MATH2130 FYTT 1

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1 Intermediate Value Theorem

Show that $f(x) = cos(x) - \sqrt{x}$ has a root in the interval $[0, 1]$.

$f(x)$ is continuous on the interval $[0, 1]$ because $cos(x)$ is continuous everywhere and $\sqrt{x}$ is continuous on the interval $[0, \infty)$.

$f(0) = cos(0) - \sqrt{0} = 1 - 0 = 1$
$f(1) = cos(1) - \sqrt{1} \approx 0.5403 - 1 \approx -0.5403$

Since $-0.5403 < 0 < 1$, by the IVT there exists a $c \in (0, 1)$ such that $f(c) = 0$.

2 Intermediate Value Theorem

Find an interval on which $f(x) = x^4 - 3x^3 + x^2 + 3x - 7$ has a root.

We must find a x_1 and a x_2 such that $f(x_1) < 0$ and $f(x_2) > 0$.

$x = 0$: $f(0) = (0)^4 - 3(0)^3 + (0)^2 + 3(0) - 7 = -7 = x_1$
$x = 3$: $f(3) = (3)^4 - 3(3)^3 + (3)^2 + 3(3) - 7 = 11 = x_2$

Since $-7 < 0 < 11$, by the IVT there exists a $c \in (0, 3)$ such that $f(c) = 0$.

3 Mean Value Theorem

Verify the Mean Value Theorem for $f(x) = \sqrt{2x-1}$ on the interval $x \in [1, 5]$.

$f(x)$ is continuous and differentiable on the interval $[\frac{1}{2}, \infty)$. The slope of the secant between $(1, f(1))$ and $(5, f(5))$ is given by:

$$\frac{f(5)-f(1)}{5-1}$$

$$= \frac{\sqrt{2(5)-1}-\sqrt{2(1)-1}}{4}$$

$$= \frac{\sqrt{9}-\sqrt{1}}{4}$$

$$= \frac{2}{4}$$

$$= \frac{1}{2}$$

The slope of the tangent of $f(x)$ is given by its derivative. $f'(x) = \frac{1}{2}(2x-1)^{-\frac{1}{2}} \cdot 2 = \frac{1}{\sqrt{2x-1}}$.

$$\frac{1}{\sqrt{2x-1}} = \frac{1}{2}$$

$$\Rightarrow \sqrt{2x-1} = 2$$

$$\Rightarrow 2x - 1 = 4$$

$$\Rightarrow 2x = 5$$

$$\Rightarrow x = 5/2$$

Since $1 < 5/2 < 5$, this is our c in the Mean Value Theorem.

4 Mean Value Theorem

Verify the Mean Value Theorem for $f(x) = \frac{1}{x}$ on the interval $x \in [2, 3]$.

$f(x)$ is continuous and differentiable on the interval $(0, \infty)$. The slope of the secant between $(2, f(2))$ and $(3, f(3))$ is given by:

$$\frac{f(3)-f(2)}{3-2}$$

$$= \frac{\frac{1}{3}-\frac{1}{2}}{1}$$

$$= -\frac{1}{6}$$

The slope of the tangent of $f(x)$ is given by its derivative. $f'(x) = -\frac{1}{x^2}$

$$-\frac{1}{x^2} = -\frac{1}{6}$$

$$\Rightarrow x^2 = 6$$

$$\Rightarrow x = \sqrt{6} \approx 2.45$$

Since $2 < \sqrt{6} < 3$, this is our c value in the Mean Value Theorem.

5 Taylor's Theorem

Some commonly-used Taylor expansions are as follows:

1. About $x = 0$, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
2. About $x = 0$, $cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
3. About $x = 0$, $sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
4. About $x = 0$, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$
5. About $x = 1$, $ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$

It is useful to have these on-hand or to have the ability to quickly derive them as needed! Research or recall how to derive each of these series using basic calculus.

1)
General form of a taylor series expansion:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...$$

Choose to expand the series around $a = 0$:

$$f(x) = f(0) + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + ...$$

Simplify:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$$

All order derivatives are equal to e^x, and plugging 0 into $f(x)$ and into all the derivatives are equal to 1:

$$f(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2)
General form of a taylor series expansion:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...$$

Choose to expand the series around $a = 0$:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$$

Take the derivatives at x = 0:

$$f(x) = cos(x) \Rightarrow f(0) = cos(0) = 1$$

$$f'(x) = -sin(x) \Rightarrow f'(0) = -sin(0) = 0$$

$$f''(x) = -cos(x) \Rightarrow f''(0) = -cos(0) = -1$$

$$f'''(x) = sin(x) \Rightarrow f'''(0) = sin(0) = 0$$

$$f^{(4)}(x) = cos(x) \Rightarrow f^{(4)}(0) = cos(0) = 1$$

This results in a cycle of $\{1, 0, -1, 0\}$ repeats itself. Resulting in the following series.

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$

3)
The taylor series expansion around for $sin(x)$ is the same as $cos(x)$ except we add one to the $2n$ terms:

$$cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$

$$sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

4)
General form of a taylor series expansion around $a = 0$:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$$

Find derivatives and evaluate at $x = 0$:

$$f(0) = 1$$

$$f'(x) = (1-x)^{-2} \Rightarrow f'(0) = 1$$

$$f''(x) = 2(1-x)^{-3} \Rightarrow f''(0) = 1$$

$$f'''(x) = 6(1-x)^{-4} \Rightarrow f'''(0) = 2$$

$$f^4(x) = 24(1-x)^{-5} \Rightarrow f^4(0) = 24$$

Plug into the general form:

$$f(x) = 1 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \frac{24}{4!}x^4$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + ...$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

5)
General form of a taylor series expansion:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...$$

Find derivatives and evaluate at $x = 0$:

$$f(x) = ln(x) \Rightarrow f(1) = ln(1) = 0$$

$$f'(x) = \frac{1}{x} \Rightarrow f'(1) = \frac{1}{1} = 1$$

$$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -\frac{1}{(1)^2} = -1$$

$$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = \frac{2}{(1)^3} = 2$$

$$f^{(4)} = -\frac{(2)(3)}{x^4} \Rightarrow f^{(4)}(1) = -\frac{(2)(3)}{(1)^4} = -(2)(3)$$

We can see a pattern forming, so we can plug in this information into our formula:

$$0 + 1(x-1) - \frac{1(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{(2)(3)(x-1)^4}{4!}...$$

Writing this as a series:

$$ln(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{((n-1)!)(x-1)^n}{n!}$$

$$\Rightarrow ln(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$$

6 Taylor's Theorem

Using the formula for error $R_n(x)$ in the Taylor Polynomial approximation which was presented in class, we know that the difference between $f(x) = sin(x)$ and $P_n(x)$ can be expressed as:

$$sin(x) - P_n(x) = R_n(x) = \frac{f^{(n+1)}(c)x^{n+1}}{(n+1)!}$$

for some c between 0 and x. Given that $x \in [0, \pi/2]$, it follows that $c \in (0, \pi/2)$. To ensure that n is sufficiently large so that $P_n(x)$ is within 10^{-5} of $sin(x)$, we aim to select n large enough so that:

$$|sin(x) - P_n(x)| = |\frac{f^{(n+1)}(c)x^{n+1}}{(n+1)!}| = \frac{|f^{(n+1)}(c)||x|^{n+1}}{(n+1)!} < 10^{-5}$$

for all $x \in [0, \pi/2]$ and $c \in (0, \pi/2)$. To establish a bound $f^{n+1}(c)$, we remark that the derivatives of $f(x) = sin(x)$ follow a pattern described by the following formula:

$$f^{(n+1)}(x) = \begin{cases} cos(x) & n = 0, 4, 8... \\ -sin(x) & n = 1, 5, 9... \\ -cos(x) & n = 2, 6, 10... \\ sin(x) & n = 3, 7, 11... \end{cases}$$

Given that $| \pm sin(x)| \leq 1$ and $| \pm cos(x)| \leq 1$ for all $x \in \mathbb{R}$, it follows that $|f^{n+1}(c)| \leq 1$ for any $c \in (0, \pi/2)$. Furthermore, for any integer $n \geq 1$, $|x|^{n+1}$ is an increasing function of x on the interval $[0, \pi/2]$, so we know that $|x|^{n+1} \leq (\pi/2)^{n+1}$ for all $x \in [0, \pi/2]$. Combining these observations, we have that:

$$|sin(x) - P_n(x)| = \frac{|f^{(n+1)}(c)||x|^{n+1}}{(n+1)!} \leq \frac{|x|^{n+1}}{(n+1)!} \leq \frac{(\pi/2)^{n+1}}{(n+1)!}$$

for all $x \in [0, \pi/2]$ and $x \in (0, \pi/2)$. We now sek the smallest value of n such that the upper bound on the right-hand side of the above inequality is less than 10^{-5}. Checking a few values, we see that:

$$\frac{(\pi/2)^{10}}{(10)!} \approx 2.5 \times 10^{-5} > 10^{-5},$$

and

$$\frac{(\pi/2)^{11}}{(11)!} \approx 3.6 \times 10^{-6} < 10^{-5},$$

so that $n = 10$ is he first value that works. To recap, we have found that for any $n \geq 10$, it holds that $P_n(x)$ is within 10^{-5} of $sin(x)$ for all $x \in [0, \pi/2]$.