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PRACTICE 13 – 2 DOF FORCED SYSTEM

VIBRATIONS

– BMEGEMMBXM4 –

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Example

In Fig. 1, a two-degree-of-freedom system is shown which consists of two rigid bodies: a disk with mass m_1 and radius R and a block with mass m_2. The disk is rolling on a horizontal surface, and its center is connected to the environment through a spring with stiffness k_1. The block is moving vertically in the gravitational field between frictionless linear guideways and it is connected to a spring with stiffness k_2. The other side of this spring is connected to an ideal (inextensible) rope which is fixed to the center of gravity of the disk. The rope is changing direction on an ideal (massless/frictionless) pully, and we assume that due to the pretension of the rope it will remain tensioned during the vibrations.

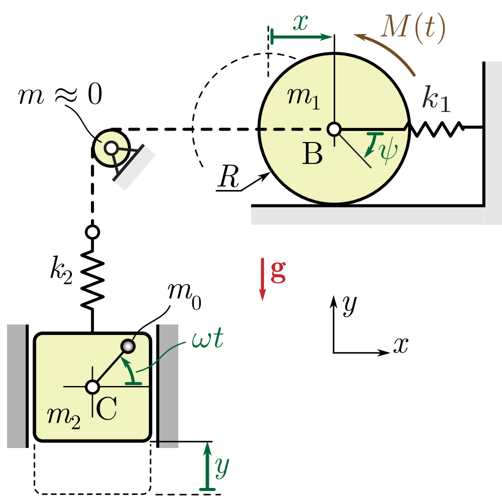

Fig. 1: Mechanical model

The forced vibration is created by the harmonically varying moment $M(t) = M_0 \cos(\omega t + \varepsilon)$ acting on the disk and the rotating eccentric mass (with angular velocity ω, eccentricity e and mass m_0) connected to the block.

In the equilibrium position ($y = 0, \psi = 0$) the springs are preloaded, which results a force acting against the gravitational force.

Data

$m_0 = 0.1\,\mathrm{kg}$	$m_1 = 1\,\mathrm{kg}$	$m_2 = 3\,\mathrm{kg}$	$R = 0.3\,\mathrm{m}$	$e = 0.01\,\mathrm{m}$
$k_1 = 100\,\mathrm{N/m}$	$k_2 = 200\,\mathrm{N/m}$	$M_0 = 3\,\mathrm{Nm}$	$\omega = 30\,\mathrm{rad/s}$	$\varepsilon = \pi/6$

Tasks

1. Derive the linear equation of motion with matrix coefficients!
2. Determine the particular part of the law of motion!
3. What is the maximum force in spring k_2 during the stationary vibration?
+4. Determine the natural angular frequencies (ω_{nk}) and the corresponding modeshapes ($\mathbf{A}_k$)!

Made by Dániel Bachrathy

Solution

Task 1

As a first step, the general coordinates have to be defined. It can be seen in Fig. 1 that the system has two degrees of freedom ($n = 2$ DoF). The vertical motion of the block can be described by its vertical coordinate y. The motion of the rolling disc can be characterized either by the horizontal coordinate x or by the rotation angle ψ. During our solution y and ψ will be used as generalized coordinates, which are measured from the equilibrium position:

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} y \\ \psi \end{bmatrix}. \tag{1}$$

The calculation begins with the determination of the kinetic energy T for the mass matrix $\mathbf{M}$, the dissipation function $\mathcal{D}$ for the damping matrix $\mathbf{C}$, the potential function U for the stiffness matrix $\mathbf{K}$ and the power of the external excitations P for the remain part of the generalized force $\mathbf{Q}$.

The kinetic energy can be calculated as follows:

$$T = \frac{1}{2}m_1 v_\mathrm{B}^2 + \frac{1}{2}\Theta_\mathrm{B}\omega_1^2 + \frac{1}{2}m_2 v_\mathrm{C}^2 + \frac{1}{2}m_0 v_0^2\,, \tag{2}$$

where

$$v_\mathrm{B} = R\dot{\psi} \quad \omega_1 = \dot{\psi} \quad v_\mathrm{C} = \dot{y} \quad \Theta_\mathrm{B} = \frac{1}{2}m_1 R^2\,, \tag{3}$$

and the velocity v_0 of the point mass m_0 can be determined via the position vector $\mathbf{r}_0$:

$$\mathbf{r}_0 = \begin{bmatrix} e\cos(\omega t) + const. \\ y + e\sin(\omega t) + const. \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_0 = \dot{\mathbf{r}}_0 = \begin{bmatrix} -e\omega\sin(\omega t) \\ \dot{y} + e\omega\cos(\omega t) \end{bmatrix}. \tag{4}$$

For the kinetic energy it is enough to compute the square of the vector

$$v_0^2 = \dot{y}^2 + e^2\omega^2 + 2\dot{y}e\omega\cos(\omega t). \tag{5}$$

Substituting Eqs. (3),(5) into Eq. (2) the final form of the kinematic energy is

$$T = \frac{1}{2}m_1 R^2\dot{\psi}^2 + \frac{1}{4}m_1 R^2\dot{\psi}^2 + \frac{1}{2}m_2\dot{y}^2 + \frac{1}{2}m_0\big(\dot{y}^2 + e^2\omega^2 + 2\dot{y}e\omega\cos(\omega t)\big). \tag{6}$$

The mass matrix cannot be determined directly form the kinetic energy due to the rotating eccentric mass, so the Lagrange's equation of the second kind must be derived and linearized (note: that there is no damping in our system $\mathcal{D} = 0$):

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k}}_{\substack{\Downarrow \\ \mathbf{M}\ddot{\mathbf{q}}\,\&\,\mathbf{Q}^{m_0}}} + \underbrace{\frac{\partial U}{\partial q_k}}_{\substack{\Downarrow \\ \mathbf{K}\mathbf{q}}} = Q_k^*, \quad k = 1, 2. \tag{7}$$

Derivation of the kinetic energy leads to

$$\frac{\partial T}{\partial \dot{y}} = m_2\dot{y} + m_0\dot{y} + m_0 e\omega\cos(\omega t), \tag{8}$$

$$\frac{\partial T}{\partial y} = 0\,, \tag{9}$$

$$\frac{\partial T}{\partial \dot{\psi}} = \frac{3}{2}m_1 R^2\dot{\psi}\,, \tag{10}$$

$$\frac{\partial T}{\partial \psi} = 0\,. \tag{11}$$

The corresponding components of the Lagrange's equation are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{y}} - \frac{\partial T}{\partial y} = (m_2 + m_0)\ddot{y} - m_0 e\omega^2 \sin(\omega t)\,, \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\psi}} - \frac{\partial T}{\partial \psi} = \frac{3}{2}m_1 R^2 \ddot{\psi}\,. \tag{13}$$

Since we have only linear terms in these formulas, we can write:

$$\begin{bmatrix} (m_2 + m_0)\ddot{y} - m_0 e\omega^2 \sin(\omega t) \\ \frac{3}{2}m_1 R^2 \ddot{\psi} \end{bmatrix} \equiv \mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}^{m_0}(t)\,, \tag{14}$$

Namely, the coefficients of the second order derivatives ($\ddot{y}$ and $\ddot{\psi}$) can be arranged into the mass matrix:

$$\mathbf{M} = \begin{bmatrix} m_2 + m_0 & 0 \\ 0 & \frac{3}{2}m_1 R^2 \end{bmatrix}. \tag{15}$$

The generalized force component representing the forcing by the rotating eccentric mass:

$$\mathbf{Q}^{m_0} = \begin{bmatrix} m_0 e\omega^2 \sin(\omega t) \\ 0 \end{bmatrix}. \tag{16}$$

The stiffness matrix can directly be derived from the potential function due to the lack of the displacement excitation:

$$U = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2(x - y)^2 + m_0 g\, e \sin(\omega t). \tag{17}$$

Here, we do not consider the pretension of the springs, neither the potential functions of the constant gravitational forces, because if the system contains linear elements only and the generalized coordinate is measured form the equilibrium position, then the resulting terms cancel each other. Note, that the last component in Eq. (17) is independent from the generalized coordinates, thus it vanishes by the derivations.

Using the geometric relation $x = R\psi$ which is related to the kinematic constraint of rolling, the potential function can be expressed as a function of the generalized coordinates:

$$U = \frac{1}{2}k_1(R\psi)^2 + \frac{1}{2}k_2(R\psi - y)^2 + m_0 g\, e \sin(\omega t). \tag{18}$$

Based on this, the elements of the stiffness matrix can be calculated as

$$k_{ij} = \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{\mathbf{q}=\mathbf{0}}. \tag{19}$$

The first derivatives of the potential function are

$$\frac{\partial U}{\partial y} = -k_2(R\psi - y)\,, \tag{20}$$

$$\frac{\partial U}{\partial \psi} = k_1 R^2 \psi + k_2(R\psi - y)R\,, \tag{21}$$

and the second derivatives:

$$\frac{\partial^2 U}{\partial y^2} = k_2\,, \tag{22}$$

$$\frac{\partial^2 U}{\partial \psi \partial y} = -k_2 R\,, \tag{23}$$

$$\frac{\partial^2 U}{\partial \psi^2} = (k_1 + k_2)R^2\,. \tag{24}$$

These elements can be arranged such that they form the stiffness matrix:

$$\mathbf{K} = \begin{bmatrix} k_2 & -k_2 R \\ -k_2 R & (k_1 + k_2)R^2 \end{bmatrix}. \tag{25}$$

The last step is to find the power of the external forces and moments as a function of the generalized velocities. In this problem only the moment $M(t)$ acts as an external moment on the disk, and its power is given by:

$$P^M = \mathbf{M} \cdot \boldsymbol{\omega}_1 = \begin{bmatrix} 0 \\ 0 \\ M(t) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -\dot{\psi} \end{bmatrix} = -M(t)\dot{\psi} \equiv Q_1^M \dot{y} + Q_2^M \dot{\psi}. \tag{26}$$

As it is highlighted, the first coordinate of the generalized force vector is the coefficient of $\dot{y}$, while the second component is the coefficient of $\dot{\psi}$, so the generalized force vector corresponding to the excitation of the moment is

$$\mathbf{Q}^M = \begin{bmatrix} 0 \\ -M(t) \end{bmatrix}. \tag{27}$$

The excitation force vector is the sum of $\mathbf{Q}^{m_0}$ and $\mathbf{Q}^M$:

$$\mathbf{Q}(t) = \mathbf{Q}^M + \mathbf{Q}^{m_0} = \begin{bmatrix} m_0 e \omega^2 \sin(\omega t) \\ -M_0 \cos(\omega t + \varepsilon) \end{bmatrix}. \tag{28}$$

For the later steps, using the trigonometrical identities

$$M_0 \cos(\omega t + \varepsilon) \equiv M_0 \cos(\omega t)\cos(\varepsilon) - M_0 \sin(\omega t)\sin(\varepsilon)\,, \tag{29}$$

the excitation vector can decomposed as

$$\mathbf{Q}(t) = \underbrace{\begin{bmatrix} m_0 e \omega^2 \\ M_0 \sin(\varepsilon) \end{bmatrix}}_{=\mathbf{F}_s} \sin(\omega t) + \underbrace{\begin{bmatrix} 0 \\ -M_0 \cos(\varepsilon) \end{bmatrix}}_{=\mathbf{F}_c} \cos(\omega t). \tag{30}$$

Using the above notations, the linear governing equation of motion with matrix coefficients is

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}_s \sin(\omega t) + \mathbf{F}_c \cos(\omega t). \tag{31}$$

Substituting the previous results, we obtain the parametric form:

$$\boxed{\begin{bmatrix} m_2 + m_0 & 0 \\ 0 & \frac{3}{2} m_1 R^2 \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\psi} \end{bmatrix} + \begin{bmatrix} k_2 & -k_2 R \\ -k_2 R & (k_1 + k_2)R^2 \end{bmatrix} \begin{bmatrix} y \\ \psi \end{bmatrix} = \begin{bmatrix} m_0 e \omega^2 \\ M_0 \sin(\varepsilon) \end{bmatrix} \sin(\omega t) + \begin{bmatrix} 0 \\ -M_0 \cos(\varepsilon) \end{bmatrix} \cos(\omega t).} \tag{32}$$

Task 2

The particular solution (periodic stationary solution) of the law of motion must be considered in a harmonic form for harmonic excitation:

$$\mathbf{q}_\mathrm{p}(t) = \mathbf{L}\cos(\omega t) + \mathbf{N}\sin(\omega t). \tag{33}$$

Using the derivatives

$$\dot{\mathbf{q}}_\mathrm{p}(t) = -\omega \mathbf{L}\sin(\omega t) + \omega \mathbf{N}\cos(\omega t), \tag{34}$$

$$\ddot{\mathbf{q}}_\mathrm{p}(t) = -\omega^2 \mathbf{L}\cos(\omega t) - \omega^2 \mathbf{N}\sin(\omega t), \tag{35}$$

in the governing equation (31) and collecting the coefficients of $\sin(\omega t)$ and $\cos(\omega t)$ we will end up with a linear equation system for $\mathbf{L}$ and $\mathbf{N}$:

$$\begin{bmatrix} -\omega^2 \mathbf{M} + \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\omega^2 \mathbf{M} + \mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{L} \\ \mathbf{N} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_c \\ \mathbf{F}_s \end{bmatrix}. \tag{36}$$

Note, that in our case there is no damping, thus the off diagonal (block) elements are zero-matrices. Using the governing equation (32), we obtain:

$$\begin{bmatrix} -\omega^2(m_2+m_0)+k_2 & -k_2R & 0 & 0 \\ -k_2R & -\omega^2\frac{3}{2}m_1R^2+(k_1+k_2)R^2 & 0 & 0 \\ 0 & 0 & -\omega^2(m_2+m_0)+k_2 & -k_2R \\ 0 & 0 & -k_2R & -\omega^2\frac{3}{2}m_1R^2+(k_1+k_2)R^2 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -M_0\cos(\varepsilon) \\ m_0e\omega^2 \\ M_0\sin(\varepsilon) \end{bmatrix} \tag{37}$$

This gives the system of equations:

$$\begin{aligned} \left(-\omega^2(m_2+m_0)+k_2\right)L_1 - k_2RL_2 =& 0\,, \\ -k_2RL_1 + \left(-\omega^2\frac{3}{2}m_1R^2+(k_1+k_2)R^2\right)L_2 =& -M_0\cos(\varepsilon)\,, \\ \left(-\omega^2(m_2+m_0)+k_2\right)N_1 - k_2RN_2 =& m_0e\omega^2\,, \\ -k_2RN_1 + \left(-\omega^2\frac{3}{2}m_1R^2+(k_1+k_2)R^2\right)N_2 =& M_0\sin(\varepsilon)\,. \end{aligned} \tag{38}$$

The system of equation can be solved separately (due to the lack of damping), namely, the first and second equations give L_1 and L_2 while the third and fourth equations leads to N_1 and N_2. For the given parameters, the equations have the following forms in SI units:

$$\begin{cases} -2590L_1 - 40L_2 = 0 \\ -40L_1 + -42L_2 = -2.598 \end{cases} \Rightarrow \quad \begin{aligned} L_1 &= -0.000970\,\text{m} \\ L_2 &= 0.062782\,\text{rad} \end{aligned} \tag{39}$$

$$\begin{cases} -2590N_1 - 40N_2 = 0.9 \\ -40N_1 + -42N_2 = -1.5 \end{cases} \Rightarrow \quad \begin{aligned} N_1 &= 0.000207\,\text{m} \\ N_2 &= -0.035912\,\text{rad} \end{aligned} \tag{40}$$

Substituting these results into Eq. (33) the particular solution can be derived as:

$$\boxed{\mathbf{q}_\text{p}(t) = \begin{bmatrix} -0.000970 \\ 0.062782 \end{bmatrix}\cos(30\,t) + \begin{bmatrix} 0.000207 \\ 0.035912 \end{bmatrix}\sin(30\,t) \quad \begin{bmatrix} \text{m} \\ \text{rad} \end{bmatrix}.} \tag{41}$$

Task 3

During the computation of the maximal force in the spring of stiffness k_2, firstly, we have to consider the statical component $F_{k_2,\text{st}}$ emerging in the steady state case, which reads

$$F_{k_2,\text{st}} = (m_0+m_2)g = 30.411\,\text{N}. \tag{42}$$

Secondly, the dynamical component $F_{k_2,\text{dyn}}$ caused by dynamical change of length of the spring can be calculated, which is given by the relative vibrations of the block and the disk, namely the extension of the spring is $\Delta l = x(t) - y(t)$, hence

$$F_{k_2,\text{dyn}}(t) = k_2(x_\text{p}(t) - y_\text{p}(t)) = k_2(R\psi_\text{p}(t) - y_\text{p}(t))\,. \tag{43}$$

As we already found the particular solution for the motion of the bodies (see Eq. (41)):

$$y_\text{p}(t) = L_1\cos(\omega t) + N_1\sin(\omega t), \tag{44}$$

$$\psi_\text{p}(t) = L_2\cos(\omega t) + N_2\sin(\omega t), \tag{45}$$

the dynamical component can be reformulated:

$$F_{k_2,\text{dyn}}(t) = k_2\big((RL_2 - L_1)\cos(\omega t) + (RN_2 - N_1)\sin(\omega t)\big). \tag{46}$$

To find the extremum of the force, we transform the dynamical component into the following form:

$$F_{k_2,\text{dyn}} = k_2A\sin(\omega t + \delta)\,, \tag{47}$$

in which the vibration amplitude of the spring can be factorized as

$$A\sin(\omega t+\delta) = A\sin(\omega t)\cos(\delta) + A\cos(\omega t)\sin(\delta). \tag{48}$$

By comparing the coefficients of the $\sin(\omega t)$ and $\cos(\omega t)$ terms in Eq. (46) and Eq. (48) we obtain

$$\begin{aligned} \sin(\omega t):\quad & RN_2 - N_1 = A\cos\delta\,, \\ \cos(\omega t):\quad & RL_2 - L_1 = A\sin\delta\,. \end{aligned} \tag{49}$$

Adding the squares of Eq. (50) results

$$(RN_2 - N_1)^2 + (RL_2 - L_1)^2 = A^2\,, \tag{50}$$

from where the amplitude is

$$A = \sqrt{(RN_2 - N_1)^2 + (RL_2 - L_1)^2} = 0.0154\,\mathrm{m}. \tag{51}$$

Since the static force $F_{k_2,\mathrm{st}}$ is positive, the extremum of the overall force in the spring can be calculated as

$$\boxed{F_{k_2} = F_{k_2,\mathrm{st}} + k_2 A = 33.494\,\mathrm{N}.} \tag{52}$$

Task 4

The natural angular frequencies and the mode shapes can be calculated using the formulas:

$$\det(-\omega_\mathrm{n}^2\mathbf{M} + \mathbf{K}) = 0 \tag{53}$$

and

$$(-\omega_{\mathrm{n}k}^2\mathbf{M} + \mathbf{K})\mathbf{A}_k = \mathbf{0} \qquad k = 1, 2. \tag{54}$$

Here, we only present the numerical results:

$$\boxed{\begin{aligned} &\omega_{\mathrm{n}1} = 4.17\,\mathrm{rad/s}\,, && \mathbf{A}_1 = \begin{bmatrix} 1 \\ 3.65 \end{bmatrix}\begin{bmatrix} \mathrm{m} \\ \mathrm{rad} \end{bmatrix}, \\ &\omega_{\mathrm{n}2} = 15.72\,\mathrm{rad/s}\,, && \mathbf{A}_2 = \begin{bmatrix} 1 \\ -14.15 \end{bmatrix}\begin{bmatrix} \mathrm{m} \\ \mathrm{rad} \end{bmatrix}. \end{aligned}} \tag{55}$$