

# wright\_fisher-hints

March 12, 2016

This is an ipython notebook. Lectures about Python, useful both for beginners and experts, can be found at <http://scipy-lectures.github.io>.

I recommend installing the [Anaconda](#) distribution. Make sure not to pay for it! Click Anaconda Academic License; it should be free for those with academic e-mail addresses.

Open the notebook by (1) copying this file into a directory, (2) in that directory typing ipython notebook and (3) selecting the notebook.

In this exercise, we will build a Wright-Fisher simulation model, which will be the basis of most of our simulation efforts.

## 1 Wright-Fisher model

### 1.1 Motivation

Population genetics seeks to describe and understand patterns of genetic diversity found in natural and artificial populations. In previous exercises, we've focused on the description part, measuring statistical differences among populations. We were able to interpret these differences in qualitative terms. Some populations seemed more related to each other, some were more distinct, and we could tell stories about that: maybe the more distinct groups were separated for a longer time. However, if we want to really understand the data, we need to go beyond describing it. We need to build models for the data that make predictions and can be falsified. The Hardy-Weinberg equilibrium model was a good start in that direction, but it only predicts a specific aspect of the data: the relationship between allele frequency and heterozygosity. It told us nothing about the number or frequency of variants in a population, or the amount of differences across populations. It didn't tell us how things change over time.

In this exercise, we will study the evolution of allele frequencies in a finite population. This notebook implements a very simple model of allele frequency evolution that we will use over and over in this course. To build an actual evolutionary model, we need to specify a number of parameters: the population size, the structure of the genome (e.g., the number of chromosomes, its ploidy), the mode of reproduction of the population (e.g., sexual vs asexual), the structure of the population, the distribution of offspring by individual, how new alleles are created, how alleles are transmitted from parent to offspring. We also have to decide on the starting point in our model, the ancestral population from which we will consider evolution. Since this is our first model, we'd like to keep everything as simple as possible:

- Population size: constant,  $N$
- Structure of the genome: one single haploid chromosome of length 1 base pair.
- Mode of reproduction: asexual
- Mutation process: No mutation
- Transmission: Asexual transmission (clonal reproduction)
- Distribution of offspring: ?

It was straightforward to come up with the simplest parameters so far, but here we need to think a bit more. Since we have a haploid population of constant size, the average number of offspring per individual must be one. If every individual has exactly one offspring, the population will never change and the model will be useless, so we need to let parents have different number of offspring.

If we let each parent independently pick a random number of offspring, the population size could change a little bit at every generation. That's not really a big deal, since real populations sizes do fluctuate, but we'd like to be able to control those fluctuations for two reasons. First, if we happen to know the size of the population we want to model, we don't want to waste time simulating populations of the wrong size. Second, if we want to model a population for a long period of time, the size fluctuations will add up and the population will either go extinct or grow uncontrollably.

If parents vary in their number of offspring, how can they synchronize to produce exactly one offspring, on average? One way would be to add feedback, reducing the mean number of offspring when the population size increases, and increasing it when the population size decreases. That could work, but there are many parameters to fix, and even there we're not immune to the occasional population extinction.

There are a few elegant but slightly strange ways of solving this problem by keeping the population size at an exact, predefined  $N$ . First, you can imagine that each parent produces very many offspring, but that only  $N$  offspring from the entire population are allowed to survive, and these are selected at random among all offspring. You can also turn the table and imagine that each of the allowed  $N$  offspring "picks" a parent at random. If the idea of children picking their parents is confusing, you can imagine that there are  $N$  child-bearing permits available, and each permit is offered to a randomly-chosen parent.

Interestingly, these approaches are exactly equivalent, in that they make the exact same predictions about the number of surviving offsprings per parent. **Take the time to convince yourself that this is the case.**

This sampling approach defines the Wright-Fisher model. We can add many features to this model, such as selection, recombination, mutation, and so forth, but as long as you have this discrete generations and random selection of parents, you're within the Wright-Fisher model.

I'll just mention one alternative, called the Moran Model, which involves replacing a single, randomly selected individual by the genotype of another, randomly selected individual, and repeating this  $N$  times per generation. The Moran model is not exactly equivalent to the Wright-Fisher model, and there are cases where it is more convenient. But we'll stick to Wright-Fisher for this notebook.

#### Optional Mathematical exercises

1. What is the distribution of offspring number per individual in the Wright-Fisher model?  
A. For each individual, a specific offspring will either be theirs, or it won't (i.e. discrete success/failure). Whether or not an individual has one offspring has no affect on whether the individual has another (i.e. independent events). There are  $N$  total offspring, and the chance of having each offspring will thus be  $1/N$ , so this will be a binomial distribution with  $n = N$ ,  $p = 1/N$ .
2. Convince yourself that, for large enough populations, this distribution is approximately Poisson with mean 1.  
A. Poisson distribution approximates binomial distributions as  $n$  goes to infinity, as long as  $np$  remains constant.  $np = N \cdot 1/N = 1$  no matter what the population size is, so Poisson will be a good approximation for large enough populations. Mean of a Poisson distribution is equal to  $np$ , which is indeed 1 as we just showed.

## 1.2 Libraries

We'll need the following python plotting libraries.

```
In [82]: ### 1
%matplotlib inline
#If you run into errors with %matplotlib, check that your version of ipython is >=1.0
import numpy as np #numpy defines useful functions for manipulating arrays and matrices.
import matplotlib.pyplot as plt #matplotlib is plotting library
```

### 1.3 Implementation

We have specified almost everything we needed in the model, except for the initial state of the population. We will suppose that the single site has two alleles labeled 0 and 1. We need to specify the number of individuals  $n_{\text{Ind}}$  in the initial population and the proportion  $p_0$  of 1 alleles

```
In [83]: ### 2
         p0 = 0.1 # initial proportion of "1" alleles
         nInd = 100 # initial population size (number of individuals)
```

Now we need to create an initial population with the given number of individuals, and the appropriate proportion of 1 alleles. We'll store the population as a `np.array` called "initial\_population", of length  $n_{\text{Ind}}$  containing 0s and 1s.

```
In [84]: ### 3
         # Initialize a population of length nInd with only 0 alleles.
         initial_population = np.zeros(nInd)

         # Set the first p0*nInd alleles to 1.

         for i in range(int(p0*nInd)):
             initial_population[i] = 1

         # For added realism, shuffle the population so that the ones are distributed across the sample
         # You can use np.random.shuffle to do this.

         np.random.shuffle(initial_population)
```

To take finite samples from this population, we can use the `np.random.choice` function. When taking a sample from a population, we can pick each sample only once—the "replace=False" option below tells us that we don't replace the sampled individual in the population before drawing a new one. Read the `np.random.choice` documentation for more detail!

```
In [85]: ### 4
         sample_size = 10
         np.random.choice(initial_population, sample_size, replace=False )
```

```
Out[85]: array([ 0.,  0.,  0.,  0.,  0.,  1.,  0.,  0.,  0.,  0.]
```

When we take repeated samples from the same population, we can find very different numbers of alternate alleles—we'll have to take this into account when looking at real data!

#### Optional Mathematical exercise

1 - What is the distribution of the number of alternate alleles if we sample  $s$  individuals in a population with allele frequency  $p$ ?

A - As we are not replacing individuals from the population between choices, the sampling will follow a hypergeometric distribution, with  $N$  (pop size) =  $n_{\text{Ind}}$ ,  $K$  (# of 1 alleles) =  $n_{\text{Ind}} * p$ ,  $n$  (# picks) =  $s$ . This certainly seems to be the case from the graph that follows.

I generated a bunch of samples below, and compare the resulting histogram to plausible probability distributions so that you can pick the one that fits best. If you don't do the math problems, read a bit about the best-fitting one and check that it makes sense.

```
In [87]: ### 5
         import scipy
         from scipy import stats

         iterations = 10000 # the number of times to draw.
```

```

sample_size = 50 # the size of each sample
alt_counts = [] # number of alternate alleles (i.e., 1's) for each draw

for i in range(iterations):
    sample=np.random.choice(initial_population, sample_size, replace=False)
    # get the number of alt alleles
    alt_counts.append(sample.sum())

# plot a histogram of sampled values
plt.hist(alt_counts, sample_size + 1, range=(-0.5, sample_size + 1 - 0.5), label="random sample")
plt.xlabel("number of alt alleles")
plt.ylabel("counts")

# Compare this to some discrete distributions
x_range = range(sample_size + 1) # all the possible values

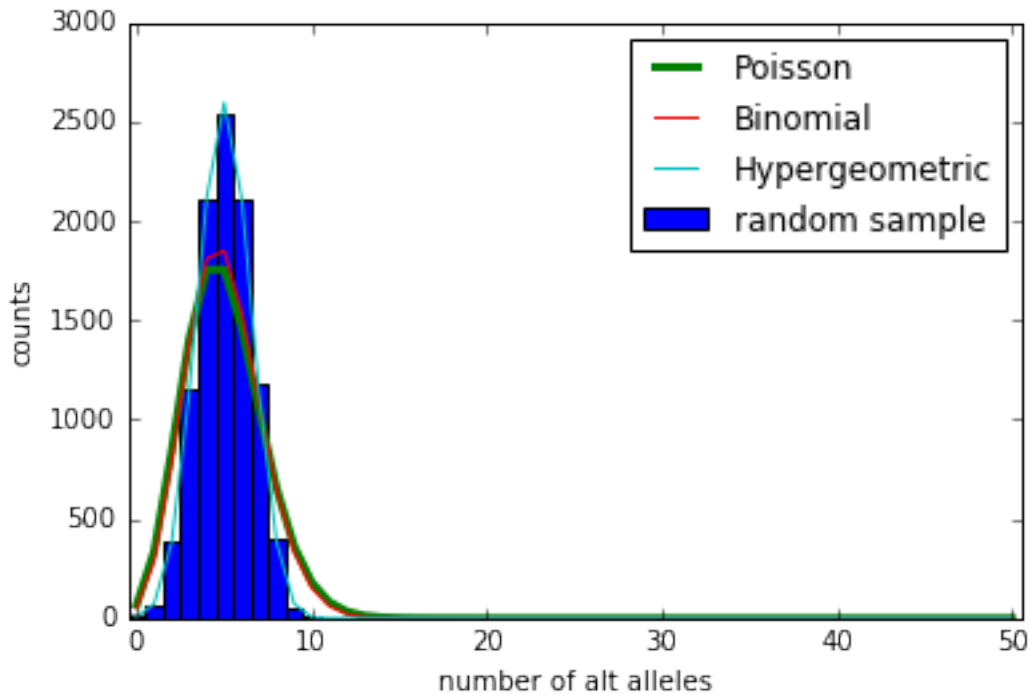
p = np.sum(initial_population) * 1. / len(initial_population) # initial fraction of alt's

# poisson with mean sample_size * p
y_poisson = stats.poisson.pmf(x_range, sample_size*p) * iterations
# binomial with probability p and sample_size draws
y_binom = stats.binom.pmf(x_range, sample_size,p) * iterations
# hypergeometric draw of sample_size from population of size len(initial_population)
# with np.sum(initial_population) ones.
y_hypergeom = stats.hypergeom.pmf(x_range, len(initial_population), np.sum(initial_population)
    * iterations

plt.plot(x_range, y_poisson, label="Poisson", lw=3)
plt.plot(x_range, y_binom, label="Binomial")
plt.plot(x_range, y_hypergeom, label="Hypergeometric")
plt.xlim(-0.5, sample_size + 0.5)
plt.legend()

```

Out[87]: <matplotlib.legend.Legend at 0x1f1950dcc50>



Now comes the time to code up the Wright-Fisher model. Remember that there were two ways of thinking about Wright-Fisher reproduction:

- 1- We generate a very large number of offspring for each parent, and then we take a sample from this very large population.
- 2- Each offspring picks a parent at random.

In 2-, each parent can be chosen multiple times. This is equivalent to taking a sample from the previous generation, but with replacement. Convince yourself that this is true. This is not optional! If you are not convinced, try programming it both ways, or ask questions.

Now code a function that takes in a parental population as an array (such as “initial\_population”, above), and returns an offspring population.

```
In [88]: """ 6
def generation(pop):
    """Takes in a list or array describing an asexual parental population.
    Return a descendant population according to Wright-Fisher dynamics with constant size"""
    nInd = len(pop) #number of individuals. We could use the global definition of nInd,

    #but it's better to use the information that is passed to the function
    #now generate the offspring population.
    return np.random.choice(initial_population, nInd)

generation(initial_population)

Out[88]: array([ 0.,  0.,  0.,  1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
                  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
                  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
                  0.,  1.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,  0.,
```

```

0., 0., 0., 0., 0., 0., 0., 1., 0., 0., 0., 0., 0.,
0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.,
0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.,
0., 0., 0., 0., 0., 0., 0., 0., 0.]

```

As for regular sampling, we get a different number of ones every time we run the “generation” function. Here again, I generated a bunch of samples just to get an idea of how much variation there is, and overlaid some plausible distribution. Which one fits best? Does it make sense to you?

In [89]: `### 7`

```

nsample = 10000 # the number of samples to draw.
alt_counts = [] # number of alternate alleles (i.e., 1's) for each draw

for i in range(nsample):
    offspring = generation(initial_population)
    alt_counts.append(offspring.sum())

hist = plt.hist(alt_counts, len(initial_population)+1, range=(0-0.5, len(initial_population)+0.5))
plt.xlabel("number of alt alleles")
plt.ylabel("counts")

#Here I just check that the initial population is still a list of length nInd
assert nInd==len(initial_population), "initial_population doesn't have the same length as nInd"

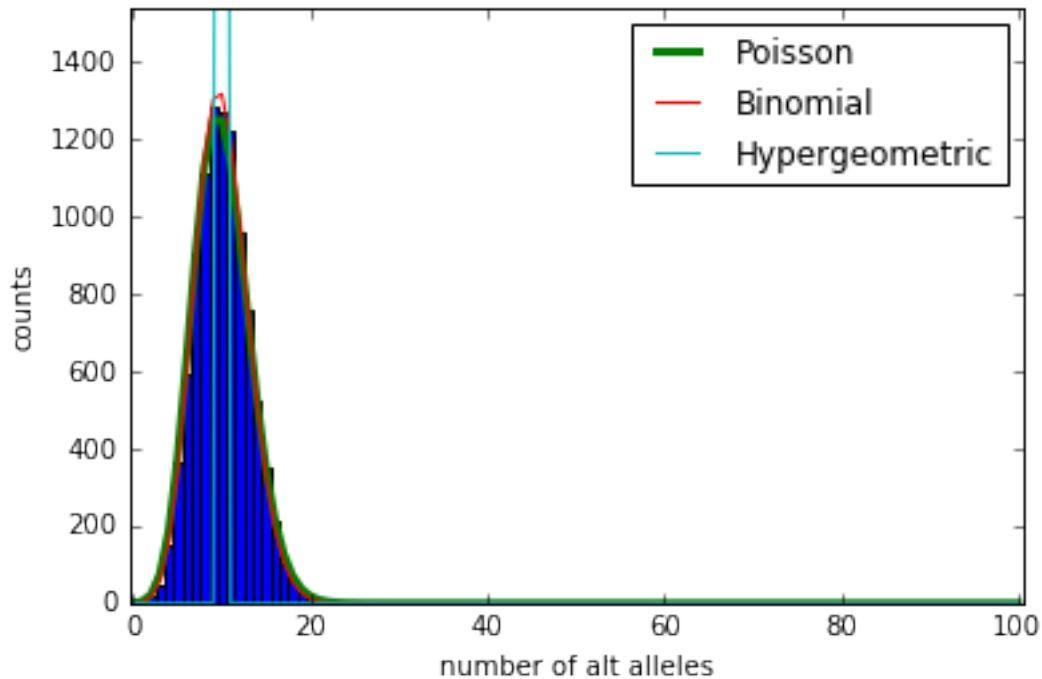
x_range=range(nInd+1) #all the possible values
p=np.sum(initial_population)*1./nInd #the initial frequency

#Compare this to some distributions
y_poisson=stats.poisson.pmf(x_range, nInd*p) * nsample
y_binom=stats.binom.pmf(x_range, nInd, p) * nsample
y_hypergeom=stats.hypergeom.pmf(x_range, nInd, np.sum(initial_population), nInd) * nsample

plt.plot(x_range, y_poisson, label="Poisson",lw=3)
plt.plot(x_range, y_binom, label="Binomial")
plt.plot(x_range, y_hypergeom, label="Hypergeometric")
plt.xlim(-0.5, nInd+0.5)
plt.ylim(0, 1.2*max(hist[0]))
plt.legend()

```

Out[89]: `<matplotlib.legend.Legend at 0x1f1952ace48>`



Now we are ready to evolve our population for 100 generations. Let's store the entire genotypes for each generation in a list.

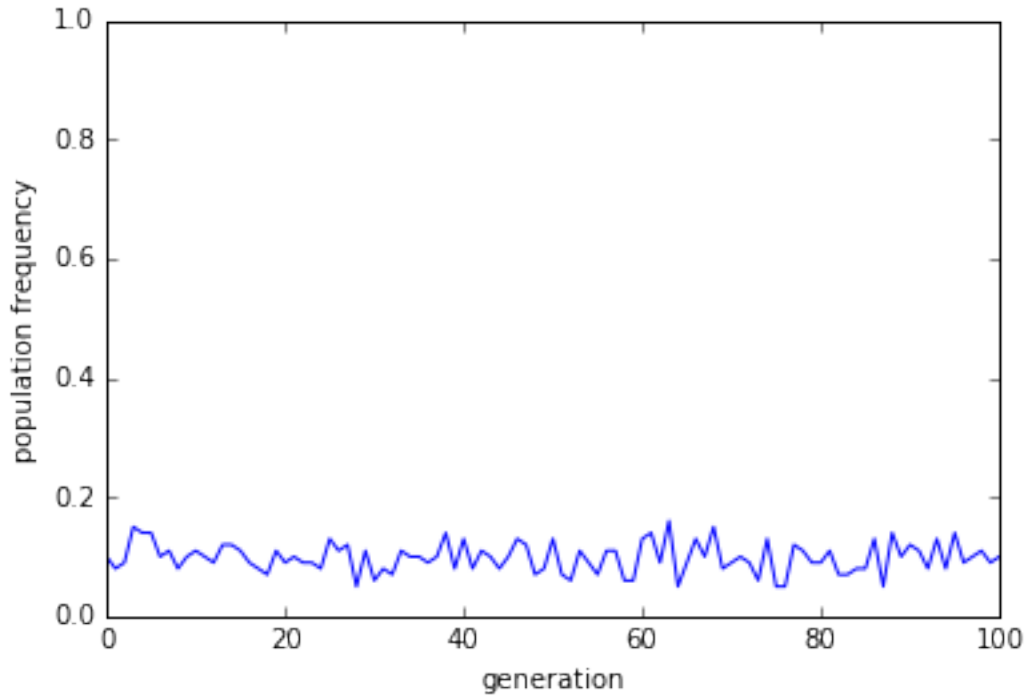
```
In [90]: ### 8
nGen = 100 # number of generations to simulate
history = [initial_population] # a container list for our simulations. It will contain the po
                                     # state after generations 0 to nGen

for i in range(nGen):
    # evolve the population for one generation, and append the result to history.
    history.append(generation(history[i]))
history = np.array(history) # convert the list into an array for convenient manipulation later
```

Now we want to look at the results. Let's compute the allele frequency at each generation and plot that as a function of time.

```
In [91]: ### 9
#compute the allele frequency at each generation.
#freqs should be a list or array of frequencies, with one frequency per generation.
#history is a np array and has two methods that can help you here: sum, and mean.
#Mean is probably the best bet here.
freqs = np.array([np.mean(history[i]) for i in range(len(history))])
plt.plot(freqs)
plt.axis([0, 100, 0, 1]);#define the plotting range
plt.xlabel("generation")
plt.ylabel("population frequency")
```

```
Out[91]: <matplotlib.text.Text at 0x1f1952f8f98>
```



Now we would like to experiment a bit with the tools that we have developed. Before we do this, we will organize them a bit better, using a Python “class” and object-oriented programming. We have defined above variables that describe a population (such as the population size `nInd`, and the ancestral frequency `p0`). We have also defined functions that apply to a population, such as “generation”. A class is used to keep track of the relation between objects, variables, and functions.

If you are not familiar with classes and are having issues, have a look at [this tutorial](#).

```
In [92]: ### 10
class population:
    """
    Initialization call:

    population(nInd,p0)
    requires a number of individuals nInd and an initial frequency p0

    Variables:
    nInd: The number of individuals
    p0: the initial allele frequency
    initial_population: an array of nInd alleles
    history: a list of genotypes for each generation
    traj: an allele frequency trajectory; only defined if getTraj is run.
    Methods:
    generation: returns the offspring from the current population, which is also the last one
    evolve: evolves the population for a fixed number of generations, stores results to self.h
    getTraj: calculates the allele frequency history for the population
    plotTraj: plots the allele frequency history for the population

    """
    def __init__(self, nInd, p0):
```



```

"""initialize the population. nInd is the number of individuals. p0 is the initial allele frequency.
__init__ is a method that, when run, creates a "population" class and defines some of its attributes.
Here we define this __init__ method but we don't run it, so there is no "population" object yet.
In the meantime, we'll refer to the eventual population object as "self".
We'll eventually create a population by stating something like
pop = population(nInd,p0)
This will call the __init__ function and pass a "population" object to it in lieu of self
"""

self.nInd = nInd
self.p0 = p0
#initialize the population
self.initial_population = np.zeros(self.nInd)
self.initial_population[0 : int(self.p0*self.nInd)] = 1
np.random.shuffle(self.initial_population)
#history is a container that records the genotype at each generation.
#we'll update this list
self.history = [self.initial_population]

def generation(self):
    """class methods need "self" as an argument in their definition to know that they apply to the class
    The class structure
    gives you a more readable way of calling this function: If we have a population "pop",
    the population as the first argument. Putting the object name upfront often makes code more readable.
    Takes the last element of the history.
    Return a descendant population according to Wright-Fisher dynamics with constant size
    """
    return np.random.choice(self.history[-1], len(self.history[-1]))

def evolve(self,nGen):
    """
    This is a method with one additional argument, the number of generations nGen.
    To call this method on a population "pop", we'd call pop.evolve(nGen).
    This function can be called many times on the same population.
    pop.evolve(2)
    pop.evolve(3)
    would evolve the population for 5 generations.
    For each step, we make a call to the function generation() and append the population to the history.
    """
    for i in range(nGen):
        self.history.append(self.generation())
    self.getTraj()

def getTraj(self):
    """
    calculates the allele frequency history for the population
    """
    history_array = np.array(self.history)
    self.traj = history_array.mean(axis=1)
    return self.traj

def plotTraj(self,ax="auto"):
    """
    plots the allele frequency history for the population
    """

```

```

plt.plot(self.traj)
if ax=="auto":
    plt.axis([0, len(self.history), 0, 1])
else:
    plt.axis(ax)

```

## 2 Exploration

### 2.1 Drift

We can now define multiple populations, and let them evolve from the same initial conditions.

```

In [93]: ### 11
         nInd = 100
         nGen = 30
         nRuns = 10
         p0 = 0.3
         # Create a list of length nRuns containing initial populations
         # with initial frequency p0 and nInd individuals.
         pops = [population(nInd, p0) for i in range(nRuns)]

```

Evolve each population for nGen generations. Because each population object has it's own internal storage for the history of the population, we don't have to worry about recording anything.

```

In [94]: ### 12
         for pop in pops:
             pop.evolve(nGen);

```

Now plot each population trajectory, using the built-in method from the population class.

```

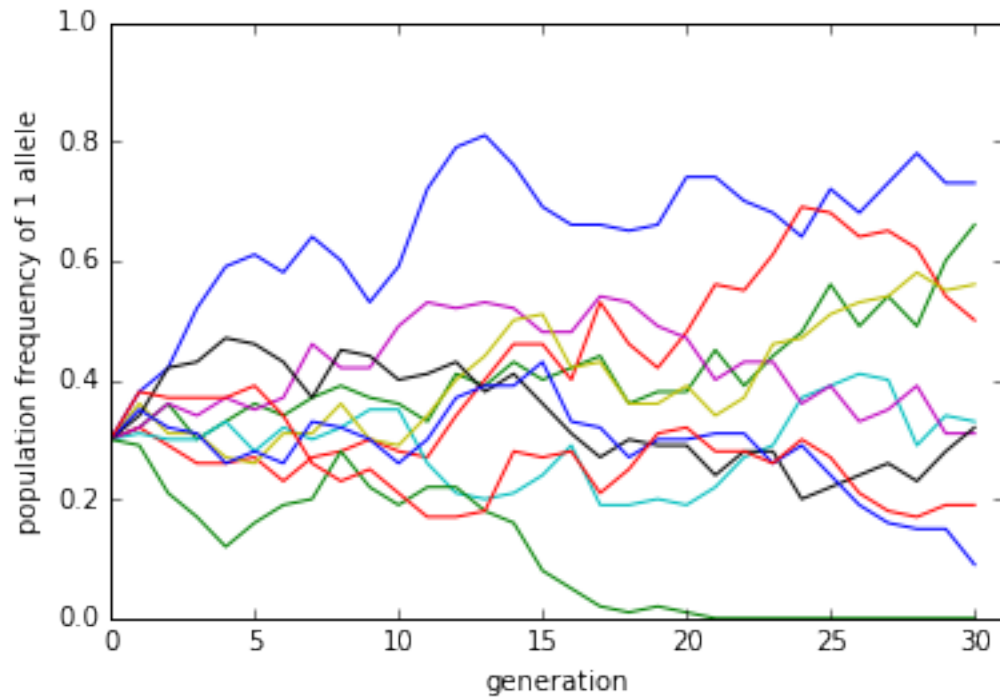
In [95]: ### 13
         for pop in pops:
             pop.plotTraj();
         plt.xlabel("generation")
         plt.ylabel("population frequency of 1 allele")

```

```

Out[95]: <matplotlib.text.Text at 0x1f194ecb438>

```

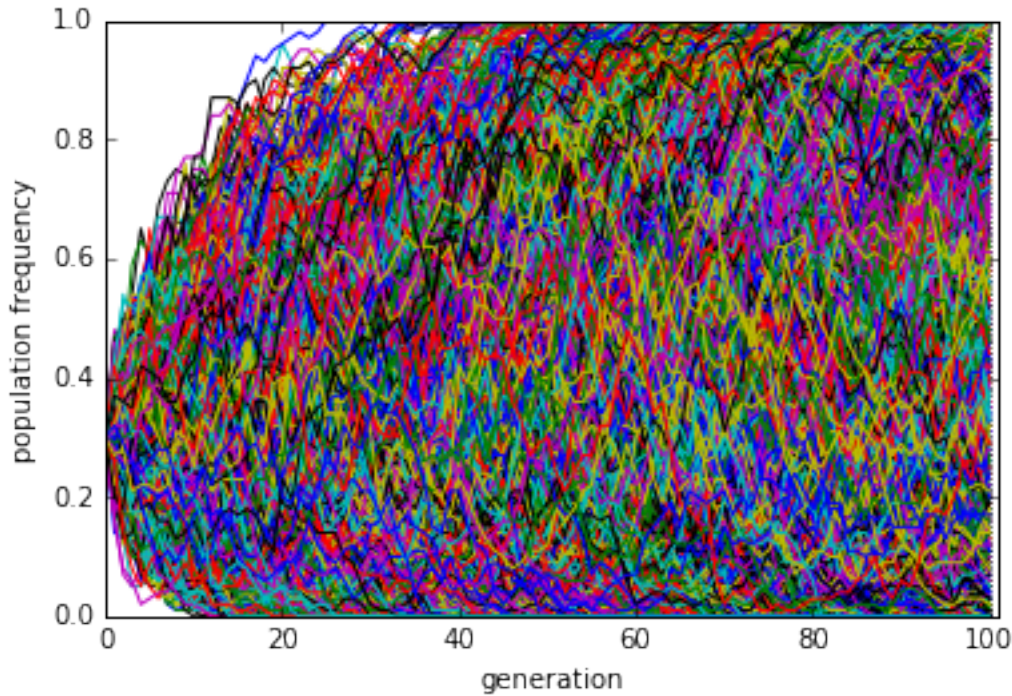


Now that we know it works, let's explore this a bit numerically. Try to get at least 1000 runs, it'll make graphs prettier down the road.

```
In [96]: ### 14
         nInd = 100
         nGen = 100
         nRuns = 2500
         p0 = 0.3
         pops = [population(nInd, p0) for i in range(nRuns)]
         for pop in pops:
             pop.evolve(nGen)
             pop.plotTraj()

         plt.xlabel("generation")
         plt.ylabel("population frequency")
```

```
Out[96]: <matplotlib.text.Text at 0x1f1934f9898>
```



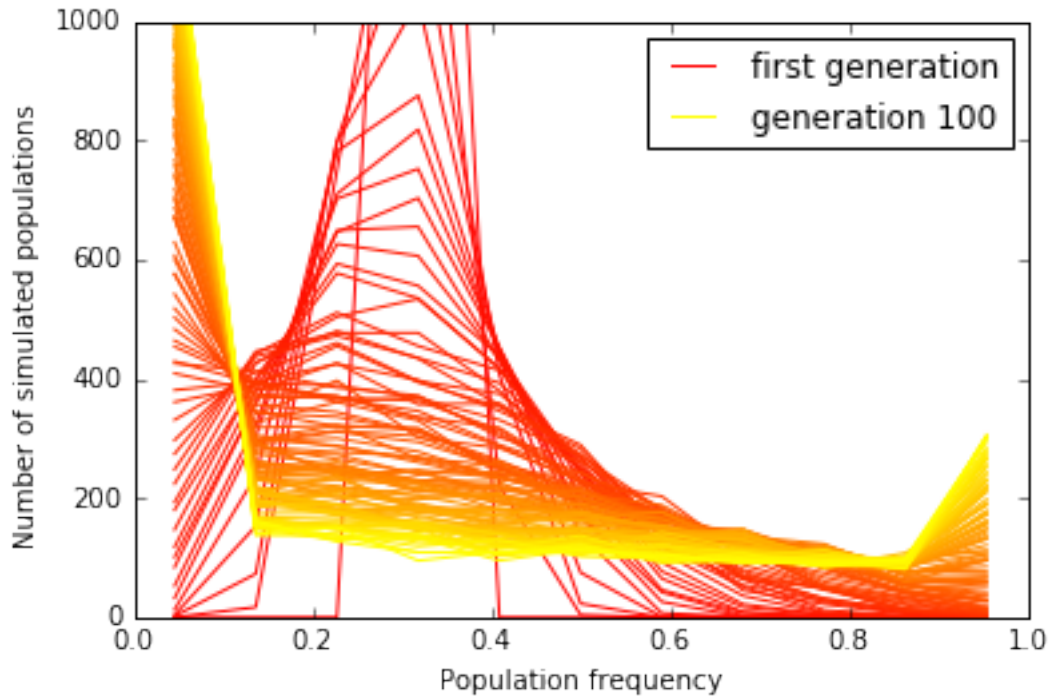
So there is a lot of randomness in there, but if you run it multiple times you should see that there is some regularity in how fast the allele frequencies depart from the initial values. To investigate this, calculate and plot the distribution of frequency at each generation.

```
In [97]: ### 15
def frequencyAtGen(generation_number, populations, nBins=11):
    """calculates the allele frequency at generation genN for a list of populations pops.
    Generates a histogram of the observed values"""
    counts_per_bin, bin_edge_positions = np.histogram([pop.traj[generation_number] for pop in populations],
                                                       bins=nBins)
    bin_centers = np.array([(bin_edge_positions[i+1] + bin_edge_positions[i]) / 2 for i in range(len(bin_edge_positions) - 1)])
    return bin_centers, counts_per_bin # Return the data from which we will generate the plot

In [98]: ### 16
nBins = 11 # The number of frequency bins that we will use to partition the data.
for i in range(nGen+1):

    bin_centers, counts_per_bin = frequencyAtGen(i, pops);
    if i==0:
        plt.plot(bin_centers, counts_per_bin, color=plt.cm.autumn(i*1./nGen), label="first generation",
                  # color with which to plot the first generation
    elif i==nGen:
        plt.plot(bin_centers, counts_per_bin, color=plt.cm.autumn(i*1./nGen), label="generation nGen",
    else:
        plt.plot(bin_centers, counts_per_bin, color=plt.cm.autumn(i*1./nGen))
plt.legend()
plt.axis([0., 1., 0, 1000])
plt.xlabel("Population frequency")
plt.ylabel("Number of simulated populations ")
```

Out[98]: <matplotlib.text.Text at 0x1f1a07dbb70>



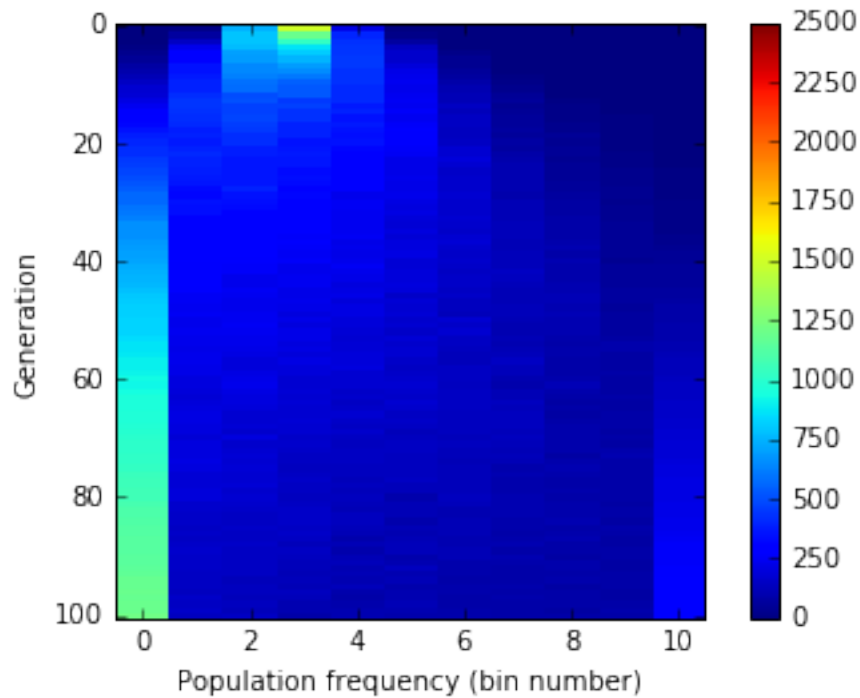
There are three important observations here:

- 1-Frequencies tend to spread out over time
- 2-Over time, there are more and more populations at frequencies 0 and 1. (Why?)
- 3-Apart from the 0 and 1 bins, the distribution becomes entirely flat.

A few alternate ways of visualizing the data: first a density map

```
In [99]: ### 17
nBins = 11
sfs_by_generation = np.array([frequencyAtGen(i, pops, nBins=nBins)[1] for i in range(0, nGen+1)
bins = frequencyAtGen(i, pops, nBins=nBins)[0]
plt.imshow(sfs_by_generation, aspect=nBins*1./nGen, interpolation='nearest')
plt.xlabel("Population frequency (bin number)")
plt.ylabel("Generation")
plt.colorbar()
```

Out[99]: <matplotlib.colorbar.Colorbar at 0x1f1a09e7cf8>



Then a 3D histogram, unfortunately a bit slow to compute.

```
In [100]: ### 18
from mpl_toolkits.mplot3d import Axes3D

fig = plt.figure(figsize=(10,10))
ax = fig.add_subplot(111, projection='3d', elev=90)
xedges = bins
yedges = np.arange(nGen+1)

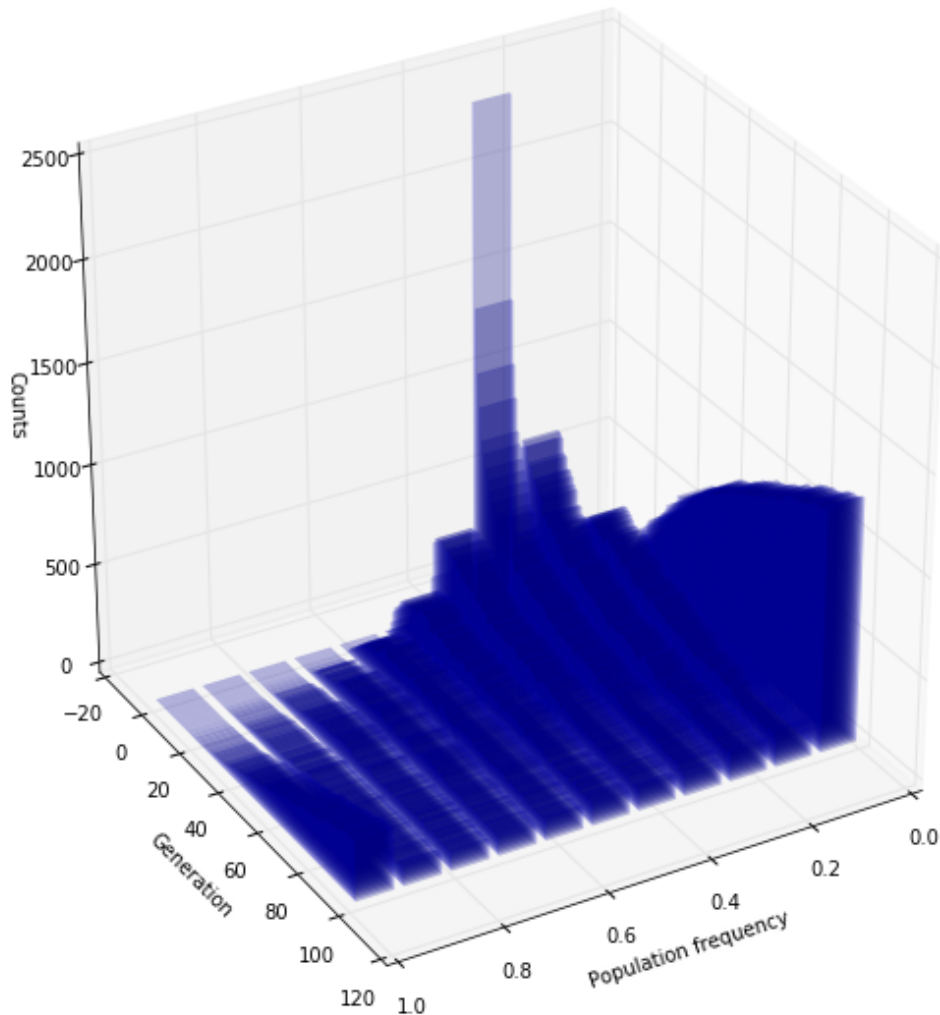
xpos, ypos = np.meshgrid(xedges-.4/nBins, yedges-0.5)

xpos = xpos.flatten()
ypos = ypos.flatten()
zpos = 0 * ypos
dx = .8 / nBins
dy = 1
dz = sfs_by_generation.flatten()

ax.bar3d(xpos, ypos, zpos, dx, dy, dz, color='b', edgecolor='none', alpha=0.15)

ax.view_init(elev=30., azimuth=60)
ax.set_xlabel("Population frequency")
ax.set_ylabel("Generation")
ax.set_zlabel("Counts")

plt.show()
```



Now let's dig into the effect of population size in a bit more detail. Consider the change in frequency after just one generation:

Mathematical exercise (NOT optional):

- What is the expected distribution of allele frequencies after one generation, if they start at frequency  $p$  in a population of size  $N$ ?

In this case, as we have two alleles, allele 1 can be picked with a probability of  $p$ , while allele 0 can be picked with a probability  $1 - p$ . Because of the model we are using, this choice happens independently for each of the  $n$  individuals in the next population. Thus, the occurrence of an allele can be modeled as a binomial distribution. Binomial distributions have an expected value of  $np$ , but we want frequency, not total occurrence, so we divide by  $n$  to get an expected value of  $p$  in the next generation as well.

- What is the variance of this distribution? (Look it up if you don't know—wikipedia is useful for that kind of stuff)

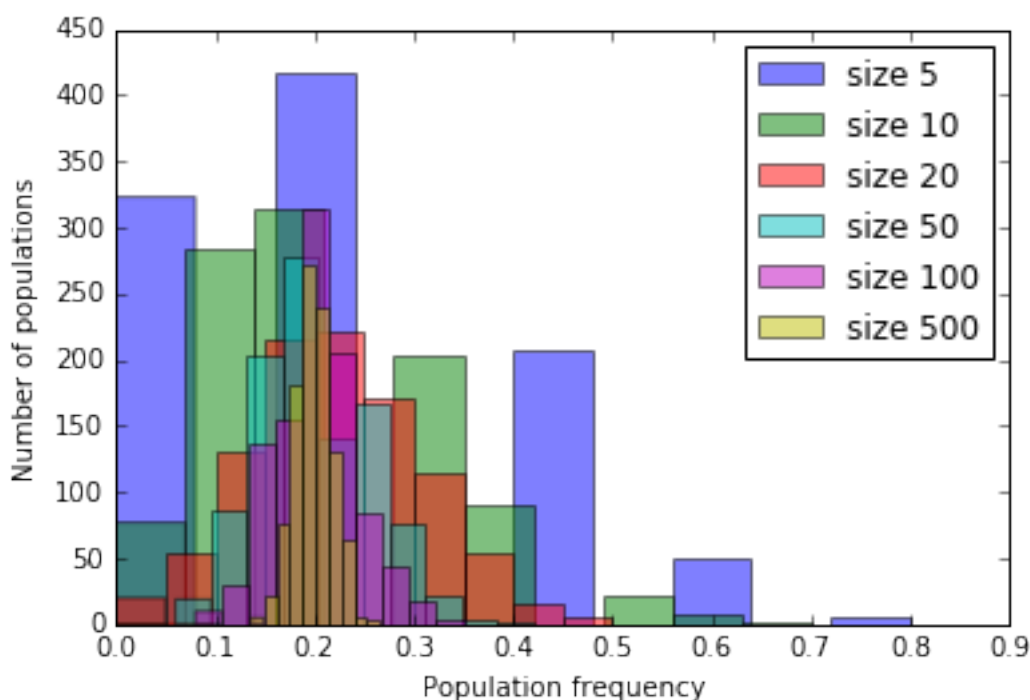
Once again, we can model allele occurrences as a binomial distribution. This distribution has a variance of  $np(1 - p)$ . However, we aren't looking at total allele occurrence, just frequency. In this case, since

variance involves squares of the observed variable, we divide by  $n^2$  to get a frequency variance of  $p(1-p)/n$ .

To study the effect of population size on the rate of change in allele frequencies, plot the distribution of allele frequencies after nGen generation. Start with nGen=1 generation.

```
In [104]: ### 19
histograms = []
variances = []
p0 = 0.2
sizes = [5, 10, 20, 50, 100, 500]
nGen = 1
for nInd in sizes:
    pops=[population(nInd,p0) for i in range(1000)]
    [pop.evolve(nGen) for pop in pops]
    sample = [pop.getTraj()[-1] for pop in pops]
    variances.append(np.var(sample))
    histograms.append(plt.hist(sample, alpha=0.5, label="size %d" % (nInd,) ))
plt.xlabel("Population frequency")
plt.ylabel("Number of populations")
plt.legend()
```

```
Out[104]: <matplotlib.legend.Legend at 0x1f1a0ceaeb8>
```



So how does population size affect the change in allele frequency after one generation? Can you give a specific function describing the relationship between variance and population size?

$$Var = \frac{p(1-p)}{POP\ SIZE}$$



You can get this relationship from the math exercise above, or just try to guess it from the data. If you want to try to guess, start by plotting the variances (stored in “variances”) against the population sizes (stored in “sizes”). Then you can either try to plot different functional forms to see if they fit, or you can change the way you plot the data such that it looks like a straight line. If you do the latter, make sure you update the labels!

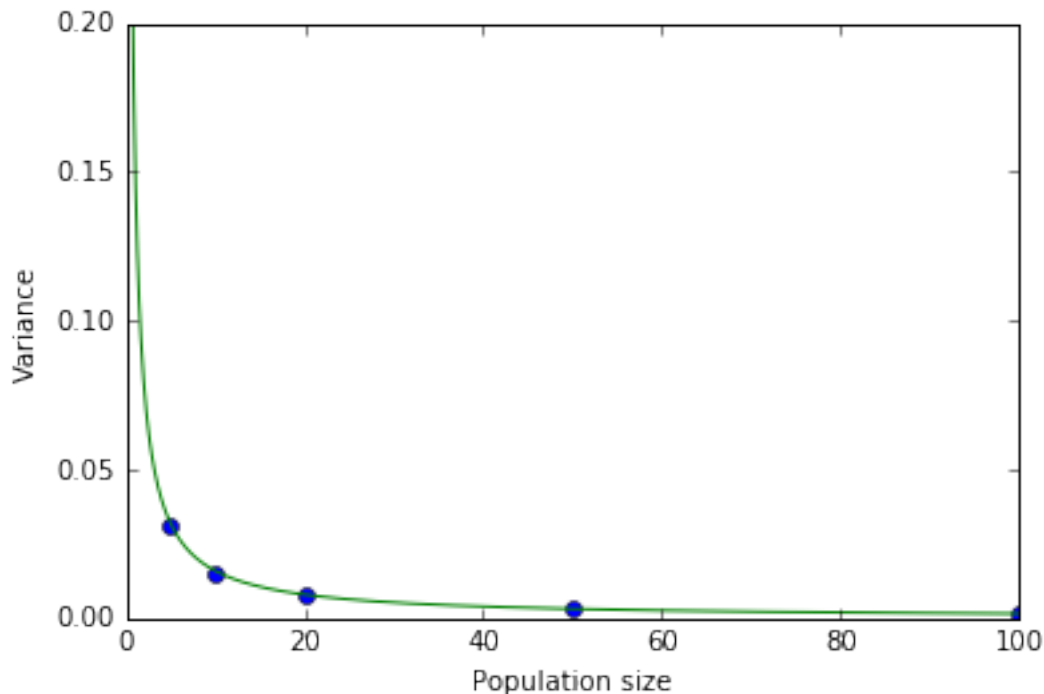
Here I’m giving you a bit more room to explore—there are multiple ways to get there.

In [106]: `### 20`

```
plt.plot(np.array(sizes), variances, 'o', label="simulation") #this is a starting point, but
# Your theory.
xvals = np.arange(0.1, 100., 0.1)
plt.plot(xvals, (p0*(1-p0))/xvals)
plt.axis([0,100,0,0.2])

plt.xlabel("Population size")
plt.ylabel("Variance")
```

Out[106]: `<matplotlib.text.Text at 0x1f1a0abe0b8>`



For short times, the expected changes in allele frequencies,  $Var[E[(x - x_0)^2]]$ , are larger for smaller population, a crucial result of population genetics.

The next question is: How does the rate of change in allele frequency depend on the initial allele frequency? We can plot the histograms of allele frequency as before:

In [108]: `### 21`

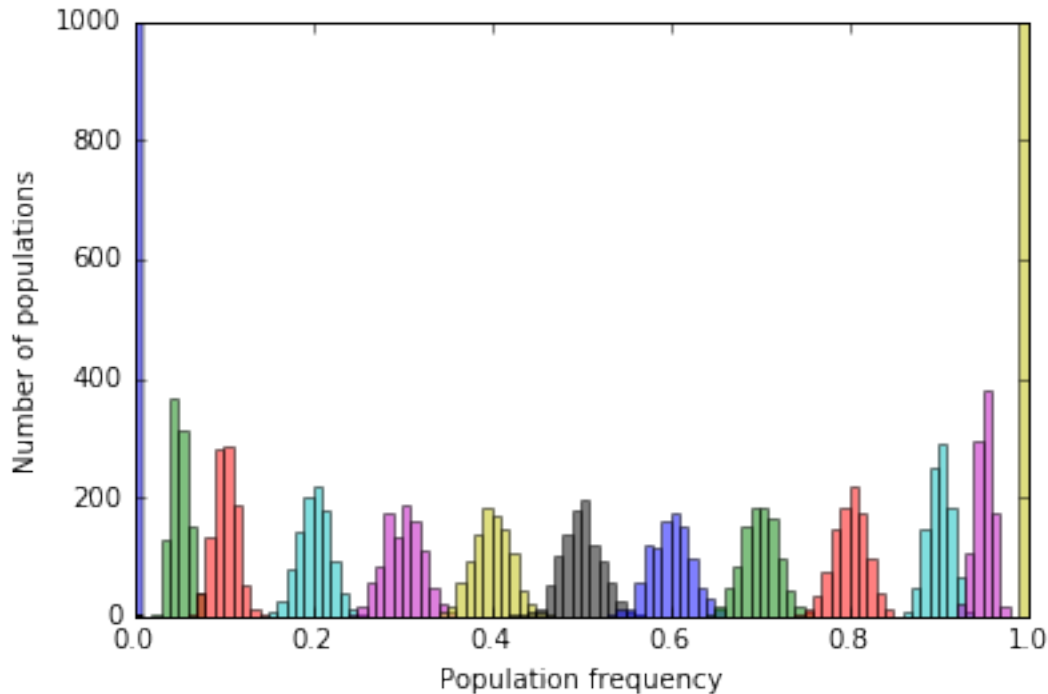
```
histograms = []
variances = []
p0_list = np.array([0, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, .6, .7, .8, 0.9, 0.95, 1])
nGen = 1
```

```

for p0 in p0_list:
    pops = [population(nInd, p0) for i in range(1000)]
    [pop.evolve(nGen) for pop in pops]
    sample = [pop.getTraj()[-1] for pop in pops]
    variances.append(np.var(sample))
    histograms.append(plt.hist(sample, 100, alpha=0.5, range=(0,1)))
plt.xlabel("Population frequency")
plt.ylabel("Number of populations")

```

Out[108]: <matplotlib.text.Text at 0x1f1a1154518>



Find the relationship between initial frequency and variance. Again, this can be from the math exercise above, from looking it up, but you can also just try to guess it from the data—it's a simple function.

Tips for guessing:

First, make the plot of variance vs frequency below

Then consider how much variance there is for  $p_0=0$  and  $p_0=1$ .

Can you come up with a simple function that has these values? Hint: it's simpler than a trigonometric function.

$$Var = \frac{p(1-p)}{POP\ SIZE}$$

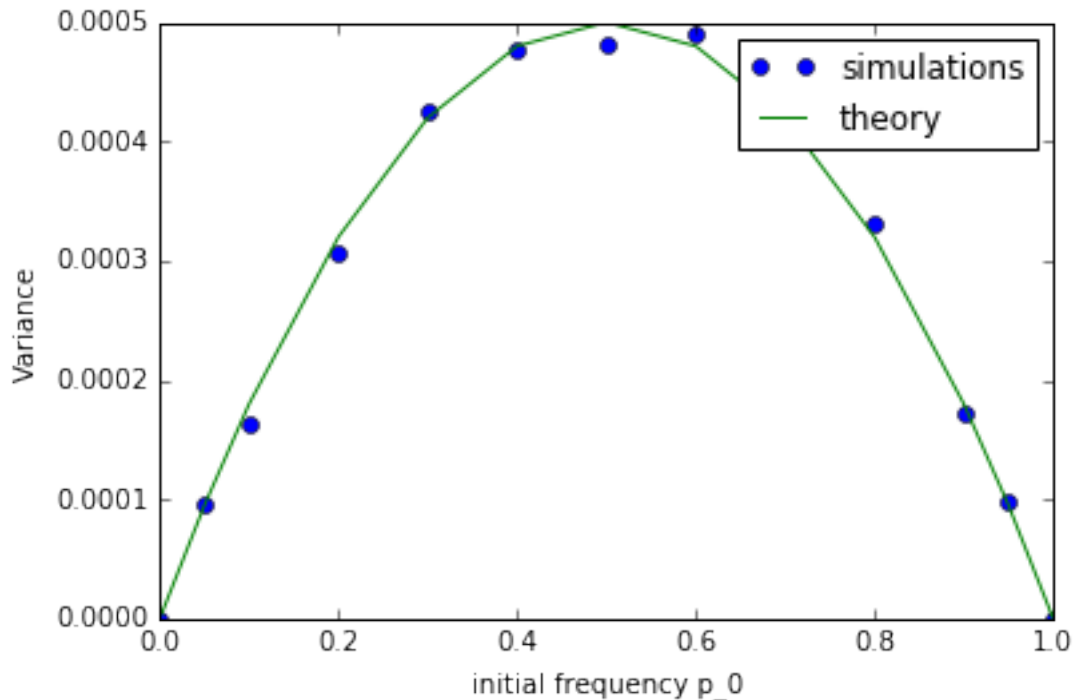
In [109]: `### 22`

```

plt.plot(np.array(p0_list), variances, 'o', label="simulations")
plt.plot(np.array(p0_list), np.array(p0_list)*(1-np.array(p0_list))/nInd, '-', label="theory")
plt.ylabel("Variance")
plt.xlabel(r"initial frequency p_0")
plt.legend()

```

Out[109]: <matplotlib.legend.Legend at 0x1f1a377abe0>



Can you explain why this function is symmetrical around  $p_0 = 0.5$ ?  
 Simple observation of the equation implies it will be symmetrical here, as it is a quadratic with zeroes at  $p = 0$  and  $p = 1$ . We could also take a derivative w.r.t  $p$  and solve for 0 to get  $(1 - p) - p = 0$ , so  $p = 0.5$  is a critical point.  
 We can also see the function must be symmetric by intuition. The function doesn't care if  $p$  is defined as the frequency of allele 0 or allele 1. Therefore it must work the same for both cases, which will only be true if it is symmetric around  $p = 0.5$ .

## 2.2 Mutation

New mutations enter the population in a single individual, and therefore begin their journey at frequency  $\frac{1}{N}$ . Numerically estimate the probability that such a new mutation will eventually fix (i.e., the probability that the mutation reaches frequency 1) in the population, if no subsequent mutations occur.

In [114]: `### 23`

```
nInd = 10
nGen = 100
nRuns = 2000
#enter the initial allele frequency for new mutations
p0 = 1/nInd
pops = [population(nInd, p0) for i in range(nRuns)]
[pop.evolve(nGen) for pop in pops];
```

We can plot the number of populations at each frequency, as we did above.

In [116]: `### 24`

```
nBins = nInd + 1 # We want to have bins for 0,1,2,...,N copies of the allele.
proportion_fixed = [] # fixation rate
```

```

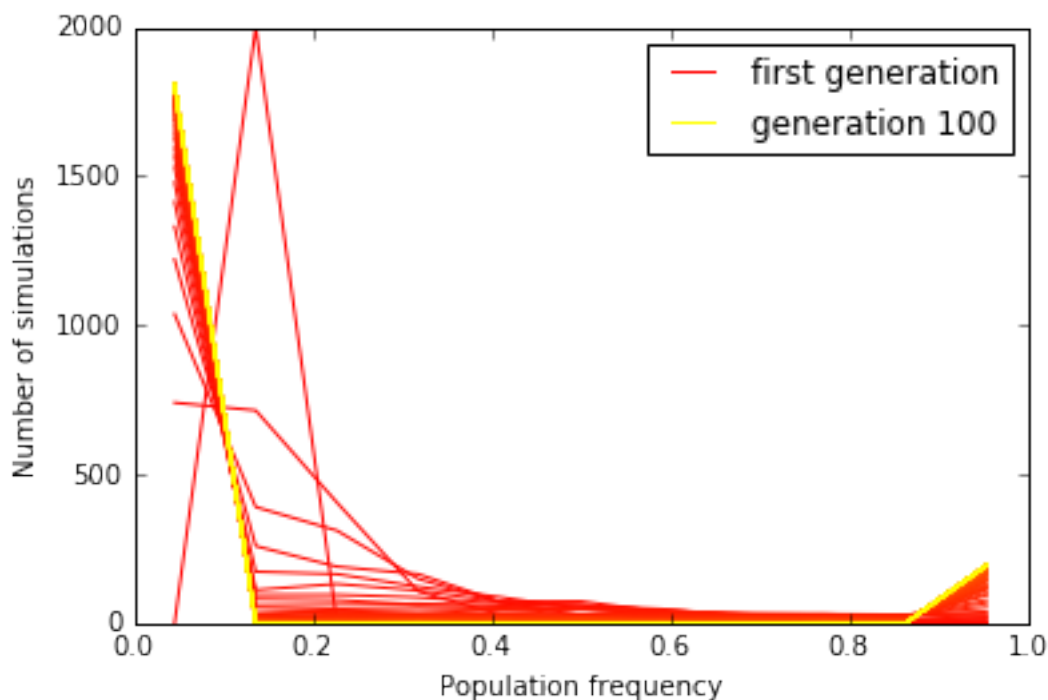
for i in range(nGen+1):
    x,y = frequencyAtGen(i, pops);
    if i==0:
        plt.plot(x, y, color=plt.cm.autumn(i*1./nGen), label="first generation") # cm.autumn
                                                                                   # color wit
    elif i==nGen:
        plt.plot(x, y, color=plt.cm.autumn(i*1./nGen), label="generation %d"% (nGen,) )
    else:
        plt.plot(x, y, color=plt.cm.autumn(i*1./nGen))

    #we'll consider a population "fixed" if it is in the highest-frequency bin. It's
    #an approximation, but not a bad one if the number of bins is comparable to the
    #population size.
    proportion_fixed.append((i, y[-1]*1./nRuns))

plt.legend()
plt.xlabel("Population frequency")
plt.ylabel("Number of simulations")

```

Out[116]: <matplotlib.text.Text at 0x1f191201198>



Here you should find that most mutations fix at zero frequency—only a small proportion survives.

What is the probability that a new mutation fixes in the population?—solve this problem both mathematically and numerically.

After running the simulation above a few times, I found that around 200 of the initial 2000 populations became fixed for the mutation. This gives a frequency of 1/10. Since  $n_{\text{Ind}}$  was equal to 10, this frequency is thus equal to the initial frequency of the mutation within the population.

The mathematical part requires almost no calculation or mathematical knowledge, once you think about it in the right way.

Your mathematical solution:

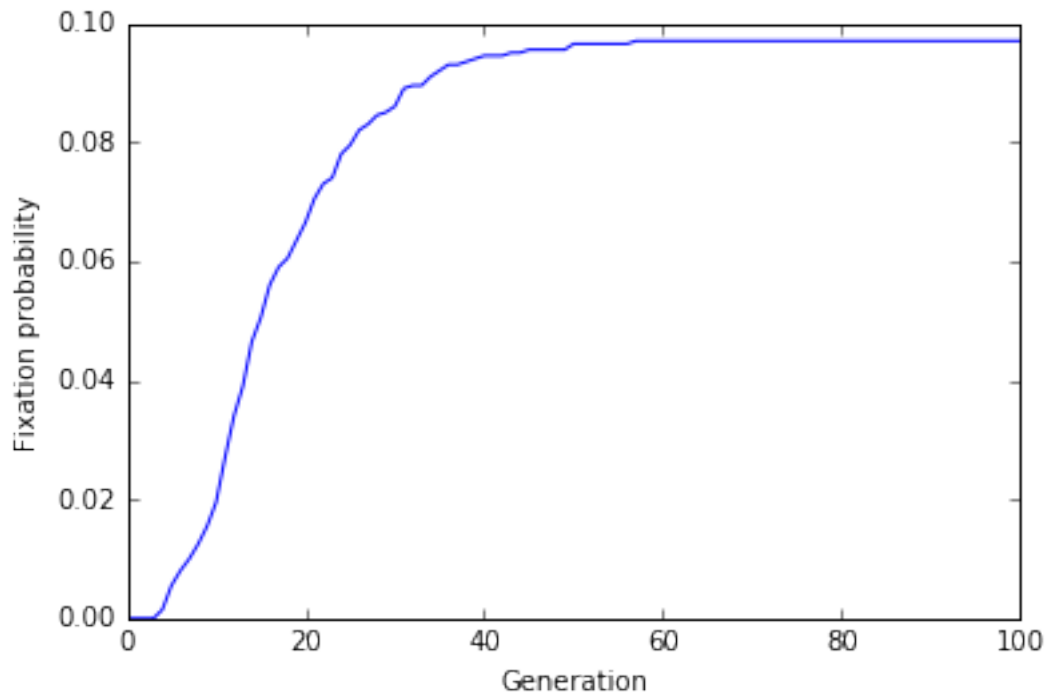
After many, many generations, all individuals will only have one common ancestor within the original parent population. This is because at each generation, some individuals will have no offspring, leading to certain lineages dying off. Because of this, and because of the way we have defined reproduction in this scenario, at some point all individuals will have the same allele as the original common ancestor. All of the individuals in the original population have an equal probability of becoming the common ancestor for this future generation. Only one of the individuals carries the mutation, so the probability it is picked will be 1 out of the population size ( $n_{Ind}$ ). Therefore:

$$P(\text{Fixation}) = 1 / n_{Ind}$$

For the computational part, note that we already computed the proportion of fixed alleles vs time in the “proportion\_fixed” variable. Make sure that the numerical value agrees with the mathematical expectation.

```
In [118]: ### 25
          proportion_fixed = np.array(proportion_fixed)
          plt.plot(proportion_fixed[:,0], proportion_fixed[:,1])
          plt.xlabel("Generation")
          plt.ylabel("Fixation probability")
```

```
Out[118]: <matplotlib.text.Text at 0x1f1a53b56d8>
```



### 3 Summary

Some important things that we’ve seen in this notebook: \* The Wright-Fisher model. Despite its simplicity, it is the basic building block of a large fraction of population genetics. \* In finite populations, sampling fluctuations are an important driver of allele frequency change. \* These sampling fluctuations cause larger frequency changes in smaller populations. \* These fluctuations mean that alleles eventually fix one way or another – We need new mutations to maintain diversity within a population. \* For neutral alleles, the probability of new mutations fixing in the population is inversely proportional to the population size

## 4 Something to think about.

We'll get to selection, recombination, and linkage in the next exercises. In the meantime, you can think about the following:

- Verify numerically that different reproductive models gives similar behavior. You may look up the Moran Model, or come up with your own evolutionary model.
- How much time will it take for a typical new mutation to reach fixation for different population sizes?
- If you add a constant influx of new mutations, how will the distribution of allele frequency look like at any given point in time?

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