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PRACTICE 5 – 1 DOF DAMPED SWINGING ARM

VIBRATIONS

– BMEGEMMBXM4 –

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Example

In Fig. 1, a swinging arm is shown that consists of two rods with different lengths and masses, and a disk with radius R. The swinging arm can only rotate along joint A, and the rods are connected to the environment through 2 spring with stiffness k_1 and k_2 and a damper with damping factor c_1. To describe the motion of the swinging arm, the angle φ measured from the horizontal axis is used as generalized coordinate. The structure is in the gravitational field and its equilibrium position is located at $\varphi = 0$. In this equilibrium position, the spring with stiffness k_2 is unloaded.

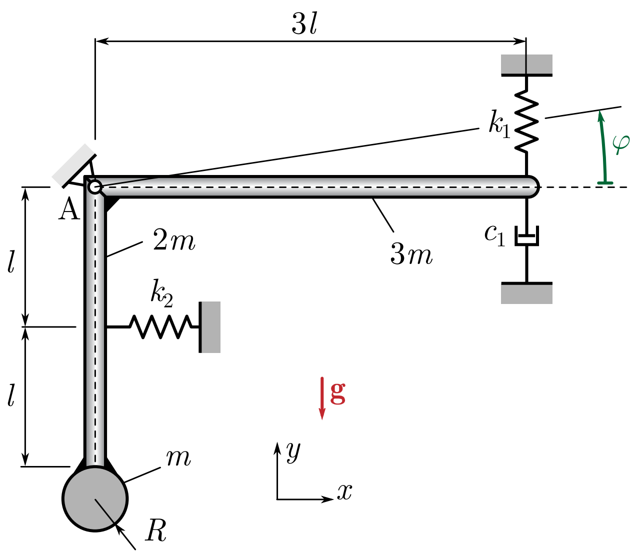

Fig. 1: The mechanical model of the swinging arm

Data

$l = 0.2$ m	$R = 0.1$ m	$m = 0.12\,$kg
$k_1 = 300\,$N/m	$k_2 = 10\,$N/m	$c_1 = 2\,$Ns/m

Tasks

1. Derive the equation of motion and calculate the natural angular frequencies of the undamped and damped system and the damping ratio! ($\omega_{\mathrm{n}} = 35.55$ rad/s, $\omega_{\mathrm{d}} = 35.31$ rad/s, $\zeta = 0.117$ [1])
2. Determine the critical damping factor in order to make the system critically damped! ($c_{1,\mathrm{cr}} = 17.10$ Ns/m)
3. Calculate the maximum force in the spring of stiffness k_1 if the initial conditions are $\varphi(t = 0) = \varphi_0 = 0.01$ rad and $\dot{\varphi}(t = 0) = 0$ rad/s! ($F_{\mathrm{r1,max}} = 3.009\,$N)

Made by Ádám Kiss

Solution

Task 1

Figure 2 shows the free body diagram of the swinging arm in a disturbed position. In a previous example (in Practice 3), we have already shown how the system can be linearized around the equilibrium and what kind of simplifications are available to provide the linear equation of motion. To summarize these assumptions:

- the gravitational force acting on the horizontal rods does not affect the natural frequency (but it affects the maximal spring forces)
- the deformation of the springs can be approximated by their arc lengths (measured from the equilibrium position).

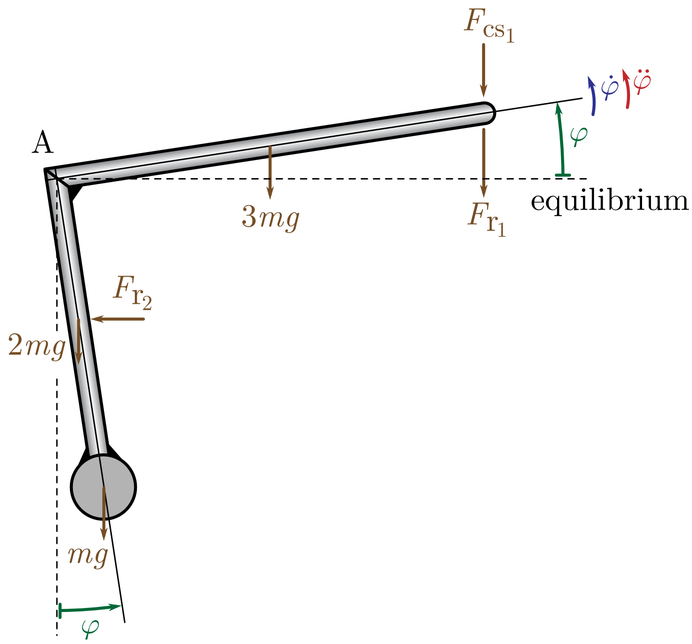

Fig. 2: Free body diagram

The equation of motion can be derived using Newton's 2nd law:

$$\dot{\mathbf{I}} = \mathbf{F}\,. \tag{1}$$

This translates into

$$\theta_A\ddot{\varphi} = -F_{\mathrm{r}_1}3l\underbrace{\cos\varphi}_{\approx 1} - F_{\mathrm{r}_2}l\underbrace{\cos\varphi}_{\approx 1} - F_{\mathrm{cs}_1}3l\underbrace{\cos\varphi}_{\approx 1} - 3mg\frac{3}{2}l\underbrace{\cos\varphi}_{\approx 1} - 2mgl\underbrace{\sin\varphi}_{\approx\varphi} - mg(2l+R)\underbrace{\sin\varphi}_{\approx\varphi}, \tag{2}$$

where the damping force F_{cs_1} and spring forces F_{r_i} are

$$F_{\mathrm{cs}_1} \cong c_1 3l\dot{\varphi} \quad \text{and} \quad F_{\mathrm{r}_1} \cong F_{\mathrm{r}_{1\mathrm{st}}} + k_1 3l\varphi \quad \text{and} \quad F_{\mathrm{r}_2} \cong k_2 l\varphi\,. \tag{3}$$

The mass moment of inertia of the swinging arm at the point A with respect to the rotational axis can be calculated by means of applying the parallel-axis theorem (Steiner's theorem) as

$$\theta_A = \frac{1}{3}3m(3l)^2 + \frac{1}{3}2m(2l)^2 + \frac{1}{2}mR^2 + m(2l+R)^2 = 0.0866\,\mathrm{kgm}^2\,. \tag{4}$$

Note that the term $-3mg\frac{3}{2}l\cos\varphi$ in Eq. (2) is balanced by the static spring force $F_{\mathrm{r}_{1\mathrm{st}}}$. These terms can be skipped together (see Practice 3), therefore, the equation of motion reads as

$$\theta_A\ddot{\varphi} = -9l^2k_1\varphi - l^2k_2\varphi - 9l^2c_1\dot{\varphi} - 2mgl\varphi - mg(2l+R)\varphi\,. \tag{5}$$

Rearranging the terms into the left-hand-side leads to

$$\boxed{\theta_A\ddot{\varphi} + 9l^2c_1\dot{\varphi} + (9l^2k_1 + l^2k_2 + 4mgl + mgR)\varphi = 0\,.} \tag{6}$$

Dividing the equation by θ_A results the general form

$$\ddot{\varphi} + \underbrace{\frac{9l^2c_1}{\theta_A}}_{=2\zeta\omega_\mathrm{n}}\dot{\varphi} + \underbrace{\frac{9l^2k_1 + l^2k_2 + 4mgl + mgR}{\theta_A}}_{=\omega_\mathrm{n}^2}\varphi = 0\,. \tag{7}$$

Substituting the parameters of the example leads to the natural angular frequency of the undamped system

$$\boxed{\omega_\mathrm{n} = \sqrt{\frac{9l^2k_1 + l^2k_2 + 4mgl + mgR}{\theta_A}} = 35.55\,\mathrm{rad/s}} \tag{8}$$

and to damping ratio

$$\boxed{\zeta = \frac{1}{2\omega_\mathrm{n}}\frac{9l^2c_1}{\theta_A} = 0.117\,(= 11.7\%)\,.} \tag{9}$$

The natural angular frequency of the damped system is

$$\boxed{\omega_\mathrm{d} = \omega_\mathrm{n}\sqrt{1-\zeta^2} = 35.31\,\mathrm{rad/s}\,,} \tag{10}$$

correspondingly, the period of oscillation

$$T_\mathrm{d} = \frac{2\pi}{\omega_\mathrm{d}} = 0.1779\,\mathrm{s} \tag{11}$$

and the frequency of the oscillation

$$f_\mathrm{d} = \frac{\omega_\mathrm{d}}{2\pi} = 5.62\,\mathrm{Hz}\,. \tag{12}$$

Task 2

In case of critically damped system, the damping ratio is set to one ($\zeta_\mathrm{cr} = 1$). Using the expression from Eq. (7)

$$2\zeta_\mathrm{cr}\omega_\mathrm{n} = \frac{9l^2c_{1,\mathrm{cr}}}{\theta_A}, \tag{13}$$

the critical damping factor can be expressed as

$$\boxed{c_{1,\mathrm{cr}} = \frac{2\zeta_\mathrm{cr}\omega_\mathrm{n}\theta_A}{9l^2} = 17.10\,\mathrm{Ns/m}\,.} \tag{14}$$

Task 3

According to Eq. (3), one can calculate the force in the spring of stiffness k_1 as

$$F_{\mathrm{r}_1}(t) = F_{\mathrm{r}_{1\mathrm{st}}} + k_1 3l\varphi(t) \tag{15}$$

since the spring is already stretched in the equilibrium at $\varphi = 0$. The static spring force $F_{\mathrm{r}_{1\mathrm{st}}}$ (resulted by the static deformation) can be calculated from static equilibrium equation as (see the free body diagram of the equilibrium in Fig. 3)

$$\sum M_A = 0: \quad -F_{\mathrm{r}_{1\mathrm{st}}}3l - 3mg\frac{3}{2}l = 0\,, \tag{16}$$

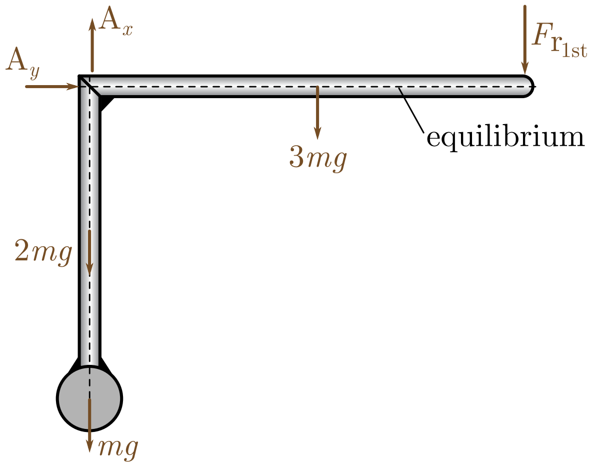

Fig. 3: Static equilibrium

from which, we obtain:

$$F_{\mathrm{r}_{1\mathrm{st}}} = -\frac{3}{2}mg = -1.7658\,\mathrm{N}\,. \tag{17}$$

In order to determine the spring force by Eq. (15), the time history of the generalized coordinate $\varphi(t)$ has to be determined. Equation (7) is a 2nd order, homogeneous, ordinary differential equation. Its general solution is of the form

$$\varphi(t) = \mathrm{e}^{-\zeta\omega_\mathrm{n}t}\left(C_1\cos(\omega_\mathrm{d}t) + C_2\sin(\omega_\mathrm{d}t)\right)\,, \tag{18}$$

$$\dot{\varphi}(t) = -\zeta\omega_\mathrm{n}\mathrm{e}^{-\zeta\omega_\mathrm{n}t}\left(C_1\cos(\omega_\mathrm{d}t) + C_2\sin(\omega_\mathrm{d}t)\right) + \omega_\mathrm{d}\mathrm{e}^{-\zeta\omega_\mathrm{n}t}\left(-C_1\sin(\omega_\mathrm{d}t) + C_2\cos(\omega_\mathrm{d}t)\right)\,. \tag{19}$$

The coefficients C_1 and C_2 can be determined from the initial conditions, which are

$$\begin{cases} \varphi(t=0) = \varphi_0 & \Rightarrow \quad C_1\cdot 1 + C_2\cdot 0 = \varphi_0\,, \\ \dot{\varphi}(t=0) = 0 & \Rightarrow \quad -\zeta\omega_\mathrm{n}\left(C_1\cdot 1 + C_2\cdot 0\right) + \omega_\mathrm{d}\left(-C_1\cdot 0 + C_2\cdot 1\right) = 0\,. \end{cases} \tag{20}$$

This leads to

$$C_1 = \varphi_0 \quad \text{and} \quad C_2 = \frac{\zeta\omega_\mathrm{n}}{\omega_\mathrm{d}}C_1 = \frac{\zeta\omega_\mathrm{n}}{\omega_\mathrm{d}}\varphi_0\,. \tag{21}$$

Since the spring force is proportional to the displacement, the maximal spring force will occur when φ reaches its maximum at $t = t^*$. In order to find t^*, we are looking for a local extremum of $\varphi(t)$. Using Eq. (19), this can be written as

$$\dot{\varphi}(t^*) = 0 \Rightarrow -\zeta\omega_\mathrm{n}\mathrm{e}^{-\zeta\omega_\mathrm{n}t^*}\big(C_1\cos(\omega_\mathrm{d}t^*) + C_2\sin(\omega_\mathrm{d}t^*)\big) + \omega_\mathrm{d}\mathrm{e}^{-\zeta\omega_\mathrm{n}t^*}\big(-C_1\sin(\omega_\mathrm{d}t^*) + C_2\cos(\omega_\mathrm{d}t^*)\big) = 0\,. \tag{22}$$

Simplifying with $\mathrm{e}^{-\zeta\omega_\mathrm{n}t^*}$ leads to

$$\tan(\omega_\mathrm{d}t^*) = \frac{-C_1\zeta\omega_\mathrm{n} + C_2\omega_\mathrm{d}}{C_2\zeta\omega_\mathrm{n} + C_1\omega_\mathrm{d}}\,, \tag{23}$$

keeping in mind that $C_2 = \frac{\zeta\omega_\mathrm{n}}{\omega_\mathrm{d}}C_1$, it can be solved for t^* as

$$t^* = \frac{1}{\omega_\mathrm{d}}\arctan 0 = 0 + j\underbrace{\frac{T_\mathrm{d}}{2}}_{=\pi/\omega_\mathrm{d}} \quad \text{for} \quad j = 0, 1, 2, \ldots\,. \tag{24}$$

Since the initial condition corresponds to a "release" from a displaced position, in other words, a start with no initial velocity ($\dot{\varphi}(0) = 0$), therefore the initial condition is a local (and also a global) extremum $\varphi_\mathrm{max} = \varphi_0 = 0.01$ rad.

But, due to the static deformation of the spring, the maximum spring force may not be at this global maximum. One have to check the maximal spring force at more local extremums (see Fig. 4). The second local extremum is located at $\varphi(t^* + T_\mathrm{d}/2) = \varphi(T_\mathrm{d}/2) = -0.007$ [rad]. The spring force at the first and the second local extremums can be calculated as

$$\begin{aligned} F_{\mathrm{r}_1}(t^*) &= F_{\mathrm{r}_{1\mathrm{st}}} + k_1 3l\varphi(t^*) = 0.034\,\mathrm{N} \\ F_{\mathrm{r}_1}(t^* + T_\mathrm{d}/2) &= F_{\mathrm{r}_{1\mathrm{st}}} + k_1 3l\varphi(t^* + T_\mathrm{d}/2) = -3.009\,\mathrm{N}, \end{aligned} \tag{25}$$

therefore, the maximal spring force is

$$\boxed{|F_{\mathrm{r1,max}}| = |F_{\mathrm{r}_1}(T_\mathrm{d}/2)| = 3.009\,\mathrm{N}.} \tag{26}$$

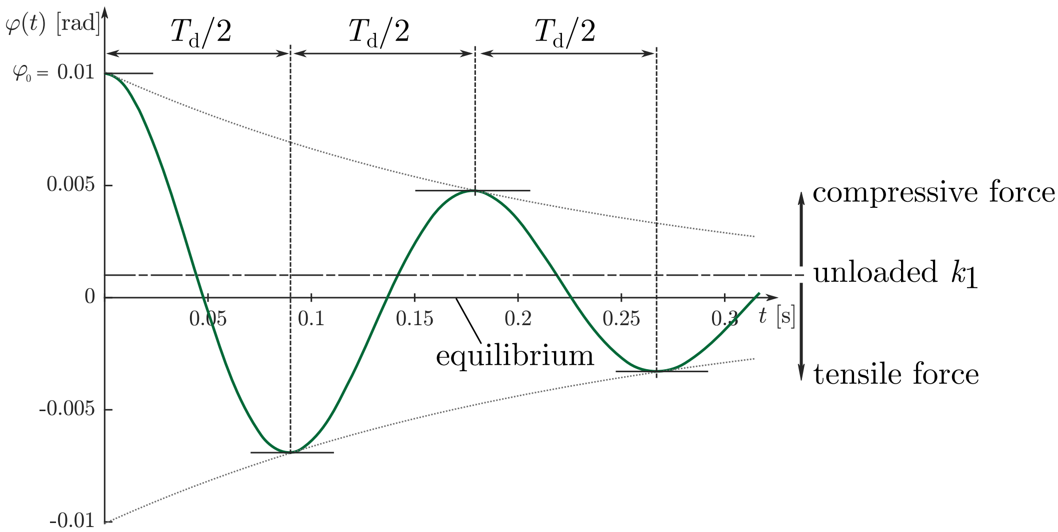

Fig. 4: Time history of generalized coordinate