Rational Numbers

1.1 Introduction

In Mathematics, we frequently come across simple equations to be solved. For example, the equation

$$x + 2 = 13 \tag{1}$$

is solved when $x = 11$, because this value of x satisfies the given equation. The solution 11 is a **natural number**. On the other hand, for the equation

$$x + 5 = 5 \tag{2}$$

the solution gives the **whole number** 0 (zero). If we consider only natural numbers, equation (2) cannot be solved. To solve equations like (2), we added the number zero to the collection of natural numbers and obtained the whole numbers. Even whole numbers will not be sufficient to solve equations of type

$$x + 18 = 5 \tag{3}$$

Do you see 'why'? We require the number –13 which is not a whole number. This led us to think of **integers, (positive and negative)**. Note that the positive integers correspond to natural numbers. One may think that we have enough numbers to solve all simple equations with the available list of integers. Consider the equations

$$2x = 3 \tag{4}$$
$$5x + 7 = 0 \tag{5}$$

for which we cannot find a solution from the integers. (Check this)

We need the numbers $\dfrac{3}{2}$ to solve equation (4) and $\dfrac{-7}{5}$ to solve equation (5). This leads us to the collection of **rational numbers**.

We have already seen basic operations on rational numbers. We now try to explore some properties of operations on the different types of numbers seen so far.

1.2 Properties of Rational Numbers

1.2.1 Closure

(i) Whole numbers

Let us revisit the closure property for all the operations on whole numbers in brief.

Operation	Numbers	Remarks
Addition	$0 + 5 = 5$, a whole number $4 + 7 = ...$. Is it a whole number? In general, $a + b$ is a whole number for any two whole numbers a and b.	Whole numbers are closed under addition.
Subtraction	$5 - 7 = -2$, which is not a whole number.	Whole numbers are **not** closed under subtraction.
Multiplication	$0 \times 3 = 0$, a whole number $3 \times 7 = ...$. Is it a whole number? In general, if a and b are any two whole numbers, their product ab is a whole number.	Whole numbers are closed under multiplication.
Division	$5 \div 8 = \dfrac{5}{8}$, which is not a whole number.	Whole numbers are **not** closed under division.

Check for closure property under all the four operations for natural numbers.

(ii) Integers

Let us now recall the operations under which integers are closed.

Operation	Numbers	Remarks
Addition	$-6 + 5 = -1$, an integer Is $-7 + (-5)$ an integer? Is $8 + 5$ an integer? In general, $a + b$ is an integer for any two integers a and b.	Integers are closed under addition.
Subtraction	$7 - 5 = 2$, an integer Is $5 - 7$ an integer? $-6 - 8 = -14$, an integer	Integers are closed under subtraction.

	$-6 - (-8) = 2$, an integer Is $8 - (-6)$ an integer? In general, for any two integers a and b, $a - b$ is again an integer. Check if $b - a$ is also an integer.	
Multiplication	$5 \times 8 = 40$, an integer Is -5×8 an integer? $-5 \times (-8) = 40$, an integer In general, for any two integers a and b, $a \times b$ is also an integer.	Integers are closed under multiplication.
Division	$5 \div 8 = \dfrac{5}{8}$, which is not an integer.	Integers are **not** closed under division.

You have seen that whole numbers are closed under addition and multiplication but not under subtraction and division. However, integers are closed under addition, subtraction and multiplication but not under division.

(iii) Rational numbers

Recall that a number which can be written in the form $\dfrac{p}{q}$, where p and q are integers

and $q \neq 0$ is called a **rational number**. For example, $-\dfrac{2}{3}$, $\dfrac{6}{7}$ are all rational

numbers. Since the numbers $0, -2, 4$ can be written in the form $\dfrac{p}{q}$, they are also

rational numbers. (Check it!)

(a) You know how to add two rational numbers. Let us add a few pairs.

$$\frac{3}{8} + \frac{(-5)}{7} = \frac{21 + (-40)}{56} = \frac{-19}{56} \qquad \text{(a rational number)}$$

$$\frac{-3}{8} + \frac{(-4)}{5} = \frac{-15 + (-32)}{40} = ... \qquad \text{Is it a rational number?}$$

$$\frac{4}{7} + \frac{6}{11} = ... \qquad \text{Is it a rational number?}$$

We find that sum of two rational numbers is again a rational number. Check it for a few more pairs of rational numbers.

We say that *rational numbers are closed under addition. That is, for any two rational numbers a and b, a + b is also a rational number.*

(b) Will the difference of two rational numbers be again a rational number?
We have,

$$\frac{-5}{7} - \frac{2}{3} = \frac{-5 \times 3 - 2 \times 7}{21} = \frac{-29}{21} \qquad \text{(a rational number)}$$

$$\frac{5}{8} - \frac{4}{5} = \frac{25 - 32}{40} = \dots \qquad \text{Is it a rational number?}$$

$$\frac{3}{7} - \left(\frac{-8}{5}\right) = \dots \qquad \text{Is it a rational number?}$$

Try this for some more pairs of rational numbers. We find that *rational numbers are closed under subtraction. That is, for any two rational numbers a and b, a – b is also a rational number.*

(c)   Let us now see the product of two rational numbers.

$$\frac{-2}{3} \times \frac{4}{5} = \frac{-8}{15}; \frac{3}{7} \times \frac{2}{5} = \frac{6}{35} \qquad \text{(both the products are rational numbers)}$$

$$-\frac{4}{5} \times \frac{-6}{11} = \dots \qquad \text{Is it a rational number?}$$

Take some more pairs of rational numbers and check that their product is again a rational number.

We say that *rational numbers are closed under multiplication. That is, for any two rational numbers a and b, a × b is also a rational number.*

(d)   We note that $\dfrac{-5}{3} \div \dfrac{2}{5} = \dfrac{-25}{6}$ (a rational number)

$\dfrac{2}{7} \div \dfrac{5}{3} = \dots$. Is it a rational number? $\dfrac{-3}{8} \div \dfrac{-2}{9} = \dots$. Is it a rational number?

Can you say that rational numbers are closed under division?

We find that for any rational number a, $a \div 0$ is **not defined**.

So rational numbers are **not closed** under division.

However, if we exclude zero then the collection of, all other rational numbers is closed under division.

TRY THESE

Fill in the blanks in the following table.

Numbers	Closed under			
	addition	**subtraction**	**multiplication**	**division**
Rational numbers	Yes	Yes	...	No
Integers	...	Yes	...	No
Whole numbers	...	...	Yes	...
Natural numbers	...	No	...	...

1.2.2 Commutativity

(i) Whole numbers

Recall the commutativity of different operations for whole numbers by filling the following table.

Operation	Numbers	Remarks
Addition	$0 + 7 = 7 + 0 = 7$ $2 + 3 = ... + ... = ...$ For any two whole numbers a and b, $a + b = b + a$	Addition is commutative.
Subtraction	....	Subtraction is not commutative.
Multiplication	....	Multiplication is commutative.
Division	....	Division is not commutative.

Check whether the commutativity of the operations hold for natural numbers also.

(ii) Integers

Fill in the following table and check the commutativity of different operations for integers:

Operation	Numbers	Remarks
Addition	....	Addition is commutative.
Subtraction	Is $5 - (-3) = -3 - 5$?	Subtraction is not commutative.
Multiplication	....	Multiplication is commutative.
Division	....	Division is not commutative.

(iii) Rational numbers

(a) Addition

You know how to add two rational numbers. Let us add a few pairs here.

$$\frac{-2}{3} + \frac{5}{7} = \frac{1}{21} \text{ and } \frac{5}{7} + \left(\frac{-2}{3}\right) = \frac{1}{21}$$

So,
$$\frac{-2}{3} + \frac{5}{7} = \frac{5}{7} + \left(\frac{-2}{3}\right)$$

Also,
$$\frac{-6}{5} + \left(\frac{-8}{3}\right) = ... \text{ and } \frac{-8}{3} + \left(\frac{-6}{5}\right) = ...$$

Is
$$\frac{-6}{5} + \left(\frac{-8}{3}\right) = \left(\frac{-8}{3}\right) + \left(\frac{-6}{5}\right)?$$

Is $$\frac{-3}{8}+\frac{1}{7}=\frac{1}{7}+\left(\frac{-3}{8}\right)?$$

You find that two *rational numbers can be added in any order. We say that addition is commutative for rational numbers. That is, for any two rational numbers a and b, a + b = b + a.*

(b) Subtraction

Is $$\frac{2}{3}-\frac{5}{4}=\frac{5}{4}-\frac{2}{3}?$$

Is $$\frac{1}{2}-\frac{3}{5}=\frac{3}{5}-\frac{1}{2}?$$

You will find that subtraction is not commutative for rational numbers.

(c) Multiplication

We have, $$\frac{-7}{3}\times\frac{6}{5}=\frac{-42}{15}=\frac{6}{5}\times\left(\frac{-7}{3}\right)$$

Is $$\frac{-8}{9}\times\left(\frac{-4}{7}\right)=\frac{-4}{7}\times\left(\frac{-8}{9}\right)?$$

Check for some more such products.

You will find that *multiplication is commutative for rational numbers*.

In general, a × b = b × a for any two rational numbers a and b.

(d) Division

Is $$\frac{-5}{4}\div\frac{3}{7}=\frac{3}{7}\div\left(\frac{-5}{4}\right)?$$

You will find that expressions on both sides are not equal.

So division is **not commutative** for rational numbers.

TRY THESE

Complete the following table:

Numbers	Commutative for			
	addition	**subtraction**	**multiplication**	**division**
Rational numbers	Yes	...	...	...
Integers	...	No	...	...
Whole numbers	...	...	Yes	...
Natural numbers	...	...	...	No

1.2.3 Associativity

(i) Whole numbers

Recall the associativity of the four operations for whole numbers through this table:

Operation	Numbers	Remarks
Addition	....	Addition is associative
Subtraction	....	Subtraction is **not** associative
Multiplication	Is $7 \times (2 \times 5) = (7 \times 2) \times 5$? Is $4 \times (6 \times 0) = (4 \times 6) \times 0$? For any three whole numbers a, b and c $a \times (b \times c) = (a \times b) \times c$	Multiplication is associative
Division	....	Division is **not** associative

Fill in this table and verify the remarks given in the last column.
Check for yourself the associativity of different operations for natural numbers.

(ii) Integers

Associativity of the four operations for integers can be seen from this table

Operation	Numbers	Remarks
Addition	Is $(-2) + [3 + (-4)]$ $= [(-2) + 3)] + (-4)$? Is $(-6) + [(-4) + (-5)]$ $= [(-6) + (-4)] + (-5)$? For any three integers a, b and c $a + (b + c) = (a + b) + c$	Addition is associative
Subtraction	Is $5 - (7 - 3) = (5 - 7) - 3$?	Subtraction is **not** associative
Multiplication	Is $5 \times [(-7) \times (-8)$ $= [5 \times (-7)] \times (-8)$? Is $(-4) \times [(-8) \times (-5)]$ $= [(-4) \times (-8)] \times (-5)$? For any three integers a, b and c $a \times (b \times c) = (a \times b) \times c$	Multiplication is associative
Division	Is $[(-10) \div 2] \div (-5)$ $= (-10) \div [2 \div (-5)]$?	Division is **not** associative

(iii) Rational numbers

(a) Addition

We have

$$\frac{-2}{3}+\left[\frac{3}{5}+\left(\frac{-5}{6}\right)\right]=\frac{-2}{3}+\left(\frac{-7}{30}\right)=\frac{-27}{30}=\frac{-9}{10}$$

$$\left[\frac{-2}{3}+\frac{3}{5}\right]+\left(\frac{-5}{6}\right)=\frac{-1}{15}+\left(\frac{-5}{6}\right)=\frac{-27}{30}=\frac{-9}{10}$$

So, $$\frac{-2}{3}+\left[\frac{3}{5}+\left(\frac{-5}{6}\right)\right]=\left[\frac{-2}{3}+\frac{3}{5}\right]+\left(\frac{-5}{6}\right)$$

Find $\dfrac{-1}{2}+\left[\dfrac{3}{7}+\left(\dfrac{-4}{3}\right)\right]$ and $\left[\dfrac{-1}{2}+\dfrac{3}{7}\right]+\left(\dfrac{-4}{3}\right)$. Are the two sums equal?

Take some more rational numbers, add them as above and see if the two sums are equal. We find that *addition is associative for rational numbers. That is, for any three rational numbers a*, *b* and *c*, *a* + (*b* + *c*) = (*a* + *b*) + *c*.

(b) Subtraction

Is $$\frac{-2}{3}-\left[\frac{-4}{5}-\frac{1}{2}\right]=\left[\frac{2}{3}-\left(\frac{-4}{5}\right)\right]-\frac{1}{2}?$$

Check for yourself.

Subtraction is **not associative** for rational numbers.

(c) Multiplication

Let us check the associativity for multiplication.

$$\frac{-7}{3}\times\left(\frac{5}{4}\times\frac{2}{9}\right)=\frac{-7}{3}\times\frac{10}{36}=\frac{-70}{108}=\frac{-35}{54}$$

$$\left(\frac{-7}{3}\times\frac{5}{4}\right)\times\frac{2}{9}=...$$

We find that $$\frac{-7}{3}\times\left(\frac{5}{4}\times\frac{2}{9}\right)=\left(\frac{-7}{3}\times\frac{5}{4}\right)\times\frac{2}{9}$$

Is $$\frac{2}{3}\times\left(\frac{-6}{7}\times\frac{4}{5}\right)=\left(\frac{2}{3}\times\frac{-6}{7}\right)\times\frac{4}{5}?$$

Take some more rational numbers and check for yourself.

We observe that *multiplication is associative for rational numbers. That is for any three rational numbers a*, *b* and *c*, *a* × (*b* × *c*) = (*a* × *b*) × *c*.

(d) Division

Let us see if $\dfrac{1}{2} \div \left[\dfrac{-1}{3} \div \dfrac{2}{5} \right] = \left[\dfrac{1}{2} \div \left(\dfrac{-1}{2} \right) \right] \div \dfrac{2}{5}$

We have, LHS $= \dfrac{1}{2} \div \left(\dfrac{-1}{3} \div \dfrac{2}{5} \right) = \dfrac{1}{2} \div \left(\dfrac{-1}{3} \times \dfrac{5}{2} \right)$ (reciprocal of $\dfrac{2}{5}$ is $\dfrac{5}{2}$)

$= \dfrac{1}{2} \div \left(-\dfrac{5}{6} \right) = \ldots$

RHS $= \left[\dfrac{1}{2} \div \left(\dfrac{-1}{3} \right) \right] \div \dfrac{2}{5}$

$= \left(\dfrac{1}{2} \times \dfrac{-3}{1} \right) \div \dfrac{2}{5} = \dfrac{-3}{2} \div \dfrac{2}{5} = \ldots$

Is LHS = RHS? Check for yourself. You will find that division is **not associative** for rational numbers.

TRY THESE

Complete the following table:

Numbers	Associative for			
	addition	subtraction	multiplication	division
Rational numbers	...	...	...	No
Integers	...	...	Yes	...
Whole numbers	Yes	...	...	...
Natural numbers	...	No	...	...

Example 1: Find $\dfrac{3}{7} + \left(\dfrac{-6}{11} \right) + \left(\dfrac{-8}{21} \right) + \left(\dfrac{5}{22} \right)$

Solution: $\dfrac{3}{7} + \left(\dfrac{-6}{11} \right) + \left(\dfrac{-8}{21} \right) + \left(\dfrac{5}{22} \right)$

$= \dfrac{198}{462} + \left(\dfrac{-252}{462} \right) + \left(\dfrac{-176}{462} \right) + \left(\dfrac{105}{462} \right)$ (Note that 462 is the LCM of 7, 11, 21 and 22)

$= \dfrac{198 - 252 - 176 + 105}{462} = \dfrac{-125}{462}$

We can also solve it as.

$$\frac{3}{7} + \left(\frac{-6}{11}\right) + \left(\frac{-8}{21}\right) + \frac{5}{22}$$

$$= \left[\frac{3}{7} + \left(\frac{-8}{21}\right)\right] + \left[\frac{-6}{11} + \frac{5}{22}\right] \qquad \text{(by using commutativity and associativity)}$$

$$= \left[\frac{9 + (-8)}{21}\right] + \left[\frac{-12 + 5}{22}\right] \quad \text{(LCM of 7 and 21 is 21; LCM of 11 and 22 is 22)}$$

$$= \frac{1}{21} + \left(\frac{-7}{22}\right) = \frac{22 - 147}{462} = \frac{-125}{462}$$

Do you think the properties of commutativity and associativity made the calculations easier?

Example 2: Find $\dfrac{-4}{5} \times \dfrac{3}{7} \times \dfrac{15}{16} \times \left(\dfrac{-14}{9}\right)$

Solution: We have

$$\frac{-4}{5} \times \frac{3}{7} \times \frac{15}{16} \times \left(\frac{-14}{9}\right)$$

$$= \left(-\frac{4 \times 3}{5 \times 7}\right) \times \left(\frac{15 \times (-14)}{16 \times 9}\right)$$

$$= \frac{-12}{35} \times \left(\frac{-35}{24}\right) = \frac{-12 \times (-35)}{35 \times 24} = \frac{1}{2}$$

We can also do it as.

$$\frac{-4}{5} \times \frac{3}{7} \times \frac{15}{16} \times \left(\frac{-14}{9}\right)$$

$$= \left(\frac{-4}{5} \times \frac{15}{16}\right) \times \left[\frac{3}{7} \times \left(\frac{-14}{9}\right)\right] \quad \text{(Using commutativity and associativity)}$$

$$= \frac{-3}{4} \times \left(\frac{-2}{3}\right) = \frac{1}{2}$$

1.2.4 The role of zero (0)

Look at the following.

$$2 + 0 = 0 + 2 = 2 \qquad \qquad \text{(Addition of 0 to a whole number)}$$
$$-5 + 0 = \dots + \dots = -5 \qquad \qquad \text{(Addition of 0 to an integer)}$$

$$\frac{-2}{7} + \dots = 0 + \left(\frac{-2}{7}\right) = \frac{-2}{7} \qquad \qquad \text{(Addition of 0 to a rational number)}$$

You have done such additions earlier also. Do a few more such additions.

What do you observe? You will find that when you add 0 to a whole number, the sum is again that whole number. This happens for integers and rational numbers also.

In general,

$$a + 0 = 0 + a = a, \quad \text{where } a \text{ is a whole number}$$

$$b + 0 = 0 + b = b, \quad \text{where } b \text{ is an integer}$$

$$c + 0 = 0 + c = c, \quad \text{where } c \text{ is a rational number}$$

Zero is called the identity for the addition of rational numbers. It is the additive identity for integers and whole numbers as well.

1.2.5 The role of 1

We have,

$$5 \times 1 = 5 = 1 \times 5 \qquad \text{(Multiplication of 1 with a whole number)}$$

$$\frac{-2}{7} \times 1 = ... \times ... = \frac{-2}{7}$$

$$\frac{3}{8} \times ... = 1 \times \frac{3}{8} = \frac{3}{8}$$

What do you find?

You will find that when you multiply any rational number with 1, you get back that rational number as the product. Check this for a few more rational numbers. You will find that, $a \times 1 = 1 \times a = a$ for any rational number a.

We say *that* 1 *is the multiplicative identity for rational numbers.*

Is 1 the multiplicative identity for integers? For whole numbers?

THINK, DISCUSS AND WRITE

If a property holds for rational numbers, will it also hold for integers? For whole numbers? Which will? Which will not?

1.2.6 Negative of a number

While studying integers you have come across negatives of integers. What is the negative of 1? It is – 1 because $1 + (-1) = (-1) + 1 = 0$

So, what will be the negative of (–1)? It will be 1.

Also, $2 + (-2) = (-2) + 2 = 0$, so we say 2 is the **negative or additive inverse** of –2 and vice-versa. In general, for an integer a, we have, $a + (-a) = (-a) + a = 0$; so, a is the negative of $-a$ and $-a$ is the negative of a.

For the rational number $\dfrac{2}{3}$, we have,

$$\frac{2}{3} + \left(-\frac{2}{3}\right) = \frac{2 + (-2)}{3} = 0$$

Also, $\left(-\dfrac{2}{3}\right) + \dfrac{2}{3} = 0$ (How?)

Similarly, $\dfrac{-8}{9} + ... = ... + \left(\dfrac{-8}{9}\right) = 0$

$$... + \left(\dfrac{-11}{7}\right) = \left(\dfrac{-11}{7}\right) + ... = 0$$

In general, for a rational number $\dfrac{a}{b}$, we have, $\dfrac{a}{b} + \left(-\dfrac{a}{b}\right) = \left(-\dfrac{a}{b}\right) + \dfrac{a}{b} = 0$. We say

that $-\dfrac{a}{b}$ *is the additive inverse of* $\dfrac{a}{b}$ *and* $\dfrac{a}{b}$ *is the additive inverse of* $\left(-\dfrac{a}{b}\right)$.

1.2.7 Reciprocal

By which rational number would you multiply $\dfrac{8}{21}$, to get the product 1? Obviously by

$\dfrac{21}{8}$, since $\dfrac{8}{21} \times \dfrac{21}{8} = 1$.

Similarly, $\dfrac{-5}{7}$ must be multiplied by $\dfrac{7}{-5}$ so as to get the product 1.

We say that $\dfrac{21}{8}$ is the reciprocal of $\dfrac{8}{21}$ and $\dfrac{7}{-5}$ is the reciprocal of $\dfrac{-5}{7}$.

Can you say what is the reciprocal of 0 (zero)?

Is there a rational number which when multiplied by 0 gives 1? Thus, zero has no reciprocal.

We say that a rational number $\dfrac{c}{d}$ is called the **reciprocal** or **multiplicative inverse** of

another rational number $\dfrac{a}{b}$ if $\dfrac{a}{b} \times \dfrac{c}{d} = 1$.

1.2.8 Distributivity of multiplication over addition for rational numbers

To understand this, consider the rational numbers $\dfrac{-3}{4}, \dfrac{2}{3}$ and $\dfrac{-5}{6}$.

$$\dfrac{-3}{4} \times \left\{\dfrac{2}{3} + \left(\dfrac{-5}{6}\right)\right\} = \dfrac{-3}{4} \times \left\{\dfrac{(4) + (-5)}{6}\right\}$$

$$= \dfrac{-3}{4} \times \left(\dfrac{-1}{6}\right) = \dfrac{3}{24} = \dfrac{1}{8}$$

Also $\qquad \dfrac{-3}{4} \times \dfrac{2}{3} = \dfrac{-3 \times 2}{4 \times 3} = \dfrac{-6}{12} = \dfrac{-1}{2}$

And
$$\frac{-3}{4} \times \frac{-5}{6} = \frac{5}{8}$$

Therefore
$$\left(\frac{-3}{4} \times \frac{2}{3}\right) + \left(\frac{-3}{4} \times \frac{-5}{6}\right) = \frac{-1}{2} + \frac{5}{8} = \frac{1}{8}$$

Thus,
$$\frac{-3}{4} \times \left\{\frac{2}{3} + \frac{-5}{6}\right\} = \left(\frac{-3}{4} \times \frac{2}{3}\right) + \left(\frac{-3}{4} \times \frac{-5}{6}\right)$$

> **Distributivity of Multiplication over Addition and Subtraction.**
> For all rational numbers a, b and c,
> $a\,(b + c) = ab + ac$
> $a\,(b - c) = ab - ac$

TRY THESE

Find using distributivity. (i) $\left\{\frac{7}{5} \times \left(\frac{-3}{12}\right)\right\} + \left\{\frac{7}{5} \times \frac{5}{12}\right\}$ (ii) $\left\{\frac{9}{16} \times \frac{4}{12}\right\} + \left\{\frac{9}{16} \times \frac{-3}{9}\right\}$

Example 3: Write the additive inverse of the following:

(i) $\frac{-7}{19}$ (ii) $\frac{21}{112}$

> When you use distributivity, you split a product as a sum or difference of two products.

Solution:

(i) $\frac{7}{19}$ is the additive inverse of $\frac{-7}{19}$ because $\frac{-7}{19} + \frac{7}{19} = \frac{-7+7}{19} = \frac{0}{19} = 0$

(ii) The additive inverse of $\frac{21}{112}$ is $\frac{-21}{112}$ (Check!)

Example 4: Verify that $-(-x)$ is the same as x for

(i) $x = \frac{13}{17}$ (ii) $x = \frac{-21}{31}$

Solution: (i) We have, $x = \frac{13}{17}$

The additive inverse of $x = \frac{13}{17}$ is $-x = \frac{-13}{17}$ since $\frac{13}{17} + \left(\frac{-13}{17}\right) = 0$.

The same equality $\frac{13}{17} + \left(\frac{-13}{17}\right) = 0$, shows that the additive inverse of $\frac{-13}{17}$ is $\frac{13}{17}$

or $-\left(\frac{-13}{17}\right) = \frac{13}{17}$, i.e., $-(-x) = x$.

(ii) Additive inverse of $x = \frac{-21}{31}$ is $-x = \frac{21}{31}$ since $\frac{-21}{31} + \frac{21}{31} = 0$.

The same equality $\frac{-21}{31} + \frac{21}{31} = 0$, shows that the additive inverse of $\frac{21}{31}$ is $\frac{-21}{31}$,

i.e., $-(-x) = x$.

Example 5: Find $\quad \dfrac{2}{5} \times \dfrac{-3}{7} - \dfrac{1}{14} - \dfrac{3}{7} \times \dfrac{3}{5}$

Solution: $\quad \dfrac{2}{5} \times \dfrac{-3}{7} - \dfrac{1}{14} - \dfrac{3}{7} \times \dfrac{3}{5} = \dfrac{2}{5} \times \dfrac{-3}{7} - \dfrac{3}{7} \times \dfrac{3}{5} - \dfrac{1}{14}$ (by commutativity)

$$= \dfrac{2}{5} \times \dfrac{-3}{7} + \left(\dfrac{-3}{7} \right) \times \dfrac{3}{5} - \dfrac{1}{14}$$

$$= \dfrac{-3}{7} \left(\dfrac{2}{5} + \dfrac{3}{5} \right) - \dfrac{1}{14} \qquad \text{(by distributivity)}$$

$$= \dfrac{-3}{7} \times 1 - \dfrac{1}{14} = \dfrac{-6-1}{14} = \dfrac{-1}{2}$$

EXERCISE 1.1

1. Using appropriate properties find.

   (i) $\quad -\dfrac{2}{3} \times \dfrac{3}{5} + \dfrac{5}{2} - \dfrac{3}{5} \times \dfrac{1}{6}$

   (ii) $\quad \dfrac{2}{5} \times \left(-\dfrac{3}{7} \right) - \dfrac{1}{6} \times \dfrac{3}{2} + \dfrac{1}{14} \times \dfrac{2}{5}$

2. Write the additive inverse of each of the following.

   (i) $\quad \dfrac{2}{8}$ $\qquad$ (ii) $\quad \dfrac{-5}{9}$ $\qquad$ (iii) $\quad \dfrac{-6}{-5}$ $\qquad$ (iv) $\quad \dfrac{2}{-9}$ $\qquad$ (v) $\quad \dfrac{19}{-6}$

3. Verify that $-(-x) = x$ for.

   (i) $\quad x = \dfrac{11}{15}$ $\qquad$ (ii) $\quad x = -\dfrac{13}{17}$

4. Find the multiplicative inverse of the following.

   (i) $\quad -13$ $\qquad$ (ii) $\quad \dfrac{-13}{19}$ $\qquad$ (iii) $\quad \dfrac{1}{5}$ $\qquad$ (iv) $\quad \dfrac{-5}{8} \times \dfrac{-3}{7}$

   (v) $\quad -1 \times \dfrac{-2}{5}$ $\qquad$ (vi) $\quad -1$

5. Name the property under multiplication used in each of the following.

   (i) $\quad \dfrac{-4}{5} \times 1 = 1 \times \dfrac{-4}{5} = -\dfrac{4}{5}$ $\qquad$ (ii) $\quad -\dfrac{13}{17} \times \dfrac{-2}{7} = \dfrac{-2}{7} \times \dfrac{-13}{17}$

   (iii) $\quad \dfrac{-19}{29} \times \dfrac{29}{-19} = 1$

6. Multiply $\dfrac{6}{13}$ by the reciprocal of $\dfrac{-7}{16}$.

7. Tell what property allows you to compute $\dfrac{1}{3} \times \left(6 \times \dfrac{4}{3} \right)$ as $\left(\dfrac{1}{3} \times 6 \right) \times \dfrac{4}{3}$.

8. Is $\dfrac{8}{9}$ the multiplicative inverse of $-1\dfrac{1}{8}$? Why or why not?

9. Is 0.3 the multiplicative inverse of $3\dfrac{1}{3}$? Why or why not?

10. Write.

   (i) The rational number that does not have a reciprocal.

  (ii) The rational numbers that are equal to their reciprocals.

 (iii) The rational number that is equal to its negative.

11. Fill in the blanks.

   (i) Zero has _____ reciprocal.

  (ii) The numbers _____ and _____ are their own reciprocals

 (iii) The reciprocal of – 5 is _____.

 (iv) Reciprocal of $\dfrac{1}{x}$, where $x \neq 0$ is _____.

  (v) The product of two rational numbers is always a _____.

 (vi) The reciprocal of a positive rational number is _____.

1.3 Representation of Rational Numbers on the Number Line

You have learnt to represent natural numbers, whole numbers, integers and rational numbers on a number line. Let us revise them.

Natural numbers

(i)
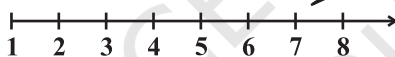
> The line extends indefinitely only to the right side of 1.
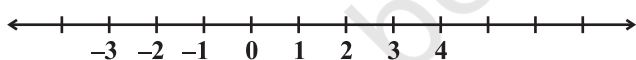
Whole numbers

(ii)
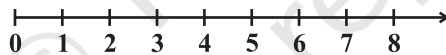
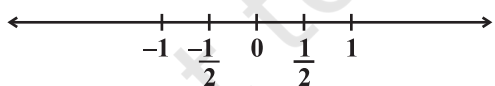> The line extends indefinitely to the right, but from 0. There are no numbers to the left of 0.

Integers

(iii)

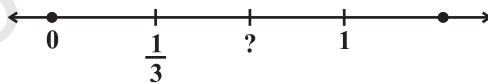> The line extends indefinitely on both sides. Do you see any numbers between –1, 0; 0, 1 etc.?

Rational numbers

(iv)

(v)

> The line extends indefinitely on both sides. But you can now see numbers between –1, 0; 0, 1 etc.
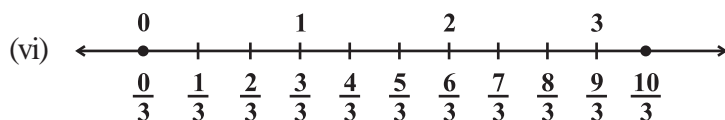
The point on the number line (iv) which is half way between 0 and 1 has been labelled $\dfrac{1}{2}$. Also, the first of the equally spaced points that divides the distance between

0 and 1 into three equal parts can be labelled $\dfrac{1}{3}$, as on number line (v). How would you label the second of these division points on number line (v)?

The point to be labelled is twice as far from and to the right of 0 as the point labelled $\frac{1}{3}$. So it is two times $\frac{1}{3}$, i.e., $\frac{2}{3}$. You can continue to label equally-spaced points on the number line in the same way. The next marking is 1. You can see that 1 is the same as $\frac{3}{3}$.
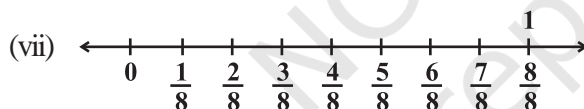
Then comes $\frac{4}{3}, \frac{5}{3}, \frac{6}{3}$ (or 2), $\frac{7}{3}$ and so on as shown on the number line (vi)

(vi)

$$\begin{array}{cccccccccccc} \mathbf{0} & & & \mathbf{1} & & & \mathbf{2} & & & \mathbf{3} & & \\ \frac{0}{3} & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} & \frac{7}{3} & \frac{8}{3} & \frac{9}{3} & \frac{10}{3} \end{array}$$

Similarly, to represent $\frac{1}{8}$, the number line may be divided into eight equal parts as shown:

$$\begin{array}{cc} \mathbf{0} & \mathbf{1} \end{array}$$
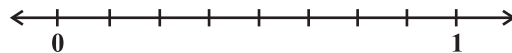
We use the number $\frac{1}{8}$ to name the first point of this division. The second point of division will be labelled $\frac{2}{8}$, the third point $\frac{3}{8}$, and so on as shown on number line (vii)

(vii)

$$\begin{array}{ccccccccc} & & & & & & & & \mathbf{1} \\ \mathbf{0} & \frac{1}{8} & \frac{2}{8} & \frac{3}{8} & \frac{4}{8} & \frac{5}{8} & \frac{6}{8} & \frac{7}{8} & \frac{8}{8} \end{array}$$
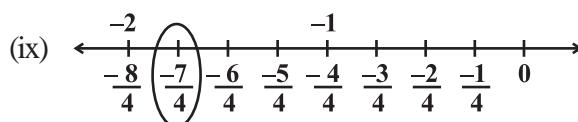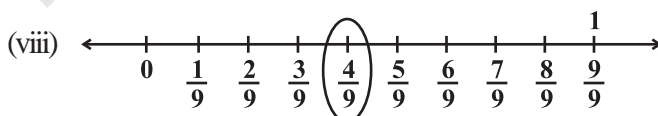
Any rational number can be represented on the number line in this way. In a rational number, the numeral below the bar, i.e., the denominator, tells the number of equal parts into which the first unit has been divided. The numeral above the bar i.e., the numerator, tells 'how many' of these parts are considered. So, a rational number such as $\frac{4}{9}$ means four of nine equal parts on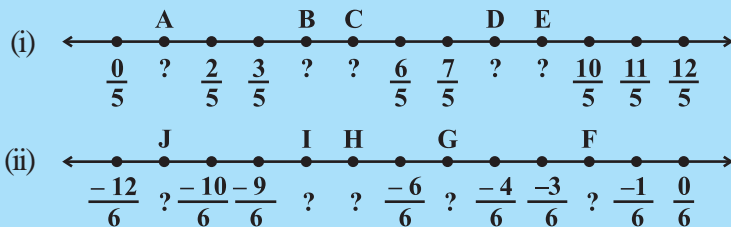 the right of 0 (number line viii) and for $\frac{-7}{4}$, we make 7 markings of distance $\frac{1}{4}$ each on the *left* of zero and starting from 0. The seventh marking is $\frac{-7}{4}$ [number line (ix)].

(viii)

$$\begin{array}{cccccccccc} & & & & & & & & & \mathbf{1} \\ \mathbf{0} & \frac{1}{9} & \frac{2}{9} & \frac{3}{9} & \boxed{\frac{4}{9}} & \frac{5}{9} & \frac{6}{9} & \frac{7}{9} & \frac{8}{9} & \frac{9}{9} \end{array}$$

(ix)

$$\begin{array}{cccccccccc} \mathbf{-2} & & & \mathbf{-1} & & & & & & \\ \frac{-8}{4} & \boxed{\frac{-7}{4}} & \frac{-6}{4} & \frac{-5}{4} & \frac{-4}{4} & \frac{-3}{4} & \frac{-2}{4} & \frac{-1}{4} & 0 \end{array}$$

TRY THESE

Write the rational number for each point labelled with a letter.

(i)

	A			B	C			D	E		

$$\frac{0}{5} \quad ? \quad \frac{2}{5} \quad \frac{3}{5} \quad ? \quad ? \quad \frac{6}{5} \quad \frac{7}{5} \quad ? \quad ? \quad \frac{10}{5} \quad \frac{11}{5} \quad \frac{12}{5}$$

(ii)

	J			I	H		G			F		

$$\frac{-12}{6} \quad ? \quad \frac{-10}{6} \quad \frac{-9}{6} \quad ? \quad ? \quad \frac{-6}{6} \quad ? \quad \frac{-4}{6} \quad \frac{-3}{6} \quad ? \quad \frac{-1}{6} \quad \frac{0}{6}$$

1.4 Rational Numbers between Two Rational Numbers

Can you tell the natural numbers between 1 and 5? They are 2, 3 and 4.

How many natural numbers are there between 7 and 9? There is one and it is 8.

How many natural numbers are there between 10 and 11? Obviously none.

List the integers that lie between –5 and 4. They are – 4, – 3, –2, –1, 0, 1, 2, 3.

How many integers are there between –1 and 1?

How many integers are there between –9 and –10?

You will find a definite number of natural numbers (integers) between two natural numbers (integers).

How many rational numbers are there between $\frac{3}{10}$ and $\frac{7}{10}$?

You may have thought that they are only $\frac{4}{10}, \frac{5}{10}$ and $\frac{6}{10}$.

But you can also write $\frac{3}{10}$ as $\frac{30}{100}$ and $\frac{7}{10}$ as $\frac{70}{100}$. Now the numbers, $\frac{31}{100}, \frac{32}{100}, \frac{33}{100}$,... $\frac{68}{100}, \frac{69}{100}$, are all between $\frac{3}{10}$ and $\frac{7}{10}$. The number of these rational numbers is 39.

Also $\frac{3}{10}$ can be expressed as $\frac{3000}{10000}$ and $\frac{7}{10}$ as $\frac{7000}{10000}$. Now, we see that the rational numbers $\frac{3001}{10000}, \frac{3002}{10000}$,..., $\frac{6998}{10000}, \frac{6999}{10000}$ are between $\frac{3}{10}$ and $\frac{7}{10}$. These are 3999 numbers in all.

In this way, we can go on inserting more and more rational numbers between $\frac{3}{10}$ and $\frac{7}{10}$. So unlike natural numbers and integers, the number of rational numbers between two rational numbers is not definite. Here is one more example.

How many rational numbers are there between $\frac{-1}{10}$ and $\frac{3}{10}$?

Obviously $\frac{0}{10}, \frac{1}{10}, \frac{2}{10}$ are rational numbers between the given numbers.

If we write $\dfrac{-1}{10}$ as $\dfrac{-10000}{100000}$ and $\dfrac{3}{10}$ as $\dfrac{30000}{100000}$, we get the rational numbers

$\dfrac{-9999}{100000}, \dfrac{-9998}{100000}, ..., \dfrac{-29998}{100000}, \dfrac{29999}{100000}$, between $\dfrac{-1}{10}$ and $\dfrac{3}{10}$.

You will find that *you get countless rational numbers between any two given rational numbers*.

Example 6: Write any 3 rational numbers between –2 and 0.

Solution: –2 can be written as $\dfrac{-20}{10}$ and 0 as $\dfrac{0}{10}$.

Thus we have $\dfrac{-19}{10}, \dfrac{-18}{10}, \dfrac{-17}{10}, \dfrac{-16}{10}, \dfrac{-15}{10}, ..., \dfrac{-1}{10}$ between –2 and 0.

You can take any three of these.

Example 7: Find any ten rational numbers between $\dfrac{-5}{6}$ and $\dfrac{5}{8}$.

Solution: We first convert $\dfrac{-5}{6}$ and $\dfrac{5}{8}$ to rational numbers with the same denominators.

$$\dfrac{-5 \times 4}{6 \times 4} = \dfrac{-20}{24} \quad \text{and} \quad \dfrac{5 \times 3}{8 \times 3} = \dfrac{15}{24}$$

Thus we have $\dfrac{-19}{24}, \dfrac{-18}{24}, \dfrac{-17}{24}, ..., \dfrac{14}{24}$ as the rational numbers between $\dfrac{-20}{24}$ and $\dfrac{15}{24}$.

You can take any ten of these.

Another Method

Let us find rational numbers between 1 and 2. One of them is 1.5 or $1\dfrac{1}{2}$ or $\dfrac{3}{2}$. This is the **mean** of 1 and 2. You have studied mean in Class VII.

We find that *between any two given numbers, we need not necessarily get an integer but there will always lie a rational number*.

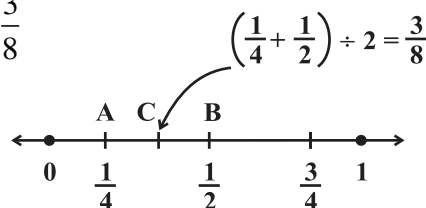We can use the idea of mean also to find rational numbers between any two given rational numbers.

Example 8: Find a rational number between $\dfrac{1}{4}$ and $\dfrac{1}{2}$.

Solution: We find the mean of the given rational numbers.

$$\left(\dfrac{1}{4} + \dfrac{1}{2}\right) \div 2 = \left(\dfrac{1+2}{4}\right) \div 2 = \dfrac{3}{4} \times \dfrac{1}{2} = \dfrac{3}{8}$$

$$\left(\dfrac{1}{4} + \dfrac{1}{2}\right) \div 2 = \dfrac{3}{8}$$

$\dfrac{3}{8}$ lies between $\dfrac{1}{4}$ and $\dfrac{1}{2}$.

This can be seen on the number line also.

We find the mid point of AB which is C, represented by $\left(\dfrac{1}{4} + \dfrac{1}{2}\right) \div 2 = \dfrac{3}{8}$.

We find that $\dfrac{1}{4} < \dfrac{3}{8} < \dfrac{1}{2}$.

If a and b are two rational numbers, then $\dfrac{a+b}{2}$ is a rational number between a and b such that $a < \dfrac{a+b}{2} < b$.

This again shows that there are countless number of rational numbers between any two given rational numbers.

Example 9: Find three rational numbers between $\dfrac{1}{4}$ and $\dfrac{1}{2}$.

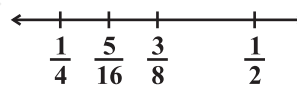**Solution:** We find the mean of the given rational numbers.

As given in the above example, the mean is $\dfrac{3}{8}$ and $\dfrac{1}{4} < \dfrac{3}{8} < \dfrac{1}{2}$.
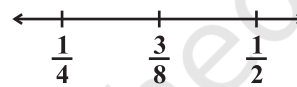
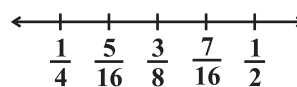We now find another rational number between $\dfrac{1}{4}$ and $\dfrac{3}{8}$. For this, we again find the mean of $\dfrac{1}{4}$ and $\dfrac{3}{8}$. That is, $\left(\dfrac{1}{4} + \dfrac{3}{8}\right) \div 2 = \dfrac{5}{8} \times \dfrac{1}{2} = \dfrac{5}{16}$

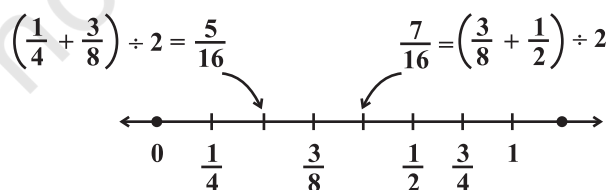$$\dfrac{1}{4} < \dfrac{5}{16} < \dfrac{3}{8} < \dfrac{1}{2}$$

Now find the mean of $\dfrac{3}{8}$ and $\dfrac{1}{2}$. We have, $\left(\dfrac{3}{8} + \dfrac{1}{2}\right) \div 2 = \dfrac{7}{8} \times \dfrac{1}{2} = \dfrac{7}{16}$

Thus we get $\dfrac{1}{4} < \dfrac{5}{16} < \dfrac{3}{8} < \dfrac{7}{16} < \dfrac{1}{2}$.

Thus, $\dfrac{5}{16}$, $\dfrac{3}{8}$, $\dfrac{7}{16}$ are the three rational numbers between $\dfrac{1}{4}$ and $\dfrac{1}{2}$.

This can clearly be shown on the number line as follows:

$$\left(\dfrac{1}{4} + \dfrac{3}{8}\right) \div 2 = \dfrac{5}{16} \qquad \dfrac{7}{16} = \left(\dfrac{3}{8} + \dfrac{1}{2}\right) \div 2$$

In the same way we can obtain as many rational numbers as we want between two given rational numbers. You have noticed that there are countless rational numbers between any two given rational numbers.

EXERCISE 1.2

1. Represent these numbers on the number line. (i) $\frac{7}{4}$ (ii) $\frac{-5}{6}$

2. Represent $\frac{-2}{11}, \frac{-5}{11}, \frac{-9}{11}$ on the number line.

3. Write five rational numbers which are smaller than 2.

4. Find ten rational numbers between $\frac{-2}{5}$ and $\frac{1}{2}$.

5. Find five rational numbers between.

   (i) $\frac{2}{3}$ and $\frac{4}{5}$ (ii) $\frac{-3}{2}$ and $\frac{5}{3}$ (iii) $\frac{1}{4}$ and $\frac{1}{2}$

6. Write five rational numbers greater than –2.

7. Find ten rational numbers between $\frac{3}{5}$ and $\frac{3}{4}$.

WHAT HAVE WE DISCUSSED?

1. Rational numbers are **closed** under the operations of addition, subtraction and multiplication.

2. The operations addition and multiplication are
   (i) **commutative** for rational numbers.
   (ii) **associative** for rational numbers.

3. The rational number 0 is the **additive identity** for rational numbers.

4. The rational number 1 is the **multiplicative identity** for rational numbers.

5. The **additive inverse** of the rational number $\frac{a}{b}$ is $-\frac{a}{b}$ and vice-versa.

6. The **reciprocal** or **multiplicative inverse** of the rational number $\frac{a}{b}$ is $\frac{c}{d}$ if $\frac{a}{b} \times \frac{c}{d} = 1$.

7. **Distributivity** of rational numbers: For all rational numbers a, b and c,
   $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$

8. Rational numbers can be represented on a number line.

9. Between any two given rational numbers there are countless rational numbers. The idea of **mean** helps us to find rational numbers between two rational numbers.