Levi Bands and Toric Richardson Varieties

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This talk is based on joint works with Pinaki Saha and Fernando Nestor Diaz Morera

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$X_w = \overline{B\,w\,B/B}$ .... Schubert variety

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Richardson varieties play a central role in the geometry of G/B. Their cohomology classes correspond to products in $H^*(G/B)$:

$$[X_u] \cdot [X^v] = [X_u^v].$$

A reference

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Toric Richardson varieties

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ABSTRACT

In this article, we provide characterizations of toric Richardson varieties across all types through three distinct approaches: 1) poset theory, 2) root theory, and 3) geometry.

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1. Introduction

Theorem A

Theorem (Can–Saha '23)

Let $u \leq v$ in W. The Richardson variety X_v^u is toric for T if and only if the Bruhat interval $[u, v]$ is a lattice. (Tenner'22: $[u, v]$ is boolean iff distributive iff modular.)

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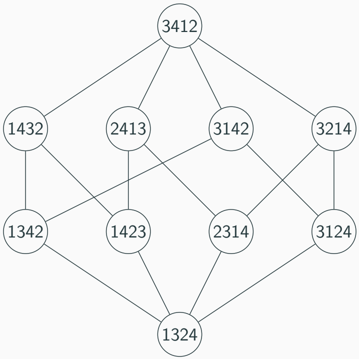

Figure 1: Hasse diagram of the interval $[\, s_2,\ s_2 s_1 s_3 s_2 \,] = [\, 1324,\ 3412 \,]$.

Theorem B

Theorem (Can–Saha '23)

Assume that G is of A-,D-, or E- type. Let $u \leq v$ in W. The Richardson variety X_v^u is a smooth toric variety for T if and only if the Bruhat interval $[u, v]$ is a boolean lattice.

Theorem B

Theorem (Can–Saha '23)

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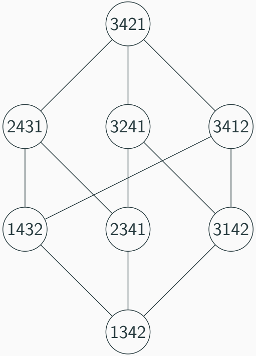

Figure 2: Hasse diagram of the interval $[\, s_3 s_2,\ s_2 s_3 s_2 s_1 s_2 \,] = [\, 1342,\ 3421 \,]$.

Theorem C

Theorem (Can–Saha '23)

Assume that G/P is minuscule, or $G = G_2$. Let $u \leq v$ in W^P. The Richardson variety X_{vP}^{uP} is a toric variety for T if and only if $v = uc$, where c is a boolean element.

Example

In type A, if $G = SL_n$ and P is the maximal parabolic subgroup stabilizing a k-plane, then G/P identifies with the Grassmannian $\mathrm{Gr}(k, n)$. A Schubert variety in $\mathrm{Gr}(k, n)$ can be described combinatorially by a monotone lattice path in a $k \times (n-k)$ grid running from $(0, 0)$ to $(k, n-k)$, and a Richardson variety X_v^u corresponds to a pair of such lattice paths with the lower path determined by u and the upper path by v. Multiplying by a boolean element c raises the lower path by a sequence of ribbons.

Here is a toric Richardson variety in $\mathrm{Gr}(6, 10)$.

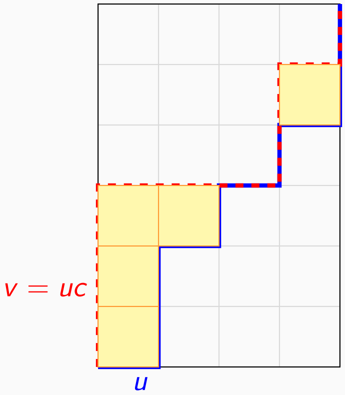

Figure 3: The lower path is u (solid blue) and the upper path is $v = uc$ (dashed red). The three shaded squares correspond to c.

Levi Bands

Relative position

Orbits of the diagonal action of G on $G/B \times G/B$ are parametrized by W. For $w \in W$, we denote by $\mathcal{O}_w$ the following orbit:

$$\mathcal{O}_w \ := \ G \cdot (B,\, wB) = \{(gB, gwB) \mid g \in G\}.$$

Hence, for $(xB, yB) \in X \times X$, there is a unique $w \in W$ such that $(xB, yB) \in \mathcal{O}_w$. This element is called the **relative position of** (x, y),

$$\mathrm{relpos}(x, y) := w.$$

Then

$$\overline{\mathcal{O}_w} = \{(gB, hB) \mid \mathrm{relpos}(g, h) \leq w\}.$$

Levi bands

We fix a Levi subgroup $L \supset T$. The L-orbit of eB in G/B is a flag variety itself:

$$Y_L := L \cdot (eB) \cong L/B_L.$$

Definition

Let $u, v \in W$. We define the **Levi band associated with** (L, u, v) as follows:

$$Z_L(u,v) := \begin{cases} \{(x,y) \in G/B \times Y_L \mid \ u \leq \mathrm{relpos}(x,y) \leq v\} & \text{if } u \leq v \\ \emptyset & \text{otherwise.} \end{cases}$$

- For $L = T$ and $u \leq v$, Y_L is a point, and $Z_L(u,v) \cong X_u^v$.
- For $L = G$ and $u \leq v$, $Y_L = G/B$ and $Z_L(u,v)$ is the union of diagonal G-orbits with $w \in [u,v]$ (the "G-**band of** $[u,v]$").

Normality

Theorem

Let $u \leq v$. Then $Z_L(u, v)$ is irreducible and normal.

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Lemma

$Z_L(u, v)$ is a locally closed, irreducible subvariety, stable for the diagonal action of L. The projection $\pi_2 : Z_L(u, v) \to Y_L$ is surjective and L-equivariant.

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Lemma

Let $U_L := B_L^- B_L / B_L \subset Y_L$ be the big B_L-cell. Then the restriction of the projection

$$\pi_2 : Z_L(u,v) \longrightarrow Y_L$$

is Zariski-locally trivial over U_L, with fiber X_u^v:

$$\Phi : \; U_L \times X_u^v \xrightarrow{\sim} Z_L(u,v) \cap \pi_2^{-1}(U_L), \qquad (l, xB) \mapsto (\, l\,xB,\; lB_L\,).$$

Positivity

Using $H^*(G/B) \otimes H^*(G/B)$ with basis $[X_a] \otimes [X_b]$, we write

$$[\overline{Z_L(u,v)}] = \sum_{a,b \in W} c_{a,b} \left([X_a] \otimes [X_b]\right), \quad c_{a,b} \in \mathbb{Z}.$$

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Theorem

All coefficients satisfy $c_{a,b} \geq 0$.

Proof. For $a \in W$, let's write $a' := w_0 a$. Then, by using Poincaré duality and Kleiman transversality on $G/B \times G/B$, we calculate $c_{a,b}$ as follows:

$$\begin{aligned} c_{a,b} &= \int_{G/B \times G/B} [\overline{Z_L(u,v)}] \smile (\pi_1^*[X^{a'}] \smile \pi_2^*[X^{b'}]) \\ &= \deg([\overline{Z_L(u,v)}] \cdot [(gX^{a'}) \times (hX^{b'})]) \end{aligned}$$

for some $(g,h) \in G \times G$. This is a nonnegative integer.

Spherical Varieties

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$$\mathrm{c}_G(X) := \operatorname{tr.deg}_k k(X)^G.$$

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- X is G-**spherical** iff $\mathrm{c}_B(X) = 0$ iff $\exists$ open B-orbit in X.

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Theorem (Brion, Vinberg)

X is G-spherical $\iff$ X contains only finitely many B-orbits.

Representation theory characterization of spherical varieties

Λ . . . . the character group of T (or, of B)

Λ^+ . . . . the monoid of dominant weights relative to B

$V(\lambda)$ . . . . the simple G-module with highest weight $\lambda \in \Lambda^+$

$M_\lambda^{(B)}$ . . . . the B-eigenspace with weight $\lambda \in \Lambda$

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A normal quasiaffine G-variety X is G-spherical iff

1. $k[X]$ **is multiplicity-free:** $k[X] = \bigoplus_{\lambda \in \Lambda^+(X)} k[X]_{(\lambda)}$ and $\dim \operatorname{Hom}_G(V(\lambda), k[X]) = 1$,
2. $\Lambda^+(X)$ **is saturated:** the weight monoid $\Lambda^+(X) := \{\lambda \in \Lambda \mid k[X]_\lambda^{(B)} \neq 0\}$ is saturated.

Stabilizers of Schubert and opposite Schubert varieties

- Schubert variety: $X_u = \overline{B\,u\,B/B}$ is stabilized by the **standard parabolic**

$$P_u := \mathrm{Stab}_G(X_u).$$

- Opposite Schubert: $X^v = \overline{B^-\,v\,B/B} = w_0\,X_{w_0 v}$ is stabilized by the **opposite parabolic**

$$\mathrm{Stab}_G(X^v) = w_0 P_v w_0^{-1}, \qquad P_v := \mathrm{Stab}_G\big(X_{w_0 v}\big).$$

- Their intersection

$$Q := P_u \cap w_0 P_v w_0^{-1}$$

need not be parabolic, but it contains the canonical Levi

$$L := L_{I_u \cap I_v} \subset Q.$$

Lemma

L fixes both X_u and X^v, hence acts on the Richardson variety $X_u^v := X_u \cap X^v$.

Band as a homogeneous fiber bundle $L \times^{B_L} X_u^v$

Theorem

There is an L-equivariant isomorphism

$$\Theta : \; L \times^{B_L} X_u^v \xrightarrow{\sim} Z_L(u,v), \qquad [l,x] \longmapsto (\, l \cdot x, \; l \cdot B_L \,).$$

Sketch of a proof.

The map $L \times X_u^v \to Z_L(u,v)$, $(l,x) \mapsto (l \cdot x, l \cdot B_L)$, is right B_L-equivariant for $(l,x) \cdot b = (lb, b^{-1} \cdot x)$, so it factors through $L \times^{B_L} X_u^v$. The fiber over eB_L is X_u^v, and L acts transitively on Y_L, giving surjectivity. The quotient relation ensures injectivity. Glueing on Bruhat "charts," we conclude it is an isomorphism. □

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This result allows us to port-over our earlier results from [Can-Saha'23].

Orbit-closure posets

Let $Z := Z_L(u, v)$ and $X := X_u^v$.

Let $\mathcal{P}_L(Z)$ be the inclusion poset of L-orbit closures in Z, and $\mathcal{P}_{B_L}(X)$ the inclusion poset of B_L-orbit closures in X.

Theorem

Suppose L has an open orbit in Z. Then the map

$$\begin{aligned} \Phi : \mathcal{P}_L(Z) &\longrightarrow \mathcal{P}_{B_L}(X), \\ \overline{\mathcal{O}} &\longmapsto \overline{\mathcal{O}} \cap \pi_2^{-1}(eB_L) \end{aligned}$$

is a well-defined poset isomorphism.

Note: $\overline{\mathcal{O}} \cap \pi_2^{-1}(eB_L) = \overline{\mathcal{O}} \cap X$.

Induction and sphericity

Theorem

Let $Z := Z_L(u, v)$ and $X := X_u^v$. Then the following are equivalent:

1. *Z is a spherical L-variety.*
2. *The fiber $X = X_u^v$ has an open B_L-toric variety.*

In particular, if X is T_L-toric, then Z is L-spherical.

Proof idea. Sphericity is preserved under parabolic induction: $L \times^{B_L} (-)$. $\square$

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Question

Is there a clean combinatorial characterization of the intervals $[u, v]$ where X_u^v is L-spherical.

Definition

A G-variety is said to be **wonderful** if it is smooth, complete, simple (unique closed G-orbit) and toroidal (no colors).

Wonderful varieties from Richardson varieties

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Corollary

Let $Z := Z_L(u, v)$ and $X := X_u^v$. Then the following are equivalent:

1. *Z is a wonderful L-variety.*
2. *The fiber $X = X_u^v$ is a smooth T_L-toric variety.*

Nearly Toric Varieties

Nearly-toric varieties

Recall that a toric variety if a normal T-variety X such that $c_T(X) = 0$.

Definition

We call a normal G-variety X a **nearly-toric G-variety** if it satisfies

1. $\mathrm{c}_T(X) = 1$,
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Example

Let $X := \mathrm{Skew}_4$, and X_{sing} denote the divisor defined by $X_{sing} := \{\det x = 0\} \cap X$. Then $\dim X = 6$ and $\dim X_{sing} = 5$. We have an action of $G := \mathbf{GL}_4$ on X and X_{sing}:

$$g \cdot A = gAg^\top \qquad (g \in G,\ A \in X).$$

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$$g \cdot A = gAg^{\top} \qquad (g \in G,\ A \in X).$$

There is a four dimensional $\mathbf{T}$-orbits in X_{sing} and $\dim X_{sing} = 5$. Hence, $c_{\mathbf{T}}(X_{sing}) = 1$. Since X_{sing} is spherical, it is a nearly-toric $\mathbf{GL}_4$-variety.

This is a non-example

Example

Let

$$\begin{aligned} G &:= \mathbf{SL}(3, \mathbb{C}) \\ B &:= \text{upper triangular Borel in } G \\ T &:= \text{the diagonal torus} \\ X &:= \mathbf{SL}(3, \mathbb{C})/\mathbf{SO}(3, \mathbb{C}) \end{aligned}$$

We consider the natural left multiplication action of G on X.

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We consider the natural left multiplication action of G on X.

Since $\dim X = 6$ and $\dim T = 2$, X is not a nearly-toric G-variety.

Conjecture on Nearly Toric Richardson Varieties in G/B

Huntch:

Let $X_u^v \subseteq G/B$ be a Richardson variety whose stabilizing Levi properly contains T. Then X_u^v is nearly toric for the L-action if and only if the Bruhat interval factors as

$$[u, v] \;\cong\; \mathcal{D} \times \mathcal{H},$$

where

- $\mathcal{D}$ is a lattice (toric part),
- $\mathcal{H} \cong S_3$.

Conjecture on Nearly Toric Richardson Varieties in Grassmannians

Huntch #1:

Assume that G/P is minuscule, or $G = G_2$. Let $u \leq v$ in W^P. The Richardson variety X_{vP}^{uP} is a toric variety for T if and only if $v = ud$, where d is a T-complexity 1 element (product of distinct simple reflections and a single braid relation).

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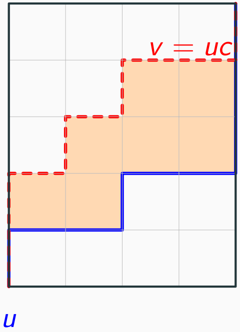
In type A, the following holds:

Huntch #2:

Let $X_{uP}^{vP} \subseteq \mathrm{Gr}(k, n)$ is nearly toric for the L-action if and only if there is a skew-shape region between the lattice paths u and v that contains one 2×2 square and all other skew-shapes between u and v are either a ribbon or contains one 2×2 square.

A nearly toric Richardson variety in $\mathrm{Gr}(6, 10)$.



one 2×2 block $\Rightarrow$ nearly toric

If Time Permits: Horospherical RVs

Horospherical Levi bands

Call Z *L-horospherical* if a maximal unipotent $U_L \leq L$ fixes a generic point of Z. Then with $Z \cong L \times^{B_L} X$ we have the following equivalent conditions:

- Z is horospherical
- U_L acts trivially on a dense open in X
- B_L-action on X factors through T_L generically.

In particular, if X is T_L-toric and its generic U_L-stabilizer is trivial, then Z is L-horospherical.

Question

Is there a clean description of the horospherical Richardson varieties.

THIS IS THE END, UNTIL NEXT TIME.

THANK YOU!