Peterson Schubert calculus

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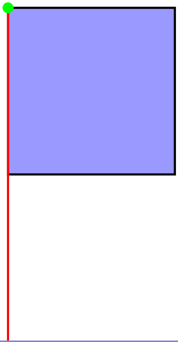
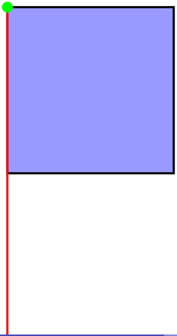


Flag varieties

$Sl(n, \mathbb{C})/B$, B upper triangular special linear matrices:
Identify with set $\{V_\bullet\}$ of nested vector spaces:

$$\{\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n\}$$

$$\dim(V_i) = i.$$



G complex semisimple Lie group, with Lie algebra $\mathfrak{g}$

B choice of Borel, with Lie algebra $\mathfrak{b}$, and B^- opposite Borel

$T = B \cap B^-$ a maximal torus, with Lie algebra $\mathfrak{t}$

G/B the *flag variety*.

Hessenberg Variety/Peterson Variety, Type A

For $h : \{1, \ldots, n\} \to \{1, \ldots, n\}$, $\quad i \le h(i) \le h(i+1)$ x $\in \mathfrak{g}$, the *Hessenberg variety* associated with x, h is

$$\mathcal{H}ess(\mathrm{x}, h) = \{V_\bullet : xV_i \subseteq V_{h(i)}\} \subset G/B.$$

Special case: $h_0 = (2, 3, \ldots, n-1, n, n)$ and

$$\mathrm{x} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 1 \ddots & 0 \\ \ddots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\mathbf{P} := \mathcal{H}ess(\mathrm{x}, h) = \{V_\bullet : xV_i \subseteq V_{i+1}\}$$

is the **Peterson variety** (in type A)

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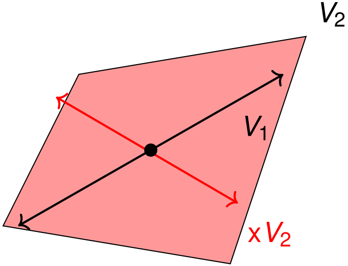

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On the cohomology

- There are distinguished classes $\sigma_w \in H^*(G/B)$ for $w \in W$ that form a basis $\{\sigma_w | w \in W\}$ of the cohomology ring. It is Poincaré dual to the basis $\{X_w = \overline{BwB/B}\}$.
- But each element σ_w is "the" Poincaré dual to the opposite Schubert variety $X^w = \overline{B^- wB/B}$.
- This holds equivaraiantly as well: the set $\{\sigma_w \in H_T^*(G/B), w \in W\}$ form a linear basis, as a module over $H_T^*(pt)$. The module structure comes from the map $G/B \to pt$ inducing a map (going the other way) on cohomology.
- The product in $H^*(G/B)$ or $H_T^*(G/B)$

$$\sigma_u \sigma_v = \sum c_{uv}^w \sigma_w$$

defines coefficients c_{uv}^w. They are **nonnegative** in $H^*(pt) \cong \mathbb{Z}$ or **Graham nonnegative** in $H_T^*(pt)$, identified with polynomials in the positive simple roots Δ.

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The Peterson variety

$$\mathbf{P} = \{V_\bullet : \mathrm{x}V_i \subset V_{i+1}\} \hookrightarrow^{\iota} Fl^*(\mathbb{C}^n),$$

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Example

Example ($n = 3$): Flags in **P** may be represented by elements:

$$\mathbf{P} = \begin{pmatrix} a & b & 1 \\ b & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cup \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & d & 1 \\ 0 & 1 & 0 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Cells $C_A = Bw_AB/B \cap \mathbf{P}$ are indexed by subsets $A \subset \Delta$, where w_A is the largest element of the Weyl group generated by the simple roots in A.
- **P** has a $S := \mathbb{C}^*$ action, by diagonal matrices w entries $(t^n, t^{n-1}, \dots, t)$.
- $\iota^* : H_S^*(G/B) \longrightarrow H_S^*(\mathbf{P})$ induced from inclusion ι is a surjection.

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Peterson Schubert calculus positivity, type A

For $A \subset \{1, 2, \ldots, n-1\}$, let $v_A = \prod_{j \in A} s_j$, Define $p_A = \iota^*(\sigma_{v_A})$, where $\sigma_{v_A} \in H_S^*(G/B)$ is the corresponding Schubert class.

Theorem (Tymoczko-Harada)

$H_S^*(\mathbf{P})$ has a linear basis $\{p_A : A \subset \{1, \ldots, n-1\}\}$, where p_A is the restriction of the Schubert class σ_{v_A} to $\mathbf{P}$.

Expand the product to get coefficients $b_{AB}^C \in \mathbb{Z}[t]$ defined by:

$$p_A p_B = \sum_C b_{AB}^C p_C.$$

Theorem (G-Gorbutt)

The coefficients b_{AB}^C are nonnegative monomials in t.

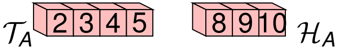
Theorem (G.-Mihalcea-Singh)

The class p_A is Poincaré dual to the varieties $\{\overline{Bw_A B/B \cap \mathbf{P}} | A \subset \{1, 2, \ldots, n-1\}\}$, where w_A is a longest element in the Weyl group associated with A.

Positive formula for Peterson Schubert calculus

$$p_A p_B = \sum_C b_{AB}^C p_C.$$

For any set $A \subset \{1, \ldots, n-1\}$ with $\mathcal{H}_A := max(A)$ and $\mathcal{T}_A := min(A)$.



Theorem (G.-Gorbutt)

Let $A, B, C \subseteq \{1, \ldots, n-1\}$ be nonempty consecutive subsets. If $C \supseteq A \cup B$ and $|C| \leq |A| + |B|$, then

$$b_{A,B}^C = d! \binom{\mathcal{H}_A - \mathcal{T}_B + 1}{d,\ \mathcal{T}_A - \mathcal{T}_C,\ \mathcal{H}_C - \mathcal{H}_B} \binom{\mathcal{H}_B - \mathcal{T}_A + 1}{d,\ \mathcal{T}_B - \mathcal{T}_C,\ \mathcal{H}_C - \mathcal{H}_A} t^d$$

for $d := |A| + |B| - |C|$.

On (forward) stability, type A

Let $\mathbf{P}_n$ denote the Peterson for $Fl(n)$. Then let $Fl(n) \hookrightarrow Fl(n+1)$ by

$$V_\bullet \mapsto (V_\bullet \oplus 0 \subset V_n \oplus \mathbb{C})$$

Let x_n be regular nilpotent $n \times n$ of Joran type (n). Then x_{n+1} has a copy of x_n inside, as the NW $n \times n$ entries. $\mathbf{P}_n \hookrightarrow \mathbf{P}_{n+1}$ under this inclusion.

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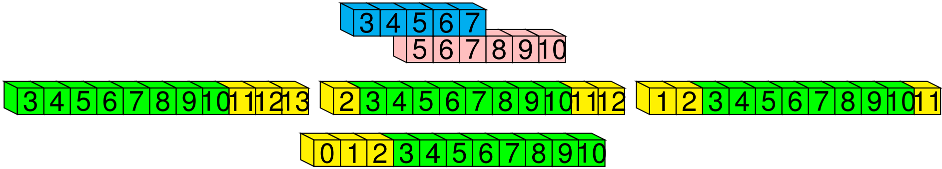
Forward stability: Lower bound for n: $max(\mathcal{H}_A, \mathcal{H}_B) + |A \cap B|$.



$$b^C_{A,B} = d!\binom{\mathcal{H}_A - \mathcal{T}_B + 1}{d,\ \mathcal{T}_A - \mathcal{T}_C,\ \mathcal{H}_C - \mathcal{H}_B}\binom{\mathcal{H}_B - \mathcal{T}_A + 1}{d,\ \mathcal{T}_B - \mathcal{T}_C,\ \mathcal{H}_C - \mathcal{H}_A} t^d,$$

WHERE $d = |A| + |B| - |C|$.

EXAMPLE

Let $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{1, 2, 3\}$, so $d = 1$.

$\mathcal{T}_A = 1$ $\mathcal{H}_A = 2$

1 2 3 4 5 A $\mathcal{T}_C = 1, \mathcal{H}_C = 3$

1 2 3 4 5 B

$\mathcal{T}_B = 2$ $\mathcal{H}_B = 3$

$$b^{123}_{12,23} = 1!\binom{2-2+1}{1,1-1,3-3}\binom{3-1+1}{1,2-1,3-2} t = \binom{1}{1}\frac{3!}{1!1!1!} = 6t.$$

Similarly, $b^{1234}_{12,23} = 3$. All other $b^C_{12,23} = 0$. Thus $p_{12}p_{23} = (6t)p_{123} + 3p_{1234}$.

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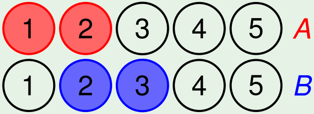

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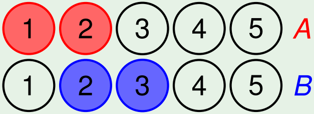

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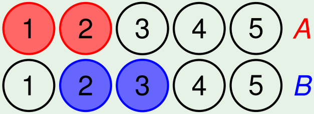

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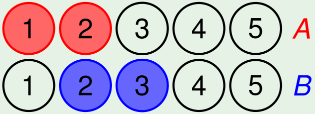

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Other Lie types

Let G be a complex algebraic Lie group, with Lie algebra $\mathfrak{g}$.
B a Borel with Lie algebra $\mathfrak{b}$.
$H \subset \mathfrak{g}$ is a **Hessenberg space** if $\mathfrak{b} \subset H$ and H is $\mathfrak{b}$-invariant: $[H, \mathfrak{b}] \subset H$.

Definition

For $\mathrm{x} \in \mathfrak{g}$ and H a Hessenberg space, the associated Hessenberg variety in G/B is

$$\mathcal{H}ess(\mathrm{x}, H) := \{gB \in G/B : Ad(g^{-1})\mathrm{x} \in H\}$$

Decompose

$$\mathfrak{g} = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}$$

Let

$$H = H_0 = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{-\alpha} \quad \text{and} \quad \mathrm{x} = \mathrm{n} = \sum_{\alpha \in \Delta} e_\alpha, \quad 0 \neq e_\alpha \in \mathfrak{g}_\alpha.$$

The **Peterson variety** is

$$\mathbf{P} := \mathcal{H}ess(\mathrm{n}, H_0)$$

Still have:

- T action on G/B. Restricts to a circle action S on $\mathbf{P}$.
- Schubert classes $\sigma_w \in H_T^*(G/B)$ forming a basis, and satisfying Graham positivity.
- Classes $\sigma_{v_A} \in H_T^*(G/B)$ for Coxeter elements v_A associated to each $A \subset \Delta$.

Theorem (Drellich)

There is a linear basis $\{p_A : A \subset \Delta\}$ of $H_S^*(\mathbf{P})$, where p_A is the restriction of the Schubert class σ_{v_A} to $\mathbf{P}$.

Theorem (G.-Mihalcea-Singh)

The coefficients b_{AB}^C are monomials with positive, integral coefficients.

Key idea

- The intersection of certain Schubert varieties with the Peterson is a smaller Peterson variety.
- There are enough of these intersections to get the whole cohomology.

W_A Weyl group generated by simple reflections from A.
w_A maximal element of W_A.

$$\mathbf{P}_A^o = \mathbf{P} \cap Bw_AB/B \qquad \mathbf{P}_A = \overline{\mathbf{P}_A^o} \subset \mathbf{P}$$

Theorem (G.-Mihalcea-Singh: Duality Theorem)

Let A, B be subsets of the set of simple roots and let $v_A \in W$ be a Coxeter element for A. Then

$$\langle \iota^* \sigma_{v_A}, [\mathbf{P}_B]_S \rangle = m(v_A)\delta_{A,B},$$

where $m(v_A)$ is the multiplicity of the (unique) point of $X^{v_A} \cap \mathbf{P}_A$.

Change of basis $\{p_A = \iota^* \sigma_{v_A}\}$ to $\{\Omega_A = \frac{p_A}{m(v_A)}\}$

Equivariant Monk Rule, all Lie types

For $\alpha \in \Phi$ (any root), $\alpha^\vee$ denotes the coroot corresponding to α, i.e.

$$\alpha^\vee = 2(\alpha|-)/(\alpha|\alpha)$$

For $A \subset \Delta$, C_A denotes the Cartan matrix of A (entries are $\langle \beta^\vee, \alpha \rangle$)
$f_A := \det(C_A)$ (the **connection index** of A).
ϖ_α (resp. $\varpi_\alpha^\vee$) for the fundamental weights (resp. coweights) for Δ.
ϖ_α^A fundamental weights, such that $\langle \beta^\vee, \varpi_\alpha^A \rangle = \delta_{\alpha\beta}$ for all $\beta \in A$.
$\varpi_\alpha^{A\vee}$ fundamental coweights dual to the roots $\alpha \in A$.

Theorem (G-Singh)

For $A \subset \Delta$, let $\Omega_A = \prod_{\beta \in A} p_{\{\beta\}}$. For any $\alpha \in \Delta$,

$$\Omega_\alpha \Omega_A = \begin{cases} \Omega_{A \cup \{\alpha\}} & \text{if } \alpha \notin A, \\ 2 \left\langle \rho_A^\vee, \varpi_\alpha^A \right\rangle t \Omega_A + \sum\limits_{\substack{\gamma \in \Delta \setminus A \\ B = A \cup \{\gamma\}}} \frac{f_B}{f_A} \left\langle \varpi_\gamma^{B\vee}, \varpi_\alpha^B \right\rangle \Omega_B & \text{if } \alpha \in A. \end{cases}$$

where $\rho_A^\vee = \frac{1}{2} \sum_{\alpha \in \Phi_A^+} \alpha^\vee$

Toward K-theory

K-theory of G/B

- 2 “natural bases” rather than 1 for $K(G/B)$. 4 natural bases rather than 2 for $K_T(G/B)$.
- For the appropriate basis $\{\mathcal{O}_w : w \in W\}$ of $K_T(G/B)$, positivity is an alternating sum:
$$\mathcal{O}_u\mathcal{O}_v = \sum_{w\in W} c_{uv}^w \mathcal{O}_w$$
where $(-1)^{\ell(w)-\ell(u)-\ell(v)} c_{uv}^w$ is positive in variables $e^{\alpha_i} - 1$ for $\alpha_i \in \Delta$.

What about for Peterson varieties?

- There is an analogous basis $\{\mathcal{O}_A\}$, $A \subset \{1, \ldots, n-1\}$
- Calculations suggest positivity
- Some standard results about vanishing of higher sheaf cohomology for G/B do not hold for **P**.

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- Some standard results about vanishing of higher sheaf cohomology for G/B do not hold for $\mathbf{P}$.

THANK YOU!

Have you seen this identity?

$m, n, w, x, y, z \in \mathbb{Z}$ with $w + x = y + z$

Theorem (G-Gorbutt)

$$\binom{w+m}{w}\binom{y+m}{x}\binom{w+n}{y}\binom{z+n}{z} = \sum_{\substack{0\le i\le m\\ 0\le j\le n}} \binom{w+m+j}{i,\ j,\ m-i,\ x-i-j,\ z-x+j,\ y-x+i}\binom{w+i+n}{n-j}.$$

(We have a bijections of sets that proves this, but maybe there's a nicer proof?)