1 Economic theory

This section covers the basics of modern portfolio theory and components of the dynamic portfolio choice problem with transaction costs. This section leans heavily on Cai, Judd and Xu (2020) and Gaegauf, Scheidegger and Trojani (2023), bridging the model from the former, with the framework of the latter.

1.1 Intertemporal portfolio choice without transaction costs

We first consider the classic portfolio optimization problem without transaction costs, as formulated by Merton (1969) and Merton (1971). For a more detailed treatment, see Björk (2019). In this setting, an investor dynamically allocates wealth between k risky assets and a risk-free asset to maximize utility over a finite horizon $[0, T]$.

The investor's wealth W_t can be allocated between a risk-free asset and k risky assets. Consumption is a non-durable good that can be purchased at each time point t. r is the risk-free rate, $\boldsymbol{\mu}$ is the vector of expected returns on the risky assets, and C_t represents consumption at time t. The investor's preferences follow a constant relative risk aversion (CRRA) utility function.

Without transaction costs, the optimal portfolio allocation, known as the Merton point is:

$$\mathbf{x}_t^* = \frac{1}{\gamma}\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r), \tag{1}$$

where γ is the coefficient of relative risk aversion, and $\mathbf{\Sigma}$ is the covariance matrix of the risky assets' returns. This provides a time-independent optimal allocation that serves as a benchmark for models incorporating frictions such as transaction costs.

1.2 Investor preferences and problem

The investor operates over a finite horizon of T years, during which she aims to maximize her expected utility. Following Cai, Judd and Xu (2013), the investment horizon is discretized into N equally spaced periods, each with a duration of $\Delta t = \frac{T}{N}$. At each time point t_j, for $j = 0, 1, \ldots, N$, where $t_0 = 0$ and $t_N = T$, the investor has the opportunity to adjust her portfolio allocation right before $t_j + \Delta t$. Reallocation is costly, and the investor is subject to proportional transaction costs. She may also choose to consume a non-durable good at each time point.

For notational simplicity, we now use t to denote these time points unless specifically referring to t_j. The investor's preferences are modeled using a constant relative risk aversion (CRRA) utility function:

$$u(C_t) = \begin{cases} \frac{C_t^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1, \\ \log(C_t) & \text{if } \gamma = 1, \end{cases} \tag{2}$$

where C_t is consumption and c_t is the fraction of wealth W_t spent on consumption at time t. Hence $c_t = C_t/W_t$, and lowercase notation is henceforth used to denote variables as fractions of wealth. γ is the coefficient of relative risk aversion. The objective is to maximize the expected utility of consumption and wealth over the investor's lifetime:

$$\max_{\mathbf{x}_t, b_t, c_t} \mathbb{E}\left[\sum_{i=0}^{N-1} \beta^i u(C_i)\Delta t + \beta^N u(W_N)\right], \tag{3}$$

where β is the discount factor, $\mathbf{x}_t$ is the allocation to risky assets, b_t is the allocation to the risk-free asset, and W_t is the investor's wealth at time t.

1.3 Asset and goods market

We consider a financial market with k risky assets and one risk-free asset, making the asset space $D = 1 + k$ dimensions. The risk-free asset, such as a bond or a bank deposit, yields a constant gross return $R_f = \mathrm{e}^{r\Delta t}$, where r is the annual interest rate and $\Delta t = \frac{T}{N}$ is the length of one investment period.

The k risky assets can be considered as listed stocks, subject to proportional transaction costs. For each reallocation of wealth in a risky asset, a transaction cost of $\tau \in [0, 1]$ is incurred as a percentage of the traded amount. The stochastic one-period gross-return vector of the risky assets is denoted as $\mathbf{R} = (R_1, R_2, \ldots, R_k)^\top$, and the corresponding net-return vector is $\mathbf{r} = (r_1, r_2, \ldots, r_k)^\top$.

In the goods market, there is a single non-durable consumption good, C, which is consumed at each time point t. The fraction of wealth allocated to consumption at time t is denoted c_t, the fraction allocated to risky assets is $\mathbf{x}_t = (x_{1,t}, x_{2,t}, \ldots, x_{k,t})^\top$, and the fraction allocated to the risk-free asset is denoted b_t. Thus, $\mathbf{x}_t \in \mathbb{R}^k$ and $b_t \in \mathbb{R}$.

1.4 Transaction costs and portfolio reallocation

Rebalancing incurs proportional transaction costs $\tau \in [0, 1]$, which are paid based on the amount bought or sold of each risky asset. Reallocation decisions are made just before $t_j + \Delta t$, such that $\mathbf{x}_t$ is the portfolio of risky assets right before realllocation. $\delta_{i,t}$ denotes the change in portfolio allocation of asset i, and $\delta_{i,t} W_t$ is thus the currency amount traded in asset i. Hence $\delta_{i,t} > 0$ implies buying asset i, and $\delta_{i,t} < 0$ implies selling asset i. Proportional transaction costs imply that the cost function associated with rebalancing is:

$$\psi(\delta_{i,t} W_t) = \tau |\delta_{i,t} W_t| \tag{4}$$

We decompose the decision variable $\delta_{i,t}$, representing the fraction of wealth used to trade risky asset i, into buying ($\delta^+_{i,t}$) and selling ($\delta^-_{i,t}$) components to ensure tractability[1]:

$$\delta_{i,t} = \delta^+_{i,t} - \delta^-_{i,t}, \quad \delta^+_{i,t}, \delta^-_{i,t} \geq 0.$$

The total transaction cost is then given by $\tau \sum_{i=1}^{k} (\delta^+_{i,t} + \delta^-_{i,t}) W_t$. And the transaction cost function is therefore a function of each trading direction:

$$\psi(\delta^+_{i,t}, \delta^-_{i,t}, W_t) = \tau(\delta^+_{i,t} + \delta^-_{i,t}) W_t \tag{5}$$

Following the reallocation, the remaining wealth is allocated between the risk-free asset and consumption. Notation of rebalancing is henceforth simplified using vectors to $\boldsymbol{\delta}_t = \boldsymbol{\delta}^+_t - \boldsymbol{\delta}^-_t$ with $\boldsymbol{\delta}^+_t = (\delta^+_{1,t}, \delta^+_{2,t}, \ldots, \delta^+_{k,t})$. We have that $\boldsymbol{\delta}_t$ is the *net change* in the risky positions, and $\boldsymbol{\delta}^+_t + \boldsymbol{\delta}^-_t$ is the *cumulative change* in the risky positions.

1.5 Asset dynamics

We follow Cai, Judd and Xu (2013) for the asset dynamics. The total composition of risky assets is assumed to follow a multivariate log-normal distribution:

$$\log(\mathbf{R}) \sim \mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t, (\boldsymbol{\Lambda}\boldsymbol{\Sigma}\boldsymbol{\Lambda})\,\Delta t\right), \tag{6}$$

where μ is the drift vector, σ^2 is a column vector of the variance σ_i^2, $\boldsymbol{\Sigma}$ is the correlation matrix, and $\boldsymbol{\Lambda} = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k)$ is the diagonal matrix of volatilities. Following Cai, Judd and Xu (2013) we utilize the Cholesky decomposition of the correlation matrix, $\boldsymbol{\Sigma} = \mathbf{L}\mathbf{L}^\top$, where $\mathbf{L} = (L_{i,j})_{k\times k}$ is a lower triangular matrix. Hence, for each risky asset i, the log-return is:

$$\log(R_i) = \left(\mu_i - \frac{\sigma_i^2}{2}\right)\Delta t + \sigma_i\sqrt{\Delta t}\sum_{j=1}^{i} L_{i,j} z_j, \tag{7}$$

where z_i are independent standard normal random variables.

1.6 No Trade Region

The no-trade-region (NTR) is a region in the asset space (risky and risk-less) where it is sub-optimal to rebalance the portfolio. Given the parameters of the model the NTR

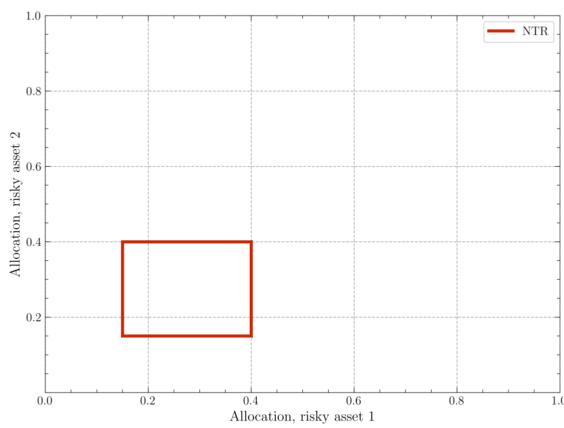
[1]Gaegauf, Scheidegger and Trojani (2023) note that this ensures differentiability. This approach is common and found in earlier work such as Akian, Menaldi and Sulem (1996), who likewise note that this ensures that the variable is continious from origin in the positive real set.

without consumption is defined as:

$$\Omega_t = \{\mathbf{x}_t : \boldsymbol{\delta}_t^+, \boldsymbol{\delta}_t^- = \mathbf{0}\} \tag{8}$$

We note that the NTR is independent of the wealth level, but only depends on the wealth allocations. The NTR stems from the transaction costs, and is a connected set. When consumption is introduced in the model **SE ANGÅENDE CONVEX HULL**: Kamin (1975), Constantinides (1976, 1979, 1986), Davis and Norman (1990), and Muthuraman and Kumar (2006). Figure 1.1 illustrates an example of a NTR with two risky assets. The square shape of the NTR occurs with proportional transaction costs and independent

Figure 1.1: Example No Trade Region with $k = 2$ risky assets.



risky assets. However the NTR is not always a square, for more on this see Dybvig and Pezzo (2020).

References

Akian, Marianne, Menaldi, José Luis and Sulem, Agnès (1996). "On an Investment-Consumption Model with Transaction Costs". In: *SIAM Journal on Control and Optimization* Vol. 34, No. 1, pp. 329–364. DOI: 10.1137/S0363012993247159.

Björk, Tomas (2019). *Arbitrage theory in continuous time.* 4. ed. Oxford Univ. Press. ISBN: 9780198851615. URL: http://gso.gbv.de/DB=2.1/CMD?ACT=SRCHA&SRT=YOP&IKT=1016&TRM=ppn+505893878&sourceid=fbw_bibsonomy.

Cai, Yongyang, Judd, Kenneth L. and Xu, Rong (2013). *Numerical Solution of Dynamic Portfolio Optimization with Transaction Costs.* Working Paper 18709. National Bureau of Economic Research. DOI: 10.3386/w18709.

Cai, Yongyang, Judd, Kenneth L. and Xu, Rong (2020). *Numerical Solution of Dynamic Portfolio Optimization with Transaction Costs.* Tech. rep. arXiv: 2003.01809 [q-fin.PM]. URL: https://arxiv.org/abs/2003.01809.

Dybvig, Philip H and Pezzo, Luca (2020). "Mean-variance portfolio rebalancing with transaction costs". In: *Available at SSRN 3373329*.

Gaegauf, Luca, Scheidegger, Simon and Trojani, Fabio (2023). *A Comprehensive Machine Learning Framework for Dynamic Portfolio Choice With Transaction Costs*. Tech. rep. 23-114. Swiss Finance Institute, p. 70. URL: https://ssrn.com/abstract=4543794.

Merton, Robert C. (1969). "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case". In: *The Review of Economics and Statistics*, pp. 247–257.

Merton, Robert C. (1971). "Optimum Consumption and Portfolio Rules in a Continuous-Time Model". In: *Journal of Economic Theory* Vol. 3, No. 4, pp. 373–413. DOI: 10.1016/0022-0531(71)90038-X.