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BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS
DEPARTMENT OF APPLIED MECHANICS

PRACTICE 8 – 1 DOF FORCED DAMPED SWINGING ARM

VIBRATIONS

– BMEGEMMBXM4 –

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Example

In Fig. 1, a welded structure is shown that consists of two rods with different lengths and masses, and a disk with radius R. The structure can only rotate along joint A. The vertical rod is connected to the environment through a spring with stiffness k_1 and a damper with c_1 damping coefficient, while at point B a harmonic force excitation is applied. On the horizontal rod a harmonic displacement excitation is applied through a spring with stiffness k_2. The deflection angle of the structure is measured from the horizontal axis by the general coordinate φ. The structure is in the gravitational field and its equilibrium position is located at $\varphi = 0$, where the spring of stiffness k_1 has static deformation only.

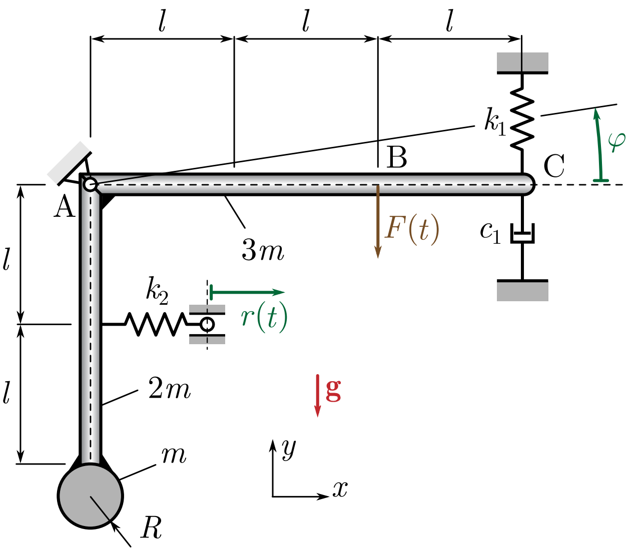

Fig. 1: Mechanical modeling of the structure

Data

$m = 0.12\,\text{kg}$	$l = 0.2$ m	$R = 0.1$ m	$r(t) = r_0 \sin(\omega t)$	$r_0 = 0.01$ m	$\omega = 20\,\text{rad/s}$
$k_1 = 300\,\text{N/m}$	$k_2 = 10\,\text{N/m}$	$c_1 = 2\,\text{Ns/m}$	$F(t) = F_0 \sin(\omega t)$	$F_0 = 2$ N	

Tasks

1. Derive the equation of motion for the small vibrations about the equilibrium using the Lagrange equation of the second kind! Calculate the undamped natural angular frequency, the damped natural angular frequency, the damping ratio and the static deformation! ($\omega_n = 35.55$ rad/s, $\omega_d = 35.31$ rad/s, $\zeta = 0.117$, $f_0 = -0.00713$ rad)
2. Calculate the stationary solution of the system! ($\varphi_p(t) = -0.0102 \sin(20t - 0.19)$)

Made by Szabolcs Berezvai

Solution

Task 1

In this exercise the equation of the motion is going to be derived using the Lagrange equation of the second kind, which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} + \frac{\partial \mathcal{D}}{\partial \dot{q}_k} + \frac{\partial U}{\partial q_k} = Q_k^*, \quad \text{with } k = 1...n, \tag{1}$$

where n denotes the number of the generalized coordinates q_k that describe the motion of the mechanical system uniquely. T stands for the kinetic energy, $\mathcal{D}$ is the Rayleigh's dissipation function, U refers to the potential function. The k^{th} component Q_k^* of the generalized force is originated in the active forces that does not have potential or dissipative function.

The mechanical model presented in Fig. 1 is a 1 DoF system, since the vibration of the structure can be described using a single general coordinate φ. This means that

$$n = 1 \text{ DoF} \quad \text{and} \quad q = \varphi, \tag{2}$$

from which the Lagrange equation of the second kind simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\varphi}} - \frac{\partial T}{\partial \varphi} + \frac{\partial \mathcal{D}}{\partial \dot{\varphi}} + \frac{\partial U}{\partial \varphi} = Q^*. \tag{3}$$

In order to have the equation of motion for small vibrations, Eq. (3) has to be linearized around the equilibrium ($\varphi \equiv 0$).

Kinetic energy

The kinetic energy can be calculated with respect to the fixed point A as

$$T = \frac{1}{2}\theta_A \dot{\varphi}^2. \tag{4}$$

Therefore, the terms in the Lagrange equation of the second kind can be obtained as

$$\frac{\partial T}{\partial \dot{\varphi}} = \theta_A \dot{\varphi} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\varphi}} = \theta_A \ddot{\varphi} \quad \text{and} \quad \frac{\partial T}{\partial \varphi} = 0\,. \tag{5}$$

The mass moment of inertia of the system with respect to the z axis at point A by means of applying the Steiner-theorem becomes

$$\theta_A = \frac{1}{3}3m(3l)^2 + \frac{1}{3}2m(2l)^2 + \frac{1}{2}mR^2 + m(2l+R)^2 = 0.0866\,\mathrm{kgm}^2. \tag{6}$$

Rayleigh's dissipative function

The dissipative function for the dashpot can be given as

$$\mathcal{D} = \frac{1}{2}c_1(\Delta v)^2\,, \tag{7}$$

where Δv refers to the deformation speed of the damper. Since we are interested in the linearized equation of motion, the terms up to second-degree are necessary only, namely, we can use the linearized deformation speed in the dissipative function. We can write

$$\mathcal{D} = \frac{1}{2}c_1(3l\dot{\varphi})^2, \tag{8}$$

from which the third term in the Lagrange equation becomes

$$\frac{\partial \mathcal{D}}{\partial \dot{\varphi}} = c_1(3l)^2\dot{\varphi} = 9l^2c_1\dot{\varphi}\,. \tag{9}$$

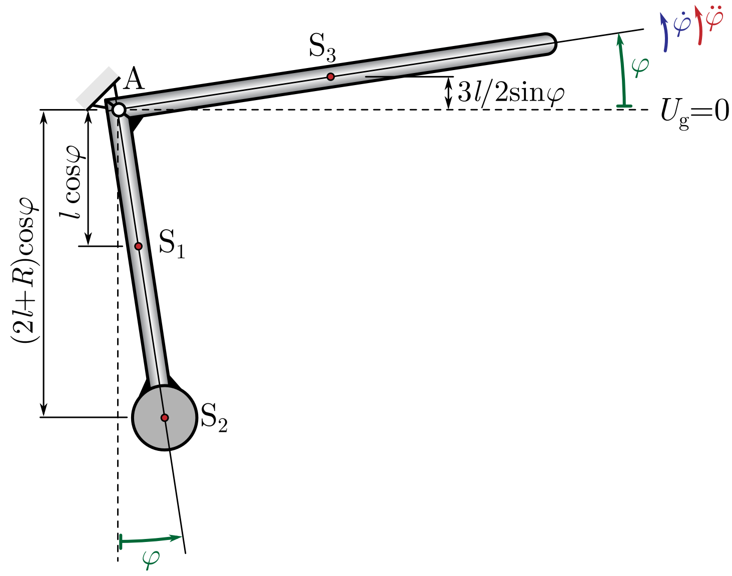

Fig. 2: Positions of the centres of mass in the gravitational potential field

Potential function

The potential function contains the strain energy of the springs and the potentials of the gravitational forces acting on the rods and the disk. Thus,

$$U = \underbrace{U_{\mathrm{r}_1} + U_{\mathrm{r}_2}}_{\text{springs}} + \underbrace{U_1 + U_2 + U_3}_{\text{gravitational potential}} \,, \tag{10}$$

where the spring potentials can be expressed as

$$U_{\mathrm{r}_1} = \frac{1}{2}k_1(\Delta l_{\mathrm{st}} + 3l\varphi)^2 \quad \text{and} \quad U_{\mathrm{r}_2} = \frac{1}{2}k_2(r(t) - l\varphi)^2, \tag{11}$$

where Δl_{st} refers to the deformation of the spring at the equilibrium. In the potential function, the linearized spring deformations are considered since the second degree terms of the spring potentials can provide terms in the linearized equation of motion only.

The gravitational potentials become

$$U_1 = -2mgl\cos\varphi, \quad U_2 = -mg(2l + R)\cos\varphi \quad \text{and} \quad U_3 = 3mg\frac{3}{2}l\sin\varphi\,, \tag{12}$$

if the gravitational potential is calculated with respect to zero potential level which is indicated in Fig. 2. Note, that the linearization of the gravitational potentials would lead incorrect terms in the equation of motion, hence, we keep the original trigonometrical functions in the expressions and we linearized the corresponding terms after the differentiation with respect to φ.

Based on these, Eq. (10) gives

$$U = \frac{1}{2}k_1(\Delta l_{\mathrm{st}} + 3l\varphi)^2 + \frac{1}{2}k_2(r(t) - l\varphi)^2 - 2mgl\cos\varphi - mg(2l + R)\cos\varphi + 3mg\frac{3}{2}l\sin\varphi, \tag{13}$$

while, the corresponding term of the Lagrange equation becomes

$$\frac{\partial U}{\partial\varphi} = k_1(\Delta l_{\mathrm{st}} + 3l)3l + k_2(r(t) - l)l + 2mgl\underbrace{\sin\varphi}_{\approx\varphi} + mg(2l + R)\underbrace{\sin\varphi}_{\approx\varphi} + \frac{9}{2}mgl\underbrace{\cos\varphi}_{\approx 1}. \tag{14}$$

After linearization this gives

$$\frac{\partial U}{\partial\varphi} = 3k_1l\Delta_{\mathrm{st}} + 9l^2k_1\varphi - k_2r(t)l + k_2l^2\varphi + mg(4l + R)\varphi + \frac{9}{2}mgl. \tag{15}$$

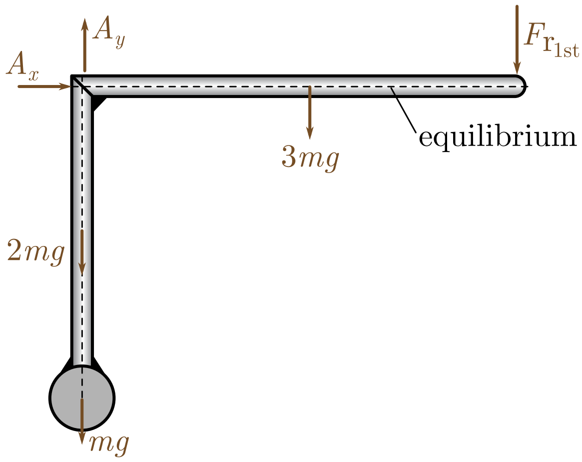

Fig. 3: Free body diagram in static equilibrium

The initial deformation Δ_{st} of the spring k_1 can be expressed from the static equilibrium equation in accordance with the free body diagram (FBD) of the structure in equilibrium, which is presented in Fig. 3. Based on these:

$$\sum M_{\mathrm{A}z} = 0: \quad \underbrace{F_{1_{\mathrm{st}}}}_{k_1\Delta_{\mathrm{st}}} 3l - 3mg\frac{3}{2}l = 0 \quad \Rightarrow \quad 3k_1 l\Delta_{\mathrm{st}} = \frac{9}{2}mgl \quad \Rightarrow \quad \Delta_{\mathrm{st}} = -\frac{3mg}{2k_1}. \tag{16}$$

After substituting the initial deformation into Eq. (15), the expression simplifies to

$$\frac{\partial U}{\partial \varphi} = 9l^2 k_1 \varphi - k_2 r(t) l + k_2 l^2 \varphi + mg(4l + R)\varphi. \tag{17}$$

Consequently, the initial deformation of the spring and the gravitational potential of the force that generates the deformation can be omitted together during the derivation. This means that we can introduce a simplified potential function

$$\tilde{U} = \frac{1}{2}k_1(3l\varphi)^2 + \frac{1}{2}k_2(r(t) - l\varphi)^2 - 2mgl\cos\varphi - mg(2l + R)\cos\varphi\,, \tag{18}$$

by which we can obtain the same result:

$$\frac{\partial \tilde{U}}{\partial \varphi} = \frac{\partial U}{\partial \varphi} = 9l^2 k_1 \varphi - k_2 r(t) l + k_2 l^2 \varphi + mg(4l + R)\varphi\,. \tag{19}$$

Power of other active forces

Since the harmonic force excitation is not included in the terms above, it should be taken into account in the generalized force component Q^*. For this, let us calculate the power of $F(t)$, which should be equal to the power of the unknown generalized force Q^*.

Based on Fig. 4, the power of $F(t)$ can be expressed as

$$P = \mathbf{F} \cdot \mathbf{v}_B = \begin{bmatrix} 0 \\ -F(t) \end{bmatrix} \cdot \begin{bmatrix} -2l\dot{\varphi}\sin\varphi \\ 2l\dot{\varphi}\cos\varphi \end{bmatrix} = -F(t)2l\dot{\varphi}\cos\varphi \overset{!}{=} Q^*\dot{\varphi}. \tag{20}$$

which yields

$$Q^* = -F(t)2l\underbrace{\cos\varphi}_{\approx 1} \approx -F(t)2l. \tag{21}$$

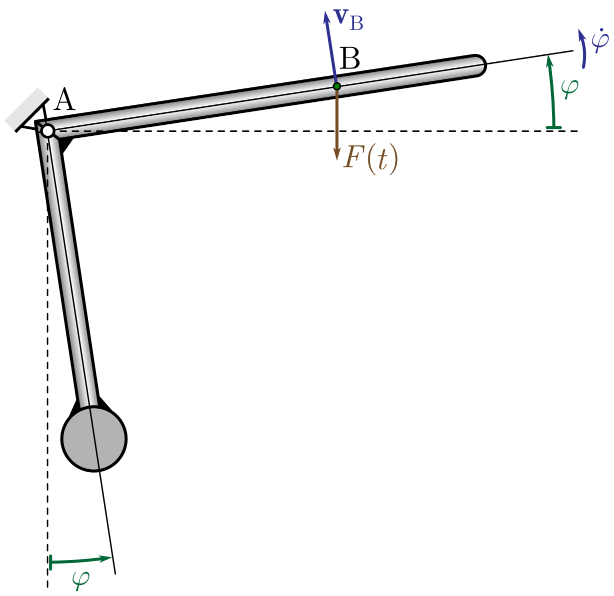

Fig. 4: Illustration of the velocity of point B

Equation of the motion

Since all terms in the Lagrange equation are expressed in Eqs. (5), (9), (19) and (21), the equation of the motion for the small vibrations around the equilibrium reads

$$\theta_A\ddot{\varphi} + 9l^2c_1\dot{\varphi} + 9l^2k_1\varphi - k_2r(t)l + k_2l^2\varphi + mg(4l+R)\varphi = -F(t)2l. \tag{22}$$

Rearranging the terms leads to

$$\theta_A\ddot{\varphi} + 9l^2c_1\dot{\varphi} + \left(9l^2k_1 + k_2l^2 + mg(4l+R)\right)\varphi = -2lF(t) + k_2lr(t). \tag{23}$$

Dividing the equation by θ_A results the general form of the equation of the motion

$$\boxed{\ddot{\varphi} + \underbrace{\frac{9l^2c_1}{\theta_A}}_{=2\zeta\omega_n}\dot{\varphi} + \underbrace{\frac{9l^2k_1 + l^2k_2 + mg(4l+R)}{\theta_A}}_{=\omega_n^2}\varphi = \underbrace{\frac{-2lF_0 + k_2lr_0}{\theta_A}}_{f_0\omega_n^2}\sin(\omega t)\,.} \tag{24}$$

The substitution of the parameters yields

$$\boxed{\omega_n = \sqrt{\frac{9l^2k_1 + l^2k_2 + mg(4l+R)}{\theta_A}} = 35.55\,\mathrm{rad/s}\,,} \tag{25}$$

$$\boxed{\zeta = \frac{1}{2\omega_n}\frac{9l^2c_1}{\theta_A} = 0.117\,(=11.7\%)\,.} \tag{26}$$

The natural angular frequency of the damped system is

$$\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2} = 35.31\,\mathrm{rad/s},} \tag{27}$$

while the static deformation

$$\boxed{f_0 = \frac{1}{\omega_n^2}\frac{-2lF_0 + k_2lr_0}{\theta_A} = -0.00713\,\mathrm{rad}.} \tag{28}$$

Task 2

The solution $\varphi(t)$ of the equation of the motion can be composed as

$$\varphi(t) = \varphi_{\mathrm{h}}(t) + \varphi_{\mathrm{p}}(t), \tag{29}$$

where $\varphi_{\mathrm{h}}(t)$ is the homogeneous solution and $\varphi_{\mathrm{p}}(t)$ is the stationary solution which assumes the form

$$\varphi_{\mathrm{p}}(t) = \Phi \sin(\omega t - \vartheta), \tag{30}$$

where Φ is the amplitude and ϑ is the phase angle, which can be calculated using the frequency ratio $\lambda = \omega/\omega_{\mathrm{n}} = 0.5625$. Using the resonance curve, we get

$$N = \frac{1}{\sqrt{(1-\lambda)^2 + 4\zeta^2\lambda^2}} = 1.436, \tag{31}$$

from which the amplitude of the stationary motion is

$$\Phi = N f_0 = -0.0102 \text{ rad}, \tag{32}$$

and the phase angle:

$$\vartheta = \arctan \frac{2\zeta\lambda}{1-\lambda^2} = 0.19 \text{ rad}. \tag{33}$$

Finally, the stationary solution in Eq. (30) reads

$$\boxed{\varphi_{\mathrm{p}}(t) = -0.0102 \sin(20t - 0.19).} \tag{34}$$