

1 Results

The theoretical framework for the dynamic portfolio choice problem with fixed and proportional transaction costs was presented in Section ???. The implementation details were covered in Section ???. The purpose of Section 1 is to present the results of the implementation, as well as the changes to the theoretical framework, when fixed costs are considered. The results are presented in the following order: First, the baseline of only proportional cost is considered. I then present the new computational scheme needed to tackle fixed costs and solve this model. Afterwards I consider the results of the fixed cost model, and how the No-Trade Region (NTR) changes when fixed costs are considered. During this I present how this new method can be adapted to other cost structures. Finally, I consider the results of the model with fixed and proportional costs.

1.1 Parameters and Model Setup

For the following results I consider 3 types of parameterizations for the problem. The first is a simple case where the assets are identically distributed (i.i.d), these parameters are seen in (Cai, Judd and Xu 2013). The second is a case where the parameters are chosen to match the parameters in (Schober, Valentin and Pflüger 2022) also seen in (Gaegau, Scheidegger and Trojani 2023). This is in order to be able to draw correct comparisons between the results. Furthermore, this case showcases assets with slight variation in the mean return and a small correlation between the returns, and there is no asset which the others. The last parameterization is a modification of the first case where the correlation between the returns is large (correlation coefficient of 0.75). This parameterization clearly displays the effect of correlation on the NTR. The parameters for the three cases are displayed in Table 1.

Table 1: Parameters for portfolio models

	i.i.d Assets	Schober Parameters	High Correlation
T	6	6	6
Δt	1	1	1
k	3	5	3
γ	3.0	3.5	3.0
τ	0.5%	0.5%	0.5%
β	0.97	0.97	0.97
r	3%	4%	3%
μ^\top	(0.07, 0.07)	μ_{Schober}	(0.07, 0.07)
Σ	$\begin{bmatrix} 0.04 & 0.00 \\ 0.00 & 0.04 \end{bmatrix}$	Σ_{Schober}	$\begin{bmatrix} 0.04 & 0.03 \\ 0.03 & 0.04 \end{bmatrix}$

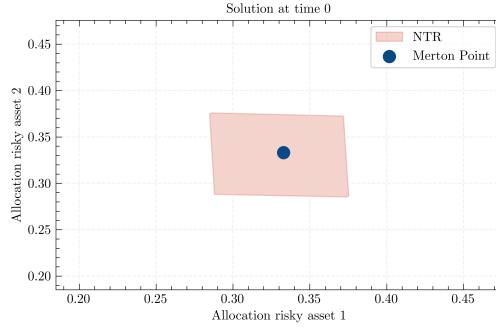
$$\mu_{\text{Schober}}^\top = [0.0572 \ 0.0638 \ 0.07 \ 0.0764 \ 0.0828]$$

$$\Sigma_{\text{Schober}} = \begin{bmatrix} 0.0256 & 0.00576 & 0.00288 & 0.00176 & 0.00096 \\ 0.00576 & 0.0324 & 0.0090432 & 0.010692 & 0.01296 \\ 0.00288 & 0.0090432 & 0.04 & 0.0132 & 0.0168 \\ 0.00176 & 0.010692 & 0.0132 & 0.0484 & 0.02112 \\ 0.00096 & 0.01296 & 0.0168 & 0.02112 & 0.0576 \end{bmatrix}$$

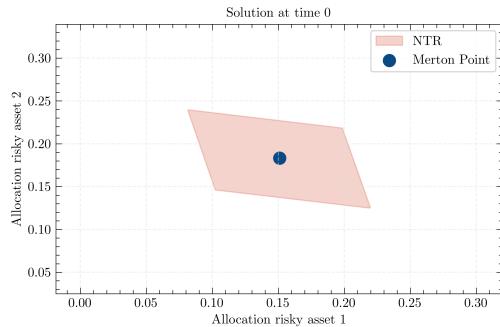
1.2 Dynamic Portfolio Choice without consumption

I first consider the base model with proportional transaction costs and no consumption. In the absence of consumption, the optimal portfolio is the Merton point, which I plot in every figure. I plot the No-trade region at time point 0 (initial time point) for each of the parameterizations in Figure 1.1. When using the Schober parameters I select the d first elements of the mean vector and truncate the covariance matrix to a $d \times d$ matrix, depending on the number of assets d in the model. If I consider larger dimensions for the other parameters, I simply add more assets with the same mean and covariance structure. I solve the models for a $T = 6$ year horizon. I note that for each of the parameterizations

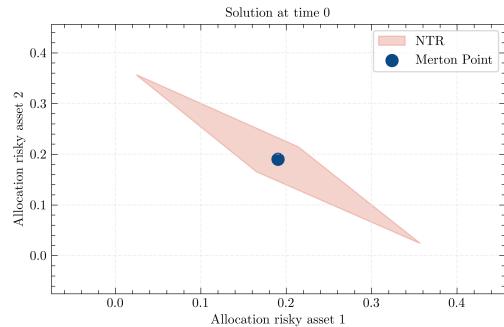
Figure 1.1: Comparison of No Trade Regions with proportional costs.



(a) No Trade Region for Independent Identically Distributed Assets.



(b) No Trade Region for Schober Parameters.



(c) No Trade Region for High Correlation.

the No-Trade region is a rectangle or parallelogram. The Merton point is located in the center of each NTR and the NTR is symmetric around the Merton point.

The i.i.d parameterization has the highest optimal asset allocations, which is due to the high returns of the risky assets. the Merton point is $(0.33, 0.33)$. The NTR is almost a perfect square, but displays slight skewness. This is numerical instability in the optimization process, and compounding value function approximations.

For the Schober parameters and the high correlation case, the No-Trade region is a parallelogram. This is due to the correlation between the returns. When some correlation is present, the No-Trade region is skewed, since some allocations that would be optimal in the absence of correlation are no longer optimal. This is because when the returns are correlated, the assets are substitutes to the degree of correlation. Large weighting in one risky asset are then only optimal, by reducing the weight in another asset, in order to diversify risk for the investor. Comparing the Schober parameters to the high correlation case, this is apparent. Likewise, if the correlation was inversed, the investor would prefer high weighting in both assets in order to hedge risk.

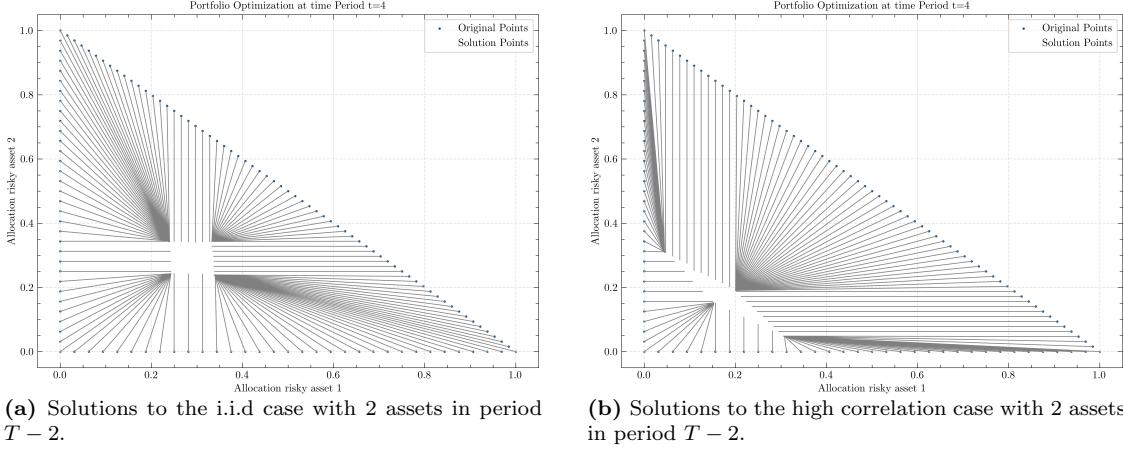
1.2.1 Verifying the geometric shape of the No-trade Region

Since much of the procedure for solving this problem, and approximating the NTR leverages the a-priori assumptions regarding the geometric shape of the NTR, I first want to verify that the NTR indeed has 4 vertices for the 2d case, and no more, and is a convex hull.

In order to do this, I do a small modification to the solution algorithm proposed earlier. Instead of computing vertices using 2^d predetermined points, I will instead sample a larger set of points, ($2^7 = 128$) covering the boundaries of the feasible state space. For each of these points, I then solve the optimization problem and plot the solution from allocations \mathbf{x}_t and their solutions to the problem $\hat{\omega}_t$. I do this by using my original sample scheme, and adding mid-points between points, which either sum to 1.0 or have 0.0 as allocation for one of the assets. I consider the i.i.d case and the high correlation case, with $\tau = 0.0075$ i.e 0.75%. I have increased the costs slightly in order to increase the size of the NTR. This is to ensure that points also converge towards the faces and not only the verticies by extending the length of the faces. This is akin to the green regions in Figure ???. Otherwise I would need more points on a finer grid. I plot the solutions for next to last period with investment decisions $T - 2$. The solutions are plotted below.

I find that the assumptions regarding the NTR are indeed correct in the two dimensional example, I constructed. Furthermore, this verifies that the assumptions also hold for correlated returns, as conjectured by (Liu 2004) and found by (Cai, Judd and Xu 2013). Furthermore, these plots also nicely confirm that the optimization process as a whole works as intended and provide an intuitive understanding of the NTR. Further verification

Figure 1.2: Verifying the assumptions of the NTR in two dimensions.



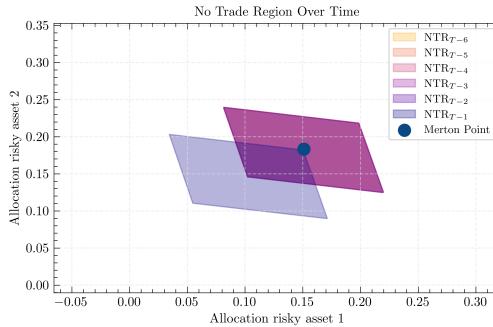
in higher dimensions are not considered. First of all (Liu 2004) confirm this formally in larger dimensions for the case of uncorrelated returns and the intuition regarding the NTR does not change when dimensionality is increased.

1.2.2 Investigating the No-Trade Region

I now look at the No-Trade region for the base model with proportional transaction costs and no consumption in more detail. Specifically I look at how the region behaves over the entire investment horizon $[0, T - 1]$, and how the region changes with different transaction cost levels. I choose to look at the model with the Schober parameters as this is a mixture of the other two parameterizations.

I first solve the model for the Schober parameters, and plot the NTR for each time point in the investment horizon.

Figure 1.3: No Trade Region for Schober Parameters over Time.

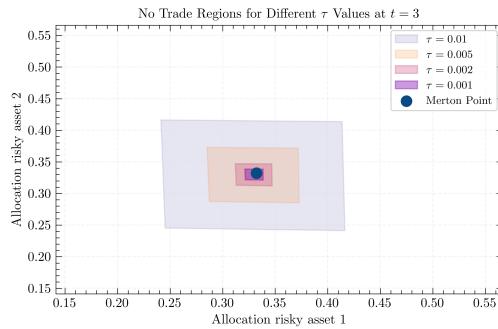


The No-Trade region is plotted for the Schober parameters over the entire investment horizon $[0, T - 1]$. For time points $t \in [0, T - 2]$ the NTRs overlap.

I note that at the last time point $t = T - 1$ the NTR moves away from the Merton point towards the origin, and the Merton point is then at the upper right corner of the NTR. For all other time periods the NTR is the same, and the Merton point is in the center. This is consistent with behaviour found by (Cai, Judd and Xu 2013; Gaegau, Scheidegger and Trojani 2023), and might suggest that I only require solutions for two periods, $T - 1$ and $T - 2$ in order to effectively cover the NTR for all periods. I can therefore use the solution for $T - 2$, which is higher quality than $t = 0$, due to numerical instabilities in the optimization process, and effectively cover the NTR for all time points.

I now investigate how the No-Trade region changes with different transaction cost levels. I do this for the i.i.d. parameters, and plot the NTR for different values of τ in Figure

Figure 1.4: No Trade Region for the i.i.d Parameters with different values of τ .



1.4 for $\tau \in \{0.01, 0.005, 0.002, 0.001\}$. When the transaction costs are increases the NTR increases aswell and vice-versa. I note that for low enough transaction costs, the NTR shrinks towards the Merton point. However, when transaction costs are low enough, the Merton point is not in the exact center, which might signify that at low enough values, some numerical instabilities from the minimizer and function approximation using Gaussian process regression (GPR), might be present.

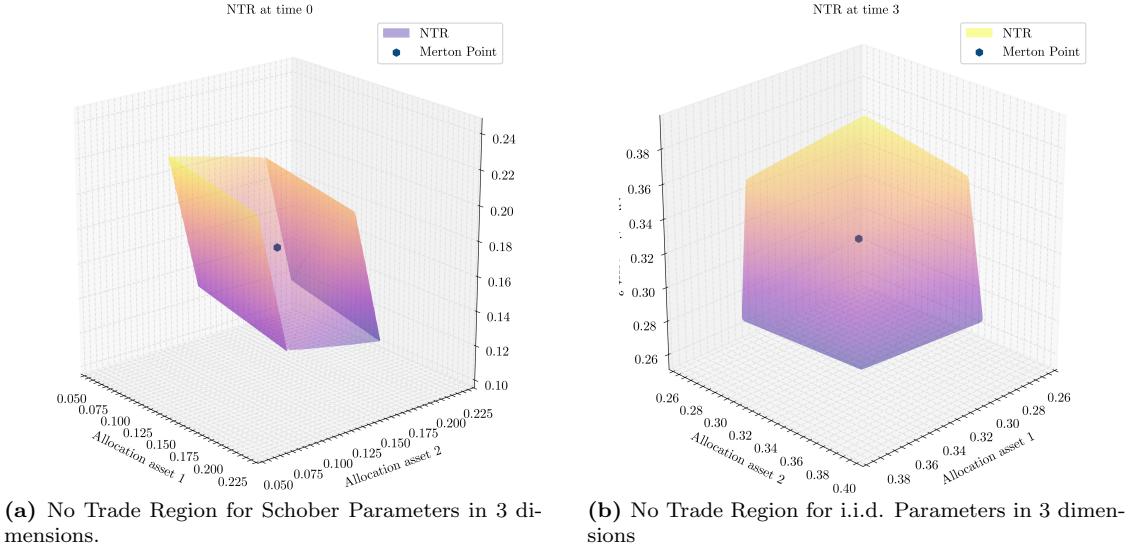
1.2.3 Increasing the dimensionality of the model

I now increase the dimensionality of the model to $D = 3$ and look at the No-Trade region for the Schober parameters and for the i.i.d parameters.

I plot the NTR with open faces, however the actual NTR has all the faces closed, and is a convex hull. I plot for time period $T - 3$. This choice is arbitrary. Note, that the i.i.d NTR looks like a skewed cube, whereas this was a perfect square in the 2 dimensional case.

Looking at the points forming the convex hull that is the NTR, it is clear that the NTR is restricted by the no-borrowing constraint. The border points, which would otherwise form the perfect cube, would outside the feasible space if borrowing was possible, and is then projected back into the feasible space, hence the skewness. The Merton point for the

Figure 1.5: Comparison of No Trade Regions with three risky assets.



i.i.d case with three assets is $(0.33, 0.33, 0.33)$, and the total risky investment portfolio is 99% of wealth, hence the Merton point is close to the border of the feasible space. Thus, when the risky returns outweigh the risk-free return, to such a degree that the Merton point moves towards the boundary of the feasible space, cube like shapes are no longer possible. In the case of two risky assets, this is akin to the NTR being close to the budget line, and the NTR would then form a triangle.

This is clear when compared to the Schober parameters, where the Merton point is in the center of the NTR, and the NTR is a skewed cube well within the feasible space. The Merton point in this case suggests lower portfolio allocations to the risky assets, and hence the NTR is not restricted by the no-trading and no-borrowing constraints. The NTR behaves similar to the 2d case, and the intuition regarding the NTR is the same.

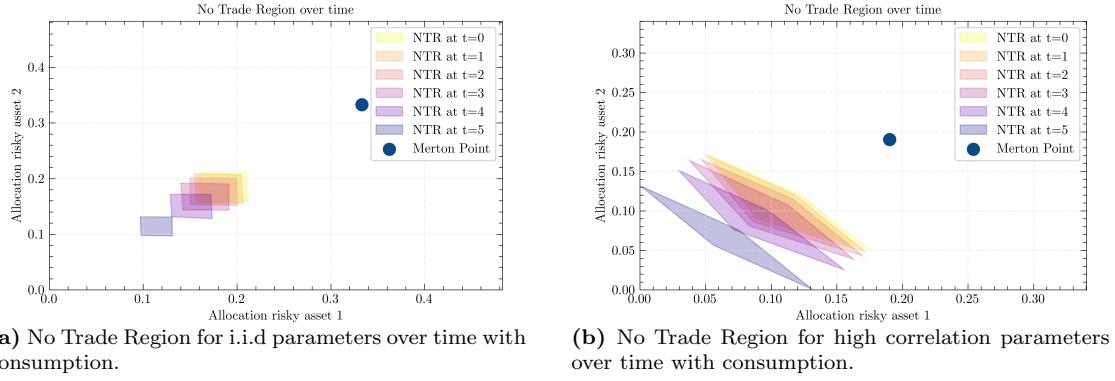
Furthermore the NTR behaves similar to the 2d case, and is only considerably different in period $T - 1$, compared to periods $T - 2, T - 3$ and so forth.

1.3 Dynamic Portfolio Choice with consumption

I now consider the base model with proportional transaction costs which now includes consumption of a non-durable good. This adds an extra decision variable which needs to be solved for, and consumption now adds immediate utility to the investor in each period.

Note, that when consumption is included, the NTR no longer encapsulates the Merton point at any time point. If the Merton point is scaled to the wealth after consumption, $x_t^{\text{Merton}} \cdot (1 - c_t^*)$ then the Merton point is encapsulated by the NTR.

Figure 1.6: Comparison of No Trade Regions over time with consumption.



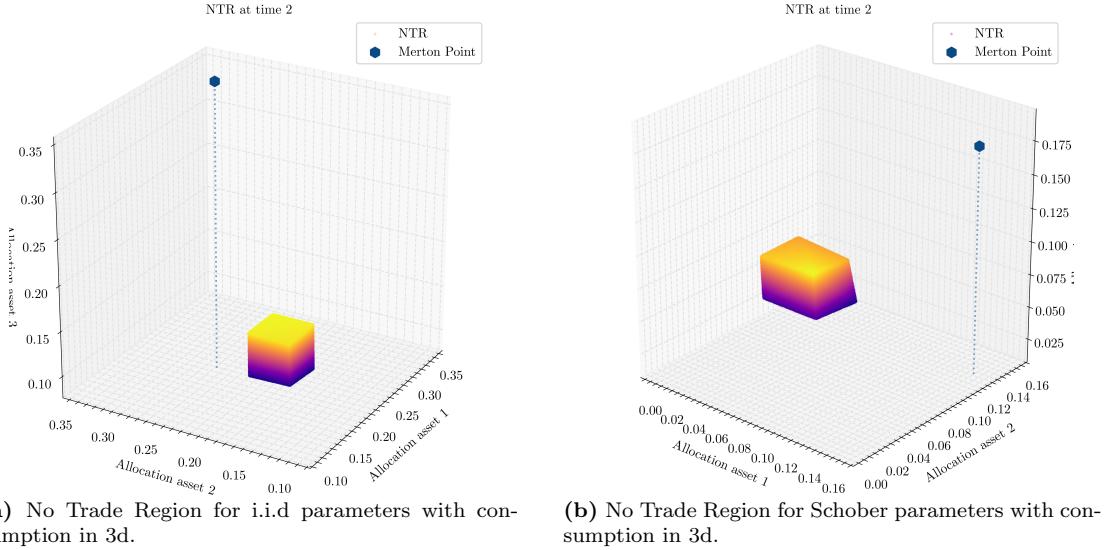
The No-Trade regions are plotted over the entire investment horizon $[0, T - 1]$.

Furthermore, the NTR now moves over time towards the origin as opposed to the case without consumption, where the NTR was static for all time points except the next to last period (last period with trading decisions). Hence, the NTR is now no longer sufficiently described by solutions to $T - 1$ and $T - 2$ as the optimal consumption decision changes over time, moving the NTR towards the origin, as $t \rightarrow 0$.

The shifts of the NTR diminish as $t \rightarrow T$, as I note that the largest movement occurs from $t = T - 1$ to $t = T - 2$. This is because the consumption decision increases exponentially over time.

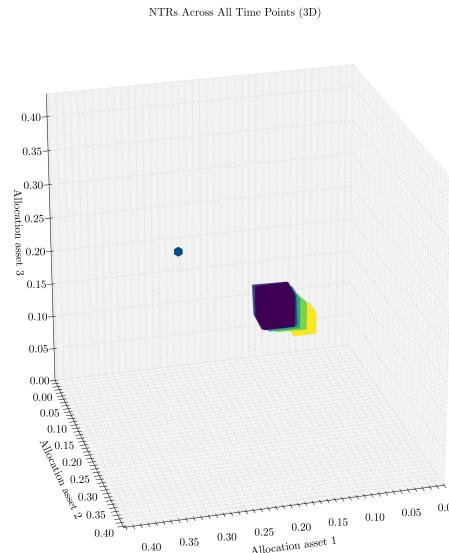
This behaviour is consistent in higher dimensions, and with results found by (Gaegauf, Scheidegger and Trojani 2023). Below are plots of the NTR for two different parameterizations of the model, with three assets, at a singular time point.

Figure 1.7: No trade regions with consumption in multiple dimensions, singular time period



I can similar to the two asset case, plot the NTR over the entire investment horizon, as seen in Figure 1.8a. Once again, I note that the NTR moves towards the Merton point as $t \rightarrow T$, but never encapsulates the Merton Point. Likewise, the movements of the NTR diminish as $t \rightarrow 0$, since the change in consumption decision between periods diminishes as $t \rightarrow 0$.

Figure 1.8: No trade regions with consumption in multiple dimensions entire horizon

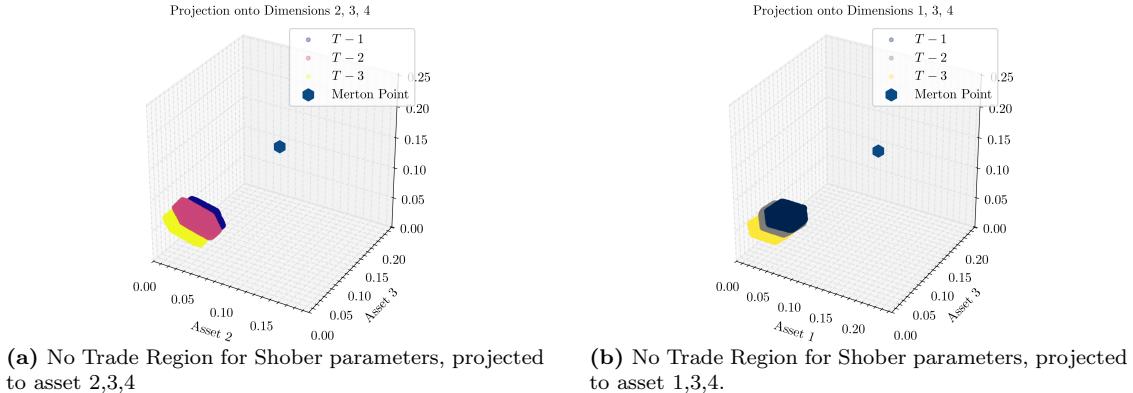


(a) NTR for 3 assets with i.i.d parameters over time with consumption.

1.3.1 Increasing the dimensionality of the model further

I now look at increasing the dimensionality of the model even further. This moves the solutions of the model to higher dimensional spaces, which have well defined mathematical properties. Illustrations will be projected to three dimensions, as the model is not easily visualized in higher dimensions. I solve the model with the Schober parameters for $d = 4$ assets with consumption. This increases the complexity of the model, and the NTR is now a hyper-cube, however solutions remain feasible, despite the algorithm running on a laptop computer. For the $d = 4$ case, the Merton point is $(0.1448, 0.0971, 0.1317, 0.1522)$ which suggests a total risky investment portfolio of 52.58% of wealth. I use $60 \cdot D = 240$ generated points and their respective solutions to train the Gaussian process (GP) in each iteration. Approximating the NTR by itself now takes considerably longer¹ than the previous cases, and the NTR is projected to three dimensions. and larger dimensions $D > 4$ are therefore not in the scope of this paper, as the computational power required is too large for me to handle . However, solving the 4 risky asset porfolio, on a personal laptop, remains a computational feat, as previous studies (Cai, Judd and Xu 2013; Schober, Valentin and Pflüger 2022) relied on super computers to solve the model. (Gaegau, Scheidegger and Trojani 2023) makes no mention of the computational setup, and a direct comparison to the results is therefore dissapointingly not possible. The NTR is plotted below, and each vertice for $T - 1$ and $T - 2$ can be found in Appendix ??

Figure 1.9: No trade regions for Shoher parameters with consumption, four assets projected to three dimensions.



1.4 Dynamic Portfolio Choice with fixed costs

I now consider the base model with fixed transaction costs and no consumption. From (Dybvig and Pezzo 2020), I know that the NTR is no longer rectangular when we only consider fixed costs, but instead circular with the Merton point in the middle when there

¹Solving for three periods took 180 minutes.

is no consumption in the model. This poses a problem for my current sampling scheme, which leverages my predetermined knowledge of the geometric shape of the NTR. As I noted in Section ??, in order to effectively sample points for the NTR approximation, given the framework for the proportional cost case, I now need to sample points, such that when they hit the NTR these points are evenly distributed on the sphere, in order to approximate the NTR correctly. However this is still not sufficient as the fixed costs pose further problems for the solution algorithm by (Gaegau, Scheidegger and Trojani 2023). In order to see this a little intuition is needed.

1.4.1 Generating a new solution algorithm for the fixed cost case

Transaction costs no longer scale in the fixed case, but are treated as a *sunk cost*, the moment the decision to trade is made. Hence, if trading is optimal the investor will trade to the optimal point, and if trading is sub-optimal then no trading will occur. The problem is therefore first of all a trading decision problem. If trading is optimal, then the investor will trade to the Merton point when no consumption is present, as this is the optimal point.

This is in stark contrast to the proportional case, where the trading trajectory from outside the NTR was to the border of the NTR, and the NTR approximation could be done by sampling points on the border of the feasible space.

Now, any point sampled outside the NTR trades to the Merton point, and I need to construct a new strategy, in order to efficiently construct the NTR, as no new information is gained by sampling multiple points outside the NTR.

Furthermore, the transaction cost function is now an indicator function, depending on a threshold, i.e $\sum_{i=1}^k \delta_{i,t}^+ + \delta_{i,t}^- > 0$. This is non-differentiable at the kink, $\sum_{i=1}^k \delta_{i,t}^+ + \delta_{i,t}^- = 0$ which is a critical point (the trading decision boundary), that I have to deal with in order to solve the optimization problem.

I therefore split the optimization process into two parts. I evaluate the objective function (value function), *conditional* on no trading ($\delta_t = \mathbf{0}$) and *conditional* on trading ($\delta_t \neq \mathbf{0}$). Since there is no consumption decision then the no-trading decision is trivial, whereas I still optimize the trading decision conditional on trading occurring in order to maximize expected utility.

By splitting the optimization process, I can avoid the the non-differentiable edge case, and the gradients are trivial for the optimizer. I then evaluate the value function for the no-trading decision, and the trading decision, and choose the decision which maximizes the value function. Furthermore when splitting the problem by the trading decision, the optimization problem is convex once again, as per (Dybvig and Pezzo 2020). Hence the issues mentioned in Section ?? are no longer present.

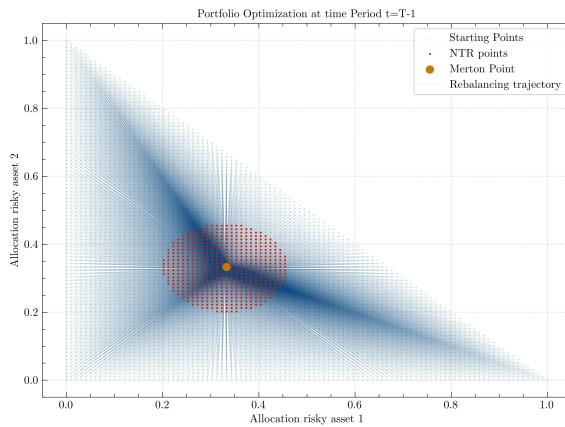
I now consider the base model with fixed transaction costs, and no consumption. I use

the simple i.i.d parameterization, with two assets and solve the optimization problem for the next to last period $T - 1$ over an evenly spaced grid of points. I do this in order to verify that the solution algorithm works as intended, and that the NTR is circular as expected, conflicting with my prior assumptions for the proportional case.

I set the fixed costs to 0.005% of the investors total wealth at any time point and solve at a very fine grid of points, in order to approximate the NTR correctly. I find that the NTR is circular as expected, and that the new solution algorithm works as intended.

I therefore proceed with generating a strategy for dealing with fixed costs, which can leverage my new found knowledge of the NTR.

Figure 1.10: Grid solution to the i.i.d case with fixed costs, two assets in period $T - 1$.



The optimization scheme ran with 5044 evenly spaced grid points. The points are plotted in the feasible space, and the NTR is the convex hull of these points.

1.4.2 Constructing a new sampling scheme and approximation method for the NTR

Noting that for each point outside the NTR, the investor will trade to the same optimal point, since the cost of trading is a *sunk cost*, I can select a single starting point at the origin and solve for the optimal trading decision. If the optimal decision is to trade, then I immediately know the center of the NTR and now only need the radius to construct the NTR. This holds for any dimensionality of the model, as any circle/sphere/hypersphere can be defined by the center point and the radius.

I find the radius, by moving towards one of the boundary starting points, from the center of the NTR, solving the optimization problem for each point, and noting at which point that trading occurs (return to the center).

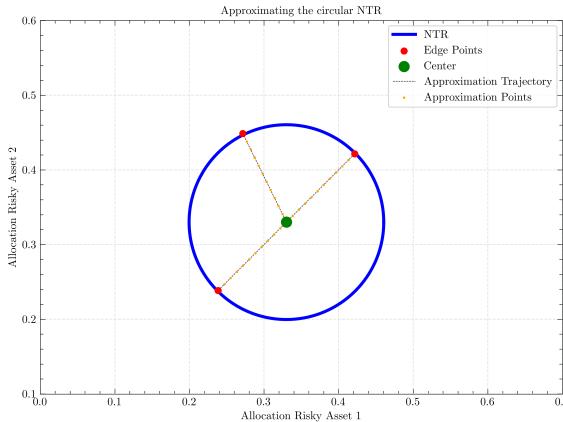
By using a bisection method², I find the border of the NTR with a tolerance of $5 \cdot 10^{-7}$,

²Which is simple to implement since we only consider allocations along the straight line from some angle

which is a tolerance equal to 0.00005% of the total wealth of the investor. I solve for multiple directions from the center, and choose the largest radius. This is because the circle might be truncated along the borders of the feasible space if the NTR is close to either of the axis or the budget constraint (no borrowing/shorting). Furthermore, by selecting directions in evenly spaced angles around the center, I ensure that one of the directions of trading, must hit the border as long as the NTR does not cover the entire feasible space. For the latter case, the algorithm would never find a center point, and the solution is trivial any how.

This optimization process can be seen in Figure 1.10. I start from a blue point and rebalance to the Merton point. Then turn to Figure 1.11, I move along a straight line in the direction of a starting point outwards and solve. If no trading occurs the point is in the NTR. If trading occurs then the point is outside the NTR, and I move backwards towards the center, and solve again. This is repeated for each trading direction. The figure below displays this specific part of the algorithm.

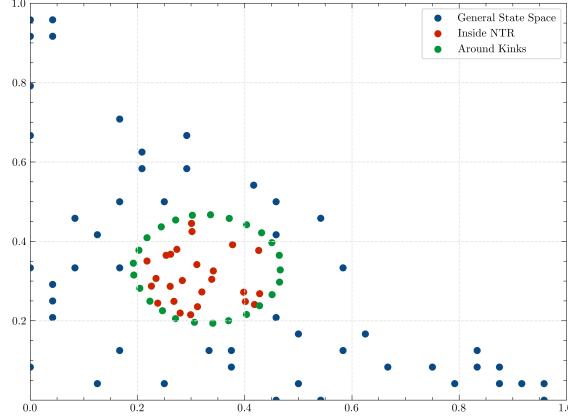
Figure 1.11: Two dimensional approximation algorithm for the fixed cost NTR with no correlation.



I also need to consider the sampling strategy for my GPR-related training points. As I mentioned previously, I need to sample three types of points. Points within the NTR, points outside the NTR, and points near the border the NTR. The last points were previously kink points, when the NTR had followed assumptions ?? and ???. For the circular case however, there are no kinks. The previous sampling strategy is otherwise easily applicable to the circular case, but for the border points, I change the strategy slightly. I sample evenly spaced points (defined by their relative angle to the center point) on the border of the approximated NTR, and add random positive noise to ensure these are outside the NTR. This effectively covers the entire circle, and I can now leverage a low amount of training points for the GPR as for the proportional cost case.

outwards from the center of the NTR.

Figure 1.12: Sampling strategy for the fixed cost NTR, with no consumption or correlation.

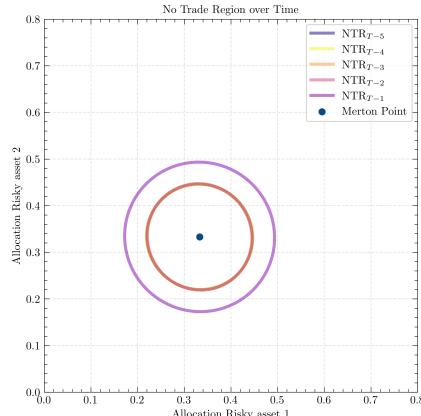


this sample strategy uses the same number of points as the schematic for the proportional cost sampling strategy

1.4.3 Fixed costs No-Trade Region without correlation

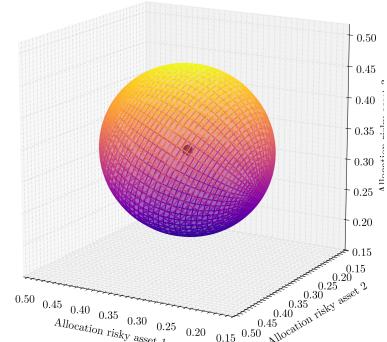
I now employ the mentioned algorithm, and solve the fixed cost case for the i.i.d parameters, with two assets, and no consumption. I solve the problem with the i.i.d parameters and a fixed cost of 0.075% of the investors total wealth at any time point. I consider an investment horizon of $T = 5$.

Figure 1.13: Fixed cost No-Trade Regions with i.i.d assets.



(a) NTR at each time step for i.i.d parameters, with two risky assets, fixed costs and no consumption.

3D Circular NTR at time $t = T-1$



(b) NTR at time step $T - 1$ for i.i.d parameters, with three risky assets, fixed costs and no consumption.

Sample points for the GPR used 210 points. Fixed costs at 0.075% of total wealth.

For the plot with two risky assets, I note that the NTR displays two distinct regions, one for $t = T - 1$ and one for $t < T - 1$. At the last period, the NTR is larger and the fixed costs affect the trading decision moreso than for periods $t < T - 1$. The Merton

point is now in the center of the NTR, and the NTR is circular as expected.

At time step $T - 1$ the radius is 0.16 whereas for time step $T - 4$ (0) the radius is 0.11. Hence, when fixed costs are considered only two periods need to be solved for, in order to approximate the NTR over the entire investment horizon. Any more solutions will only add to the approximation error, and the true NTR will not change significantly. I likewise solve the model for $D = 3$. I note for the three risky assets case that the NTR is now a sphere, which is to be expected, as the NTR is extended to higher dimensions. In order to approximate the higher order NTRs, I use a similar fitting scheme as in Figure 1.11. This is displayed in Figure ?? located in Appendix ???. The fixed costs problem can likewise be solved in higher dimensions, however since this poses no changes to the proposed solution methods, I do not consider this further, and continue to the correlated case, in order to generate a more general fixed cost solution.

1.5 Dynamic Portfolio Choice with fixed costs and correlation

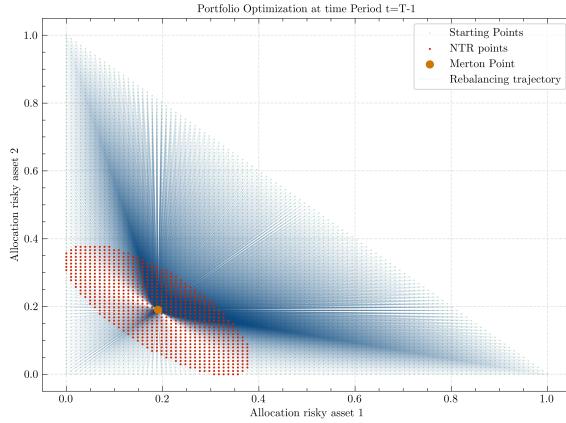
I now solve the model for correlated returns, that is, I solve for the Schober parameterization and for the high correlation parameterization. I set the fixed costs to 0.005% of the investors total wealth at any time point and consider an investment horizon of $T = 5$. For the proportional cost case, when asset returns were correlated, the square was shifted into a parallelogram shape, and I expect the same to happen for the fixed cost case, shifting the circle into an ellipse. I first solve for two assets for the high correlation case, as this parameterization should have the most pronounced effect on the geometric shape of the NTR. Furthermore I want to verify that the NTR solution, once again, is defined by two distinct regions, one for $t = T - 1$ and one for $t < T - 1$, as noted in Figure 1.13a.

I first verify the shape of the NTR for the high correlation case by solving over a fine grid as previously mentioned.

I note from Figure 1.14 that the NTR is now an ellipse, as expected. This new shape is due to the correlation between the returns, and the NTR is now skewed, as the returns are correlated. This is because the assets are now substitutes to some degree given by their positive correlation. I then need to reformulate the solution algorithm for this case, as the NTR is no longer circular, and the solution algorithm for the fixed costs case with i.i.d assets is no longer applicable, since an ellipse is not defined by a center point and a radius. Thus, I need to adapt the solution algorithm, in order to approximate the NTR correctly.

An ellipse in two dimensions is defined by its *foci*. For any point on the ellipse, the sum of the distances to the foci is constant. The ellipse has a major diameter (major axis) and a minor diameter (minor axis), respectively the longest and shortest distance between two points on the ellipse (Ivanov 2020). Given a center point and enough points

Figure 1.14: Grid solution to the high correlation case with fixed costs and two assets in period $T - 1$.



The optimization scheme ran with 5253 evenly spaced grid points. The points are plotted in the feasible space, and the NTR is the convex hull of these points. Fixed costs at 0.0005.

on the border of the ellipse (which may be noisy) I can approximate the ellipse by a least squares algorithm (Gander, Golub and Strelbel 1994). This requires enough points in order to solve the the problem sufficiently.

For two dimensions the minimum required points are 5 points with no three points collinear. For higher dimensions the required points are $d(d + 3)/2$ points, however otherwise the same procedure can be applied (Bertoni 2010).

I modify the solution algorithm in the following manner. I solve the optimization problem for a single point outside the NTR, and find the optimal trading decision towards the center³. I then sample $d(d + 3)/2 + 2^d + d + 1$ points, on the borders of the NTR. The 2^d points are the border points used for the square NTR sampling scheme. I then add $d(d + 3)/2 + d + 1$ random points, which are still on the border of the feasible space, by drawing random points on the border⁴. This should leave me with enough points which are not collinear, to approximate the ellipse. I then proceed with the bisection algorithm as previously mentioned, until I for each direction from the center, find the border point of the NTR.

I then apply the least squares algorithm and solve for the parametric equation of the ellipse (Gander, Golub and Strelbel 1994; Bertoni 2010). This method has the advantage that I can still solve for the ellipse using relatively few points, and these points need not cover the ellipse evenly, as the least squares algorithm will find the best fit ellipse for the

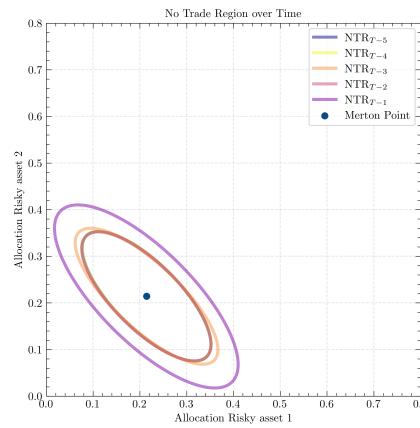
³I do this for a few points in order to ensure that the point is outside the unknown NTR. However, a singular point is all that is needed. For example full investment into one of the risky assets, will most likely fall outside the NTR.

⁴I constrain these points so they are sufficiently distanced from my previously sampled points. This ensures that the resulting directions from the center are unique, and border points are not identical.

points given⁵. The rest of the circular algorithms can be used as before. Hence the ellipse NTR is slightly more complex, given the fitting scheme and points required, but the rest of the solution algorithm is the same.

I do a slight modification to the high correlation μ vector in order to move the Merton point and make space for the resulting NTR. The new mean asset return is now $\mu = 0.075$ for each asset. This moves the Merton point to $(0.2143, 0.2143)$ from the previous $(0.1905, 0.1905)$. The solution for each time point is plotted below: As expected, the

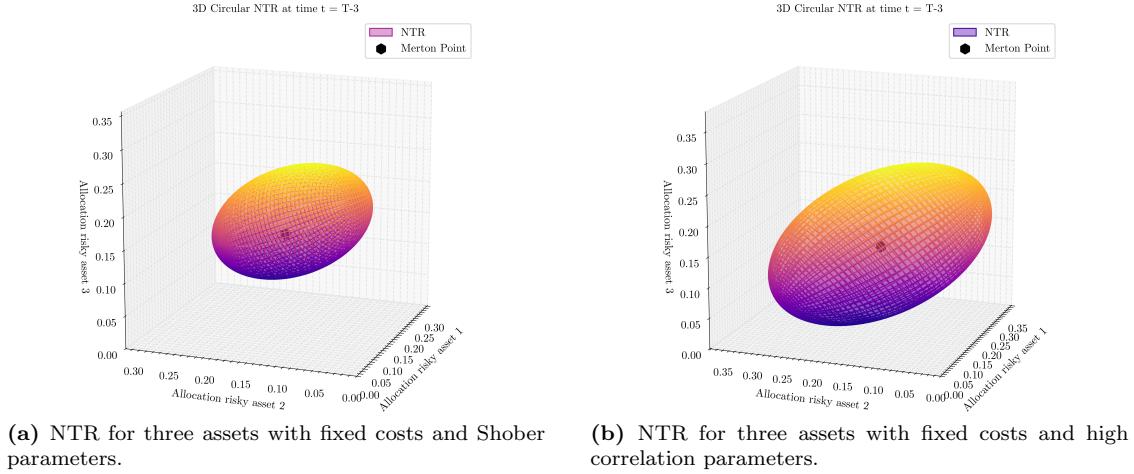
Figure 1.15: NTR for two assets with fixed costs and high correlation parameters.



NTR adjusts in the same manner as for the i.i.d case, and shrinks from period $T - 1$ to $T - 2$. Following this the NTR is constant, with some compounding approximation error across periods. Therefore, in order to approximate the NTR over the entire investment horizon, only two periods need to be solved for, as the NTR does not change significantly over time. Also note that the NTR still displays the same logic as for proportional costs, in regard to the substitutability of the assets, but slightly larger weighting of the risky assets is now preferred, compared to the case for proportional costs. I now increase the dimensionality of the model to $d = 3$ and look at the No-Trade region for the Schober parameters and for the high correlation parameters. The resulting NTRs are plotted below.

⁵This is also why small noise is not an issue.

Figure 1.16: Three asset No-Trade Regions with fixed costs and correlation



The No-Trade regions are plotted at time $T - 3$ which for 5 periods is $t = 2$.

The resulting NTRs are now ellipsoids, and the intuition from the two dimensional case carries over to the three dimensional case. The shape is now similar to an american football, and the high correlation case has more pronounced skewness as expected due to substitutability between the assets.

1.6 Adjusting the algorithm to new cost structures or assets

Given the methods I used, to adjust the original solution algorithm to the fixed cost case, I can now propose a general plan for adapting my framework to new cost structures. The framework applied followed these steps:

1. Solve the optimization problem over a fine grid, in order to verify the shape of the NTR.
2. Construct a new approximation scheme for the NTR given the new shape.
3. Construct a new sampling scheme for the GPR in order to approximate the NTR. Points near the NTR and its vertices, if it has any are of high importance.
4. Solve the dynamic portfolio choice problem, and approximate the NTR over the entire investment horizon.

Given this framework, the solution algorithm can be adapted to a wide range of cost structures, not considered. These cost functions could be: Quadratic costs, asset specific proportional costs, asset specific fixed costs and combinations of these. Price impact

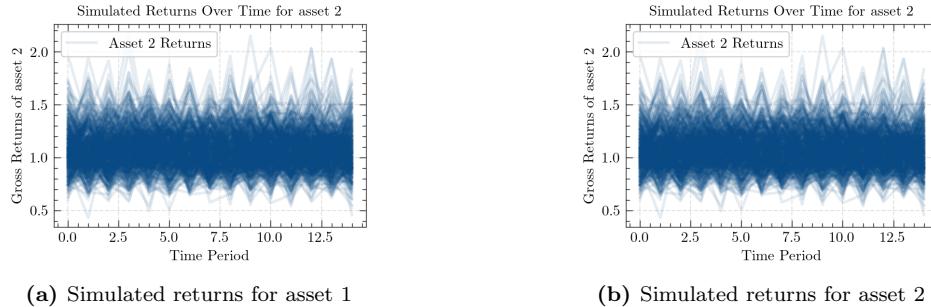
could likewise also be considered. These costs structures are solved in the static case by (Dybvig and Pezzo 2020).

This method of adapting the solution algorithm can also be applied to new asset structures. I consider this proposed framework to be a contribution to the literature, as it allows for much future research to be conducted.

1.7 Portfolio simulations

I now simulate actual investor behavior, given the NTR and the optimal trading decisions and consumption decisions. I also simulate a competing strategy, which is to trade with no regard to transaction costs⁶, and consume the optimal amount of wealth in each period. This is done in order to compare strategies which take transaction costs into account when rebalancing and strategies which do not. For each simulation I draw 499 random paths for the risky asset returns, and simulate my investor, according to the chosen rebalancing strategy and consumption decisions. I simulate $T = 15$ time periods. Simulated returns are plotted below:

Figure 1.17: Simulated returns for the two assets over time.



1.7.1 Portfolio simulations with fixed costs

I solve the portfolio problem for the high correlation parameters for two assets and fixed costs at $fc = 0.0005$ i.e 0.05%. I solve the model for $T = 15$ time periods. I then draw 499 random paths for the risky asset returns, and simulate my investor, according to the chosen rebalancing strategy and consumption decisions. This is done by:

1. Setting initial allocations⁷ $\mathbf{x}_{t=0} = (0.5, 0.5)$ and initial wealth at $W_{t=0} = 100$
2. For each time period t from 0 to $T - 1$ do:
 - (a) Solve the optimization problem for the given time period, and find δ_t^*, c_t^* (b_t^* residual)

⁶This is to the Merton point when consumption is not included. When consumption is included this is to the Merton point, multiplied by $(1 - c_t^*)$.

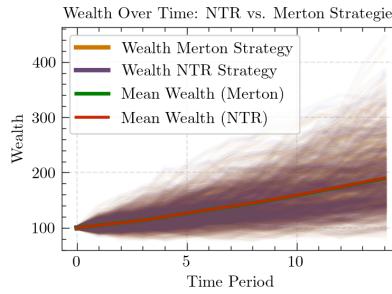
⁷The choice of starting allocation is somewhat arbitrary. I choose a naive 50% split.

- (b) Use drawn asset returns and state dynamics to compute next periods states
- (c) repeat until $t = T - 1$

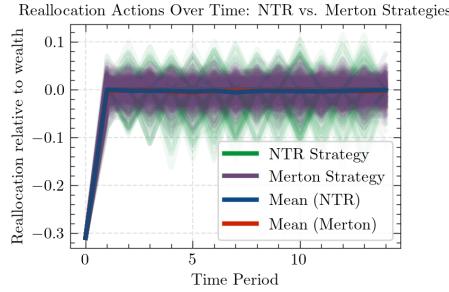
For the rebalancing strategy which trades to the Merton point, I set $f_c = 0.0$ when solving for the optimal trading and consumption decision. However, f_c is non-zero in the actual state dynamics. This is equivalent to not incorporating transaction costs in the rebalancing strategy. For the NTR strategy, I set $f_c = 0.0005$ when solving for the optimal trading and consumption decision, as in the actual state dynamics.

I first simulate the model with no consumption, and plot relevant variables over time for each simulation, and the mean of the simulations. Figure 1.18 displays the simulated

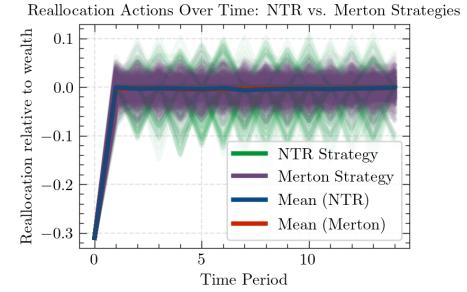
Figure 1.18: Simulated variables over time for the fixed cost case.



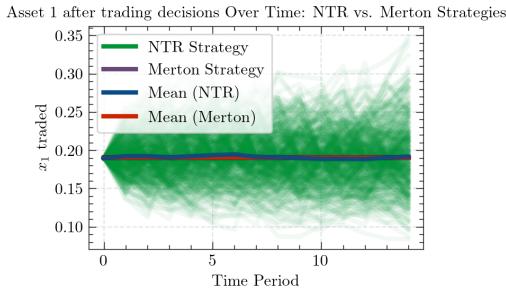
(a) Wealth over time for the two strategies.



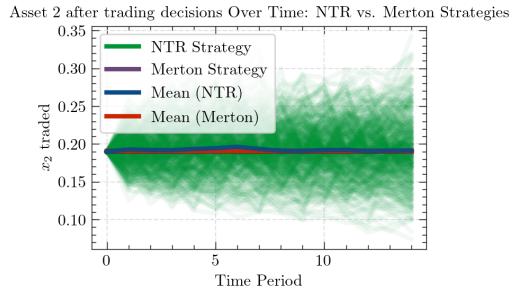
(b) Simulated δ_1 over time



(c) Simulated δ_2 over time



(d) Simulated $x_1 + \delta_1$ over time



(e) Simulated $x_2 + \delta_2$ over time

variables over time for the fixed cost case. I track the total wealth of the investor,

the trading decisions, and the new allocations after trading but before they are subject to returns. This is to display the strategies before random shocks (returns). I note firstly that the NTR strategy outperforms the no transaction cost strategy, as expected. The final mean total wealth of the NTR strategy is $W_{T-1} = 190.039$, whereas the no transaction cost strategy has a final mean total wealth of $W_{T-1} = 189.289$. Thus, the performance is negligibly better for the NTR strategy.

I note from the allocations after trade, that the NTR strategy contains many different allocations. These are allocations which are within the NTR, and the investor does not trade, which would otherwise be to the Merton point. This can also be seen in the plots of the δ variables. For the Merton strategy, there is only one asset allocation combination which is traded to, as to be expected.

For the NTR strategy I see that trades are larger, as this only occurs sufficiently far from the Merton point. Trades are also less frequent, which is due to the the NTR. Trades are generally smaller and more frequent for the Merton point strategy, as the investor trades to the Merton point, and the Merton point is close to the border of the NTR.

1.8 Dynamic Portfolio Choice with fixed and proportional costs

I now consider the model with both fixed and proportional transaction costs and no consumption. I first solve the model with no prior knowledge of the geometric shape of the NTR, in order to verify the shape of the NTR and the solution algorithm.

(Dybvig and Pezzo 2020) solve this in the static case, with asset specific costs, and find a hexagonal shape with an inner and outer NTR. I expect something similar to happen in the dynamic case, however whether the lines connecting the vertices are straight or not is unknown. The hexagonal shape found by (Dybvig and Pezzo 2020), seems to stem from asset specific costs, and not from the combination.

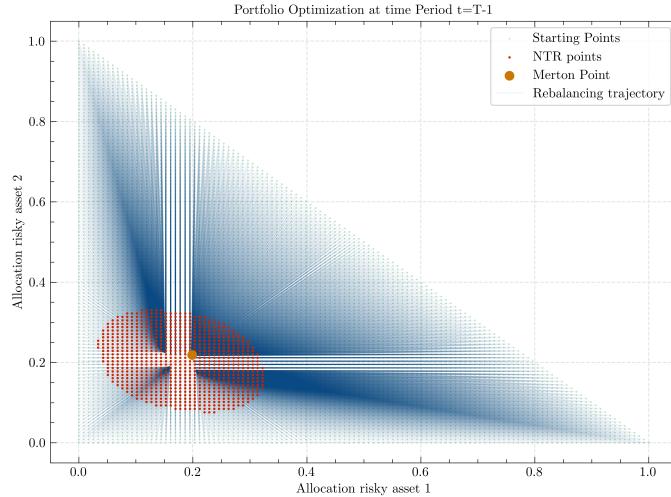
I would therefore expect an inner and outer NTR, each stemming from each type of transaction cost. I consider the Schober parameters and solve for the next to last period $T - 1$ with 2 assets. However, since the Merton point is close to the origin, and the borders of the feasible space, I add 0.005 to the mean asset return of each asset, in order to move the Merton point away from the borders. This moves the Merton point from (0.1508, 0.1831) to (0.1986, 0.2187).

I set the fixed costs to $fc = 0.0003$, which is 0.03% of the investors total wealth at any time point, and the proportional costs at 0.002, i.e 0.2% of the traded amount of wealth in each asset.

I follow my procedure and solve over a fine grid of points in order to figure out the shape of the resulting NTR.

The figure displays the crude solution method, i.e the entirely grid based method, of the proportional and fixed cost NTR. The resulting trade decisions and NTR are a mixture

Figure 1.19: Grid solution for two assets with fixed and proportional costs and Schober parameters.



The optimization scheme ran with 7140 evenly spaced grid points.

of the findings with either proportional or fixed costs, however the NTR is not a hexagon. The hexagonal shape found in (Dybvig and Pezzo 2020) does indeed seem to stem from the asset specific costs.

Instead the NTR now consists of two distinct shapes. The red points concern the ellipsoid, resulting from the fixed costs. For these points, the decision to trade is determined by the fixed cost, and the investor does not trade at all given the costs. This shape has some relation to the findings of (Liu 2004), as an inner and outer NTR is found, however the shape is not two squares, and is therefore different from the findings of (Liu 2004), but in line with findings of (Dybvig and Pezzo 2020).

When trading is optimal, the investor trades to the boundary of the proportional cost NTR, which has a parallelogram shape, and is inside the ellipsoid NTR. Optimal trading occurs to the vertices of this, and the Merton Point is at the right most border of the NTR, as previously seen in solutions to period $T - 1$ for proportional costs. Note that it is therefore no longer in the center of the ellipsoid, as for the case of only fixed costs. Hence while the ellipsoid stems from the fixed costs, the position of the ellipsoid is connected to the proportional costs, and the center of the proportional cost NTR is the center of the ellipsoid.

I propose, that for specific fractions of fixed cost and proportional cost, the vertices of the proportional cost NTR, will be on the border of the fixed cost NTR and the NTR will be a combination of the two shapes.

The NTR would in this case have vertices, which are the intersection of the two NTRs, but these would be connected by curved lines, from the fixed cost ellipsoid.

Whether the proportional NTR or the fixed cost NTR forms the outer most NTR is unknown a priori, without further investigation.

Interestingly, the proportional cost part of the NTR, is inside the fixed cost NTR, despite the proportional cost being higher than the fixed cost.

It is therefore not trivial to determine parameterizations where the two NTRs would form a cohesive shape, and this is left for future research.

As the optimal trades are no longer trivial in any manner, when the location of the NTR, its skewness and angles are all unknown a new tailored solution algorithm would be needed to effectively cover this case.

The case of the proportional cost NTR being the outermost NTR at any point is identical to the case of only proportional costs, and is therefore easily solvable with the previously mentioned solution algorithm. In this case fixed costs are of no concern.

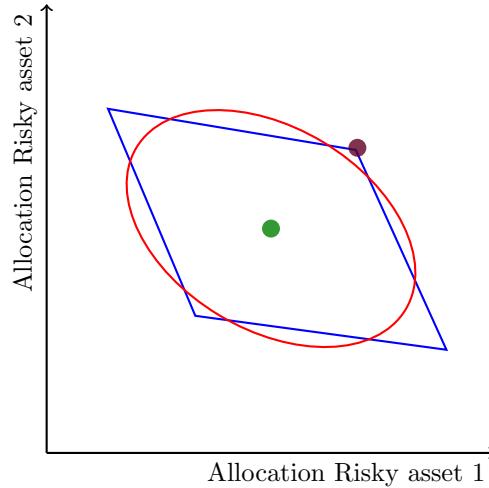
However when the reverse case is true, as in Figure 1.19 the solution algorithm is not applicable, and a new solution algorithm is needed.

Solutions return the vertices of the inner NTR, and the bisection algorithm is therefore not immediately applicable either.

Firstly the center point would have to be found, as the center of the inner NTR, and the border points would have to be found, in order to approximate the NTR. Then the bisection algorithm could be applied to find the border, and then I would have to determine the final NTR.

I note that this is even more complex. Furthermore, an even worse outcome can occur, if the proportional cost NTR is sometimes the outermost NTR, and sometimes the innermost NTR. This is displayed in the schematic figure below:

Figure 1.20: Schematic of the complex shaped NTR with both fixed and proportional costs.



Blue parallelogram stems from proportional costs, red ellipse stems from fixed costs. The purple dot is the placement of the Merton point, relative to the two NTRs at $t = T - 1$, known to be at the upper rightmost corner. The green dot is the placement of the Merton point, relative to the two NTRs at $t < T - 1$.

This NTR requires solutions of each separate NTR, the parallelogram and the ellipse that is, and then forming a convex hull as the outermost combination of these. The Angle of the fixe cost NTR and the proportional cost NTR stem from the correlation, so they are at least identical. However the proportional cost NTR could have no vertices outside the fixed cost NTR. Maybe it could have just two vertices outside the fixed cost NTR, or all the vertices could be outside the fixed cost NTR.

I consider this to be beyond the scope of this thesis, and do not consider this further, leaving this to future research in the field of dynamic portfolio choice with transaction costs.