

1 Results

For the following results we consider 3 types of parameterizations for the portfolio problem. The first is a simple case where the assets are identically distributed as seen in (Cai, Judd and Xu 2013), the second is a case where the parameters are chosen to match the parameters in (Schober, Valentin and Pflüger 2022) also seen in (Gaegauf, Scheidegger and Trojani 2023). This is in order to be able to draw correct comparisons between the results. Furthermore this case, displays assets with slight variation in the mean and a small correlation between the assets, and no asset is dominating the others. The last parameterization is a modification of the first case where the correlation between the assets is larger (correlation coefficient of 0.75).

Table 1: Parameters for Examples of Portfolio Problems

	i.i.d Assets	Schober Parameters	High Correlation
T	6	6	6
k	3	5	3
γ	3.0	3.5	3.0
τ	0.5%	0.5%	0.5%
β	0.97	0.97	0.97
r	3%	4%	3%
μ^\top	(0.07, 0.07)	μ_{Schober}	(0.07, 0.07)
Σ	$\begin{bmatrix} 0.04 & 0.00 \\ 0.00 & 0.04 \end{bmatrix}$	Σ_{Schober}	$\begin{bmatrix} 0.04 & 0.03 \\ 0.03 & 0.04 \end{bmatrix}$

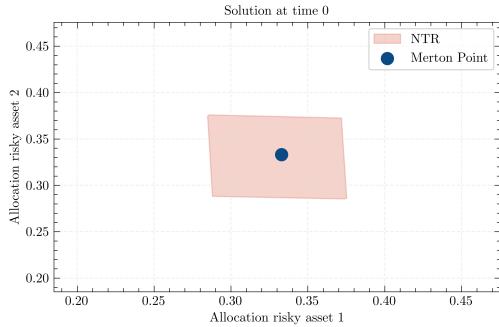
$$\mu_{\text{Schober}}^\top = [0.0572 \ 0.0638 \ 0.07 \ 0.0764 \ 0.0828]$$

$$\Sigma_{\text{Schober}} = \begin{bmatrix} 0.0256 & 0.00576 & 0.00288 & 0.00176 & 0.00096 \\ 0.00576 & 0.0324 & 0.0090432 & 0.010692 & 0.01296 \\ 0.00288 & 0.0090432 & 0.04 & 0.0132 & 0.0168 \\ 0.00176 & 0.010692 & 0.0132 & 0.0484 & 0.02112 \\ 0.00096 & 0.01296 & 0.0168 & 0.02112 & 0.0576 \end{bmatrix}$$

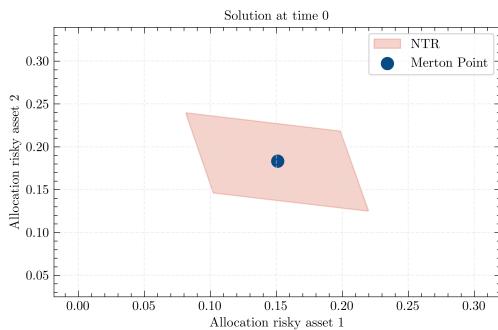
1.1 Dynamic Portfolio Choice without consumption

I first consider the base model with proportional transaction costs and no consumption. In the absence of consumption, the optimal portfolio is the merton point, which we plot in every figure. I plot the No-trade region at time point 0 (initial time point) for each of the parameterizations in figure 1.1. When using the Schober parameters we select the d first elements of the mean vector, and truncate the covariance matrix to a $d \times d$ matrix, depending on the number of assets d in the model. I note that for each of the

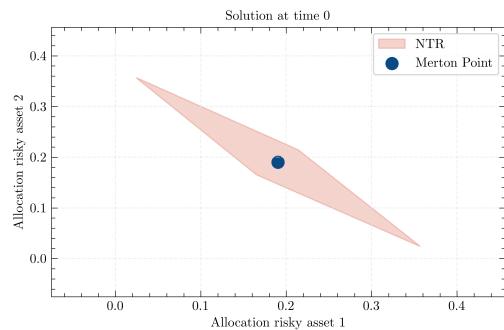
Figure 1.1: Comparison of No Trade Regions.



(a) No Trade Region for Independent Identically Distributed Assets.



(b) No Trade Region for Schober Parameters.



(c) No Trade Region for High Correlation.

parameterizations the No-Trade region is a rectangle or parallelogram. For the case of identical and independent assets, the No-Trade region is a perfect square, whereas for the Schober parameters and the high correlation case, the No-Trade region is a parallelogram. This is due to the correlation between the assets. When some correlation is present, the No-Trade region is skewed, since some allocations which would be optimal in the absence of correlation are no longer optimal.

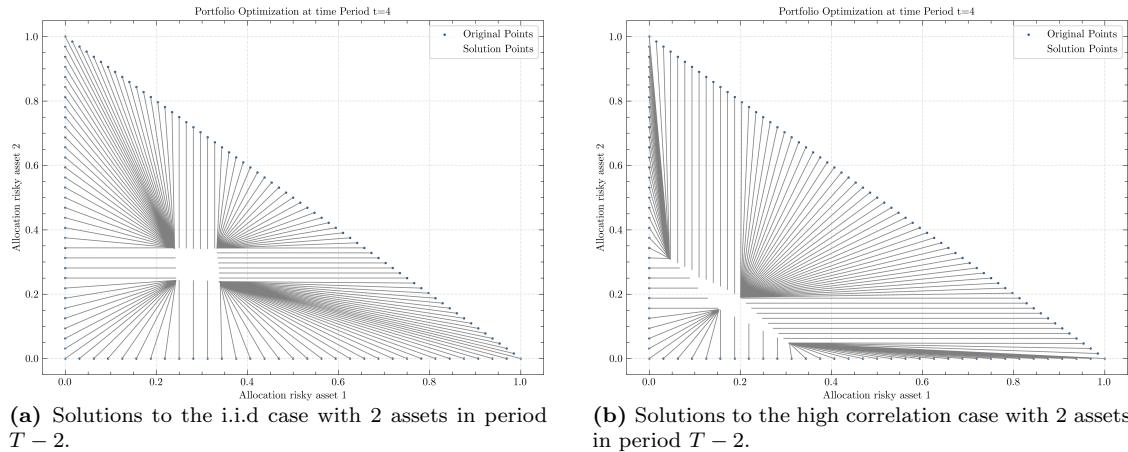
1.1.1 Verifying the geometric shape of the No-trade Region

Since much of the procedure for solving this problem, and approximating the NTR, leverages the a-priori assumptions regarding the geometric shape of the NTR, i first want to verify that the NTR indeed has 4 vertices for the 2d case and is a convex hull.

In order to do this, i do a small modification to the solution algorithm proposed earlier. Instead of computing vertices using 2^d predetermined points, i will instead sample a larger set of points, ($2^7 = 128$) covering the boundaries of the feasible state space. For each of these points i then solve the optimization problem, and plot the solution, from allocations \mathbf{x}_t and their solutions to the problem $\hat{\omega}_t$. I do this by using my original sample scheme, and adding mid-points between points, which either sum to 1.0 or have 0.0 as allocation for one of the assets. I consider the i.i.d case and the high correlation

case, with $\tau = 1\%$. I have increased the costs slightly in order to increase the size of the NTR. This is to ensure that points also converge towards the faces and not only the vertices. This is akin to the green regions in figure ???. Otherwise I would need more points. I plot the solutions for next to last period with investment decisions $T - 2$. The solutions are plotted below.

Figure 1.2: Verifying the assumptions of the NTR in 2 dimensions.



I find that the assumptions regarding the no-trade-region (NTR) are indeed correct in the two dimensional examples I have constructed. Furthermore this verifies that the assumptions also hold for correlated assets, which was only postulated by (Liu 2004). Furthermore these plots also nicely confirm that the optimization process as a whole works as intended. Further verification in higher dimensions are not considered. First of all (Liu 2004) confirms this formally in larger dimensions, for the case of uncorrelated assets, and the intuition regarding the NTR does not change when dimensionality is increased.

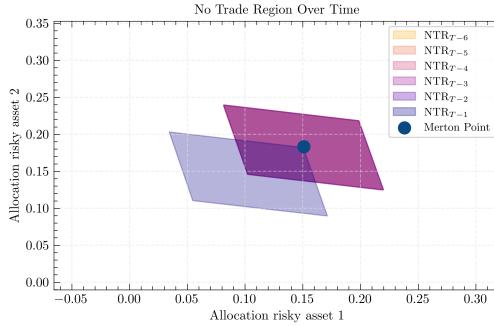
1.1.2 Investigating the No-Trade Region

We now look at the No-Trade region for the base model with proportional transaction costs and no consumption in more detail. Specifically we look at how the region behaves over the entire investment horizon $[0, T]$, and how the region changes with different transaction cost levels. We choose to look at the model with the Schober parameters, as this is a mixture of the other two parameterizations.

I note that at the last time point $t = T - 1$ the NTR moves away from the Merton point towards the origin, and the Merton point is now the upper right corner of the NTR. For all other time periods the NTR is the same, and the Merton point is in the center.

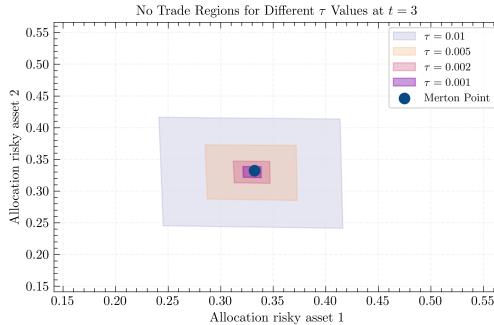
I now investigate how the No-Trade region changes with different transaction cost

Figure 1.3: No Trade Region for Schober Parameters over Time.



The No-Trade region is plotted for the Schober parameters over the entire investment horizon $[0, T - 1]$. For time points $t \in [0, T - 2]$ the NTRs overlap.

Figure 1.4: No Trade Region for the iid Parameters with different values of τ .



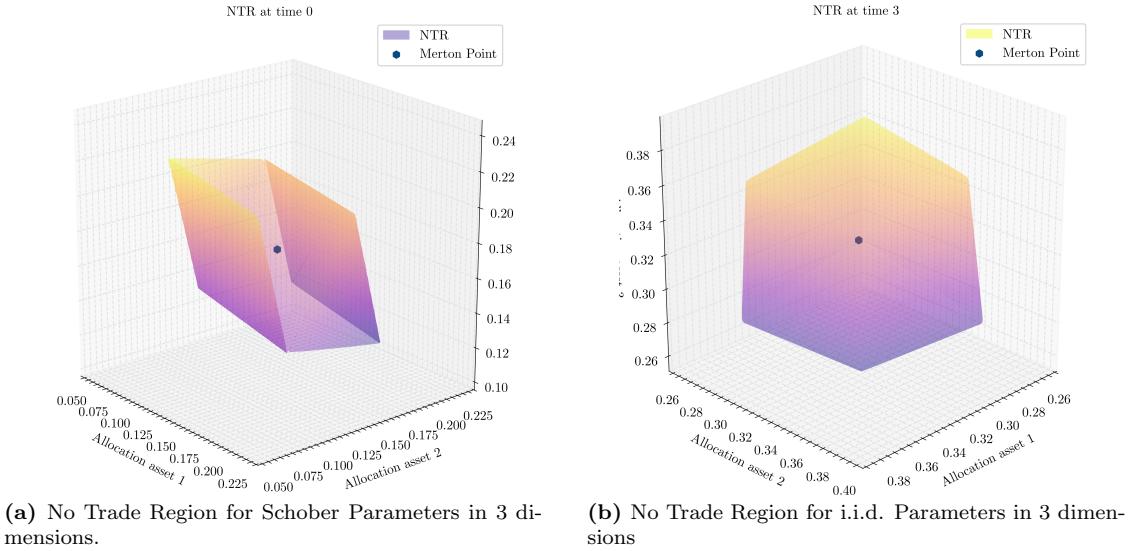
levels. I do this for the i.i.d. parameters, and plot the NTR for different values of τ in [Figure 1.4](#). When the transaction costs are increased, the NTR increases as well and vice-versa. I note that for low enough transaction costs, the NTR shrinks towards the Merton point. However when transaction costs are low enough, the Merton point is not in the exact center, which might signify that at low enough values, some numerical instabilities from the minimizer, and function approximation using Gaussian process regression (GPR) might be present.

1.1.3 Increasing the dimensionality of the model

We now increase the dimensionality of the model to $d = 3$ and look at the No-Trade region for the Schober parameters and for the i.i.d parameters.

Note that the i.i.d NTR looks like a skewed cube, whereas this was a perfect square in the 2 dimensional case. Looking that the points forming the convex hull that is the NTR, it is clear that the NTR is restricted by the no-borrowing constraint, since one of the border points, which would otherwise form the perfect cube, would be outside the feasible

Figure 1.5: Comparison of No Trade Regions.



space if this was possible, and is then projected into the feasible space. Hence when the risky returns outweigh the risk-free return, to such a degree that the merton point moves towards the boundary of the feasible space, cube like shapes are no longer possible. In the 2 dimension case, this is akin to the NTR being close to the budget line, and the NTR would then form a triangle.

This is clear when compared to the Schober parameters, where the merton point is in the center of the NTR, and the NTR is a skewed cube. The merton point in this case suggest lower portfolio allocations to the risky assets, and hence the NTR is not restricted by the no-trading and no-borrowing constraints.

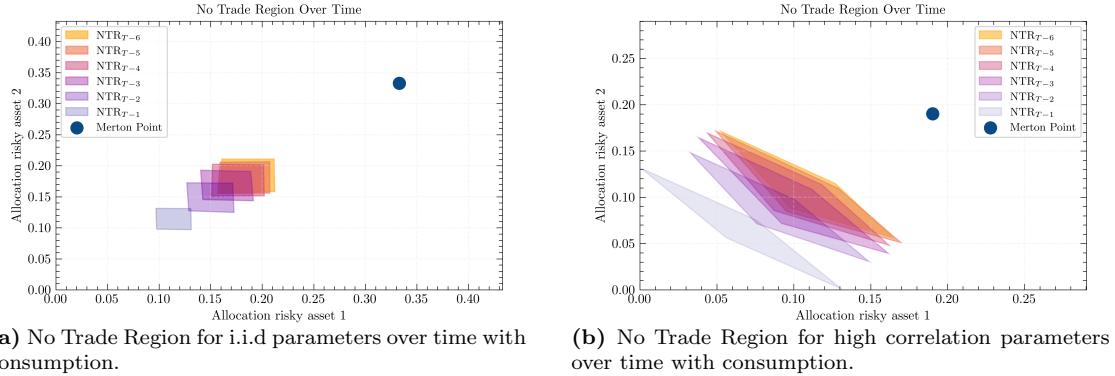
1.2 Dynamic Portfolio Choice with consumption

I now consider the base model with proportional transaction costs which now includes consumption of a non-durable good. This adds an extra decision variable which needs to be solved for, and consumption now adds immediate utility to the investor, in each period.

Note that when consumption is included, the NTR no longer encapsulates the Merton point at any time point. Furthermore the NTR now moves over time, towards the origin, as opposed to the case without consumption, where the NTR was static for all time points except the next to last period (last period with trading decisions).

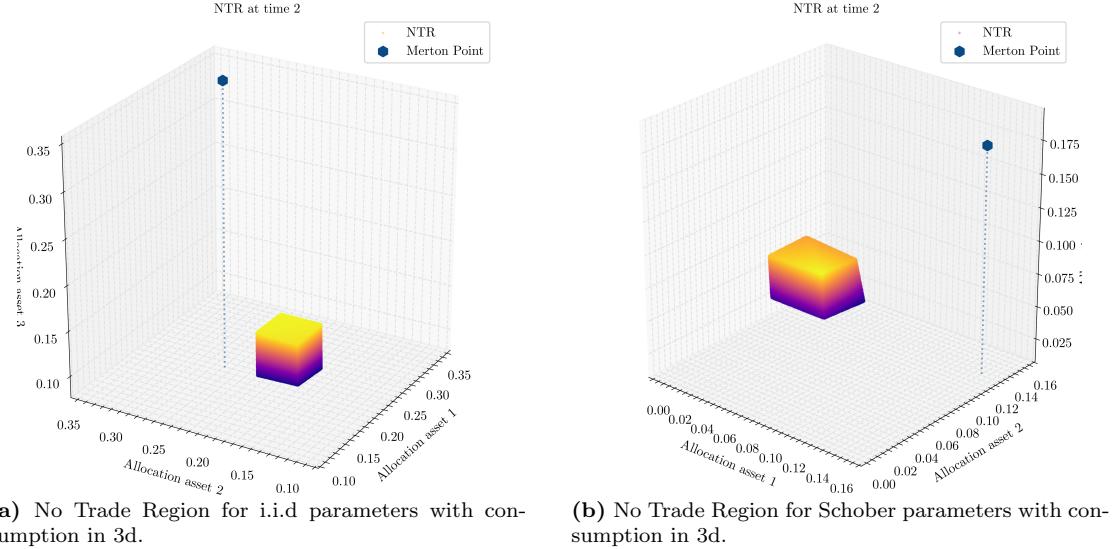
This behaviour is consistent in higher dimensions.

Figure 1.6: Comparison of No Trade Regions over time with consumption.



The No-Trade regions are plotted over the entire investment horizon $[0, T - 1]$.

Figure 1.7: No trade regions with consumption in multiple dimensions.



The No-Trade regions are plotted at time $t = 2$.

1.3 Dynamic Portfolio Choice with fixed costs

I now consider the base model with fixed transaction costs, and no consumption. From (Dybvig and Pezzo 2020) I know that the NTR is no longer rectangular when we only consider fixed costs, but instead circular with the merton point in the middle when there is no consumption in the model. This poses a problem for my current sampling scheme, which leverages our predetermined knowledge of the geometric shape of the NTR. As I noted in Section ??, in order to effectively sample points for the NTR approximation, given the framework for the proportional cost case, I now need to sample points, such

that when they hit the NTR these points are evenly distributed on the sphere, in order to approximate the NTR correctly.

However this is still not sufficient as the fixed costs pose further problems for the solution algorithm. In order to see this a little intuition is needed.

Transaction costs no longer scale in the fixed case, but are treated as a *sunk cost*, the moment the decision to trade is made. Hence if trading is optimal, the investor will trade to the optimal point, and if trading is sub-optimal then no trading will occur. The problem is therefore first of all a trading decision problem, and if trading is optimal, then the investor will trade to the merton point when no consumption is present, as this is the optimal point.

This is in stark contrast to the proportional case, where the trading trajectory from outside the NTR was to the border of the NTR, and the NTR approximation could be done by sampling points on the border of the feasible space.

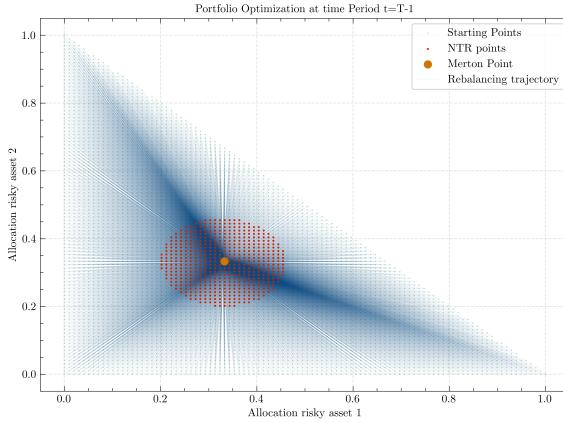
Now, any point sampled outside the NTR trades to the merton point, and i need to construct a new strategy, in order to efficiently construct the NTR, as no new information is gained by sampling points multiple points outside the NTR.

Furthermore, the transaction cost function is now an indicator function, depending on a threshhold, i.e $\sum_{i=1}^k \delta_{i,t}^+ + \delta_{i,t}^- > 0$. This is non-differentiable at the kink, $\sum_{i=1}^k \delta_{i,t}^+ + \delta_{i,t}^- = 0$ which is a critical point (the trading decision boundary), which i have to deal with, in order to solve the optimization problem.

I therefore split the optimization process into two parts. I evaluate the objective function (value function), *conditional* on no trading ($\delta_t = \mathbf{0}$), and *conditional* on trading ($\delta_t \neq \mathbf{0}$). Since there is no consumption decision the no-trading decision is trivial, whereas i still optimize the trading decision in the trading problem, in order to maximize expected utility. By splitting the optimization process, i can avoid the the non-differentiable edge case, and the derivative with regard to fixed costs i trivial for the optimizer. I then evaluate the value function for the no-trading decision, and the trading decision, and choose the decision which maximizes the value function.

I now consider the base model with fixed transaction costs, and no consumption. I use the simple i.i.d parameterization, with 2 assets and solve the optimization problem for the next to last period $T - 1$, over an evenly spaced grid of points. I do this in order to verify that the solution algorithm works as intended, and that the NTR is circular as expected, conflicting with my prior assumptions for the proportional case. I set the fixed costs to 0.005% of the investors total wealth, at any time point, and solve at a very fine grid of points, in order to approximate the NTR correctly. I find that the NTR is circular as expected, and the solution algorithm works as intended. I therefore proceed with generating a strategy for dealing with fixed costs, which can leverage my new found knowledge of the NTR.

Figure 1.8: Solution to the i.i.d case with fixed costs, 2 assets in period $T - 1$.



The optimization scheme ran with 5044 evenly spaced grid points. The points are plotted in the feasible space, and the NTR is the convex hull of these points.

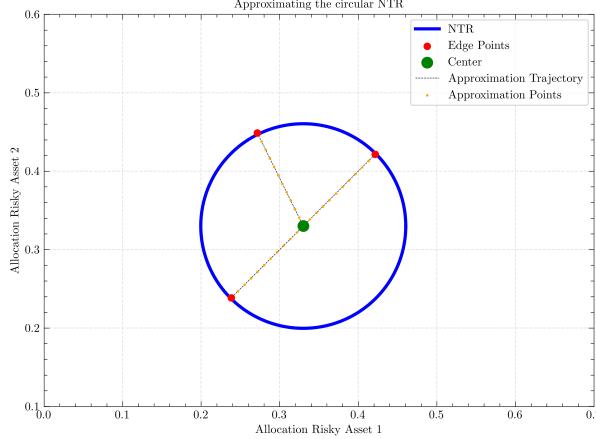
1.3.1 Constructing a new sampling scheme for the fixed cost NTR

Noting that for each point outside the NTR, the investor will trade to the same optimal point, since the cost of trading is a *sunk cost*, i can select a single starting point, at the origin for example, and solve for the optimal trading decision. If this is to trade, then i immediatly know the center of the NTR, and now only need the radius to construct the NTR. This holds for any dimensionality of the model, as any circle/sphere/hyper-sphere can be defined by the center point and the radius.

I find the radius, by slowly moving towards one of the boundary starting points, from the center of the NTR, solving the optimization problem for each point, and noting when trading occurs (return to the center). By using a bisection method¹, i find the border of the NTR with a tolerance of 10^{-8} , which is a tolerance of 0.000001% of the total wealth of the investor. I solve for multiple directions from the center, and choose the largest radius. This is because the circle might be truncated along the borders of the feasible space, if the NTR is close to either of the axis or the budget constraint (no borrowing/shorting). Furthermore, by selecting directions in evenly spaced angles around the center, i ensure that one of the directions of trading, must hit the border, as along as the NTR does no cover the entire feasible space. This optimization process can be seen in figure 1.8. I start from a blue point and rebalance to the Merton point. Following this, i move along a straight line outwards, and solve. If no trading occurs the point is red, and is in the NTR. This is repeated for each trading direction. The figure below displays this specific part of the algorithm:

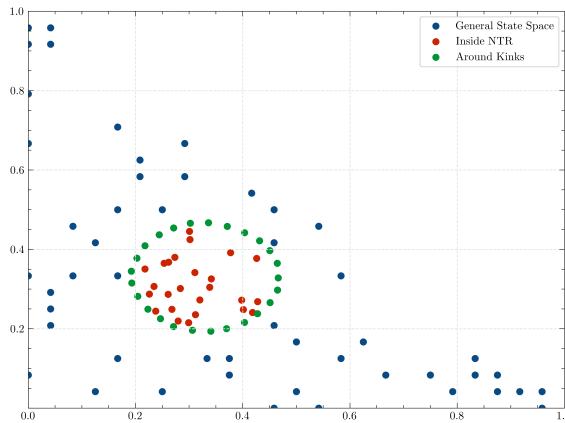
¹Which is simple to implement since we only consider allocations along the straight line from some angle, outwards from the center of the NTR

Figure 1.9: 2-Dimensional approximation algorithm for the fixed cost NTR with no correlation.



I also need to consider the sampling strategy for my GPR-related training points. As i mentioned previously, i need to sample three types of points. Points within the NTR, points outside the NTR, and points near the border border the NTR. The last points where previously kink points, when the NTR had followed assumptions 1 and 2, however for the circular case there are no kinks. The previous sampling strategy is easily applicable to the circlur case, however for the border points, i change the strategy slightly. I sample evenly spaced points (defined by their relative angle to the center point) on the border of the approximated NTR, and add a slight pertubation to ensure these are outside the NTR. This effectively covers the entire circle, and i can now leverage a low amount of training points for the GPR, as for the proportional cost case. The fixed costs problem

Figure 1.10: 2-Dimensional sampling strategy for the fixed cost NTR, with no consumption or correlation.

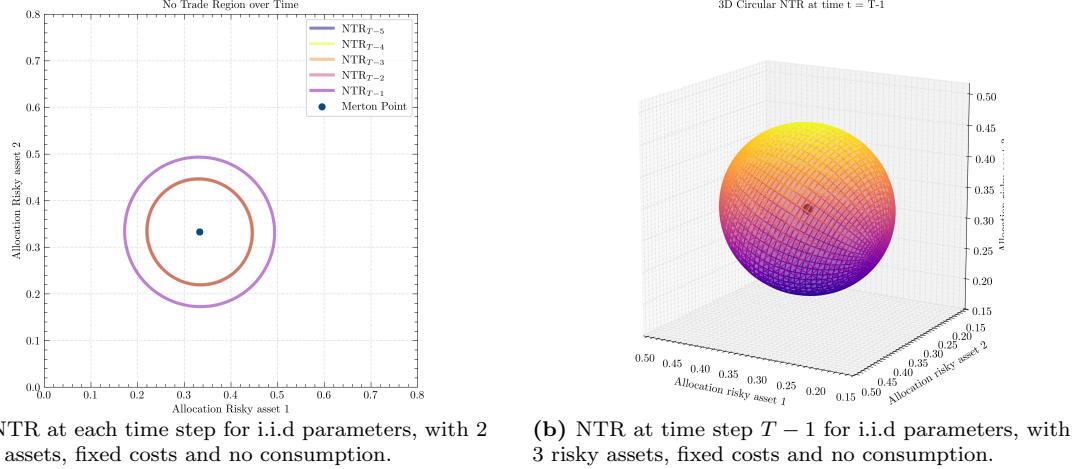


this sample strategy uses the same number of points as the schematic for the proportional cost sampling strategy

can likewise be solved in higher dimensions, however since this poses no changes to the

proposed solution methods, i do not consider this further, and continue to the correlated case. I solve the problem with the i.i.d parameters and a fixed cost of 0.075% of the investors total wealth at any time point. I consider an investment horizon of $T = 5$.

Figure 1.11: No trade regions with i.i.d assets.



Sample points for the GPR used 210 points. Fixed costs at 0.075% of total wealth.

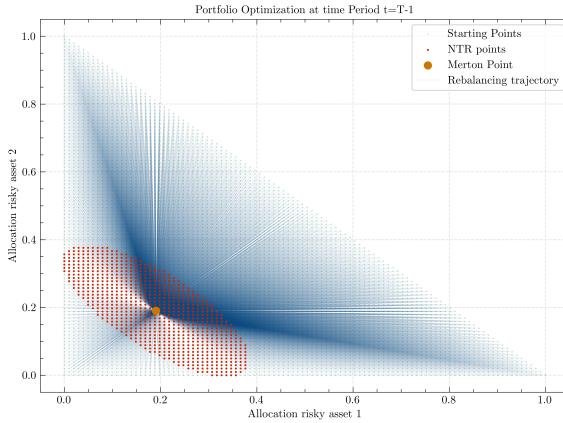
For the 2D plot i note that the NTR displays slight jitter, shrinking slightly over time. This is due to slight numerical instability over time. As function approximators are used iteratively to approximate the value function, the approximation error is compounded over time. I Note that the NTR at the terminal decision period $T - 1$ is slightly larger than for other periods. At time step $T - 1$ the radius is 0.16 whereas for time step $T - 5$ (0) the radius is 0.11. Hence when fixed costs are considered only two periods need to be solved for, in order to approximate the NTR over the entire investment horizon. Any more solutions will only add to the approximation error, and the NTR will not change significantly. I note for the 3D case that the NTR is now a sphere, which is to be expected, as the NTR has extended to higher dimensions before, keepings its original shape. In order to approximate the higher order NTRs, i use a similar fitting scheme as in figure 1.9. This is displayed in figure ?? located in appendix ??.

1.4 Dynamic Portfolio Choice with fixed costs and correlation

I now solve the model for correlated assets, that is, i solve for the Schober parametrization and for the high correlation parameterization. I set the fixed costs to 0.005% of the investors total wealth at any time point, and consider an investment horizon of $T = 5$. For the proportional cost case, when assets where correlated, the square was shifted into a parallelogram shape, and i expect the same to happen for the fixed cost case, shifting the circle into an ellipse. I first solve the 2D case for the high correlation case, as this

parameterization should have the most pronounced effect on the geometric shape of the NTR. Furthermore i want to verify that the NTR solution, once again, is defined by two distinct regions, one for $t = T - 1$ and one for $t < T - 1$, as noted in figure 1.11a. I first verify the shape of the NTR for the high correlation case, by solving over a fine grid as previously mentioned.

Figure 1.12: Solution to the high correlation case with fixed costs and 2 assets in period $T - 1$.



The optimization scheme ran with 5253 evenly spaced grid points. The points are plotted in the feasible space, and the NTR is the convex hull of these points. Fixed costs at 0.0005.

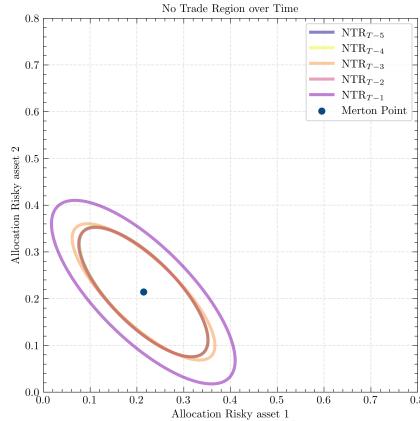
I note from figure 1.12 that the NTR is now an ellipse, as expected. This new shape is due to the correlation between the assets, and the NTR is now skewed, as the correlation between the assets is not 0. I then need to reformulate the solution algorithm for this case, as the NTR is no longer circular, and the solution algorithm for the fixed costs case is no longer applicable, since an ellipse is not defined by a center point and a radius. I need to reformulate the solution algorithm, in order to approximate the NTR correctly. An ellipse in 2 dimensions is defined by its *foci*. For any point on the ellipse, the sum of the distances to the foci is constant. The ellipse has a major diameter (major axis), and a minor diameter (minor axis), respectively the longest and shortest distance between two points on the ellipse (Ivanov 2020). Given a center point and enough points on the border of the ellipse (which may be noisy) i can approximate the ellipse by a least squares algorithm (Gander, Golub and Streb 1994). This requires enough points in order to solve the the problem sufficiently, for 2 dimensions the minimum required points is 5 points with no three points collinear. For higher dimensions the required points are $d(d + 3)/2$ points, however otherwise the same procedure can be applied (Bertoni 2010).

I modify the solution algorithm in the following manner. I solve the optimization problem for a single point outside the NTR, and find the optimal trading decision towards

the center². I then sample $d(d+3)/2 + 2^d + d + 1$ points, on the borders of the NTR. The 2^d points are the border points used for the square NTR sampling scheme. I then add $d(d+3)/2 + d + 1$ random points, which are still on the border of the feasible space, by drawing random points on the border³. This should leave me with enough points to approximate the ellipse, which are not collinear. I then proceed with the bisection algorithm as previously mentioned, until for each outwards direction from the center, find the border point of the NTR.

I then apply the least squares algorithm and solve for the parametric equation of the ellipse (Gander, Golub and Strelitz 1994; Bertoni 2010). This algorithm has the advantage that it can still solve for the ellipse using relatively few points, and these points need not cover the ellipse evenly, as the least squares algorithm will find the best fit ellipse for the points given. The rest of the circular algorithms can be used as before. Hence the ellipse NTR is slightly more complex, given the fitting scheme and points required, but the rest of the solution algorithm is the same. I do a slight modification to the high correlation μ vector, in order to move the merton point and make space for the resulting NTR. The new mean asset return is now $\mu = 0.075$ for each asset. This moves the merton point to $(0.2143, 0.2143)$ from the previous $(0.1905, 0.1905)$. The solution for each time point is plotted below: As expected, the NTR adjusts in the same manner as for the i.i.d case,

Figure 1.13: NTR for 2 assets with fixed costs and high correlation parameters.



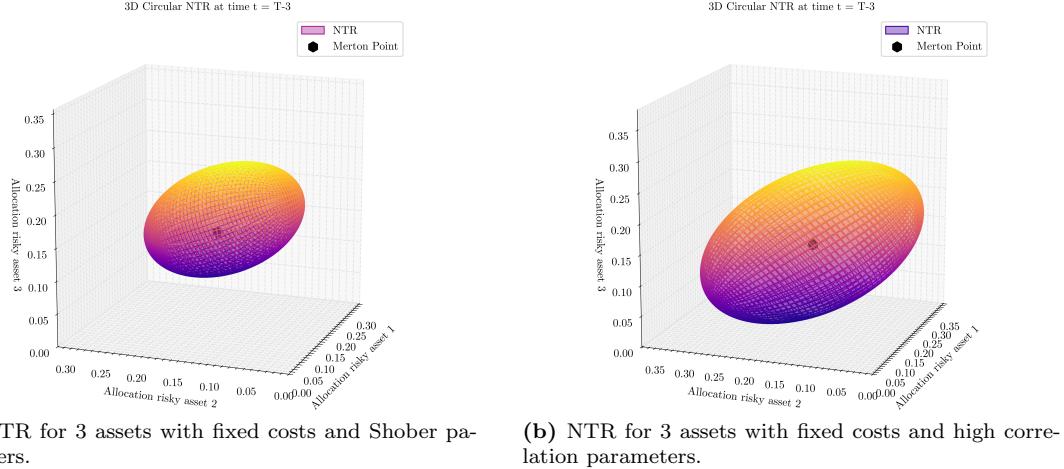
and shrinks from period $T - 1$ to $T - 2$. Following this the NTR is constant, with some compounding approximation error across periods. Therefore, in order to approximate the NTR over the entire investment horizon, only two periods need to be solved for, as the NTR does not change significantly over time.

²I do this for a few points in order to ensure that the point is outside the unknown NTR. However, a singular point is all that is needed. For example full investment into one of the risky assets, will most likely fall outside the NTR.

³I constrain these points so they are sufficiently distanced from my previously sampled points. This ensures that the resulting directions from the center are unique, and border points are not identical.

I now increase the dimensionality of the model to $d = 3$ and look at the No-Trade region for the Schober parameters and for the high correlation parameters. The resulting NTRs are plotted below. The resulting NTRs are now ellipsoids, and the intuition from

Figure 1.14: No trade regions with fixed costs and correlation.



The No-Trade regions are plotted at time $T - 3$ which for 5 periods is $t = 2$.

the 2d case carries over to the 3d case.

1.5 Dynamic Portfolio Choice with fixed and proportional costs