Modelling Population Dynamics using Field-Theoretic Techniques

A thesis submitted to the School of Physics at the University of East Anglia in partial fulfilment of the requirements for the degree of Bachelors of Science

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Abstract

Classical population dynamics models such as the Malthusian model have been used to describe general deterministic exponential population growth. These classical deterministic systems are restricted by, and have a strong dependency on the precision of measurement of initial conditions. This research project looks to develop a Markovian, i.e. memory-less probabilistic system that encapsulates birth-death processes in a population to bypass these hard constraints.

At present, stochastic processes that realise this classically are difficult to model and solve. The Doi-Peliti formalism provides a succinct basis for quantising, and solving these processes. This formalism has recently been applied to fields such as mathematical biology, and quantum field theory.

On the basis of a budding branching process, we look to present a systematic derivation of the coherent-state path integral via intricate field-theoretic machinery, to provide analytic solutions to the birth-death processes. This will produce a continuous population-dependent stochastic process. These are then compared to their corresponding Malthusian results. The research project generalises a stochastic process with respect to a specified master equation to advance collation, to the results derived from utilising the Doi-Peliti formalism. The results and comparisons grant an intuition of the coherent state path integral, and modelling of binary fission processes with comparison of deterministic population dynamics to population-dependent stochastic systems.

Keywords: *quantum field theory, birth-death processes, generating functions, coherent states*

Contents

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1

Introduction

Populations can be divided into two categories; biological, or chemical. Populations dynamics share proclivities with decay processes. Just as beta-decay splinters a neutron into three offspring known to be a proton, electron and neutrino, biological equivalencies can be made. From the scale of microscopic cell division such as mitosis, the process of division of a cell into two identical clones, to macroscopic scales such as evolution of a population of organisms over time. Quantitative predictions based on mathematical models of said populations is paramount, as for example, smaller populations may have a death rate exceeding the birth, rapidly declining the population to extinction. Prediction of evolution of these populations allows for observation, and potential intervention to resuscitate a species.

However classical models historically, are deterministic[22], and have hard constraints on initialising the system with known quantities, whilst some remain semi-Markovian; we seek to develop a non-deterministic description. Previous models include the predator-prey relationship, in which the parameterisation of predator and prey occupation, birth-death rates, and the interaction of species, was inducted by Lotka-Volterra equations[26]. Another rudimentary model considered is the logistic growth, which accounts for the carrying capacity of the ecosystem, developed by Verhulst[35]. The quintessence of this being the McKendrick-Von Foerster equation[19]; a deterministic model with age-structured population dynamics applied on a lattice. However, none of the

models above account for stochastic fluctuation. Randomness in nature is a fundamental property and must be accounted for. Contagion pandemics occur spontaneously, cosmological impacts and natural disasters cause population fluctuation and speciation.[20]

Heuristic models leveraging quantum field-theoretic techniques have been implemented in field subsets of theoretical ecology, mathematical biology and statistical physics. Quantisation of classical analogues such as the Doi-Peliti Formalism [9][30], provide a basis to formulate an effective field theory.

As a consequence, path integral methods to model birth-death processes on a lattice with Fock-space formalism, where pragmatic implications can be drawn to diffusion-reaction processes were developed; also accounting for memory. Alternate examples include Jun Ohkubo[28], where use of extended duality relations form descriptions of continuous and discrete state stochastic processes of birth-death processes led to differential equations, with the motivation of accelerating the mathematical viewpoint within the field.

As previous literature is refined to a high level of specificity, research will focus towards a generic model to circumvent charging the paper with direct application, rather to show a pedagogical approach. Developing a sequence in which one is able to follow the trajectory from *ab initio*, leading to the classical forward time-continuous master equation, to a coherent state path integral. Which can be considered adjacent to the realisation of a large scale system diffusion process using coherent states, in combination with an effective field theory, as shown by Weise[37]. We will attempt to provide physical or non-physical intuition where possible to give a tangible understanding of the processes.

We will consider an *equilibrium* statistical mechanics route, in which the concept stems from branching processes. Instantiations include, positron-electron annihilation [16], where two objects sum into a null. This would be described by our *death* process. Contrasting this; one object becoming two, as observed in mitosis cell division[1], would be adequately described by

our *birth* process. To provide intuition for an equilibrium neutral evolution, consider a single object, producing another single object over time, uniformly incrementing the population by one per cycle; this is the reproduction cycle of a Hydra[34], a microscopic aquatic organism. This is a *budding* process, and will represent our state of equilibrium.

We still start by mathematically defining these processes classically, and applying Doi-Peliti Formalism to produce the functional coherent state path integral. One hopes to present a modicum of physical intuition, where the architecture has intention of accommodating a broad spectrum of applications with slight modification to the original rudimentary model, to fulfil said niche.

2

Theoretical Framework

2.1 Birth-Death Processes

2.1.1 *Ab Initio*

To begin delving into population dynamics using field-theoretic techniques, the classical analogue must briefly be introduced. One may begin to develop mathematical intuition behind birth-death processes using *ab initio* methods. Ab initio methods and first principles were the backbone of noteworthy concepts such as the Hartree-Fock method[12], or quantum Monte Carlo[18], providing solutions simulating complex quantum many-body *Schrödinger* equation. However this research will begin with logical arguments derived using probability theory.

Consider, a population of As, where n is the population size, denoting the number of individuals in a population, at a time, t. To emulate the branching process, we formally define the binary fission, or *simple* budding process and annihilation processes, respectively. Let A $\rightarrow$ A + A be a birth process, and A+A$\rightarrow$ ϕ be a death process where, ϕ is the null state. Let $P_n(t)$ be the probability of a size, n, at a time t. Let β = birth rate, per individual; μ = the death rate, per individual. Then, in a time dt, the probability of birth and death are given by, $\beta_n dt$ *and* $\mu_n dt$ conditioned over the infinitesimal period dt, respectively.

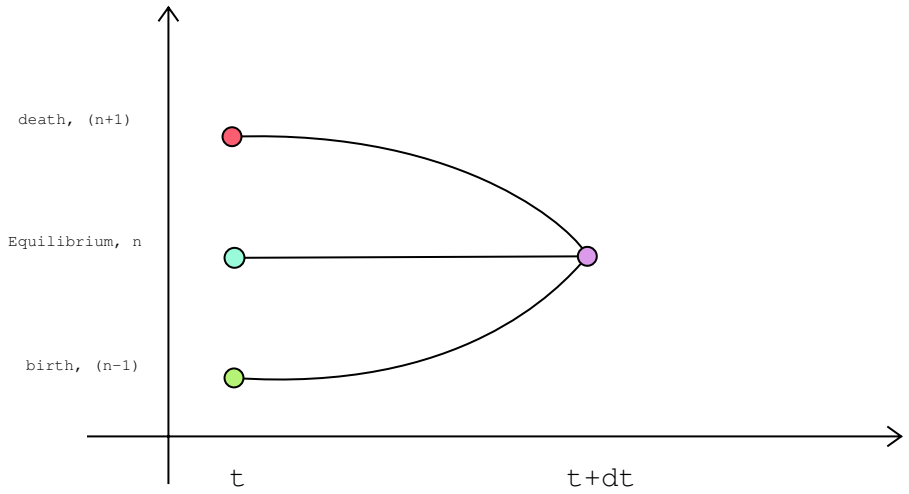

Figure 2.1.1: All possibilities in an interval, t to t+dt

As we are considering an equilibrium system, we require that the number of individuals per unit time remains a constant, i.e. $\frac{dn}{dt} = 0$. Then, in a period t+dt, as shown in Figure 2.1.1 the sum of the probabilities, of the possibilities available to return the system to a state of equilibrium must be given by, P_n(t+dt)=Pr(n+1, at a time t)Pr(1 death, in dt)+Pr(n-1, at a time, t)Pr(1 birth, in dt)+Pr(n, at a time, t)Pr(no change, in dt). Where the birth, death, and no change possibilities are shown by the green, red, and blue dots in Figure 2.1.1 respectively.

Computing the derivative with respect to time of this probability, produces,

$$\frac{P_n(t+dt) - P_n(t)}{dt} = \mu(n+1)P_{n+1}(t)dt + \beta(n-1)P_{n-1}(t)dt - \left(\frac{\beta n dt - \mu n dt}{dt}\right) \tag{2.1.1}$$

Then taking the limit dt $\to$ 0,

$$\frac{dP_n}{dt} = \beta(n-1)P_{n-1} - \beta_n P_n + \mu(n+1)P_{n+1} - \mu n P_n \tag{2.1.2}$$

Where $\sum_{n=0}^{\infty} P_n(t) = 1$ i.e. conservation of probability.

This is the standard forward time-continuous master equation. We will note that $P_n(t) = \psi(t)$ is our probability density function, where the marginal densities are given by $\psi_n(t) = \int \psi(n_1, ..., n_n) dn_1..dn_{-1}dn_{n+1}...dn_n$ for n = 1,2,3...n. Useful

for modelling time evolution of Markov[10] systems where the system at a given time, can be described by a combination of probabilistic states. However, we seek to develop a stochastic model that consolidates fluctuations with number of individuals. Intuitively, each individual term is structured such that the system returns to a population size n, at a time period t+dt, from a previous time t, with all possibilities accounted for.

2.1.2 Expectation of the probability density function

To find the expected value of our probability density function, $P_n(t)$ there are two available routes. The approach chosen is an ordinary generating function; for simplicity we will only consider the death process $\frac{dP_n(t)}{dt} = -\beta P_n + \beta(n+1)P_{n+1}$ and by symmetry assume similar result for the birth process.

The ordinary generating function defined to be $G(x) = \sum_n P_n(t)x^n$ yields,

$$\mathbb{E}[(N),(t)] = \sum_{n=1}^{\infty} nP_n(t) = \frac{dG}{dx}$$

when differentiated. Applying this to equation (2.1.2),

$$\sum_n n\frac{dP_n}{dt} = -\sum_n \beta n^2 P_n + \sum_n \beta n(n+1)P_{n+1}$$

$$= \frac{d}{dt}\sum_n nP_n = -\sum_n \beta n^2 P_n + \sum_m \beta(m-1)mP_m$$

$$= -\sum_n \beta n^2 P_n + \sum_m \beta m^2 P_m - \sum_m \beta m P_m$$

where $m = (n+1)$ is true.

Hence,

$$\frac{d}{dt}\mathbb{E}(t) = -\beta\mathbb{E}(t)$$

Subject to the conditions,

$$G(x,0) = \sum_{n=0}^{\infty} P_n(0)x^n = x^{n_0} \text{ where } n_0 = \mathbb{E}(0)$$

yields our Malthusian exponential growth, $\mathbb{E}(t) = \mathbb{E}(0)e^{-\beta t}$, considered to be the Newton's first law of population modelling. Similarly, the variance of the probability density function can be defined by taking the second derivative of the generating function. In the simplistic case of the death process, the variance is given by, for a more rigorous formulation see Subsection A.1.1.

$$\mathbb{V}(N(t)) = \mathbb{E}(0)e^{-\beta t}(e^{-\beta t} - 1)$$

Both of these functions are of particular interest as they allow for comparison to the density functions derived via functional differentiation of the coherent state path integral.

2.1.3 Quantum Interpretation

To convey a quantum formulation of the classical analogue above, one may seek to employ the *Schrödinger picture*[33]. For employment of the formulation we must first consider our probability density function as a superposition of base vectors, $|n\rangle$ corresponding a state with exactly n individuals. In a memoryless process we can assume the states depend only on the number of individuals, and potentially time. Therefore we portray our macroscopic system as a state $|\psi(t)\rangle = \sum_n P_n |n\rangle$. In Schrödingers framework, a state vector can be described as superposition or, linear combination of base vectors that form a complete set. Said states evolve under regulation of the unitary-time evolution operators, where the operators themselves remain a constant with respect to time; therefore utilisation is ideal. The operator itself retains the property of unitarity, such that,

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi_0| U^\dagger(t,t_0)U(t,t_0) |\psi_0\rangle = \langle\psi_0|\psi_0\rangle$$

if

$$\langle\psi(t)| = \langle\psi_0| U^\dagger(t,t_0) \; ; \; |\psi_t\rangle = U(t,t_0)\,|\psi_0\rangle$$

then $U^\dagger(t,t_0)U(t,t_0) = I$ where I, is identity, and the ket is an element of Hilbert space, must be true. The nuance of the Schrödinger picture allows for handling of a time-independent Hamiltonian, where the operator has the form $U(t,t_0) = e^{Lt}$ in our case, and allows for supplication of a Taylor series expansion around the Hamiltonian. Note the state vector was adapted from Peliti[30], where originally applied to a lattice structure with spatial coordinate definitions, however we will not be incorporating spatial dimensions. Our macroscopic system can be given in the form,

$$|\psi_t\rangle = \sum_n P_n(t)\,|n\rangle = e^{\mathcal{L}t}\,|\psi_0\rangle \tag{2.1.3}$$

Proposing the inner product, specifying two states orthogonality property when occupation number varies (*Grassberger and Scheunert*[14]). We refer to Peliti[30], for clarification and use; to prove commutation relations in Subsection 2.1.4.

$$\langle n|m\rangle = n!\delta_{nm}$$

We take brief entr'acte to pursue tangent to prove this. Grassberger and Scheunert propose convenience of inclusive product representation as such. Exploitation follows,

$$\langle n|m\rangle = \sum_k \frac{1}{k!}(n)_k(m)_k$$

Where $(n)_k$ and $(m)_k$ are Pochammer symbols such that $(n)_k = n(n-1)(n-k+1)\cdots = \frac{n!}{(n-k)!}$. Then by definition of arbitrary vector products,

$$\langle\phi|\chi\rangle_{in} = \langle\phi|\, e^{a^\dagger} e^a\,|m\rangle = \sum_k \frac{1}{k!}\,\langle n|\, e^{a^\dagger} a^k\,|m\rangle$$

$$= \sum_{k,l} \frac{1}{k!}\frac{1}{l!}(m)_k\,\langle n|\,(a^\dagger)^l\,|m-k\rangle$$

$$= \sum_k \frac{1}{k!}(m)_k(n)_k = \langle n|m\rangle_{in}$$

Resuming, our classical mirror can be recovered by differentiating the superposition with respect to time in conjunction with the appropriate commutation relations, operator definitions and re-normalisation properties. Where the differential equation for our operator can be seen to have the form,

$$\frac{\partial}{\partial t}|\psi_t\rangle = \mathcal{L}e^{\mathcal{L}t}|\psi_0\rangle$$

$$\frac{\partial}{\partial t}|\psi_t\rangle = \mathcal{L}\sum_n P_n(t)|n\rangle$$

Under the canvas of the Schrödinger interpretation. For an explicit demonstration please see Subsection A.1.2.

2.1.4 Doi-Peliti formalism and Second Quantisation

Now that we have begun to play the game of the quantum interpretation, we can start by defining some rules to play by. The canonical principle that sets the playing field, is how our system physically changes. State transitions are governed by our annihilation and creation operators. They lower, or raise the macroscopic state, respectively, which implicitly raises or lowers our number of individuals. Therefore, we inaugurate these operators mathematically, in the form,

$$a|n\rangle = n|n-1\rangle\,, \quad a^\dagger|n\rangle = |n+1\rangle \tag{2.1.4}$$

Where a is the annihilation operator, and $a^\dagger$, the *conjugate transpose* also known as the *adjoint* is the creation operator. These correspond to implicit reduction of the number of individuals of the population, or increase, respectively. In quantum mechanics, $a^\dagger$ is known as the *hermitian conjugate*, equivalent to applying conjugation, and then transposition of the appropriate

creation matrix.

This is analogous to a reaction-diffusion system, where applications of these operators move a particle left or right on a lattice. Second quantisation has seen utilisation in the *tight-binding model*[3] in solid-state physics, where the free-space Hamiltonian in the position basis of a Bravais lattice has remarkable resemblance to the Hamiltonian in play with respect to our system.

In order to observe the symbiosis of the two operators, we can note the constructed commutation relation ($[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = 1$) provided by the Doi-Peliti formalism. Note the commutation equates to one, as we don't want our population to fall into negative values. We introduce our relations as such,

$$[a, a^\dagger] = 1 \tag{2.1.5}$$

For renormalisation property refer to Equation 2.1.3. Action of annihilation operator acting upon n'th state can be demonstrated via induction. Let $P(n_i)$ be precedent such that $P(n_i) \forall\, n_i \in \mathbb{N}$; for the base-step, $n_i = 0$. Hence, $a_i \left|\phi\right\rangle = 0$ by definition as P(0) remains factual. Assuming, $P(n_i)$ to be correct,

$$a_i \left|n_i\right\rangle = n_i \left|n_i - 1\right\rangle$$

Then, if $[a_i, a_i^\dagger] = 1 = a_i a_i^\dagger = a_i a_i^\dagger - a_i^\dagger a_i = 1$ can be rearranged, $a_i a_i^\dagger = a_i^\dagger a_i + 1$ It can be shown,

$$a_i \left|n_i + 1\right\rangle = a_i (a_i^\dagger)^{n_i+1} \left|\phi\right\rangle = a_i a_i^\dagger (a_i^\dagger)^{n_i} \left|\phi\right\rangle = (a_i^\dagger a_i + 1) = \left|n_i\right\rangle$$

By inductive assumption $a_i \left|n_i\right\rangle = n_i \left|n_i - 1\right\rangle$, it must hold that,

$$(a_i^\dagger a_i + 1) \left|n_i\right\rangle = \left|n_i\right\rangle + a_i^\dagger n_i \left|n_i - 1\right\rangle = \left|n_i\right\rangle + n_i \left|n_i\right\rangle = (n_i + 1) \left|n_i\right\rangle$$

Thus, $P(n) \forall\, n \in \mathbb{N}$ is held.

Following the Doi-Peliti formalism, we choose an appropriate "Hamiltonian" such that $\mathcal{L}_{\mathcal{T}} = \mathcal{L}_{\mathcal{B}} + \mathcal{L}_{\mathcal{D}} + \mathcal{L}_{\mathcal{O}}$, where $\mathcal{L}_{\mathcal{T}}$ is the sum of the processes in the system, $\mathcal{L}_{\mathcal{B}}$ is the birth term, $\mathcal{L}_{\mathcal{D}}$ is the death term, $\mathcal{L}_{\mathcal{O}}$ retains the equilibrium of the system. Then,

$$\mathcal{L} = \mathcal{L}_{\mathcal{B}} + \mathcal{L}_{\mathcal{D}} + \mathcal{L}_{\mathcal{O}}$$

$$\mathcal{L} = \beta((a^{\dagger})^2 a - a^{\dagger}a) + \mu(a - a^{\dagger}a) \tag{2.1.6}$$

Where the notation $\mathcal{L}_{\mathcal{T}}$ has been dropped for simplicity. The *do nothing* term, or $\mathcal{L}_{\mathcal{O}}$ has been disregarded, due to zero contribution.

2.1.5 Coherent States

Coherent states possess three fundamental properties which implicate an ideal suiting for an effective field theory. These properties span the classicality[23], the eigenstates formed by the field operators, and from Doi-Peliti, the overcompleteness stated by unity[30]. Coherent states can be considered to have value when you have a system of two or more interacting quantum fields, then the time evolution operator is unable to generate a state, and produces a vacuum. For completeness, we will briefly state the overcomplete property, where rigorous derivation can be found in Peliti [30].

$$\mathbb{1} = \int \frac{dzdz*}{2\pi} \left|z\right\rangle \left\langle iz\right| e^{-izz*}$$

The utilisation of the resolution of identity above plays an integral part in the formulation of the path integral. Here, the identity is proposed in polar coordinates, where z is complex, and $z = |z|e^{i\theta}$; We refer to general completeness of coherent states, for more information one may refer to a lecture series presented by John Cardy [6].

Integrals are in polar coordinates given by $d^2\phi = d(R\phi)d(Im\phi)$.

$$\mathbb{1} = \int \frac{d\phi d\phi^*}{\pi} e^{-\phi^*\phi} e^{\phi a^\dagger} \left|0\right\rangle \left\langle 0\right| e^{\phi^* a} \tag{2.1.7}$$

Where $|0\rangle$ is the null-state under standard notation; please refer to Subsection A.1.7 for a more satisfying proof.

The presupposition of coherent states may seem abstract, however you can consider these as a wavefunction, or wave packet, with the optimal spread. What this essentially means is that when we think of the Heisenberg uncertainty principle, for all coherent states, the inequality transforms to equality. Mechanically, given rise by the Poissonian distributive behaviour of these states, shown in Subsection A.1.3. Where the distribution follows,

$$P(n) = e^{-\phi} \frac{\phi^n}{n!}$$

Alternatively, we introduce the functional coherent states, which provide a succinct generalisation of the coherent states defined in Doi[9]. Constructed as,

$$\left|u\right\rangle = e^{ua^\dagger} \left|\phi\right\rangle \quad ; \quad a\left|u\right\rangle = ua^\dagger \left|\phi\right\rangle = a\left|u\right\rangle = u\left|u\right\rangle \tag{2.1.8}$$

Where the *Hermitian conjugate* is constructed as,

$$\left\langle v\right| = \left\langle \phi\right| e^{va} \tag{2.1.9}$$

With the renormalisation property established using the Baker-Campbell Hausdorff theorem[29],

$$\left\langle \phi\right| e^{va^\dagger} e^{ua} \left|\phi\right\rangle = \left\langle \phi\right| e^{uv} e^{ua} e^{a^\dagger v} \left|\phi\right\rangle \tag{2.1.10}$$

$$\langle v|u\rangle = e^{uv} \tag{2.1.11}$$

Furthermore, the average of some function $\mathcal{A}(t)$ must be given by,

$$\left\langle \mathcal{A}\, n, t \right\rangle = \sum_{n} a(n) P_n(t)$$

by definition. It is noteworthy that this is linear in probabilistic terms, P(t). By comparison, the Doi state $|\psi_t\rangle$, is linear in probabilistic terms. Conversely, unable to implicate the quantum mechanical definition of a density matrix $\langle\psi_t|\,\hat{\mathcal{A}}\,|\psi_t\rangle$ due to quadratic behaviours. Hence we require $\langle\mathcal{Z}|\,\hat{\mathcal{A}}\,|\psi_t\rangle$ where $\langle\mathcal{Z}|$ considered by Doi as projective state. By Doi definition,

$$\langle\mathcal{Z}| = \langle\phi| \sum_{i=1}^{N} e^{a_i} = \langle\phi|\, e^{\sum_i a_i}$$

We can expand exponential via Taylor series invoking action of a_i such that the original state acted upon returns. Before proceeding let us observe this demonstration provides alternate arrival of $\langle 1|$ the left eigenstate of u, where $u = 1$. Let us remark, $\langle\mathcal{Z}| \equiv \langle u(u=1)| \equiv \langle 1|$. Continuing,

$$\begin{aligned} a\,|\mathcal{Z}\rangle &= a e^{a^\dagger}\,|\phi\rangle \\ &= a(1 + a + \frac{a^2}{2!}\frac{a^3}{3!} + ...)\,|\phi\rangle \\ &= a + aa^\dagger + \frac{a(a^\dagger)^2}{2!} + \frac{a(a^\dagger)^3}{3!} + ... \end{aligned} \tag{2.1.12}$$

By instantiation of definition $a\,|n\rangle = n\,|n-1\rangle$,

$$\begin{aligned} &= (0 + 1\,|\phi\rangle + a\,|1\rangle + \frac{a\,|2\rangle}{2!}... \\ &= (1 + a^\dagger + \frac{(a^\dagger)^2}{2!} + \frac{(a^\dagger)^3}{3!} + ...)\,|\phi\rangle \\ &= e^{a^\dagger}\,|\phi\rangle = |\mathcal{Z}\rangle \end{aligned} \tag{2.1.13}$$

Reduction of $\langle\mathcal{Z}|$ is possible, limiting available occupations to n, and implicitly

limiting $a^\dagger$. Further Taylor expansion is plausible to yield,

$$\begin{aligned} \langle \mathcal{Z}| &= \langle \phi| \, 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + ... \\ &= \langle \phi| \sum_{n=0}^{\infty} \frac{(\sum_{i=1}^{K} a_i)^n}{n!} \\ &= \sum_{n=0}^{\infty} \langle \mathcal{Z}| \end{aligned} \tag{2.1.14}$$

Then,

$$\langle \mathcal{Z}| = \langle \phi| \, \frac{1}{n!} (\sum_{i}^{K} a_i)^n$$

we have introduced a projection superposition state of exactly n individuals.

Let us remark, $\langle \mathcal{Z}|\psi_t\rangle = 1$; conservation of probability insists,

$$0 = \frac{d}{dt}(1) = \langle \mathcal{Z}| \, \mathcal{L} \, |\psi_t\rangle$$

So "Hamiltonian" must obey,

$$\langle \mathcal{Z}| \, \mathcal{L} = 0$$

Let us note an observation initially drawn by Weise[37]. Defining the expectation of an *observable*, also known as a *self adjoint operator*. Observables contain a feature such that the eigenvalues correspond to real numbers, which we shall remark on later when formulating the path integral. Weise noticed that,

$$\left\langle \mathcal{A} \right\rangle_{\psi t} = \sum_{n=0}^{\infty} A(n) P_t(n) \tag{2.1.15}$$

$$= \langle 0| \, e^a \mathcal{A}(a^\dagger a) \, |\psi_t\rangle$$

$$= \langle 1| \, a^\dagger a \, |\psi_t\rangle$$

$$= \sum_{n} n P_n(t) \, \langle 0| \, e^a (a^\dagger)^n \, |n\rangle$$

$$= a^{\dagger}a \left|n\right\rangle = n \left|n\right\rangle$$

Hence,

$$\mathcal{A}(a^{\dagger}a) \left|n\right\rangle = A(n) \left|n\right\rangle$$

Where the full computation can be found in Subsection A.1.4. This can be compared to the original realisation of Doi[9]. Specifically, the correlation density function has the equivalence of the expectation of the operators defined above, on the functional coherent state and the probability density function noted in Subsection 2.1.3. We will not be considering spatial dimensions integrated in Peliti[30].

$$\left\langle \mathcal{A} \right\rangle_{\psi t} = \left\langle 1\right| a \left|\psi_t\right\rangle \tag{2.1.16}$$

Which from the Schrödinger representation can be given in the form,

$$\left\langle 1\right| a \left|\psi_t\right\rangle = \left\langle 1\right| ae^{\mathcal{L}t} \left|\psi_0\right\rangle \tag{2.1.17}$$

Here the value of the coherent state u(n), has been set to 1, and a conjugate application of the eigenstate property stated in Equation 2.1.8. We seek to draw comparison between the classical expectation given in Subsection 2.1.2 where the expectation was defined in time-forward continuous framework. We notice that the expectation above, mirrors the canonical density matrix detailing the statistical state of a system in quantum mechanics[17], known to produce expectation values, from discrete states using a superposition of ensembled *kets*. However, these should not be considered identical, due to quadratic behaviours stated previously.

3

Path Integral Formulation

Here we propose path integral formulation, on the basis of a pedagogical approach. Hence we will outline the sequence of action leading to formulation. We plan to decompose the expectation of the correlation density function into the representation of the initial macroscopic state, given by Equation 2.1.17. We seek to *split* time, into *slices*, ϵ i.e. disintegrate the time term in Equation 2.1.17 into N discrete intervals, where time exists in an infinitesimal sense. Upon this, we place resolution of identity between these slices. After extensive mathematical machinery, we conclude the final coherent state path integral. It is noteworthy that annihilation operators provide eigenvalues of coherent states, meaning eigenstates will be provided, implicitly, however creation operators do not. This is one of the intrinsic driving forces for application of coherent states, in Subsection 2.1.5

This can be demonstrated by using an arbitrary state, $|\psi\rangle = c_n |n\rangle$ and application of the creation operator defined for quantum harmonic oscillators[16].

If,

$$a |n\rangle = \sqrt{n+1}\, |n+1\rangle$$

Let the arbitrary state be given by,

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

Then applying the raising *ladder operator*,

$$a\,|\psi\rangle = a\sum_{n=0}^{\infty} c_n\,|n\rangle$$

$$= \sum_{n=0}^{\infty} c_n\sqrt{n+1}\,|n+1\rangle$$

$$= \sum_{n=0}^{\infty} c_{n-1}\sqrt{n}\,|n\rangle$$

Which $\forall$ n, $c_n = 0$.

Note, if eigenstates, χ, of the creation operator exist, they must satisfy[16],

$$|\chi\rangle = e^{-\frac{|\chi|^2}{2}}\sum_{n=0}^{\infty}\frac{\chi^n}{n!}\,|n\rangle$$

Observing the Poissonian distribtution term, we have shown this to be our coherent state. Hence we are required to have the machinery accompany our pedagogical trajectory, where we require a coherent resolution of identity.

3.1 Resolution of Identity

To draw intuition, we make comparison to the resolution of identity developed for *Spectral theory* borrowed from Borel functional analysis[31]. The resolution of identity can be considered as a mapping technique, where a projection valued measure has self-adjoint projections onto a fixed Hilbert space. If a spatial dimension such as Fock space was considered, as in Peliti[30], we may be able to map the projection. However no spatial dimensions are considered, and we regard this intuition as assisting the slices to resolve to a continuum basis.

We progress then, with the elucidation of the coherent resolution of identity, in the form,

$$\mathcal{I} = \mathbb{1} = \int \mathcal{D}u\mathcal{D}v \; e^{-i\int uv} \left|iv\right\rangle \left\langle u\right| \tag{3.1.1}$$

Where the symbol $\int \mathcal{D}(\bullet)$ is a succinct way of conveying an integral over infinite dimensions in all possible configurations. For proof of equivalency to Equation 2.1.7 see Subsection A.1.8. Furthermore,

$$\int \mathcal{D}u\mathcal{D}v \equiv \lim_{\substack{\epsilon\to \\ N\to\infty}} \prod_{i=1/\epsilon}^{N-1} \int \frac{\epsilon}{2\pi} \, dudv$$

Where these are functional integrals over complex numbers u and v, where more information about coherent states and the rigorous formulation of the coherent state identity can be found in [15].

3.2 Path integral architecture

We now deconstruct the expectation of the correlation function density, defined in Equation 2.1.17. We anticipate that individual matrix elements require decomposition and solution under renormalisation properties, to congregate individual products for summation. We now perform the decomposition such that, $t = \sum_n \epsilon_1 + \epsilon_2 \cdots + \epsilon_n$, i.e. *time slices*.

$$\left\langle 1\right| a \left|\psi_t\right\rangle = \left\langle 1\right| ae^{\mathcal{L}t} \left|\psi_0\right\rangle = \left\langle 1\right| ae^{\mathcal{L}\epsilon} e^{\mathcal{L}\epsilon} \cdots e^{\mathcal{L}\epsilon} \left|\psi_0\right\rangle \tag{3.2.1}$$

Let us intermission for a moment, and consider the proposition conceptually. We visualise the probability of our state propagating from a state x, to a state y, over a continuum interval of time, as recreated in Figure 3.2.1. We then pose the question of how to mathematically describe the sum of the spaces, to find a

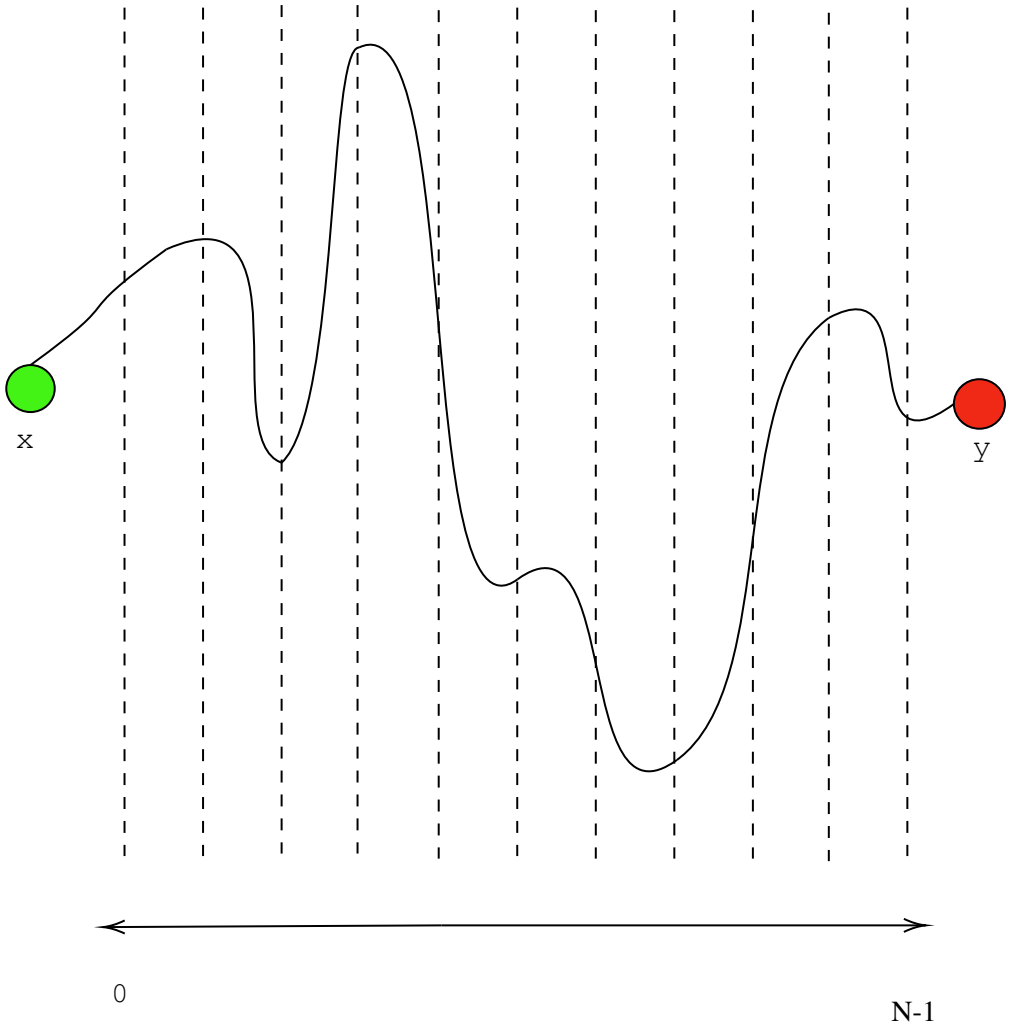

Figure 3.2.1: visualisation of time slices

rendition of the total trajectory. We can draw comparison to the *action principles*[13] in Lagrangian mechanics, where we might ask, how we would span a path with the highest efficiency. Now we can now propose, to summate infinitely many possibilities of the action trajectory, and the probability of traversing associated with each. This provides an adequate cognitive image of the ambition of quantum field theory. We now require a means of description, of infinitely many discrete points over the space the state is traversing. To achieve this we partition the space infinitely many times, into slices, N-1 of them for specificity, and integrate along said slices. You can think of this intuitively, as almost a generalisation of the *double-slit experiment*[38], and except for having a singular 'slice' with two slits in it, we require infinitely many slices and

infinitely many slits, over all possible configurations. The path integral will tell us the probabilities associated with specified trajectories through an instance of those slits through the infinite slices.

Satisfying intuition, we progress with the mathematical machinery. Post-decomposition of expectation and time slicing, we insert resolutions preceding the exponents, such that,

$$\left\langle \mathcal{A} \right\rangle_{\psi t} = \langle 1 |\, a \mathcal{I} e^{\mathcal{L}\epsilon} \mathcal{I} e^{\mathcal{L}\epsilon} \mathcal{I} e^{\mathcal{L}\epsilon} \cdots \mathcal{I} e^{\mathcal{L}\epsilon} \mathcal{I} \,| \psi_0 \rangle \tag{3.2.2}$$

Then substituting for the resolutions, gives the expectation in the form,

$$\langle 1 |\, a \int \mathcal{D}u\mathcal{D}v \; e^{-i\int uv} \,| iv \rangle \, \langle u |\, e^{\mathcal{L}\epsilon} \int \mathcal{D}u\mathcal{D}v \; e^{-i\int uv} \,| iv \rangle \, \langle u | \ldots \langle u | \psi_0 \rangle \tag{3.2.3}$$

We now implement bookkeeping for convenience, such that we are able to differentiate between the terms, therefore,

$$\begin{aligned} \left\langle \mathcal{A} \right\rangle_{\psi t} = \langle 1 |\, a \int \mathcal{D}u\mathcal{D}v e^{-\int uv} \,| iv_N \rangle \\ \langle u_N |\, e^{\mathcal{L}\epsilon} \int \mathcal{D}u\mathcal{D}v e^{-\int uv} \,| iv_{N-1} \rangle \, \langle u_{N-1} | \\ e^{\mathcal{L}\epsilon} \int \mathcal{D}u\mathcal{D}v e^{-\int uv} \,| iv_0 \rangle \, \langle u_0 | \int \mathcal{D}u\mathcal{D}v e^{-\int uv} \,| \psi_0 \rangle \end{aligned} \tag{3.2.4}$$

We factor out the $\int \mathcal{D}u\mathcal{D}v$ to reduce to the form,

$$\begin{aligned} \left\langle \mathcal{A} \right\rangle_{\psi t} = \int \mathcal{D}u\mathcal{D}v \, \langle 1 |\, a e^{-\int uv} \,| iv_N \rangle \\ \langle u_N |\, e^{\mathcal{L}\epsilon} e^{-\int uv} \,| iv_{N-1} \rangle \, \langle u_{N-1} | \\ e^{\mathcal{L}\epsilon} e^{-\int uv} \,| iv_0 \rangle \, \langle u_0 |\, e^{-\int uv} \,| \psi_0 \rangle \end{aligned} \tag{3.2.5}$$

We look to represent our terms in a simplistic form, by withdrawing $e^{\int uv}$, which will require additional parenthesis. We choose to represent our integral as a sum of all individual terms with their respective matrices, $e^{\sum_{i=1}^{N} u_i v_i}$. We introduce this to hope to solve the matrices individually.

Hence,

$$\int \mathcal{D}u\mathcal{D}v \, \langle 1| \, a \, |iv_N\rangle \, \langle u_N| \, e^{\mathcal{L}\epsilon} \, |iv_{N-1}\rangle \, \langle u_{N-1}| \, e^{\mathcal{L}\epsilon} \, |iv_0\rangle \, \langle u_0|\psi_0\rangle \\ \ldots \; e^{\sum_{i=1}^{N} u_i v_i} \tag{3.2.6}$$

We arrive at the four matrix terms as a result of decomposition of the expectation. The first matrix term, $\langle 1| \, a \, |iv_N\rangle$, and the fourth matrix term, $\langle u_0|\psi_0\rangle$, as well as the functional integral, and exponential term, we will stagnate addressing for the immediate future, and direct our efforts towards solutions of the second and third matrix terms, respectively.

$$\langle u_N| \, e^{\mathcal{L}\epsilon} \, |iv_{N-1}\rangle \tag{3.2.7}$$

$$\langle u_{N-1}| \, e^{\mathcal{L}\epsilon} \, |iv_0\rangle \tag{3.2.8}$$

We are able to generalise the solution for these intermediate terms, by the following,

$$\prod_{i=0}^{N-1} \langle u_i| \, e^{\mathcal{L}\epsilon} \, |iv_{i-1}\rangle \tag{3.2.9}$$

Where we are collecting all individual states defined in the interval [i=0,N-1], over a time period [0,T], under a product. For a visual analogy please defer back to Figure 3.2.1.

We recall our Subsection 2.1.3. The elegance of the Schrödinger picture canvas allows for a Taylor series expansion about the unitary time evolution operator, $U^{\dagger}(t,t_0)U(t,t_0) = \mathcal{I} = e^{\mathcal{L}t}$ as it remains a constant through time. As we sliced time into infinitesimal intervals, ϵ, we assume that implicitly these intervals will too, remain a constant. Hence, we apply a Taylor series expansion about $e^{\mathcal{L}\epsilon}$.

$$\begin{aligned} &\langle u_i|\, e^{\mathcal{L}\epsilon}\, |iv_{i-1}\rangle \\ &\langle u_i|\, 1 + \mathcal{L}\epsilon + \frac{1}{2}\mathcal{L}^2\epsilon^2 + \cdots .. |iv_{i-1}\rangle \\ &\langle u_i|\, 1 + \mathcal{L}\epsilon\, |iv_{i-1}\rangle \end{aligned} \tag{3.2.10}$$

As we have defined ϵ to be an infinitesimally small value of time, we only consider first order term, as the compound of the ϵ as $\epsilon \to\ 0$ will be negligible.

As we distributed the terms through the states, we note that $\langle u_i|\, 1\, |iv_{i-1}\rangle = \langle u_i|iv_{i-1}\rangle$. We drop the bookkeeping notation briefly for simplicity, then express the above in the form,

$$\langle u|iv\rangle \ + \ \epsilon\, \langle u|\, \mathcal{L}\, |iv\rangle \tag{3.2.11}$$

We re-instantiate the formal definition of our Hamiltonian and look for a novel, simplistic derivation, were we consider the death term of the Hamiltonian only, as one could extrapolate the process with similar logic, to produce the complete description. We absorb ϵ back into the expression,

$$\langle u|iv\rangle + \epsilon\, \langle u|\, \beta(iv - uiv)\, |iv\rangle = \langle u|\, 1 + \epsilon\beta(iv - uiv)\, |iv\rangle \tag{3.2.12}$$

Allowing us to revert to the definition of an exponent,

$$\langle u|\, e^{\epsilon\beta(iv-uiv)}\, |iv\rangle = e^{\epsilon\beta(iv-uiv)}\, \langle u|iv\rangle \tag{3.2.13}$$

We apply the Baker-Campbell Hausdorff theorem to the coherent state properties Equation 2.1.8, Equation 2.1.9, for an intermediate representation of the second

matrix term:

$$\langle u|\, e^{\mathcal{L}\epsilon}\, |iv\rangle = e^{\epsilon\beta(iv-uiv)}e^{iuv} \tag{3.2.14}$$

We previously disregarded the bookkeeping temporarily. We are at an adequate place to correct this in order to introduce the necessary mathematical machinery. We seek to express the $e^{\epsilon\beta(iv-uiv)}$, as the product of all possibilities in all configurations to account for the generalisation made previously in Equation 3.2.9. Disregarding e^{iuv} shortly,

$$\prod_{k=0}^{N} exp\Big\{\epsilon\beta(iv_k - iv_{k+1}v_k)\Big\} \tag{3.2.15}$$

$$exp\Big\{\epsilon\beta \prod_{k=0}^{N}(iv_k - iv_{k+1}v_k)\Big\} \tag{3.2.16}$$

Where we have dropped the β term for simplicity and will carry through informally. Splitting the product into the parenthesis k=0, k=1 ; N:

$$\prod_{k=0}^{N} exp\Big\{iv_{k+1}u_k\Big\} \prod_{k=1}^{N} exp\Big\{-iv_k u_k\Big\} \tag{3.2.17}$$

Yields,

$$\sum_{k=0}^{N} exp\Big\{\frac{i\epsilon(v_{k+1} - iv_k)u_k}{\epsilon}\Big\} \tag{3.2.18}$$

Where the equation above is in a notable form, the *Riemann sum*. We expressed a desire previously to morph the exponent in a familiar form. Accounting for β, upon operation we obtain:

$$exp\Big\{i\int_0^T dt(\frac{du}{dt}v - \beta(v - uv)\Big\} \tag{3.2.19}$$

This is a satisfactory representation of the matrix element. We have obtained this result using the *Riemann sum* for conversion of the summation to an integral, accounting for the initial e^{iuv} dropped for convenience and unifcation of exponents, by $e^a + e^b = e^{ab}$, where integration by parts was performed on the

pre-existing $\int \frac{dv}{dt} u$ term. Let us remark, some identities established from the renormalisation properties and the coherent state properties constructed in Subsection 2.1.5 and constructed over our finite interval [0,T]; for a more rigorous formulation please refer to Subsection A.1.5.

$$\langle 1|\, a\, |iv\rangle = e^{iv_T} \;;\; \langle \phi | iv_T \rangle = 1. \tag{3.2.20}$$

Let us refer to our original 4-part matrix decomposition Equation 3.2.22; three of the four matrix elements have known solutions however the fourth matrix element $\langle u_0|\psi_0\rangle$ has eluded us. We need to ponder on the definition of our fundamental coherent states. To condense our formalism, we must consider ψ_0, our fundamental state, to be a coherent state. This has the implication, that the initial fundamental state must obey a Poissonian distribution, to satiate this proposition, please see Subsection A.1.3. We require our initial distribution such that, each individual has the initial condition ω, where they are accounted with a Poisson parameter, α. Mathematically we compel the initial distribution to be given by $|\psi_0\rangle = e^{-\alpha}\,|\alpha\omega\rangle$, and hence,

$$\langle u_0|\psi_0\rangle = e^{\alpha\omega(u_0-1)} \tag{3.2.21}$$

For completeness, we will consider the summation of our identities, and derived expressions to stimulate a metamorphosis of Equation 3.2.22 under the boundary constraints. After a strenuous pedagogical sequence of events, we have arrived at our penultimate step; the path integral representation of the correlation density function.

$$\begin{aligned}\int \mathcal{D}u\mathcal{D}v\, \langle 1|\, a\, |iv_N\rangle\, \langle u_N|\, e^{\mathcal{L}_\epsilon}\, |iv_{N-1}\rangle\, \langle u_{N-1}|\, e^{\mathcal{L}_\epsilon}\, |iv_0\rangle\, \langle u_0|\psi_0\rangle \\ \ldots\; e^{\sum_{i=1}^{N} u_i v_i}\end{aligned} \tag{3.2.22}$$

Furthermore, we substitute our derived representations of the matrix elements defined previously; as a result of this, we present the *functional coherent state path integral*:

$$\int \mathcal{D}u\mathcal{D}v\, iv_T\, exp\Big\{ i\int_0^T dt[\tfrac{du}{dt}v - \beta(v-uv)] - iu_T v_T + iv_T + \alpha\omega(u_0-1) \Big\} \tag{3.2.23}$$

3.2.1 Doi-Shift and perturbative expansion

The path integral given above, is a novel construction as only half of the Hamiltonian considered, for simplicity. When considering the full Hamiltonian $\mathcal{L} = \beta((a^\dagger)^2 a - a^\dagger a) + \mu(a - a^\dagger a)$ in coherent state morph $\mathcal{L} = \mathcal{L}[u,v]$ we have a compact path integral formulation. Where formal derivation can be found via substitution using principles outlined in this research. However we choose to represent in a succinct form, to outline the macroscopic approach of formulation.

$$\Big\langle \mathcal{A} \Big\rangle_{\psi t} = \int \mathcal{D}u\mathcal{D}v\, e^{i\int dt\{\frac{du}{dt}v - \mathcal{L}[u,v]\}} \tag{3.2.24}$$

We draw comparison from the canonical path integral formulation in quantum field theory, found by Richard Feynman[11] specified for a wavefunction in position representation[2]. Note, Z is the normalisation factor; S is the action of trajectory from a point x to a point y, mirroring previous description. The canonical path integral is given by:

$$\psi(x,t) = \frac{1}{Z}\int_{\mathbf{X}(0)=x} \mathcal{D}\mathbf{X} e^{i\mathcal{S}[x,\dot{x}]}\psi_0(x,t) \tag{3.2.25}$$

Where $\mathcal{S}[x,\dot{x}] = \int dt\mathcal{L}(x(t),\dot{x}(t))$ Therefore,

$$\psi(x,t) = \frac{1}{Z}\int_{\mathbf{X}(0)=x} \mathcal{D}\mathbf{X}\, exp\Big\{ i\int dt[\mathcal{L}(x(t),\dot{x}(t))]\,\psi_0(x(t)) \Big\} \tag{3.2.26}$$

We can conduct some analysis by inspection. We draw parallels between the overarching forms of the equations, where they both follow the general form $\mathcal{D}\bullet e^{i\int dt\mathcal{L}[\bullet,\dot{\bullet}\cdots]}\psi_0$. We have generalised coordinates with $\bullet$. Note, both equations are constrained upon the Hamiltonian of the system, in physical systems this can be absorbed into the action. We recognise both have a constraint on the initial state of the system; both initial states are defined as a probabilistic distribution to account for stochastic fluctuations. The account of fluctuations is represented in the criticality, which we shall discuss later. The canonical path integral can be solved via utilisation of a propagator, we will draw upon this due to symmetry.

We establish the concept of a propagator in a functional path integral via functional differentiation. In completion of this, we can take the lim $J(T) \rightarrow\ 0$, applying the appropriate conditions, to recover our Malthusian exponential, for completeness. We perform this to prove equivalence between the quantum mechanical formulation, and the classical interpretation under specific constraints.

For pedagogical simplicity we will revert to the novel expression Equation 3.2.23. Before functional differentation can be applied, we require a Doi shift[9], [30]; a field shift in which u $\rightarrow$ u+1:

$$\int \mathcal{D}u\mathcal{D}v\, iv_T\, exp\Big\{ i\int_0^T dt[\frac{du}{dt}v - \beta(v-uv)] - iu_Tv_T + iv_T + \alpha\omega(u_0-1)\Big\}$$

Under the shift, terms $u' \rightarrow u'$; $-iu_Tv_T + iv_T \rightarrow -iu_Tv_T$; $u'v \rightarrow u'v$; $\beta(v - uv) \rightarrow \beta(-uv)$; resulting in the following,

$$(u \rightarrow u+1) = \int \mathcal{D}u\mathcal{D}v\, iv_T\, exp\Big\{ i\int_0^T dt[\dot{u}v - \beta uv] - iu_Tv_T + \alpha\omega u_0\Big\} \tag{3.2.27}$$

Then we are able to define an indeterminate function J(T) dependent on the final destination of our interval [0,T] stated previously. We merge J with our field shifted path integral to produce,

$$\mathcal{Z}(J) = \int \mathcal{D}u\mathcal{D}v\, exp\Big\{ i\int_0^T dt[\dot{u}v - \beta uv + Jv] - iu_T v_T + \alpha\omega u_0 \Big\} \tag{3.2.28}$$

Computing the $\int \mathcal{D}v$ integral, the integrand arrives at $\dot{u} - \beta u + J = 0$; $u_T = 0$, we seek solutions for the differential integrand,

$$\begin{aligned} dt(ue^{-\beta t}) &= -Je^{-\beta t} \\ u_T e^{\beta t} - u_t e^{\beta t} &= -\int_t^T Je^{-\beta s} ds \\ u_t &= \int_t^T e^{\beta(s-t)} J(s) ds \\ u_0 &= \int_0^T e^{-\beta s} J(s) \end{aligned} \tag{3.2.29}$$

$\mathcal{Z}(J)$ can be expressed as,

$$\mathcal{Z}(J) = \int \mathcal{D}u\ \hat{u}(t)\, e^{\alpha\omega u_0} = e^{\alpha\omega u_0} = e^{\alpha\omega \int_0^T e^{-\beta s} J(s)} \tag{3.2.30}$$

We are ready to perform the functional differentiation of $\mathcal{Z}$ with respect to our indeterminate function, J(T), yielding,

$$\frac{\delta\mathcal{Z}(J)}{\delta J(T)}\bigg|\bigg|_{J=0} = \omega e^{-\beta T} e^{\alpha\omega \int_0^T e^{-\beta s} J(s)} \tag{3.2.31}$$

Where our initial population distribution is ω. The limit ensures the integral term has no contribution.

$$\frac{\delta\mathcal{Z}(J)}{\delta J(T)} = \mathbb{E}(N(T)) = \omega e^{-\beta t} \tag{3.2.32}$$

Thus, our propagator $\frac{\delta\mathcal{Z}(J)}{\delta J(T)} = G(T;T')$ is shown in Equation 3.2.31. For a complete rigorous computation of the functional derivative please refer to Equation A.1.18.

3.3 Conclusion

In this research, we sought to construct a classical population model via probability theory in Section 2.1. Time forward-continuous master equations were established successfully in Subsection 2.1.2. Birth-death rates were in the parameter inclusion; in reality this is a simplistic, but effective model of the macroscopic temporal transformation of populations, it should however not be considered rigorous. Furthermore, for increased accuracy, age parameterisation should be included, however this yields a partial differential equation in which resultant kinetic equations are renowned for difficulty, comparable to the BBGKY Hierarchy equations [5]; therefore neglected.

We also utilised techniques to quantise the system via Schrödinger picture Subsection 2.1.3 and Doi-Peliti formalism Subsection 2.1.4. The Schrödinger picture provides excellent framework. Evolution operators were constant with respect to time evolution, sanctioning governing operators to act without change. Alternatively, systems where unitary time evolution operators evolve with state can be examined in the interaction picture [27]. The Doi-Peliti formalism provided a linear trajectory of events, however in real populations this is not always the case; for example genetic bottleneck phenomena i.e. natural disasters, these are non-linear by definition.

As noted in Chapter 1, more intricate modelling would include in-depth domain knowledge and statistical inference such that carrying capacity, predator-prey relations, and multi-species interaction terms included. However, domain would require simplification of vastly complicated many body systems, many of which remain unsolved analytically.

The implementation of Doi-Peliti formalism [30] in this research did not assume spatial dimensions. This allowed for a novel mathematical framework to be developed and intricacy to be supplicated. Note, for ideal spatial consideration, topological sensitivity analysis may be included [4]. We can remark supplication

of lattice-techniques [30] would aid the description of propagation of phenotypic diffusion through spatial plane. We may note Subsection 2.1.5 includes application of coherent states, where an assumed property is Poissonian distribution. Alteration of initial conditions of the fundamental state resolves this; however, with loss of completeness in exchange for generality.

We hoped to provide path integral formulation in Chapter 3 such that stochastic fluctuation perceived. The derivation provides field-theoretic techniques to describe relationships between the probability density function, contrasting to classical formulation, where results lack clarity. The coherent state path integral accounted for all possibilities of events in all configurations, as a probabilistic distribution in the intervals defined [i=1, N-1] ; [0,T].

Lastly we compare the formulations. The time derivative in the master equation for our system Equation 2.1.2, has constraint on occupation number, n. Whilst state evolution is formalised rigorously, occupation number is cumbersome to address directly, with respect to the master equation. The master equation is a linear differential equation such that solution requires initial condition instantiation. It contains a concoction of states, which would constitute off-diagonal terms in matrix notation, this is less than ideal. The master equation also fails to directly describe temporal evolution of the systems density. We hope to provide an indirect equivalency between the density fluctuations of the master equation and associate them with the coherent state path integral to solve the inadequacy.

Although one may construct an imprecise model description of a real system, the machinery and formal solution of said differentials is a demanding due to the intrinsic stochastic behaviours and the magnitude of fluctuation consequences in real systems. The integration of the master equation for real systems is generally impractical and is a rudimentary simplistic approximation; often requiring neglect of fluctuations and correlations i.e. mean-field theory. Explicit solutions are not able to give quantitative representation of the system, and processes occurring

within it. Even if spatial dimensions were applied, the prior logic would still hold true at a distance exceeding the critical domain.

We use Doi representation of the master equation to remove explicit dependence on the number of individuals, or, occupation number. The derivation of Doi can be found in [9] and provides a succinct basis of quantisation. Utilisation of a , $a^{\dagger}$ operators provides machinery with constant coefficients with respect to fundamental state vector, n. For more information one can refer to [30].

In this comparison the Doi-Peliti formalism provides an eloquent basis of how one could include effects of fluctuations in the analytical description of the process. The self-organised criticality employed via field-theoretic techniques cannot be understated. The temporal scale invariance of transitions without the constraint of fine-tuned control parameters to specificity is remarkable. Thus, we have removed the hard dependencies we hoped to achieve. Note; one may also achieve criticality via stochastic coarse-grained sand-piles, in conjunction.

For more information on alternate formulations; one may seek conversion of partial differential equations to extended duality relations [28]. If unsatisfactory, refer to statistical mechanical approach of coarse-grained sand-piling implemented in Weise[36]. Conversely, an intricate approach to account for age-dependent populations, in which kinetic equations derived; see Greenman[15].

We hope to have provided motivation to physicists and mathematicians alike, to advance the framework provided to design more complex systems. Note the functional differentiation performed may be extrapolated via perturbative expansion, yielding propagators to construct suitable Feynman Diagrams [11] that allow elegant visual presentation of system evolution and formal definitions of interaction terms. This research hopes to provide a pedagogical approach to the derivation of coherent state path integrals, where intuition was provided when feasible. Visualisation aided where deemed possible and uniform conceptual explanations were accommodated throughout.

Bibliography

[1] Bruce Alberts, Alexander Johnson, Julian Lewis, Martin Raff, Keith Roberts, and Peter Walter. Molecular biology of the cell. garland science. *New York*, 1392, 2007.

[2] Sergio Albeverio, Rafael Høegh-Krohn, and Sonia Mazzucchi. *Mathematical theory of Feynman path integrals: an introduction*, volume 523. Springer Science & Business Media, 2008.

[3] Neil W Ashcroft. Nd mermin solid state physics. *Saunders College, Philadelphia*, 120, 1976.

[4] Ann C Babtie, Paul Kirk, and Michael PH Stumpf. Topological sensitivity analysis for systems biology. *Proceedings of the National Academy of Sciences*, 111(52):18507–18512, 2014.

[5] Nikolaĭ Bogoliubov. Problems of dynamic theory in statistical physics; problemy dinamicheskoi teorii v statisticheskoi fiziki. Technical report.

[6] John Cardy. Reaction-diffusion processes. *A+ A*, 100:26, 2006.

[7] Richard Courant and David Hilbert. Chapter iv. the calculus of variations. *Methods of Mathematical Physics. New York: Interscience Publishers*, pages 164–274, 1953.

[8] Paul Adrien Maurice Dirac. A new notation for quantum mechanics. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 35, pages 416–418. Cambridge University Press, 1939.

[9] Masao Doi. Second quantization representation for classical many-particle system. *Journal of Physics A: Mathematical and General*, 9(9):1465, 1976.

[10] Stewart N Ethier and Thomas G Kurtz. *Markov processes: characterization and convergence*, volume 282. John Wiley & Sons, 2009.

[11] Richard Phillips Feynman. Space-time approach to non-relativistic quantum mechanics. In *Feynman's Thesis—A New Approach To Quantum Theory*, pages 71–109. World Scientific, 2005.

[12] Charlotte Froese Fischer. General hartree-fock program. *Computer physics communications*, 43(3):355–365, 1987.

[13] Herbert Goldstein, Charles Poole, and John Safko. Classical mechanics, 2002.

[14] Peter Grassberger and M Scheunert. Fock-space methods for identical classical objects. *Fortschritte der Physik*, 28(10):547–578, 1980.

[15] Chris D Greenman. A path integral approach to age dependent branching processes. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(3):033101, 2017.

[16] David Griffiths. *Introduction to elementary particles*. John Wiley & Sons, 2008.

[17] Brian C Hall. Systems and subsystems, multiple particles. In *Quantum Theory for Mathematicians*, pages 419–440. Springer, 2013.

[18] Brian L Hammond, William A Lester, and Peter James Reynolds. *Monte Carlo methods in ab initio quantum chemistry*, volume 1. World Scientific, 1994.

[19] Alan Hastings. Mckendrick von foerster models for patch dynamics. In *Differential Equations Models in Biology, Epidemiology and Ecology*, pages 189–199. Springer, 1991.

[20] John Hawks, Keith Hunley, Sang-Hee Lee, and Milford Wolpoff. Population bottlenecks and pleistocene human evolution. *Molecular biology and evolution*, 17(1):2–22, 2000.

[21] Alan Isaacs. Oxford dictionary of physics. *Oxford University Press. Oxford, New-York. 474 pp., ISBN 0-19-280030-2*, 1996.

[22] Sharon E Kingsland and Sharon E Kingsland. *Modeling nature*. University of Chicago Press, 1995.

[23] D Markham and V Vedral. Classicality of spin-coherent states via entanglement and distinguishability. *Physical Review A*, 67(4):042113, 2003.

[24] William G McCallum, Deborah Hughes-Hallett, Brad G Osgood, and Daniel Flath. *Calculus: multivariable calculus*. Wiley, 2005.

[25] David AB Miller. *Quantum mechanics for scientists and engineers*. Cambridge University Press, 2008.

[26] James D Murray. *Mathematical biology: I. An introduction*, volume 17. Springer Science & Business Media, 2007.

[27] John W Negele. *Quantum many-particle systems*. CRC Press, 2018.

[28] Jun Ohkubo. Extended duality relations between birth–death processes and partial differential equations. *Journal of Physics A: Mathematical and Theoretical*, 46(37):375004, 2013.

[29] JA Oteo. The baker–campbell–hausdorff formula and nested commutator identities. *Journal of mathematical physics*, 32(2):419–424, 1991.

[30] L Peliti. Path integral approach to birth-death processes on a lattice. *Journal de Physique*, 46(9):1469–1483, 1985.

[31] Michael Reed and Barry Simon. I: Functional analysis, volume 1 (methods of modern mathematical physics)(vol 1). *Academic Press*, 1:40170–110, 1981.

[32] Erwin Schrödinger. An undulatory theory of the mechanics of atoms and molecules. *Physical review*, 28(6):1049, 1926.

[33] Ramamurti Shankar. *Principles of quantum mechanics*. Springer Science & Business Media, 2012.

[34] Szymon Tomczyk, Kathleen Fischer, Steven Austad, and Brigitte Galliot. Hydra, a powerful model for aging studies. *Invertebrate reproduction & development*, 59(sup1):11–16, 2015.

[35] Pierre-François Verhulst. Notice sur la loi que la population suit dans son accroissement. *Corresp. Math. Phys.*, 10:113–126, 1838.

[36] Eric W Weisstein. *CRC concise encyclopedia of mathematics*. CRC press, 2002.

[37] Kay Jörg Wiese. Coherent-state path integral versus coarse-grained effective stochastic equation of motion: From reaction diffusion to stochastic sandpiles. *Physical Review E*, 93(4):042117, 2016.

[38] Thomas Young. Ii. the bakerian lecture. on the theory of light and colours. *Philosophical transactions of the Royal Society of London*, (92):12–48, 1802.

A

Appendices

A.1 Proofs and Identities

A.1.1 Variance

From Subsection 2.1.2, we know $\frac{\partial G}{\partial x} = \sum_{n=0}^{\infty}$, and $\frac{\partial G}{\partial t}$ as probability only time dependent term; then,

$$\frac{\partial G}{\partial t} = -\beta x \frac{\partial G}{\partial x} + \beta x^2 \frac{\partial G}{\partial x} = \beta x(x-1)\frac{\partial G}{\partial x} \tag{A.1.1}$$

Defining a function, f, such that,

$$\begin{aligned} \frac{dx}{df} = \beta x(x-1) \rightarrow \int \frac{dx}{x(x-1)} &= \int \beta df = \frac{1}{x-1} - \frac{1}{x} dx = \beta f \\ &= ln(\frac{x-1}{x}) = \beta f = (\frac{x-1}{x}) = e^{\beta f} \\ &= (1 - e^{\beta f})^{-1} \end{aligned} \tag{A.1.2}$$

Then, let F(f,t)=G(x,t);

$$\begin{aligned} \frac{\partial F}{\partial t} = \frac{\partial G}{\partial t} &= \beta x(x-1)\frac{\partial G}{\partial x} \\ &= \frac{dx}{df}\frac{dG}{dx} = \frac{\partial G}{\partial f} = \frac{\partial F}{\partial f} \end{aligned} \tag{A.1.3}$$

by Chain rule. We look for a solution in the form F(f,t)=h(f+t), such that the boundary conditions imply $h(f) = F(f,0) = G(x,0) = \sum_{n=0}^{\infty} P_n(0)x^n = x^{n_0}$, therefore;

$$If \;\; x = (1-e^{\beta f})^{-1} \rightarrow h(f) = (1-e^{\beta f})^{-1} \tag{A.1.4}$$
$$F(f,t) = h(f+t) = (1-e^{\beta(f+t)})^{-n_0}$$

then as F(f,t)=G(x,t), substituting for f,

$$G(x,t) = (1-e^{\beta f}e^{\beta t})^{-n} = (1-(1-\frac{1}{x})e^{\beta t})^{-n_0-2} \tag{A.1.5}$$

Thus, $G_{xx}(1,t)$ given by $G_x(1,t) - G_x(1,t)^2$ i.e. the Variance $\mathbb{V} = Var = E[X^2] - E[X]^2$;

$$\frac{\partial^2 G}{\partial x^2} = -n_0(1-(1-\frac{1}{x})e^{\beta t})^{-n_0-1}e^{\beta t}\frac{2}{x^3} + n_0(n_0+1)(1-(1-\frac{1}{x})e^{\beta t})^{-n_0-2}e^{2\beta t}\frac{1}{x^4} \tag{A.1.6}$$

$$= -2n_0e^{\beta t} + n_0(n_0+1)e^{2\beta t}$$
$$= -2n_0e^{\beta t} + n_0(n_0+1)e^{2\beta t} + n_0e^{\beta t} - n_0^2e^{2\beta t}$$
$$= \mathbb{E}(0)e^{\beta t}(e^{\beta t}-1)$$

A.1.2 Master Equation recovery

Recovery of Master equation from state interpretation defined in, Subsection 2.1.3;

Assuming $\frac{\partial}{\partial t}|\psi_t\rangle = \sum_n P_n(t)|n\rangle$ is true, then we may substitute formal definition of Hamiltonian $\mathcal{L} = \beta((a^\dagger)^2a - a^\dagger a) + \mu(a - a^\dagger a)$ into our macroscopic state, then;

$$|\psi_t\rangle = \sum_n P - n[\beta((a^\dagger)^2a - a^\dagger a) + \mu(a - a^\dagger a)]|n\rangle \tag{A.1.7}$$
$$= \sum_n P_n[\beta n|n+1\rangle - \beta n|n\rangle + \mu n|n-1\rangle - \mu n|n\rangle]$$

From application of operators. We seek to find $\sum_n \frac{dP_n}{dt} |n\rangle$ Therefore;

$$\sum_n \frac{dP_n}{dt} |n\rangle = \sum_{n'} P_{n'-1}\beta(n'-1) |n'\rangle - \sum_n P_n \beta n |n\rangle + \sum_{n'} P_{n+1}\mu(n+1) |n\rangle \tag{A.1.8}$$

$$\begin{aligned} &- \sum_n P_n \mu n |n\rangle \\ &= \sum_n P_{n-1}\beta(n-1) |n\rangle - \sum_n P_n \beta n |n\rangle + \sum_n P_{n+1}\mu(n+1) |n\rangle \\ &- \sum_n P_n \mu n |n\rangle \end{aligned}$$

Multiplying through by $\langle m|n\rangle = \delta_{mn} n!$ where n! is the normalisation factor;

$$\frac{dP_n}{dt} m! = \langle m| \left(\sum_n P_{n-1}\beta(n-1) |n\rangle - \sum_n P_n \beta n |n\rangle + \sum_n P_{n+1}\mu(n+1) |n\rangle \right. \tag{A.1.9}$$

$$\begin{aligned} &\left. - \sum_n P_n \mu n |n\rangle \right) \\ &= P_{m-1}\beta(m-1)m! - P_m \beta m\, m! + P_{m+1}\breve{(}m+1)m! - P_m \mu m\, m! \end{aligned}$$

Dividing through by $m!$;

$$\frac{dP_m}{dt} = P_{m-1}\beta(m-1) - P_m \beta m + P_{m+1}\breve{(}m+1) - P_m \mu m \tag{A.1.10}$$

Hence, we have recovered the original master equation, under the m'th state.

A.1.3 Poissonian Distribution

If we consider an arbitrary coherent state, with identical definition to the coherent state constructed in Subsection 2.1.5, $|\omega\rangle = e^{a^\dagger \omega} |\phi\rangle$, we apply our

initial distribution $e^{-\omega}$, we arrive at the conclusion;

$$e^{-\omega}\left|\omega_0\right\rangle = e^{-\omega}e^{a^\dagger\omega}\left|\phi\right\rangle \tag{A.1.11}$$
$$= e^{-\omega}\sum_{n}^{-\omega}\frac{(a^\dagger\omega)^n}{n!}\left|\phi\right\rangle$$

From our definition of a coherent state distribution, derived from definition of arbitrary eigenstate identity revised in Chapter 3. Then,

$$= e^{-\omega}\sum_{n}^{\omega}\frac{\omega^n}{n!}(a^\dagger)^n\left|\phi\right\rangle \tag{A.1.12}$$
$$= w^{-\omega}\sum_{n}^{\omega}\frac{\omega^n}{n!}\left|n\right\rangle = \sum_{n}\frac{(e^{-\omega}\omega^n)}{n!}\left|n\right\rangle$$
$$\rho_n(0) = e^{\omega}\frac{\omega^n}{n!}$$

Where $\rho_n(0)$ denotes the initialised probability density, hence we have recovered a Poisson distribution with respect to an arbitrary coherent state, which implicitly should hold for the functional coherent states used, as constructed definitions are mirrored.

A.1.4 Expected value of probability density via Baker-Campbell Hausdorff Theorem

Here we will show, using standard commutation relations, and the Baker-Campbell Hausdorff theorem, why Equation 2.1.15 is true in a rigorous fashion. We supplicate our definition $\langle 0|\, e^a\mathcal{A}(a^\dagger a)\left|\psi_t\right\rangle$ suggested by Wiese[37]. We will conduct a two-part analysis, on the $A(a^\dagger a)$ and e^a terms, respectively.

We ignore the arbitrary operator $\mathcal{A}$, for simplicity.

$$
\begin{aligned}
|1\rangle\, a^\dagger a\, |\psi_t\rangle &= |1\rangle\, a^\dagger a \sum_n P_n(t)\, |n\rangle \qquad \text{(A.1.13)}\\
&= |1\rangle \sum_n P_n(t) a^\dagger a\, |n\rangle \\
&= |1\rangle \sum_n P_n(t) n\, |n\rangle \\
&= \sum_n n P_n(t)\, \langle 1|n\rangle \\
&= \sum_n n P_n(t)\, |0\rangle\, e^a (a^\dagger)^n\, |n\rangle \\
&= a^\dagger a\, |n\rangle \\
&= a^\dagger a (a^\dagger)^n\, |\phi\rangle \\
&= a^\dagger a a^\dagger \cdots a^\dagger\, |\phi\rangle \\
&= a^\dagger a^\dagger a^\dagger \cdots a^\dagger\, |\phi\rangle \\
&+ a^\dagger a^\dagger a a^\dagger a^\dagger \cdots a^\dagger\, |\phi\rangle \\
&\textit{commutation relation applied to move} \\
&\textit{a across until } \cdots a\, |\phi\rangle \\
&= (a^\dagger)^n\, |\phi\rangle + a^\dagger a^\dagger a^\dagger \cdots a^\dagger\, |\phi\rangle \\
&+ a^\dagger a^\dagger a^\dagger a a^\dagger a^\dagger \cdots a^\dagger\, |\phi\rangle \\
&= (a^\dagger)^n\, |\phi\rangle \\
&+ (a^\dagger)^n\, |\phi\rangle \;\; + (a^\dagger)^3 a (a^\dagger)^{n+1-3} \\
&= (a^\dagger)^n\, |\phi\rangle + (a^\dagger)^n\, |\phi\rangle + \cdots + (a^\dagger)^n\, |\phi\rangle + a^\dagger a^\dagger \cdots a^\dagger a\, |\phi\rangle \\
&= |n\rangle + |n\rangle + \cdots + |n\rangle + 0 \\
&= n\, |n\rangle \\
&\rightarrow a^\dagger a\, |n\rangle = n\, |n\rangle \\
&\mathcal{A}(a^\dagger a)\, |n\rangle = \mathcal{A}(n)\, |n\rangle
\end{aligned}
$$

We now use the Baker-Campbell Hausdorff(BCH) to compact, where the BCH is given by $e^{sX} Y e^{-sX} = Y + s[X, Y]$; with respect to annihilation-creation operators

where s=1;

$$e^a a^\dagger e^{-a} = a^\dagger + [a, a^\dagger] \rightarrow e^a a^\dagger e^{-a} = a^\dagger + 1 \tag{A.1.14}$$
$$e^a a^\dagger = a^\dagger e^a + e^a \rightarrow [e^a, a^\dagger] = e^a$$

Where we have used the definition $e^a a^\dagger = (a^\dagger + 1)e^a$. Then we can presume;

$$e^a a = a e^a \rightarrow e^a a^\dagger = (a^\dagger + 1)e^a \tag{A.1.15}$$
$$e^a (a^\dagger a)^2 = e^a a^\dagger a a^\dagger a$$
$$= (a^\dagger + 1)a(a^\dagger + 1)a e^a$$
$$= ((a^\dagger + 1)a)^2 e^a$$
$$\textit{Applying to arbitrary function}$$
$$= e^a f(a^\dagger a) = f((a^\dagger + 1)a)e^a$$
$$= \sum_n n P_n(t) \left|\phi\right\rangle (1 + a + \frac{a^2}{a} + \cdots + (a^\dagger)^n \left|\phi\right\rangle$$
$$= \sum_n n P_n(t) \left|\phi\right\rangle (a^\dagger)^n + a(a^\dagger)^n + \cdots + \frac{a^n}{n!}(a^\dagger)^n \left|\phi\right\rangle$$
$$= \sum_n n P_n(t) \left|\phi\right\rangle \frac{a^n (a^\dagger)^n}{n!} \left|\phi\right\rangle$$
$$= \sum_n n P_n(t) \frac{\langle n|n\rangle}{n!}$$
$$= \sum n P_n(t) \frac{n!}{n!}$$
$$= \sum n P_n(t)$$

Hence, we have recovered our expectation, defined by our generating function.

A.1.5 Matrix Identities

We utilise the identity stated in Equation 3.2.20, Chapter 3, without rigorous formulation. Here, we show this to be factual, using definitions of coherent states

and standard commutation relations.

$$\begin{aligned}\langle 1|\, a\, |iv\rangle &= \langle iv|\, e^{iv_T a^\dagger} a e^{iva}\, |v\rangle \\ &= \langle \phi|\, e^{iv_T a^\dagger} a\, v_T\, |v\rangle \\ &= e^{iv_T} v_T\, \langle \phi|v\rangle \\ &= e^{iv_T} v_T * 1 \\ &= e^{iv_T}\end{aligned} \tag{A.1.16}$$

For the above to be true, we must also infer $\langle \phi|v\rangle = 1$.

A.1.6 Functional Differentiation

A general functional with the definition $F[K(x)] = e^{\int K(x)g(x)dx}$, can be shown to have the general solution as follows; $\frac{\delta F[K(x)]}{\delta K(y)]} = e^{\int K(x)g(x)dx} g(y)$ given in [7]; with assistance of delta functions, then application to our functional derivative can be shown to be;

$$\mathcal{Z}(J) = \int \mathcal{D}u\ \hat{u}(t) e^{\alpha\omega u_0} = e^{\alpha\omega \int_0^T e^{-\beta s} J(s)} \tag{A.1.17}$$

$$\begin{aligned}\frac{\delta \mathcal{Z}(J)}{\delta J(T)} &= \lim_{\epsilon\to 0} \frac{\mathcal{Z}[J(s) + \epsilon\delta(s-T) - \mathcal{Z}(J)]}{\epsilon} \\ &= \lim_{\epsilon\to 0} \frac{e^{\int J(s) + \epsilon\delta(s-T)\omega e^{-\beta s} - \int_0^T e^{-\beta s} J(s)}}{\epsilon} \\ &= e^{\alpha\omega \int_0^T e^{-\beta s} J(s)} \lim_{\epsilon\to 0} \frac{e^{\epsilon \int \delta(s-T)\omega e^{-\beta s}} - 1}{\epsilon} \\ &= e^{\alpha\omega \int_0^T e^{-\beta s} J(s)} \lim_{\epsilon\to 0} \frac{e^{\epsilon\omega e^{-\beta T}} - 1}{\epsilon} \\ &= \omega e^{-\beta T} e^{\alpha\omega \int_0^T e^{-\beta s} J(s)} \\ &= \left.\frac{\delta \mathcal{Z}(J)}{\delta J(T)}\right|_{J=0} \\ &= \omega e^{-\beta T}\end{aligned} \tag{A.1.18}$$

A.1.7 Completeness

We seek to demonstrate proof of the complete set. We expand polar integral over complex plane yielding,

$$\int \frac{d\phi^* i}{\pi} e^{|\phi|^2} e^{\phi a^\dagger} |0\rangle \langle 0| e^{a\phi^*} = \int \frac{d\phi d\phi^* i}{\pi} e^{|\phi|^2} \sum_{m,n}^{\infty} \frac{\phi^m \phi^{*n}}{m!n!} |m\rangle \langle n| \tag{A.1.19}$$

Where m and n are integer signatures. Then, by definition of the complex conjugates $\phi = re^{i\theta}$; $\phi^* = re^{-i\theta}$, where in polar terms r denotes magnitude, and ϕ and θ specify phase information in the complex plane. Then,

$$\int \frac{d\phi^* i}{\pi} e^{|\phi|^2} e^{\phi a^\dagger} |0\rangle \langle 0| e^{a\phi^*} = \int \frac{d^2 r dr d\theta}{\pi} e^{r^2} \sum_{m,n}^{\infty} \frac{(re^{i\theta})^m (re^{-i\theta})^n}{m!n!} |m\rangle \langle n|$$

$$= \int_0^\infty dr \int_0^{2\pi} d\theta \frac{1}{\pi} e^{r^2} \sum_{m,n}^{\infty} \frac{r^{m+n+1} e^{i(m-n)\theta}}{m!\, n!} |m\rangle \langle n| \tag{A.1.20}$$

Remark, $e^{ik\theta}$ produces zero when evaluated, however, when $k = 0$ $e^{i0} = 1$; therefore θ integral zero where m=n case is exempt. Then gives, $2\pi\delta_{m,n}$. From this,

$$= \int_0^\infty dr \frac{1}{\pi} e^{r^2} \sum_{m,n} \frac{r^{m+n+1} 2\pi \delta_{m,n}}{m!n!} |m\rangle \langle n| \tag{A.1.21}$$

$$= \int_0^\infty dr 2e^{r^2} \sum_n \frac{r^{2n+1}}{(n!)^2} |n\rangle \langle n| \tag{A.1.22}$$

Where the Kronecker delta has been applied. Performing integration by parts by parts on the latter integral, such that $u = r^2$ and $du = 2rdr$;

$$= \int_0^\infty du \sum_n \frac{e^{-u} u^n}{(n!)^2} |n\rangle \langle n| = \sum_n \frac{1}{n!} |n\rangle \langle n| \tag{A.1.23}$$

This has correspondence to standard definition of quantum harmonic oscillators. Therefore,

$$\mathbb{1} = \sum_n \frac{1}{n!} |n\rangle \langle n| = \int \frac{d\phi d\phi^* i}{\pi} e^{-\phi^* \phi} e^{\phi a^\dagger} |0\rangle \langle 0| e^{a\phi^*} \tag{A.1.24}$$

A.1.8 Resolution of Identity equivalency

We seek to show equivalency between Equation 3.1.1 and the standard definition quantum harmonic oscillator complete set, implicitly showing via transitivity the general coherent state resolution of identity ≡ Equation 3.1.1. Mathematically,

$$\int \frac{1}{m!} |m\rangle \langle m| = \int \mathcal{D}u\mathcal{D}v e^{i\int uv} |iv\rangle \langle u|$$

Where,

$$\int \mathcal{D}u\mathcal{D}v \equiv \lim_{\substack{\epsilon\to \\ N\to\infty}} \prod_{i=1/\epsilon}^{N-1} \int \frac{\epsilon}{2\pi} dudv$$

Where u and v are complex numbers. Substituting our definition of a general coherent state into our resolution of identity above, we obtain,

$$\mathbb{1} = \int e^{-iuv} |iv\rangle \langle u| \frac{dudv}{2\pi} \tag{A.1.25}$$

$$\mathbb{1} = \int e^{-iuv} \sum_{m,n} \frac{(iv)^m (a^\dagger)^n}{m!} |0\rangle \langle 0| \frac{a^n u^n}{n!} \frac{dudv}{2\pi} \tag{A.1.26}$$

Application of operators on their respective states allows the form,

$$\mathbb{1} = \sum_{m,n} \frac{1}{m!\, n!} \int e^{-iuv} (iv)^m u^n |m\rangle \langle n| \frac{dudv}{2\pi} \tag{A.1.27}$$

Where we present the above as the following,

$$= \sum_{m,n} \frac{1}{m!\, n!} \left[\int \frac{dudv}{2\pi} e^{-iuv} (iv)^m u^n \right] |m\rangle \langle n| \tag{A.1.28}$$

Then, we may compute integration by parts on the bracketed term[30],

$$\int du\, u^n \left(-\frac{\partial}{\partial u} \right)^m \delta(u) = m!\, \delta_{mn} \tag{A.1.29}$$

Substitution in the above allows,

$$\sum_{m,n} \frac{1}{m!\, n!}\, m!\, \delta_{mn}\ |m\rangle \langle n| \tag{A.1.30}$$

Where instantiation of the Kronecker delta leads us to conclude,

$$\sum_{m} \frac{1}{m!} = |m\rangle \langle m| \tag{A.1.31}$$

We have derived our standard definition of the quantum harmonic oscillator. Thus, proving implicitly the validity of the coherent state resolution of identity applied.

B

Glossary

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ab initio from the beginning, or from first principles[21]

adjoint also known as Hermitian adjoint in Physics; adjoints of operators are defined as the conjugate transpose of a square matrix[25]

conjugate transpose If a matrix, A, has dimensions n x n, the conjugate transpose is the result of taking the conjugate, and then transposing each element[36]

double slit experiment An experiment performed in 1801 by Thomas Young, to show the behaviour of light[38]

eigenstate Solutions to a wavefunction of a quantum mechanical state[16]

eigenvalue Reference to coefficients of the solutions of the wavefunction[16]

ket bra-ket notation developed for the mathematics of quantum mechanics[8]

observable An operator defined to a real physical quantity, equivalent to the hermitian conjugate

Riemann Sum Mathematical formulation of conversion of a sum to an integral [24].

Schrodinger equation An equation describing the wavefunction of a quantum mechanical system[32].