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Master's Thesis

Dynamic Portfolio Choice Under Various Transaction Costs and Asset Structures

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Abstract

This thesis

[*]I thank my supervisor Bertel Schjerning for...

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Abbreviations

GP	Gaussian process
GPR	Gaussian process regression
LDS	low-discrepancy sequences
LOVE	LanczOS Variance estimates
MC	Monte Carlo
MPT	randomized quasi-Monte Carlo
MPT	quasi-Monte Carlo
MPT	modern portfolio theory
NTR	no-trade-region
SKIP	Structured Kernel Interpolation for Products

1 What has been the contributions in this paper for now (Not in final paper)

- Bridging Cai, Judd and Xu (2020) and Gaegauf, Scheidegger and Trojani (2023). By adding the following to the framework of Gaegauf, Scheidegger and Trojani (2023):
  - By adding time period spacing Δt. Adds dimensions from behavioural finance, and continous finance theory.
  - Adding θ_t **maybe** (Not coded!)
  - By adding option pricing (Not coded!). Adds dimensions from derivatives pricing. Complex asset dynamics.
- By adding other transaction cost types. Such as:
  - Fixed transaction costs (Not coded!)
  - Asset specific transaction costs (Not coded!)
- Computationally:
  - Using LOVE approximation for the GP (Faster Variance estimation)
  - Using Kernel interpolation for the GP (Speeding up Kernel Matrix Computations)

2 To-Do

- Introduction
- Economic theory
  - NTR Theoy (Dybvig etc)
  - Option Price theory (Black-Scholes).
  - Transaction cost theory (Fixed, asset specific (Just formulations))
  - Price impact theory **maybe** (See Garlenau etc).
- Portfolio choice models
  - Model with options (After option theory)
  - Model with other transaction costs
- Skrive implementation details

  - No trade region approximation (sample etc.)
  - Final Algorithms

- Kode:
  - Option pricing, new: Exotic options.
  - Implement other transaction cost types
  - Implement θ_t (Maybe not. not that interesting)

3 Introduction

This is the introduction content

4 Literature review

The purpose of this section is to review relevant literature to help understand the contributions made in this thesis. This review covers modern portfolio theory (MPT), from its foundations and into the 21st century.

Modern theory on portfolio choice can be traced back to the mean-variance framework of Harry Markowitz, who constructed and solved the now well established, static and single period, portfolio optimization problem, Markowitz (1952). This covers the mean-variance framework which is the foundation of MPT, suggesting investors should allocate wealth in order to maximize expected return, while minimizing exposure to risk. Following this, the mean-variance framework has since been extended to a continious time setting, most notably by Robert Merton, who introduced a solution to the intertemporal portfolio choice problem in frictionless markets, Merton (1969), and later adding consumption rules aswell Merton (1971). This solution is known as the Merton point in the asset allocation space, or the Merton portfolio. Mertons closed form solution suggests optimal asset allocations based on the asset return dynamics (mean-variance), and the risk aversion of the investor (preferences). Hence in a continious time setting, the optimal allocation changes if the asset dynamics change.

Multiple extensions have been made to the classical dynamic portfolio choice problem, such as the introduction of transaction costs, adding realistic constraints to the problem, since trading assets incurs costs in the real world, and markets are not frictionless. Zabel (1973) adresses transaction costs with CRRA preferences, but is limited to a discrete time setting, a single risky asset and a small horizon.

Constantinides (1976) and Constantinides (1986) returns to the continious time setting, and find that for multiple preference types, under proportional transaction costs. The investors decision then depends on the the remaining life span, wealth and current allocation. Trading costs create a no-trade-region (NTR), where the optimal reallocation

decision for portfolios inside this, is do to nothing, and for portfolios outside this region, the optimal decision is to trade towards the boundary of the NTR. This is a shift from Mertons framework, where constant trading toward the Merton allocation, which is the optimal allocation in the absence of transaction costs, is optimal. Hence transaction costs restrain investors from acting optimally in the classical sense.

Numerical examples only cover the case of one risky asset, with restrictions on the decision space, and results remain qualitative or approximate. Notably Davis and Norman (1990) derive explicit solutions for the case of a single risky asset. They similarly find that proportional transaction costs lead to a NTR around the Merton point, and provide a solution algorithm for the stochastic control problem. This has later been made more rigorous such as Akian, Menaldi and Sulem (1996) who use a Hamilton-Jacobi-Bellman equation in the N-dimensional asset space, and provide further insight to the properties of the NTR, however the problem is only solved for the case of $k = 2$ risky assets with one risk free asset. Further analysis of this has been conducted extensively, e.g see Shreve and Soner (1994), Oksendal and Sulem (2002), Janeček and Shreve (2004), however the asset space is still constrained or solutions remain asymptotic. Muthuraman and Kumar (2006) and Muthuraman and Kumar (2008) tackle a $D = 3$ risky asset space, and provide a numerical solution to the problem, using a finites differences.

The seminal paper by Cai, Judd and Xu (2013) consider a more general setting, with multiple risky assets and a risk-free asset, and provides a solution algorithm, based on dynamic programming, numerical integration and polynomial approximation, to solve the dynamic problem for up to $k = 6$ risky assets and thus $D = 7$ assets in total, and later introduce and solve the problem with novelties, such as stochastic asset parameters or an option on an underlying asset in the portfolio Cai, Judd and Xu (2020). The curse of dimensionality, which haunts the prior methods applied, is somewhat tackled by the use of adaptive sparse grid methods, and sparse quadratuture rules by Schober, Valentin and Pflüger (2022).

Gaegauf, Scheidegger and Trojani (2023) furthrt reduces the computational burden by using a Gaussian process regression to approximate value functions, and a problem specific point sampling strategy to reduce the number of points in the state space needed to characterize the NTR. Increasing the dimensions of the asset space does still increase the dimensionality of the problem, and the computational burden, however this is at a much lower extent than previous methods.

Beyond the analysis conducted by the authors above, several related avenues of reseatch have been conducted on the dynamic portfolio choice problemn. Garleanu and Pedersen (2013) remains an influential paper, which aims to derive optimal closed form portfolio policy, when returns are driven by signals with mean reversion. This provides an insightful analysis of how to trade towards the optimal portfolio, given quadratic transaction costs, within a set scope of serially correlated assets. Dybvig and Pezzo (2020) provides

a comprehensive overview on the usage of different transaction cost functions, hedging with futures and security specific costs. Dybvig find that by changing the transaction cost function, the properties of the NTR is altered.[1]

5 Economic theory

This section covers the basics of modern portfolio theory and components of the dynamic portfolio choice problem with transaction costs. This section leans heavily on Cai, Judd and Xu (2020) and Gaegauf, Scheidegger and Trojani (2023), bridging the model from the former, with the framework of the latter.

5.1 Intertemporal portfolio choice without transaction costs

We first consider the classic portfolio optimization problem without transaction costs, as formulated by Merton (1969) and Merton (1971). For a more detailed treatment, see Björk (2019). In this setting, an investor dynamically allocates wealth between k risky assets and a risk-free asset to maximize utility over a finite horizon $[0, T]$.

The investor's wealth W_t can be allocated between a risk-free asset and k risky assets. Consumption is a non-durable good that can be purchased at each time point t. r is the risk-free rate, $\boldsymbol{\mu}$ is the vector of expected returns on the risky assets, and C_t represents consumption at time t. The investor's preferences follow a constant relative risk aversion (CRRA) utility function.

Without transaction costs, the optimal portfolio allocation, known as the Merton point is:

$$\mathbf{x}_t^* = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r), \tag{1}$$

where γ is the coefficient of relative risk aversion, and $\boldsymbol{\Sigma}$ is the covariance matrix of the risky assets' returns. This provides a time-independent optimal allocation that serves as a benchmark for models incorporating frictions such as transaction costs.

5.2 Investor preferences and problem

The investor operates over a finite horizon of T years, during which she aims to maximize her expected utility. Following Cai, Judd and Xu (2013), the investment horizon is discretized into N equally spaced periods, each with a duration of $\Delta t = \frac{T}{N}$. At each time point t_j, for $j = 0, 1, \ldots, N$, where $t_0 = 0$ and $t_N = T$, the investor has the opportunity to adjust her portfolio allocation right before $t_j + \Delta t$. Reallocation is costly, and the investor is subject to proportional transaction costs. She may also choose to consume a non-durable good at each time point.

[1] Gaegauf, Scheidegger and Trojani (2023), also note that their framework is applicable to different transaction cost functions.

For notational simplicity, we now use t to denote these time points unless specifically referring to t_j. The investor's preferences are modeled using a constant relative risk aversion (CRRA) utility function:

$$u(C_t) = \begin{cases} \frac{C_t^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1, \\ \log(C_t) & \text{if } \gamma = 1, \end{cases} \tag{2}$$

where C_t is consumption and c_t is the fraction of wealth W_t spent on consumption at time t. Hence $c_t = C_t/W_t$, and lowercase notation is henceforth used to denote variables as fractions of wealth. γ is the coefficient of relative risk aversion. The objective is to maximize the expected utility of consumption and wealth over the investor's lifetime:

$$\max_{\mathbf{x}_t, b_t, c_t} \mathbb{E}\left[\sum_{i=0}^{N-1} \beta^i u(C_i)\Delta t + \beta^N u(W_N)\right], \tag{3}$$

where β is the discount factor, $\mathbf{x}_t$ is the allocation to risky assets, b_t is the allocation to the risk-free asset, and W_t is the investor's wealth at time t.

5.3 Asset and goods market

We consider a financial market with k risky assets and one risk-free asset, making the asset space $D = 1 + k$ dimensions. The risk-free asset, such as a bond or a bank deposit, yields a constant gross return $R_f = \mathrm{e}^{r\Delta t}$, where r is the annual interest rate and $\Delta t = \frac{T}{N}$ is the length of one investment period.

The k risky assets can be considered as listed stocks, subject to proportional transaction costs. For each reallocation of wealth in a risky asset, a transaction cost of $\tau \in [0, 1]$ is incurred as a percentage of the traded amount. The stochastic one-period gross-return vector of the risky assets is denoted as $\mathbf{R} = (R_1, R_2, \ldots, R_k)^\top$, and the corresponding net-return vector is $\mathbf{r} = (r_1, r_2, \ldots, r_k)^\top$.

In the goods market, there is a single non-durable consumption good, C, which is consumed at each time point t. The fraction of wealth allocated to consumption at time t is denoted c_t, the fraction allocated to risky assets is $\mathbf{x}_t = (x_{1,t}, x_{2,t}, \ldots, x_{k,t})^\top$, and the fraction allocated to the risk-free asset is denoted b_t. Thus, $\mathbf{x}_t \in \mathbb{R}^k$ and $b_t \in \mathbb{R}$.

5.4 Transaction costs and portfolio reallocation

Rebalancing incurs proportional transaction costs $\tau \in [0, 1]$, which are paid based on the amount bought or sold of each risky asset. Reallocation decisions are made just before $t_j + \Delta t$, such that $\mathbf{x}_t$ is the portfolio of risky assets right before realllocation. $\delta_{i,t}$ denotes the change in portfolio allocation of asset i, and $\delta_{i,t}W_t$ is thus the currency amount traded in asset i. Hence $\delta_{i,t} > 0$ implies buying asset i, and $\delta_{i,t} < 0$ implies selling asset i.

Proportional transaction costs imply that the cost function associated with rebalancing is:

$$\psi(\delta_{i,t}W_t) = \tau|\delta_{i,t}W_t| \tag{4}$$

We decompose the decision variable $\delta_{i,t}$, representing the fraction of wealth used to trade risky asset i, into buying ($\delta^+_{i,t}$) and selling ($\delta^-_{i,t}$) components to ensure tractability[2]:

$$\delta_{i,t} = \delta^+_{i,t} - \delta^-_{i,t}, \quad \delta^+_{i,t}, \delta^-_{i,t} \geq 0.$$

The total transaction cost is then given by $\tau \sum_{i=1}^{k}(\delta^+_{i,t} + \delta^-_{i,t})W_t$. And the transaction cost function is therefore a function of each trading direction:

$$\psi(\delta^+_{i,t}, \delta^-_{i,t}, W_t) = \tau(\delta^+_{i,t} + \delta^-_{i,t})W_t \tag{5}$$

Following the reallocation, the remaining wealth is allocated between the risk-free asset and consumption. Notation of rebalancing is henceforth simplified using vectors to $\boldsymbol{\delta}_t = \boldsymbol{\delta}^+_t - \boldsymbol{\delta}^-_t$ with $\boldsymbol{\delta}^+_t = (\delta^+_{1,t}, \delta^+_{2,t}, \ldots, \delta^+_{k,t})$. We have that $\boldsymbol{\delta}_t$ is the *net change* in the risky positions, and $\boldsymbol{\delta}^+_t + \boldsymbol{\delta}^-_t$ is the *cumulative change* in the risky positions.

5.5 Asset dynamics

We follow Cai, Judd and Xu (2013) for the asset dynamics. The total composition of risky assets is assumed to follow a multivariate log-normal distribution:

$$\log(\mathbf{R}) \sim \mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t, (\boldsymbol{\Lambda}\boldsymbol{\Sigma}\boldsymbol{\Lambda})\,\Delta t\right), \tag{6}$$

where μ is the drift vector, σ^2 is a column vector of the variance σ_i^2, $\boldsymbol{\Sigma}$ is the correlation matrix, and $\boldsymbol{\Lambda} = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k)$ is the diagonal matrix of volatilities. Following Cai, Judd and Xu (2013) we utilize the Cholesky decomposition of the correlation matrix, $\boldsymbol{\Sigma} = \mathbf{L}\mathbf{L}^\top$, where $\mathbf{L} = (L_{i,j})_{k\times k}$ is a lower triangular matrix. Hence, for each risky asset i, the log-return is:

$$\log(R_i) = \left(\mu_i - \frac{\sigma_i^2}{2}\right)\Delta t + \sigma_i\sqrt{\Delta t}\sum_{j=1}^{i} L_{i,j}z_j, \tag{7}$$

where z_i are independent standard normal random variables.

[2]Gaegauf, Scheidegger and Trojani (2023) note that this ensures differentiability. This approach is common and found in earlier work such as Akian, Menaldi and Sulem (1996), who likewise note that this ensures that the variable is continious from origin in the positive real set.

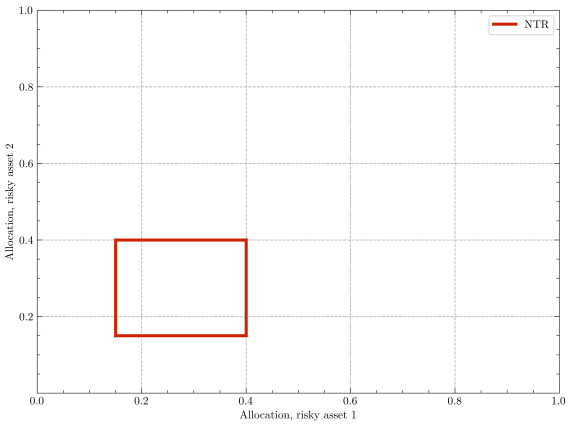
5.6 No Trade Region

The NTR is a region in the asset space (risky and risk-less) where it is sub-optimal to rebalance the portfolio. Given the parameters of the model the NTR without consumption is defined as:

$$\Omega_t = \{\mathbf{x}_t : \boldsymbol{\delta}_t^+, \boldsymbol{\delta}_t^- = \mathbf{0}\} \tag{8}$$

We note that the NTR is independent of the wealth level, but only depends on the wealth allocations. The NTR stems from the transaction costs, and is a connected set. When consumption is introduced in the model **SE ANGÅENDE CONVEX HULL**: Kamin (1975), Constantinides (1976, 1979, 1986), Davis and Norman (1990), and Muthuraman and Kumar (2006). Figure 5.1 illustrates an example of a NTR with two risky assets. The square shape of the NTR occurs with proportional transaction costs and independent

Figure 5.1: Example No Trade Region with $k = 2$ risky assets.



risky assets. However the NTR is not always a square, for more on this see Dybvig and Pezzo (2020).

6 Portfolio choice models

This section covers the different portfolio choice models. First the basic model is presented. This is followed by the most straightforward extensions, and finally models with novel extensions are introduced.

6.1 The general class of dynamic portfolio choice with transaction costs and intertemporal consumption

Considering the components presented in Section 5, the class of dynamic portfolio optimization problems, given one risk free asset and k risky assets, can be formulated by the

following Bellman equation, Bellman (1958)[3]:

$$V_t(W_t, \mathbf{x}_t, \theta_t) = \max_{c_t, \boldsymbol{\delta}_t^+, \boldsymbol{\delta}_t^-} \{u(c_t W_t)\Delta t + \beta \mathbb{E}_t \left[V_{t+\Delta t}(W_{t+\Delta t}\mathbf{x}_{t+\Delta t}, \theta_{t+\Delta t})\right]\}, \quad t < T \tag{9}$$

Given some initial level of wealth W_0 and portfolio allocation $\mathbf{x}_0$. θ_t is a vector of stochastic variables, which the gross one period risk free return, and risky return depends on, i.e $\mathbf{R}(\theta_t)$ and $R_f(\theta_t)$. These could cover the drift μ, volatiliy σ^2, correlation of the risky assets Σ, and the risk free return r or only some of these, dependent on the model. Notice that future wealth and allocations are stochastic, as they depend on the future realization of θ_t.

Notice that consumption and reallocation are decision variables, whereas bond holding are not (Explicitly). This is because bond holdings can be determined as the residual wealth, after consumption and reallocation decisions are made:

$$b_t W_t = \left(1 - \mathbf{1}^\top \cdot \mathbf{x_t}\right) W_t - \mathbf{1}^\top \cdot \boldsymbol{\delta}_t W_t - \psi(\boldsymbol{\delta}_t^+, \boldsymbol{\delta}_t^-, W_t) - c_t W_t \tag{10}$$

Where $\psi(\cdot)$ is the transaction cost function, and $\mathbf{1}$ is a vector of ones.

The dynamics of the state variables follow Schober, Valentin and Pflüger (2022) and are given by:

$$W_{t+\Delta t} = b_t W_t R_f(\theta_t) + \left([\mathbf{x_t} + \boldsymbol{\delta}_t] W_t\right)^\top \cdot \mathbf{R}(\theta_t) \tag{11}$$

$$\mathbf{x}_{t+\Delta t} = \frac{((\mathbf{x}_t + \boldsymbol{\delta}_t) W_t) \odot \mathbf{R}_t(\theta_t)}{W_{t+\Delta t}} \tag{12}$$

Where $\odot$ is the elementwise product (Hadamand product). The terminal value function is given by[4]:

$$V_T(W_T, \mathbf{x}_T, \theta_T) = u((W_T - \psi(\mathbf{x}_T W_T)) \cdot (1 - R_f(\theta_T))) \cdot \Delta t \cdot (1 - \beta)^{-1} \tag{13}$$

Which implies that the investor transfers all wealth to the bank account at the terminal period, and consumes out of the interest returns[5]. Finally we note that the optimization

[3] This is consolidated model of the base model, and with consumption model, of Cai, Judd and Xu (2020), however the cost function is generalized and correlation of returns is included.

[4] Stemming from the infinite sum of discounted utility of interest payments.

[5] This formulation stems from Cai, Judd and Xu (2013) and assumes that the investor lives forever. Gaegauf, Scheidegger and Trojani (2023) assumes that the investor consumes everything at the terminal period.

problem is subject to the following constraints:

$$\boldsymbol{\delta}_t W_t \geq -\mathbf{x}_t W_t \tag{14}$$

$$b_t W_t \geq 0 \tag{15}$$

$$\mathbf{1}^\top \mathbf{x}_t \leq 1 \tag{16}$$

The first constraint ensures that the investor does not short sell risky assets, The second constraint is also a no shorting constraint and the third is a no-borrowing constraint. Hence This formulation does not consider leveraged investments.
Furhtermore we can note that the rebalancing decision (in each direction), is only feasible in the space:

$$\delta^+_{i,t} \in [0, 1 - x_{i,t}] \tag{17}$$

$$\delta^-_{i,t} \in [0, x_{i,t}] \tag{18}$$

This is a direct formulation of the constraints, already captured in the equations above. The problem can be simplified by normalizing wrt. wealth, and removing wealth as a state variable, since wealth is seperable from the rest of the state space $\mathbf{x}_t, \theta_t$ as noted by Cai, Judd and Xu (2013).
This is because portfolio optimality is independent of wealth for CRRA utility function. The Bellman equation is then:

$$v_t(\mathbf{x}_t, \theta_t) = \max_{c_t, \boldsymbol{\delta}^+_t, \boldsymbol{\delta}^-_t} \{u(c_t)\Delta t + \beta \mathbb{E}_t \left[\pi^{1-\gamma}_{t+\Delta t} v_{t+\Delta t}(\mathbf{x}_{t+\Delta t}, \theta_{t+\Delta t})\right]\}, \quad t < T \tag{19}$$

The normalized bond holdings are then:

$$b_t = 1 - \mathbf{1}^\top \cdot (\mathbf{x_t} - \boldsymbol{\delta}_t - \psi(\boldsymbol{\delta}^+_t, \boldsymbol{\delta}^-_t)) - c_t \Delta t \tag{20}$$

We see that these are still the residual of the wealth after the rebalancing and consumption decision. Where we formulate the transaction cost function $\psi(\cdot)$ in terms of the buying and selling components, and using changes to allocations proportional to wealth, instead of the prior formulations, where wealth was a direct input. The dynamics are then:

$$\pi_{t+\Delta t} = b_t R_f(\theta_t) + (\mathbf{x}_t + \boldsymbol{\delta}_t)^\top \cdot \mathbf{R}(\theta_t) \tag{21}$$

$$\mathbf{x}_{t+\Delta t} = \frac{(\mathbf{x}_t + \boldsymbol{\delta}_t) \odot \mathbf{R}_t(\theta_t)}{\pi_{t+\Delta t}} \tag{22}$$

$$W_{t+\Delta t} = \pi_{t+\Delta t} W_t \tag{23}$$

Where we now formulate the problem with regard to the proportional wealth change $\pi_{t+\Delta t} = \frac{W_{t+\Delta t}}{W_t}$. The terminal value function is:

$$v_T(\mathbf{x}_T, \theta_T) = u((1 - \psi(\mathbf{x}_T)) \cdot (1 - R_f(\theta_T))) \cdot \Delta t \cdot (1 - \beta)^{-1} \tag{24}$$

The constrains are likewise normalized:

$$\boldsymbol{\delta}_t \geq -\mathbf{x}_t \tag{25}$$

$$b_t \geq 0 \tag{26}$$

$$\mathbf{1}^\top \mathbf{x}_t \leq 1 \tag{27}$$

This class of dynamic portfolio choice problems covers any formulation of the problem, where the transaction cost specification is differentiable, and the utility function allows for seperability of wealth and remaining state variables. Later formulations will be based on this class structure, covering the necessary Bellman equaiton, state dynamics, preferences and transaction costs functions as well as the constraints and any extensions not yet presented.

6.2 Base problem: Portfolio with proportional costs and consumption

Considering the class of problems constructed in the prior section, we can now quickly introduce the basic problem formulation. We consider an investor with CRRA utility function. She can invest in one risk free asset and k risky assets. Trading is subject to proportional transaction costs hence we have the following cost function (in cumulative terms):

$$\psi(\boldsymbol{\delta}^+_{i,t}, \boldsymbol{\delta}^-_{i,t}) = \tau(\boldsymbol{\delta}^+_{i,t} + \boldsymbol{\delta}^-_{i,t}) \tag{28}$$

We do not assume that returns are dependent on stochastic parameters, but instead are drawn from a distribution with known parameters. Hence we assume $\theta_t = \theta$ for all t. That is that we assume a constant return on the risk free asset, hence $R_f(\theta_t) = R_f$, and the risky assets follow a multivariate log-normal distribution, with some mean and covariance matrix. We can now formulate the entire problem giventhe class structure from section 6.1. The terminal value function is given by equation (24). The system is subject to the constraints of equations (25), (26) and (27), as well as a simple constrain on consumption, $c_t \geq 0$. We assume that the position in bond holdings is the residual wealth, and they therefore follow the process in (20). The Bellman equation is therefore:

$$v_t(\mathbf{x}_t, \theta_t) = \max_{c_t, \boldsymbol{\delta}^+_t, \boldsymbol{\delta}^-_t}, \{u(c_t)\Delta t + \beta \mathbb{E}_t \left[\pi^{1-\gamma}_{t+\Delta t} v_{t+\Delta t}(\mathbf{x}_{t+\Delta t})\right]\}, \quad t < T$$

With same terminal condition as before, where wealth is transfered to the bond account at the terminal period, and consumption is financed by the interest returns.

$$v_T(\mathbf{x}_T) = u((1 - \psi(\mathbf{0}, \mathbf{x}_T)) \cdot (1 - R_f)) \cdot \Delta t \cdot (1 - \beta)^{-1}$$

6.3 Portfolio with fixed and proportional costs

6.4 Portfolio with asset specific costs

7 Numerical implementation details

This section covers detail regarding the solution algorithm and numerical implementation. Each method is presented in a separate subsection, and the final solution algorithm is presented in the last subsection, which combines each of the methods. These span points sampling, numerical integration techniques, function approximation methods and solution techniques specific to this class of problemns.

7.1 Numerical integration

Consider the basic problem with proportional transaction costs, basic risky assets and a risk-free asset and no stochastic parameters. We need to evaluate the expectation of the value function: $\mathbb{E}\left[v_{t+\Delta t}(\mathbf{x}_{t+\Delta t})\right]$. In order to compue this expectation, we need to evaluate the integral:

$$\mathbb{E}_t\left[\pi_{t+1}^{1-\gamma} v_{t+1}(x_{t+1})\right] = \int \pi_{t+1}^{1-\gamma} v_{t+1}(x_{t+1}) f(R_{t+1}), dR_{t+1} \tag{29}$$

where $f(R_{t+1})$ is the probability density function of the risky asset returns. If we look at the case of stochastic parameters, would need to evaluate the conditional expectation with regard to these aswell, given some distributional assumption on the parameters. The integral can be computed using Monte-carlo methods or by using quadrature rules.

7.1.1 Gauss-Hermite quadrature

Gaussian quadrature is a numerical integration method based on approximation and interpolation theory. Gaussian quadrature can be used to approximate integrals using the following form, Judd (1998):

$$\int_a^b f(x)w(x)dx \approx \sum_{i=1}^{n} \omega_i f(x_i), \tag{30}$$

Where ω_i are quadrature weights, x_i are quadrature nodes and $w(x)$ is a weighting function. This approximation is exact when $f(x)$ is a polynomial of degree $2n - 1$ or less.

Then we can approximate the integral using n points x_i and n weights ω_i. There are many different Gaussian quadrature schemes, with differering intervals $[a, b]$ and weighting functions $w(x)$. We consider the use of a Gauss-Hermite quadrature rule, for a comprehensive review on Gaussian quadrature rules, see Judd (1998). Gauss-Hermite quadrature is used to approximate integrals of the form:

$$\int_{-\infty}^{\infty} f(x)e^{-x^2}dx \approx \sum_{i=1}^{n} \omega_i f(x_i) + \frac{n!\sqrt{\pi}}{2^n} \cdot \frac{f^{(2n)}(\zeta)}{(2n)!}, \tag{31}$$

Where $\zeta \in (-\infty, \infty)$. If a random variable X is normally distributed, i.e $X \sim \mathcal{N}(\mu, \sigma^2)$, then we can compute the expectation, $\mathbb{E}[f(X)]$, which is given by:

$$\mathbb{E}[f(X)] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} f(x)e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx \tag{32}$$

Using a change of variables $y = \frac{x-\mu}{\sqrt{2}\sigma}$, then we can rewrite the expectation on the form of the Gauss-Hermite quadrature rule:

$$\begin{aligned}
\mathbb{E}[f(X)] &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} f(\sqrt{2}\sigma y + \mu)e^{-y^2}\sqrt{2}\sigma dy && (33)\\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} f(\sqrt{2}\sigma y + \mu)dy && (34)\\
&\approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^{n} \omega_i f(\sqrt{2}\sigma x_i + \mu) && (35)
\end{aligned}$$

Where ω_i are the quadrature weights, x_i are the quadrature nodes over the interval $(-\infty, \infty)$.

When X is log-normal, i.e $\log X \sim \mathcal{N}(\mu, \sigma^2)$, then we can use a variable change once again: $X = e^Y$ and $Y \sim \mathcal{N}(\mu, \sigma^2)$. Then we can rewrite the expectation as:

$$\mathbb{E}[f(X)] = \mathbb{E}[f(e^Y)] \approx \pi^{-\frac{1}{2}} \sum n_{i=1}\omega_i f\left(e^{\sqrt{2}\sigma x_i + \mu}\right) \tag{36}$$

If we want to extend this framework to multiple dimensions we can use product rules as noted by Cai, Judd and Xu (2013). Consider Y which is multivariate normal, i.e $Y \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$, where μ is the drift vector and Σ is the covariance matrix. Let L be a

lower-triangular matrix such that $LL^\top = \Sigma$ (Cholesky factorisation). Then we have that:

$$\mathbb{E}\{f(Y)\} = \left((2\pi)^d \det(\Sigma)\right)^{-\frac{1}{2}} \int_{\mathbb{R}^d} f(y)\, e^{-\frac{1}{2}(y-\mu)^\top \Sigma^{-1}(y-\mu)}\, dy \tag{37}$$

$$= \left((2\pi)^d \det(L)^2\right)^{-\frac{1}{2}} \int_{\mathbb{R}^d} f\left(\sqrt{2}Ly + \mu\right)\, e^{-\frac{1}{2}y^\top y}\, dy \tag{38}$$

$$\approx \pi^{-\frac{d}{2}} \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} \omega_{i_1} \cdots \omega_{i_d}\, f\Bigg(\sqrt{2}L_{1,1}y_{i_1} + \mu_1,$$

$$\sqrt{2}(L_{2,1}y_{i_1} + L_{2,2}y_{i_2}) + \mu_2,\ \ldots,\ \sqrt{2}\Big(\sum_{j=1}^{d} L_{d,j}y_{i_j}\Big) + \mu_d\Bigg) \tag{39}$$

Where d refers to the number of dimensions, n is the number of quadrature points, ω_i are the quadrature weights and y_i are the quadrature nodes. $L_{i,j}$ is the ith row and jth column of the Cholesky factorisation matrix L. det is the matrix determinant. We note that the use of product rules suffers from the curse of dimensionality, as the complexity scales exponentionally with the number of dimensions. This is because the quadrature points with the product rule, normally use a tensor product grid, which is constructed using the Cartesian product of the quadrature points in each dimension. We can use sparse grid methods to partially tackle this. One common method is the Smolyak method, Smolyak (1963). Smolyaks sparse grid method approximates multidimensional integrals, over dimesion d while limiting the amount of points used. The method is composed of the following:

1. **Univariate Quadrature Rules**: Each dimension of the integration domain is assigned a univariate quadrature rule, which provides both nodes (quadrature points) and weights for numerical integration in that dimension. The accuracy of each rule is determined by its *level*, denoted by i_d for each dimension d. The level determines the number of quadrature points in that dimension, which improves the accuracy of the quadrature rule.

2. **Approximation Level (μ)**: The accuracy of the Smolyak sparse grid is controlled by the *approximation level* μ. This parameter sets a limit on the sum of levels across all dimensions, controlling the total number of grid points. Higher values of μ result in more accurate approximations but increase computational complexity.

3. **Multi-Index and Combination of Levels**: In a d-dimensional integral, the Smolyak method uses a *multi-index* $i = (i_1, i_2, \ldots, i_d)$ to represent the level of the quadrature rule in each dimension. The multi-index specifies a unique combination of quadrature levels for each dimension, where i_d denotes the level for dimension d. To construct a sparse grid, Smolyak's method restricts the sum of these levels

using the following condition:

$$d \leq i_1 + i_2 + \cdots + i_d \leq d + \mu$$

This constraint on the sum of levels, reduces the number of tensor products. We denote the sum of multi indicies: $|i| = i_1 + i_2 + \ldots i_d$.

4. **Tensor Product of Univariate Rules**: The Smolyak grid is formed by taking the *tensor product* of univariate quadrature rules that satisfy the multi-index constraint. Each univariate quadrature rule, represented by Q_{i_d} at level i_d in dimension d, is combined across dimensions according to the set of multi-indices i. This combination is given by:

$$A(\mu, d) = \sum_{d \leq |i| \leq d+\mu} (-1)^{\mu+d-|i|} \binom{d-1}{\mu+d-1-|i|} \bigotimes_{d=1}^{d} Q_{i_d}$$

where:

- Q_{i_d} is the univariate quadrature rule at level i_d in dimension d,
- $\bigotimes$ denotes the tensor product, and
- $\binom{d-1}{\mu+d-1-|i|}$ is a combinatorial coefficient that assigns weights to each tensor product, for accurate integration up to the specified approximation level μ.

By resticting the multi indicies i with the approximation level μ, the Smolyak method reduces the number of points needed for numerical integration in higher dimensions. Tensor grid methods grows exponentially with the number of dimensions d, the Smolyak grid grows polynomially, Judd et al. (2014), hence it directly combats the curse of dimensionality. For more on this see Smolyak (1963), Judd et al. (2014) and Horneff, Maurer and Schober (2016).

7.1.2 Monte Carlo integration (MC)

Monte Carlo integration is a numerical integration method based on *sampling*, as opposed to quadrature rules which are based on interpolation.
The convergence of Monte Carlo integration is generally slower than some quadrature methods; however, its convergence rate is independent of the dimensionality of the integral, making it well-suited for high-dimensional problems. Monte Carlo integration breaks the curse of dimensionality. Monte Carlo (MC) integration is based on random sampling[6] over the domain of the integral, and then computing the sample average of the

[6]Strictly speaking the samples are not random, but pseudo-random, meaning that deterministic samples are used, which appear random. For more in this see Judd (1998) or Glasserman (2004)

function to be integrated. Assume we wish to approximate the d-dimensional integral:

$$I = \int_{\Omega} f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x} = \mathbb{E}[f(\mathbf{x})], \tag{40}$$

where $g(\mathbf{x})$ is the probability density function of the random variable $\mathbf{x}$ over its support Ω, we approximate I as:

$$Q_N = \frac{1}{N} \sum_{i=1}^{N} f(\mathbf{X}_i), \tag{41}$$

where $\mathbf{X}_i$ are independent samples drawn from $g(\mathbf{x})$. The procedure is then:

1. Sample N points $\mathbf{x}_1, \ldots, \mathbf{x}_N$ from $g(\mathbf{x})$.

2. Approximate the expectation $\mathbb{E}[f(\mathbf{x})]$ by the sample average:

$$I \approx Q_N = \frac{1}{N} \sum_{i=1}^{N} f(\mathbf{x}_i).$$

The Law of Large Numbers ensures that the sample average converges to the mean as $N \to \infty$:

$$\lim_{N \to \infty} Q_N = \mathbb{E}[f(\mathbf{x})] = I.$$

And by the Central Limit Theorem, we have:

$$\sqrt{N} \left(Q_N - I \right) \xrightarrow{d} N \left(0, \sigma^2 \right),$$

where $\sigma^2 = \mathrm{Var}[f(\mathbf{x})]$ does not depend on N or d. The standard error of Q_N is:

$$\sigma_{Q_N} = \frac{\sigma}{\sqrt{N}}.$$

The convergence rate of $1/\sqrt{N}$ is independent of the dimension.

7.1.3 Quasi-Monte Carlo integration (QMC)

Quasi-Monte Carlo integration substitutes the 'random' samples in Monte Carlo integration with specific deterministic sequences such as equidistributed sequences, low-discrepancy sequences (LDS) or Lattice point rules etc. We will focus on the use of low discrepancy sequences. For a comprehesive review of sequences and rules see Judd (1998). LDS are deterministic sequences which cover the domain of the integral more evenly than random samples. Discrepancy is in this case a measure of deviation from perfect uniformity over the domain of the integral. Thus to go from MC in (41) to QMC, we replace the random samples $\mathbf{X}_i$ with LDS samples. We note that the sampling of the QMC is now dependent on the dimensionality of the integral, as opposed to MC, as the LDS samples have to be drawn with respect to the dimensionality of the integral.

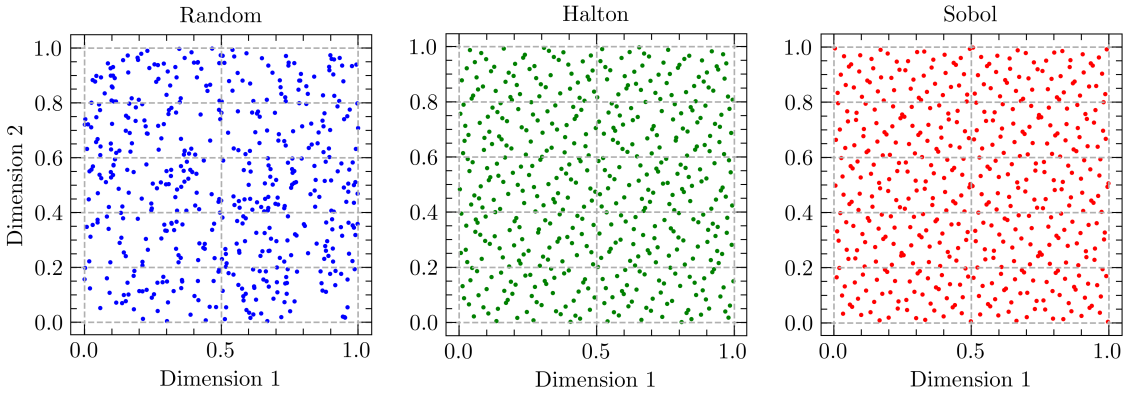
We consider two different types of LDS sequences, the Halton sequence and the Sobol sequence. Both sequences are popular LDS sequences, which are used in quasi-Monte Carlo (MPT) applications, (Glasserman 2004).

The convergence rate of MPT is:

$$\frac{(\log N)^d}{N} \tag{42}$$

Hence QMC is generally faster than MC, e.g $\frac{(\log N)^d}{N} < \frac{1}{\sqrt{N}}$ for large N and small d. We note that as dimensionality d increases, the quality of the Halton sequence decreases, as the dimensions become more correlated, Glasserman (2004). Specifically the Halton seuqnce will produce diagonal points when projected onto a 2D plane. This is displayed in figure 7.1. We therefore prefer the Sobol sequence when the dimensionality is sufficiently high, and as not to complicate matters, also use the Sobol sequence in lower dimensions, when MPT schemes are used. Figures below shows Random samples, Halton samples and Sobol samples in 2d. Second figure shows the same in 18 dimensions. Halton shows that dimension 17 and 18 are correlated.

Figure 7.1: Comparison of sample generation for Monte Carlo and Quasi-Monte Carlo

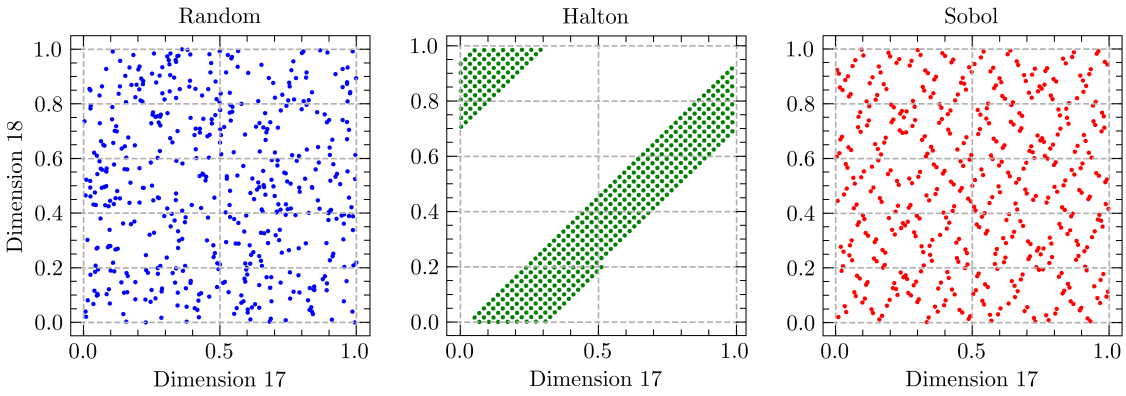


Note: Each sequence was generated using $N = 500$ samples and $d = 2$ dimensions.

QMC is generally found to be more efficient than MC, as noted by Glasserman (2004), Judd (1998), and notably Glasserman find that dimensionality has to be quite large before the Monte Carlo method is favorable to the quasi Monte Carlo method. Furthermore Glasserman find that while we generally might assume that N must by increase a lot when d is increased, this is not always the case in classic financial applications, as the integrals employed in these examples can often be approximated by integrals of much lower dimension. QMC therefore performs better than to be expected.

However we note that MPT lacks a straightforward variance estimator, a feature

Figure 7.2: Comparison of sample generation for Monte Carlo and Quasi-Monte Carlo with increased dimensionality



Note: Each sequence was generated using $N = 500$ samples and $d = 18$ dimensions.

recovered through *randomized QMC*, which will be discussed in the next section.

7.1.4 Randomized Quasi-Monte carlo integration (RQMC)

Randomized quasi-Monte Carlo integration (RQMC) is a combination of MPT and MC integration. We consider the the QMC integral, i.e the equaiton of (41), using an LDS sequence. The point of randomized quasi-Monte Carlo (MPT) is then to introduce randomness to the sequence: $P_n = \{x_1, \ldots, x_n\}$. We will cover the most simple case, *Random shift* and *Scrambling* methods, however for a comprehensive review of randomization methods see Glasserman (2004). The most simple method of randomizing P_n is to add a *random shift* to each point in the sequence, using random numbers drawn from a uniform distribution of the same dimensionality as the sequence, wrapped to the interval of P_n. Hence if $x_i \in [0,1)^d$ then we add a random shift $u_i \bmod 1$, where $\bmod 1$ keeps the shift within the interval $[0,1)$. A major disadvantage of the random shift is that is changes the discrepancy properties of the sequence, and hence the quality of the sequence is lost. Scrambled nets is a method of randomization which can be applied to LDS sequences specifically. Scrambling works by applying a sequence of random permutations to the digits in the base-b representation of each coordinate in the LDS. Each digit is permuted based on the values of the digits that came before it. This structure retains the low-discrepancy properties while introducing a controlled level of randomness, which enables the calculation of variance for RQMC estimates. In multi-dimensional settings, this scrambling is applied independently to each coordinate of the sequence, allowing us to estimate variance across the entire space. Scrambling the Sobol sequence has been found to be particularily effective in financial applications, as noted by Hok and Kucherenko (2023). QMC is generally more efficient than MC, and RQMC increases the rate of

converence of QMC and allows for the estimation of variance.

7.2 Value function approximation

This section covers the necessary function approximation methods used in the solution algorithm. We will cover the use of Gaussian process regression (GPR) and Baysian optimization, in order to maximize the value function of the dynamic portfolio allocation problem.

7.2.1 Gaussian process regressions (GPR)

A Gaussian process (GP) is a probabilistic model that defines a distribution over functions used to make predictions based on available data. It is specified by two functions: the mean function and the covariance function, also called the kernel. The mean function, $m(\mathbf{x})$, represents the expected value of the function at a given input $\mathbf{x}$, and the covariance function, $k(\mathbf{x}, \mathbf{x}')$, captures the covariance between function values at different input points $\mathbf{x}$ and $\mathbf{x}'$. In a GP, any finite set of input points $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$ within the domain $\mathbb{R}^d$ results in the function values $\mathbf{f} = (f(\mathbf{x}_1), \ldots, f(\mathbf{x}_N))$ having a joint multivariate Gaussian distribution. This property enables a GP to provide a prior distribution over functions based on the defined mean and covariance.

We use GPR to estimate the value function in the dynamic portfolio allocation problem, when we are not at the terminal period, i.e., $t < T$, following Gaegauf, Scheidegger and Trojani (2023). The GP is formulated by the previously mentioned mean and covariance functions:

$$f(\mathbf{x}) \sim \mathcal{GP}(m(\mathbf{x}), k(\mathbf{x}, \mathbf{x}')), \tag{43}$$

The covariance kernel function $k(\mathbf{x}, \mathbf{x}')$ can be any Mercer kernel, i.e., positive definite (Murphy 2023). Common kernel choices include the Radial Basis Function (RBF) kernel, the Matern kernel, and the Exponential kernel. We employ a Matern kernel, which, depending on the parameter ν, can be a generalization of the RBF kernel or the Exponential kernel. This choice follows Gaegauf, Scheidegger and Trojani (2023). The Matern kernel is given by:

$$k_{\text{Matern}}(\mathbf{x}, \mathbf{x}') = \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{\sqrt{2\nu}\|\mathbf{x} - \mathbf{x}'\|_2}{\ell} \right) K_\nu \left(\frac{\sqrt{2\nu}\|\mathbf{x} - \mathbf{x}'\|_2}{\ell} \right), \tag{44}$$

where $\|\cdot\|_2$ is the Euclidean norm, Γ is the gamma function, and K_ν is the modified Bessel function. The length scale ℓ and smoothness parameter ν are both positive. As $\nu \to \infty$, the Matern kernel converges to the RBF kernel (Gonzalvez et al. 2019). Functions from this class are k-times differentiable when $\nu > k$. When $\nu = 1/2$, the Matern kernel corresponds to the Ornstein-Uhlenbeck process (Murphy 2023), which is commonly used

in financial applications, such as models of interest rates (Glasserman 2004).

Consider a training dataset $\{\mathbf{X}, \mathbf{y}\}$ with N states $\mathbf{x}_i$ and observed values $\mathbf{y}$. We assume that the observations $\mathbf{y}$ are generated by an unknown function f, such that

$$y_i = f(\mathbf{x}_i) + \varepsilon_i, \quad \varepsilon_i \sim \mathcal{N}(0, \sigma_\varepsilon^2),$$

where σ_ε^2 represents the observational noise[7]. The goal is to train a GP on this dataset and then use it to predict the value function at a new state $\mathbf{x}_*$, yielding a new predicted output f_*.

The training observations $\mathbf{y}$ and the predicted noise-free function f_* have a joint Gaussian distribution:

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{f}_* \end{bmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} k(\mathbf{X}, \mathbf{X}) + \sigma_\varepsilon^2 \mathbf{I} & k(\mathbf{X}, \mathbf{x}_*) \\ k(\mathbf{x}_*, \mathbf{X}) & k(\mathbf{x}_*, \mathbf{x}_*) \end{bmatrix}\right) \tag{45}$$

Here i have assumed a zero mean function[8], and the kernel function is the Matern kernel. The posterior distribution of the predicted value function f_* given the training data is then a multivariate normal (Murphy 2023), with mean:

$$\tilde{\mu}(\mathbf{x}) = k(\mathbf{x}_*, \mathbf{X})[k(\mathbf{X}, \mathbf{X}) + \sigma_\varepsilon^2 \mathbf{I}]^{-1} \mathbf{y}, \tag{46}$$

And covariance:

$$\tilde{k}(\mathbf{x}_*, \mathbf{x}_*^{'}) = k(\mathbf{x}_*, \mathbf{x}_*^{'}) - k(\mathbf{x}_*, \mathbf{X})[k(\mathbf{X}, \mathbf{X}) + \sigma_\varepsilon^2 \mathbf{I}]^{-1} k(\mathbf{X}, \mathbf{x}_*^{'}) \tag{47}$$

Therefore in order to predict the value function at a new state $\mathbf{x}_*$, we need to compute the mean and covariance. This step is computationally burdensome as we have to compute the four covariance matrices in the joint distribution (45). Afterwards we can compute predictions using the mean function (46) and the covariance function (47) can be used to compute error bands on our predictions.

As noted, training and predicting with a GP is computationally expensive. I will therefore introduce the methods employed to reduce the computational burden of the GP.

We use automatic relevance detection (ARD) which is a modification to the Matern kernel to use a length scale for each dimension, ℓ_i. Dimensions with low impact has a high length scale, and are effectively ignored. Note that this is not the same as Lasso, as these coefficients are not set to 0.

[7] The noise assumption implies that the GP model does not interpolate the data but rather fits a smooth function. This results in computational costs of $O(N)$ for the mean prediction and $O(N^2)$ for the variance prediction. For more details, see (Murphy 2023).

[8] Zero mean ... XXXX

We use LanczOS Variance estimates (LOVE) to reduce the computational burden of estimating the variance of the GP.

We use Structured Kernel Interpolation for Products (SKIP) to reduce the computational burden of computen the matrices in the joint distribution (45).

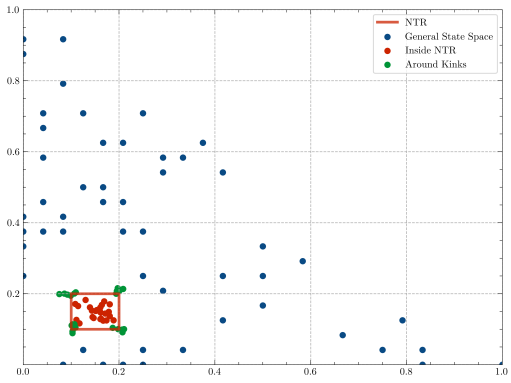
Use LancZos Variance Estimate (Love) and SKIP to reduce computational burden.

Here is the documentation in my package https://docs.gpytorch.ai/en/stable/examples/02_Scalable_Exact_GPs/index.html

And here is a paper on the subject https://arxiv.org/pdf/1803.06058.

7.2.2 Strategic point sampling

Figure 7.3: A designed sampling strategy

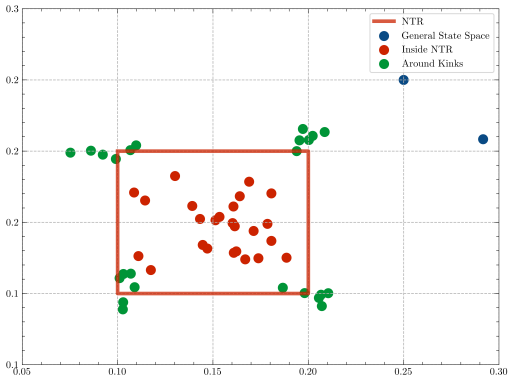


Note: Sample consists of $N = 100$ points, with 51 points in the general state space, 25 points inside the NTR and 24 points around the NTR kinks.

7.3 Leveraging the no trade region

7.4 Final solution algorithm

Figure 7.4: Zoom in on the NTR



Note: Sample consists of $N = 100$ points, with 51 points in the general state space, 25 points inside the NTR and 24 points around the NTR kinks.

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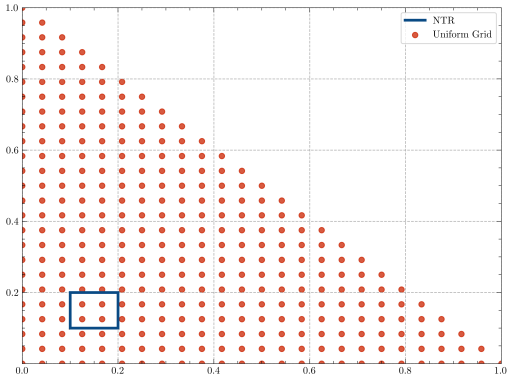
Appendices

A

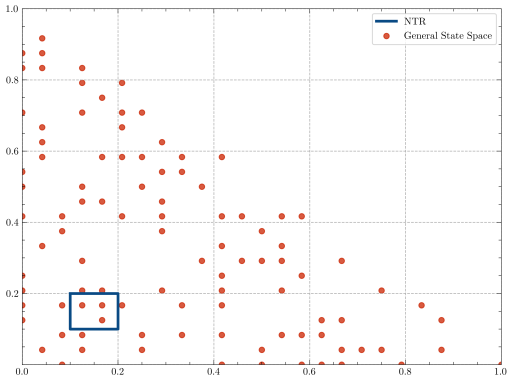
B Other sampling strategies

Figure B.1: Uniform grid sampling strategy



Note: Sample consists of $N = 200$ points.

Figure B.2: Naive random sampling strategy



Note: Sample consists of $N = 200$ points.