1 Literature review

The purpose of this section is to review relevant literature to help understand the contributions made in this thesis, and their relation to the existing literature. This review covers dynamic portfolio choice problems, the introduction of transaction costs, and most notable contributions to the field.

Modern theory on portfolio choice can be traced back to the mean-variance framework of Harry Markowitz, who constructed and solved the static and single period, portfolio optimization problem, (Markowitz 1952).

This covers the mean-variance framework which is the foundation of Modern Portfolio Theory (MPT), suggesting investors should allocate wealth in order to maximize expected return, while minimizing exposure to risk. Following this, the mean-variance framework has since been extended, most notably by Robert Merton, who introduced a solution to the intertemporal portfolio choice problem in frictionless markets, (Merton 1969). This solution is known as the Merton point in the asset allocation space or the Merton portfolio. Merton's closed form solution suggests optimal asset allocations based on the asset return dynamics (mean-variance), and the risk aversion of the investor (preferences). Later, an optimal consumption rule was found aswell (Merton 1971).

Multiple extensions have been made to the dynamic portfolio choice problem, such as the introduction of transaction costs, adding realistic frictions to the problem, since trading assets incurs costs in the real world, and markets are not frictionless. Most of the literature find that transaction costs create a region in the asset space, where it is sub-optimal to trade, known as the No-Trade Region (NTR).

The literature on proportional costs in the dynamic portfolio choice problem is vast, whereas the fixed costs problem is less explored. (Morton and Pliska 1995) analyse the problem with a fixed cost, relative to the investors wealth, and solve the problem numerically for two correlated risky assets. They find a NTR which is similar to an ellipse but with vertices. They conclude that the NTR is an ellipse. (Liu 2004) solves the problem for uncorrelated assets with proportional and fixed costs and consumption. With fixed costs, No-Trade bounds are found for one risky asset in the shape of a conic. Results differ from (Morton and Pliska 1995), as the fixed cost NTR is not an ellipse but has corners. They conjecture this to be the case for correlated assets as well but skewed. For proportional and fixed costs, multiple target portfolios are found inside the NTR, one for each corner, and the shape of the NTR is square. (Altarovici, Muhle-Karbe and Soner 2015) solve the dynamic problem for two uncorrelated risky assets with fixed costs. They find that the NTR is a slightly angled ellipsoid, using a differential equation approach. (Dybvig and Pezzo 2020) provide a comprehensive review of different transaction costs, and the implications of these on the optimal portfolio choice problem, however the setting is static. They find that the NTR from fixed costs with no correlation is circular, similar to the results of (Morton and Pliska 1995). From this, the excact shape of the fixed cost NTR is not entirely clear. Most find an ellipsoid, but the skewness, connection to the correlation of the asset returns, and whether the NTR has corners or not, is not entirely clear. Furthermore, solutions in the literature are limited to two risky assets, and the solution methods for the dynamic setting has not followed the same advances as the proportional costs problem.

(Zabel 1973; Constantinides 1976; Constantinides 1986) find that for multiple preference types, under proportional transaction costs, the investors decision depends on the the remaining life span, wealth and current allocation. Transaction costs entail an NTR, where the optimal reallocation decision for portfolios inside, is to do nothing, and for portfolios outside this region, the optimal decision is to trade towards the boundary of the NTR. This is a shift from Mertons framework, where constant trading toward the Merton allocation, which is the optimal allocation in the absence of transaction costs, is optimal. Thus, transaction costs restrain investors from acting optimally in the classical sense. Numerical examples only cover the case of one risky asset with restrictions on the decision space and results remain qualitative or approximate.

Notably, (Davis and Norman 1990) derive explicit solutions for the case of a single risky asset. They similarly find that proportional transaction costs lead to a NTR around the Merton point and provide a solution algorithm for the stochastic control problem.

(Akian, Menaldi and Sulem 1996) use a Bellman equation in the N-dimensional asset space, and provide further insight to the properties of the NTR, however the problem is only solved for the case of two risky assets with one risk free asset. Further analysis of this has been conducted extensively, e.g see (Shreve and Soner 1994; Oksendal and Sulem 2002; Janeček and Shreve 2004), however the asset space is still constrained or solutions remain asymptotic. (Muthuraman and Kumar 2006; Muthuraman and Kumar 2008) tackle a three risky asset space and provide a numerical solution to the problem, using a finites differences method.

The paper by (Cai, Judd and Xu 2013), which is central to this thesis, considers a more general setting with multiple risky assets and a risk-free asset and provide a solution algorithm based on Dynamic Programming (DP), numerical integration and polynomial approximation. They solve the dynamic problem for up to six risky assets, and later introduce and solve the problem with novelties, such as stochastic asset parameters or an option on a underlying asset in the portfolio (Cai, Judd and Xu 2020). This work only considers proportional transaction costs and relies on a super computer to solve the problem.

The curse of dimensionality, which haunts the prior methods applied, is somewhat tackled by the use of adaptive sparse grid methods and sparse quadratuture rules by (Schober, Valentin and Pflüger 2022). However, results require the use of super computers. (Gaegauf, Scheidegger and Trojani 2023) further reduce the computational burden by using a Gaussian process regression (GPR), to approximate value functions and provide a problem specific point-sampling strategy to reduce the number of points in the state space needed to approximate the NTR. Increasing the dimensions of the asset space does still increase the dimensionality of the problem and the computational burden, however this is at a much lower extent than previous methods. This is the most recent computational advance currently in the field, and is basis for the computational framework in this thesis.

References

Akian, Marianne, Menaldi, José Luis and Sulem, Agnès (1996). "On an Investment-Consumption Model with Transaction Costs". In: *SIAM Journal on Control and Optimization* Vol. 34, No. 1, pp. 329–364. DOI: 10.1137/S0363012993247159.

Altarovici, Albert, Muhle-Karbe, Johannes and Soner, Halil Mete (2015). "Asymptotics for fixed transaction costs". In: *Finance and Stochastics* Vol. 19, No. 2, pp. 363–414.

Cai, Yongyang, Judd, Kenneth L. and Xu, Rong (2013). *Numerical Solution of Dynamic Portfolio Optimization with Transaction Costs*. Working Paper 18709. National Bureau of Economic Research. DOI: 10.3386/w18709.

Cai, Yongyang, Judd, Kenneth L. and Xu, Rong (2020). *Numerical Solution of Dynamic Portfolio Optimization with Transaction Costs*. Tech. rep. arXiv: 2003.01809 [q-fin.PM].

Constantinides, George M. (1976). "Note—Optimal Portfolio Revision with Proportional Transaction Costs: Extension to Hara Utility Functions and Exogenous Deterministic Income". In: *Management Science* Vol. 22, No. 8, pp. 921–923.

Constantinides, George M. (1986). "Capital Market Equilibrium with Transaction Costs". In: *Journal of Political Economy* Vol. 94, No. 4, pp. 842–862. ISSN: 00223808, 1537534X.

Davis, M. H. A. and Norman, A. R. (1990). "Portfolio Selection with Transaction Costs". In: *Mathematics of Operations Research* Vol. 15, No. 4, pp. 676–713. DOI: 10.1287/moor.15.4.676.

Dybvig, Philip H and Pezzo, Luca (2020). "Mean-variance portfolio rebalancing with transaction costs". In: *Available at SSRN 3373329*.

Gaegauf, Luca, Scheidegger, Simon and Trojani, Fabio (2023). *A Comprehensive Machine Learning Framework for Dynamic Portfolio Choice With Transaction Costs*. Tech. rep. 23-114. Swiss Finance Institute, p. 70.

Janeček, Karel and Shreve, Steven E. (2004). "Asymptotic analysis for optimal investment and consumption with transaction costs". In: *Finance and Stochastics* Vol. 8, No. 2, pp. 181–206. DOI: 10.1007/s00780-003-0113-4.

Liu, Hong (2004). "Optimal Consumption and Investment with Transaction Costs and Multiple Risky Assets". In: *The Journal of Finance* Vol. 59, No. 1, pp. 289–338. ISSN: 00221082, 15406261.

Markowitz, Harry (1952). "Portfolio Selection". In: *The Journal of Finance* Vol. 7, No. 1, pp. 77–91. ISSN: 00221082, 15406261.

Merton, Robert C. (1969). "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case". In: *The Review of Economics and Statistics*, pp. 247–257.

Merton, Robert C. (1971). "Optimum Consumption and Portfolio Rules in a Continuous-Time Model". In: *Journal of Economic Theory* Vol. 3, No. 4, pp. 373–413. DOI: 10.1016/0022-0531(71)90038-X.

Morton, Andrew J and Pliska, Stanley R (1995). "Optimal portfolio management with fixed transaction costs". In: *Mathematical Finance* Vol. 5, No. 4, pp. 337–356.

Muthuraman, Kumar and Kumar, Sunil (2006). "Multidimensional Portfolio Optimization with Proportional Transaction Costs". In: *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics* Vol. 16, No. 2, pp. 301–335.

Muthuraman, Kumar and Kumar, Sunil (2008). "Solving Free-boundary Problems with Applications in Finance". In: *Foundations and Trends® in Stochastic Systems* Vol. 1, No. 4, pp. 259–341. DOI: 10.1561/0900000006.

Oksendal, Bernt and Sulem, Agnès (2002). "Optimal Consumption and Portfolio with Both Fixed and Proportional Transaction Costs". In: *SIAM Journal on Control and Optimization* Vol. 40, No. 6, pp. 1765–1790. DOI: 10.1137/S0363012900376013.

Schober, Peter, Valentin, Johannes and Pflüger, Dirk (2022). "Solving High-Dimensional Dynamic Portfolio Choice Models with Hierarchical B-Splines on Sparse Grids". In: *Computational Economics* Vol. 59, No. 1, pp. 185–224. DOI: 10.1007/s10614-021-10118-9.

Shreve, S. E. and Soner, H. M. (1994). "Optimal Investment and Consumption with Transaction Costs". In: *The Annals of Applied Probability* Vol. 4, No. 3, pp. 609–692. DOI: 10.1214/aoap/1177004966.

Zabel, Edward (1973). "Consumer Choice, Portfolio Decisions, and Transaction Costs". In: *Econometrica* Vol. 41, No. 2, pp. 321–335.