Budapest University of Technology and Economics
Department of Applied Mechanics

PRACTICE 11 – 3 DOF LINEAR SYSTEM

VIBRATIONS

– BMEGEMMBXM4 –

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Example

In Fig. 1, a 3 DoF system can be seen that consists of three masses m_1, m_2 and m_3. Between the masses there are two springs with a stiffness of k_1 and k_2. The displacements of the corresponding masses are described by the generalized coordinates x_1, x_2 and x_3.

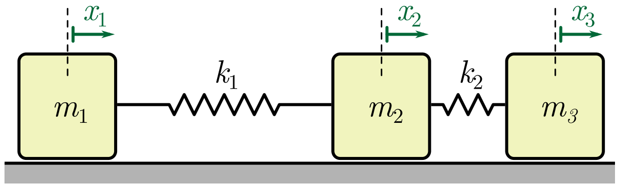

Fig. 1: Mechanical model of a 3 DoF oscillator

Data

$m_1 = 2\,\mathrm{kg}$ $\quad m_2 = 4\,\mathrm{kg}$ $\quad m_3 = 5\,\mathrm{kg}$
$k_1 = 200\,\mathrm{N/m}$ $\quad k_2 = 500\,\mathrm{N/m}$

Tasks

1. Derive the equations of motion of the system!
2. Calculate the natural angular frequencies of the multi degree-of-freedom system and determine the corresponding mode shapes!

Made by Bálint Bodor

Solution

Task 1

In general, the liearised equations of motion of a multi degree-of-freedom mechanical system can be directly determined if the system is only exposed to geometric constraints not depending on time. For the small vibrations around a certain equilibrium point, the equations of motion have the form:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q}^*, \tag{1}$$

where $\mathbf{q}$ is the vector of the generalized coordinates. The constant matrix coefficients are the mass matrix $\mathbf{M} = [m_{ij}]$, the damping matrix $\mathbf{C} = [c_{ij}]$ and the stiffness matrix $\mathbf{K} = [k_{ij}]$. The vector $\mathbf{Q}^*$ denotes those components of the generalized forces which cannot be calculated from the terms $\mathbf{C}\dot{\mathbf{q}}$ and $\mathbf{K}\mathbf{q}$. The m_{ij}, c_{ij} and k_{ij} elements of the matrices can be calculated with the following formulas (assuming $\mathbf{q} = \mathbf{0}$ is an equilibrium state):

$$m_{ij} = \left.\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right|_{\mathbf{q}=\mathbf{0}}, \tag{2}$$

$$c_{ij} = \left.\frac{\partial^2 \mathcal{D}}{\partial \dot{q}_i \partial \dot{q}_j}\right|_{\mathbf{q}=\mathbf{0}}, \tag{3}$$

$$k_{ij} = \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{\mathbf{q}=\mathbf{0}}. \tag{4}$$

Here T denotes the kinetic energy of the system, U is the potential function, and $\mathcal{D}$ refers to Rayleigh's dissipation function.

In the present example to derive the equations of motion we express the kinetic energy and the potential function first:

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2, \tag{5}$$

$$U = \frac{1}{2}k_1(x_2 - x_1)^2 + \frac{1}{2}(x_3 - x_2)^2. \tag{6}$$

The elements of the matrix coefficients can be computed by evaluating the partial derivatives in Eqs. (2)-(4). In our case, the mass matrix has non-zero elements only in the diagonal:

$$m_{11} = \left.\frac{\partial^2 T}{\partial \dot{x}_1^2}\right|_{\mathbf{q}=\mathbf{0}} = m_1\,, \tag{7}$$

$$m_{22} = \left.\frac{\partial^2 T}{\partial \dot{x}_2^2}\right|_{\mathbf{q}=\mathbf{0}} = m_2\,, \tag{8}$$

$$m_{33} = \left.\frac{\partial^2 T}{\partial \dot{x}_3^2}\right|_{\mathbf{q}=\mathbf{0}} = m_3\,. \tag{9}$$

The elements in the stiffness matrix that differ from zero are:

$$k_{11} = \left.\frac{\partial^2 U}{\partial x_1^2}\right|_{\mathbf{q}=\mathbf{0}} = k_1, \tag{10}$$

$$k_{22} = \left.\frac{\partial^2 U}{\partial x_2^2}\right|_{\mathbf{q}=\mathbf{0}} = k_1 + k_2, \tag{11}$$

$$k_{33} = \left.\frac{\partial^2 U}{\partial x_3^2}\right|_{\mathbf{q}=\mathbf{0}} = k_2, \tag{12}$$

$$k_{12} = k_{21} = \left.\frac{\partial^2 U}{\partial x_1 \partial x_2}\right|_{\mathbf{q}=\mathbf{0}} = -k_1, \tag{13}$$

$$k_{23} = k_{32} = \left.\frac{\partial^2 U}{\partial x_2 \partial x_3}\right|_{\mathbf{q}=\mathbf{0}} = -k_2. \tag{14}$$

As there is no damping element in the system, therefore we have $\mathcal{D} = 0$ and $\mathbf{C} = \mathbf{0}$ as well. Furthermore $\mathbf{Q}^* = \mathbf{0}$ holds too, because we do not have any external forces or excitations.

Finally the equations of motion can be expressed in the form:

$$\boxed{\underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}}_{=\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}}_{=\ddot{\mathbf{q}}} + \underbrace{\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}}_{=\mathbf{K}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{=\mathbf{q}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.} \tag{15}$$

Task 2

As the equations of motion have the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}\,, \tag{16}$$

therefore we can assume the following trial solution:

$$\mathbf{q}(t) = \mathbf{A}e^{\mathrm{i}\omega_\mathrm{n} t}\,, \tag{17}$$

where $\mathbf{A}$ is a mode shape vector. Substituting this into Eq. (16) we obtain:

$$-\omega_\mathrm{n}^2 \mathbf{M}\mathbf{A}e^{\mathrm{i}\omega_\mathrm{n} t} + \mathbf{K}\mathbf{A}e^{\mathrm{i}\omega_\mathrm{n} t} = \mathbf{0}, \tag{18}$$

which leads to:

$$(-\omega_\mathrm{n}^2 \mathbf{M} + \mathbf{K}) \underbrace{\mathbf{A}}_{\neq \mathbf{0}} \underbrace{e^{\mathrm{i}\omega_\mathrm{n} t}}_{\neq 0} = \mathbf{0}\,. \tag{19}$$

In this equation $e^{\mathrm{i}\omega_\mathrm{n} t} \neq 0$, furthermore, $\mathbf{A} \neq \mathbf{0}$ should hold as well, as we aim to determine the non-trivial solution. Namely, the homogenous linear equation (19) implies that a non-trivial solution exists if and only if

$$\det(-\omega_\mathrm{n}^2 \mathbf{M} + \mathbf{K}) = 0\,. \tag{20}$$

Eq. (20) is called frequency equation as the solutions $\omega_{\mathrm{n}i}$ are the natural angular frequencies that are ordered in an ascending sequence by convention, that is: $0 \leq \omega_\mathrm{n1} < \omega_\mathrm{n2} < ... < \omega_{\mathrm{n}n}$ (with n denoting the number of DoF of the system). Substituting the i-th solution to the homogenous equation (19) we get:

$$(-\omega_{\mathrm{n}i}^2 \mathbf{M} + \mathbf{K})\mathbf{A}_i = \mathbf{0}\,. \tag{21}$$

From this equation the mode shape vector $\mathbf{A}_i$ can be calculated. The solution of the equations of motion can be constructed by a linear combination of the mode shapes:

$$\mathbf{q}(t) = \sum_{i=1}^{n} \mathbf{A}_i \big(C_{i1} \cos(\omega_{\mathrm{n}i} t) + C_{i2} \sin(\omega_{\mathrm{n}i} t) \big). \tag{22}$$

Here, the constants C_{i1} and C_{i2} (with $i = 1, ..., n$) can be calculated from the the initial conditions.

In the present example the frequency equation (20) takes the form:

$$\begin{vmatrix} -m_1\omega_\mathrm{n}^2 + k_1 & -k_1 & 0 \\ -k_1 & -m_2\omega_\mathrm{n}^2 + k_1 + k_2 & -k_2 \\ 0 & -k_2 & -m_3\omega_\mathrm{n}^2 + k_2 \end{vmatrix} = 0\,. \tag{23}$$

After the substitution of the numerical values of the parameters the determinant reads

$$\omega_\mathrm{n}^2(\omega_\mathrm{n}^4 - 375\omega_\mathrm{n}^2 + 27500) = 0\,. \tag{24}$$

The solutions of the equation are the natural angular frequencies:

$$\boxed{\omega_{\mathrm{n}1} = 0 \text{ rad/s},} \tag{25}$$

$$\boxed{\omega_{\mathrm{n}2} = 10 \text{ rad/s},} \tag{26}$$

$$\boxed{\omega_{\mathrm{n}3} = 16.58 \text{ rad/s}.} \tag{27}$$

To calculate the mode shapes we solve Eq. (21) using all the natural frequencies. Because Eq. (20) holds, therefore the equations are not independent. In other words the solution of Eq. (21) is not unique. Hence, we select the first component of all mode shape vectors $\mathbf{A}_i$ to be 1, namely:

$$\mathbf{A}_i = \begin{bmatrix} A_{i1} \\ A_{i2} \\ A_{i3} \end{bmatrix} = \begin{bmatrix} 1 \\ A_{i2} \\ A_{i3} \end{bmatrix}. \tag{28}$$

Using this in Eq. (21) lead to

$$\begin{bmatrix} -m_1\omega_{\mathrm{n}i}^2 + k_1 & -k_1 & 0 \\ -k_1 & -m_2\omega_{\mathrm{n}i}^2 + k_1 + k_2 & -k_2 \\ 0 & -k_2 & -m_3\omega_{\mathrm{n}i}^2 + k_2 \end{bmatrix} \begin{bmatrix} A_{i1} \\ A_{i2} \\ A_{i3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{29}$$

For $i = 1$, the first two equations give

$$k_1 A_{11} - k_1 A_{12} = 0\,, \tag{30}$$

$$-k_1 A_{11} + (k_1 + k_2) A_{12} - k_2 A_{13} = 0\,. \tag{31}$$

With $A_{11} = 1$ these lead to $A_{12} = 1$ and $A_{13} = 1$. Therefore the first mode is

$$\boxed{\mathbf{A}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ m.}} \tag{32}$$

The second mode shape can be calculated using Eq. (29) with $i = 2$. Now, we use the first and the third equations:

$$(-m_1\omega_{\mathrm{n}2}^2 + k_1) A_{21} - k_1 A_{22} = 0\,, \tag{33}$$

$$-k_2 A_{22} + (-m_3\omega_{\mathrm{n}2}^2 + k_2) A_{23} = 0\,. \tag{34}$$

With $A_{21} = 1$ these result $A_{22} = 0$ and $A_{23} = -0.4$, namely:

$$\boxed{\mathbf{A}_2 = \begin{bmatrix} 1 \\ 0 \\ -0.4 \end{bmatrix} \text{ m.}} \tag{35}$$

Similarly, the third mode shape can be calculated as

$$\boxed{\mathbf{A}_3 = \begin{bmatrix} 1 \\ -1.75 \\ 1 \end{bmatrix} \text{ m.}} \tag{36}$$

Based on Eq. (22), for specific initial conditions, the system is oscillating with the i^{th} natural angular frequency, and the values of the generalized coordinates will be proportional to the elements of i^{th} mode shape in any arbitrary time instant. This can be illustrated for our mechanical model in a specific way that is shown in Fig. 2.

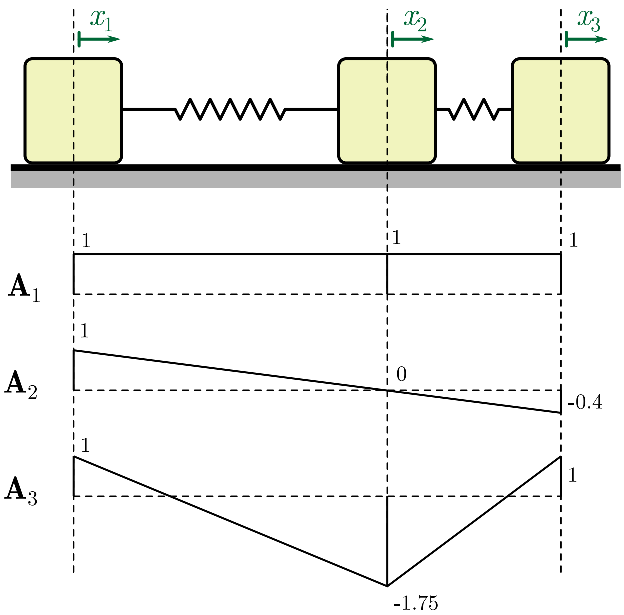

Fig. 2: Illustration of the mode shapes