1 Results

For the following results we consider 3 types of parameterizations for the portfolio problem. The first is a simple case where the assets are identically distributed as seen in (Cai, Judd and Xu 2013), the second is a case where the parameters are chosen to match the parameters in (Schober, Valentin and Pflüger 2022) also seen in (Gaegauf, Scheidegger and Trojani 2023). This is in order to be able to draw correct comparisons between the results. Furthermore this case, displays assets with slight variation in the mean and a small correlation between the assets, and no asset is dominating the others. The last parameterization is a modification of the first case where the correlation between the assets is larger (correlation coefficient of 0.75).

Table 1: Parameters for Examples of Portfolio Problems

	i.i.d Assets	Schober Parameters	High Correlation
T	6	6	6
k	3	5	3
γ	3.0	3.5	3.0
τ	0.5%	0.5%	0.5%
β	0.97	0.97	0.97
r	3%	4%	3%
$\mu^\top$	(0.07, 0.07)	μ_{Shober}	(0.07, 0.07)
Σ	$\begin{bmatrix} 0.04 & 0.00 \\ 0.00 & 0.04 \end{bmatrix}$	Σ_{Schober}	$\begin{bmatrix} 0.04 & 0.03 \\ 0.03 & 0.04 \end{bmatrix}$

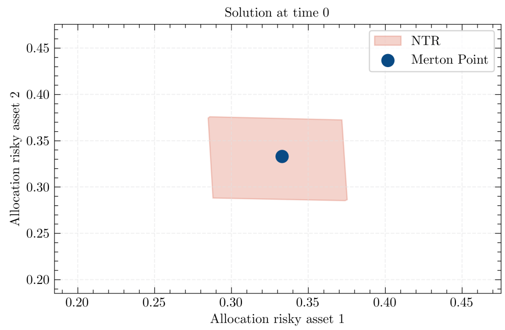
$$\mu_{\text{Schober}}^\top = \begin{bmatrix} 0.0572 & 0.0638 & 0.07 & 0.0764 & 0.0828 \end{bmatrix}$$

$$\Sigma_{\text{Schober}} = \begin{bmatrix} 0.0256 & 0.00576 & 0.00288 & 0.00176 & 0.00096 \\ 0.00576 & 0.0324 & 0.0090432 & 0.010692 & 0.01296 \\ 0.00288 & 0.0090432 & 0.04 & 0.0132 & 0.0168 \\ 0.00176 & 0.010692 & 0.0132 & 0.0484 & 0.02112 \\ 0.00096 & 0.01296 & 0.0168 & 0.02112 & 0.0576 \end{bmatrix}$$

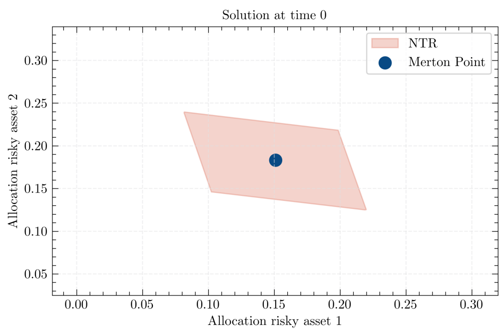
1.1 Dynamic Portfolio Choice without consumption

I first consider the base model with proportional transaction costs and no consumption. In the absence of consumption, the optimal portfolio is the merton point, which we plot in every figure. I plot the No-trade region at time point 0 (initial time point) for each of the parameterizations in 1.1. When using the Schober parameters we select the d first elements of the mean vector, and truncate the covariance matrix to a $d \times d$ matrix, depending on the number of assets d in the model. I note that for each of the

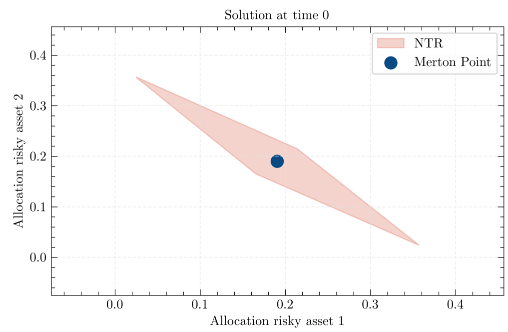
Figure 1.1: Comparison of No Trade Regions.



(a) No Trade Region for Independent Identically Distributed Assets.



(b) No Trade Region for Schober Parameters.



(c) No Trade Region for High Correlation.

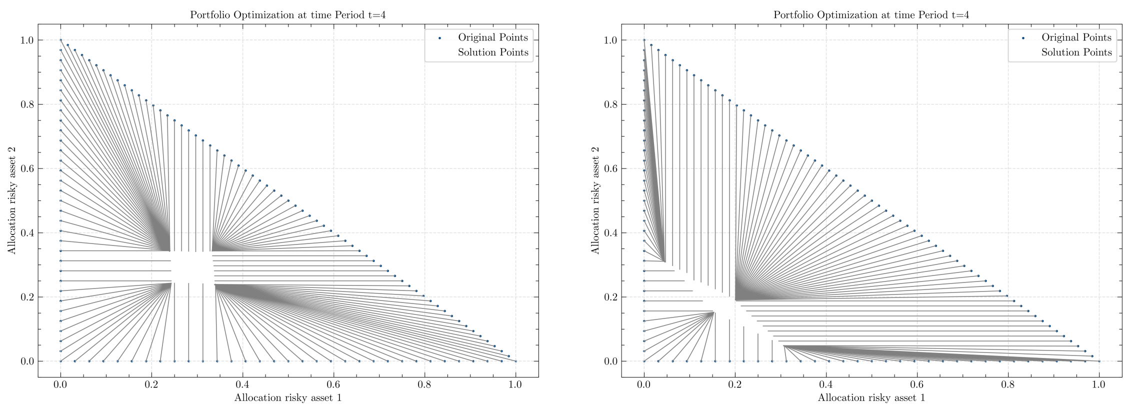
parameterizations the No-Trade region is a rectangle or parallelogram. For the case of identical and independent assets, the No-Trade region is a perfect square, whereas for the Schober parameters and the high correlation case, the No-Trade region is a parallelogram. This is due to the correlation between the assets. When some correlation is present, the No-Trade region is skewed, since some allocations which would be optimal in the absence of correlation are no longer optimal.

1.1.1 Verifying the geometric shape of the No-trade Region

Since much of the procedure for solving this problem, and approximating the NTR, leverages the a-priori assumptions regarding the geometric shape of the NTR, i first want to verify that the NTR indeed has 4 vertices for the 2d case and is a convex hull.
In order to do this, i do a small modification to the solution algorithm proposed earlier. Instead of computing vertices using 2^d predetermined points, i will instead sample a larger set of points, ($2^7 = 128$) covering the boundaries of the feasible state space. For each of these points i then solve the optimization problem, and plot the solution, from allocations $\mathbf{x}_t$ and their solutions to the problem $\hat{\boldsymbol{\omega}}_t$. I do this by using my original sample scheme, and adding mid-points between points, which either sum to 1.0 or have 0.0 as allocation for one of the assets. I consider the i.i.d case and the high correlation

case, with $tau = 1\%$. I have increased the costs slightly in order to increase the size of the NTR. This is to ensure that points also converge towards the faces and not only the verticies. This is akin the the green regions in Figure **??**. Otherwise i would need more points. I plot the solutions for next to last period with investment decisions $T - 2$. The solutions are plotted below.

Figure 1.2: Verifying the assumptions of the NTR iin 2 dimensions.



(a) Solutions to the i.i.d case with 2 assets in period $T - 2$.

(b) Solutions to the high correlation case with 2 assets in period $T - 2$.

I find that the assumptions regarding the no-trade-region (NTR) are indeed correct in the two dimensional examples i have constructed. Furthermore this verifies that the assumptions also hold for correlated asset, which was only postulated by (Liu 2004). Furthermore these plots also nicely confirm that the optimization process as a whole works as intended. Further verification in higher dimensions are not considered. First of all (Liu 2004) confirms this formally in larger dimensions, for the case of uncorrelated assets, and the intuition regarding the NTR does not change when dimensionality is increased.

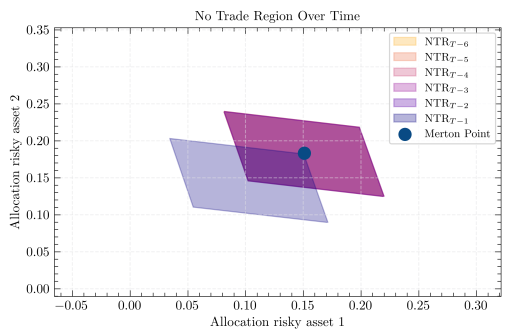
1.1.2 Investigating the No-Trade Region

We now look at the No-Trade region for the base model with proportional transaction costs and no consumption in more detail. Specifically we look at how the region behaves over the entire investment horizon $[0, T]$, and how the region changes with different transaction cost levels. We choose to look at the model with the Schober parameters, as this is a mixture of the other two parameterizations.

I note that at the last time point $t = T - 1$ the NTR moves away from the merton point towards the origin, and the Merton point is is now the upper right corner of the NTR. For all other time periods the NTR is the same, and the Merton point is in the center.

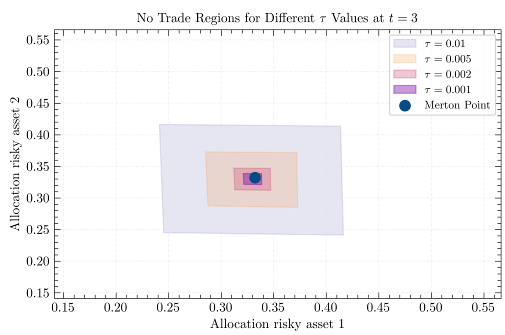
I now investigate how the No-Trade region changes with different transaction cost

Figure 1.3: No Trade Region for Schober Parameters over Time.



The No-Trade region is plotted for the Schober parameters over the entire investment horizon $[0, T-1]$ For time points $t \in [0, T-2]$ the NTRs overlap.

Figure 1.4: No Trade Region for the iid Parameters with different values of τ.



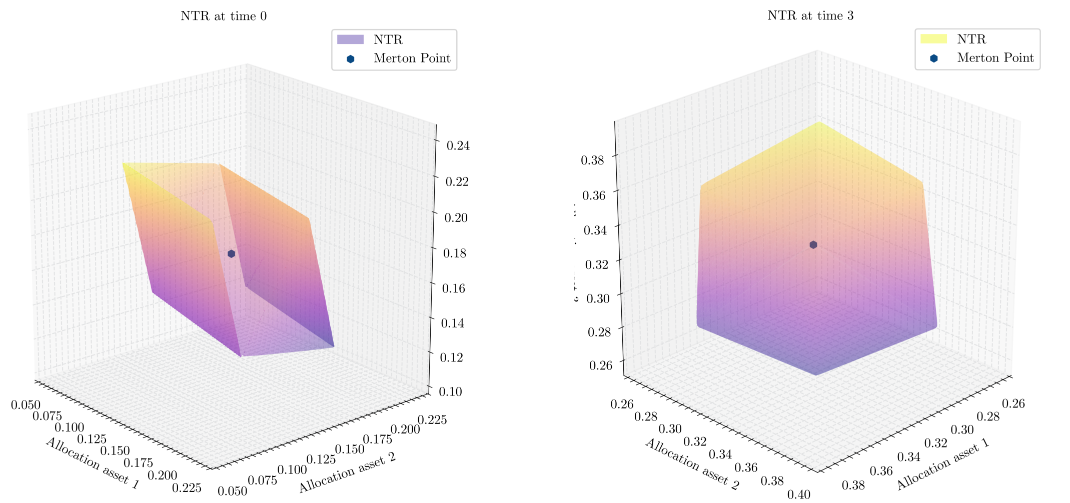
levels. I do this for the i.i.d. parameters, and plot the NTR for different values of τ in Figure 1.4. When the transaction costs are increased, the NTR increases aswell and vice-versa. I note that for low enough transaction costs, the NTR shrinks towards the Merton point. However when transaction costs are low enough, the Merton point is not in the excact center, which might signify that at low enough values, some numerical instabilities from the minimizer, and function approximation using Gaussian process regression (GPR) might be present.

1.1.3 Increasing the dimensionality of the model

We now increase the dimensionality of the model to $d = 3$ and look at the No-Trade region for the Schober parameters and for the i.i.d parameters.

Note that the i.i.d NTR looks like a skewed cube, whereas this was a perfect square in the 2 dimensional case. Looking that the points forming the convex hull that is the NTR, it is clear that the NTR is restricted by the no-borrowing constraint, since one of the border points, which would otherwise form the perfect cube, would outside the feasible

Figure 1.5: Comparison of No Trade Regions.



(a) No Trade Region for Schober Parameters in 3 dimensions.

(b) No Trade Region for i.i.d. Parameters in 3 dimensions

space if this was possible, and is then projected into the feasible space. Hence when the risky returns outweigh the risk-free return, to such a degree that the merton point moves towards the boundary of the feasible space, cube like shapes are no longer possible. In the 2 dimension case, this is akin to the NTR being close to the budget line, and the NTR would then form a triangle.

This is clear when compared to the Schober parameters, where the merton point is in the center of the NTR, and the NTR is a skewed cube. The merton point in this case suggest lower portfolio allocations to the risky assets, and hence the NTR is not restricted by the no-trading and no-borrowing constraints.

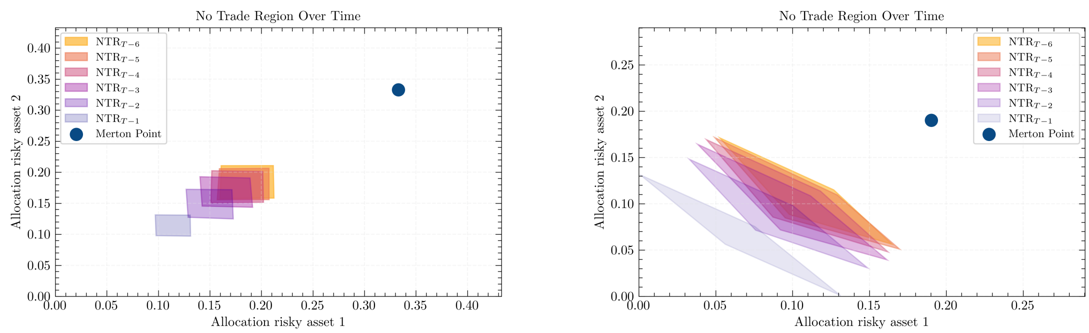
1.2 Dynamic Portfolio Choice with consumption

I now consider the base model with proportional transaction costs which now includes consumption of a non-durable good. This adds an extra decision variable which needs to be solved for, and consumption now adds immediate utility to the investor, in each period.

Note that when consumption is included, the NTR no longer encapulates the Merton point at any time point. Furthermore the NTR now moves over time, towards the origin, as opposed to the case without consumption, where the NTR was static for all time points except the next to last period (last period with trading decisions).

This behaviour is consistent in higher dimensions.

Figure 1.6: Comparison of No Trade Regions over time with consumption.

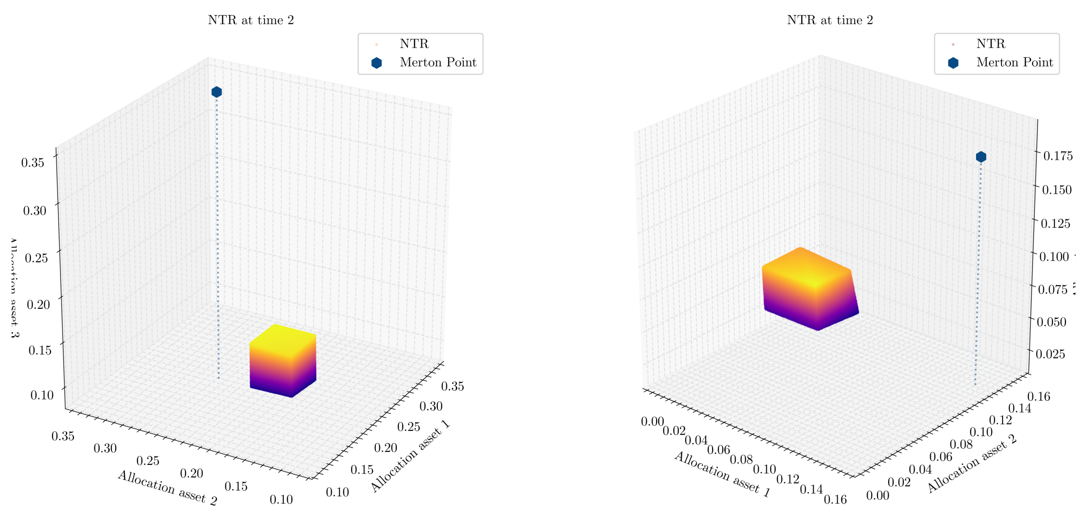


(a) No Trade Region for i.i.d parameters over time with consumption.

(b) No Trade Region for high correlation parameters over time with consumption.

The No-Trade regions are plotted over the entire investment horizon $[0, T-1]$.

Figure 1.7: No trade regions with consumption in multiple dimensions.



(a) No Trade Region for i.i.d parameters with consumption in 3d.

(b) No Trade Region for Schober parameters with consumption in 3d.

The No-Trade regions are plotted at time $t = 2$.

1.3 Dynamic Portfolio Choice with fixed costs

I now consider the base model with fixed transaction costs, and no consumption. From (Dybvig and Pezzo 2020) i know that the NTR is no longer rectangular when we only consider fixed costs, but instead circular with the merton point in the middle. This poses a problem for my current sampling scheme, which leverages our predetermined knowledge of the geometric shape of the NTR. As i noted in Section **??**, in order to effectively sample points for the NTR approximation, i now need to sample points, such that when they hit the NTR these points are evenly distributed on the sphere, in order to approximate the

NTR correctly.
The fixed costs pose further problems for the solution algorithm. In order to see this a little intuition is needed.

Transaction costs no longer scale, but are treated as a *sunk cost*, the moment the decision to trade is made. Hence if trading is optimal, the investor will trade to the optimal point, and if trading is sub-optimal then no trading will occur. The problem is therefore first of all a trading decision problem, and if trading is optimal, then the investor will trade to the merton point.

This is in stark contrast to the proportional case, where the trading trajectory from outside the NTR was to the border of the NTR, and the NTR approximation could be done by sampling points on the border of the feasible space.

Now, any point sampled outside the NTR trades to the merton point, and i need to construct a new strategy, in order to efficiently construct the NTR.

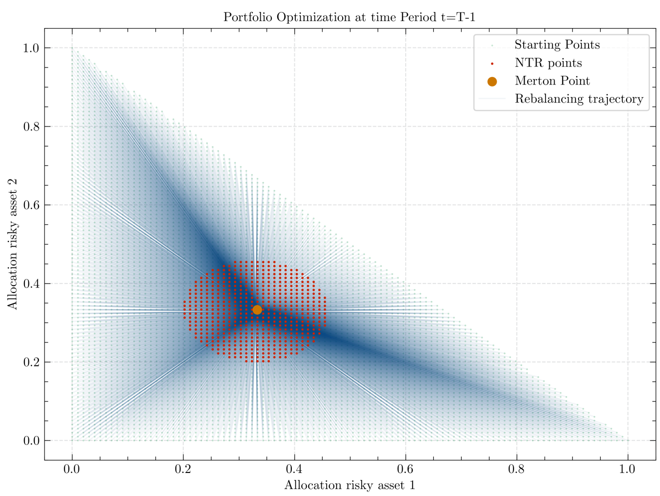
Furthermore, the transaction cost function is now an indicator function, depending on a threshhold, i.e $\sum_{i=1}^{k} \delta_{i,t}^{+} + \delta_{i,t}^{-} > 0$. This is non-differentiable at the kink, $\sum_{i=1}^{k} \delta_{i,t}^{+} + \delta_{i,t}^{-} = 0$, which is a critical point, which i have to deal with, in order to solve the optimization problem.

I therefore split the optimization process into two parts. I evaluate the objective function (value function), conditional on no trading ($\boldsymbol{\delta}_t = \mathbf{0}$), and conditional on trading ($\boldsymbol{\delta}_t \neq \mathbf{0}$). Since there is no consumption decision the no-trading decision is trivial, whereas i still optimize the trading decision in order to maximize expected utility. By splitting the optimization process, i can avoid the the non-differentiable edge case, and the derivative with regard to fixed costs i trivial for the optimizer. I then evaluate the value function for the no-trading decision, and the trading decision, and choose the decision which maximizes the value function.

I now consider the base model with fixed transaction costs, and no consumption. I use the simple i.i.d parameterization, with 2 assets and solve the optimization problem for the next to last period $T-1$, over an evenly spaced grid of points. I do this in order to verify that the solution algorithm works as intended, and that the NTR is circular as expected, conflicting with my prior assumptions for the proportional case. I set the fixed costs to 0.005% of the investors total wealth, at any time point, and solve at a very fine grid of points, in order to approximate the NTR correctly. I find that the NTR is circular as expected, and the solution algorithm works as intended. I therefore proceed with generating a strategy for dealing with fixed costs, which can leverage my new found knowledge of the NTR.

1.3.1 Constructing a new sampling scheme for the fixed cost NTR

Figure 1.8: Solution to the i.i.d case with fixed costs, 2 assets in period $T-1$.



The optimization scheme ran with 5044 evenly spaced grid points. The points are plotted in the feasible space, and the NTR is the convex hull of these points.