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PRACTICE 12 – 2 DOF FORCED SYSTEM

VIBRATIONS

– BMEGEMMBXM4 –

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Example

In Fig. 1, a 2 DoF system is shown, which consists of a cart and a pendulum. The cart is modelled by a rigid body that can move in the horizontal direction only, its mass is denoted by m_1 and the pendulum is modelled by a rod of length l and mass m_2. The rod can only rotate about the joint B. Two different types of excitations are applied: harmonic force excitation $F(t) = F_0 \sin(\omega t + \varepsilon)$ at the lower endpoint C of the pendulum and harmonic displacement excitation $r(t) = r_0 \cos(\omega t)$ through a spring connected to the cart of stiffness k. To describe the motion of the corresponding 2 DoF system, the generalised coordinates x_1 and x_2 are introduced. Here, x_1 is the displacement of the centre of gravity of the cart and x_2 is the displacement of the point C of the rod. The structure is in the vertical plane (the gravitational acceleration is g) and its equilibrium position is located at $x_1 = 0$ and $x_2 = 0$.

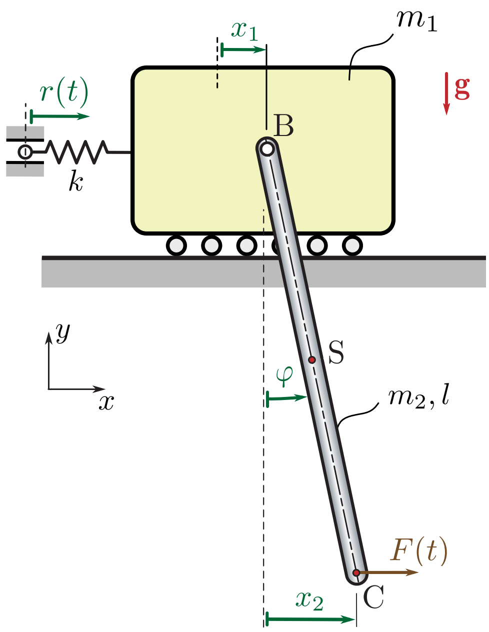

Fig. 1: Mechanical model of the 2 DoF system

Data

$m_1 = 2\,\mathrm{kg}$	$l = 0.5\,\mathrm{m}$	$\omega = 20\,\mathrm{rad/s}$
$m_2 = 1\,\mathrm{kg}$	$F_0 = 5\,\mathrm{N}$	$\varepsilon = \pi/3$
$k = 100\,\mathrm{N/m}$	$r_0 = 0.01\,\mathrm{m}$	

Tasks

1. Derive the matrix formulation of the equation of motion for the model assuming small oscillations around the equilibrium!
2. Determine the stationary solution of the system!
3. Calculate the maximal excursion of the rod in the stationary solution!

Made by Csenge Andrea Molnár

Solution

Task 1

The matrix formulation of the linearized equation of motion is

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{Q}(t), \tag{1}$$

where $\mathbf{M}$ stands for the mass matrix, $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ refers to the stiffness matrix, $\mathbf{Q}(t)$ contains the excitations and $\mathbf{q}$ is the vector of generalized coordinates. For systems with n DoF, $\mathbf{q} \in \mathbb{R}^{\mathrm{n}}$, and $\mathbf{M}, \mathbf{C}, \mathbf{K} \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$. Generally, Eq. (1) can be obtained by linearization of the Lagrange's equation of the second kind. When the system has time-independent constraints only, i.e., the system is holonomic and scleronomous, the linearized equation of motion can be calculated directly by the next method.

First, we have to determine the kinetic energy T, the Rayleigh's dissipation function $\mathcal{D}$ and the potential function U. After that, the elements of the mass, damping and stiffness matrices of the linearized equation of motion can be calculated by differentiating the kinetic energy, the Rayleigh's dissipation function and the potential function with respect to the corresponding generalized coordinates/velocities and taking these derivatives at the equilibrium ($\mathbf{q} = \mathbf{0}$), namely,

$$m_{\mathrm{ij}} = \left. \frac{\partial^2 T}{\partial \dot{q}_{\mathrm{i}} \partial \dot{q}_{\mathrm{j}}} \right|_{\mathbf{q}=\mathbf{0}}, \tag{2}$$

$$c_{\mathrm{ij}} = \left. \frac{\partial^2 \mathcal{D}}{\partial \dot{q}_{\mathrm{i}} \partial \dot{q}_{\mathrm{j}}} \right|_{\mathbf{q}=\mathbf{0}}, \tag{3}$$

$$k_{\mathrm{ij}} = \left. \frac{\partial^2 U}{\partial q_{\mathrm{i}} \partial q_{\mathrm{j}}} \right|_{\mathbf{q}=\mathbf{0}}. \tag{4}$$

The vector $\mathbf{Q}$ is determined from the power of the active forces that do not have a potential or a dissipation function.

When time-dependent constraint (e.g. harmonic displacement excitation through a spring, rotating mass excitation) is applied in the system, the equation of motion can not be calculated directly as shown above. To obtain the correct equation of motion, Lagrange's equation of the second kind has to be also applied.

If the kinetic energy T depends explicitly on the time (e.g. in case of excitation due to rotating unbalanced mass) then the kinetic energy related parts of Lagrange's equation have to be calculated in order to obtain the excitation in the linearized equation of motion:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k}}_{\substack{\Downarrow \\ \mathbf{M}\ddot{\mathbf{q}}\,\text{and a part of}\,-\mathbf{Q}(t)}} + \underbrace{\frac{\partial \mathcal{D}}{\partial \dot{q}_k}}_{\substack{\Downarrow \\ \mathbf{C}\dot{\mathbf{q}}}} + \underbrace{\frac{\partial U}{\partial q_k}}_{\substack{\Downarrow \\ \mathbf{K}\mathbf{q}}} = \underbrace{Q_k^*}_{\substack{\Downarrow \\ \text{a part of}\,\mathbf{Q}(t)}}, \quad k = 1, \ldots, n. \tag{5}$$

Similarly, if the potential function U depends explicitly on the time (e.g. in our case where we have displacement excitation) then the potential function related part of Lagrange's equation has to be calculated:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k}}_{\substack{\Downarrow \\ \mathbf{M}\ddot{\mathbf{q}}}} + \underbrace{\frac{\partial \mathcal{D}}{\partial \dot{q}_k}}_{\substack{\Downarrow \\ \mathbf{C}\dot{\mathbf{q}}}} + \underbrace{\frac{\partial U}{\partial q_k}}_{\substack{\Downarrow \\ \mathbf{K}\mathbf{q}\,\text{and a part of}\,-\mathbf{Q}(t)}} = \underbrace{Q_k^*}_{\substack{\Downarrow \\ \text{a part of}\,\mathbf{Q}(t)}}, \quad k = 1, \ldots, n. \tag{6}$$

Kinetic energy and mass matrix

The system has two DoF ($n = 2$), therefore the vector of generalized coordinates is

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{7}$$

where x_1 stands for the horizontal displacement of the cart, and x_2 refers to the horizontal displacement of the lowest point of the pendulum.

The kinetic energy of the system is the sum of the kinetic energy of the cart and the pendulum:

$$T = \underbrace{\frac{1}{2}m_1 v_\mathrm{B}^2}_{T_\mathrm{cart}} + \underbrace{\frac{1}{2}m_2 v_\mathrm{S}^2 + \frac{1}{2}\theta_\mathrm{S}\omega_2^2}_{T_\mathrm{pendulum}}, \tag{8}$$

where the velocity of the centre of gravity of the cart is

$$v_\mathrm{B} = \dot{x}_1\,, \tag{9}$$

the mass moment of inertia of the rod with respect to the center of gravity is

$$\theta_\mathrm{S} = \frac{1}{12}m_2 l^2, \tag{10}$$

the angular velocity ω_2 of the rod can be expressed with the generalized coordinate x_2 as

$$x_2 = l\sin\varphi \approx l\varphi \quad \Longrightarrow \quad \varphi \approx \frac{x_2}{l} \quad \Longrightarrow \quad \omega_2 = \dot{\varphi} \approx \frac{\dot{x}_2}{l}. \tag{11}$$

The velocity of the centre of gravity of the rod can be determined as

$$\mathbf{v}_\mathrm{S} = \mathbf{v}_\mathrm{B} + \boldsymbol{\omega}_2 \times \mathbf{r}_\mathrm{BS} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\varphi} \end{bmatrix} \times \begin{bmatrix} \frac{l}{2}\sin\varphi \\ -\frac{l}{2}\cos\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 + \frac{l}{2}\dot{\varphi}\cos\varphi \\ \frac{l}{2}\dot{\varphi}\sin\varphi \\ 0 \end{bmatrix}, \tag{12}$$

from which

$$v_\mathrm{S}^2 = \dot{x}_1^2 + \frac{l^2}{4}\dot{\varphi}^2 + l\dot{\varphi}\dot{x}_1\cos\varphi\,. \tag{13}$$

Finally, the kinetic energy can be written as

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left(\dot{x}_1^2 + \frac{\dot{x}_2^2}{4} + \dot{x}_1\dot{x}_2\cos\frac{x_2}{l}\right) + \frac{1}{24}m_2 l^2\left(\frac{\dot{x}_2}{l}\right)^2. \tag{14}$$

Since, the kinetic energy does not depend explicitly on the time t, the mass matrix $\mathbf{M}$ can be directly determined from the kinetic energy using Eq. (2). The first and second derivatives of the kinetic energy with respect to the corresponding generalized velocities are

$$\frac{\partial T}{\partial \dot{x}_1} = m_1\dot{x}_1 + m_2\dot{x}_1 + \frac{1}{2}m_2\dot{x}_2\cos\frac{x_2}{l}, \tag{15}$$

$$\frac{\partial T}{\partial \dot{x}_2} = m_2\frac{\dot{x}_2}{4} + \frac{1}{2}m_2\dot{x}_1\cos\frac{x_2}{l} + \frac{1}{12}m_2\dot{x}_2, \tag{16}$$

$$\frac{\partial^2 T}{\partial \dot{x}_1^2} = m_1 + m_2, \tag{17}$$

$$\frac{\partial^2 T}{\partial \dot{x}_1\partial \dot{x}_2} = \frac{\partial^2 T}{\partial \dot{x}_2\partial \dot{x}_1} = \frac{1}{2}m_2\cos\frac{x_2}{l}, \tag{18}$$

$$\frac{\partial^2 T}{\partial \dot{x}_2^2} = \frac{1}{3}m_2. \tag{19}$$

After substituting the equilibrium positions ($x_1 = 0$, $x_2 = 0$) into these derivatives, the mass matrix becomes

$$\mathbf{M} = \begin{bmatrix} m_1 + m_2 & \frac{m_2}{2} \\ \frac{m_2}{2} & \frac{m_2}{3} \end{bmatrix}. \tag{20}$$

Potential energy function and stiffness matrix

Using Eq. (11), the potential function U reads

$$U = \frac{1}{2}k(x_1 - r(t))^2 - m_2 g\frac{l}{2}\cos\varphi \approx \frac{1}{2}k(x_1 - r(t))^2 - m_2 g\frac{l}{2}\cos\frac{x_2}{l}\,. \tag{21}$$

As it can be seen, the potential function depends explicitly on time through the displacement excitation $r(t)$, and the stiffness matrix and the related excitation force can be calculated as shown in Eq. (6).

Thus, we need the first derivatives of the potential function with respect to the generalized coordinates:

$$\frac{\partial U}{\partial x_1} = kx_1 - kr(t)\,, \tag{22}$$

$$\frac{\partial U}{\partial x_2} = \frac{m_2 g}{2}\sin\frac{x_2}{l}\,. \tag{23}$$

By the linearization of these around the equilibrium, we obtain the terms that appear in the linearized equation of motion, namely, they are given by $\mathbf{Kq}$ and $-\mathbf{Q}^r(t)$ (the superscript r relates to the displacement excitation):

$$\begin{bmatrix} \frac{\partial U}{\partial x_1} \\ \frac{\partial U}{\partial x_2} \end{bmatrix} \approx \begin{bmatrix} kx_1 - kr(t) \\ \frac{m_2 g}{2l}x_2 \end{bmatrix} \equiv \mathbf{Kq} - \mathbf{Q}^r(t)\,, \tag{24}$$

from where the generalized force related to the displacement excitation can be identified as

$$\mathbf{Q}^{\mathrm{r}}(t) = \begin{bmatrix} kr_0\cos(\omega t) \\ 0 \end{bmatrix}. \tag{25}$$

and the stiffness matrix is

$$\mathbf{K} = \begin{bmatrix} k & 0 \\ 0 & \dfrac{m_2 g}{2l} \end{bmatrix}. \tag{26}$$

Of course, the elements of the stiffness matrix can be also calculated by means of Eq. (4):

$$\left.\frac{\partial^2 U}{\partial x_1^2}\right|_{\mathbf{q}=\mathbf{0}} = k\,, \tag{27}$$

$$\left.\frac{\partial^2 U}{\partial x_1 \partial x_2}\right|_{\mathbf{q}=\mathbf{0}} = 0\,, \tag{28}$$

$$\left.\frac{\partial^2 U}{\partial x_2^2}\right|_{\mathbf{q}=\mathbf{0}} = \frac{m_2 g}{2l}\,. \tag{29}$$

Excitation forces

As we calculated one part of the excitation force vector $\mathbf{Q}(t)$ corresponds to the displacement excitation. The other part of it is related to the harmonic force excitation. The power of $F(t)$ can be obtained as

$$P_{\mathrm{F}} = \mathbf{F}(t)\cdot\mathbf{v}_{\mathrm{C}} = \begin{bmatrix} F(t) \\ 0 \end{bmatrix}\cdot\begin{bmatrix} \dot{x}_1 + \dot{x}_2 \\ v_{\mathrm{Cy}} \end{bmatrix} = F(t)(\dot{x}_1 + \dot{x}_2) = \underbrace{F(t)}_{=Q_1^{\mathrm{F}}}\dot{x}_1 + \underbrace{F(t)}_{=Q_2^{\mathrm{F}}}\dot{x}_2 \equiv \sum_{k=1}^{n} Q_{\mathrm{k}}^{\mathrm{F}}\dot{q}_{\mathrm{k}}\,, \tag{30}$$

from which

$$\mathbf{Q}^{\mathrm{F}}(t) = \begin{bmatrix} Q_1^{\mathrm{F}} \\ Q_2^{\mathrm{F}} \end{bmatrix} = \begin{bmatrix} F(t) \\ F(t) \end{bmatrix}. \tag{31}$$

Finally, the excitation force is

$$\mathbf{Q}(t) = \mathbf{Q}^{\mathrm{r}}(t) + \mathbf{Q}^{\mathrm{F}}(t) = \begin{bmatrix} kr_0\cos(\omega t) + F_0\sin(\omega t + \varepsilon) \\ F_0\sin(\omega t + \varepsilon) \end{bmatrix}. \tag{32}$$

In the following task, we will calculate the stationary solution of the system, therefore we need to write the excitation force in the form of

$$\mathbf{Q}(t) = \mathbf{F}_{\mathrm{S}}\sin(\omega t) + \mathbf{F}_{\mathrm{C}}\cos(\omega t). \tag{33}$$

Using trigonometric identities we can write

$$\mathbf{Q}(t) = \begin{bmatrix} kr_0\cos(\omega t) + F_0\sin(\omega t)\cos\varepsilon + F_0\cos(\omega t)\sin\varepsilon \\ F_0\sin(\omega t)\cos\varepsilon + F_0\cos(\omega t)\sin\varepsilon \end{bmatrix} = \underbrace{\begin{bmatrix} F_0\cos\varepsilon \\ F_0\cos\varepsilon \end{bmatrix}}_{\mathbf{F}_{\mathrm{S}}}\sin(\omega t) + \underbrace{\begin{bmatrix} kr_0 + F_0\sin\varepsilon \\ F_0\sin\varepsilon \end{bmatrix}}_{\mathbf{F}_{\mathrm{C}}}\cos(\omega t). \tag{34}$$

The linearized equation of motion

Substituting the mass matrix (20), the stiffness matrix (26) and the excitation (34) into Eq. (1) one can obtain the equation of motion:

$$\boxed{\begin{bmatrix} m_1 + m_2 & \frac{m_2}{2} \\ \frac{m_2}{2} & \frac{m_2}{3} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & \frac{m_2 g}{2l} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_0\cos\varepsilon \\ F_0\cos\varepsilon \end{bmatrix}\sin(\omega t) + \begin{bmatrix} kr_0 + F_0\sin\varepsilon \\ F_0\sin\varepsilon \end{bmatrix}\cos(\omega t)}. \tag{35}$$

Note, that there is no damping in the system, i.e. $\mathbf{C} = \mathbf{0}$.

Task 2

Equation of motion of multi-DoF, harmonically excited linear system has the form of

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{F}_{\mathrm{S}}\sin(\omega t) + \mathbf{F}_{\mathrm{C}}\cos(\omega t). \tag{36}$$

The stationary or particular solution of Eq. (36) reads

$$\mathbf{q}_{\mathrm{p}} = \mathbf{L}\cos(\omega t) + \mathbf{N}\sin(\omega t). \tag{37}$$

Substituting Eq. (37) and its first and second derivative with respect to time into Eq. (36), we obtain an expression, which can be divided into two equations by collecting the coefficients of $\cos(\omega t)$ and $\sin(\omega t)$ on both sides. Finally, we obtain the following linear algebraic equation

$$\begin{bmatrix} -\omega^2\mathbf{M} + \mathbf{K} & \omega\mathbf{C} \\ -\omega\mathbf{C} & -\omega^2\mathbf{M} + \mathbf{K} \end{bmatrix}\begin{bmatrix} \mathbf{L} \\ \mathbf{N} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{\mathrm{C}} \\ \mathbf{F}_{\mathrm{S}} \end{bmatrix}. \tag{38}$$

For our system this yields

$$\begin{bmatrix} -(m_1+m_2)\omega^2 + k & -\frac{m_2}{2}\omega^2 & 0 & 0 \\ -\frac{m_2}{2}\omega^2 & -\frac{m_2}{3}\omega^2 + \frac{m_2 g}{2l} & 0 & 0 \\ 0 & 0 & -(m_1+m_2)\omega^2 + k & -\frac{m_2}{2}\omega^2 \\ 0 & 0 & -\frac{m_2}{2}\omega^2 & -\frac{m_2}{3}\omega^2 + \frac{m_2 g}{2l} \end{bmatrix}\begin{bmatrix} L_1 \\ L_2 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} kr_0 + F_0\sin\varepsilon \\ F_0\sin\varepsilon \\ F_0\cos\varepsilon \\ F_0\cos\varepsilon \end{bmatrix}, \tag{39}$$

which can be divided into two independent equation system with two unknowns:

$$\begin{cases} \Big(-(m_1+m_2)\omega^2 + k\Big)L_1 - \dfrac{m_2}{2}\omega^2 L_2 = kr_0 + F_0\sin\varepsilon\,, \\[2ex] -\dfrac{m_2}{2}\omega^2 L_1 + \Big(-\dfrac{m_2}{3}\omega^2 + \dfrac{m_2 g}{2l}\Big)L_2 = F_0\sin\varepsilon\,, \end{cases} \tag{40}$$

$$\begin{cases} \Big(-(m_1+m_2)\omega^2+k\Big)N_1-\dfrac{m_2}{2}\omega^2 N_2=F_0\cos\varepsilon\,,\\[2ex] -\dfrac{m_2}{2}\omega^2 N_1+\Big(-\dfrac{m_2}{3}\omega^2+\dfrac{m_2 g}{2l}\Big)N_2=F_0\cos\varepsilon\,. \end{cases} \tag{41}$$

By substituting the numerical data into Eq. (40) and (41), it yields

$$\begin{cases} -1100L_1-200L_2=5.33\,,\\[2ex] -200L_1-123.52L_2=4.33\,, \end{cases} \implies \boxed{L_1=0.00217\text{ m},\quad L_2=-0.03856\text{ m},} \tag{42}$$

$$\begin{cases} -1100N_1-200N_2=2.5\,,\\[2ex] -200N_1-123.52N_2=2.5\,, \end{cases} \implies \boxed{N_1=0.00199\text{ m},\quad N_2=-0.02347\text{ m}.} \tag{43}$$

Finally, the stationary solution of Eq. (37) reads

$$\boxed{\mathbf{q}_\mathrm{p}=\begin{bmatrix}0.00217\\-0.03856\end{bmatrix}\cos(20t)+\begin{bmatrix}0.00199\\-0.02347\end{bmatrix}\sin(20t)\text{ m}}. \tag{44}$$

Task 3

The maximal deflection angle φ_max of the rod belongs to the maximal displacement of point C, referring to Eq. (11)

$$\varphi_\mathrm{max}\approx\frac{x_{2\mathrm{max}}}{l}. \tag{45}$$

Displacement of the lower end of the rod can be expressed from the second row of Eq. (44) as

$$x_2(t)=L_2\cos(\omega t)+N_2\sin(\omega t)=X\sin(\omega t+\delta), \tag{46}$$

where $x_{2\mathrm{max}}=X$ and

$$X=\sqrt{L_2^2+N_2^2}=0.0451\text{ m}, \tag{47}$$

and finally

$$\boxed{\varphi_\mathrm{max}=0.090\text{ rad.}} \tag{48}$$