1 The Dynamic Portfolio Choice Setting

This section covers the basics of modern portfolio theory and components of the dynamic portfolio choice problem with transaction costs. This section leans heavily on Cai, Judd and Xu (2020) and Gaegauf, Scheidegger and Trojani (2023), bridging the model from the former, with the framework of the latter.

1.1 Asset and goods market

We consider a financial market with k risky assets and one risk-free asset, making the asset space $D = 1 + k$ dimensions. The risk-free asset, such as a bond or a bank deposit, yields a constant gross return $R_f = \mathrm{e}^{r\Delta t}$, where r is the annual interest rate and $\Delta t = \frac{T}{N}$ is the length of one investment period.

The k risky assets can be considered as listed stocks, subject to proportional transaction costs. For each reallocation of wealth in a risky asset, a transaction cost of $\tau \in [0, 1]$ is incurred as a percentage of the traded amount. The stochastic one-period gross-return vector of the risky assets is denoted as $\mathbf{R} = (R_1, R_2, \ldots, R_k)^\top$, and the corresponding net-return vector is $\mathbf{r} = (r_1, r_2, \ldots, r_k)^\top$.

In the goods market, there is a single non-durable consumption good, C, which is consumed at each time point t. The fraction of wealth allocated to consumption at time t is denoted c_t, the fraction allocated to risky assets is $\mathbf{x}_t = (x_{1,t}, x_{2,t}, \ldots, x_{k,t})^\top$, and the fraction allocated to the risk-free asset is denoted b_t. Thus, $\mathbf{x}_t \in \mathbb{R}^k$ and $b_t \in \mathbb{R}$.

1.2 Asset dynamics

I follow Cai, Judd and Xu (2013) for the asset dynamics. The total composition of risky assets is assumed to follow a multivariate log-normal distribution:

$$\log(\mathbf{R}) \sim \mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t, (\mathbf{\Lambda\Sigma\Lambda})\,\Delta t\right), \tag{1}$$

where μ is the drift vector, σ^2 is a column vector of the variance σ_i^2, $\mathbf{\Sigma}$ is the correlation matrix, and $\mathbf{\Lambda} = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k)$ is the diagonal matrix of volatilities. Following Cai, Judd and Xu (2013) we utilize the Cholesky decomposition of the correlation matrix, $\mathbf{\Sigma} = \mathbf{L}\mathbf{L}^\top$, where $\mathbf{L} = (L_{i,j})_{k\times k}$ is a lower triangular matrix. Hence, for each risky asset i, the log-return is:

$$\log(R_i) = \left(\mu_i - \frac{\sigma_i^2}{2}\right)\Delta t + \sigma_i\sqrt{\Delta t}\sum_{j=1}^{i} L_{i,j} z_j, \tag{2}$$

where z_i are independent standard normal random variables.

1.3 Transaction costs and portfolio reallocation

Rebalancing incurs proportional transaction costs $\tau \in [0,1]$, which are paid based on the amount bought or sold of each risky asset. Reallocation decisions are made just before $t_j + \Delta t$, such that $\mathbf{x}_t$ is the portfolio of risky assets right before realllocation. $\delta_{i,t}$ denotes the change in portfolio allocation of asset i, and $\delta_{i,t}W_t$ is thus the currency amount traded in asset i. Hence $\delta_{i,t} > 0$ implies buying asset i, and $\delta_{i,t} < 0$ implies selling asset i. Proportional transaction costs imply that the cost function associated with rebalancing is:

$$\psi(\delta_{i,t}W_t) = \tau|\delta_{i,t}W_t| \tag{3}$$

I decompose the decision variable $\delta_{i,t}$, representing the fraction of wealth used to trade risky asset i, into buying ($\delta_{i,t}^+$) and selling ($\delta_{i,t}^-$) components to ensure tractability[1]:

$$\delta_{i,t} = \delta_{i,t}^+ - \delta_{i,t}^-, \quad \delta_{i,t}^+, \delta_{i,t}^- \geq 0.$$

The total transaction cost is then given by $\tau \sum_{i=1}^{k}(\delta_{i,t}^+ + \delta_{i,t}^-)W_t$. And the transaction cost function is therefore a function of each trading direction:

$$\psi(\delta_{i,t}^+, \delta_{i,t}^-, W_t) = \tau(\delta_{i,t}^+ + \delta_{i,t}^-)W_t \tag{4}$$

Following the reallocation, the remaining wealth is allocated between the risk-free asset and consumption. Notation of rebalancing is henceforth simplified using vectors to $\boldsymbol{\delta}_t = \boldsymbol{\delta}_t^+ - \boldsymbol{\delta}_t^-$ with $\boldsymbol{\delta}_t^+ = (\delta_{1,t}^+, \delta_{2,t}^+, \ldots, \delta_{k,t}^+)$. We have that $\boldsymbol{\delta}_t$ is the *net change* in the risky positions, and $\boldsymbol{\delta}_t^+ + \boldsymbol{\delta}_t^-$ is the *cumulative change* in the risky positions.

1.4 Investor preferences and problem

The investor operates over a finite horizon of T years, during which the aim is to maximize expected utility. Following Cai, Judd and Xu (2013), the investment horizon is discretized into N equally spaced periods, each with a duration of $\Delta t = \frac{T}{N}$. At each time point t_j, for $j = 0, 1, \ldots, N$, where $t_0 = 0$ and $t_N = T$, the investor has the opportunity to adjust the portfolio allocations right before $t_j + \Delta t$. Reallocation is costly, and the investor is subject to proportional transaction costs. If consumption is included the investor may also choose to consume a non-durable good at each time point.

For notational simplicity, i now use t to denote these time points unless specifically referring to t_j. The investor's preferences are modeled using a constant relative risk

[1] Gaegauf, Scheidegger and Trojani (2023) note that this ensures differentiability. This approach is common and found in earlier work such as Akian, Menaldi and Sulem (1996), who likewise note that this ensures that the variable is continious from origin in the positive real set.

aversion (CRRA) utility function:

$$u(C_t) = \begin{cases} \frac{C_t^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1, \\ \log(C_t) & \text{if } \gamma = 1, \end{cases} \tag{5}$$

where C_t is consumption and c_t is the fraction of wealth W_t spent on consumption at time t. Hence $c_t = C_t/W_t$, and lowercase notation is henceforth used to denote variables as fractions of wealth. γ is the coefficient of relative risk aversion. The objective is to maximize the expected utility of consumption and wealth over the investor's lifetime:

$$\max_{\mathbf{x}_t, b_t, c_t} \mathbb{E}\left[\sum_{i=0}^{N-1} \beta^i u(C_i)\Delta t + \beta^N u(W_N)\right], \tag{6}$$

where β is the discount factor, $\mathbf{x}_t$ is the allocation to risky assets, b_t is the allocation to the risk-free asset, and W_t is the investor's wealth at time t.

1.5 Intertemporal portfolio choice without transaction costs

When there are no transaction costs (no market frictions) the investor can freely rebalance the portfolio. This reduces the problem to a classic portfolio optimization problem formulated by Merton (1969) and Merton (1971). For a more detailed treatment, see Björk (2019). In this setting, the investor dynamically allocates wealth between k risky assets and a risk-free asset to maximize utility over a finite horizon $[0, T]$.

The investor's wealth W_t can be allocated between a risk-free asset and k risky assets. Consumption is a non-durable good that can be purchased at each time point t. r is the risk-free rate, $\boldsymbol{\mu}$ is the vector of expected returns on the risky assets, and C_t represents consumption at time t. The investor's preferences follow a constant relative risk aversion (CRRA) utility function.

Without transaction costs, the optimal portfolio allocation, known as the Merton point is:

$$\mathbf{x}_t^* = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r), \tag{7}$$

where γ is the coefficient of relative risk aversion, and $\boldsymbol{\Sigma}$ is the covariance matrix of the risky assets' returns. This provides a time-independent optimal allocation that serves as a benchmark for models incorporating frictions such as transaction costs.

1.6 The general class of dynamic portfolio choice with transaction costs and intertemporal consumption

Now consider when transaction costs are present, and the investor can consume a non-durable good at each time point. The solution to the dynamic portfolio choice problem is no longer given by the closes form solution of the Merton point. Considering the components presented in this section, the class of dynamic portfolio optimization problems, given one risk free asset and k risky assets, can be formulated by the following Bellman equation, Bellman (1958)[2]:

$$V_t(W_t, \mathbf{x}_t, \theta_t) = \max_{c_t, \boldsymbol{\delta}_t^+, \boldsymbol{\delta}_t^-} \{u(c_t W_t)\Delta t + \beta \mathbb{E}_t \left[V_{t+\Delta t}(W_{t+\Delta t}\mathbf{x}_{t+\Delta t}, \theta_{t+\Delta t})\right]\}, \quad t < T \tag{8}$$

Given some initial level of wealth W_0 and portfolio allocation $\mathbf{x}_0$. θ_t is a vector of stochastic variables, which the gross one period risk free return, and risky return depends on, i.e $\mathbf{R}(\theta_t)$ and $R_f(\theta_t)$. These could cover the drift μ, volatiliy σ^2, correlation of the risky assets Σ, and the risk free return r or only some of these, dependent on the model. Notice that future wealth and allocations are stochastic, as they depend on the future realization of θ_t.
Notice that consumption and reallocation are decision variables, whereas bond holding are not (Explicitly). This is because bond holdings can be determined as the residual wealth, after consumption and reallocation decisions are made:

$$b_t W_t = \left(1 - \mathbf{1}^\top \cdot \mathbf{x_t}\right) W_t - \mathbf{1}^\top \cdot \boldsymbol{\delta}_t W_t - \psi(\boldsymbol{\delta}_t^+, \boldsymbol{\delta}_t^-, W_t) - c_t W_t \tag{9}$$

Where $\psi(\cdot)$ is the transaction cost function, and $\mathbf{1}$ is a vector of ones.
The dynamics of the state variables follow Schober, Valentin and Pflüger (2022) and are given by:

$$W_{t+\Delta t} = b_t W_t R_f(\theta_t) + ([\mathbf{x_t} + \boldsymbol{\delta}_t] W_t)^\top \cdot \mathbf{R}(\theta_t) \tag{10}$$

$$\mathbf{x}_{t+\Delta t} = \frac{((\mathbf{x}_t + \boldsymbol{\delta}_t) W_t) \odot \mathbf{R}_t(\theta_t)}{W_{t+\Delta t}} \tag{11}$$

Where $\odot$ is the elementwise product (Hadamand product). The terminal value function is given by[3]:

$$V_T(W_T, \mathbf{x}_T, \theta_T) = u((W_T - \psi(\mathbf{x}_T W_T)) \cdot (1 - R_f(\theta_T))) \cdot \Delta t \cdot (1 - \beta)^{-1} \tag{12}$$

Which implies that the investor transfers all wealth to the bank account at the terminal period, and consumes out of the interest returns[4]. Finally we note that the optimization

[2] This is consolidated model of the base model, and with consumption model, of Cai, Judd and Xu (2020), however the cost function is generalized and correlation of returns is included.
[3] Stemming from the infinite sum of discounted utility of interest payments.
[4] This formulation stems from Cai, Judd and Xu (2013) and assumes that the investor lives forever. Gaegauf, Scheidegger and Trojani (2023) assumes that the investor consumes everything at the terminal period.

problem is subject to the following constraints:

$$\boldsymbol{\delta}_t W_t \geq -\mathbf{x}_t W_t \tag{13}$$

$$b_t W_t \geq 0 \tag{14}$$

$$\mathbf{1}^\top \mathbf{x}_t \leq 1 \tag{15}$$

The first constraint ensures that the investor does not short sell risky assets, The second constraint is also a no shorting constraint and the third is a no-borrowing constraint. Hence This formulation does not consider leveraged investments.

Furhtermore we can note that the rebalancing decision (in each direction), is only feasible in the space:

$$\delta^+_{i,t} \in [0, 1 - x_{i,t}] \tag{16}$$

$$\delta^-_{i,t} \in [0, x_{i,t}] \tag{17}$$

This is a direct formulation of the constraints, already captured in the equations above. The problem can be simplified by normalizing wrt. wealth, and removing wealth as a state variable, since wealth is seperable from the rest of the state space $\mathbf{x}_t, \theta_t$ as noted by Cai, Judd and Xu (2013).

This is because portfolio optimality is independent of wealth for CRRA utility function. The Bellman equation is then:

$$v_t(\mathbf{x}_t, \theta_t) = \max_{c_t, \boldsymbol{\delta}^+_t, \boldsymbol{\delta}^-_t} \{u(c_t)\Delta t + \beta \mathbb{E}_t \left[\pi^{1-\gamma}_{t+\Delta t} v_{t+\Delta t}(\mathbf{x}_{t+\Delta t}, \theta_{t+\Delta t})\right]\}, \quad t < T \tag{18}$$

The normalized bond holdings are then:

$$b_t = 1 - \mathbf{1}^\top \cdot (\mathbf{x_t} - \boldsymbol{\delta}_t - \psi(\boldsymbol{\delta}^+_t, \boldsymbol{\delta}^-_t)) - c_t \Delta t \tag{19}$$

We see that these are still the residual of the wealth after the rebalancing and consumption decision. Where we formulate the transaction cost function $\psi(\cdot)$ in terms of the buying and selling components, and using changes to allocations proportional to wealth, instead of the prior formulations, where wealth was a direct input. The dynamics are then:

$$\pi_{t+\Delta t} = b_t R_f(\theta_t) + (\mathbf{x}_t + \boldsymbol{\delta}_t)^\top \cdot \mathbf{R}(\theta_t) \tag{20}$$

$$\mathbf{x}_{t+\Delta t} = \frac{(\mathbf{x}_t + \boldsymbol{\delta}_t) \odot \mathbf{R}_t(\theta_t)}{\pi_{t+\Delta t}} \tag{21}$$

$$W_{t+\Delta t} = \pi_{t+\Delta t} W_t \tag{22}$$

Where we now formulate the problem with regard to the proportional wealth change $\pi_{t+\Delta t} = \frac{W_{t+\Delta t}}{W_t}$. The terminal value function is:

$$v_T(\mathbf{x}_T, \theta_T) = u((1 - \psi(\mathbf{x}_T)) \cdot (1 - R_f(\theta_T))) \cdot \Delta t \cdot (1 - \beta)^{-1} \tag{23}$$

The constrains are likewise normalized:

$$\boldsymbol{\delta}_t \geq -\mathbf{x}_t \tag{24}$$

$$b_t \geq 0 \tag{25}$$

$$\mathbf{1}^\top \mathbf{x}_t \leq 1 \tag{26}$$

This class of dynamic portfolio choice problems covers any formulation of the problem, where the transaction cost specification is differentiable, and the utility function allows for seperability of wealth and remaining state variables. Later formulations will be based on this class structure, covering the necessary Bellman equaiton, state dynamics, preferences and transaction costs functions as well as the constraints and any extensions not yet presented.

The non-normalized optimal choices can be obtained by multiplying the normalized choices with the wealth level W_t at a given time point t. The no-trade-region (NTR) is in this framework the set of asset allocations where it is sub-optimal to rebalance the portfolio, and is defined as:

$$\Omega_t = \{\mathbf{x}_t : \boldsymbol{\delta}_t^{+,*}, \boldsymbol{\delta}_t^{-,*} = \mathbf{0}\} \tag{27}$$

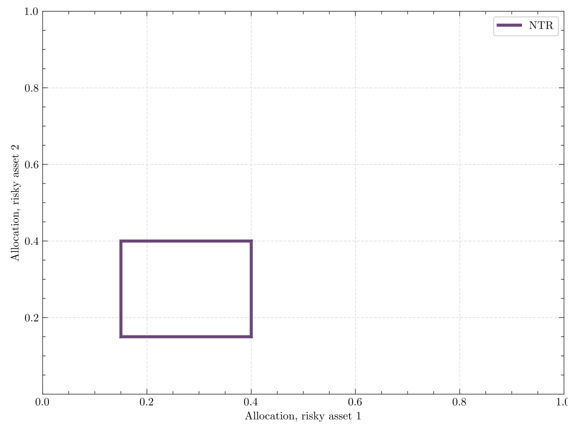
Where $\boldsymbol{\delta}_t^{+,*}, \boldsymbol{\delta}_t^{-,*}$ are the optimal buying and selling policies at time t. The next section will cover the NTR in more detail.

1.7 No Trade Region

The NTR is a region in the asset space where it is sub-optimal to rebalance the portfolio. Given the parameters of the model the NTR without consumption is defined as in equation (27). If consumption is included, this definiton remains the same, but the consumption decision varies within the NTR. Note that the NTR is independent of the wealth level, but only depends on the wealth allocations. The NTR stems from the introduction of transaction costs, and is a connected set.

SE ANGÅENDE CONVEX HULL: Kamin (1975), Constantinides (1976, 1979, 1986), Davis and Norman (1990), and Muthuraman and Kumar (2006). Figure 1.1 illustrates an example of a NTR with two risky assets. The square shape of the NTR occurs with proportional transaction costs and independent (i.i.d) risky assets. However the NTR is not always a perfect square, for more on this see (Dybvig and Pezzo 2020).

Figure 1.1: Example No Trade Region with $k = 2$ risky assets.



1.8 Base problem: Portfolio with proportional costs and consumption

Considering the class of problems constructed in the prior section, we can now quickly introduce the basic problem formulation. We consider an investor with CRRA utility function. She can invest in one risk free asset and k risky assets. Trading is subject to proportional transaction costs hence we have the following cost function (in cumulative terms):

$$\psi(\boldsymbol{\delta}^+_{i,t}, \boldsymbol{\delta}^-_{i,t}) = \tau(\boldsymbol{\delta}^+_{i,t} + \boldsymbol{\delta}^-_{i,t}) \tag{28}$$

We do not assume that returns are dependent on stochastic parameters, but instead are drawn from a distribution with known parameters. Hence we assume $\theta_t = \theta$ for all t. That is that we assume a constant return on the risk free asset, hence $R_f(\theta_t) = R_f$, and the risky assets follow a multivariate log-normal distribution, with some mean and covariance matrix. We can now formulate the entire problem giventhe class structure from section 1.6. The terminal value function is given by equation (23). The system is subject to the constraints of equations (24), (25) and (26), as well as a simple constrain on consumption, $c_t \geq 0$. We assume that the position in bond holdings is the residual wealth, and they therefore follow the process in (19). The Bellman equation is therefore:

$$v_t(\mathbf{x}_t, \theta_t) = \max_{c_t, \boldsymbol{\delta}^+_t, \boldsymbol{\delta}^-_t}, \{u(c_t)\Delta t + \beta \mathbb{E}_t \left[\pi^{1-\gamma}_{t+\Delta t} v_{t+\Delta t}(\mathbf{x}_{t+\Delta t})\right]\}, \quad t < T$$

With same terminal condition as before, where wealth is transfered to the bond account at the terminal period, and consumption is financed by the interest returns.

$$v_T(\mathbf{x}_T) = u((1 - \psi(\mathbf{0}, \mathbf{x}_T)) \cdot (1 - R_f)) \cdot \Delta t \cdot (1 - \beta)^{-1}$$

1.9 Portfolio with fixed and proportional costs

1.10 Portfolio with asset specific costs

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