Chapter 13

Lab 7: Epidemic Model

In this lab, we use optimal control techniques to find a vaccination schedule for an epidemic disease. A micro-parasitic infectious disease is considered. Permanent immunity to the disease can be achieved through natural recovery or immunization. Immunity is not passed on during birth, so that everyone is born susceptible. Our goal is to minimize the number of infectious persons and the overall cost of the vaccine during a fixed time period.

To model the dynamics of the disease in a population, we use a standard SEIR (or SEIRN) model. Let $S(t)$, $I(t)$, and $R(t)$ represent number of susceptible, infectious, and recovered (immune) individuals at time t. The model allows for an incubation period for the disease inside its host, where an infected person remains latent for some time before becoming infectious, creating an exposed class. Let $E(t)$ be the number of exposed or latent individuals at time t. Let $N(t)$ be the total number of people in the population, so that $N(t) = S(t) + E(t) + I(t) + R(t)$.

Let $u(t)$, the control, be the percentage of susceptible individuals being vaccinated per unit of time. As vaccination of the entire susceptible population is impossible, we bound the control with $0 \leq u(t) \leq 0.9$. Let b be the natural exponential birth rate of the population and d the natural exponential death rate. The incidence of the disease is described by the term $cS(t)I(t)$. The parameter e is the rate at which the exposed individuals become infectious, and g is the rate at which infectious individuals recover. Therefore, $\frac{1}{e}$ is the mean latent period, and $\frac{1}{g}$ is the mean infectious period before recovery, if recovery occurs. The death rate due to the disease in infectious individuals is a. The optimal control problem is as follows,

$$\min_u \int_0^T AI(t) + u(t)^2 \, dt$$

$$\begin{aligned}
\text{subject to} \quad & S'(t) = bN(t) - dS(t) - cS(t)I(t) - u(t)S(t),\ S(0) = S_0 \geq 0, \\
& E'(t) = cS(t)I(t) - (e+d)E(t),\ E(0) = E_0 \geq 0, \\
& I'(t) = eE(t) - (g+a+d)I(t),\ I(0) = I_0 \geq 0, \\
& R'(t) = gI(t) - dR(i) + u(t)S(t),\ R(0) = R_0 \geq 0, \\
& N'(t) = (b-d)N(t) - aI(t),\ N(0) = N_0, \\
& 0 \leq u(t) \leq 0.9.
\end{aligned}$$

See [93] for results on a similar problem using incidence term $\frac{cSI}{N}$. The left-hand side of the differential equations gives us the name of the type of model (SEIR). A flow chart of the model is given in Figure 13.1. Also, observe the variable R appears only in the R' differential equation. So, the other variables do not depend on R, and we can ignore R when we solve the optimality system. Specifically, as you see in the code, only S, E, I, and N are solved forward in time, and the four associated adjoints are solved backward in time. Once convergence has been achieved, R^* is solved using its differential equation. Refer back to Example 12.3.

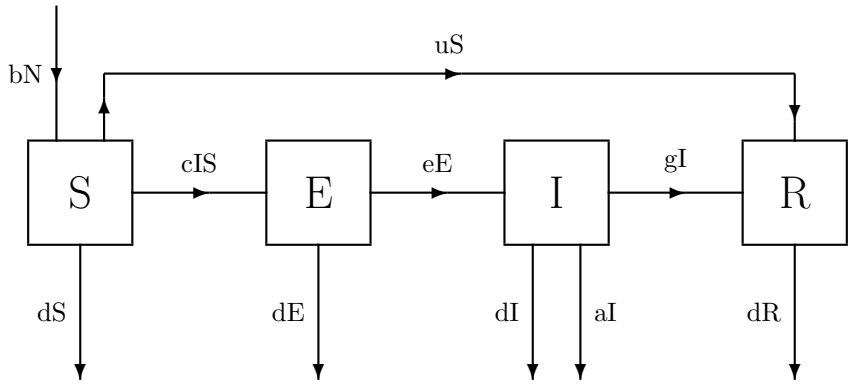

FIGURE 13.1: This is a flow chart for our model. The four boxes represent the four groups of individuals. The arrows show the movement between groups, and into and out of the population.

Type *lab7* at the prompt and press enter. Start with the values

$$\boxed{\begin{array}{cccc} b = 0.525 & d = 0.5 & c = 0.0001 & e = 0.5 \\ g = 0.1 & a = 0.2 & S_0 = 1000 & E_0 = 100 \\ I_0 = 50 & R_0 = 15 & A = 0.1 & T = 20 \end{array}} \quad . \tag{13.1}$$

This is a simulation of a disease with a low incidence measure. The optimal vaccination schedule is one of containment. An early round of vaccinations is used to shield the susceptible population from the initially significant exposed and infectious populations. This, combined with the low incidence level, results in the virtual end of disease spread. Exposed and infectious populations quickly disappear (through death and recovery). By year 5, the disease is essentially wiped out and vaccination ends. The small number of people who do carry the disease pose little threat of spreading it. The recovered group increases rapidly at first due to vaccinations, but slowly disappears when vaccination ends. By the end of the time period, susceptible people make up

almost the entire population. Notice, the susceptible population decreases slightly at the beginning of the time interval. Here, the vaccination rate is greater than the overall growth. See Figure 13.2 to compare this simulation with that of no vaccination.

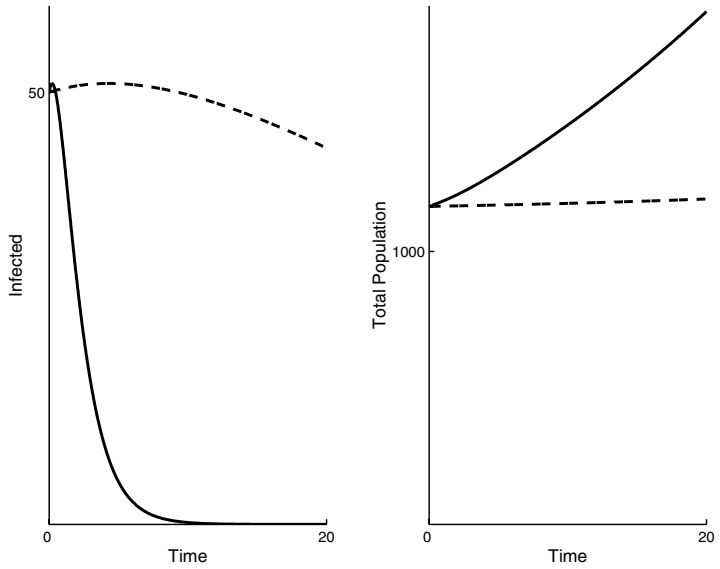

FIGURE 13.2: Results from optimal vaccination from (13.1), in solid, compared with that of no vaccination (dashed). Infectious I and total population N are pictured. With no vaccination, the number of infectious decreases little and total population is nearly constant.

Now, vary (13.1) with $c = 0.001$, a much higher, and more realistic, incidence level. Here, the threat of disease spread is much more serious, and a more aggressive plan is needed. Maximum vaccination is used initially, followed by a reduction, but a slower reduction than in the first system. With a low incidence, there was no need to vaccinate once the exposed and infectious populations were reduced, as almost no one would contract the disease. In the system with $c = 0.001$, though, we see it is advantageous to continue vaccinating almost 40% of the population, even after exposed and infectious populations are reduced. The susceptible population is reduced by half in the first two years, as the great majority are being vaccinated and many of the others are exposed. The infectious population even sees a slight initial increase. However, after the first several years, the same dynamic of the other simulation returns. Susceptible begins to steadily climb, almost reaching the levels of those in the first system. Exposed and infectious almost disappear,

and vaccination levels do eventually reach 0. Total population is hardly affected by the increase in incidence. It is worth noting in the second system, at the end of the period, the recovered group is still a significant portion of the total population.

This time, enter

$$\boxed{\begin{array}{cccc} b = 0.525 & d = 0.5 & c = 0.001 & e = 0.5 \\ g = 0.1 & a = 0.2 & S_0 = 1000 & E_0 = 100 \\ I_0 = 50 & R_0 = 15 & A = 0.1 & T = 20 \end{array}} \quad . \tag{13.2}$$

and vary A using as the second choice $A = 2$. With a higher weight parameter, we can vaccinate at the maximum level for a longer period of time. This change greatly decreases the susceptible population and increases the recovered population. However, the exposed and infectious populations are reduced, but the change is marginal. Total population seems unaffected in any way, which seems to suggest early vaccinations are the key to disease management. Later vaccinations, while effective and helpful, become increasingly less efficient as time passes. To verify this, vary $A = 0.1$ versus $A = 200$. Here, with such a high A value, vaccination cost is of virtually no importance. As such, maximum vaccination is used almost exclusively. The effects on the susceptible and recovered populations are pronounced, but the change in the number of exposed and infectious people is small.

Enter (13.2) varying with $g = 0$, representing a disease where no recovery can occur. A higher vaccination rate must be used, as immunity is no longer achieved naturally. The second system has a higher infectious population throughout. This makes sense, as no one is recovering. The reduction in infectious people in the second system is due only to death. Now try (13.2) varying with $g = 0.4$. Here, one stands a much better chance of recovering from the disease. The infectious population reduces more rapidly, meaning a less aggressive vaccination routine can be used. Note, even though natural recovery is occurring more often in the second system, there are fewer recovered people. The shift in vaccination outweighs the shift in recovery rate.

We might suspect a higher disease-related mortality would necessitate a greater immunization rate. However, the opposite is actually true. Enter (13.2), this time varying with $a = 0.4$. In the second system, a slightly less powerful, but still aggressive initial immunization strategy is used. The infectious population reduces more rapidly due to the greater mortality rate, and fewer vaccinations are needed. Notice, however, that the total population in the second system is lowered slightly. If you try a disease mortality rate as high as $a = 1.5$, you will see very little vaccination is used, as the infectious population rapidly declines. The effect on total population also becomes more severe. Recall our objective functional minimizes the number of infectious people only. This simulation suggests we might also consider the total population in our goals.

Vary the latency of the disease, using (13.2) and varying with $e = 0.1$. In the second system, the disease has an incubation period five times as long. So, a large initial round of immunizations is not needed. The longer incubation period allows the immunizations to be spread out over the first several years. Also, the infectious population receives no initial boost and reduces at a greater speed. The recovery and death rates of infectious individuals are now larger than the rate at which susceptible people become exposed, then infectious.

Consider the relationship between the management of the disease and the effective growth of the population as a whole. In all the previous simulations, we considered a population with moderate growth. We now turn our attention to a simulation with rapid growth. Enter (13.2), varying with $b = 0.55$. Here, we have doubled the effective growth rate $(b - d)$. With so many more susceptible people, the disease can spread more easily. Thus, a more stringent schedule of immunizations must be used to balance out the population growth. Infectious and exposed populations are similar in both systems for the majority of the time interval. However, as immunization is decreased, both begin to rise at approximately 15 years. At this point, there are so many susceptible people, even a few infectious individuals are enough to restart the epidemic if immunizations are not continued. Conversely, consider a population with small or no effective growth, i.e., $b = d$. Enter the same (13.2) values as before, this time varying d to $d = 0.525$. The initial immunization blitz reduces the number of susceptible people as normal, but as the growth rate is zero, the susceptible population will have a slower increase. Thus, fewer vaccinations are needed after the first few years. The exposed and infectious populations are reduced in the usual way. However, the total population, without disease, is naturally static as $b = d$. Thus, the disease-caused deaths cause the total population to reduce in size.

To this point, we have examined populations where susceptible people were in the majority. Now, let us consider a case where the infection has been spreading unchecked for some time before intervention occurs. Enter the values

$$\boxed{\begin{array}{cccc} b = 0.525 & d = 0.5 & c = 0.001 & e = 0.5 \\ g = 0.1 & a = 0.2 & S_0 = 1000 & E_0 = 1000 \\ I_0 = 2000 & R_0 = 500 & A = 0.1 & T = 20 \end{array}} \quad . \tag{13.3}$$

Here, maximum vaccination is almost the entire period. The number of exposed and infectious people are reduced by the end of the period, but not nearly to the levels we have been observing. The number of exposed people actually begins to increase again in the last year. Most troublesome, the total population drastically falls in the first five years, before stabilizing and then increasing. Now try

$$\boxed{\begin{array}{cccc} b = 0.525 & d = 0.5 & c = 0.001 & e = 0.5 \\ g = 0.1 & a = 0.2 & S_0 = 1000 & E_0 = 2000 \\ I_0 = 5000 & R_0 = 1000 & A = 0.1 & T = 20 \end{array}} \quad . \tag{13.4}$$

Here, vaccination has begun too late. Even with maximum vaccination for more than 19 of the 20 years, total population steadily falls. This again establishes the importance of early vaccinations. Treatment must begin before the infection gets out-of-hand. Try to create a set of parameters where the population has moderate growth but eventually dies out, despite the immunization tactics.

On your own, examine a few special cases of the initial conditions. Run a simulation where immunization begins before the disease becomes infectious, namely $I_0 = R_0 = 0$. Consider a closed environment, such as a cruise ship, where a few infectious individuals enter an uncontaminated population, specifically, $E_0 = R_0 = 0$. Also, try $E_0 = I_0 = R_0 = 0$.

Vary each of the initial conditions one by one to see their effect on the optimal immunization treatment. How does shortening the time interval alter the execution and efficiency of the immunization schedule?

Exercise 13.1 Consider this model with an objective functional that instead maximizes N. For example,

$$\max_u \int_0^T AN(t) - u(t)^2 \, dt$$

$$\begin{aligned} \text{subject to} \quad & S'(t) = bN(t) - dS(t) - cS(t)I(t) - u(t)S(t),\ S(0) = S_0 \geq 0, \\ & E'(t) = cS(t)I(t) - (e+d)E(t),\ E(0) = E_0 \geq 0, \\ & I'(t) = eE(t) - (g+a+d)I(t),\ I(0) = I_0 \geq 0, \\ & R'(t) = gI(t) - dR(i) + u(t)S(t),\ R(0) = R_0 \geq 0, \\ & N'(t) = (b-d)N(t) - aI(t), N(0) = N_0, \\ & 0 \leq u(t) \leq 0.9. \end{aligned}$$

Write a code for this problem (or alter code7.m), and examine the differences between the two problems.