Budapest University of Technology and Economics
Department of Applied Mechanics

PRACTICE 9 – VIBRATION EXCITER

VIBRATIONS

– BMEGEMMBXM4 –

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Example

In Fig. 1, a vibration exciter is shown, which consists of a rigid body with total mass m, four symmetrically rotating eccentric masses with angular velocity ω, eccentricity r, and mass m_0. According to the simplified mechanical model, the moving rigid body is connected to the base with an ideal spring and damper with stiffness k and damping factor c, respectively. To describe the motion of the single-degree-of-freedom vibration exciter, the vertical displacement y measured from the static equilibrium is used as a generalized coordinate. In the equilibrium position ($y = 0$) the spring is preloaded, which results a force acting against the gravity.

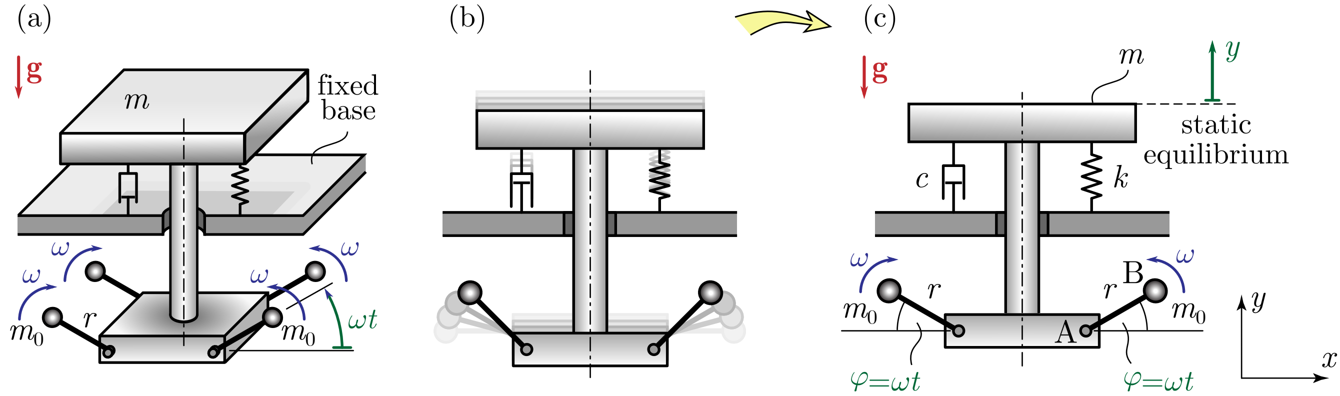

Fig. 1: Mechanical model of a vibration exciter. (a) 3D view. (b) Vibration exciter in operation. (c) Frontal view.

Data

$m = 60\,\mathrm{kg}$	$m_0 = 0.5\,\mathrm{kg}$ ($\times 4$ pieces)	$r = 0.1\,\mathrm{m}$
$\omega = 75\,\mathrm{rad/s}$ ($=$ const.)	$k = 25000\,\mathrm{N/m}$	

Tasks

1. Determine the damping factor c for which the relative damping ratio of the system is $\zeta = 0.05$!
2. Determine the amplitude Y of displacement of the stationary solution $y_\mathrm{p}(t)$!
3. What is the maximum of the force $F_{\mathrm{b,max}}$, which acts on the fixed base during the stationary motion?

Made by Dávid Hajdu

Solution

Task 1

Figure 1 shows the front view of the dynamical model. To construct the equation of motion, Lagrange's equation of the second kind is applied in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} + \frac{\partial \mathcal{D}}{\partial \dot{q}_k} + \frac{\partial U}{\partial q_k} = Q_k^*, \quad k = 1, \ldots, n, \tag{1}$$

where n denotes the number of the generalized coordinates q_k, T stands for the kinetic energy, $\mathcal{D}$ is the Rayleigh's dissipation function, U refers to the potential energy function, and Q_k^* are the generalized forces, which are calculated form the power of the active forces that do not have a potential or a dissipation function. In the present example, due to the fact that the motions of the point masses relative to the rigid body are constrained with non-stationary geometric constraints, the number of degrees of freedom is only $n = 1$ ($k = 1$). Thus, the vertical displacement y of the table can be chosen as a generalized coordinate, i.e., $q_1 = y$. Moreover, there acts no force on the body, which are not included in the terms on the left-hand-side in Eq. (1), therefore the generalized force is $Q_1^* = 0$ and the equation of motion simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{y}} - \frac{\partial T}{\partial y} + \frac{\partial \mathcal{D}}{\partial \dot{y}} + \frac{\partial U}{\partial y} = 0. \tag{2}$$

Kinetic energy

The total kinetic energy is the sum of the kinetic energy of the moving rigid body and the kinetic energy of the point masses, i.e.,

$$T = \underbrace{\frac{1}{2}m\dot{y}^2}_{T_m} + 4\underbrace{\frac{1}{2}m_0 v_0^2}_{T_{m_0}}, \tag{3}$$

where v_0 is the speed of the eccentric masses, which are identical for each point mass, due to the symmetric structure of the exciter table. The velocity of the point masses must be determined as a function of the generalized velocity $\dot{y}$, which can be done in various ways. For example, the absolute velocity can be written by determining the position vector, i.e.,

$$\mathbf{r}_0 = \begin{bmatrix} r\cos(\omega t) + C_1 \\ y + r\sin(\omega t) + C_2 \end{bmatrix} \Rightarrow \mathbf{v}_0 = \dot{\mathbf{r}}_0 = \begin{bmatrix} -r\omega\sin(\omega t) \\ \dot{y} + r\omega\cos(\omega t) \end{bmatrix} \Rightarrow v_0^2 = \dot{y}^2 + r^2\omega^2 + 2\dot{y}\omega\cos(\omega t), \tag{4}$$

where C_1 and C_2 are arbitrary constants, that depend on the choice of the origin of the coordinate system, but vanishes with differentiation with respect to time.

Alternatively, using the known formulae for rigid body motions, one can calculate the velocity $\mathbf{v}_0$ of the eccentric mass by using the kinematic quantities in the form

$$\mathbf{v}_0 = \mathbf{v}_\mathrm{B} = \mathbf{v}_\mathrm{A} + \boldsymbol{\omega} \times \mathbf{r}_\mathrm{AB} = \begin{bmatrix} 0 \\ \dot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \times \begin{bmatrix} r\cos(\omega t) \\ r\sin(\omega t) \\ 0 \end{bmatrix} = \begin{bmatrix} -r\omega\sin(\omega t) \\ \dot{y} + r\omega\cos(\omega t) \\ 0 \end{bmatrix}, \tag{5}$$

where the points A and B are shown in Fig. 1.

A third way to determine the velocity $\mathbf{v}_0$ is the application of the formulae used in relative kinematics. The absolute velocity $\mathbf{v}_\mathrm{B} = \mathbf{v}_\mathrm{B,abs}$ is determined as

$$\mathbf{v}_0 = \mathbf{v}_\mathrm{B} = \mathbf{v}_\mathrm{B,abs} = \mathbf{v}_\mathrm{B,carr} + \mathbf{v}_\mathrm{B,rel} = \begin{bmatrix} 0 \\ \dot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} -r\omega\sin(\omega t) \\ r\omega\cos(\omega t) \\ 0 \end{bmatrix}, \tag{6}$$

see Fig.2 for a graphical representation.

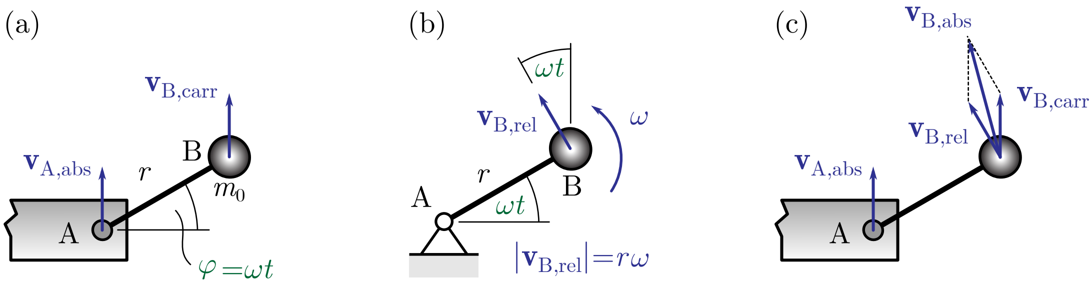

Fig. 2: Relative kinematics. (a) Carrier velocity. (b) Relative velocity. (c) Absolute velocity.

Finally, the kinetic energy can be written as

$$T = \frac{1}{2}(m + 4m_0)\dot{y}^2 + 2m_0\big(r^2\omega^2 + 2\dot{y}r\omega\cos(\omega t)\big), \tag{7}$$

and the deriatives as

$$\frac{\partial T}{\partial \dot{y}} = (m + 4m_0)\dot{y} + 4m_0 r\omega\cos(\omega t), \tag{8a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{y}} = (m + 4m_0)\ddot{y} - 4m_0 r\omega^2\sin(\omega t), \tag{8b}$$

$$\frac{\partial T}{\partial y} = 0\,. \tag{8c}$$

Rayleigh's dissipative function

The dissipative function for the dashpot and its partial derivative with respect to the general velocity is given as

$$\mathcal{D} = \frac{1}{2}c(\Delta v)^2 = \frac{1}{2}c\dot{y}^2 \;\Rightarrow\; \frac{\partial \mathcal{D}}{\partial \dot{y}} = c\dot{y}, \tag{9}$$

where Δv refers to the deformation speed of the damper.

Potential function

The spring is preloaded at the static equilibrium state, therefore a new coordinate z can be introduced from the unloaded state, such that

$$z = y + z_{\mathrm{st}} \;\Rightarrow\; z = y - z_{\mathrm{st}}\,, \tag{10}$$

where z_{st} denotes the initial deformation, while z gives the actual deformation of the spring, see Fig. 3. Using the new coordinate, the potential function can be written as

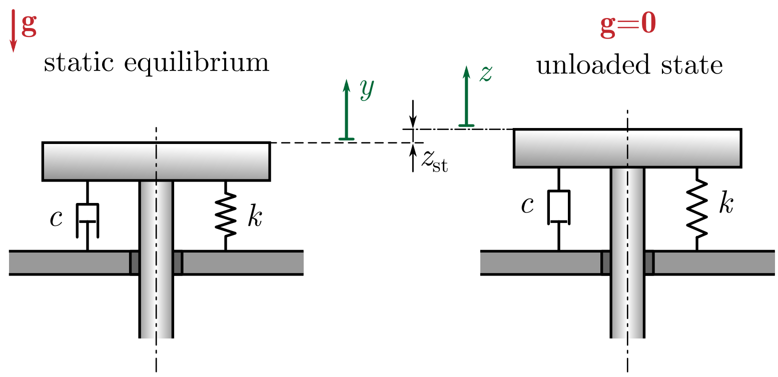

Fig. 3: Static deformation of the spring.

$$U = \frac{1}{2}kz^2 + mgz + 4m_0 g\big(z + r\sin(\omega t)\big) + C_0\,, \tag{11}$$

where C_0 can be chosen arbitrarily, since the reference point of the potential function of the gravitational force U_g can be shifted. Differentiation of U with respect to y gives

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial z}\underbrace{\frac{\partial z}{\partial y}}_{1} = \frac{\partial U}{\partial z} = kz + mg + 4m_0 g \overset{z = y - z_{\mathrm{st}}}{\stackrel{\downarrow}{=}} ky + \underbrace{(m + 4m_0)g - kz_{\mathrm{st}}}_{=\,0,\ \text{static equil.}}\,, \tag{12}$$

where the last term is equal to zero, since it gives the the equation for the static equilibrium, where the static spring force keeps equilibrium with the gravitational force. One can obtain the similar result for a simplified potential function $\tilde{U}$, which does not include the terms that eventually vanish after differentiation, for example

$$U = \underbrace{\frac{1}{2}ky^2}_{\tilde{U}} + \underbrace{\big((m+4m_0)g - kz_{\mathrm{st}}\big)}_{=0} y + \underbrace{4m_0 gr\sin(\omega t) + C_0}_{\text{const.}}, \Rightarrow \frac{\partial U}{\partial y} = \frac{\partial \tilde{U}}{\partial y} = ky. \tag{13}$$

Equation of motion

Finally, the equation of motion based on Eq. (1) can be written as

$$\underbrace{(m+4m_0)\ddot{y} - 4m_0 r\omega^2\sin(\omega t)}_{=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{y}} - \frac{\partial T}{\partial y}} + \underbrace{c\dot{y}}_{=\frac{\partial \mathcal{D}}{\partial \dot{y}}} + \underbrace{ky}_{=\frac{\partial U}{\partial y}} = \underbrace{0}_{=Q^*}. \tag{14}$$

Reformulating the equation, the standard form is given as

$$\ddot{y} + \underbrace{\frac{c}{m+4m_0}}_{=2\zeta\omega_{\mathrm{n}}}\dot{y} + \underbrace{\frac{k}{m+4m_0}}_{=\omega_{\mathrm{n}}^2} y = \underbrace{\frac{4m_0}{m+4m_0} r\omega^2}_{=f_0\omega_{\mathrm{n}}^2}\sin(\omega t) \tag{15}$$

and the coefficients lead to

$$\omega_{\mathrm{n}}^2 = \frac{k}{m+4m_0} \quad \Rightarrow \quad \boxed{\omega_{\mathrm{n}} = \sqrt{\frac{k}{m+4m_0}} = 20.08\,\mathrm{rad/s}\,,} \tag{16}$$

$$2\zeta\omega_{\mathrm{n}} = \frac{c}{m+4m_0} \quad \Rightarrow \quad \boxed{c = 2\zeta\omega_{\mathrm{n}}(m+4m_0) = 124.5\,\mathrm{Ns/m}\,,} \tag{17}$$

$$f_0\omega_{\mathrm{n}}^2 = \frac{4m_0}{m+4m_0} r\omega^2 \quad \Rightarrow \quad \boxed{f_0 = \frac{1}{\omega_{\mathrm{n}}^2}\frac{4m_0}{m+4m_0} r\omega^2 = 46.60\cdot 10^{-3}\,\mathrm{m}\,.} \tag{18}$$

Task 2

We are interested in the particular solution, which is searched in the form

$$y_{\mathrm{p}}(t) = Y\sin(\omega t - \vartheta), \tag{19}$$

where Y is the amplitude of displacement and ϑ is the phase angle between the excitation and the response of the system, which can be calculated using the frequency ratio $\lambda = \omega/\omega_{\mathrm{n}} \approx 3.735$. Using the resonance curve, we obtain

$$N = \frac{1}{\sqrt{(1-\lambda^2)^2 + 4\zeta^2\lambda^2}} = 0.0772\,, \tag{20}$$

from which the amplitude of the stationary motion is

$$\boxed{Y = Nf_0 = 3.46\cdot 10^{-3}\ \mathrm{m},} \tag{21}$$

and the phase angle is

$$\boxed{\vartheta = \arctan\frac{2\zeta\lambda}{1-\lambda^2} = -28.83\cdot 10^{-3} + \pi = 3.1128\ \mathrm{rad}.} \tag{22}$$

Task 3

The maximum force which is transmitted to the base is the sum of the static force, which is constant in time and the force of the dashpot $F_\mathrm{d}(t)$ and spring $F_\mathrm{s}(t)$, which not only depends on time, but also shifted in phase. Therefore, the actual time-dependent force is decomposed as

$$F_\mathrm{b}(t) = F_\mathrm{st} + \underbrace{F_\mathrm{s}(t) + F_\mathrm{d}(t)}_{F_\mathrm{dyn}(t)}, \tag{23}$$

where the static force F_st is equal to the force in the spring, when the excitation is zero, i.e.,

$$\boxed{F_\mathrm{st} = kz_\mathrm{st} = (m + 4m_0)g = 607.6\ \mathrm{N},} \tag{24}$$

and the dynamic part is written as

$$F_\mathrm{dyn}(t) = F_\mathrm{s}(t) + F_\mathrm{d}(t) = ky_\mathrm{p}(t) + c\dot{y}_\mathrm{p}(t) = kY\sin(\omega t - \vartheta) + cY\omega\cos(\omega t - \vartheta). \tag{25}$$

Eq. (25) gives the function of the dynamic force transmitted to the base, but in order to determine its maximum, one needs to reformulate it in the form

$$F_\mathrm{dyn}(t) = F_\mathrm{A}\cos(\omega t + \delta), \tag{26}$$

where F_A is the amplitude of the dynamic part of the force. Using trigonometric identities, Eq. (26) can be rewritten as

$$\begin{aligned} F_\mathrm{A}\cos(\omega t + \delta) &= F_\mathrm{A}\cos(\omega t)\cos\delta - F_\mathrm{A}\sin(\omega t)\sin\delta = \\ &= \underbrace{kY\sin(\omega t)\cos\vartheta - kY\cos(\omega t)\sin\vartheta}_{= kY\sin(\omega t - \vartheta)} + \underbrace{cY\omega\cos(\omega t)\cos\vartheta + cY\omega\sin(\omega t)\sin\vartheta}_{= cY\omega\cos(\omega t - \vartheta)}, \end{aligned} \tag{27}$$

where the coefficients of the harmonic functions can be separated:

$$\cos(\omega t): \quad F_\mathrm{A}\cos\delta = -kY\sin\vartheta + cY\omega\cos\vartheta, \tag{28}$$
$$\sin(\omega t): \quad -F_\mathrm{A}\sin\delta = kY\cos\vartheta + cY\omega\sin\vartheta. \tag{29}$$

Taking the square and the sum of Eqs. (28-29) gives

$$\underbrace{\big(-kY\sin\vartheta + cY\omega\cos\vartheta\big)^2 + \big(kY\cos\vartheta + cY\omega\sin\vartheta\big)^2}_{= k^2Y^2 + c^2Y^2\omega^2} = F_\mathrm{A}^2\underbrace{\big(\cos^2\delta + \sin^2\delta\big)}_{=1}, \tag{30}$$

from which the force amplitude F_A can easily be obtained as

$$F_\mathrm{A} = \sqrt{k^2Y^2 + c^2Y^2\omega^2} = Y\sqrt{k^2 + c^2\omega^2} = 92.337\ \mathrm{N}. \tag{31}$$

Therefore, the maximum of the force which acts on the base is

$$\boxed{F_\mathrm{b,max} = F_\mathrm{st} + F_\mathrm{A} = (m + 4m_0)g + Y\sqrt{k^2 + c^2\omega^2} = 699.9\ \mathrm{N}.} \tag{32}$$