MŰEGYETEM 1782

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PRACTICE 4 – RAILWAY BUFFER STOP

VIBRATIONS

– BMEGEMMBXM4 –

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Example

In Fig. 1, a railroad car of mass m crashes into a buffer stop with the initial velocity v_0. The buffer stop is assumed to be immovable. In order to consider the elasticity and the energy dissipation of the buffer stop during the impact, we us the simplified mechanical model shown in the figure. While the wagon is touching the buffer, the spring stiffness and damping coefficients of the buffers stops can be combined into the equivalent stiffness k and equivalent damping factor $2c$.

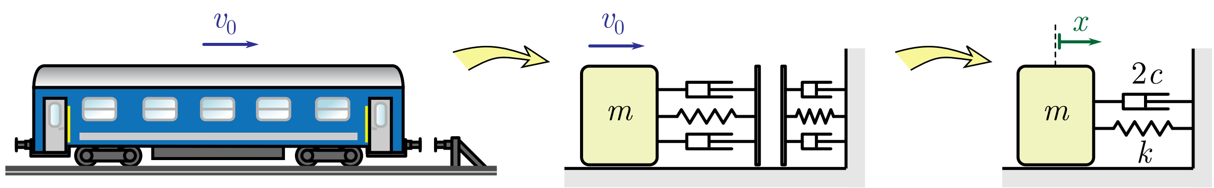

Fig. 1: Mechanical modeling of the impact as a damped oscillator

Data

$m = 5 \cdot 10^4\,\mathrm{kg}$ $\quad v_0 = 1\,\mathrm{m/s}$
$k = 10^6\,\mathrm{N/m}$ $\quad c = 10^5\,\mathrm{Ns/m}$

Tasks

1. Calculate the maximum spring force arising during the impact! ($F_{\mathrm{r,max}} = 128.55\,\mathrm{kN}$)
2. Determine the amount of energy dissipated by the impact! ($E^{\mathrm{diss}} = 22.27\,\mathrm{kJ}$)

Made by Illés Vörös

Solution

Task 1

Figure 2 shows the free body diagram of the 1 DoF damped oscillator, which models the wagon touching the buffer stop. The only degree of freedom is the horizontal displacement x of the mass. The spring is unstretched in the position $x = 0$.

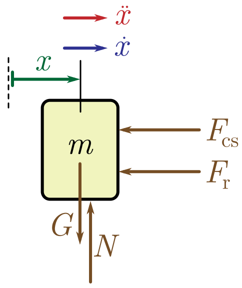

Fig. 2: Free body diagram of the 1 DoF damped oscillator

The equation of motion can be derived using Newton's 2nd law

$$\dot{\mathbf{I}} = \mathbf{F}\,. \tag{1}$$

In the horizontal direction, this translates into

$$m\ddot{x} = -F_{cs} - F_r\,, \tag{2}$$

where the damping force F_{cs} and spring force F_r are

$$F_{cs} = 2c\dot{x} \quad \text{and} \quad F_r = kx\,. \tag{3}$$

This leads to the equation of motion

$$m\ddot{x} + 2c\dot{x} + kx = 0\,. \tag{4}$$

Dividing the equation by m leads to the general form

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0\,, \tag{5}$$

where the damping ratio ζ and the natural angular frequency of the undamped system ω_n are defined according to

$$2\zeta\omega_n = \frac{2c}{m} \quad \text{and} \quad \omega_n^2 = \frac{k}{m}\,. \tag{6}$$

Substituting the parameters of the example leads to

$$\omega_n = \sqrt{\frac{k}{m}} = 4.47\,\mathrm{rad/s}\,, \tag{7}$$

$$\zeta = \frac{1}{2\omega_n}\frac{2c}{m} = 0.45\,(= 45\%)\,. \tag{8}$$

The natural frequency of the damped system is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4\,\mathrm{rad/s}\,, \tag{9}$$

correspondingly, the period of oscillation

$$T_d = \frac{2\pi}{\omega_d} = 1.57\,\mathrm{s} \tag{10}$$

and frequency

$$f_\mathrm{d} = \frac{\omega_\mathrm{d}}{2\pi} = 0.64\,\mathrm{Hz}\,. \tag{11}$$

Equation (5) is a 2nd order, homogeneous, ordinary differential equation. Considering that $\zeta < 1$, its general solution is of the form

$$x(t) = \mathrm{e}^{-\zeta\omega_\mathrm{n}t}\left(C_1\cos(\omega_\mathrm{d}t) + C_2\sin(\omega_\mathrm{d}t)\right)\,, \tag{12}$$

$$\dot{x}(t) = -\zeta\omega_\mathrm{n}\mathrm{e}^{-\zeta\omega_\mathrm{n}t}\left(C_1\cos(\omega_\mathrm{d}t) + C_2\sin(\omega_\mathrm{d}t)\right) + \omega_\mathrm{d}\mathrm{e}^{-\zeta\omega_\mathrm{n}t}\left(-C_1\sin(\omega_\mathrm{d}t) + C_2\cos(\omega_\mathrm{d}t)\right)\,. \tag{13}$$

The coefficients C_1 and C_2 can be determined from the initial conditions, which are

$$\begin{cases} x(t=0) = 0 & \Rightarrow \quad C_1\cdot 1 + C_2\cdot 0 = 0\,, \\ \dot{x}(t=0) = v_0 & \Rightarrow \quad -\zeta\omega_\mathrm{n}\left(C_1\cdot 1 + C_2\cdot 0\right) + \omega_\mathrm{d}\left(-C_1\cdot 0 + C_2\cdot 1\right) = v_0\,. \end{cases} \tag{14}$$

This leads to

$$C_1 = 0 \quad \text{and} \quad C_2 = \frac{v_0}{\omega_\mathrm{d}} = 0.25\,\mathrm{m}\,, \tag{15}$$

thus the displacement of the system is given by

$$x(t) = 0.25\mathrm{e}^{-2t}\sin(4t)\,\mathrm{m}\,. \tag{16}$$

Equation (16) is valid up until the point when the wagon detaches from the buffer. During the impact, the wagon first reaches a maximum displacement in the positive x direction, then the spring (i.e. the combined elasticity of the wagon and the buffer) starts pushing it back from the wall. Since the spring force is proportional to the displacement, the maximal spring force will occur when x reaches its maximum at $t = t^*$:

$$F_\mathrm{r}(t) = kx(t) \Rightarrow F_\mathrm{r,max} = kx_\mathrm{max} = kx(t^*)\,. \tag{17}$$

Note, that because of the exponential term in the solution, this occurs before $t = T_\mathrm{d}/4$ (see Fig. 3).

In order to find t^*, we are looking for a local extremum of $x(t)$. Using Eq. (13) and keeping in mind that $C_1 = 0$, this can be written as

$$\dot{x}(t^*) = 0 \quad \rightarrow \quad -\zeta\omega_\mathrm{n}\mathrm{e}^{-\zeta\omega_\mathrm{n}t^*}C_2\sin(\omega_\mathrm{d}t^*) + \omega_\mathrm{d}\mathrm{e}^{-\zeta\omega_\mathrm{n}t^*}C_2\cos(\omega_\mathrm{d}t^*) = 0\,. \tag{18}$$

Simplifying with $\mathrm{e}^{-\zeta\omega_\mathrm{n}t^*}C_2$ leads to

$$\frac{\omega_\mathrm{d}}{\zeta\omega_\mathrm{n}} = \frac{\sin(\omega_\mathrm{d}t^*)}{\cos(\omega_\mathrm{d}t^*)} \quad \Rightarrow \quad \tan(\omega_\mathrm{d}t^*) = \frac{\omega_\mathrm{d}}{\zeta\omega_\mathrm{n}}\,, \tag{19}$$

which can be solved for t^* as

$$t^* = \frac{1}{\omega_\mathrm{d}}\arctan\left(\frac{\omega_\mathrm{d}}{\zeta\omega_\mathrm{n}}\right) = 0.277\,\mathrm{s} < \frac{\mathrm{T_d}}{4}\,. \tag{20}$$

Although the tangent function is periodic with π, we are only looking for the first extremum of $x(t)$, which occurs before $t = T_\mathrm{d}/4$. The substitution of $t = t^*$ into Eq. (16) leads to the maximal displacement

$$x_\mathrm{max} = x(t^*) = 0.129\,\mathrm{m}\,, \tag{21}$$

namely, the maximal spring force is

$$\boxed{F_\mathrm{r,max} = kx_\mathrm{max} \cong 128.55\,\mathrm{kN}\,.} \tag{22}$$

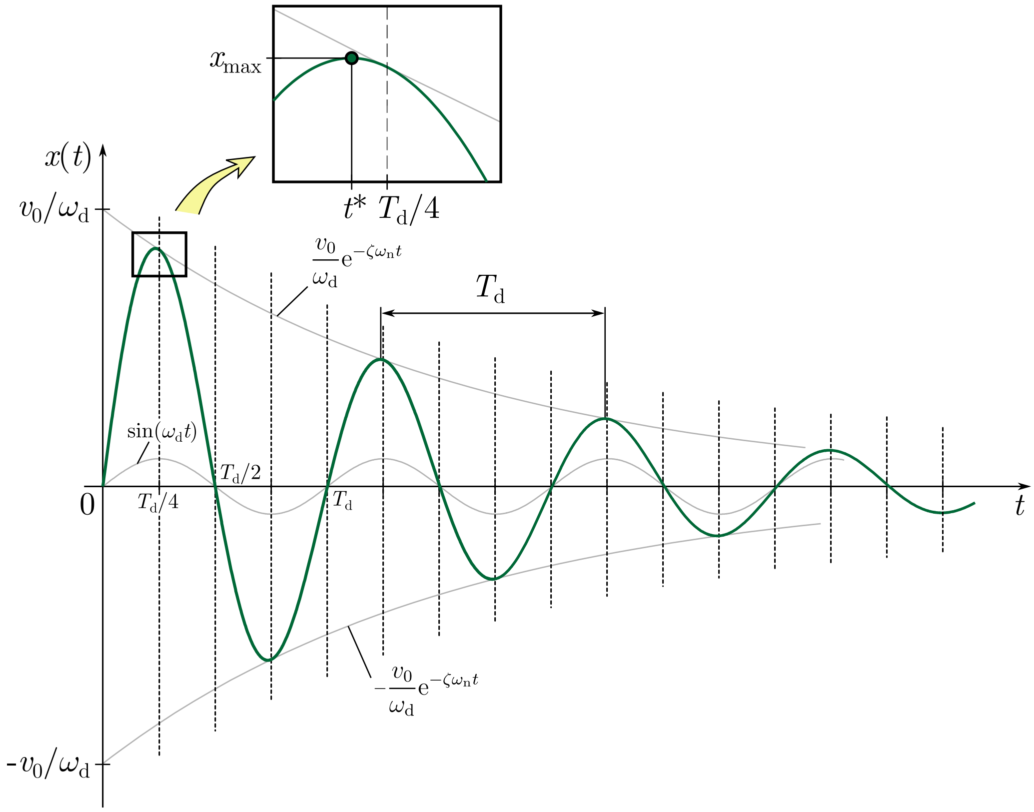

Fig. 3: Displacement response of the weakly damped 1 DoF oscillator

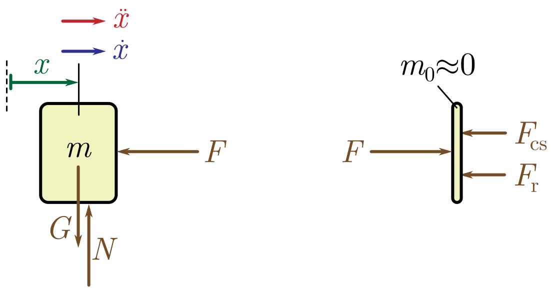

Fig. 4: Separated free body diagram of the wagon and the buffer stop

Task 2

The contact force between the wagon and the buffer stop can only be positive, it can only push the two objects away from each other. Consequently, contact between them will end when the contact force would change its direction.

Figure 4 shows the separated free body diagram of the wagon and the buffer stop with the contact force F acting between them. F can be determined by applying Newton's 2nd law either to the wagon, or to the buffer:

$$m\ddot{x} = -F \quad \text{or} \quad 0 = F - F_{\mathrm{cs}} - F_{\mathrm{r}}\,, \tag{23}$$

which leads to the following two different ways to express F:

$$F(t) = -m\ddot{x}(t) \quad \text{or} \quad F(t) = kx(t) + 2c\dot{x}(t)\,. \tag{24}$$

We are looking for the time $t = t^{**}$, when $F(t)$ changes signs, i.e. $F(t^{**}) = 0$. Based on Eq. (24), this can either be calculated from $\ddot{x}(t^{**}) = 0$ or from $kx(t^{**}) + 2c\dot{x}(t^{**}) = 0$. We are going to use the second option, leading to

$$k\mathrm{e}^{-\zeta\omega_{\mathrm{n}}t^{**}}C_2\sin(\omega_{\mathrm{d}}t^{**}) + 2c\mathrm{e}^{-\zeta\omega_{\mathrm{n}}t^{**}}\left(C_2\omega_{\mathrm{d}}\cos(\omega_{\mathrm{d}}t^{**}) - C_2\zeta\omega_{\mathrm{n}}\sin(\omega_{\mathrm{d}}t^{**})\right) = 0\,. \tag{25}$$

After simplifying with $C_2 \mathrm{e}^{-\zeta\omega_\mathrm{n} t^{**}}$, this can be written as

$$\tan(\omega_\mathrm{d} t^{**}) = \frac{2\omega_\mathrm{d} c}{2\omega_\mathrm{n}\zeta c - k} = \frac{\frac{2c}{m}\omega_\mathrm{d}}{\frac{2c}{m}\omega_\mathrm{n}\zeta - \frac{k}{m}} = \frac{2\zeta\omega_\mathrm{n}\omega_\mathrm{d}}{2\zeta^2\omega_\mathrm{n}^2 - \omega_\mathrm{n}^2} = \frac{2\zeta\omega_\mathrm{d}}{\omega_\mathrm{n}(2\zeta^2 - 1)} \,. \tag{26}$$

From this, the time duration of the impact can be expressed as

$$t^{**} = \frac{1}{\omega_\mathrm{d}} \arctan\left(\frac{2\zeta\omega_\mathrm{d}}{\omega_\mathrm{n}(2\zeta^2 - 1)}\right) = -0.232\,\mathrm{s} + \frac{T_\mathrm{d}}{2} = 0.553\,\mathrm{s}\,. \tag{27}$$

Taking into account the periodicity of the tangent function, we are looking for the smallest positive solution for t^{**}. The velocity of the wagon at the point of detachment is

$$v_1 = \dot{x}(t^{**}) = -\zeta\omega_\mathrm{n}\mathrm{e}^{-\zeta\omega_\mathrm{n} t^{**}} C_2 \sin(\omega_\mathrm{d} t^{**}) + \omega_\mathrm{d}\mathrm{e}^{-\zeta\omega_\mathrm{n} t^{**}} C_2 \cos(\omega_\mathrm{d} t^{**}) = -0.33\,\mathrm{m/s}\,. \quad (\leftarrow) \tag{28}$$

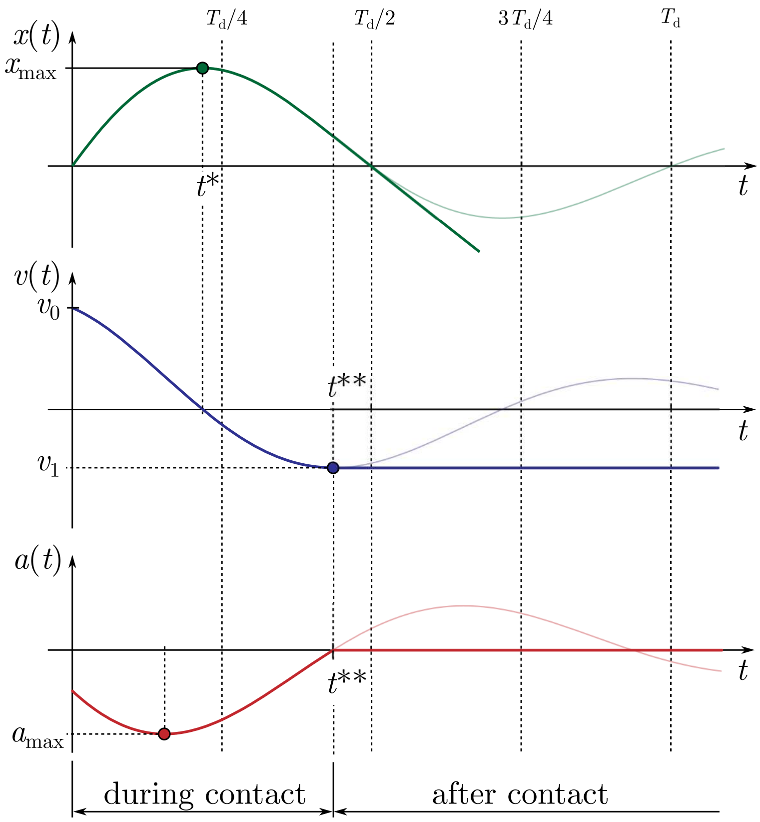

Fig. 5: Displacement, velocity and acceleration curves of the impact

Figure 5 also illustrates the motion of the wagon after the release (i.e. $t^{**} < t$). Since no horizontal forces are acting on the wagon anymore, its acceleration will stay zero and its velocity will remain v_1. Naturally, this will lead to a linearly changing displacement.

Since the velocity of the wagon is known before and after the impact, the change in kinetic energy can be calculated that is equal to the work of the dissipative forces:

$$W^\mathrm{diss} = T(t^{**}) - T(0) = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 \cong -22.27\,\mathrm{kJ}\,. \tag{29}$$

Thus, the amount of dissipated energy is

$$\boxed{E^\mathrm{diss} = -W^\mathrm{diss} = 22.27\,\mathrm{kJ}\,.} \tag{30}$$