Discrete Structures

CSE 2315 (Spring 2018)

Lecture 17 Set

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Set Fundamentals

- A set is an **unordered** collection of objects
- The fundamental question in set theory is membership, i.e., does object x belong to set A. This is denoted as: does $x \in A$?
- Two sets are equal, if they contain the same elements. Logically,

$$A = B \Rightarrow (\forall x)[x \in A \leftrightarrow x \in B]$$

Fundamentals

- Representing Sets
  - The extensional method - Explicitly enumerate all the elements of the set; e.g., A = {1, 5, 7}, B = {1, 2, 3, …,100}, C = {red, white, blue}.
  - The intensional method - Specify a property P that characterizes the set elements; e.g., A = {x| x is an integer less than 7, but at least 3}.
  - Recursion - We can describe the set of all even positive integers as follows: (a) $2 \in S$. (b) if $x \in S$, then so is $x + 2$.
- Three ways to describe a sets?
  - List its elements
  - Use recursion to describe how to generate the set elements
  - Describe a property P that characterizes the set elements

Fundamentals

- Some important sets
  - N - The set of non-negative integers {0, 1, …… }.
  - Z - The set of all integers {…, -1, 0, 1, …}.
  - Q - The set of all rational numbers.
  - R - The set of all real numbers.
  - C - The set of all complex numbers.
  - {} or $\emptyset$ - The set with no elements or null set.

- $\{\emptyset\} = \emptyset$?

Relationships

- A is said to be a subset of B, denoted by

$$A \subseteq B, \text{ if } (\forall x)[x \in A \rightarrow x \in B]$$

- A is said to be a proper subset of B, denoted by

$$A \subset B, \text{ if } A \subseteq B, \text{ but } A \neq B$$

- Example
  - The statement $\emptyset \subseteq C$ is always **true**, since the statement $(\forall x)(x \in \phi \rightarrow x \in C)$ is vacuously **true**.
  - Let A = {x | x is a multiple of 8} and B = {x | x is a multiple of 4}. Show that A $\subseteq$ B.

Power Set

- The set of all possible subsets of a set S is called its **power set** and denoted by P(S)

- Example

  Let $S = \{0, 1\}$. $P(S) = \{\phi, \{0\}, \{1\}, \{0, 1\}\}$.

- Exercise

  Show that if a set has n elements, then its power set will have 2^n elements

Binary Operations

- $\circ$ is a binary operation on a set S, if for every ordered pair (x, y) of S, $x \circ y$ exists, is unique, and is a member of S. The properties "exists" and "is unique" are collectively referred to as the property of being "**well-defined**"; the property that $x \circ y \in S$ is called the **closure** property.
- Example

Is + an operation on N?

Is – an operation on N? Z?

Is ÷ an operation on N? $x \circ y = 1$, if $x>=5$; $x \circ y = 0$, if $x<=5$

Unary Operations

- # is said to be a unary operation on S, if for all $x \in S$, $x^{\#}$ is well-defined and S is closed under #.

- The operation $x^{\#} = -x$ is a unary operation on Z, but not on N.

- The operation $x^{\#} = (x)^{1/2}$ is not a unary operation on N, Z; but it is a unary operation on R_{+}.

Operations on Sets

- For discussing operations on sets, we assume the existence of a ground set S and its power set P(S). All operations are defined on the elements of P(S); P(S) is called the universal set or the universe of discourse.

- Principal Operations

Let $A, B \in \mathcal{P}(S)$, i.e., A and B are subsets of S.

(i) $A \cup B$ (**union**) is defined as: $\{x \mid x \in A$ **or** $x \in B\}$.

(ii) $A \cap B$ (**intersection**) is defined as: $\{x \mid x \in A$ **and** $x \in B\}$.

(iii) A' (**complement**) is defined as : $\{x \mid x \in S$ **and** $x \notin A\}$.

(iv) $A - B$ (**difference**) is defined as: $\{x \mid x \in A$ **and** $x \notin B\}$.

(v) $A \times B$ (**Cartesian Product**) is defined as: $\{(x, y) \mid x \in A$ **and** $y \in B\}$.

Examples

Let $A = \{1, 2, 3\}$ and $B = \{a, b, 1\}$. Compute $A \cup B$, $A \cap B$, $A - B$, $A \times B$ and $B \times A$.

What is $(A \cup B)$, $(A \cap B)$, $(A\text{-}B)$, $(A \times B)$, $(B \times A)$?

$$
\begin{aligned}
&A \cup B = \{1, 2, 3, a, b\}, \\
&A \cap B = \{1\}, \\
&A - B = \{2, 3\}, \\
&A \times B = \{(1, a), (1, b), (1, 1), (2, a), (2, b), (2, 1), (3, a), (3, b), (3, 1)\}, \\
&B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (1, 1), (1, 2), (1, 3)\}.
\end{aligned}
$$

- Note
  - Two set A and B such that $A \cap B = \emptyset$ are said to be **disjoint**.
  - $A \times A$ is referred to as A^2, $A \times A \times A$ as A^3 and so on.

Set Identities

- Recall that all sets under discussion are subsets of the ground set S.

$$\text{Commutative}: \begin{cases} A \cup B = B \cup A \\ A \cap B = B \cap A \end{cases}$$

$$\text{Associative}: \begin{cases} (A \cup B) \cup C = A \cup (B \cup C) \\ (A \cap B) \cap C = A \cap (B \cap C) \end{cases}$$

$$\text{Distributive}: \begin{cases} A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \end{cases}$$

$$\text{Identity}: \begin{cases} A \cup \emptyset = A \\ A \cap S = A \end{cases}$$

$$\text{Complement}: \begin{cases} A \cup A' = S \\ A \cap A' = \emptyset \end{cases}$$

Proving Set Identities

Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Observe that,

$$
\begin{aligned}
x \in A \cup (B \cap C) \quad &\rightarrow \quad x \in A \textbf{ or } x \in (B \cap C) \\
&\rightarrow \quad (x \in A) \textbf{ or } (x \in B \textbf{ and } x \in C) \\
&\rightarrow \quad (x \in A \textbf{ or } x \in B) \textbf{ and } (x \in A \textbf{ or } x \in C) \\
&\rightarrow \quad (x \in A \cup B) \textbf{ and } (x \in A \cup C) \\
&\rightarrow \quad x \in (A \cup B) \cap (A \cup C)
\end{aligned}
$$

Simply reverse the argument to show that every element in the set represented by the RHS is also an element of the set represented by the LHS.

Proving Set Identities

- Show that

$$[A \cup (B \cap C)] \cap ([A' \cup (B \cap C)] \cap (B \cap C)') = \emptyset$$

- Solution

$$\begin{aligned}
& [A \cup (B \cap C)] \cap ([A' \cup (B \cap C)] \cap (B \cap C)') \\
&= ([A \cup (B \cap C)] \cap [A' \cup (B \cap C)]) \cap (B \cap C)' \text{ Associativity} \\
&= ([(B \cap C) \cup A] \cap [(B \cap C) \cup A']) \cap (B \cap C)' \text{ Commutativity} \\
&= ([(B \cap C) \cup (A \cap A')]) \cap (B \cap C)' \text{ Distributivity} \\
&= [(B \cap C) \cup \emptyset] \cap (B \cap C)' \text{ complement} \\
&= (B \cap C) \cap (B \cap C)' \text{ identity} \\
&= \emptyset \text{ complement}
\end{aligned}$$

Three approaches to proving set identities

- Draw a Venn diagram
- Establish set inclusion in each direction
  - take an arbitrary member of one side and show it belongs to the other side, and conversely.
- Use already proved identities
  - be sure to match the pattern of the identity you want to use.

Countable and Uncountable Sets

- The number of elements in a set S is called its **cardinality**.
- A set S is said to be **finite**, if $|S| = k$, for some $k \in N$.
- A set S is said to be **denumerable**, if its cardinality is ∞, but its elements can be enumerated in some order. e.g., N, Z^+, Z^-, Z and so on.
- A set S is said to be **countable** if it is either finite or denumerable. Otherwise, it is said to be **uncountable**.

Countability

- Is the set Q^+ (positive rationals) countable?

- Solution

$$\begin{bmatrix} 1/1, & 1/2, & 1/3, & 1/4, \ldots \\ 2/1, & 2/2, & 2/3, & 2/4, \ldots \\ 3/1, & 3/2, & 3/3, & 3/4, \ldots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$