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PRACTICE 10 – 2 DOF NONLINEAR SYSTEM

VIBRATIONS

– BMEGEMMBXM4 –

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Example

In Fig. 1, a 2 DoF dynamical system is shown. The disk of mass m_1 and radius r rolls on the moving surface of radius R. The body of mass m_2 is supported by the springs of stiffnesses k_1, k_2 and k_3. The vertical displacement of the mass m_2 is described by the generalized coordinate y and the position of the disk is described by the generalized coordinate φ. The system is in the gravitational field and its equilibrium position is located at $y = 0$ and $\varphi = 0$, where the springs have static deformations only.

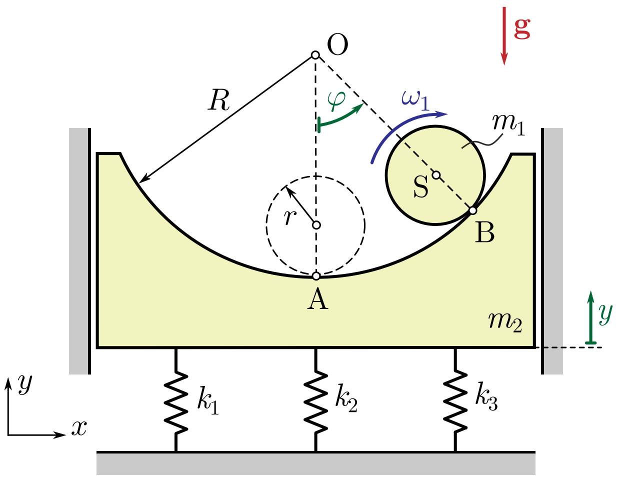

Fig. 1: Mechanical model of the 2 DoF system

Data

The masses m_1 and m_2, the radii r and R and the stiffnesses k_1, k_2 and k_3 are known.

Tasks

1. Determine the nonlinear equations of motion of the system using the Lagrange equation of the second kind and linearize them about the equilibrium $y = 0$ and $\varphi = 0$!
2. Determine the linearized matrix differential equation of the system!

Solution

Task 1

The mechanical model presented in Fig. 1 is a 2 DoF system, since the motion can be described using the two generalized coordinates y and φ. This means that

$$n = 2 \text{ DoF} \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} y \\ \varphi \end{bmatrix} . \tag{1}$$

The kinetic energy can be calculated as

$$T = \frac{1}{2} m_2 \dot{y}^2 + \frac{1}{2} m_1 v_{\mathrm{S}}^2 + \frac{1}{2} \Theta_{1\mathrm{Sz}} \omega_{1z}^2 , \tag{2}$$

where $\Theta_{1\mathrm{Sz}} = \frac{1}{2} m_1 r^2$ is the mass moment of inertia of the disk about the z axis with respect to the center of gravity S.

First, we have to express the kinetic energy (2) in terms of the generalized velocities and generalized coordinates. The velocity of the point S can be expressed as

$$\mathbf{v}_{\mathrm{S}} = \mathbf{v}_{\mathrm{O}} + \begin{bmatrix} 0 \\ 0 \\ \dot{\varphi} \end{bmatrix} \times \mathbf{r}_{\mathrm{OS}} , \tag{3}$$

and since the disk rolls on the surface:

$$\mathbf{v}_{\mathrm{S}} = \mathbf{v}_{\mathrm{B}} + \boldsymbol{\omega}_1 \times \mathbf{r}_{\mathrm{BS}} . \tag{4}$$

The velocity of the points O and B are $\mathbf{v}_{\mathrm{O}} = \mathbf{v}_{\mathrm{B}} = \begin{bmatrix} 0 & \dot{y} & 0 \end{bmatrix}^T$, therefore $\begin{bmatrix} 0 & 0 & \dot{\varphi} \end{bmatrix}^T \times \mathbf{r}_{\mathrm{OS}} = \boldsymbol{\omega}_1 \times \mathbf{r}_{\mathrm{BS}}$. Expanding the cross products yields

$$(R - r)\dot{\varphi} = -r\omega_{1z} \quad \Rightarrow \quad \omega_{1z} = -\frac{R - r}{r} \dot{\varphi} . \tag{5}$$

Using Eq. (4), the velocity of the center of gravity S:

$$\mathbf{v}_{\mathrm{S}} = \begin{bmatrix} 0 \\ \dot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \omega_{1z} \end{bmatrix} \times \begin{bmatrix} -r \sin\varphi \\ r\cos\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} -r\omega_{1z}\cos\varphi \\ \dot{y} - r\omega_{1z}\sin\varphi \\ 0 \end{bmatrix} . \tag{6}$$

Based on Eq. (5), we obtain

$$v_{\mathrm{S}}^2 = \dot{y}^2 + 2(R - r)\dot{y}\dot{\varphi}\sin\varphi + (R - r)^2 \dot{\varphi}^2 . \tag{7}$$

Finally, the kinetic energy:

$$T = \frac{1}{2}(m_1 + m_2)\dot{y}^2 + \frac{1}{2} m_1 \left(2(R - r)\dot{y}\dot{\varphi}\sin\varphi + \frac{3}{2}(R - r)^2 \dot{\varphi}^2 \right) . \tag{8}$$

The potential function contains the strain energy of the springs and the potentials of the gravitational forces. Let us introduce the vertical displacement z measured from the unloaded state of the springs, see Fig. 2. Namely, $z = y - z_{\mathrm{st}}$, where z_{st} is the static deformation of the springs, correspondingly $(k_1 + k_2 + k_3) z_{\mathrm{st}} = (m_1 + m_2) g$. Thus,

$$U = \frac{1}{2} \underbrace{(k_1 + k_2 + k_3)}_{=: \, k_{\mathrm{e}}} z^2 + m_1 g (z - (R - r)\cos\varphi) + m_2 g z . \tag{9}$$

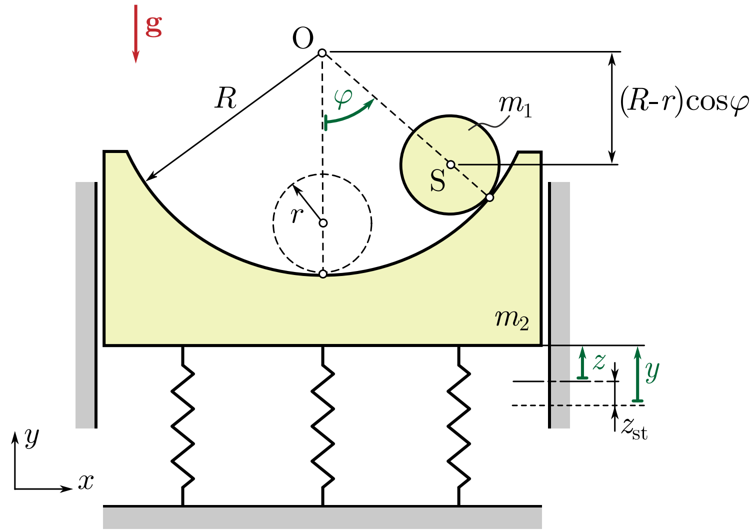

Fig. 2: The illustration of the static deformation

After substitution the coordinate transformation $z = y - z_{\mathrm{st}}$, we obtain

$$U = \frac{1}{2}k_{\mathrm{e}}y^2 + \underbrace{\left((m_1+m_2)g - k_{\mathrm{e}}z_{\mathrm{st}}\right)}_{=0} y - m_1 g(R-r)\cos\varphi + \mathrm{const.}\,. \tag{10}$$

A simplified potential function $\widetilde{U}$ can also be composed:

$$\widetilde{U} = \frac{1}{2}k_{\mathrm{e}}y^2 - m_1 g(R-r)\cos\varphi\,, \tag{11}$$

which results the same terms in the Lagrange equation as our original potential function U.

Now, we can calculate the terms of the Lagrange equation:

$$\left.\begin{aligned}
\frac{\partial T}{\partial \dot{y}} &= (m_1+m_2)\dot{y} + m_1(R-r)\dot{\varphi}\sin\varphi\,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{y}} &= (m_1+m_2)\ddot{y} + m_1(R-r)\ddot{\varphi}\sin\varphi + m_1(R-r)\dot{\varphi}^2\cos\varphi\,,\\
\frac{\partial T}{\partial y} &= 0\,,\\
\frac{\partial \widetilde{U}}{\partial y} &= k_{\mathrm{e}}y\,,
\end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned}
\frac{\partial T}{\partial \dot{\varphi}} &= \frac{3}{2}m_1(R-r)^2\dot{\varphi} + m_1(R-r)\dot{y}\sin\varphi\,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\varphi}} &= \frac{3}{2}m_1(R-r)^2\ddot{\varphi} + m_1(R-r)\ddot{y}\sin\varphi + m_1(R-r)\dot{y}\dot{\varphi}\cos\varphi\,,\\
\frac{\partial T}{\partial \varphi} &= m_1(R-r)\dot{y}\dot{\varphi}\cos\varphi\,,\\
\frac{\partial \widetilde{U}}{\partial \varphi} &= m_1 g(R-r)\sin\varphi\,.
\end{aligned}\right\} \tag{13}$$

Then, the nonlinear equations of motion read

$$\boxed{\begin{aligned}
(m_1+m_2)\ddot{y} + m_1(R-r)\ddot{\varphi}\sin\varphi + m_1(R-r)\dot{\varphi}^2\cos\varphi + k_{\mathrm{e}}y &= 0\,,\\
\frac{3}{2}m_1(R-r)^2\ddot{\varphi} + m_1(R-r)\ddot{y}\sin\varphi + m_1 g(R-r)\sin\varphi &= 0\,.
\end{aligned}} \tag{14}$$

For the linearization about the equilibrium, one can use

$$\sin\varphi \approx \varphi \quad \text{and} \quad \cos\varphi \approx 1\,, \tag{15}$$

and after substitution of these into the equations of motion, the secondary small terms (e.g. $\ddot{\varphi}\varphi \approx 0$) have to be omitted. After linearization the equations (14) can be written as

$$\boxed{\begin{aligned} (m_1+m_2)\ddot{y}+k_\mathrm{e}y &= 0\,,\\ \frac{3}{2}m_1(R-r)^2\ddot{\varphi}+m_1g(R-r)\varphi &= 0\,. \end{aligned}} \tag{16}$$

It is worth to note that the resulted linearized equations of motion of our system are not coupled, namely, the small vibrations described by y and φ do not influence each other.

Task 2

The linearized matrix differential equation of the system has the form

$$\boxed{\mathbf{M}\ddot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{0}\,,} \tag{17}$$

where the elements of the mass matrix $\mathbf{M}=[m_{ij}]$ and stiffness matrix $\mathbf{K}=[k_{ij}]$ can be identified from the linearized equation of motion (16), or they can also be calculated directly from the kinetic energy and from the potential function as

$$m_{ij}=\left.\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right|_{\mathbf{0}} \quad \text{and} \quad k_{ij}=\left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{\mathbf{0}}\,. \tag{18}$$

Using the kinetic energy (8) and the potential function (10), the elements of the matrices are

$$m_{11}=\left.\frac{\partial^2 T}{\partial \dot{y}^2}\right|_{\substack{y=0\\ \varphi=0}}=m_1+m_2\,, \quad m_{22}=\left.\frac{\partial^2 T}{\partial \dot{\varphi}^2}\right|_{\substack{y=0\\ \varphi=0}}=\frac{3}{2}m_1\left(R-r\right)^2\,, \tag{19}$$

$$m_{12}=m_{21}=\left.\frac{\partial^2 T}{\partial \dot{y}\partial \dot{\varphi}}\right|_{\substack{y=0\\ \varphi=0}}=m_1(R-r)\sin(0)=0\,, \tag{20}$$

$$k_{11}=\left.\frac{\partial^2 U}{\partial y^2}\right|_{\substack{y=0\\ \varphi=0}}=k_\mathrm{e}\,, \quad k_{22}=\left.\frac{\partial^2 U}{\partial \varphi^2}\right|_{\substack{y=0\\ \varphi=0}}=m_1g\left(R-r\right)\cos(0)=m_1g\left(R-r\right)\,, \tag{21}$$

$$k_{12}=k_{21}=\left.\frac{\partial^2 U}{\partial y\partial \varphi}\right|_{\substack{y=0\\ \varphi=0}}=0\,. \tag{22}$$

Therefore, the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ are

$$\boxed{\mathbf{M}=\begin{bmatrix} m_1+m_2 & 0\\ 0 & \frac{3}{2}m_1\left(R-r\right)^2 \end{bmatrix}} \tag{23}$$

and

$$\boxed{\mathbf{K}=\begin{bmatrix} k_\mathrm{e} & 0\\ 0 & m_1g\left(R-r\right) \end{bmatrix}\,,} \tag{24}$$

respectively.