Then we have unconditional stability for
$\theta = 1$ and $f = 0$ (for (BE) method)

1.2.2

2nd case $f \neq 0$

r.h.s.
$$\int_\Omega f^{n+1} v_h \rightsquigarrow \int_\Omega f^{n+1} u_h^{n+1} \underset{\substack{\text{c.c.}\\ \text{cauchy}}}{\leq} \|f^{n+1}\|_{L^2(\Omega)} \|u_h^{n+1}\|_{L^2(\Omega)}$$

$$\underset{\text{young}}{\leq} \alpha \|u_h^{n+1}\|_V^2 + \frac{1}{4\alpha} \|f^{n+1}\|_{L^2(\Omega)}^2$$

$$\text{r.h.s} \Rightarrow \|u_h^{n+1}\|_{L^2(\Omega)}^2 + 2\alpha\Delta t \|u_h^{n+1}\|_V^2 \leq \|u_h^n\|_{L^2(\Omega)}^2 + \alpha\Delta t \|u_h^{n+1}\|_V^2 + \frac{\Delta t}{4\alpha} \|f^{n+1}\|_{L^2(\Omega)}^2$$

$$\Rightarrow \underbrace{\|u_h^{n+1}\|_{L^2(\Omega)}^2 - \|u_h^n\|_{L^2(\Omega)}^2}_{\sum_{n=0}^{m}(\;\cdot\;)\ \text{(telescopic Sum)}} + \underbrace{\alpha\Delta t \|u_h^{n+1}\|_V^2}_{\sum_{n=0}^{m}(\;\cdot\;)} \leq \underbrace{\frac{\Delta t}{4\alpha} \|f^{n+1}\|_{L^2(\Omega)}^2}_{\sum_{n=0}^{m}(\;\cdot\;)}$$

$$\hookrightarrow \sum_{n=0}^{m} (a^{n+1} - a^n) = a^{m+1} - a^0$$

$$\Rightarrow \|u_h^{m+1}\|_{L^2(\Omega)}^2 + \alpha \sum_{n=0}^{m} \Delta t \|u_h^{n+1}\|_V^2 \leq \underbrace{\|u_{0h}\|_{L^2(\Omega)}^2 + \frac{1}{4\alpha} \sum_{n=0}^{m} \Delta t \|f^{n+1}\|_{L^2(\Omega)}^2}_{\text{data}}$$

Have stability