Solution of the Two-Body Free Fall Problem
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Suppose two particles are initially at rest, at locations x_A and x_B. The particles have masses M_A and M_B, respectively. Here I provide the analytical solution for the free fall. The center of mass of this system stays stationaary in this problem, since there are no external forces. Therefore, if I denote by $r(t)$ the present separation of the two particles, the equation of motion is

$$\ddot{r}(t) = -\frac{GM}{r^2}, \tag{1}$$

where M is the total mass of the system $M = M_A + M_B$ and the initial condition is $r_0 = x_{A0} + x_{B0}$.

This can be solved by recognizing that

$$\frac{d^2r}{dt^2} = \frac{dv}{dt} = v(r)\frac{dv(r)}{dr} \tag{2}$$

where $v(r)$ is the instantaneous velocity at particle separation $r(t)$. The equation of motion is then separable:

$$\begin{aligned} v\,dv &= -GM\frac{dr}{r^2} && (3)\\ \Rightarrow \frac{1}{2}v(r)^2 &= -GM\int_{r_0}^{r}\frac{dr}{r^2} && (4)\\ &= GM\left[\frac{1}{r} - \frac{1}{r_0}\right] && (5)\\ \Rightarrow v(r) &= \sqrt{\frac{2GM}{r_0}\left[\frac{r_0}{r} - 1\right]}. && (6) \end{aligned}$$

Since $v(r) = dr/dt$, this equation is also separable:

$$t(r) = \sqrt{\frac{r_0}{2GM}}\int_{r_0}^{r}\frac{dr}{\sqrt{\frac{r_0}{r} - 1}}. \tag{7}$$

Here I use the variable substitution

$$u(r) = \frac{r}{r_0}; \quad u(r_0) = 1; \quad du = \frac{1}{r_0}dr. \tag{8}$$

Then the integral takes the form

$$t(r) = \sqrt{\frac{r_0^3}{2GM}}\int_{1}^{r/r_0}\frac{du}{\sqrt{\frac{1}{u} - 1}}. \tag{9}$$

Here a trigonometric substitution makes sense:

$$u(\theta) = \cos^2(\theta); \quad \theta(u = 1) = 0; \quad du = 2\cos(\theta)\sin(\theta)\,d\theta \tag{10}$$

Then, using the trigonometric identity

$$\frac{1}{\cos^2(\theta)} - 1 = \tan^2(\theta),$$

the integral gets transformed into the final form

$$\begin{aligned}
t(r) &= 2\sqrt{\frac{r_0^3}{2GM}} \int_0^{\theta(r)} d\theta \, \frac{\sin(\theta)\cos(\theta)}{\tan(\theta)} && (11)\\
&= 2\sqrt{\frac{r_0^3}{2GM}} \int_0^{\theta(r)} d\theta \, \cos^2(\theta) && (12)\\
&= \sqrt{\frac{r_0^3}{2GM}} \Big[\theta + \sin(\theta)\cos(\theta)\Big]_0^{\theta(r)} && (13)\\
&= \sqrt{\frac{r_0^3}{2GM}} \Big[\theta + \cos(\theta)\sqrt{1-\cos^2(\theta)}\Big]_0^{\theta(r)} && (14)\\
&= \sqrt{\frac{r_0^3}{2GM}} \left[\arccos\left(\sqrt{\frac{r}{r_0}}\right) + \sqrt{\frac{r}{r_0}\left(1 - \frac{r}{r_0}\right)}\,\right]. && (15)
\end{aligned}$$