1 The Dynamic Portfolio Choice Setting

This section covers the theoretical setting and components of the dynamic portfolio choice problem with transaction costs. This section leans heavily on (Cai, Judd and Xu 2020) and (Gaegauf, Scheidegger and Trojani 2023), bridging both models to create a comprehensive framework for the dynamic portfolio choice problem with transaction costs. I first cover the individual components of the model, and then present the general class of dynamic portfolio choice problems with transaction costs. The baseline model with proportional costs is covered and extended to include fixed costs.

1.1 Asset and goods market

I consider a financial market with D risky assets and one risk-free asset, making the asset space $1 + D$ dimensions. The risk-free asset, such as a bond or a bank deposit, yields a constant gross return $R_f = \mathrm{e}^{r\Delta t}$, where r is the annual interest rate and $\Delta t = \frac{T}{N}$ is the length of one investment period. The risk-free asset is assumed to be liquid and can be traded without transaction costs. The investor has wealth W_t at time t, which is allocated between the risk-free asset, the risky assets and consumption. For each time period all wealth must be allocated to either of these.

The D risky assets can be considered as listed stocks, subject to proportional transaction costs. For each reallocation of wealth in a risky asset, a transaction cost of $\tau \in [0, 1]$ is incurred as a percentage of the traded amount. The stochastic one-period gross-return vector of the risky assets is denoted as $\mathbf{R} = (R_1, R_2, \ldots, R_D)^\top$, and the corresponding net-return vector is $\mathbf{r} = (r_1, r_2, \ldots, r_D)^\top$.

In the goods market, there is a single non-durable consumption good, C, which is consumed at each time point t. The fraction of wealth allocated to consumption at time t is denoted c_t, the fraction allocated to risky assets is $\mathbf{x}_t = (x_{1,t}, x_{2,t}, \ldots, x_{k,D})^\top$, and the fraction allocated to the risk-free asset is denoted[1] b_t. I assume that no shorting of assets, or borrowing is allowed, thus the variables are constrained by $\sum_{i=1}^{D} x_{i,t} + b_t \leq 1$ and $\mathbf{x}_t \in [0, 1]^D$ and $b_t \in [0, 1]$, whereas actual amounts denoted in currency exists in $\mathbb{R}^+$.

1.2 Asset dynamics

I follow Cai, Judd and Xu (2013) for the asset dynamics. The total composition of risky assets is assumed to follow a multivariate log-normal distribution:

$$\log(\mathbf{R}) \sim \mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t, (\mathbf{\Lambda\Sigma\Lambda})\,\Delta t\right), \tag{1}$$

[1] This notations stems from earlier literature, where the risk free asset is a risk free bond.

where μ is the drift vector, σ^2 is a column vector of the variance σ_i^2, $\mathbf{\Sigma}$ is the correlation matrix, and $\mathbf{\Lambda} = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k)$ is the diagonal matrix of volatilities. Following Cai, Judd and Xu (2013) I utilize the Cholesky decomposition of the correlation matrix, $\mathbf{\Sigma} = \mathbf{L}\mathbf{L}^\top$, where $\mathbf{L} = (L_{i,j})_{k\times k}$ is a lower triangular matrix. Hence, for each risky asset i, the log-return is:

$$\log(R_i) = \left(\mu_i - \frac{\sigma_i^2}{2}\right)\Delta t + \sigma_i\sqrt{\Delta t}\sum_{j=1}^{i} L_{i,j} z_j, \tag{2}$$

where z_i are independent, standard normal random variables.

1.3 Transaction costs and portfolio reallocation

Rebalancing is subject to proportional transaction costs $\tau \in [0,1]$, which are paid based on the amount bought or sold of each risky asset. Reallocation decisions are made just before $t_j + \Delta t$, such that $\mathbf{x}_t$ is the portfolio of risky assets right before realllocation, and before portfolio returns are incurred, this is akin to trading decisions being made before the market opens. $\delta_{i,t}$ denotes the change in portfolio allocation of asset i, and $\delta_{i,t}W_t$ is thus the currency amount traded in asset i. Hence $\delta_{i,t} > 0$ implies buying asset i, and $\delta_{i,t} < 0$ implies selling asset i. Proportional transaction costs imply that the cost function associated with rebalancing is:

$$\psi(\delta_{i,t}W_t) = \tau|\delta_{i,t}W_t| \tag{3}$$

I decompose the decision variable $\delta_{i,t}$, representing the fraction of wealth used to trade risky asset i, into buying ($\delta_{i,t}^+$) and selling ($\delta_{i,t}^-$) components to ensure tractability[2]:

$$\delta_{i,t} = \delta_{i,t}^+ - \delta_{i,t}^-, \quad \delta_{i,t}^+, \delta_{i,t}^- \geq 0.$$

The total transaction cost is then given by $\tau\sum_{i=1}^{k}(\delta_{i,t}^+ + \delta_{i,t}^-)W_t$. The transaction cost function is therefore a function of each trading direction:

$$\psi(\delta_{i,t}^+, \delta_{i,t}^-, W_t) = \tau(\delta_{i,t}^+ + \delta_{i,t}^-)W_t \tag{4}$$

Following the reallocation, the remaining wealth is allocated between the risk-free asset and consumption. Notation of rebalancing is henceforth simplified using vectors to $\boldsymbol{\delta}_t = \boldsymbol{\delta}_t^+ - \boldsymbol{\delta}_t^-$ with $\boldsymbol{\delta}_t^+ = (\delta_{1,t}^+, \delta_{2,t}^+, \ldots, \delta_{D,t}^+)$ and $\boldsymbol{\delta}_-^+ = (\delta_{1,t}^-, \delta_{2,t}^-, \ldots, \delta_{D,t}^-)$. $\boldsymbol{\delta}_t$ is the *net change* in the risky positions, and $\boldsymbol{\delta}_t^+ + \boldsymbol{\delta}_t^-$ is the *total trading volume* in the risky positions. Total trading volume is linked to the transaction costs, and the net change is linked to

[2] Gaegauf, Scheidegger and Trojani (2023) note that this ensures differentiability. This approach is common and found in earlier work such as Akian, Menaldi and Sulem (1996), who likewise note that this ensures that the variable is continious from origin in the positive real set.

the portfolio allocation.

1.4 Investor preferences and problem

The investor operates over a finite horizon of T years, during which the aim is to maximize expected utility. Following Cai, Judd and Xu (2013), the investment horizon is discretized into N equally spaced periods, each with a duration of $\Delta t = \frac{T}{N}$. Hence this is a discrete time model, which converges to the continuous time model as $\Delta t \to 0$. At each time point t_j, for $j = 0, 1, \ldots, N$, where $t_0 = 0$ and $t_N = T$, the investor has the opportunity to adjust the portfolio allocations right before $t_j + \Delta t$. Reallocation is costly, and the investor is subject to proportional transaction costs. If consumption is included the investor may also choose to consume a non-durable good at each time point.

For notational simplicity, I now use t to denote these time points unless specifically referring to t_j. The investor's preferences are isoelastic, i.e I use a constant relative risk aversion (CRRA) utility function[3]:

$$u(C_t) = \begin{cases} \frac{C_t^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1, \\ \log(C_t) & \text{if } \gamma = 1, \end{cases} \tag{5}$$

where C_t is consumption and c_t is the fraction of wealth W_t spent on consumption at time t. Hence $c_t = C_t/W_t$, and lowercase notation is henceforth used to denote variables as fractions of wealth. γ is the coefficient of relative risk aversion. The objective is to maximize the expected utility of consumption and wealth over the investor's lifetime:

$$\max_{\mathbf{x}_t, b_t, c_t} \mathbb{E}\left[\sum_{i=0}^{N-1} \beta^i u(C_i)\Delta t + \beta^N u(W_N)\right], \tag{6}$$

where β is the discount factor, $\mathbf{x}_t$ is the allocation to risky assets, b_t is the allocation to the risk-free asset, and W_t is the investor's wealth at time t.

1.5 Intertemporal portfolio choice without transaction costs

When there are no transaction costs (no market frictions) the investor can freely rebalance the portfolio. This reduces the problem to the classic portfolio optimization problem formulated by (Merton 1969; Merton 1971). For a more detailed treatment, see (Björk 2019).

In this setting, the investor dynamically allocates wealth between D risky assets and a risk-free asset to maximize utility over a finite horizon $[0, T]$. The investor's wealth W_t can be allocated between a risk-free asset and D risky assets. Consumption is a

[3] This is common in the literature. Other utility functions such as Epstein-zin preferences have been used but are less common.

non-durable good that can be purchased at each time point t. r is the risk-free rate, $\boldsymbol{\mu}$ is the vector of expected returns on the risky assets, and C_t represents consumption at time t. The investor's preferences follow a constant relative risk aversion (CRRA) utility function.

Without transaction costs, the optimal portfolio allocation, known as the Merton point is:

$$\mathbf{x}_t^* = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r), \tag{7}$$

where γ is the coefficient of relative risk aversion, and $\boldsymbol{\Sigma}$ is the covariance matrix of the risky assets' returns. This provides a time-independent optimal allocation that serves as a benchmark for models incorporating frictions such as transaction costs. This fraction is the optimal investment allocation, after consumption, to each risky asset and is independent of the wealth level. Merton also provides a closed form solution to the optimal consumption level. However this depends on wealth, and I find no relation to this and my solutions later, so this is not covered.

1.6 The general class of dynamic portfolio choice with transaction costs and intertemporal consumption

Now consider when transaction costs are present, and the investor can consume a non-durable good at each time point. The solution to the dynamic portfolio choice problem is no longer given by the closed form solution of the Merton point. Considering the components presented in this section, the class of dynamic portfolio optimization problems, given one risk free asset and D risky assets, can be formulated by the following Bellman equation[4]:

$$V_t(W_t, \mathbf{x}_t, \theta_t) = \max_{c_t, \boldsymbol{\delta}_t^+, \boldsymbol{\delta}_t^-} \{u(c_t W_t)\Delta t + \beta \mathbb{E}_t \left[V_{t+\Delta t}(W_{t+\Delta t}\mathbf{x}_{t+\Delta t}, \theta_{t+\Delta t})\right]\}, \quad t < T \tag{8}$$

Given some initial level of wealth W_0 and portfolio allocation $\mathbf{x}_0$. θ_t is a vector of stochastic variables, which the gross one period risk free return, and risky return depends on, i.e $\mathbf{R}(\theta_t)$ and $R_f(\theta_t)$. These could cover the drift μ, volatiliy σ^2, correlation of the risky assets Σ, and the risk free return r or only some of these, dependent on the model. Notice that future wealth and allocations are stochastic, as they depend on the future realization of θ_t.

Consumption and reallocation are decision variables, whereas bond holdings are not (explicitly). This is because bond holdings can be determined as the residual wealth, after

[4] This is consolidated model of the base model, and with consumption model, of (Cai, Judd and Xu 2020), however the cost function is generalized and correlation of returns is included. For more on Bellman equations see (Bellman 1958)

consumption and reallocation decisions are made:

$$b_t W_t = \left(1 - \mathbf{1}^\top \cdot \mathbf{x_t}\right) W_t - \mathbf{1}^\top \cdot \boldsymbol{\delta}_t W_t - \psi(\boldsymbol{\delta}_t^+, \boldsymbol{\delta}_t^-, W_t) - c_t W_t \tag{9}$$

Where $\psi(\cdot)$ is the transaction cost function, and $\mathbf{1}$ is a vector of ones.
The dynamics of the state variables follow Schober, Valentin and Pflüger (2022) and are given by:

$$W_{t+\Delta t} = b_t W_t R_f(\theta_t) + ([\mathbf{x_t} + \boldsymbol{\delta}_t] W_t)^\top \cdot \mathbf{R}(\theta_t) \tag{10}$$

$$\mathbf{x}_{t+\Delta t} = \frac{((\mathbf{x}_t + \boldsymbol{\delta}_t) W_t) \odot \mathbf{R}_t(\theta_t)}{W_{t+\Delta t}} \tag{11}$$

Where $\odot$ is the elementwise product (Hadamand product). The terminal value function is given by[5]:

$$V_T(W_T, \mathbf{x}_T, \theta_T) = u(W_T - \psi(\mathbf{x}_T W_T)) \tag{12}$$

Which implies that the investor consumes everything at the terminal period, after moving investments to the deposit (bonds) account. Finally I note that the optimization problem is subject to the following constraints:

$$\boldsymbol{\delta}_t W_t \geq -\mathbf{x}_t W_t \tag{13}$$

$$b_t W_t \geq 0 \tag{14}$$

$$\mathbf{1}^\top \mathbf{x}_t \leq 1 \tag{15}$$

The first constraint ensures that the investor does not short sell risky assets, The second is a no borrowing constraint constraint, or no shorting if b_t is viewed as a bond. The third is a no-leverage constraint (and no shorting / borrowing).
Furhtermore I note that the rebalancing decision (in each direction), is only feasible in the space, given the current allocation.:

$$\delta_{i,t}^+ \in [0, 1 - x_{i,t}] \tag{16}$$

$$\delta_{i,t}^- \in [0, x_{i,t}] \tag{17}$$

This is a direct formulation of the constraints, already captured in the prior constraints.
The problem can be simplified by normalizing with regard to wealth W_t, which removes W_t as a state variable, since wealth is seperable from the rest of the state space $\mathbf{x}_t, \theta_t$ as noted in (Cai, Judd and Xu 2013).
This is because portfolio optimality is independent of wealth for CRRA utility function

[5] Stemming from the sum of discounted utility.

under proportional costs[6]. The Bellman equation is then:

$$v_t(\mathbf{x}_t, \theta_t) = \max_{c_t, \boldsymbol{\delta}_t^+, \boldsymbol{\delta}_t^-} \{u(c_t)\Delta t + \beta \mathbb{E}_t \left[\pi_{t+\Delta t}^{1-\gamma} v_{t+\Delta t}(\mathbf{x}_{t+\Delta t}, \theta_{t+\Delta t})\right]\}, \quad t < T \tag{18}$$

The normalized bond holdings are then:

$$b_t = 1 - \mathbf{1}^\top \cdot (\mathbf{x_t} - \boldsymbol{\delta}_t - \psi(\boldsymbol{\delta}_t^+, \boldsymbol{\delta}_t^-)) - c_t \Delta t \tag{19}$$

This is still the residual of the wealth after the rebalancing and consumption decision. I still formulate the transaction cost function $\psi(\cdot)$ in terms of the buying and selling components, and use changes to allocations proportional to wealth, instead of the prior formulations, where wealth was a direct input. The state dynamics are then:

$$\pi_{t+\Delta t} = b_t R_f(\theta_t) + (\mathbf{x}_t + \boldsymbol{\delta}_t)^\top \cdot \mathbf{R}(\theta_t) \tag{20}$$

$$\mathbf{x}_{t+\Delta t} = \frac{(\mathbf{x}_t + \boldsymbol{\delta}_t) \odot \mathbf{R}_t(\theta_t)}{\pi_{t+\Delta t}} \tag{21}$$

$$W_{t+\Delta t} = \pi_{t+\Delta t} W_t \tag{22}$$

I now formulate the problem with regard to the proportional wealth change $\pi_{t+\Delta t} = \frac{W_{t+\Delta t}}{W_t}$. The terminal value function is:

$$v_T(\mathbf{x}_T, \theta_T) = u(1 - \psi(\mathbf{x}_T)) \tag{23}$$

The constrains are likewise normalized:

$$\boldsymbol{\delta}_t \geq -\mathbf{x}_t \tag{24}$$

$$b_t \geq 0 \tag{25}$$

$$\mathbf{1}^\top \mathbf{x}_t \leq 1 \tag{26}$$

This class of dynamic portfolio choice problems covers most formulations of the problem, where the transaction cost specification is differentiable, and the utility function allows for seperability of wealth and remaining state variables. If seperability is not feasible, then the formulation with wealth as a state variable must be used, and the problem is more complex to solve. Since wealth denoted variables depend on prior wealth levels, and the problem is not seperable over time.

Later formulations will be based on this class structure, covering the necessary Bellman equation, state dynamics, preferences and transaction costs functions as well as the constraints and any extensions not yet presented.

The non-normalized optimal choices can be obtained by multiplying the normalized

[6] This will also be the case for my formulation of fixed costs.

choices with the wealth level W_t at a given time point t.
No closed form solution has been found for this class of problems with more than two risky assets. Numerical methods are required to solve the problem. The solutions to this problem are the optimal trading decisions and consumption, $\boldsymbol{\delta}_t^{+,*}, \boldsymbol{\delta}_t^{-,*}, c_t^*$. Optimal trading decisions for the problem, have been found to be the trading trajectory towards a region known as the No-Trade Region (NTR), which minimizes the euclidian distance between the allocation $\mathbf{x}_t$ and the NTR (Cai, Judd and Xu 2013). The NTR is in this framework the set of asset allocations where it is sub-optimal to rebalance the portfolio, and is defined as:

$$\Omega_t = \{\mathbf{x}_t : \boldsymbol{\delta}_t^{+,*}, \boldsymbol{\delta}_t^{-,*} = \mathbf{0}\} \tag{27}$$

Where $\boldsymbol{\delta}_t^{+,*}, \boldsymbol{\delta}_t^{-,*}$ are the optimal buying and selling policies at time t. The NTR is central to this problem, as this region covers optimal trading decisions, and therefore every solution, when consumption is not included. When consumption is included, optimal consumption levels still need to be solved for and the problem is more complex. The next section will cover the NTR in more detail.

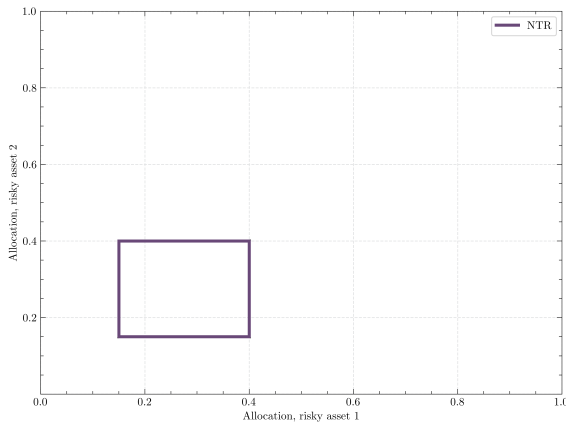
1.7 No Trade Region (NTR)

The NTR is a region in the risky asset space where it is sub-optimal to rebalance the portfolio. Given the parameters of the model the NTR without consumption is defined as in equation (27). If consumption is included, this definiton remains the same, but the consumption decision may vary within the NTR. Note that the NTR is independent of the wealth level, but only depends on the wealth allocations. The NTR stems from the introduction of transaction costs, and is a connected set when the utility function is positively homogenous (Abrams and Karmarkar 1980). With proportional transaction costs, among others, the NTR has been found to be a convex set for static models (Dybvig and Pezzo 2020), and the same has been verified for dynamic models with proportional costs (Cai, Judd and Xu 2013).

The shape of the NTR is dependent on the parameters of the model, and can be a complex shape. With only proportional transaction costs and independent risky asset returns, the NTR for two risky assets is a square, as shown by Akian, Menaldi and Sulem (1996) among others. For one asset the NTR represent bounds on the relative allocation to the risky asset, as lines (Davis and Norman 1990).

The NTR may extend out of the feasible region, in which case the shape is truncated to the feasible space given the restrictions I set. For most cases seen in the literature, when consumption is not included, the NTR encapsulates the Merton point. Figure 1.1 illustrates an example of a NTR with two risky assets. The NTR is a key component of the model, and illustrates the trade-off between the utility of rebalancing the portfolio and the cost of doing so. I will go into further detail on the NTR when covering the

Figure 1.1: Example No Trade Region with two risky assets.



solution algorithm, where I will discuss the formal properties of the NTR, and how I can leverage key findings from the NTR to solve the dynamic portfolio choice problem with transaction costs efficiently.

1.8 Base problem: Portfolio choice with proportional costs and consumption

Considering the class of problems constructed in the prior section, I can now quickly introduce the basic problem formulation. I consider an investor with CRRA utility function. The investor can invest in one risk free asset and D risky assets. Trading is subject to proportional transaction costs hence I have the following cost function (in total trading volume):

$$\psi(\boldsymbol{\delta}_{i,t}^{+}, \boldsymbol{\delta}_{i,t}^{-}) = \tau(\boldsymbol{\delta}_{i,t}^{+} + \boldsymbol{\delta}_{i,t}^{-}) \tag{28}$$

I do not assume that returns are dependent on stochastic parameters, but instead that they are drawn from a distribution with known parameters. Hence, I assume $\theta_t = \theta$ for all t. I assume a constant return on the risk free asset, i.e $R_f(\theta_t) = R_f$, and the risky assets follow a multivariate log-normal distribution, with some mean and covariance matrix. I can now formulate the entire problem given the class structure from section 1.6. The terminal value function is given by equation (23). The system is subject to the constraints of equations (24), (25) and (26), as well as a simple constrain on consumption, $c_t \geq 0$. I assume that the position in bond holdings is the residual wealth, and they therefore follow (19). The Bellman equation is therefore:

$$v_t(\mathbf{x}_t, \theta_t) = \max_{c_t, \boldsymbol{\delta}_t^{+}, \boldsymbol{\delta}_t^{-}}, \{u(c_t)\Delta t + \beta \mathbb{E}_t \left[\pi_{t+\Delta t}^{1-\gamma} v_{t+\Delta t}(\mathbf{x}_{t+\Delta t})\right]\}, \quad t < T$$

With the same terminal condition as before, where investments are sold and wealth is consumed:

$$v_T(\mathbf{x}_T) = u(1 - \psi(\mathbf{0}, \mathbf{x}_T))$$

1.9 Portfolio choice with fixed costs

I now consider the model, where the investor faces fixed costs when rebalancing the portfolio, instead of proportional costs. Fixed costs are common in practice, and can be seen as a fixed fee for trading, regardless of the traded amount. I consider a slight modification to the purely fixed costs, and instead consider fixed costs as a percentage of the wealth. I do this to be able to use the same model structure as in section 1.8, where variables are in fractions of wealth, in order to drop wealth as a state variable.

This is seen previously in (Morton and Pliska 1995), who note that such a fixed cost can be seen as a portfolio management fee. In practice, when setting the level of the fixed cost, I make an implicit assumption on the wealth of the investor, if I want to draw comparisons with pure fixed costs.

The cost function is then given by:

$$\psi(\boldsymbol{\delta}_t^+, \boldsymbol{\delta}_t^-) = \mathbf{1}\left(\sum_{i=1}^{k} \delta_{i,t}^+ + \delta_{i,t}^- > 0\right) \cdot \mathrm{fc} \tag{29}$$

Where fc is the fixed cost, and $\mathbf{1}(\cdot)$ is the indicator function. The fixed cost is only incurred if the investor rebalances the portfolio, and the cost is independent of the traded amount. The normalized bond holdings are therefore given by:

$$b_t = 1 - \mathbf{1}^\top \cdot (\mathbf{x_t} - \boldsymbol{\delta}_t) - \psi(\boldsymbol{\delta}_t^+, \boldsymbol{\delta}_t^-) - c_t \Delta t \tag{30}$$

The model otherwise remains the same as in section 1.8, with the same constraints and dynamics, while using the new cost function. Note, that in the terminal period, when all investments are sold, the fixed cost is incurred, unless the investor holds no risky assets. Note also, that the fixed cost function is not differentiable. Furthermore (Dybvig and Pezzo 2020) notes that the fixed cost problem is not a convex optimization problem, and is therefore not as easily solved as the proportional cost problem. I will deal with these issues individually when implementing the model.

1.10 Portfolio choice with fixed and proportional costs

The last model I consider is a combination of the two previous models, where the investor faces both fixed and proportional costs. This is a more realistic model, as it combines the two most common types of transaction costs an individual investor face on the financial markets. These could for example be a fixed brokerage fee and a percentage of the traded amount stemming from bid ask spreads, taxes or commisions (Lesmond, Ogden and Trzcinka 1999). The cost function is then given by:

$$\psi(\boldsymbol{\delta}_t^+, \boldsymbol{\delta}_t^-) = \mathbf{1}\left(\sum_{i=1}^{k} \delta_{i,t}^+ + \delta_{i,t}^- > 0\right) \cdot \mathrm{fc} + \tau(\boldsymbol{\delta}_t^+ + \boldsymbol{\delta}_t^-) \tag{31}$$

The normalized bond holdings are therefore given by:

$$b_t = 1 - \mathbf{1}^\top \cdot (\mathbf{x_t} - \boldsymbol{\delta}_t -) - \psi(\boldsymbol{\delta}_t^+, \boldsymbol{\delta}_t^-) - c_t \Delta t \tag{32}$$

The model otherwise remains the same as in section 1.8, with the same constraints and dynamics, while using the new cost function.

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