1 Literature review

The purpose of this section is to review relevant literature to help understand the contributions made in this thesis. This review covers modern portfolio theory (MPT), from its foundations and into the 21st century.

Modern theory on portfolio choice can be traced back to the mean-variance framework of Harry Markowitz, who constructed and solved the now well established, static and single period, portfolio optimization problem, Markowitz (1952). This covers the mean-variance framework which is the foundation of MPT, suggesting investors should allocate wealth in order to maximize expected return, while minimizing exposure to risk. Following this, the mean-variance framework has since been extended to a continious time setting, most notably by Robert Merton, who introduced a solution to the intertemporal portfolio choice problem in frictionless markets, Merton (1969), and later adding consumption rules aswell Merton (1971). This solution is known as the Merton point in the asset allocation space, or the Merton portfolio. Mertons closed form solution suggests optimal asset allocations based on the asset return dynamics (mean-variance), and the risk aversion of the investor (preferences). Hence in a continious time setting, the optimal allocation changes if the asset dynamics change.

Multiple extensions have been made to the classical dynamic portfolio choice problem, such as the introduction of transaction costs, adding realistic constraints to the problem, since trading assets incurs costs in the real world, and markets are not frictionless. Zabel (1973) adresses transaction costs with CRRA preferences, but is limited to a discrete time setting, a single risky asset and a small horizon.

Constantinides (1976) and Constantinides (1986) returns to the continious time setting, and find that for multiple preference types, under proportional transaction costs. The investors decision then depends on the the remaining life span, wealth and current allocation. Trading costs create a no-trade-region (NTR), where the optimal reallocation decision for portfolios inside this, is do to nothing, and for portfolios outside this region, the optimal decision is to trade towards the boundary of the NTR. This is a shift from Mertons framework, where constant trading toward the Merton allocation, which is the optimal allocation in the absence of transaction costs, is optimal. Hence transaction costs restrain investors from acting optimally in the classical sense.

Numerical examples only cover the case of one risky asset, with restrictions on the decision space, and results remain qualitative or approximate. Notably Davis and Norman (1990) derive explicit solutions for the case of a single risky asset. They similarly find that proportional transaction costs lead to a NTR around the Merton point, and provide a solution algorithm for the stochastic control problem. This has later been made more rigorous such as Akian, Menaldi and Sulem (1996) who use a Hamilton-Jacobi-Bellman equation in the N-dimensional asset space, and provide further insight to the properties of the NTR, however the problem is only solved for the case of $k = 2$ risky assets with one risk free asset. Further analysis of this has been conducted extensively, e.g see Shreve and Soner (1994), Oksendal and Sulem (2002), Janeček and Shreve (2004), however the asset space is still constrained or solutions remain asymptotic. Muthuraman and Kumar (2006) and Muthuraman and Kumar (2008) tackle a $D = 3$ risky asset space, and provide a numerical solution to the problem, using a finites differences.

The paper by Cai, Judd and Xu (2013), which is central to this thesis, consider a more general setting, with multiple risky assets and a risk-free asset, and provides a solution algorithm, based on dynamic programming, numerical integration and polynomial approximation, to solve the dynamic problem for up to $k = 6$ risky assets and thus $D = 7$ assets in total, and later introduce and solve the problem with novelties, such as stochastic asset parameters or an option on an underlying asset in the portfolio Cai, Judd and Xu (2020). The curse of dimensionality, which haunts the prior methods applied, is somewhat tackled by the use of adaptive sparse grid methods, and sparse quadratuture rules by Schober, Valentin and Pflüger (2022).

Gaegauf, Scheidegger and Trojani (2023) further reduces the computational burden by using a Gaussian process regression to approximate value functions, and a problem specific point sampling strategy to reduce the number of points in the state space needed to characterize the NTR. Increasing the dimensions of the asset space does still increase the dimensionality of the problem, and the computational burden, however this is at a much lower extent than previous methods.

Beyond the analysis conducted by the authors above, several related avenues of reseatch have been conducted on the dynamic portfolio choice problemn. Garleanu and Pedersen (2013) remains an influential paper, which aims to derive optimal closed form portfolio policy, when returns are driven by signals with mean reversion. This provides an insightful analysis of how to trade towards the optimal portfolio, given quadratic transaction costs, within a set scope of serially correlated assets. Dybvig and Pezzo (2020) provides a comprehensive overview on the usage of different transaction cost functions, hedging with futures and security specific costs. Dybvig find that by changing the transaction cost function, the properties of the NTR is altered.[1]

References

Akian, Marianne, Menaldi, José Luis and Sulem, Agnès (1996). “On an Investment-Consumption Model with Transaction Costs”. In: *SIAM Journal on Control and Optimization* Vol. 34, No. 1, pp. 329–364. DOI: 10.1137/S0363012993247159.

[1]Gaegauf, Scheidegger and Trojani (2023), also note that their framework is applicable to different transaction cost functions.

Cai, Yongyang, Judd, Kenneth L. and Xu, Rong (2013). *Numerical Solution of Dynamic Portfolio Optimization with Transaction Costs*. Working Paper 18709. National Bureau of Economic Research. DOI: 10.3386/w18709.

Cai, Yongyang, Judd, Kenneth L. and Xu, Rong (2020). *Numerical Solution of Dynamic Portfolio Optimization with Transaction Costs*. Tech. rep. arXiv: 2003.01809 [q-fin.PM]. URL: https://arxiv.org/abs/2003.01809.

Constantinides, George M. (1976). "Note—Optimal Portfolio Revision with Proportional Transaction Costs: Extension to Hara Utility Functions and Exogenous Deterministic Income". In: *Management Science* Vol. 22, No. 8, pp. 921–923.

Constantinides, George M. (1986). "Capital Market Equilibrium with Transaction Costs". In: *Journal of Political Economy* Vol. 94, No. 4, pp. 842–862. ISSN: 00223808, 1537534X. URL: http://www.jstor.org/stable/1833205.

Davis, M. H. A. and Norman, A. R. (1990). "Portfolio Selection with Transaction Costs". In: *Mathematics of Operations Research* Vol. 15, No. 4, pp. 676–713. DOI: 10.1287/moor.15.4.676.

Dybvig, Philip H and Pezzo, Luca (2020). "Mean-variance portfolio rebalancing with transaction costs". In: *Available at SSRN 3373329*.

Gaegauf, Luca, Scheidegger, Simon and Trojani, Fabio (2023). *A Comprehensive Machine Learning Framework for Dynamic Portfolio Choice With Transaction Costs*. Tech. rep. 23-114. Swiss Finance Institute, p. 70. URL: https://ssrn.com/abstract=4543794.

Garleanu, Nicolae and Pedersen, Lasse Heje (2013). "Dynamic Trading with Predictable Returns and Transaction Costs". In: *The Journal of Finance* Vol. 68, No. 6, pp. 2309–2340.

Janeček, Karel and Shreve, Steven E. (2004). "Asymptotic analysis for optimal investment and consumption with transaction costs". In: *Finance and Stochastics* Vol. 8, No. 2, pp. 181–206. DOI: 10.1007/s00780-003-0113-4.

Markowitz, Harry (1952). "Portfolio Selection". In: *The Journal of Finance* Vol. 7, No. 1, pp. 77–91. ISSN: 00221082, 15406261. URL: http://www.jstor.org/stable/2975974.

Merton, Robert C. (1969). "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case". In: *The Review of Economics and Statistics*, pp. 247–257.

Merton, Robert C. (1971). "Optimum Consumption and Portfolio Rules in a Continuous-Time Model". In: *Journal of Economic Theory* Vol. 3, No. 4, pp. 373–413. DOI: 10.1016/0022-0531(71)90038-X.

Muthuraman, Kumar and Kumar, Sunil (2006). "Multidimensional Portfolio Optimization with Proportional Transaction Costs". In: *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics* Vol. 16, No. 2, pp. 301–335.

Muthuraman, Kumar and Kumar, Sunil (2008). “Solving Free-boundary Problems with Applications in Finance”. In: *Foundations and Trends® in Stochastic Systems* Vol. 1, No. 4, pp. 259–341. DOI: 10.1561/0900000006.

Oksendal, Bernt and Sulem, Agnès (2002). “Optimal Consumption and Portfolio with Both Fixed and Proportional Transaction Costs”. In: *SIAM Journal on Control and Optimization* Vol. 40, No. 6, pp. 1765–1790. DOI: 10.1137/S0363012900376013.

Schober, Peter, Valentin, Johannes and Pflüger, Dirk (2022). “Solving High-Dimensional Dynamic Portfolio Choice Models with Hierarchical B-Splines on Sparse Grids”. In: *Computational Economics* Vol. 59, No. 1, pp. 185–224. DOI: 10.1007/s10614-021-10118-9.

Shreve, S. E. and Soner, H. M. (1994). “Optimal Investment and Consumption with Transaction Costs”. In: *The Annals of Applied Probability* Vol. 4, No. 3, pp. 609–692. DOI: 10.1214/aoap/1177004966.

Zabel, Edward (1973). “Consumer Choice, Portfolio Decisions, and Transaction Costs”. In: *Econometrica* Vol. 41, No. 2, pp. 321–335. URL: http://www.jstor.org/stable/1913492.