Geometry terms:

in 1-d spherical, the equations appear as

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{1}{r^2}\frac{\partial r^2 F}{\partial r} = 0$$

in 2-d axisymmetry, the equations appear as

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{1}{r}\frac{\partial r F^{(r)}}{\partial r} + \frac{\partial F^{(z)}}{\partial z} = 0$$

For just the "radial" part, we have something of the form

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{1}{r^\alpha}\frac{\partial r^\alpha F^{(r)}}{\partial r} = 0$$

where $\alpha = 1$ for 2-d axisymmetric
$\alpha = 2$ for 1-d spherical

We can write this to look like the usual Cartesian form by expanding out the derivative

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{1}{r^\alpha}\left[r^\alpha \frac{\partial F^{(r)}}{\partial r} + F^{(r)}\,\alpha\, r^{\alpha-1}\right] = 0$$

or $$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial F^{(r)}}{\partial r} = -\frac{\alpha F^{(r)}}{r}$$

where $A = r^\alpha$

the combination $\frac{\alpha}{r}$ is sometimes expressed as $d\log A$

Now what do the primitive variables look like?

$$\mathcal{U} = (\rho, \rho u, \rho v, \rho w, \rho E, \rho e, \rho X_k)^T$$

we'll just worry about the 'r' flux

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial r} = -\frac{\alpha \rho u}{r}$$

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u u)}{\partial r} + \frac{\partial p}{\partial r} = -\frac{\alpha \rho u u}{r}$$

$$\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v u)}{\partial r} = -\frac{\alpha \rho v u}{r}$$

$$\frac{\partial (\rho w)}{\partial t} + \frac{\partial (\rho w u)}{\partial r} = -\frac{\alpha \rho w u}{r}$$

$$\frac{\partial (\rho E)}{\partial t} + \frac{\partial}{\partial r}(\rho u E + u p) = -\frac{\alpha (\rho u E + u p)}{r}$$

$$\cancel{\frac{\partial (\rho e)}{\partial t} + \frac{\partial}{\partial r}(\rho u e) + p \frac{\partial u}{\partial r}}$$

$$\frac{\partial (\rho X_k)}{\partial t} + \frac{\partial}{\partial r}(\rho u X_k) = -\frac{\alpha \rho u X_k}{r}$$

3.

Look at the velocity equations

u:

$$u\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial t} + u\frac{\partial \rho u}{\partial r} + \rho u\frac{\partial u}{\partial r} + \frac{\partial p}{\partial r} = -\frac{\alpha\rho u u}{r}$$

now subtract u · continuity

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial r} + \frac{\partial p}{\partial r} = 0$$ — the source cancels

v:

$$v\frac{\partial \rho}{\partial t} + \rho\frac{\partial v}{\partial t} + v\frac{\partial \rho u}{\partial r} + \rho u\frac{\partial v}{\partial r} = -\frac{\alpha\rho v u}{r}$$

subtract v · continuity

$$\rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial r} = 0$$ — source cancels

w: same as v:

$$\rho\frac{\partial w}{\partial t} + \rho u\frac{\partial w}{\partial r} = 0$$

What about species?

$$\rho \frac{\partial X_k}{\partial t} + X_k \frac{\partial \rho}{\partial t} + X_k \frac{\partial \rho u}{\partial t} + \rho u \frac{\partial X_k}{\partial r} = - \frac{\alpha \rho u X_k}{r}$$

subtrack $X_k \cdot$ continuity

$$\rho \frac{\partial X_k}{\partial t} + \rho u \frac{\partial X_k}{\partial r} = 0$$

or $$\frac{\partial X_k}{\partial t} + u \frac{\partial X_k}{\partial r} = 0$$ — again no source

Energy:

$$\frac{\partial(\rho e)}{\partial t} + \frac{1}{2}\frac{\partial(\rho u^2)}{\partial t} + \frac{1}{2}\frac{\partial(\rho v^2)}{\partial t} + \frac{1}{2}\frac{\partial(\rho w^2)}{\partial t}$$

$$+ \frac{\partial}{\partial r}(\rho u e) + \frac{1}{2}\frac{\partial}{\partial r}(\rho u u^2) + \frac{1}{2}\frac{\partial}{\partial r}(\rho u v^2) + \frac{1}{2}\frac{\partial}{\partial r}(\rho u w^2)$$

$$+ \frac{\partial}{\partial r}(up) = -\frac{\alpha(\rho u E + up)}{r}$$

expanding out some derivatives

$$\frac{\partial(\rho e)}{\partial t} + \Big[\frac{1}{2}u^2\frac{\partial(\rho)}{\partial t} + \frac{1}{2}\rho\frac{\partial u^2}{\partial t}$$

$$+\frac{1}{2}v^2\frac{\partial\rho}{\partial t} + \frac{1}{2}\rho\frac{\partial v^2}{\partial t}$$

$$+\frac{1}{2}w^2\frac{\partial\rho}{\partial t} + \frac{1}{2}\rho\frac{\partial w^2}{\partial t}\Big]$$

$$+ \frac{\partial}{\partial r}(\rho u e) + \Big[\frac{1}{2}\rho u\frac{\partial u^2}{\partial r} + \frac{1}{2}u^2\frac{\partial\rho u}{\partial r}$$

$$+\frac{1}{2}\rho u\frac{\partial v^2}{\partial r} + \frac{1}{2}v^2\frac{\partial\rho u}{\partial r}$$

$$+\frac{1}{2}\rho u\frac{\partial w^2}{\partial r} + \frac{1}{2}w^2\frac{\partial\rho u}{\partial r}\Big]$$

$$+ \frac{\partial}{\partial r}(up) = -\frac{\alpha(\rho u E + up)}{r}$$

subtracting off $\frac{1}{2}u^2$ [continuity] $+\frac{1}{2}v^2$ [continuity] $+\frac{1}{2}w^2$ [continuity]

we have

$$\frac{\partial(\rho e)}{\partial t}+\frac{1}{2}\rho\frac{\partial u^2}{\partial t}+\frac{1}{2}\rho\frac{\partial v^2}{\partial t}+\frac{1}{2}\rho\frac{\partial w^2}{\partial t}+\frac{\partial}{\partial r}(\rho u e)$$
$$+\frac{1}{2}\rho u\frac{\partial u^2}{\partial r}+\frac{1}{2}\rho u\frac{\partial v^2}{\partial r}+\frac{1}{2}\rho u\frac{\partial w^2}{\partial r}\Big]+\frac{\partial}{\partial r}(up)$$
$$=-\frac{\alpha(\rho uE+up)}{r}+\frac{1}{2}(u^2+v^2+w^2)\frac{\alpha\rho u}{r}$$

source term:

$$-\frac{\alpha(\rho ue+\frac{1}{2}\rho u(u^2+\cancel{v^2+w^2})+up)}{r}+\cancel{\frac{1}{2}(u^2+v^2+w^2)\frac{\alpha\rho u}{r}}$$
$$=-\frac{\alpha(\rho ue+up)}{r}$$

We can also expand out the $\frac{\partial u^2}{\partial t}, \dots$

$$\frac{\partial(\rho e)}{\partial t}+\rho u\frac{\partial u}{\partial t}+\rho v\frac{\partial v}{\partial t}+\rho w\frac{\partial w}{\partial t}+\rho u u\frac{\partial u}{\partial r}+\rho u v\frac{\partial v}{\partial r}+\rho u w\frac{\partial w}{\partial r}$$
$$+\frac{\partial}{\partial r}(\rho u e)+\frac{\partial(up)}{\partial r}=-\frac{\alpha(\rho ue+up)}{r}$$

now grouping,

$$\frac{\partial(\rho e)}{\partial t}+\rho u\Big[\underbrace{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}}_{=-\frac{1}{\rho}\frac{\partial p}{\partial r}}\Big]+\rho v\Big[\cancel{\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}}\to 0\Big]+\rho w\Big[\cancel{\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}}\to 0\Big]$$
$$+\frac{\partial}{\partial r}(\rho u e)+\frac{\partial(up)}{\partial r}=-\frac{\alpha(\rho ue+up)}{r}$$

7.

This gives

$$\frac{\partial(\rho e)}{\partial t} + \frac{\partial}{\partial r}(\rho v e) + \frac{\partial(v p)}{\partial r} - v\frac{\partial p}{\partial r} = -\frac{\alpha(\rho v e + v p)}{r}$$

or

$$\frac{\partial(\rho e)}{\partial t} + \frac{\partial}{\partial r}(\rho v e) + p\frac{\partial v}{\partial r} = -\frac{\alpha(\rho v e + v p)}{r}$$

in primitive form:

$$\frac{\partial(\rho e)}{\partial t} + v\frac{\partial \rho e}{\partial r} + \rho e\frac{\partial v}{\partial r} + p\frac{\partial v}{\partial r} = -\frac{\alpha(\rho v e + v p)}{r}$$

this is sometimes written in terms of enthalpy, $h = e + p/\rho$

$$\frac{\partial(\rho e)}{\partial t} + v\frac{\partial \rho e}{\partial r} + \rho h\frac{\partial v}{\partial r} = -\frac{\alpha \rho v h}{r}$$

What is the p evolution?

A. Consider γ-law EoS, then $p = \rho e(\gamma-1)$

$$\frac{1}{\gamma-1}\frac{\partial p}{\partial t} + \frac{1}{\gamma-1} u \frac{\partial p}{\partial r} + \frac{p}{\gamma-1}\frac{\partial u}{\partial r} + p\frac{\partial u}{\partial r} = -\frac{\alpha(\rho u e + u p)}{r}$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r} + \gamma p \frac{\partial u}{\partial r} = -\frac{\alpha \rho u h(\gamma-1)}{r} = -\frac{\alpha u}{r}\left[(\gamma-1)(\rho e + p)\right]$$

$$= -\frac{\alpha u(\gamma-1)}{r}\left[\frac{p}{\gamma-1}+p\right] = -\frac{\alpha\gamma p u}{r}$$

B. non-constant γ

consider $p = p(\rho, s)$

$$\frac{Dp}{Dt} = \left.\frac{\partial p}{\partial \rho}\right|_s \frac{D\rho}{Dt} + \left.\frac{\partial p}{\partial s}\right|_\rho \cancelto{0}{\frac{Ds}{Dt}}$$

0 if there are no heat sources

$$\Gamma_1 = \left.\frac{\partial \ln p}{\partial \ln \rho}\right|_s = \frac{\rho}{p}\left.\frac{\partial p}{\partial\rho}\right|_s$$

then

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r} = \frac{\Gamma_1 p}{\rho}\frac{D\rho}{Dt}$$

$$= \frac{\Gamma_1 p}{\rho}\left[-\rho\frac{\partial u}{\partial r} - \frac{\alpha\rho u}{r}\right]$$

$\Leftarrow$

$\therefore$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r} + \Gamma_1 p\frac{\partial u}{\partial r} = -\frac{\alpha\Gamma_1 p u}{r}$$

this agrees w/ A

Summary ;

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} = -\frac{\alpha \rho u}{r}$$

$$\frac{\partial (\rho e)}{\partial t} + u\frac{\partial \rho e}{\partial r} + \rho h\frac{\partial u}{\partial r} = -\frac{\alpha \rho u h}{r}$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r} + \Gamma_1 p\frac{\partial u}{\partial r} = -\frac{\alpha \Gamma_1 p u}{r} = -\frac{\alpha \rho c_s^2 u}{r}$$

no geometric sources for velocity or species