Discrete Structures

CSE 2315 (Spring 2018)

Lecture 19 Counting

Dajiang Zhu, Ph.D.
Department of Computer Science and Engineering

Inclusion and Exclusion

- Give a formula for $|A \cup B|$ in terms of $|A|$, $|B|$ and $|A \cap B|$?
- Give a formula for $|A \cup B \cup C|$ in terms of $|A|$, $|B|$, $|C|$ and related sets?
- Solution

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

$$A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$

- The Inclusion-Exclusion Principle

$$|A_1 \cup A_2 \cup \ldots A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \ldots (-1)^{n+1} |A_1 \cap A_2 \ldots A_n|$$

- Can be proved using induction. Beyond scope of course.

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One More Example

- 35 voters were queried about their opinions regarding two referendums. 14 supported referendum 1 and 26 supported referendum 2. How many voters supported both, assuming that every voter supported either referendum 1 or referendum 2 or both?
- Solution

Let $A \equiv$ voters who supported referendum 1 and $B \equiv$ voters who supported referendum 2. Then, we have, $|A \cup B| = 35$, $|A| = 14$ and $|B| = 26$. Using the Inclusion-Exclusion principle,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ \rightarrow |A \cap B| &= |A| + B| - |A \cup B| \\ &= 14 + 26 - 35 \\ &= 5 \end{aligned}$$

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Applying the Inclusion-Exclusion Principle

- Consider a group of students ordering pizza. 13 will eat sausage topping, 10 will eat pepperoni, 12 will eat pineapple, 4 will eat both sausage and pepperoni, 5 will eat both pepperoni and pineapple, 7 will eat both sausage and pineapple and 3 will eat all three toppings. How many students are there in the group?
- Solution

Let $A \equiv$ students who eat sausage, $B \equiv$ students who eat pepperoni and $C \equiv$ students who eat pineapple. Accordingly,

$|A| = 13,\ |B| = 10,\ |C| = 12,\ |A \cap B| = 4,\ |B \cap C| = 5,\ |A \cap C| = 7, |A \cap B \cap C| = 3.$

Using the Inclusion-Exclusion principle,

$$|A \cup B \cup C| = 13 + 10 + 12 - (4 + 5 + 7) + 3 = 22$$

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Pigeonhole Principle

- If more than k items are placed in k bins, then at least one bin contains more than one item.
- How many times should a die be tossed before you can be certain that the same value shows up twice?

  Solution: 7.
- Show that if 51 positive integers between 11 and 100 are chosen, then one of them must divide the other.

  Solution: Every number can be expressed as a product of prime numbers. Let $n_1, n_2, \ldots, n_{51}$ denote the chosen numbers. Therefore, each $n_i = 2^{ki} . b_i$, where b_i is some odd number, such that $1 <= b_i <= 99$.

  But there are exactly 50 odd numbers between 1 and 99.

  Therefore, $b_i = b_j$, for some pair (n_i, n_j) (pigeonhole principle).

  In other words, we must have $n_i = 2^{ki} . b_i$ and $n_j = 2^{kj} . b_j$.

  Depending on whether $k_i >= k_j$ or vice versa, one of n_i and n_j must divide the other.