}, \\
v = g_D & \text{on } \partial\Omega \times [0,T],
\end{cases}
$$

With $c(x,y) = 1 + x + y$



L.-M. Imbert-Gérard, Andrea Moiola, PS, *A space-time quasi-Trefftz DG method for the wave equation [...]* , arXiv preprint, arXiv:2011.04617, 2021

Example code

Require: Basis functions $\{\phi_i\}_i$, DG formulation (a_h, l), operators $\mathcal{L}$, $\tilde{\mathcal{L}}$, truncation parameter ε, r.h.s. f

function DG MATRIX
  $(\mathbf{A})_{ij} = a_h(\phi_j, \phi_i)$
  $(\mathbf{l})_i = \ell(\phi_i)$
for $K \in \mathcal{T}_h$ **do**
  $(\mathbf{W}_K)_{ij} = \langle \mathcal{L}\phi_j, \tilde{\mathcal{L}}\phi_i \rangle_{0,h}$
  $\mathbf{T}_K = \ker_h(\varepsilon; \mathbf{W}_K)$
  if $f \neq 0$ **then**
    $(\mathbf{w}_K)_i = \langle f, \tilde{\mathcal{L}}\phi_i \rangle_{0,h}$
    $(\mathbf{u}_f)_K = \mathbf{W}_K^{\dagger} \mathbf{w}_K$
Solve $\mathbf{T}^T \mathbf{A} \mathbf{T}\, \mathbf{u}_{\mathbb{T}} = \mathbf{T}^T (\mathbf{l} - \mathbf{A}\mathbf{u}_f)$
$\mathbf{u}_h = \mathbf{T}\mathbf{u}_{\mathbb{T}} + \mathbf{u}_f$
output $\mathbf{u}_h$

```
def Solve(mesh, order, dgscheme,
          L, Ltilde, eps,
          rhs):
  fes = L2(mesh,order=order,dgjumps=True)
  uh = GridFunction(fes)
  a,f = dgscheme(fes)
  u,v = fes.TnT()
  W = L(u)*Ltilde(v)*dx
  w = rhs*Ltilde(v)*dx
  T, uf = TrefftzEmbedding(W,fes,eps,w)
  Tt = T.CreateTranspose()
  TA = Tt@a.mat@T
  ut = TA.Inverse()*(Tt*(f.vec-a.mat*uf))
  uh.vec.data = T*ut + uf
  return uh
```

Algorithmic complexity: A rough comparison

- direct solver
- $N_{\text{el}} := \#\mathcal{T}_h \sim h^{-d}$
- p-scaling (no constants)

Costs:	Standard DG	Trefftz DG	Embedded Trefftz DG	Hybrid DG
Vector representation:				
total `ndofs` stored	$\sim N_{\text{el}} p^d$	$\sim N_{\text{el}} p^{d-1}$	$\sim N_{\text{el}} p^d$	$\sim N_{\text{el}} p^d$
globally coupled `ndofs`	$\sim N_{\text{el}} p^d$	$\sim N_{\text{el}} p^{d-1}$	$\sim N_{\text{el}} p^{d-1}$	$\sim N_{\text{el}} p^{d-1}$
Setup linear systems:				
`nnzes` $\mathbf{A}$	$\sim N_{\text{el}} p^{2d}$	$\sim N_{\text{el}} p^{2d-2}$	$\sim N_{\text{el}} p^{2d}$	$\sim N_{\text{el}} p^{2d}$
Additional costs:			Setup $\mathbf{T}$:	static cond.:
	—	—	$\sim N_{\text{el}} p^{3d}$	$\sim N_{\text{el}} p^{3d}$
Solving linear systems:				
global matrix	$\mathbf{A}$	$\mathbf{A}$	$\mathbf{T}^T\mathbf{A}\mathbf{T}$	$\mathbf{S}$
`nnzes`	$\sim N_{\text{el}} p^{2d}$	$\sim N_{\text{el}} p^{2d-2}$	$\sim N_{\text{el}} p^{2d-2}$	$\sim N_{\text{el}} p^{2d-2}$
arithmetic ops. ($\mathcal{O}(N^\alpha)$)	$\sim (N_{\text{el}} p^{2d})^\alpha$	$\sim (N_{\text{el}} p^{2d-2})^\alpha$	$\sim (N_{\text{el}} p^{2d-2})^\alpha$	$\sim (N_{\text{el}} p^{2d-2})^\alpha$